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CRYSTAL BASES FOR THE QUANTUM SUPERALGEBRA $U_q(\mathfrak{gl}(m,n))$

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Contents

1. Introduction

The quantized enveloping algebras $U_q(\mathfrak{g})$ of symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$ play a prominent role in two-dimensional solvable lattice models. The parameter q corresponds to the temperature in the lattice model. Since $q = 0$ corresponds to absolute zero temperature, one expects special behavior at this particular value.

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Associated with each integrable $U_q(\mathfrak{g})$-module M, there is a remarkable basis at $q=0$, the *crystal base*, which was introduced by Kashiwara in [7]. If A denotes the local ring of all rational functions $f/g \in \mathbf{Q}(q)$ with $g(0) \neq 0$, then M contains an A-lattice L, called the *crystal lattice*. The crystal base is a certain basis B for the $\mathbf{Q}$-vector space L/qL which possesses many striking features. It is preserved under the action of the modified root vector operators $\tilde{e}_i$ and $\tilde{f}_i$ (what are often called *Kashiwara operators*). It is well-behaved with respect to tensor products, and it has important connections with combinatorial bases of tableaux (see [9], [13], [6], and [12]).

Our goal in this work is to develop a crystal base theory for one of the most fundamental Lie superalgebras—the general linear Lie superalgebra $\mathfrak{gl}(m,n)$. Suppose $V = V_0 \oplus V_1$ is a $\mathbf{Z}_2$-graded vector space such that $\dim V_0 = m$ and $\dim V_1 = n$. For $a = 0, 1$, let

$$\mathrm{End}(V)_a = \{x \in \mathrm{End}(V) \mid xV_b \subseteq V_{a+b}\}$$

(subscripts are read mod 2). Then $\mathfrak{gl}(m,n)$ is $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ regarded as a Lie superalgebra under the supercommutator product

$$[x,y] = xy - (-1)^{ab}yx, \qquad x \in \mathrm{End}(V)_a,\ y \in \mathrm{End}(V)_b,$$

and V is the simplest representation of $\mathfrak{gl}(m,n)$. Tensor powers of V have been shown to be completely reducible $\mathfrak{gl}(m,n)$-modules (see [2]). In that same paper, Berele and Regev introduced tableau bases for the simple summands and showed that the characters of these simple modules have a combinatorial interpretation as hook Schur functions.

Corresponding to the Lie superalgebra $\mathfrak{gl}(m,n)$ is its quantized enveloping algebra $U_q(\mathfrak{gl}(m,n))$ which is a Hopf superalgebra. The fundamental representation of $U_q(\mathfrak{gl}(m,n))$ is its $(m+n)$-dimensional vector representation $\mathbf{V}$ which is the analogue of the $\mathfrak{gl}(m,n)$-module V. We prove that the tensor powers of the $U_q(\mathfrak{gl}(m,n))$-module $\mathbf{V}$ are completely reducible, and their irreducible summands are indexed by partitions having what is called an (m,n)-hook shape. Such a partition corresponds to a frame or Young diagram Y, and a crystal base for the module is indexed by the set $B(Y)$ consisting of the semistandard tableaux with diagram Y. We give $B(Y)$ a crystal structure by an admissible reading and show the crystal is connected. We obtain an explicit description of the isomorphism $\mathbf{B} \otimes B(Y) \cong B(Y) \otimes \mathbf{B}$ (here $\mathbf{B}$ is the crystal base for $\mathbf{V}$) by the bumping procedure (and its reverse) described in Section 4.

Our approach to developing the crystal base theory of $\mathfrak{gl}(m,n)$ is closely akin to that adopted in [9], [13], [6], and [12]. Explicit crystal bases are given in terms of tableaux for the quantized enveloping algebras of Lie algebras of types A_n, B_n, C_n, and D_n (in [9]), of type G_2 (in [6]), and for the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ (in [13]). The crystal construction in [9] has enabled Nakashima [15] to prove generalized Littlewood-Richardson rules for tensor product decompositions. Littelmann's realization of crystal bases in terms of generalized tableaux for the Lie algebras of types A_n, B_n, C_n, D_n, E_6, and G_2 has also yielded generalized Littlewood-Richardson rules for these algebras (see [12]).

The superalgebra case addressed in this work presents new and challenging difficulties not encountered in the Lie algebra case. In general, representations for Lie superalgebras need not be completely reducible. In order to overcome this obstacle, we restrict our study to a certain class of representations of $U_q(\mathfrak{gl}(m,n))$ stable under tensor products. The existence of what we term "fake" highest and lowest weight vectors creates additional problems. In [19], Zou has constructed a crystal base theory for the quantum superalgebra $U_q(\mathfrak{sl}(2,1))$. However, it should be noted that Zou's notion of a crystal base in that paper, which was designed to circumvent some of the superalgebra difficulties, differs from the one adopted here (and in [7], [8], [9], [13], [6], and [3]), since his base is invariant under some but not all of the Kashiwara operators.

In recent work [14], Musson and Zou have developed a comprehensive crystal base theory for the orthosymplectic Lie superalgebras $\mathfrak{osp}(1,2r)$ using the more standard definition of a crystal base, but they do not adopt a tableau approach in their construction. Tableau bases for irreducible $\mathfrak{osp}(1,2r)$-modules are known (see [1], [11]), and it seems likely this case could also be handled by the same methods as in our paper. The algebras $\mathfrak{osp}(1,2r)$ are singular in superalgebra theory, because they are the only simple Lie superalgebras whose finite-dimensional modules are completely reducible. It was observed in [16] that the finite-dimensional irreducible modules for $\mathfrak{osp}(1,2r)$ have many similarities with the nonspinor irreducible modules of the orthogonal Lie algebra $\mathfrak{o}(2r+1)$ (of type B_r). In fact, the tableaux defined by Sundaram in [17] can be used to index a basis of both. It is interesting to ask if the tableaux developed in [1] (which reduce to those in [17] when $m=1$) can be used to construct a crystal base for tensor representations of the orthosymplectic Lie superalgebras $\mathfrak{osp}(m,2r)$.

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2. Quantum superalgebras

2.1. **Definition.** We begin by introducing the q-analogue of the universal enveloping algebra for a Lie superalgebra in terms of the Chevalley generators.

The set I will be the index set for the simple roots. It is assumed to be divided into two parts corresponding to the even simple roots and the odd simple roots:

$$I = I_{\text{even}} \sqcup I_{\text{odd}}.$$

Set $p(i) = 0$ or 1 according to whether $i \in I_{\text{even}}$ or $i \in I_{\text{odd}}$.

Let P be a free $\mathbf{Z}$-module (of integral weights) with a $\mathbf{Q}$-valued symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$. To each $i \in I$, the simple root $\alpha_i \in P$ and the simple coroot $h_i \in P^*$ are given as data, and, relative to the natural pairing $\langle \cdot, \cdot \rangle$ between P and P^*, they are assumed to satisfy

$$\begin{aligned} &\langle h_i, \alpha_i \rangle = 2 \quad \text{if } i \in I_{\text{even}}, \\ &\langle h_i, \alpha_i \rangle = 0 \text{ or } 2 \quad \text{if } i \in I_{\text{odd}}, \\ &\langle h_i, \alpha_j \rangle \le 0 \quad \text{if } j \ne i. \end{aligned} \tag{2.1}$$

We suppose that there are nonzero integers ℓ_i so that

$$\ell_i \langle h_i, \lambda \rangle = (\alpha_i, \lambda) \quad \text{for any } \lambda \in P. \tag{2.2}$$

In particular, since $\ell_i\langle h_i, \alpha_j\rangle = (\alpha_i, \alpha_j) = (\alpha_j, \alpha_i) = \ell_j\langle h_j, \alpha_i\rangle$, the Cartan matrix of values $\langle h_i, \alpha_j\rangle$, $i, j \in I$, is symmetrizable.

Let $\mathfrak{g}$ denote the contragredient Lie superalgebra corresponding to this data as in [4] and [5]. We now introduce the q-analogue of the universal enveloping algebra of $\mathfrak{g}$ (compare [10] and [18]). Assume q is an indeterminate, and set $q_i = q^{\ell_i}$. The associated quantized enveloping algebra $U_q'(\mathfrak{g})$ is the unital associative algebra over $\mathbf{Q}(q)$ with generators e_i, f_i $(i \in I)$, q^h $(h \in P^*)$, which satisfy the following defining relations:

$$\begin{aligned}
&q^h = 1 \quad \text{for } h = 0,\\
&q^{h_1+h_2} = q^{h_1}q^{h_2} \quad \text{for } h_1, h_2 \in P^*,\\
&q^h e_i = q^{\langle h,\alpha_i\rangle} e_i q^h \quad \text{for } h \in P \text{ and } i \in I,\\
&q^h f_i = q^{-\langle h,\alpha_i\rangle} f_i q^h \quad \text{for } h \in P \text{ and } i \in I,\\
&e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1}) \quad \text{for } i, j \in I,\\
&\text{where } t_i = q^{\ell_i h_i}.
\end{aligned} \tag{2.3}$$

We assume further:

$$\begin{aligned}
&\text{If } a \in U_q^{++}(\mathfrak{g}) \text{ satisfies } f_i a \in U_q^{+}(\mathfrak{g}) f_i \text{ for all } i, \text{ then } a = 0.\\
&\text{If } a \in U_q^{--}(\mathfrak{g}) \text{ satisfies } e_i a \in U_q^{-}(\mathfrak{g}) e_i \text{ for all } i, \text{ then } a = 0.
\end{aligned} \tag{2.4}$$

Here $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) is the subalgebra of $U_q'(\mathfrak{g})$ generated by the e_i's (resp. f_i's), and $U_q^{++}(\mathfrak{g})$ (resp. $U_q^{--}(\mathfrak{g})$) is the ideal of $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) generated by the e_i's (resp. f_i's).

In order to define the Hopf algebra structure, we introduce the parity operator σ on $U_q'(\mathfrak{g})$, which is defined by $\sigma(e_i) = (-1)^{p(i)} e_i$, $\sigma(f_i) = (-1)^{p(i)} f_i$, for all $i \in I$, and $\sigma(q^h) = q^h$ for all $h \in P^*$. It is easily seen from (2.3) that σ extends to an automorphism of $U_q'(\mathfrak{g})$ with $\sigma^2 = 1$. Then $U_q(\mathfrak{g}) = U_q'(\mathfrak{g}) \oplus U_q'(\mathfrak{g})\sigma$ is the algebra (the skew group algebra over $U_q'(\mathfrak{g})$) with multiplication given by $\sigma^2 = 1$ and $\sigma x \sigma = \sigma(x)$ for any $x \in U_q'(\mathfrak{g})$. Now $U_q(\mathfrak{g})$ has a Hopf algebra structure whose comultiplication is the algebra homomorphism $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ specified by

$$\begin{aligned}
\Delta(\sigma) &= \sigma \otimes \sigma,\\
\Delta(q^h) &= q^h \otimes q^h,\\
\Delta(e_i) &= e_i \otimes t_i^{-1} + \sigma^{p(i)} \otimes e_i,\\
\Delta(f_i) &= f_i \otimes 1 + \sigma^{p(i)} t_i \otimes f_i.
\end{aligned} \tag{2.5}$$

The antipode S is therefore given by

$$\begin{aligned}
S(\sigma) &= \sigma,\\
S(q^h) &= q^{-h},\\
S(e_i) &= -\sigma^{p(i)} e_i t_i,\\
S(f_i) &= -\sigma^{p(i)} t_i^{-1} f_i,
\end{aligned} \tag{2.6}$$

and the counit by

$$\begin{aligned}
\varepsilon(\sigma) &= 1 = \varepsilon(q^h),\\
\varepsilon(e_i) &= 0 = \varepsilon(f_i).
\end{aligned} \tag{2.7}$$

2.2. **Polarization.** The anti-automorphism η of $U_q(\mathfrak{g})$ determined by

$$\begin{aligned}
\eta(\sigma) &= \sigma,\\
\eta(q^h) &= q^h,\\
\eta(e_i) &= q_i f_i t_i^{-1},\\
\eta(f_i) &= q_i^{-1} t_i e_i
\end{aligned} \tag{2.8}$$

satisfies $\eta^2 = \mathrm{id}$. We say that a symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ on a $U_q(\mathfrak{g})$-module M is a *polarization* if $(au, v) = (u, \eta(a)v)$ holds for any $u, v \in M$ and $a \in U_q(\mathfrak{g})$.

The next lemma is an easy consequence of the following relation:

$$\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta. \tag{2.9}$$

Lemma 2.1. *Let M_1 and M_2 be two $U_q(\mathfrak{g})$-modules with polarizations. Then the symmetric bilinear form $(\cdot\,,\cdot)$ on $M_1 \otimes M_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2)$ is a polarization.*

2.3. **Crystal base.** We restrict ourselves to the case that $\langle h_i, \alpha_i\rangle = 0$ for any $i \in I_{\mathrm{odd}}$. Note that for such an i, we have $e_i^2 = f_i^2 = 0$. Indeed, it follows from (2.3) that $[f_j, e_i^2] = [e_j, f_i^2] = 0$ for any $j \in I$, and then (2.4) implies $e_i^2 = f_i^2 = 0$.

For $i \in I_{\mathrm{even}}$, let $U_q(\mathfrak{g})_i$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i and t_i. This algebra is isomorphic to the quantized enveloping algebra $U_{q_i}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. We consider the following class of $U_q(\mathfrak{g})$-modules.

Definition 2.2. $\mathcal{O}_{\mathrm{int}}$ is the category of $U_q(\mathfrak{g})$-modules M and $U_q(\mathfrak{g})$-linear homomorphisms satisfying the following conditions:

(i) M has a weight decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$, where $M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda\rangle} u \quad \text{for any } h \in P^*\}$.
(ii) $\dim M_\lambda < \infty$ for any $\lambda \in P$.
(iii) For any $i \in I_{\mathrm{even}}$, M is locally $U_q(\mathfrak{g})_i$-finite (i.e. $\dim U_q(\mathfrak{g})_i u < \infty$ for any $u \in M$).
(iv) For any $i \in I_{\mathrm{odd}}$ and $\mu \in P$, $M_\mu \neq 0$ implies $\langle h_i, \mu\rangle \geq 0$.
(v) $e_i M_\mu = f_i M_\mu = 0$ for any $\mu \in P$ and $i \in I_{\mathrm{odd}}$ such that $\langle h_i, \mu\rangle = 0$.

The category $\mathcal{O}_{\mathrm{int}}$ is stable under taking subquotients and tensor products.

We conjecture that modules in $\mathcal{O}_{\mathrm{int}}$ are completely reducible whenever I is finite.

As in the Lie algebra case, the weights of the module in $\mathcal{O}_{\mathrm{int}}$ are invariant under the action of the Weyl group W. Here the Weyl group W is the subgroup of $\mathrm{Aut}(P)$ generated by the simple reflections r_i ($i \in I_{\mathrm{even}}$), where

$$r_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i.$$

We now define the modified operators (often referred to as Kashiwara operators) $\tilde{e}_i$ and $\tilde{f}_i$ on the modules M in $\mathcal{O}_{\mathrm{int}}$. They are defined so that $\tilde{e}_i$ and $\tilde{f}_i$ are transpose to each other at $q = 0$ with respect to a polarization (see Proposition 2.9).

First let us consider the case $i \in I_{\mathrm{even}}$. For any $u \in M$ of weight $\lambda \in P$, there is a unique expression

$$u = \sum_{k \geq 0, -\langle h_i, \lambda\rangle} f_i^{(k)} u_k$$

with $e_i u_k = 0$ for each k. Here

$$f_i^{(n)} = \frac{1}{[n]_i!} f_i^n, \tag{2.10}$$

where

$$\begin{aligned} [n]_i &= (q_i^n - q_i^{-n})/(q_i - q_i^{-1}), \\ [n]_i! &= \prod_{k=1}^{n} [k]_i \quad \text{for } n \geq 1, \quad \text{and} \quad [0]! = 1. \end{aligned}$$

Case (1): *i even and $(\alpha_i, \alpha_i) > 0$ (equivalently, $\ell_i > 0$).* We define

$$\begin{aligned} \tilde{e}_i u &= \textstyle\sum_k f_i^{(k-1)} u_k, \\ \tilde{f}_i u &= \textstyle\sum_k f_i^{(k+1)} u_k. \end{aligned} \tag{2.11}$$

It is to be understood that

$$f_i^{(n)} = 0 \quad \text{for } n < 0.$$

Case (2): *i even and* $(\alpha_i, \alpha_i) < 0$ (*i.e.* $\ell_i < 0$). Assume that u has weight λ. Then u_k has weight $\lambda + k\alpha_i$. Set $l_k = \langle h_i, \lambda + k\alpha_i \rangle$, and define

$$\begin{aligned} \tilde{e}_i u &= \textstyle\sum_k q_i^{l_k - 2k + 1} f_i^{(k-1)} u_k, \\ \tilde{f}_i u &= \textstyle\sum_k q_i^{-l_k + 2k + 1} f_i^{(k+1)} u_k. \end{aligned} \tag{2.12}$$

Hence we have

$$\tilde{f}_i^n u_k = q_i^{-n(l_k - n)} f_i^{(n)} u_k \quad \text{and} \quad \tilde{e}_i^n f_i^{(l_k)} u_k = q_i^{-n(l_k - n)} f_i^{(l_k - n)} u_k. \tag{2.13}$$

Case (3): *i odd and* $(\alpha_i, \alpha_i) = 0$. In this final case we define

$$\begin{aligned} \tilde{e}_i u &= \begin{cases} q_i^{-1} t_i e_i u & \text{if } \ell_i > 0, \\ e_i u & \text{if } \ell_i < 0, \end{cases} \\ \tilde{f}_i u &= \begin{cases} f_i u & \text{if } \ell_i > 0, \\ q_i f_i t_i^{-1} u & \text{if } \ell_i < 0. \end{cases} \end{aligned} \tag{2.14}$$

Suppose u is a weight vector of weight λ and set $\lambda_i = \langle h_i, \lambda \rangle$. If $e_i u = 0$ and $\ell_i > 0$, then

$$\tilde{e}_i(\tilde{f}_i u) = \tilde{e}_i(f_i u) = \frac{1 - q_i^{2\lambda_i}}{1 - q_i^2} u.$$

On the other hand, if $f_i u = 0$ and $\ell_i < 0$, then

$$\tilde{f}_i(\tilde{e}_i u) = \tilde{f}_i(e_i u) = \frac{1 - q_i^{-2\lambda_i}}{1 - q_i^{-2}} u.$$

Hence, $\tilde{e}_i$ and $\tilde{f}_i$ are almost inverses of each other at $q = 0$.

Let us denote by A the subring of $\mathbf{Q}(q)$ consisting of all rational functions $f/g \in \mathbf{Q}(q)$ such that $g(0) \neq 0$. Observe that inverses of elements of $1 + qA$ belong to $1 + qA$.

Definition 2.3. Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. A free A-submodule L is called a ***crystal lattice*** if

(i) L generates M as a vector space over $\mathbf{Q}(q)$.
(ii) $\sigma L = L$ and L has a weight decomposition $L = \bigoplus_{\lambda \in P} L_\lambda$ with $L_\lambda = L \cap M_\lambda$.
(iii) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any $i \in I$.

This brings us to the notion of a crystal base. In the super case, anti-commutativity forces us to relax one of the conditions that a crystal base in the non-super case satisfies (see postulate (iii) below).

Definition 2.4. Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. A ***crystal base*** of M is a pair (L, B) such that

(i) L is a crystal lattice.
(ii) B is a subset of L/qL such that $\sigma b = \pm b$ for any $b \in B$, and B has a weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$ with $B_\lambda = B \cap (L_\lambda / qL_\lambda)$.
(iii) B is a pseudo-base of L/qL (i.e. $B = B^\bullet \cup (-B^\bullet)$ for a $\mathbf{Q}$-basis $B^\bullet$ of L/qL).
(iv) $\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B \sqcup \{0\}$.
(v) For any $b, b' \in B$ and $i \in I$, the condition $b = \tilde{f}_i b'$ is equivalent to $b' = \tilde{e}_i b$.

For a crystal base (L,B), its associated crystal is $B/\{\pm 1\}$ with the structure of a colored oriented graph: b, $b' \in B/\{\pm 1\}$ are joined by the i-arrow, $b \xrightarrow{i} b'$, if $\tilde{f}_i b = b'$.

Lemma 2.5. *Let (L,B) be a crystal base of a $U_q(\mathfrak{g})$-module M in $\mathcal{O}_{\rm int}$, and suppose $b \in B$.*

(i) *If $i \in I_{\rm even}$ and $(\alpha_i,\alpha_i) > 0$, then there is $u \in L_\mu$ for some $\mu \in P$ and an integer k such that $e_i u = 0$ and $b = f_i^{(k)} u \bmod qL$. Moreover, B contains $\{f_i^{(\nu)} u \bmod qL \mid 0 \le \nu \le \langle h_i, \mu\rangle\}$.*

(ii) *If $i \in I_{\rm even}$ and $(\alpha_i,\alpha_i) < 0$, then there is $u \in L_\mu$ for some $\mu \in P$ and an integer k such that $e_i u = 0$ and $b = q_i^{-k(l-k)} f_i^{(k)} u \bmod qL$, where $l = \langle h_i, \mu\rangle$. Moreover, B contains $\{q_i^{-\nu(l-\nu)} f_i^{(\nu)} u \bmod qL \mid 0 \le \nu \le l\}$.*

(iii) *Assume $i \in I_{\rm odd}$ and $\langle h_i, {\rm wt}(b)\rangle > 0$. Then there is $u \in L_\mu$ with $e_i u = 0$ such that $b \equiv u \bmod qL$ or $b \equiv \tilde{f}_i u \bmod qL$. Accordingly, B contains $\tilde{f}_i b$ or $\tilde{e}_i b$.*

Proof. Case (i) is already known ([7, 8]). In case (ii), the elements

$$E_i = e_i, \quad F_i = f_i, \quad K_i = t_i^{-1} \quad \text{and} \quad Q = q^{-\ell_i} \tag{2.15}$$

satisfy the commutation relations

$$\begin{aligned} K_i E_i K_i^{-1} &= Q^2 E_i, \\ K_i F_i K_i^{-1} &= Q^{-2} F_i, \\ [E_i, F_i] &= \frac{K_i - K_i^{-1}}{Q - Q^{-1}}. \end{aligned} \tag{2.16}$$

Hence they generate a subalgebra isomorphic to the quantized enveloping algebra $U_Q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. Then $\tilde{e}_i$ and $\tilde{f}_i$ coincide with the operators $\tilde{E}_i$ and $\tilde{F}_i$ defined in [7, (2.4)] or [8, §2.4] (this is the modified action of E_i and F_i for the upper crystal setting, up to a multiple from $1 + qA$). Therefore the crystal base is the same as the upper crystal base, and the assertion holds by [7].

Now let us prove (iii). We can write $b = u \bmod qL$ for $u \in L_\mu$, and then express u as $u = u_0 + \tilde{f}_i u_1$ where $e_i u_0 = e_i u_1 = 0$. Then $\tilde{f}_i u = f_i u_0 \in L$ and $\tilde{e}_i \tilde{f}_i u \in (1+qA) u_0$. Hence, $u_0 \in L$ since elements of $1 + qA$ are invertible. If $\tilde{e}_i \tilde{f}_i b \neq 0$, then $b = \tilde{e}_i \tilde{f}_i b = u_0 \bmod qL$ and $\tilde{f}_i b \in B$. Alternately, if $\tilde{e}_i \tilde{f}_i b = 0$, then $u_0 \in qL$ and $b = \tilde{f}_i u_1 \bmod qL$. Moreover, $u_1 \equiv \tilde{e}_i \tilde{f}_i u_1 \bmod qL$ implies that B contains $u_1 \bmod qL$. Q.E.D.

For $b \in B$ and $i \in I$, we set

$$\begin{aligned} \varepsilon_i(b) &= \max\{n \in \mathbf{Z}_{\ge 0} \mid \tilde{e}_i^n b \neq 0\}, \\ \varphi_i(b) &= \max\{n \in \mathbf{Z}_{\ge 0} \mid \tilde{f}_i^n b \neq 0\}. \end{aligned} \tag{2.17}$$

Then from the representation theory of $U_Q(\mathfrak{sl}_2)$, we have

$$\langle h_i, {\rm wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b) \quad \text{for } i \in I_{\rm even}. \tag{2.18}$$

For $i \in I_{\rm odd}$, we have $\varphi_i(b) + \varepsilon_i(b) = 0$ or 1 according to whether $\langle h_i, {\rm wt}(b)\rangle = 0$ or not.

Lemma 2.6. *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\rm int}$ with two crystal bases (L,B) and (L',B'). Assume λ is a weight such that $\dim M_\lambda = 1$. Then the connected component of B containing B_λ is isomorphic to the connected component of B' containing B'_λ.*

Proof. We may assume that $L_\lambda = L'_\lambda$ and $B_\lambda = B'_\lambda$. Set $L'' = L + L'$. Then L'' is a crystal lattice of M. Let $\psi : L/qL \to L''/qL''$ and $\psi' : L'/qL' \to L''/qL''$ be the induced homomorphisms. Let $\tilde{B}$ (resp. $\tilde{B}'$) be the connected component of B (resp. B') containing B_λ (resp. B'_λ). Then the map $\tilde{B} \to \psi(\tilde{B})$ commutes with $\tilde{e}_i$ and $\tilde{f}_i$. Moreover it is bijective by Definition 2.4 (v); similarly so is $\tilde{B}' \to \psi'(\tilde{B}')$. Since $\psi(\tilde{B})$ and $\psi'(\tilde{B}')$ are connected with nonempty intersection, they must coincide. Q.E.D.

Lemma 2.7. *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\rm int}$ with a crystal base (L, B). Assume that*

(a) *the associated crystal is connected, and*
(b) *there is a weight λ such that* $\dim M_\lambda = 1$.

Then

(i) *L/qL is an irreducible module over the algebra generated by the $\tilde{e}_i$'s and the $\tilde{f}_i$'s.*
(ii) *M is irreducible.*
(iii) *For any crystal lattice L', the condition $L'_\lambda = L_\lambda$ implies $L' = L$.*
(iv) *The crystal base of M is unique up to a constant multiple.*

Proof. (i) Let K be a nonzero subspace of L/qL stabilized by the $\tilde{e}_i$'s and the $\tilde{f}_i$'s. Choose a nonzero weight vector $v \in K$, and write $v = \sum_{b \in C} a_b b$, where C is a linearly independent subset of B and the a_b are nonzero scalars. Take a product x of $\tilde{e}_i$'s and $\tilde{f}_i$'s and $b \in C$ such that $xb \in B_\lambda$. Since $\dim M_\lambda = 1$, by Definition 2.4 (iii), xb' is equal to either $\pm xb$ or 0 for any $b' \in C$. This shows that $xb' = 0$ for any $b' \in C$ other than $\pm b$. Hence we have $xv = a_b xb$, which implies $B_\lambda \subset K$. Since B is connected, $B \subset K$. Consequently, $K = L/qL$.

(ii) Let N be a nonzero $U_q(\mathfrak{g})$-submodule of M. Set $L(N) = L \cap N$ and $\overline{L(N)} = L(N)/qL(N) \subset L/qL$. Then $\overline{L(N)} \neq 0$, and, as a result, $\overline{L(N)} = L/qL$ by (i). This implies $N = M$.

(iii) Assume first $L' \subset L$. Then the map $\psi : L'/qL' \to L/qL$ is well-defined and injective. Since $\psi(L'/qL')$ contains B_λ, it contains B. Therefore ψ is surjective, and Nakayama's lemma implies $L' = L$. For an arbitrary L', we apply the preceding argument to $L \cap L'$ and obtain $L \subset L'$. Let K be the kernel of $\psi : L/qL \to L'/qL'$. Since K is invariant under the $\tilde{e}_i$'s and the $\tilde{f}_i$'s, and since $K \neq L/qL$, (i) implies $K = 0$. This says ψ is injective, and, therefore, bijective by comparing the dimension of each weight space. Consequently, $L' = L$ by Nakayama's lemma.

(iv) This follows easily from (iii). Q.E.D.

2.4. **Tensor products.** Let M_1 and M_2 be $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\rm int}$, and let (L_1, B_1) and (L_2, B_2) be their crystal bases. Set $L = L_1 \otimes_A L_2$ and $B = B_1 \otimes B_2 \subset (L_1/qL_1) \otimes (L_2/qL_2) = L/qL$.

Proposition 2.8. (i) *(L, B) is a crystal base of $M_1 \otimes M_2$.*

(ii) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $b_1 \otimes b_2$ ($b_1 \in B_1$ and $b_2 \in B_2$) are given as follows.*

(a) *If i is even and $\ell_i > 0$, then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

(b) *If i is even and $\ell_i < 0$, then*

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases} \end{aligned}$$

(c) *If i is odd, $(\alpha_i, \alpha_i) = 0$ and $\ell_i > 0$, then*

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \langle h_i, \mathrm{wt}(b_1) \rangle > 0, \\ \sigma b_1 \otimes \tilde{e}_i(b_2) & \text{if } \langle h_i, \mathrm{wt}(b_1) \rangle = 0, \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \langle h_i, \mathrm{wt}(b_1) \rangle > 0, \\ \sigma b_1 \otimes \tilde{f}_i(b_2) & \text{if } \langle h_i, \mathrm{wt}(b_1) \rangle = 0. \end{cases} \end{aligned}$$

(d) *If i is odd, $(\alpha_i, \alpha_i) = 0$ and $\ell_i < 0$, then*

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \sigma b_1 \otimes \tilde{e}_i(b_2) & \text{if } \langle h_i, \mathrm{wt}(b_2) \rangle > 0, \\ \tilde{e}_i(b_1) \otimes b_2 & \text{if } \langle h_i, \mathrm{wt}(b_2) \rangle = 0, \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \sigma b_1 \otimes \tilde{f}_i(b_2) & \text{if } \langle h_i, \mathrm{wt}(b_2) \rangle > 0, \\ \tilde{f}_i(b_1) \otimes b_2 & \text{if } \langle h_i, \mathrm{wt}(b_2) \rangle = 0. \end{cases} \end{aligned}$$

Proof. It is enough to verify these relations for each $i \in I$. In particular, for $i \in I_{\text{even}}$ with $\ell_i > 0$, this is already known ([7, 8]).

Let us consider the case $i \in I_{\text{even}}$ with $\ell_i < 0$. We may assume that M_1 and M_2 are irreducible modules over $U_q(\mathfrak{g})_i$. With the notation given in (2.15), E_i, F_i and K_i generate $U_Q(\mathfrak{sl}_2)$. Then $\tilde{e}_i$ and $\tilde{f}_i$ coincide with the operators $\tilde{E}_i$ and $\tilde{F}_i$ defined in [7, (2.4)] or [8, §2.4], which give the modified action of E_i and F_i for the upper crystal setting, up to a multiple in $1 + qA$. Hence the crystal bases are the same as the upper crystal base. Moreover, we have

$$\begin{aligned} \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, \\ \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i. \end{aligned} \tag{2.19}$$

After exchanging the first and second factors in the tensor product, we see that this comultiplication is the same as that employed in [7], which behaves well for upper crystal bases. (This just amounts to twisting the comultiplication by the automorphism ω which interchanges E_i and F_i and maps K_i to K_i^{-1}, so that the new comultiplication is $(\omega \otimes \omega) \circ \Delta \circ \omega$.) Hence (b) is obtained by exchanging the first and the second factors in the action in (a).

Now let us consider the case when i is odd. We may assume that M_1 and M_2 are irreducible over $U_q(\mathfrak{g})_i$. Then they are one- or two-dimensional, and we can check the assertions easily. Q.E.D.

Proposition 2.9. *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}$ with a crystal lattice L, and let $(\,\cdot\,,\,\cdot\,)$ be a polarization of M. Assume $(L, L) \subset A$. Let $(\,\cdot\,,\,\cdot\,)_0$ be the induced* **Q***-valued symmetric bilinear form on L/qL. Then $(\tilde{e}_i b, b')_0 = (b, \tilde{f}_i b')_0$ for any b, $b' \in L/qL$.*

Proof. The case $i \in I_{\text{odd}}$ is obvious since $\eta(\tilde{e}_i) = \tilde{f}_i$. Let us consider the case $i \in I_{\text{even}}$. We can reduce to the case $b = f_i^{(k+1)} u$ and $b' = f_i^{(k)} u'$ for $u, u' \in L$ with $e_i u = e_i u' = 0$. Furthermore, we can assume that u and u' have the same weight, say λ. Set $l = \langle h_i, \lambda\rangle$. Then we have

$$\begin{aligned}(f_i^{(k)}u, f_i^{(k)}u') &= \frac{1}{[k]_i!}((q_i^{-1}t_ie_i)^k f_i^{(k)}u, u') \\ &= q_i^{-k+k(k+1)}(e_i^{(k)}t_i^k f_i^{(k)}u, u') \\ &= q_i^{k^2+k(l-2k)}\begin{bmatrix} l \\ k \end{bmatrix}_i (u, u') \\ &\in (1+qA)q_i^{k(l-k)}q^{-|\ell_i|k(l-k)}(u, u').\end{aligned}$$

Now assume $\ell_i > 0$. Then $q_i^{k(l-k)}q^{-|\ell_i|k(l-k)} = 1$ and

$$\begin{aligned}(\tilde{e}_i f_i^{(k+1)}u, f_i^{(k)}u') &= (f_i^{(k)}u, f_i^{(k)}u') \ \in \ (1+qA)(u, u'), \\ (f_i^{(k+1)}u, \tilde{f}_i f_i^{(k)}u') &= (f_i^{(k+1)}u, f_i^{(k+1)}u') \ \in \ (1+qA)(u, u').\end{aligned}$$

Consequently we have $(\tilde{e}_i f_i^{(k+1)}u, f_i^{(k)}u') \in (1+qA)(f_i^{(k+1)}u, \tilde{f}_i f_i^{(k)}u')$.

If $\ell_i < 0$, then

$$\begin{aligned}(\tilde{e}_i f_i^{(k+1)}u, f_i^{(k)}u') &= (q_i^{l-2k-1}f_i^{(k)}u, f_i^{(k)}u') \\ &\in (1+qA)q_i^{l-2k-1+2k(l-k)}(u, u') \\ &\quad = (1+qA)q_i^{l-1+2k(l-k-1)}(u, u'),\end{aligned}$$

and

$$\begin{aligned}(f_i^{(k+1)}u, \tilde{f}_i f_i^{(k)}u') &= (f_i^{(k+1)}u, q_i^{-l+2k+1}f_i^{(k+1)}u') \\ &\in (1+qA)q_i^{-l+2k+1+2(k+1)(l-k-1)}(u, u') \\ &\quad = (1+qA)q_i^{l-1+2k(l-k-1)}(u, u').\end{aligned}$$

Hence we obtain the desired result. Q.E.D.

Definition 2.10. We say that a crystal base (L, B) for a $U_q(\mathfrak{g})$-module M is *polarizable* if there exists a polarization $(\,\cdot\,,\,\cdot\,)$ of M such that $(L, L) \subset A$, and if with respect to the induced $\mathbf{Q}$-valued symmetric bilinear form $(\,\cdot\,,\,\cdot\,)_0$ on L/qL,

$$(b, b')_0 = \begin{cases} \pm 1 & \text{if } b' = \pm b, \\ 0 & \text{otherwise} \end{cases}$$

for all $b, b' \in B$

The following is an immediate consequence of Lemma 2.1 and Proposition 2.8.

Lemma 2.11. *Let (L_ν, B_ν) be a polarizable crystal base of $M_\nu \in \mathcal{O}_{\rm int}$ $(\nu = 1, 2)$. Then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a polarizable crystal base.*

The next theorem on complete reducibility follows from the positive definiteness of the polarization at $q = 0$.

Theorem 2.12. *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\rm int}$ with a polarizable crystal base. Then M is completely reducible.*

Proof. Let us argue that any submodule N of M is a direct summand. Now $N^\perp = \{u \in M \mid (u, N) = 0\}$ is a $U_q(\mathfrak{g})$-module since $(au, v) = (u, \eta(a)v) = 0$ for all $u \in N^\perp$, $v \in N$, and $a \in U_q(\mathfrak{g})$. Since $\dim N_\lambda + \dim(N^\perp)_\lambda = \dim M_\lambda$ for any $\lambda \in P$, it is enough to show that $K \overset{\rm def}{=} N \cap N^\perp = 0$. Let (L, B) be a polarizable crystal base of M and let $(\,\cdot\,,\,\cdot\,)_0$ be the induced form on L/qL. Then $(\,\cdot\,,\,\cdot\,)_0$ is a positive-definite symmetric form by Definition 2.10. Since $(\cdot\,,\,\cdot)_0$ vanishes on $(K \cap L)/q(K \cap L) \subset L/qL$, it must be that $(K \cap L)/q(K \cap L) = 0$. Then $K = 0$ follows from Nakayama's lemma applied to each weight space. Q.E.D.

This theorem along with Lemma 2.11 gives the following result.

Corollary 2.13. *Let M_ν be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$ with a polarizable crystal base $(\nu = 1, \dots, N)$. Then $M_1 \otimes \cdots \otimes M_N$ is completely reducible.*

3. The quantum superalgebra $U_q(\mathfrak{gl}(m,n))$

3.1. **Definition.** For the general linear superalgebra $\mathfrak{g} = \mathfrak{gl}(m,n)$, we assume the index set $I = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$ of simple roots is given by

$$\begin{aligned} I_{\mathrm{even}} &= \{\overline{m-1}, \dots, \overline{1}, 1, \dots, n-1\}, \\ I_{\mathrm{odd}} &= \{0\}. \end{aligned} \tag{3.1}$$

The lattice P of integral weights is

$$P = \bigoplus_{b \in \mathbf{B}} \mathbf{Z}\epsilon_b, \tag{3.2}$$

where $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, $\mathbf{B}_+ = \{\overline{m}, \dots, \overline{1}\}$, and $\mathbf{B}_- = \{1, \dots, n\}$, and the corresponding symmetric form on P is defined by

$$(\epsilon_a, \epsilon_{a'}) = \begin{cases} 1 & \text{if } a = a' \in \mathbf{B}_+, \\ -1 & \text{if } a = a' \in \mathbf{B}_-, \\ 0 & \text{otherwise.} \end{cases}$$

The simple roots are given by

$$\alpha_i = \begin{cases} \epsilon_{\overline{a+1}} - \epsilon_{\overline{a}} & \text{if } i = \overline{a} \text{ with } a = m-1, \dots, 1, \\ \epsilon_{\overline{1}} - \epsilon_1 & \text{if } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{if } i = 1, \dots, n-1. \end{cases} \tag{3.3}$$

We set

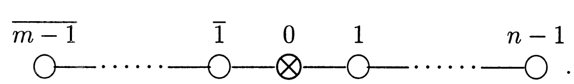
$$\ell_i = \begin{cases} 1 & \text{if } i = \overline{m-1}, \dots, \overline{1} \text{ or } 0, \\ -1 & \text{if } i = 1, \dots, n-1. \end{cases} \tag{3.4}$$

Then the coroot corresponding to α_i is the unique $h_i \in P^*$ satisfying

$$\ell_i \langle h_i, \lambda \rangle = (\alpha_i, \lambda) \quad \text{for any } \lambda \in P. \tag{3.5}$$

Relative to this indexing of simple roots, the Dynkin diagram is given by

(3.6)



The Weyl group W, which is generated by the reflections in the even simple roots, is isomorphic to $S_m \times S_n$ for $\mathfrak{gl}(m,n)$.

3.2. **Vector representation.** The simplest representation of $U_q(\mathfrak{g})$ is its $(m+n)$-dimensional vector representation $\mathbf{V}$. The underlying space is $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$, where $\mathbf{V}_\pm = \bigoplus_{b\in\mathbf{B}_\pm} \mathbf{Q}(q)v_b$, and the action is specified by

$$\begin{aligned}
&\sigma|_{\mathbf{V}_\pm} = \pm\,\mathrm{id}_{\mathbf{V}_\pm},\\
&q^h v_b = q^{\epsilon_b(h)} v_b,\\
&e_i v_b = \begin{cases} v_{\overline{k+1}} & \text{if } i=\overline{k} \text{ and } b=\overline{k} \text{ with } k=1,\dots,m-1,\\ v_{\overline{1}} & \text{if } i=0 \text{ and } b=1,\\ v_k & \text{if } i=k \text{ and } b=k+1 \text{ with } k=1,\dots,n-1,\\ 0 & \text{otherwise,}\end{cases}\\
&f_i v_b = \begin{cases} v_{\overline{k}} & \text{if } i=\overline{k} \text{ and } b=\overline{k+1} \text{ with } k=1,\dots,m-1,\\ v_1 & \text{if } i=0 \text{ and } b=\overline{1},\\ v_{k+1} & \text{if } i=k \text{ and } b=k \text{ with } k=1,\dots,n-1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{3.7}$$

The $U_q(\mathfrak{g})$-module $\mathbf{V}$ belongs to the category $\mathcal{O}_{\mathrm{int}}$, and $\mathbf{L} = \bigoplus_{b\in\mathbf{B}} Av_b$ is a crystal lattice. The set $\{\pm v_b \bmod \mathbf{L} \mid b\in\mathbf{B}\}$ determines a crystal base of $\mathbf{V}$ with associated crystal graph:

$$\begin{aligned}
&\boxed{\overline{m}} \xrightarrow{\overline{m-1}} \boxed{\overline{m-1}} \xrightarrow{\overline{m-2}} \cdots\\
&\qquad \cdots \xrightarrow{\overline{2}} \boxed{\overline{2}} \xrightarrow{\overline{1}} \boxed{\overline{1}} \xrightarrow{0} \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots\\
&\qquad\qquad \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n}\,.
\end{aligned}$$

Note in displaying the crystal graph we write just the subscripts of the crystal base elements not the vectors themselves, and picture only $\mathbf{B}$ not the pseudobase $\mathbf{B}\sqcup(-\mathbf{B})$.

With respect to the symmetric bilinear form on $\mathbf{V}$ which has $\{v_b\}$ as an orthonormal basis, $(\mathbf{L}, \mathbf{B}\sqcup(-\mathbf{B}))$ is a polarizable crystal base. Therefore, by Corollary 2.13, we have

Proposition 3.1. *The $U_q(\mathfrak{g})$-module $\mathbf{V}^{\otimes k}$ is completely reducible for all $k\geq 1$.*

3.3. **Poincaré-Birkhoff-Witt basis.** The set of positive odd roots of $\mathfrak{gl}(m,n)$ is given by

$$\Delta_1^+ = W\alpha_0 = \{\epsilon_a - \epsilon_{a'} \mid a\in\mathbf{B}_+ \text{ and } a'\in\mathbf{B}_-\}. \tag{3.8}$$

Suppose $\Delta_1^+ = \{\beta_1,\dots,\beta_{mn}\}$ is any enumeration of the roots in Δ_1^+. Then we have the following proposition (e.g. see [18, Prop. 10.4.1]).

Proposition 3.2. *Assume $\Delta_1^+ = \{\beta_1,\dots,\beta_{mn}\}$. Then there exist $x_\nu \in U_q^-(\mathfrak{g})$ of weight $-\beta_\nu$ ($\nu = 1,\dots,mn$) such that*

$$U_q^-(\mathfrak{g}) = \sum_{1\leq i_1<\cdots<i_k\leq mn} x_{i_1}\cdots x_{i_k} U_q^-(\mathfrak{g}_0), \tag{3.9}$$

where $U_q^-(\mathfrak{g}_0)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's ($i\in I_{\mathrm{even}}$).

For a dominant integral weight $\lambda\in P$ (i.e. $\langle h_i,\lambda\rangle\geq 0$ for any $i\neq 0$), let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module with highest weight λ, and let u_λ be the highest weight vector of $V(\lambda)$. By definition, u_λ satisfies $\sigma u_\lambda = u_\lambda$, $q^h u_\lambda = q^{\langle h,\lambda\rangle} u_\lambda$ for all $h \in P$, $e_i u_\lambda = 0$ for all $i \in I$, and $V(\lambda) = U_q(\mathfrak{g})u_\lambda$. Then we have

$$V(\lambda) = \sum_{1\le i_1<\cdots<i_k\le mn} x_{i_1}\cdots x_{i_k} U_q^-(\mathfrak{g}_0)u_\lambda. \tag{3.10}$$

Since $U_q^-(\mathfrak{g}_0)u_\lambda$ is finite-dimensional, $V(\lambda)$ is also finite-dimensional. Let w_0 be the longest element of the Weyl group W. Then the lowest weight of $U_q^-(\mathfrak{g}_0)u_\lambda$ is $w_0\lambda$. Thus, we obtain the following lemma.

Lemma 3.3. *The lowest weight μ of $V(\lambda)$ satisfies*

$$\mu \in \Big(w_0\lambda - \sum_{\beta\in\Delta_1^+}\beta\Big) + \sum_{i\in I}\mathbf{Z}_{\ge 0}\alpha_i. \tag{3.11}$$

Note that when λ is what is called a typical weight of $\mathfrak{gl}(m,n)$, i.e. when $(\beta,\lambda+\rho)\neq 0$ for any $\beta\in\Delta_1^+$, we have $\mu = w_0\lambda - \sum_{\beta\in\Delta_1^+}\beta$ (see [5]). Here ρ is an element of P satisfying $(\alpha_i,\rho) = (\alpha_i,\alpha_i)/2$ for any $i \in I$.

Proposition 3.4. *Assume that the irreducible $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight λ belongs to $\mathcal{O}_{\mathrm{int}}$. Set $\lambda_i = \langle h_i,\lambda\rangle$ for $i \in I$. Then*

(i) $\lambda_0 \ge \lambda_1 + \cdots + \lambda_{n-1}$.
(ii) *If $\lambda_k > 0$ for some $k \in \{1,\ldots,n-1\}$, then $\lambda_0 - \lambda_1 - \cdots - \lambda_k \ge k$.*

Proof. Our proof of (i) and (ii) will invoke the following properties of the weights of M:

$$\text{For } \beta\in\Delta_1^+ \text{ and } \mu\in \mathrm{Wt}(M), \text{ we have } (\beta,\mu)\ge 0. \tag{3.12}$$

$$\text{For } \beta\in\Delta_1^+ \text{ and } \mu\in \mathrm{Wt}(M), \text{ if } (\beta,\mu)\neq 0 \text{ and } \mu+\beta\notin \mathrm{Wt}(M), \text{ then } \mu-\beta\in\mathrm{Wt}(M). \tag{3.13}$$

Indeed, since $\mathrm{Wt}(M)$ is invariant under the Weyl group W and $\Delta_1^+ = W\alpha_0$, we can assume $\beta = \alpha_0$. Then (3.12) is nothing but the fourth condition in the definition of $\mathcal{O}_{\mathrm{int}}$ (Definition 2.2). In order to prove (3.13), let us take a nonzero $u \in M_\mu$. Then

$$0 \neq [(\alpha_0,\mu)]u = \frac{t_0 - t_0^{-1}}{q-q^{-1}}u = e_0f_0u + f_0e_0u = e_0f_0u. \tag{3.14}$$

We have used $e_0u \in M_{\mu+\alpha_0} = 0$ in the last equality. By (3.14), f_0u is a nonzero vector of $M_{\mu-\alpha_0}$.

Set $\beta_i = \epsilon_{\bar 1} - \epsilon_{i+1} = \alpha_0 + \cdots + \alpha_i$ for $i = 0,\ldots,n-1$. These are positive odd roots, and their inner products are given by

$$(\beta_i,\beta_j) \quad = \quad \begin{cases} 0 & \text{for } i = j,\\ 1 & \text{for } i\neq j.\end{cases}$$

Now (i) follows from

$$0 \le (\beta_{n-1},\lambda) = \lambda_0 - \lambda_1 - \cdots - \lambda_{n-1}.$$

Let us prove (ii). Suppose that $\lambda_k > 0$. By (ii), $\lambda_0 \ge \lambda_1 + \cdots + \lambda_{n-1} > 0$. Since $\lambda+\alpha_0$ is not a weight of M, property (3.13) implies $\lambda-\alpha_0 = \lambda-\beta_0$ is a weight of M, and

$$(\beta_k, \lambda-\beta_0) = \lambda_0 - \lambda_1 - \cdots - \lambda_k - 1 \ge 0.$$

Hence, $\lambda_0 \geq \lambda_1 + \cdots + \lambda_k + 1 > \lambda_1 + 1$, and $(\beta_1, \lambda - \beta_0) = \lambda_0 - \lambda_1 - 1 > 0$. Since $(\lambda - \beta_0) + \beta_1 = \lambda + \alpha_1$ is not a weight of M, we conclude $\lambda - \beta_0 - \beta_1$ is a weight. The whole argument now can be iterated — the inductive step being the following. Suppose we know that $\lambda - \beta_0 - \beta_1 - \cdots - \beta_{j-1}$ is a weight of M for $j \leq k$. Then

$$0 \leq (\beta_k, \lambda - \beta_0 - \beta_1 - \cdots - \beta_{j-1}) = \lambda_0 - \lambda_1 - \cdots - \lambda_k - j. \tag{3.15}$$

Thus if $j < k$, we see that $\lambda_0 > \lambda_1 + \cdots + \lambda_j - j$. Then

$$(\beta_j, \lambda - \beta_0 - \beta_1 - \cdots - \beta_{j-1}) = \lambda_0 - \lambda_1 - \cdots - \lambda_j - j > 0\,.$$

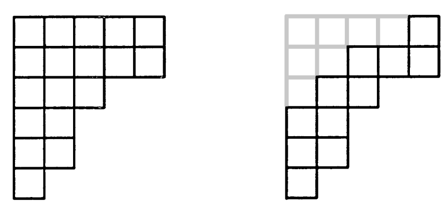
Since $\beta_0 + \cdots + \beta_{j-1} - \beta_j \notin Q_+ = \sum_{\alpha \in \Delta^+} \mathbf{Z}_{\geq 0}\alpha$, we have

$$(\lambda - \beta_0 - \beta_1 - \cdots - \beta_{j-1}) + \beta_j \notin \mathrm{Wt}(M).$$

Then (3.13) shows that $\lambda - \beta_0 - \beta_1 - \cdots - \beta_j$ is a weight. When $j = k$ is reached, we obtain (ii) from (3.15). Q.E.D.

4. Tableaux and crystals

4.1. **Semistandard tableaux.** Recall that a *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. A *skew Young diagram* is a diagram obtained by removing a smaller Young diagram from a larger one that contains it. Thus a Young diagram can be considered as a special case of a skew Young diagram.



Young diagram skew Young diagram

A box in a diagram is said to be a *corner* if there are no boxes in the diagram to its right or beneath it. Removing such a box gives a (skew) Young diagram. A place where a box can be adjoined to a diagram to create a corner of a larger diagram is called a *co-corner*. The diagrams pictured above have co-corners at the right ends of rows 1, 3, 4, 6 and 7.

We assign an ordering on $\mathbf{B} = \{\overline{m}, \overline{m-1}, \ldots, \overline{2}, \overline{1}, 1, 2, \ldots, n-1, n\}$ by saying

$$\overline{m} < \overline{m-1} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < n-1 < n.$$

Definition 4.1. A *semistandard skew tableau* is a tableau obtained from a skew Young diagram by filling the boxes with elements of $\mathbf{B}$ subject to the following two constraints:

(i) the entries in each row are increasing, allowing the repetition of elements in $\mathbf{B}_+ = \{\overline{m}, \overline{m-1}, \ldots, \overline{2}, \overline{1}\}$, but not permitting the repetition of elements in $\mathbf{B}_- = \{1, 2, \ldots, n-1, n\}$,

(ii) the entries in each column are increasing, allowing the repetition of elements in $\mathbf{B}_-$, but not permitting the repetition of elements in $\mathbf{B}_+$.

A Young diagram Y is called an (m,n)-*hook Young diagram* if the number of boxes in the $(m+1)$-st row is less than or equal to n, or equivalently, Y does not have a box at the intersection of the $(m+1)$-st row and the $(n+1)$-st column. Thus an (m,n)-hook Young diagram lies inside the (m,n)-hook as we see in Figure 4.1. For an (m,n)-hook Young diagram, the portion of the diagram consisting of the boxes inside the first m rows and also inside the first n columns is called the *body* of the diagram. The boxes inside the first m rows but not in the body constitute the *arm*, and the part consisting of the boxes in the first n-columns but not in the body is called the *leg* of the diagram.

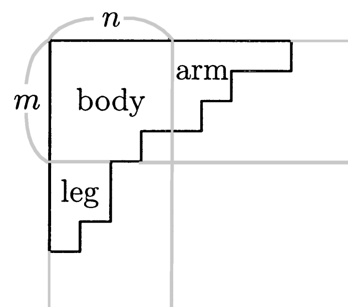


FIGURE 4.1. (m,n)-hook Young diagram

The notion of an (m,n)-hook Young diagram plays an important role in our paper because of the following lemma.

Lemma 4.2. *A Young diagram can be made into a semistandard tableau with entries in* $\mathbf{B}$ *if and only if it is an* (m,n)*-hook Young diagram.*

Proof. In a semistandard Young tableau the entry in the $(m+1)$st box in the leftmost column must be in $\mathbf{B}_-$ by (ii) of Definition 4.1. Then Definition 4.1 (i) implies that all the elements in the $(m+1)$st row must belong to $\mathbf{B}_-$ and that the length of the $(m+1)$st row must be less than or equal to n. For the opposite implication, see Section 4.2 below. Q.E.D.

Berele and Regev [2] have shown that the irreducible summands of tensor powers of the natural $(m+n)$-dimensional representation of $\mathfrak{gl}(m,n)$ can be indexed by the (m,n)-hook Young diagrams Y. A basis for such a summand is in one-to-one correspondence with the semistandard Young tableaux of shape Y.

Let Y be a skew Young diagram and let $B(Y)$ be the set of all semistandard tableaux of shape Y. Let N be the number of boxes in Y. For a given listing of the boxes in Y, we can embed $B(Y)$ into $\mathbf{B}^{\otimes N}$. More precisely, let $T = \{b_1, \ldots, b_N\}$ be a semistandard tableau of shape Y with $b_i \in \mathbf{B}$ in the i-th box of Y with respect to a given listing. Then we identify the semistandard tableau T with the tensor $b_1 \otimes \cdots \otimes b_N \in \mathbf{B}^{\otimes N}$. Such an embedding of $B(Y)$ into $\mathbf{B}^{\otimes N}$ will be called a *reading* of $B(Y)$.

Definition 4.3. (a) A *Japanese reading* (or *Chinese reading*) proceeds down columns from top to bottom and from right to left. That is, we start with the rightmost column reading the entries from top to bottom, then read the next column to the left from top to bottom, and continue this process until we read the bottom box in the leftmost column.

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = \boxed{\bar{2}} \otimes \boxed{1} \otimes \boxed{\bar{3}} \otimes \boxed{\bar{1}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \boxed{1} \otimes \boxed{2}$$

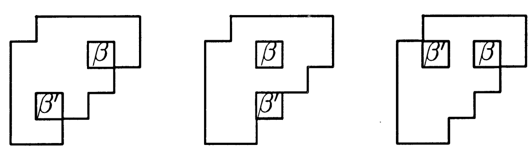
Japanese reading

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = \boxed{\bar{2}} \otimes \boxed{\bar{3}} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

Arabic reading

(b) An *Arabic reading* (or *Hebrew reading*) moves across the rows from right to left and from top to bottom. That is, we begin with the top row reading the entries from right to left, then read the next row from right to left, and continue this process until we read the leftmost box in the bottom row.

More generally, we define the notion of an *admissible reading*. Let β and β' be boxes of a skew tableau T. Suppose that β is in position (i, j) (i.e. at the i-th row from the top and the j-th column from the left) and β' lies in position (i', j'). We say that β is *strictly higher than* β' if $\beta \neq \beta'$ and $i \leq i'$ and $j \geq j'$. Then a box β is strictly higher than a box β' if β lies in the upper right corner of β'. In this case, we also say that β' is *strictly lower than* β. Whenever β is strictly higher or strictly lower than β', then β and β' are in *comparable positions*. For example, in the following figure, β is strictly higher than β'.



A reading (i.e. a listing of the boxes) of a skew tableau is said to be *admissible* if the box β is read before the box β' whenever β is strictly higher than β'. For instance, the Japanese and Arabic readings are admissible. Note that in any admissible reading, the top rightmost box is read first and the bottom leftmost box is read last.

Theorem 4.4. *Let Y be a skew Young diagram.*

(a) *For any admissible reading $\psi : B(Y) \to \mathbf{B}^{\otimes N}$ of Y, $\psi(B(Y))$ is stable under the operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$. Hence an admissible reading induces a crystal structure on $B(Y)$.*

(b) *The induced crystal structure on $B(Y)$ does not depend on the choice of the admissible reading.*

Proof. We identify a tableau T with its image $\psi(T)$ in $\mathbf{B}^{\otimes N}$. Thus, we need to argue that $\tilde{f}_i T = \tilde{f}_i \psi(T) = \psi(T') = T'$ for some semistandard tableau T' independent of ψ, and the analogous result for $\tilde{e}_i$. We begin by proving our assertion for $\tilde{f}_0$ and $\tilde{e}_0$. Let T be a semistandard tableau in $B(Y)$. First note that any two boxes in T containing $\boxed{\bar{1}}$ or $\boxed{1}$ are necessarily in comparable positions, because Y is a skew Young diagram and T is a semistandard skew tableau.

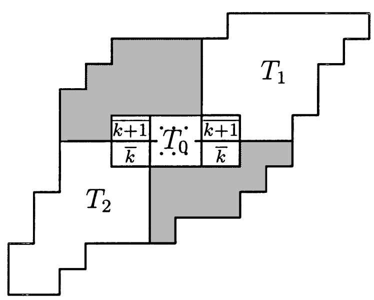
If $\boxed{\overline{1}}$ does not appear in T, then $\tilde{f}_0T = 0$ for any admissible reading. If $\boxed{\overline{1}}$ appears in T, let β be the first box among the $\boxed{\overline{1}}$'s and the $\boxed{1}$'s in some admissible reading. Then β comes first among the $\boxed{\overline{1}}$'s and the $\boxed{1}$'s in any admissible reading of T.

If $\beta = \boxed{\overline{1}}$, then $\tilde{f}_0T$ is the tableau obtained from T by replacing β by $\boxed{1}$. Clearly, $\tilde{f}_0T$ is also semistandard and is the same for any admissible reading. If $\beta = \boxed{1}$, then $\tilde{f}_0T$ vanishes in any admissible reading.

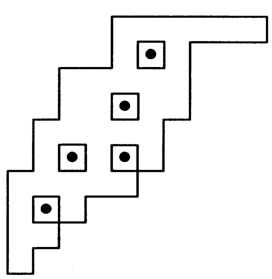
By a similar argument, we can verify that $\tilde{e}_0T$ is the same for any admissible reading and $B(Y)$ is stable under $\tilde{e}_0$.

For $\overline{k} \in I^+_{\text{even}} = \{\overline{m-1}, \cdots, \overline{2}, \overline{1}\}$, we next prove the assertions for $\tilde{f}_{\overline{k}}$. Suppose that the semistandard tableau T contains a rectangular subtableau T_0 with two rows such that its top row consists of $\boxed{\overline{k+1}}$ and the bottom row consists of $\boxed{\overline{k}}$. We assume that T_0 has maximal size among such rectangles. Such a rectangle is called a $\overline{k}$-trivial rectangle.

Let T_1 be the subtableau of T consisting of the boxes that are strictly higher than the box $\boxed{\overline{k+1}}$ which lies in the upper-right corner of T_0, and let T_2 be the subtableau of T consisting of the boxes that are strictly lower than the box $\boxed{\overline{k}}$ in the lower-left corner of T_0.



Since T is semistandard and T_0 is maximal, there are no boxes $\boxed{\overline{k+1}}$ and $\boxed{\overline{k}}$ in the shaded region of T. Hence for any admissible reading of T, T can be regarded as $T_1 \otimes T_0 \otimes T_2$ as a $\{\overline{k}\}$-crystal. Since $\varepsilon_{\overline{k}}(T_0) = \varphi_{\overline{k}}(T_0) = 0$, T can also be regarded as $T_1 \otimes T_2$ as a $\{\overline{k}\}$-crystal. By repeating the above argument, we may assume that the boxes $\boxed{\overline{k+1}}$ and $\boxed{\overline{k}}$ appear only in the dotted sites, i.e., in comparable positions, except in $\overline{k}$-trivial rectangles (see the diagram below).



$\boxed{\overline{k+1}}$ and $\boxed{\overline{k}}$ appear only in comparable positions

Hence, for any admissible reading, T can be considered the same vector as a $\{\overline{k}\}$-crystal, and clearly, $\tilde{e}_{\overline{k}}T$ and $\tilde{f}_{\overline{k}}T$ are semistandard tableaux or 0.

The assertions for $i \in \{1, 2, \ldots, n-1\}$ can be argued analogously. Q.E.D.

For any skew Young diagram Y, the set $B(Y)$ of all semistandard tableaux of shape Y has the canonical structure of a crystal by this theorem.

4.2. **Genuine highest weight vectors.** There is a partial ordering on the integral weights $P = \bigoplus_{b \in \mathbf{B}} \mathbf{Z}\epsilon_b$ of $\mathfrak{gl}(m,n)$ which is defined as follows: for $\mu, \nu \in P$ say $\mu \geq \nu$ if and only if $\mu - \nu \in Q_+ = \sum_{\alpha \in \Delta^+} \mathbf{Z}_{\geq 0}\, \alpha$. Write $\mu = \mu_1 \epsilon_{\overline{m}} + \cdots + \mu_m \epsilon_{\overline{1}} + \mu_{m+1}\epsilon_1 + \cdots + \mu_{m+n}\epsilon_n$ and $\nu = \nu_1 \epsilon_{\overline{m}} + \cdots + \nu_m \epsilon_{\overline{1}} + \nu_{m+1}\epsilon_1 + \cdots + \nu_{m+n}\epsilon_n$. Then it is easy to see that $\mu \geq \nu$ if and only if $\mu_1 + \cdots + \mu_{m+n} = \nu_1 + \cdots + \nu_{m+n}$ and $\mu_1 + \cdots + \mu_k \geq \nu_1 + \cdots + \nu_k$ for all $k = 1, \ldots, m+n$.

For a crystal B over $U_q(\mathfrak{gl}(m,n))$, we say that an element $b \in B_\lambda$ is a *genuine highest weight vector* of B if $B_\lambda = \{b\}$ and $\mathrm{Wt}(B) \subset \lambda + Q_-$, where $\mathrm{Wt}(B)$ denotes the set of all the weights of the crystal B. In this case, the weight λ is called a *genuine highest weight* of B. Similarly, $b \in B_\mu$ is termed a *genuine lowest weight vector* of B if $B_\mu = \{b\}$ and $\mathrm{Wt}(B) \subset \mu + Q_+$. The weight μ is referred to as a *genuine lowest weight* of B in this case. It is obvious that a genuine highest (resp. lowest) weight vector is unique whenever it exists.

Recall that an element $b \in B$ is said to be a *highest weight vector* (resp. *lowest weight vector*) if $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) for all $i \in I$. Clearly, a genuine highest (resp. lowest) weight vector is a highest (resp. lowest) weight vector. But in general, B may have highest (resp. lowest) weight vectors which are not genuine highest (resp. lowest) weight vectors. Those vectors will be called the *fake* highest (resp. lowest) weight vectors. In the following figure we display examples of genuine highest (resp. lowest) weight vectors and fake highest (resp. lowest) weight vectors of $B(Y)$ when $m = n = 2$ and the shape of Y is given by the partition $(3,2,1)$ of 6:

genuine highest weight vector: $\begin{array}{|c|c|c|}\hline \overline{2} & \overline{2} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$

fake highest weight vectors: $\begin{array}{|c|c|c|}\hline \overline{2} & \overline{2} & \overline{2} \\ \hline \overline{1} & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline \overline{2} & \overline{2} & 2 \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$

genuine lowest weight vector: $\begin{array}{|c|c|c|}\hline \overline{1} & 1 & 2 \\ \hline 1 & 2 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$

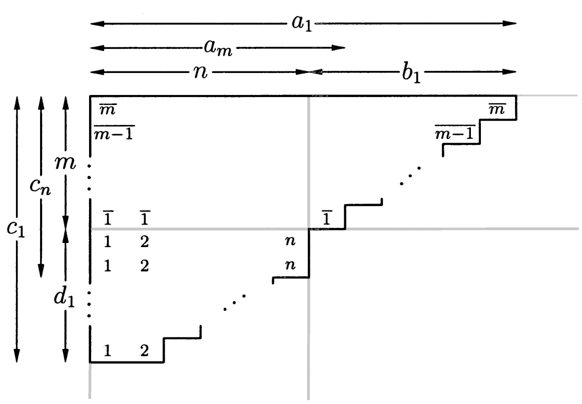
fake lowest weight vectors: $\begin{array}{|c|c|c|}\hline \overline{2} & 1 & 2 \\ \hline \overline{1} & 2 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline \overline{2} & 1 & 2 \\ \hline \overline{1} & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$

Every element in B can be moved to some highest (resp. lowest) weight vector by applying $\tilde{e}_i$'s (resp. $\tilde{f}_i$'s). If there is a unique highest (resp. lowest) weight vector in B, then it must be the genuine highest (resp. lowest) weight vector, and the crystal B is connected in this situation.

Let Y be an (m,n)-hook Young diagram, and let $B(Y)$ be the set of semistandard tableaux of shape λ with a crystal structure given by an admissible reading as in Theorem 4.4. For a semistandard tableau $T \in B(Y)$, its weight $\mathrm{wt}(T)$ is equal to the sum of ϵ_b's where b ranges over the entries of T.

For $i = 1, 2, \dots, m$, let a_i denote the number of boxes in the i-th row of Y, and let $b_i = \max(a_i - n, 0)$. Also, for $j = 1, 2, \dots, n$, let c_j denote the number of boxes in the j-th column of Y, and let $d_j = \max(c_j - m, 0)$. Then the tableau H_Y described in the following picture is the unique genuine highest weight vector of $B(Y)$ and

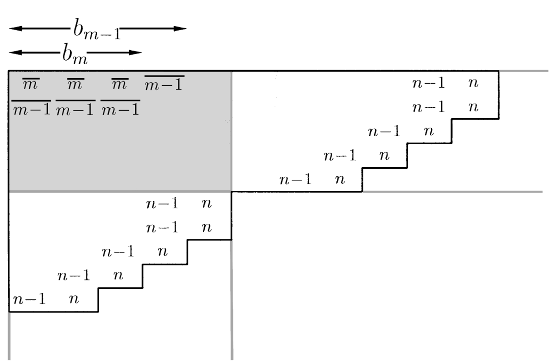
$$\mathrm{wt}(H_Y) = a_1\epsilon_{\overline{m}} + a_2\epsilon_{\overline{m-1}} + \cdots + a_m\epsilon_{\overline{1}} + d_1\epsilon_1 + \cdots + d_n\epsilon_n. \tag{4.1}$$

$H_Y =$ 

Indeed, we can easily check that every entry of a semistandard tableau T of shape Y is greater than or equal to the corresponding entry of H_Y at the same position.

Similarly, the tableau L_Y described in the following picture is the unique genuine lowest weight vector of $B(Y)$ and

$$\mathrm{wt}(L_Y) = b_m\epsilon_{\overline{m}} + b_{m-1}\epsilon_{\overline{m-1}} + \cdots + b_1\epsilon_{\overline{1}} + c_n\epsilon_1 + \cdots + c_1\epsilon_n. \tag{4.2}$$

$L_Y =$ 

Let us denote by $\mathrm{ghwt}(Y)$ (resp. $\mathrm{glwt}(Y)$) the genuine highest weight (resp. genuine lowest weight) of Y. Then there is an injective mapping $\mathcal{Y}(m,n) \to P$

from the set $\mathcal{Y}(m,n)$ of (m,n)-hook Young diagrams to the integral weight lattice P given by

$$Y \mapsto \operatorname{ghwt}(Y). \tag{4.3}$$

Let us determine its image. Let $\widetilde{P}$ be the set of $\lambda \in P$ such that $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I$ and $\langle h_0 - h_1 - \cdots - h_k, \lambda\rangle \geq k$ for $k \in \{1, \ldots, n-1\}$ with $\langle h_k, \lambda\rangle > 0$. As seen in Proposition 3.4, the highest weight of an irreducible module in $\mathcal{O}_{\mathrm{int}}$ must belong to $\widetilde{P}$. Set $\widetilde{P}^+ = \widetilde{P} \cap \bigoplus_{b\in\mathbf{B}} \mathbf{Z}_{\geq 0}\epsilon_b$, and let $\delta = \sum_{b\in\mathbf{B}_+} \epsilon_b - \sum_{b\in\mathbf{B}_-} \epsilon_b$. Then δ has the property that

$$\{\lambda \in P | \langle h_i, \lambda\rangle = 0 \quad \text{for any } i \in I\} = \mathbf{Z}\delta. \tag{4.4}$$

Proposition 4.5. (i) *The map in (4.3) is a bijection from $\mathcal{Y}(m,n)$ to $\widetilde{P}^+$.*
(ii) *$\widetilde{P} = \widetilde{P}^+ + \mathbf{Z}\delta$.*

The proof is straightforward and is omitted.

For $\lambda \in \widetilde{P}^+$, let us denote by Y_λ the (m,n)-hook Young diagram with λ as the genuine highest weight. Let ρ_- be an element of P such that

$$(\rho_-, \alpha_i) = \begin{cases} -1 & \text{if } i = 1, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

As before, w_0 denotes the longest element of the Weyl group W.

Proposition 4.6. *For $\lambda \in \widetilde{P}^+$, the genuine lowest weight of Y_λ is equal to*

$$w_0\big(\lambda - \sum_{\substack{\beta\in\Delta_1^+,\\ (\lambda+\rho_-,\beta)>0}} \beta\big).$$

Proof. For the proof we use formulas (4.1) and (4.2). Let μ be the genuine lowest weight of Y_λ. Then observing $a_k - b_k = \min(n, a_k)$ and $c_j - d_j = \sharp\{k | a_k \geq j\}$, we have

$$\begin{aligned} \lambda - w_0\mu &= \sum_{k=1}^{m} (a_k - b_k)\epsilon_{\overline{m+1-k}} - \sum_{j=1}^{n} (c_j - d_j)\epsilon_j \\ &= \sum_{k=1}^{m} \min(n, a_k)\epsilon_{\overline{m+1-k}} - \sum_{j=1}^{n} \sharp\{k|a_k \geq j\}\epsilon_j \\ &= \sum_{a_k\geq j} (\epsilon_{\overline{m+1-k}} - \epsilon_j). \end{aligned}$$

By virtue of the fact that $\Delta_1^+ = \{\epsilon_{\overline{k}} - \epsilon_j \mid 1 \leq k \leq m, 1 \leq j \leq n\}$, it is enough to show the equivalence

$$a_k \geq j \Longleftrightarrow (\lambda + \rho_-, \epsilon_{\overline{m+1-k}} - \epsilon_j) > 0. \tag{4.6}$$

It is immediate to see

$$(\lambda + \rho_-, \epsilon_{\overline{m+1-k}} - \epsilon_j) = a_k + d_j - j + 1.$$

Therefore, (4.6) is obvious in the case $d_j = 0$, and both sides in (4.6) are true in the case $d_j > 0$. Q.E.D.

Corollary 4.7. *For $\lambda \in \widetilde{P}^+$ assume that Y_λ has a full body (i.e. it contains an $m\times n$ rectangle). Then the genuine lowest weight of Y_λ is equal to $w_0\lambda - \sum_{\beta\in\Delta_1^+}\beta$ (cf. Lemma 3.3).*

Indeed under the given assumption, $(\lambda+\rho_-,\beta)>0$ for any $\beta\in\Delta_1^+$ by (4.6).

4.3. **Connectedness of the crystal** $B(Y)$**.** Even though the crystal $B(Y)$ has a unique genuine highest weight vector and a unique genuine lowest weight vector, it doesn't follow immediately that the crystal $B(Y)$ is connected because $B(Y)$ can have many fake highest and lowest weight vectors. The next theorem shows that the crystal $B(Y)$ is in fact connected.

Theorem 4.8. *The crystal $B(Y)$ associated to any (m,n)-hook Young diagram Y is connected.*

Proof. If $n=0$ or 1, one can easily show that $B(Y)$ has a unique highest weight vector, and hence $B(Y)$ is connected. Similarly, if $m=0$ or 1, then $B(Y)$ has a unique lowest weight vector and hence it is connected. Thus we may assume that $m,n\geq 2$. We will show that every semistandard tableau $T\in B(Y)$ can be moved to the genuine highest weight vector by $\tilde{e}_i$'s and $\tilde{f}_i$'s ($i\in I$).

Let T be a semistandard tableau in $B(Y)$. We will proceed by induction on n and the number p of $\boxed{n}$'s in the first m rows of T. Note that each row of T contains at most one $\boxed{n}$. If there is no $\boxed{n}$ in T, then T is a semistandard tableau over $U_q(\mathfrak{gl}(m,n-1))$, and, by induction on n, T is connected to the genuine highest weight vector H_Y.

Suppose that there is at least one $\boxed{n}$ in T. Let T' be the tableau obtained by removing all $\boxed{n}$ from T. Then T' is a semistandard tableau with respect to $U_q(\mathfrak{gl}(m,n-1))$. By induction on n, the tableau T' can be connected to the genuine highest weight vector for $U_q(\mathfrak{gl}(m,n-1))$. Thus we may assume that T' is the genuine highest weight vector.

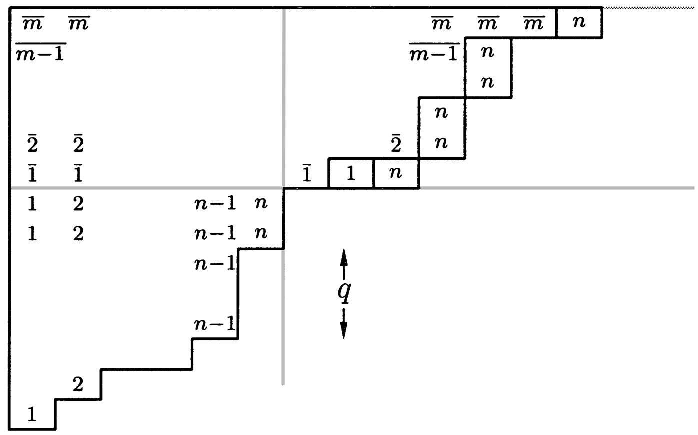
Suppose T has an empty leg. We may assume that T is a highest weight vector. Since T does not have a leg, all the entries of T' are contained in $\mathbf{B}_+=\{\overline{m},\dots,\overline{1}\}$. Then in order for T to be a highest weight vector for $U_q(\mathfrak{gl}(m,n))$, T cannot contain the box $\boxed{n}$, which is a contradiction. Therefore, T must have a nonempty leg. Since only $1,\dots,n$ can lie in the leg, and since we assumed that T is a highest weight vector, the leg of T is the same as the leg of the genuine highest weight vector H_Y. Hence, except possibly for $\boxed{n}$'s in the first m rows, T is the same as the genuine highest weight vector.

Recall that p is the number of $\boxed{n}$'s in the first m rows of T. Let q be the difference between the number of $\boxed{n-1}$'s (in the $(n-1)$-st column) and the number of $\boxed{n}$'s (in the n-th column) in the leg. If $q<p$, then by applying $\tilde{e}_{n-1}$'s we can change at least one $\boxed{n}$ to $\boxed{n-1}$ in the first m rows of T. Hence by induction on p, our assertion follows.

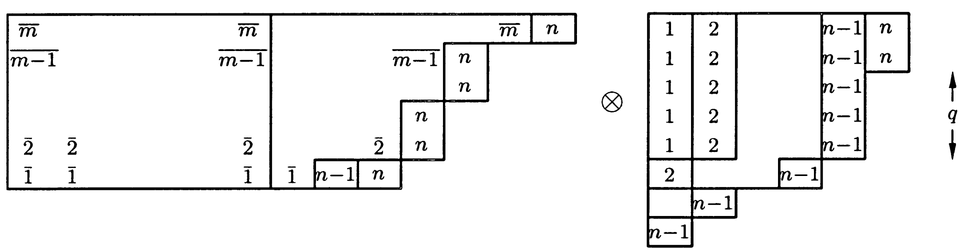
Suppose that we have $q\geq p$ and set $T'=\tilde{f}_0T$. Then T' is the semistandard tableau obtained from T by changing the rightmost $\boxed{\overline{1}}$ in the m-th row to $\boxed{1}$ (see the figure below).

By using the Arabic reading, we may view T' as the tensor product $T_0\otimes T_1$, where T_0 corresponds to the first m rows of T' and T_1 to the leg of T'.

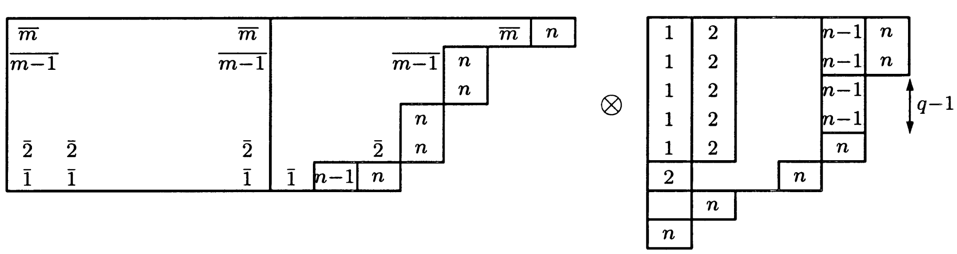
Viewed as a semistandard tableau over $U_q(\mathfrak{gl}(0,n-1))$, T' can be regarded as the vector $\boxed{1}\otimes T_1$. In general for $U_q(\mathfrak{gl}(0,n-1))$, the tensor product of two highest

$T' = \tilde{f}_0 T =$ 

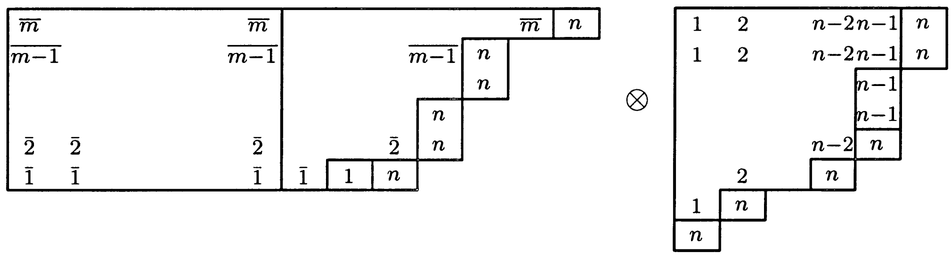
weight vectors is connected to the tensor product of lowest weight vectors. To see this, let $u \in B_1$ and $v \in B_2$ be the highest weight vectors for $U_q(\mathfrak{gl}(0, n-1))$ with weights λ and μ, respectively. Then the connected component B of $B_1 \otimes B_2$ containing $u \otimes v$ is the crystal for the irreducible highest weight module over $U_q(\mathfrak{gl}(0, n-1))$ with weight $\lambda + \mu$. Hence the lowest weight of the crystal B is the same as $w_0(\lambda + \mu)$, where w_0 is the longest element in the Weyl group of $U_q(\mathfrak{gl}(0, n-1))$. Therefore the vector $u \otimes v$ is connected to the vector $u' \otimes v'$, where u' (resp. v') is the lowest weight vector for $U_q(\mathfrak{gl}(0, n-1))$ of weight $w_0\lambda$ (resp. $w_0\mu$). Thus $\boxed{1} \otimes T_1$ is connected to the lowest vector $\boxed{n-1} \otimes T_2$ for $U_q(\mathfrak{gl}(0, n-1))$, where T_2 is also a lowest weight vector for $U_q(\mathfrak{gl}(0, n-1))$. As a result we obtain the tableau:



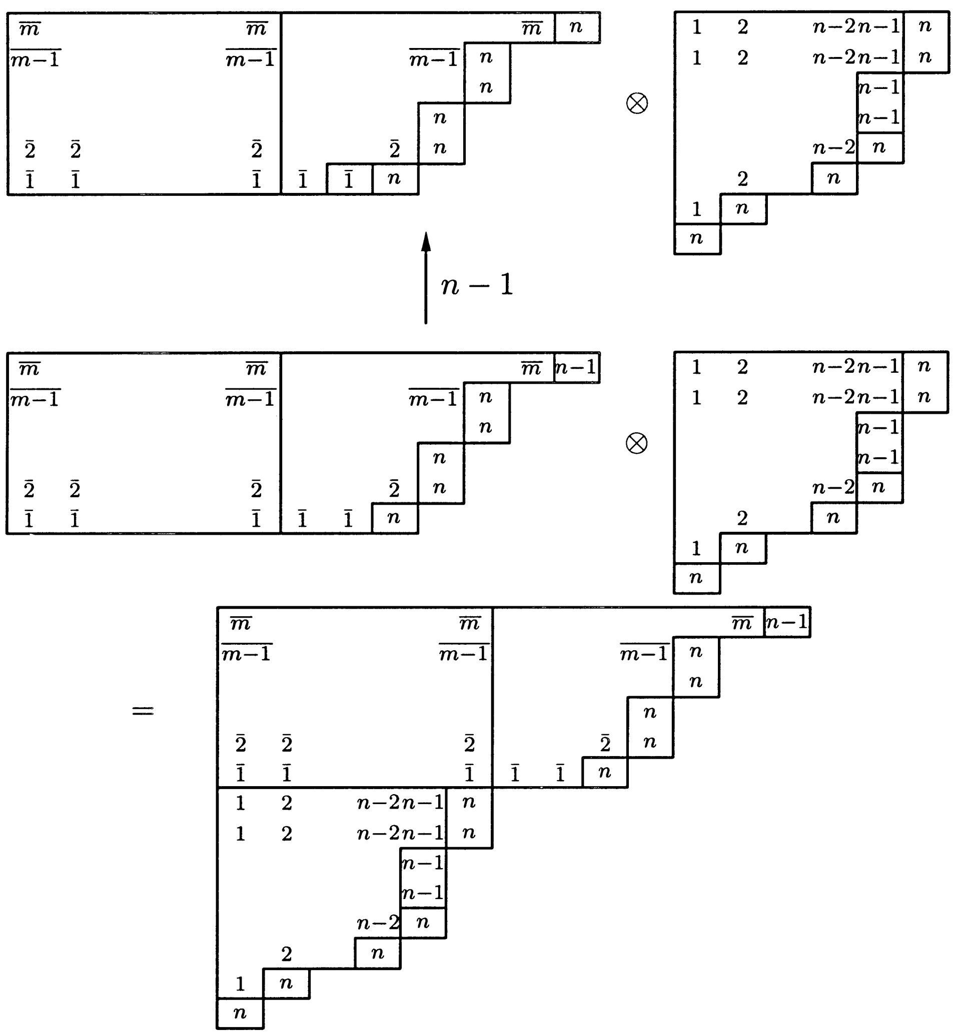
Observe that $\boxed{1}$ in the m-th row in T' has changed to $\boxed{n-1}$. Now, by applying $\tilde{f}_{n-1}$'s, we get:



Viewed as a semistandard tableau over $U_q(\mathfrak{gl}(0,n-1))$, this can be connected to the highest weight vector over $U_q(\mathfrak{gl}(0,n-1))$ which is:



By applying $\tilde{e}_0$ and $\tilde{e}_{n-1}$, we obtain:



Since the first $\boxed{n}$ in the first m rows has changed to $\boxed{n-1}$, the induction on p implies that the tableau can be connected to the genuine highest weight vector H_Y. Q.E.D.

4.4. **Knuth relation.** Let Y_0 be a skew Young diagram, and let $B(Y_0)$ be the set of semistandard tableaux of shape Y_0 which is given a canonical crystal structure by an admissible reading. In this subsection, we will describe the procedure of decomposing the tensor product of crystals $B(Y_0) \otimes \mathbf{B}$ into its connected components. Let B_j be a crystal and $b_j \in B_j$ $(j = 1, 2)$. Let C_j denote the connected component of B_j containing b_j. We say that b_1 is *equivalent* to b_2 and write $b_1 \equiv b_2$ if there is a crystal isomorphism $\psi : C_1 \xrightarrow{\sim} C_2$ sending b_1 to b_2.

For any $a, b \in \mathbf{B}$, one and only one of the following two cases occurs:

(i) $\begin{array}{|c|c|}\hline b & a \\ \hline\end{array}$ is semistandard,

(ii) $\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$ is semistandard.

In other words, the following holds.

Lemma 4.9. *We have a connected component decomposition*

$$\mathbf{B} \otimes \mathbf{B} \simeq B\left(\begin{array}{|c|c|}\hline & \\ \hline\end{array}\right) \oplus B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right).$$

The next lemma, which is known as the *Knuth relation* for the $\mathfrak{gl}(n)$-case, is the fundamental tool for describing the tensor product decomposition.

Lemma 4.10. *There is a crystal isomorphism* $\psi : B\left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\right) \xrightarrow{\sim} B\left(\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{} \\ \hline \multicolumn{1}{|c|}{} & \\ \hline\end{array}\right)$ *given by*

$$\psi\left(\begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}\right) = \begin{cases} \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a} \\ \hline \multicolumn{1}{|c|}{c} & b \\ \hline\end{array} & \text{if } \begin{array}{|c|c|}\hline c & b \\ \hline\end{array} \text{ is semistandard}, \\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{b} \\ \hline \multicolumn{1}{|c|}{a} & c \\ \hline\end{array} & \text{if } \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} \text{ is semistandard}. \end{cases} \tag{4.7}$$

The inverse isomorphism $\psi^{-1} : B\left(\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{} \\ \hline \multicolumn{1}{|c|}{} & \\ \hline\end{array}\right) \xrightarrow{\sim} B\left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\right)$ *is given by*

$$\psi^{-1}\left(\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a} \\ \hline \multicolumn{1}{|c|}{c} & b \\ \hline\end{array}\right) = \begin{cases} \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} & \text{if } \begin{array}{|c|}\hline a \\ \hline c \\ \hline\end{array} \text{ is semistandard}, \\ \begin{array}{|c|c|}\hline c & a \\ \hline b \\ \cline{1-1}\end{array} & \text{if } \begin{array}{|c|c|}\hline c & a \\ \hline\end{array} \text{ is semistandard}. \end{cases} \tag{4.8}$$

Proof. It is easy to see that the maps ψ and ψ^{-1} are inverses of each other. We need to prove that they are crystal morphisms. Our claim can be verified in a straightforward manner by a case-by-case check. For example, if $\boxed{a} = \boxed{\overline{i+1}}$ and $\begin{array}{|c|c|}\hline c & b \\ \hline\end{array}$ is semistandard, then we have $a = \overline{i+1} < c \le b$, and hence

$$\tilde{f}_{\bar{i}}\left(\begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}\right) = \begin{cases} \begin{array}{|c|c|}\hline \bar{i} & b \\ \hline c \\ \cline{1-1}\end{array} & \text{if } c \neq \bar{i}, \\ 0 & \text{if } c = \bar{i}. \end{cases}$$

On the other hand,

$$\tilde{f}_{\bar{\imath}}\left(\ytableaushort{\none a, cb}\right) = \begin{cases} \ytableaushort{\none {\bar{\imath}}, cb} & \text{if } c \neq \bar{\imath}, \\ 0 & \text{if } c = \bar{\imath}. \end{cases}$$

Therefore we get $\psi \tilde{f}_{\bar{\imath}}\left(\ytableaushort{ab, c}\right) = \tilde{f}_{\bar{\imath}}\psi\left(\ytableaushort{\none a, cb}\right)$ in this case. The rest of the cases can be verified in a similar way. Q.E.D.

The equivalences given by crystal isomorphisms ψ and ψ^{-1} in Lemma 4.10 can be expressed in the following way:

$$\ytableaushort{ab, c} \equiv \begin{cases} \ytableaushort{\none a, cb} & \text{if } \ytableaushort{cb} \text{ is semistandard}, \\ \ytableaushort{\none b, ac} & \text{if } \ytableaushort{b, c} \text{ is semistandard}, \end{cases} \tag{4.9}$$

and

$$\ytableaushort{\none a, cb} \equiv \begin{cases} \ytableaushort{ab, c} & \text{if } \ytableaushort{a, c} \text{ is semistandard}, \\ \ytableaushort{ca, b} & \text{if } \ytableaushort{ca} \text{ is semistandard}. \end{cases} \tag{4.10}$$

4.5. **Bumping procedure.**

Theorem 4.11. *There exist crystal isomorphisms*

$$\Psi : B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right) \otimes \mathbf{B} \xrightarrow{\sim} \mathbf{B} \otimes B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right)$$

and

$$\Psi^{-1} : \mathbf{B} \otimes B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right) \xrightarrow{\sim} B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right) \otimes \mathbf{B},$$

which are defined as follows:

$$\Psi\left(\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}\otimes\boxed{b}\right)=\begin{cases}\boxed{a_1}\otimes\begin{array}{|c|}\hline a_2\\\hline a_3\\\hline \vdots\\\hline a_r\\\hline b\\\hline\end{array} & \textit{if } \begin{array}{|c|}\hline a_r\\\hline b\\\hline\end{array} \textit{ is semistandard},\\[2ex] \boxed{a_\nu}\otimes\begin{array}{|c|}\hline a_1\\\hline \vdots\\\hline b\\\hline \vdots\\\hline a_r\\\hline\end{array} & \begin{array}{l}\textit{if } \begin{array}{|c|c|}\hline b & a_r\\\hline\end{array} \textit{ is semistandard, and}\\ \nu \textit{ is the smallest integer such}\\ \textit{that } \begin{array}{|c|c|}\hline b & a_\nu\\\hline\end{array} \textit{ is semistandard},\end{array}\end{cases}$$

$$\Psi^{-1}\left(\boxed{b}\otimes\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}\right)=\begin{cases}\begin{array}{|c|}\hline b\\\hline a_1\\\hline \vdots\\\hline a_{r-1}\\\hline\end{array}\otimes\boxed{a_r} & \textit{if } \begin{array}{|c|}\hline b\\\hline a_1\\\hline\end{array} \textit{ is semistandard},\\[2ex] \begin{array}{|c|}\hline a_1\\\hline \vdots\\\hline b\\\hline \vdots\\\hline a_r\\\hline\end{array}\otimes\boxed{a_\nu} & \begin{array}{l}\textit{if } \begin{array}{|c|c|}\hline a_1 & b\\\hline\end{array} \textit{ is semistandard, and}\\ \nu \textit{ is the largest integer such}\\ \textit{that } \begin{array}{|c|c|}\hline a_\nu & b\\\hline\end{array} \textit{ is semistandard}.\end{array}\end{cases}$$

Therefore we have crystal isomorphisms

$$B\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \vdots\\\hline \ \\\hline\end{array}\right)\otimes\mathbf{B}\cong\mathbf{B}\otimes B\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \vdots\\\hline \ \\\hline\end{array}\right)\cong B\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \vdots\\\hline \ \\\hline \ \\\hline\end{array}\right)\oplus B\left(\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1} \vdots\\\cline{1-1} \ \\\cline{1-1}\end{array}\right).$$

Proof. We will prove our assertion for the crystal isomorphism Ψ only. The assertion for Ψ^{-1} can be proved in a similar manner.

If $\begin{array}{|c|}\hline a_r\\\hline b\\\hline\end{array}$ is semistandard, we have

$$\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}\otimes\boxed{b}\equiv\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline b\\\hline\end{array}\equiv\boxed{a_1}\otimes\begin{array}{|c|}\hline a_2\\\hline a_3\\\hline \vdots\\\hline a_r\\\hline b\\\hline\end{array}\in\mathbf{B}\otimes B\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \vdots\\\hline \ \\\hline\end{array}\right).$$

If $\begin{array}{|c|c|}\hline b & a_r\\\hline\end{array}$ is standard, then

$$\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}\otimes\boxed{b}\equiv\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a_1}\\\cline{2-2} & \multicolumn{1}{|c|}{a_2}\\\cline{2-2} & \multicolumn{1}{|c|}{\vdots}\\\hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{a_r}\\\hline\end{array}.$$

Let ν be the smallest integer such that $\begin{array}{|c|c|}\hline b & a_\nu \\ \hline\end{array}$ is semistandard. Then by Lemma 4.10, it follows that

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a_1} \\ \cline{2-2} & \multicolumn{1}{|c|}{a_2} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{a_r} \\ \hline \end{array} \equiv \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a_1} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{a_{r-1}} \\ \hline \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array} \equiv \cdots \equiv \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a_1} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{a_\nu} \\ \hline \multicolumn{1}{|c|}{a_{\nu+1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array}.$$

Since $\begin{array}{|c|c|}\hline b & a_{\nu-1} \\ \hline\end{array}$ is not semistandard, $\begin{array}{|c|}\hline a_{\nu-1} \\ \hline b \\ \hline\end{array}$ must be semistandard, and so Lemma 4.10 yields

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a_1} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \cline{2-2} & \multicolumn{1}{|c|}{a_{\nu-1}} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{a_\nu} \\ \hline \multicolumn{1}{|c|}{a_{\nu+1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array} \equiv \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a_1} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \cline{2-2} & \multicolumn{1}{|c|}{a_{\nu-2}} \\ \hline \multicolumn{1}{|c|}{a_{\nu-1}} & \multicolumn{1}{c|}{a_\nu} \\ \hline \multicolumn{1}{|c|}{b} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_{\nu+1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array} \equiv \cdots \equiv \begin{array}{cc} \hline \multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{a_\nu} \\ \hline \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_{\nu-1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{b} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_{\nu+1}} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array} \equiv \boxed{a_\nu} \otimes \begin{array}{|c|} \hline a_1 \\ \hline \vdots \\ \hline a_{\nu-1} \\ \hline b \\ \hline a_{\nu+1} \\ \hline \vdots \\ \hline a_r \\ \hline \end{array} \in \mathbf{B} \otimes B\left(\begin{array}{|c|} \hline \\ \hline \vdots \\ \hline \\ \hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right),$$

which proves our claim

For the tensor product decomposition, our assertion follows from the following observation: the vector $\begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_r \\ \hline \end{array} \otimes \boxed{b}$ corresponds to the semistandard tableau

$$\begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_r \\ \hline b \\ \hline \end{array} \in B\left(\begin{array}{|c|} \hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \\ \hline \end{array}\right)$$

if $\begin{array}{|c|}\hline a_r \\ \hline b \\ \hline\end{array}$ is semistandard, and it corresponds to the semistandard tableau

$$\begin{array}{cc} \hline \multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{a_\nu} \\ \hline \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{b} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{a_r} & \\ \cline{1-1} \end{array} \in B\left(\begin{array}{cc} \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{} & \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} & \\ \cline{1-1} \multicolumn{1}{|c|}{} & \\ \cline{1-1} \end{array}\right)$$

if $\begin{array}{|c|c|}\hline b & a_r \\ \hline\end{array}$ is semistandard and ν is the smallest integer such that $\begin{array}{|c|c|}\hline b & a_\nu \\ \hline\end{array}$ is semistandard.

Q.E.D.

The above procedure giving Ψ in Theorem 4.11 can be rephrased as follows. Let

$$\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array} \otimes \boxed{b} \in B\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \vdots\\\hline \ \\\hline\end{array}\right) \otimes \mathbf{B}$$

and try to insert the box $\boxed{b}$ into the tableau $\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}$ from the bottom. If $\begin{array}{|c|}\hline a_r\\\hline b\\\hline\end{array}$ is semistandard, then the box $\boxed{b}$ *bumps out* the box $\boxed{a_1}$ and we get $\boxed{a_1} \otimes \begin{array}{|c|}\hline a_2\\\hline \vdots\\\hline a_r\\\hline b\\\hline\end{array}$. If $\begin{array}{|c|}\hline a_r\\\hline b\\\hline\end{array}$ is not semistandard, then $\boxed{b}$ slides into the tableau from the bottom until it reaches the point $\nu \geq 1$ where the column tableau remains semistandard after replacing a_ν with b. Then $\boxed{b}$ *bumps out* $\boxed{a_\nu}$ to yield $\boxed{a_\nu} \otimes \begin{array}{|c|}\hline a_1\\\hline \vdots\\\hline b\\\hline \vdots\\\hline a_r\\\hline\end{array}$. For this reason, the procedure giving the crystal isomorphism Ψ is called the *bumping procedure.*

Similarly, there is the *reverse bumping procedure* for the crystal isomorphism Ψ^{-1} in Theorem 4.11. The only difference is that, when we consider the vector $\boxed{b} \otimes \begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}$, we slide the box $\boxed{b}$ into the tableau $\begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_r\\\hline\end{array}$ from the top.

The crystal isomorphisms

$$\Phi : B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \otimes \mathbf{B} \xrightarrow{\sim} \mathbf{B} \otimes B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right)$$

and

$$\Phi^{-1} : \mathbf{B} \otimes B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \xrightarrow{\sim} B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \otimes \mathbf{B}$$

can also be described using the bumping procedure as can be seen in the following theorem.

Theorem 4.12. *There exist crystal isomorphisms*

$$\Phi : B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \otimes \mathbf{B} \xrightarrow{\sim} \mathbf{B} \otimes B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right)$$

and

$$\Phi^{-1} : \mathbf{B} \otimes B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \xrightarrow{\sim} B\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\\hline\end{array}\right) \otimes \mathbf{B},$$

which are defined as follows:

$$\Phi\left(\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}\otimes\boxed{b}\right)=\begin{cases}\boxed{a_r}\otimes\boxed{b}\boxed{a_1}\boxed{\cdots}\boxed{a_{r-1}} \\ \quad \textit{if } \boxed{b}\boxed{a_1} \textit{ is semistandard}, \\ \boxed{a_\nu}\otimes\boxed{a_1}\boxed{\cdots}\boxed{b}\boxed{\cdots}\boxed{a_r} \\ \quad \textit{if } \begin{array}{|c|}\hline a_1\\ \hline b\\ \hline\end{array} \textit{ is semistandard}, \\ \quad \textit{and } \nu \textit{ is the largest integer} \\ \quad \textit{such that } \begin{array}{|c|}\hline a_\nu\\ \hline b\\ \hline\end{array} \textit{ is semistandard}.\end{cases}$$

$$\Phi^{-1}\left(\boxed{b}\otimes\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}\right)=\begin{cases}\boxed{a_2}\boxed{\cdots}\boxed{a_r}\boxed{b}\otimes\boxed{a_1} \\ \quad \textit{if } \boxed{a_r}\boxed{b} \textit{ is semistandard}, \\ \boxed{a_1}\boxed{\cdots}\boxed{b}\boxed{\cdots}\boxed{a_r}\otimes\boxed{a_\nu} \\ \quad \textit{if } \begin{array}{|c|}\hline b\\ \hline a_r\\ \hline\end{array} \textit{ is semistandard}, \\ \quad \textit{and } \nu \textit{ is the smallest integer} \\ \quad \textit{such that } \begin{array}{|c|}\hline b\\ \hline a_\nu\\ \hline\end{array} \textit{ is semistandard}.\end{cases}$$

Therefore, we have crystal isomorphisms:

$$\begin{aligned} B\left(\boxed{}\boxed{}\boxed{\cdots}\boxed{}\right)\otimes\mathbf{B} &\cong \mathbf{B}\otimes B\left(\boxed{}\boxed{}\boxed{\cdots}\boxed{}\right) \\ &\cong B\left(\boxed{}\boxed{}\boxed{\cdots}\boxed{}\boxed{\blacksquare}\right)\oplus B\left(\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline \blacksquare \\ \cline{1-1}\end{array}\right). \end{aligned}$$

The bumping procedure for the crystal isomorphism Φ can be summarized as follows. For $\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}\otimes\boxed{b}\in B\left(\boxed{}\boxed{}\boxed{\cdots}\boxed{}\right)\otimes\mathbf{B}$, we slide the box $\boxed{b}$ into the tableau $\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}$ from the left-hand side. If $\boxed{b}\boxed{a_1}$ is semistandard, then the box $\boxed{b}$ bumps out the box $\boxed{a_r}$ and we get $\boxed{a_r}\otimes\boxed{b}\boxed{a_1}\boxed{\cdots}\boxed{a_{r-1}}$. If $\boxed{b}\boxed{a_1}$ is not semistandard and ν is the largest integer such that the row tableau remains semistandard after replacing a_ν with b, then $\boxed{b}$ bumps out $\boxed{a_\nu}$ to yield $\boxed{a_\nu}\otimes\boxed{a_1}\boxed{\cdots}\boxed{b}\boxed{\cdots}\boxed{a_r}$.

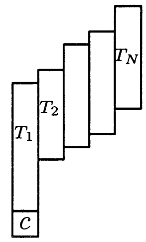
Similarly, there is the *reverse bumping procedure* for the crystal isomorphism Φ^{-1}. The only difference is that, when considering the vector $\boxed{b}\otimes\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}$, we slide the box $\boxed{b}$ into the tableau $\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_r}$ from the right.

Now, we will describe the procedure to decompose the tensor product $B(Y_0)\otimes\mathbf{B}$ for a general skew-Young diagram Y_0. Let T be a semistandard tableau of shape Y_0 and consider the vector $T\otimes\boxed{c}\in B(Y_0)\otimes\mathbf{B}$. Suppose T has N columns denoted by $T_1,\ldots,T_N$ from left to right. Then, by the Japanese reading, we have

$$T\otimes\boxed{c}=T_N\otimes T_{N-1}\otimes\cdots\otimes T_2\otimes T_1\otimes\boxed{c}. \tag{4.11}$$

If $\begin{array}{|c|}\hline T_1 \\ \hline c \\ \hline\end{array}$ is semistandard, then

$$T \otimes \boxed{c} \;\equiv\; T_N \otimes T_{N-1} \otimes \cdots \otimes T_2 \otimes \begin{array}{|c|}\hline T_1 \\ \hline c \\ \hline\end{array} \;\equiv$$



$$= T'$$

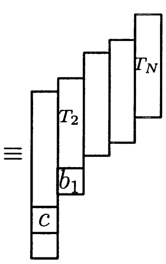
gives a semistandard tableau T'.

If $\boxed{c}$ cannot be placed at the bottom of T_1, then $\boxed{c}$ slides into the tableau T_1 and bumps out some entry $\boxed{b_1}$ from T_1:

$$T \otimes \boxed{c} \equiv T_N \otimes T_{N-1} \otimes \cdots \otimes T_2 \otimes \boxed{b_1} \otimes T_1'.$$

We now try to place $\boxed{b_1}$ at the bottom of T_2. If this is possible, there should exist a *co-corner* between T_1' and T_2. That is, there should be a space at the bottom of T_2 and next to T_1' so that we would still have a skew Young diagram after adding a box to the bottom of T_2. Recall that $\boxed{b_1}$ is the smallest entry in T_1 such that $\begin{array}{|c|c|}\hline c & b_1 \\ \hline\end{array}$ is semistandard. Thus, in order for $\boxed{b_1}$ to be placed at the bottom of T_2, $\boxed{c}$ must lie lower than $\boxed{b_1}$ (see the following figure), and the resulting tableau T'', which is obtained from $T' = T_N \otimes \cdots \otimes T_2 \otimes T_1'$ by adjoining the box $\boxed{b_1}$ at the bottom of T_2, is still semistandard:

$$T \otimes \boxed{c} \equiv T_N \otimes \cdots \otimes T_2 \otimes \boxed{b_1} \otimes T_1' \equiv$$

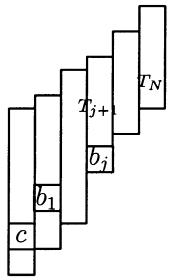


$$= T''.$$

If $\boxed{b_1}$ cannot be placed at the bottom of T_2, then $\boxed{b_1}$ slides into T_2 and bumps out $\boxed{b_2}$ from T_2, and we keep repeating the same procedure until we can add a box, say, $\boxed{b_j}$ to T_{j+1}. (Note that j could be N, in which case we create a new column

T_{N+1} which consists of one box $\boxed{b_N}$.) Thus, we obtain

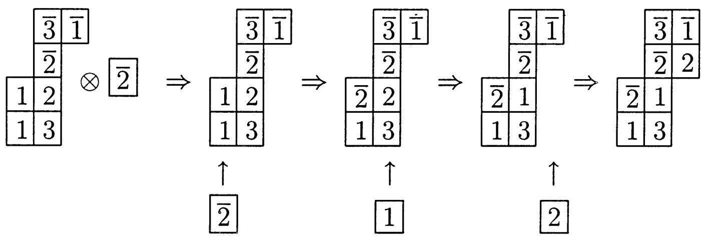
$$T \otimes \boxed{c} \equiv T_N \otimes \cdots \otimes T_{j+1} \otimes \boxed{b_j} \otimes T'_j \otimes \cdots \otimes T'_1$$

$\equiv$ 

$= T^{(j+1)}$,

and the resulting tableau $T^{(j+1)}$ is semistandard.

For example, if $m = n = 3$, we have:



Consequently, we obtain the map

$$B(Y_0) \otimes \mathbf{B} \to \bigoplus_{Y \in \mathcal{Y}} B(Y), \tag{4.12}$$

where Y runs over the set $\mathcal{Y}$ of all skew Young diagrams obtained from Y_0 by adding a box to a co-corner of Y_0, which is a crystal morphism.

Now assume that Y_0 is an (m,n)-hook Young diagram. Then the connected components of the tensor product of crystals $B(Y_0) \otimes \mathbf{B}$ have the form $B(Y)$ for some $Y \in \mathcal{Y}$. The diagrams $Y \in \mathcal{Y}$ are (m,n)-hook Young diagrams, since the tableaux produced by the crystal morphism are semistandard (compare Lemma 4.2). Conversely, let T' be a semistandard tableau of shape Y, where Y is a Young diagram in $\mathcal{Y}$. Then, starting with the box in Y outside Y_0, by reversing the above procedure, one can see that there exist a unique semistandard tableau $T \in B(Y_0)$ and a box $\boxed{c} \in \mathbf{B}$ such that $T \otimes \boxed{c} \equiv T'$. Hence we have constructed the inverse of the crystal morphism given in (4.12). As a consequence, we obtain:

Theorem 4.13. *Let Y_0 be an (m,n)-hook Young diagram and let $B(Y_0)$ be the set of all semistandard tableaux of shape Y_0 endowed with a crystal structure by an admissible reading. Then the tensor product of crystals $B(Y_0) \otimes \mathbf{B}$ has the following decomposition into connected components:*

$$B(Y_0) \otimes \mathbf{B} \cong \bigoplus_{Y \in \mathcal{Y}} B(Y), \tag{4.13}$$

where Y runs over the set $\mathcal{Y}$ of all (m,n)-hook Young diagrams obtained from Y_0 by adding a box to a co-corner of Y_0.

As an immediate corollary we have the following theorem.

Theorem 4.14. *Any connected component of the tensor product* $\mathbf{B}^{\otimes k}$ *of* k *copies of* $\mathbf{B}$ *is isomorphic to* $B(Y)$ *for some* (m,n)*-hook Young diagram* Y *with* k *boxes. Moreover, for any skew Young diagrams* Y_1 *and* Y_2*, the connected components of the tensor product of crystals* $B(Y_1) \otimes B(Y_2)$ *have the form* $B(Y)$*, where* Y *is an* (m,n)*-hook Young diagram.*

5. Existence of the crystal base

5.1. **Main results.** In this section, we shall prove that any irreducible module in $\mathcal{O}_{\mathrm{int}}$ for $U_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{gl}(m,n)$, has a crystal base, and its associated crystal base is parameterized by semistandard tableaux. Since a general theory of crystal bases is not available in the super case, the proof relies on the crystal base theory for $\mathfrak{gl}(m,0)$ and $\mathfrak{gl}(0,n)$ and the combinatorics of Young tableaux developed in the previous section.

Recall that $\widetilde{P}$ is the set of weights $\lambda \in \bigoplus_{b\in\mathbf{B}} \mathbf{Z}\epsilon_b$ such that $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I$ and $\langle h_0 - h_1 - \cdots - h_k, \lambda\rangle \geq k$ for $k \in \{1,\ldots,n-1\}$ with $\langle h_k, \lambda\rangle > 0$, and $\widetilde{P}^+ = \widetilde{P} \cap \bigoplus_{b\in\mathbf{B}} \mathbf{Z}_{\geq 0}\epsilon_b$. As we have already seen in Proposition 3.4, the highest weight of any irreducible $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$ belongs to $\widetilde{P}$.

As before, let Y_λ denote the (m,n)-hook Young diagram whose genuine highest weight is λ for $\lambda \in \widetilde{P}^+$ (see Proposition 4.5).

Theorem 5.1. *For* $\lambda \in \widetilde{P}$*, the irreducible* $U_q(\mathfrak{g})$*-module* $V(\lambda)$ *with highest weight* λ *is a* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{\mathrm{int}}$ *with a polarizable crystal base. Moreover, if* $\lambda \in \widetilde{P}^+$*, the associated crystal is isomorphic to* $B(Y_\lambda)$*.*

Proposition 5.2. *For* $\lambda \in \widetilde{P}$*, we have a direct sum decomposition:*

$$V(\lambda) \otimes \mathbf{V} \cong \bigoplus_b V(\lambda + \epsilon_b)$$

as a $U'_q(\mathfrak{g})$*-module. Here the sum ranges over* $b \in \mathbf{B}$ *such that* $\lambda + \epsilon_b \in \widetilde{P}$*.*

The proof will be given in the subsequent subsections.

As a corollary of these results along with Proposition 3.4 and Proposition 4.5, we obtain the following:

Proposition 5.3. *Any irreducible* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{\mathrm{int}}$ *is isomorphic to a direct summand of* $\mathbf{V}^{\otimes k} \otimes S$ *for some integer* k *and some one-dimensional* $U_q(\mathfrak{g})$*-module* S *in* $\mathcal{O}_{\mathrm{int}}$*.*

Every one-dimensional $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$ must be of the form $\mathbf{Q}(q)v$, where $e_i v = 0 = f_i v$ for all $i \in I_{\mathrm{even}}$, and $\sigma v = \pm v$. The vector v must have weight $\mathrm{wt}(v) = a\delta$ for some $a \in \mathbf{Z}$ (see (4.4) and Proposition 4.5).

Now Proposition 4.6 can be rephrased as follows.

Proposition 5.4. *Let* M *be an irreducible* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{\mathrm{int}}$*. Then for its highest weight* λ *and its lowest weight* μ *the following relation holds:*

$$\mu = w_0\big(\lambda - \sum_{\substack{\beta\in\Delta_1^+,\\ (\lambda+\rho_-,\beta)>0}} \beta\big).$$

5.2. **Technical lemma.** In order to prove Theorem 5.1, we may assume from the outset that $\lambda \in \widetilde{P}^+$ by Proposition 4.5. We shall first prove a lemma which is a weaker statement than Proposition 5.2.

Lemma 5.5. *Let Y_0 be an (m,n)-hook Young diagram. Assume that there is an irreducible $U_q(\mathfrak{g})$-module M with a polarizable crystal base (L,B) such that the associated crystal is isomorphic to $B(Y_0)$.*

(i) *The tensor product $M \otimes \mathbf{V}$ has the direct sum decomposition*

$$M \otimes \mathbf{V} = \bigoplus_j M_j,$$

where the M_j's are mutually nonisomorphic irreducible $U_q(\mathfrak{g})$-modules.

(ii) *We have $L \otimes \mathbf{L} = \bigoplus_j L_j$ and $B \otimes \mathbf{B} = \bigsqcup_j B_j$, where $L_j = L \cap M_j$ and $B_j = B \cap (L_j/qL_j)$. In particular, (L_j, B_j) is a crystal base of M_j.*

(iii) *For each j, one of the following holds:*

(a) *The associated crystal of B_j is isomorphic to $B(Y')$ for an (m,n)-hook Young diagram Y' obtained from Y_0 by adding a box.*

(b) *The associated crystal of B_j is isomorphic to $B(Y_1) \sqcup B(Y_2)$. Here Y_1 is an (m,n)-hook Young diagram obtained from Y_0 by adding a box in the arm, and Y_2 is an (m,n)-hook Young diagram obtained from Y_0 by adding one box in the leg of Y_0. Moreover, the highest weight of M_j is the genuine highest weight of Y_1, and the lowest weight of M_j is the genuine lowest weight of Y_2.*

Proof. Note that $M \otimes \mathbf{V}$ is completely reducible by Corollary 2.13.

Let λ be the genuine highest weight of Y_0 and μ its genuine lowest weight. By Theorem 4.13, the associated crystal graph $B(Y_0) \otimes \mathbf{B}$ has the decomposition

$$B(Y_0) \otimes \mathbf{B} \cong \bigoplus_{Y \in \mathcal{Y}} B(Y), \tag{5.1}$$

where $\mathcal{Y}$ is the set of (m,n)-hook Young diagrams obtained from Y_0 by adding one box.

The genuine highest weights of Y in $\mathcal{Y}$ are mutually different and of the form $\lambda + \epsilon_b,\ \ b \in B$. The genuine lowest weights of Y in $\mathcal{Y}$ are also mutually different and of the form $\mu + \epsilon_b$, $b \in B$. Therefore,

$$\mathcal{Y} \;=\; \mathcal{Y}_+ \cup \mathcal{Y}_-,$$

where

$$\mathcal{Y}_+ = \{Y \in \mathcal{Y} \mid \mathrm{ghwt}(Y) = \lambda + \epsilon_b \text{ for some } b \in \mathbf{B}_+\},$$
$$\mathcal{Y}_- = \{Y \in \mathcal{Y} \mid \mathrm{glwt}(Y) = \mu + \epsilon_b \text{ for some } b \in \mathbf{B}_-\}.$$

If $\mathcal{Y}_0 = \mathcal{Y}_+ \cap \mathcal{Y}_-$, then $\mathcal{Y}_0$, $\mathcal{Y}_+ \setminus \mathcal{Y}_0$, and $\mathcal{Y}_- \setminus \mathcal{Y}_0$ are the sets of Young diagrams obtained from Y_0 by adding a box to the body, arm, or leg, respectively.

Let $Y \in \mathcal{Y}_+$. Then $\mathrm{ghwt}(Y) = \lambda + \epsilon_b$ for some $b \in \mathbf{B}_+$. Suppose u_λ is the highest weight vector of M. By the representation theory of $U_q(\mathfrak{gl}_m)$, the module $U_q(\mathfrak{gl}(m,0))u_\lambda \otimes \mathbf{V}_+$ has a highest weight vector v_Y of weight $\lambda + \epsilon_b$ with respect to $U_q(\mathfrak{gl}(m,0))$. In addition we may assume that $v_Y \in L \otimes \mathbf{L}$ and $v_Y = u_\lambda \otimes b$ modulo $q(L \otimes \mathbf{L})$. The relation $e_i(U_q(\mathfrak{gl}(m,0))u_\lambda \otimes \mathbf{V}_+) = 0$ for $i = 0, 1, 2, \ldots, n-1$ implies that v_Y is a highest weight vector with respect to $\mathfrak{gl}(m,n)$. Set $V_Y = U_q(\mathfrak{g})v_Y$. Since $M \otimes \mathbf{V}$ is completely reducible, V_Y is an irreducible $U_q(\mathfrak{g})$-module with highest weight $\lambda + \epsilon_b$, and $L(V_Y) = V_Y \cap (L \otimes \mathbf{L})$ is a crystal lattice of V_Y. Moreover $\overline{L(V_Y)} = L(V_Y)/qL(V_Y)$ contains $B(Y)$.

Now consider the case that $Y \in \mathcal{Y}_-$. Then $\mathrm{glwt}(Y) = \mu + \epsilon_b$ for some $b \in \mathbf{B}_-$. Let w_μ be the lowest weight vector of M. By the representation theory of $U_q(\mathfrak{gl}_n)$, the module $U_q(\mathfrak{gl}(0,n))w_\mu \otimes \mathbf{V}_-$ has a lowest weight vector w_Y of weight $\mu + \epsilon_b$ with respect to $U_q(\mathfrak{gl}(0,n))$. We may further suppose that $w_Y \in L \otimes \mathbf{L}$ and $w_Y = w_\mu \otimes b$ modulo $q(L \otimes \mathbf{L})$. The relation $f_i(U_q(\mathfrak{gl}(0,n))w_\mu \otimes \mathbf{V}_-) = 0$ for $i = \overline{m-1}, \dots, \overline{1}, 0$ implies that w_Y is a lowest weight vector with respect to $\mathfrak{gl}(m,n)$. Then $W_Y = U_q(\mathfrak{g})w_Y$ is an irreducible $U_q(\mathfrak{g})$-module with lowest weight $\mathrm{glwt}(Y)$. Furthermore, $L(W_Y) = V_Y \cap (L \otimes \mathbf{L})$ is a crystal lattice of W_Y, and for $\overline{L(W_Y)} = L(W_Y)/qL(W_Y)$, we have $B(Y) \subset \overline{L(W_Y)}$.

Set

$$Q_-(\mathfrak{gl}(m,0)) = \sum_{i=\overline{m-1},\dots,\overline{2},\overline{1}} \mathbf{Z}_{\leq 0}\alpha_i,$$
$$Q_+(\mathfrak{gl}(0,n)) = \sum_{i=1,2,\dots,n-1} \mathbf{Z}_{\geq 0}\alpha_i.$$

Claim.

(i) The V_Y's ($Y \in \mathcal{Y}_+$) are mutually nonisomorphic.
(ii) The W_Y's ($Y \in \mathcal{Y}_-$) are mutually nonisomorphic.
(iii) If an irreducible $U_q(\mathfrak{g})$-submodule N of M has a highest weight belonging to $\lambda + \epsilon_{\overline{m}} + Q_-(\mathfrak{gl}(m,0))$, then N is equal to V_Y for some $Y \in \mathcal{Y}_+$.
(iv) If an irreducible $U_q(\mathfrak{g})$-submodule has a lowest weight belonging to $\mu + \epsilon_n + Q_+(\mathfrak{gl}(0,n))$, then it is equal to W_Y for some $Y \in \mathcal{Y}_-$.
(v) $V_Y = W_Y$ for $Y \in \mathcal{Y}_0$.

Let us verify these assertions.

(i) follows from the fact that their highest weights are distinct, and similarly for (ii).

(iii) A highest weight vector of N must belong to $U_q(\mathfrak{gl}(m,0))u_\lambda \otimes \mathbf{V}_+$, and hence it must coincide with v_Y for some $Y \in \mathcal{Y}_+$ by the representation theory of $U_q(\mathfrak{gl}(m,0))$. Hence $N \supset U_q(\mathfrak{g})v_Y = V_Y$.

The proof of (iv) is similar.

(v) For $Y \in \mathcal{Y}_0$, $B(Y) \subset \overline{L(W_Y)}$. Hence there is b in $\overline{L(W_Y)}$ corresponding to the genuine highest weight vector of $B(Y)$. Since $\tilde{e}_i b = 0$ for $i = \overline{m-1}, \dots, \overline{1}$, its representative $v \in L(W_Y)$ satisfies $e_i v = 0$ for $i = \overline{m-1}, \dots, \overline{1}$. Since $\mathrm{wt}(v) = \mathrm{ghwt}(B(Y))$ belongs to $\lambda + \mathrm{Wt}(\mathbf{B}_+)$, it must be that $v \in U_q(\mathfrak{gl}(m,0))u_\lambda \otimes \mathbf{V}_+$. Hence v coincides with v_Y, and $V_Y \subset W_Y$.

Now let us resume the proof of Lemma 5.5. Let $\mathcal{Y}'_-$ be the set of $Y \in \mathcal{Y} \setminus \mathcal{Y}_+$ such that W_Y is not equal to any of $V_{Y'}$ ($Y' \in \mathcal{Y}_+$). For $Y \in \mathcal{Y}'_-$, set $V_Y = W_Y$, and let $\mathcal{Y}' = \mathcal{Y}_+ \sqcup \mathcal{Y}'_-$. Then the modules $\{V_Y\}_{Y \in \mathcal{Y}'}$ are mutually nonisomorphic. For $Y \in \mathcal{Y}'$, we set

$$B_Y = \begin{cases} B(Y) \sqcup B(Y') & \text{if } Y \in \mathcal{Y}_+ \text{ and } V_Y = W_{Y'} \\ & \qquad \text{for some } Y' \in \mathcal{Y}_-, \\ B(Y) & \text{otherwise.} \end{cases} \tag{5.2}$$

Then we have

$$B \otimes \mathbf{B} = \bigsqcup_{Y \in \mathcal{Y}'} B_Y, \tag{5.3}$$

and

$$B_Y \subset (B \otimes \mathbf{B}) \cap \overline{L(V_Y)} \quad \text{for any } Y \in \mathcal{Y}'. \tag{5.4}$$

Hence $\sum_{Y\in\mathcal{Y}'} \overline{L(V_Y)}$ contains $B \otimes \mathbf{B}$. Therefore Nakayama's lemma implies that $L \otimes \mathbf{L} = \sum_{Y\in\mathcal{Y}'} L(V_Y)$ and $V(\lambda) \otimes V = \sum_{Y\in\mathcal{Y}'} V_Y$. Since the modules $\{V_Y\}_{Y\in\mathcal{Y}'}$ are mutually nonisomorphic, we have

$$M \otimes \mathbf{V} = \bigoplus_{Y\in\mathcal{Y}'} V_Y \quad \text{and} \quad L \otimes \mathbf{L} = \bigoplus_{Y\in\mathcal{Y}'} L(V_Y)\,.$$

Moreover equality holds instead of inclusion in (5.4). This completes the proof of Lemma 5.5.

5.3. **Proof of Theorem 5.1.** The proof of Theorem 5.1 proceeds by induction on $\mathfrak{b}(\lambda) = (\delta, \lambda)$ (the number of boxes of Young diagram Y_λ). If $\mathfrak{b}(\lambda) = 1$, then $\lambda = \epsilon_{\overline{m}}$ and $V(\lambda)$ is isomorphic to the vector representation $\mathbf{V}$. Hence we assume $\mathfrak{b}(\lambda) > 1$. At this stage it is convenient to divide the considerations into steps.

Step 1. *Theorem 5.1 holds if there is a corner of Y_λ in the body* (*see Figure 4.1*).

Let Y_0 be a Young diagram obtained from Y_λ by removing the box from such a corner. Then the induction hypothesis asserts that there is an irreducible $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$ with a polarizable crystal base whose associated crystal is isomorphic to $B(Y_0)$. Then by Lemma 5.5, there is an irreducible $U_q(\mathfrak{g})$-module N whose associated crystal contains $B(Y_\lambda) \subset B(Y_0) \otimes \mathbf{B}$. By the assumption, case (b) in Lemma 5.5 cannot occur, and N has a crystal base isomorphic to $B(Y)$. Hence, the theorem holds in this case.

Next we shall prove the main theorem in the following special case:

Step 2. *The theorem holds if Y_λ has a full body.*

Recall that this condition means that Y_λ contains a rectangle of size $m \times n$, or, in terms of λ, that $(\epsilon_{\overline{1}} - \epsilon_n, \lambda) \geq n$. Since we can exclude the case considered in Step 1, we may suppose that there is a corner either in the arm or in the leg. Since the two proofs are quite similar, we shall only treat the first possibility. Let Y_0 be the Young diagram obtained from Y_λ by removing a corner in the arm. Then by the induction hypothesis there is an irreducible $U_q(\mathfrak{g})$-module M with a polarizable crystal base isomorphic to $B(Y_0)$. By Lemma 5.5, there exists an irreducible submodule N of $M \otimes \mathbf{V}$ with a polarizable crystal base (L, B) having highest weight λ. Such a crystal base B contains $B(Y_\lambda)$. The genuine lowest weight of Y_λ is $\mu \overset{\mathrm{def}}{=} w_0\lambda - \sum_{\beta\in\Delta_{\overline{1}}^+} \beta$ by Corollary 4.7. On the other hand, the lowest weight μ of N is in $\mu + Q^+$ by Lemma 3.3. Since μ is a weight of N, μ must be the lowest weight of N. Hence case (b) in Lemma 5.5 cannot occur, and the crystal for N is $B(Y_\lambda)$. Consequently, the main theorem is true in this case.

Final Step. Now we shall prove the main theorem in the general case. For this we proceed by induction on the number k of boxes in the leg of Y_λ. Assume first that $k > 0$. Then there is a corner in the leg of Y_λ. Suppose Y_0 is a Young diagram obtained from Y_λ by removing such a corner. Let M be an irreducible $U_q(\mathfrak{g})$-module with a polarizable crystal base isomorphic to $B(Y_0)$. There is an irreducible submodule N of $M \otimes \mathbf{V}$ with a polarizable crystal base containing $B(Y_\lambda)$. Moreover, the lowest weight vector of N is the genuine lowest weight of $B(Y_\lambda)$. If the crystal base of N is $B(Y_\lambda)$, we are done. Otherwise there is a Young diagram Y_1 obtained from Y_0 by adding a box to a co-corner in the arm such that the crystal of N is $B(Y_\lambda) \sqcup B(Y_1)$. The highest weight of N is the genuine highest weight of $B(Y_1)$. Since the number of boxes in the leg of Y_1 is smaller than the corresponding number in Y_λ by one, the main theorem holds for $\mathrm{ghwt}(Y_1)$, which is a contradiction.

Thus we may assume that there are no boxes in the leg of Y_λ, which means that there are at most m rows in Y_λ. We can assume there is no corner in the body. Hence any row of Y_λ has length at most $n+1$. We can further suppose that Y_λ does not have a full body. Consequently, there are at most $m-1$ rows in Y_λ. Let Y_0 be the Young diagram obtained from Y_λ by removing a box from a corner. Then Y_0 has no co-corner in its leg. Hence Lemma 5.5 implies that there is an irreducible module N with a polarizable crystal base isomorphic to $B(Y_\lambda)$. This finishes the proof of the main theorem.

Proposition 5.2 now follows from the main theorem and Lemma 5.5.

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