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METRIC AND ISOPERIMETRIC PROBLEMS IN SYMPLECTIC GEOMETRY

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Contents

Introduction

This paper is devoted to establishing a number of theorems at the interface of symplectic and Riemannian geometry. We try to relate the symplectic way of measuring size, using so-called capacities, to the classical Riemannian approach, using the volume. One could and should try to extend these connections to other metric invariants, for instance those involving curvature.

After recalling some known facts about capacities, we state and prove the main technical result in the paper: an inequality between the capacity of a set and the capacities of its symplectic reductions.

The next section is devoted to proving isoperimetric inequalities for submanifolds of nonzero codimension. It turns out that for Lagrangian submanifolds we have a

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remarkable isoperimetric inequality, relating capacity with n-dimensional volume. This is the main result in the paper.

We prove this inequality in three different settings, closed Lagrangian submanifolds in Euclidean space, complex projective space, and Hamiltonian deformations of the zero section in cotangent bundles.

As an application, we get an estimate between the L^∞ norm of a function, and the L^1 norm of the determinant of the Hessian. This is related to the Alexandroff-Bakelman-Pucci inequality. It thus appears that for the Monge-Ampère equation, such an inequality is essentially equivalent to an isoperimetric inequality in symplectic geometry.

As another application of our main result, we get an upper bound for the shortest length of a closed billiard trajectory in a bounded subset of $\mathbb{R}^n$ in terms of the volume of the domain.

We would like to point out that our results connect Riemannian properties to symplectic properties, and are not applications of symplectic geometry to Riemannian geometry (good examples of this are in [Fer] and [Rez1] for instance).

It is a "widespread belief" that capacity is maximal, the volume being given, for the unit ball.[1]

We indicate that this may well be the case among convex sets, using John's ellipsoid, and prove that there is an isoperimetric inequality relating these two quantities. Our constant falls short of being an equality in case of the ball, even though we have no counterexamples. In fact, one of the open problems is to improve the isoperimetric constant γ_n in Theorem 5.1 from $\gamma_n = \sqrt{n}$ to $\gamma_n = 1$. In the general case, we give examples of domains with contact type boundary of arbitrarily small volume and capacity bounded from below. So the usual generalisation from convex to contact type does not work here, leaving open the question of finding a simple characterization of domains for which the above isoperimetric inequality holds (this class of domains includes all symplectic images of convex sets, and possibly more). The question of whether such an inequality holds for a domain with restricted contact type is left open.[2]

Gromov and Eliashberg proved that the set of symplectic diffeomorphisms is closed for the C^0 topology. We may ask what are the topologies on the space of diffeomorphisms, such that this still holds. The first case that occurs naturally is for Sobolev norms. Clearly whenever we have the Sobolev embedding $W^{k,p} \subset C^0$ (i.e. for $p > n/k$), such a statement holds.

We prove here that it holds for $(k,p) = (1/2, 2)$. This is a simple application of integration by parts, and seems related to "Compensated compactness" of Murat and Tartar. The main interest of this section is probably to connect two apparently unrelated questions.

I wish to thank P. Gérard for useful explanations on this last subject. He pointed out to me that the above result holds not only for $(k,p) = (1,2)$ but also for $(k,p) = (1/2, 2)$. I also thank David Hermann and David Théret for listening to and commenting on preliminary (and incorrect) versions of the results stated here.

[1] Of course this depends on the choice of the symplectic capacity. In fact $c(U) = \max\{\pi r^2 \mid \exists \phi \text{ symplectic } \phi(B^{2n}(r)) \subset U\}$ trivially satisfies an isoperimetric inequality, but this is not an interesting invariant: it is neither connected with periodic orbits, nor with holomorphic curves, or any other geometrically interesting property. Moreover it vanishes on Lagrangians.

[2] David Hermann gave a counterexample to this conjecture, thus destroying the "widespread belief" (see [H3]). There are starshaped domains with arbitrarily small volume and large capacity.

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1. Some basic results in symplectic topology

Let U be a domain in $\mathbb{R}^{2n}$, with the standard symplectic form $\omega = \sum_{j=1}^{n} dp_j \wedge dq^j$.

There are several ways to measure its symplectic size, usually through so-called *symplectic capacities*. These are invariants associated to each subset U of $\mathbb{R}^{2n}$ satisfying the following properties:

1. $c(U) \leq c(V)$ for $U \subset V$.
2. $c(\psi(U)) = \rho c(U)$ for any map ψ such that $\psi^*(\omega) = \rho\omega$.
3. $c(B^2(r) \times \mathbb{R}^{2n-2}) = c(B^{2n}(r)) = \pi r^2$.

Gromov invented the first capacity, the symplectic width, in [G], even though the term *capacity* was invented only later by Ekeland and Hofer (see [EH1]). It is defined as follows.

Let $\mathcal{J}(\omega)$ be the set of almost complex structures on $\mathbb{R}^{2n}$ such that:

- $\omega(x, Jx) > 0 \ \forall x \neq 0$,
- $\omega(Jx, Jy) = \omega(x, y) \ \forall x, y \in \mathbb{R}^{2n}$.

For given J, let $C(J, U, x)$ be the set of J-holomorphic curves which are closed in U and go through the point $x \in U$.

Definition 1.1.

$$w(U) = \sup_{J \in \mathcal{J}(\omega)} \inf_{\Sigma \in C(J,U,x)} \int_\Sigma \omega.$$

Note that the transitivity of the group of symplectic diffeomorphisms implies that for U connected, the definition does not depend on the choice of x.

There are also the capacities defined using periodic orbits of Hamiltonian systems, first discovered by Ekeland and Hofer in [EH1].

A hypersurface in $\mathbb{R}^{2n}$ is said to be of *contact type* if there is a transverse conformal vector field (i.e. $\mathcal{L}_\xi \omega = \omega$) defined near the hypersurface. It is said to be of *restricted contact type* if ξ extends to an everywhere defined conformal vector field.

Given a hypersurface, we define the characteristic line field to be the line field $\ker(\omega_{T\Sigma})$. The integral curves of the characteristic line field are also the integral curves on Σ of the Hamiltonian vector field associated to H, where H is a smooth function having Σ as a regular level.

We refer to [EH1, V2] for the definition of the Ekeland-Hofer capacity, c_{EH}, but we shall mainly use the following property:

Proposition 1.2. *For U an "open set with smooth boundary" of restricted contact type, there exists a closed characteristic γ on ∂U such that $c_{EH}(U) = \int_\gamma \lambda$ (where $\lambda = \sum_{j=1}^{n} p_j dq^j$ is a primitive of ω).*

Note that this property is also shared by the *generating function capacity* defined in [V1].

We will also use the displacement energy, defined as follows in [Ho]: set

$$\|H\| = \int_0^1 \sup_{\mathbb{R}^{2n}} |H(t,x)| dt;$$

then

$$d(U) = \inf\{\|H\| \mid \phi_1(U) \cap U = \emptyset\}.$$

Here ϕ_t is the flow of H. The main point is that this number is nonzero (which was proved in [Ho] and in [V1] independently).

The above invariants are related by the following:

Proposition 1.3. *For each open set U with restricted contact type boundary, we have*

$$d(U) \geq c_{EH}(U) \geq w(U).$$

The first inequality is due to Hofer ([Ho]), the second one to David Hermann ([H1]).

Note that for F a compact set, we have that $d(F) < c$ implies $d(U) < c$ for some neighborhood U of F. In other words $\lim_{F_n \to F} d(F_n) \leq d(F)$, where convergence is for the Hausdorff distance, and d is upper semi-continuous on the set of compact subsets.

Finally we have the symplectic invariants defined in [V1] for Lagrange submanifolds. Let $L \subset T^*N$ be a Lagrangian submanifold and let $S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$ be a smooth function. We say that S is a generating function of L if

$$L = \{(x, \frac{\partial S}{\partial x}(x,\xi)) \mid \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

We shall say that S is quadratic at infinity, if it coincides with a nondegenerate quadratic form in ξ for ξ large enough.

According to [LauS], any Lagrangian submanifold Hamiltonianly isotopic to the zero section has such a generating function. Minimax techniques, as invented by Lusternik and Schnirelman, allow us to define for each cohomology class α on N a critical value $c(\alpha, S)$. It is proved in [V1] and [Th] that $c(\alpha, S)$ only depends on L, and we denote it by $c(\alpha, L)$. For $N = S^n$, we have only two invariants, $c(1, L)$ and $c(\mu, L)$ ($\mu \in H^n(S^n)$ is the generator), and we define an invariant $\gamma(L) = c(\mu, L) - c(1, L)$. We refer to [V1] and Appendix A for a study of such invariants. Note that for H a compact supported Hamiltonian, with time one flow ϕ, the graph of ϕ, Γ_ϕ, is a Lagrangian submanifold of $\mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$.

2. Capacity and symplectic reduction

We shall first estimate the displacement energy of an open set using the displacement energy of its reductions. We shall say that a hypersurface has *proper characteristics* if the characteristics are proper curves in the hypersurface.

Proposition 2.1. *Let U be a compact set in $\mathbb{R}^{2n}$, Σ_x a foliation of $\mathbb{R}^{2n}$ by a family of hypersurfaces with proper characteristics, and U_x the symplectic reduction of U by Σ_x. Then the following inequality holds:*

$$d(U) \leq 2 \cdot \sup_x d(U_x).$$

Proof. We fix some strictly positive δ and $c > \sup_x d(U_x)$. For each x, let us choose $\varepsilon(x)$ such that on $\xi \in]x - \varepsilon(x), x + \varepsilon(x)[$ we have $\phi_x(U_\xi) \cap U_\xi = \emptyset$ where ϕ_x is the time one map of a Hamiltonian $H_x(t, z)$ with norm less than c.

By compactness of the interval, we may find a finite number of points, x_i, such that the $]x_i - \varepsilon_i, x_i + \varepsilon_i[$ cover the projection of U on the x-axis. We may moreover assume, by suitably reducing the ε_i, that $\mathbb{R}$ is covered by a finite number of intervals $I_1, \dots, I_N$ such that:

1. $I_j \cap I_k = \emptyset$ for $|k-j| > 1$.
2. $\mathrm{meas}(I_j \cap I_{j+1}) = \delta_j \le \delta$.
3. For each j there exists ϕ_j generated by H_j of norm less than c, such that for each $x \in I_j$, $\phi_j(U_x) \cap U_x = \emptyset$.

For simplicity we shall assume that $I_1 =]-\infty, a[$ and $I_N =]b, +\infty[$.

We take coordinates z in the symplectic reduction of Σ_x and y will be a coordinate dual to x.

Let H be the Hamiltonian

$$H(t,x,y,z) = \sum_{j=1}^{N} \chi_j(x) H_j(t,z) + f(t,x)$$

where χ_j has support in I_j, equals 1 in $I_j - (I_{j+1} \cup I_{j-1})$, and f is yet to be determined.

The flow of H has the following properties:

1. It preserves the hypersurfaces $\Sigma_{x_0} = \{x = x_0\}$.
2. For $x \in I_j - (I_{j+1} \cup I_{j-1})$ it coincides with ϕ_j and thus maps the projection of $U \cap \Sigma_x$ away from itself, hence $U \cap \Sigma_x$ away from itself.
3. On $I_j \cap I_{j+1}$, the coefficient of $\frac{\partial}{\partial y}$ in X_H is $H_j \chi'(x) + \frac{\partial f}{\partial x}(t,x)$.

We may assume that on $I_j \cap I_{j+1}$ we have $\chi'(x)$ bounded by some constant as close as we wish from $\frac{1}{\delta_j}$.

From the last point, we see that if $\int_0^1 \frac{\partial f}{\partial x}(t,x)dt \ge K + \frac{c}{\delta_j}$, then Σ_x is translated by at least K in the direction $\frac{\partial}{\partial y}$. Let us choose K large enough, so that $U \subset \{(x,y,z) \mid |y| \le K/2\}$; then the flow of H still maps $U \cap \Sigma_x$ away from itself. We only need that $\int_0^1 \frac{\partial f}{\partial x}(t,x)dt$ be of the order of magnitude of $K + \frac{c}{\delta_j}$, for x in an interval of length δ_j, and this may be achieved with $\|f\|$ of the order of $K\delta_j + c \le K\delta + c$, because outside such intervals f is arbitrary (hence can go down to zero). Since δ is as small as we please, we see that $\|H\|$ is bounded by $2c$. $\square$

The above argument may be improved to show:

Proposition 2.2. *Let U be such that $U \subset \mathbb{R}^{2n-1} \times [0,1]$ and let $d(U)$ be the displacement energy of U. Let $\Sigma_x = \mathbb{R}^{2n-2} \times \{x\} \times \mathbb{R}$ and let U_x be as above. We have*

$$d(U)^{k+1} \le C_{k+1} \int d(U_x)^k dx$$

where $C_{k+1} \le 2^k \frac{(k+1)^{k+1}}{(k)^k}$.

Proof. Suppose $\int d(U_x)^k dx < a$. Since $x \to d(U_x)$ is upper semi-continuous, we have that $d(U_x)^k = \inf g_n^k$, where the g_n are continuous functions. Since $\int d(U_x)^k dx = \inf \int g_n^k(x)dx$, we have for n large enough that $\mathrm{meas}\{x \mid g_n^k(x) \ge c^{1/k}\} < \frac{a}{c}$ and, being a compact set, is covered by a finite number of intervals of length at most $\frac{a}{c}$.

Since $\{x \mid d(U_x) \ge c^{1/k}\} \subset \{x \mid g_n(x) \ge c^{1/k}\}$, we have that $\{x \mid d(U_x) \ge c^{1/k}\}$ is covered by a finite set of intervals of length at most $\frac{a}{c}$. This implies as before that $d(U) \le (\frac{a}{c} + 2 \cdot c^{1/k})$. Indeed, for x such that $d(U_x) \le c^{1/k}$ we move U in the z direction, and apply 2.1, and for the other values of x, we move U in the y direction of at most 1. For this it is enough that $\frac{\partial H}{\partial x}$ be of the order of 1, hence the increase of H in this region is at most $\frac{a}{c}$.

The total increase of H will then be $\frac{a}{c}+2\cdot c^{1/k}$. But $\frac{a}{c}+2\cdot c^{1/k}$ equals $(k+1)\cdot k^{-\frac{k}{k+1}}\cdot a^{\frac{1}{k+1}}$ for $c=(ka/2)^{\frac{k}{k+1}}$. Now $C_{k+1}=\frac{1}{a^{k+1}}(\frac{a}{c}+2\cdot c^{1/k})=2^k\cdot\frac{(k+1)^{k+1}}{(k)^k}$. $\square$

Remarks. 1. Note that our statement is **not** homogeneous, and thus does not hold without the assumption $U\subset\mathbb{R}^{2n-2}\times\mathbb{R}\times[0,1]$.

2. The proposition extends to the case of a family of hypersurfaces Σ_x with proper characteristic. We must then replace the condition on U by the assumption that the intersection of any characteristics of Σ_x with U has length less than 1 (this means that the Riemannian length of the portion of the characteristic curve of Σ_x contained in U is less than 1).

3. The limiting case $k\to\infty$ again yields Proposition 2.1, using the fact that $\lim_{k\to\infty}\left(\int d(U_x)^k\right)^{1/k}=\sup_x d(U_x)$.

3. Volume estimates for Lagrange submanifolds

Let L be a Lagrange submanifold in the symplectic manifold (M,ω).

The aim of this section is to find a lower bound for the volume of L. Of course this volume depends on the choice of the metric. We shall in fact be interested in two types of results.

-the existence of some constant c such that $\mathrm{vol}_n(L)\geq c$. This is a statement independent of the metric. Of course the value of the constant $c=c(g)$ does depend on the metric, g.

-trying to either compute or estimate $c(g)$ for certain special metrics. We are looking for an estimate invariant by Hamiltonian isotopy, the constant $c(g)$ in fact depends only on the orbit of g under $Ham(M)$.

The first result in this direction is due to Givental-Kleiner-Oh, and proves that for any Hamiltonian deformation L of $\mathbb{R}P^n$ in $\mathbb{C}P^n$, the volume of L is greater than the volume of $\mathbb{R}P^n$. The proof follows from the fact that such a Lagrangian submanifold meets $A(\mathbb{R}P^n)$ (the image by an isometry, A, of $\mathbb{C}P^n$ of the standard real projective space). A generalization of the Arnold conjecture, due to Givental, shows that $L\cap A(\mathbb{R}P^n)$ is nonempty and thus Kleiner and Oh conclude, by a Crofton type formula, that $\mathrm{vol}(L)$ is bounded from below by $\mathrm{vol}(\mathbb{R}P^n)$.

However, the above result uses some peculiar features of the symplectic pair $(\mathbb{C}P^n,\mathbb{R}P^n)$. We shall see that the existence of a lower bound for the volume of Lagrangian submanifolds, invariant by Hamiltonian isotopy of the submanifold, is a general phenomenon.

We shall first prove that in $\mathbb{R}^{2n}$ the image by a symplectic map of a given Lagrangian has its volume bounded from below. We then extend this result to other symplectic manifolds, as well as to the case of Hamiltonian deformations of the zero section in $T^*\mathbb{R}^n$. Before we state our main result, we would like to mention that the definition of the displacement energy may be extended to immersed Lagrange submanifolds without change, and it is still nonzero. One can again apply the Chekanov result, to show that for immersed L, $d(L)$ is bounded from below by the smallest area of a holomorphic disc with boundary in L (note that for immersed L, the disc may well go through double points). Our estimate below still holds for such immersed manifolds.

3.1. **The case of $\mathbb{R}^{2n}$.** We first prove the following consequence of Proposition 2.1:

Theorem 3.1. *Let L be a Lagrangian submanifold in $\mathbb{R}^{2n}$. Then we have*

$$d(L)^n \leq \bar{\rho}_n \operatorname{vol}^2(L)$$

where $d(L)$ is the displacement energy of L, and $\bar{\rho}_n \leq 2^{\frac{n(n-3)}{2}} n^n$.

The proof is based on the following result. We denote by π_q (resp. π_p) the projection on the q (resp. p) coordinates, and by $\operatorname{vol}(\pi_q(L))$ (resp. $\operatorname{vol}(\pi_p(L))$) the volume of this projection. We denote by $\operatorname{vol}(L;q)$ (resp. $\operatorname{vol}(L;p)$) the integral of the density $|dq_1 \wedge \ldots \wedge dq_n|$ (resp. $|dp^1 \wedge \ldots \wedge dp^n|$) on L, that is, the volume of the projection on the q (resp. p) coordinates *counting multiplicities*. Clearly we have $\operatorname{vol}(\pi_q(L)) \leq \operatorname{vol}(L;q)$ (in fact because the projection has degree zero, we have $\operatorname{vol}(\pi_q(L)) \leq \frac{1}{2}\operatorname{vol}(L;q)$).

Our first step towards the proof of Theorem 3.1 is

Proposition 3.2. *Let L be a Lagrangian submanifold in $U \times \mathbb{R}^n$. Then we have*

$$d(L)^n \leq \rho_n(U) \operatorname{vol}(\pi_q(L)).$$

Here $\rho_n(U)$ is bounded for U bounded.

Proof. We consider coordinates $(p,q) \in \mathbb{R}^n \times \mathbb{R}^n$.

Note that it is enough to prove our proposition for a Lagrangian submanifold contained in $[0,1]^n \times \mathbb{R}^n$, since by rescaling we can always reduce ourselves to this case. Clearly the result holds for $n=1$. Set $L(x) = L \cap \{(p,q) \mid q_n = x\}$ and let L_x be the reduction of L by $\{(p,q) \mid q_n = x\}$. Then according to Proposition 2.1, arguing by induction, we get the inequalities

$$\int_{\mathbb{R}} \operatorname{vol}(\pi_q(L_x))dx = \int_{\mathbb{R}} \operatorname{vol}(\pi_q(L(x)))dx \leq \operatorname{vol}(\pi_q(L)).$$

Now, since $L_x \subset [0,1]^{n-1} \times \mathbb{R}^{n-1}$, we may use the induction hypothesis, and we obtain:

$$d(L)^n \leq C_n \cdot \int_{\mathbb{R}} d(L_x)^{n-1} dx \leq C_n \cdot \rho_{n-1} \cdot \int_{\mathbb{R}} \operatorname{vol}_{n-1}(\pi_q(L_x))dx$$

$$\leq \rho_n \cdot \int_{\mathbb{R}} \operatorname{vol}_{n-1}(\pi_q(L(x)))dx \leq \rho_n \cdot \operatorname{vol}_n(\pi_q(L)).$$

Here $\rho_n = \rho_n([0,1]^n) \leq \prod_{j=1}^n C_j = 2^{\frac{n(n-1)}{2}} n^n$. $\square$

Proof of Theorem 3.1. First notice that the inequality of the above proposition clearly implies

$$d(L)^n \leq \rho_n(U) \operatorname{vol}(L;q).$$

Let $\tilde{\rho}(U)$ be defined as the best constant in the above inequality. In other words,

$$\tilde{\rho}(U) = \sup\{\frac{d(L)^n}{\operatorname{vol}(L;q)} \mid \pi_p(L) \subset U\}.$$

We just proved that $\tilde{\rho}(U)$ is finite. It satisfies $\tilde{\rho}(U) \leq \tilde{\rho}(V)$ for $U \subset V$ and $\tilde{\rho}(\lambda \cdot U) = \lambda^n \tilde{\rho}(U)$.

We now prove that $\tilde{\rho}(U)$ is bounded by a constant times the volume of U.

Let ψ be a diffeomorphism of $\mathbb{R}^n$. It naturally extends to a symplectic diffeomorphism Ψ of $\mathbb{R}^n \times \mathbb{R}^n$. Provided ψ is volume preserving in a neighborhood of $U = \pi_p(L)$, we have $\tilde{\rho}(\Psi(U)) = \tilde{\rho}(U)$. Indeed, $d(\Psi(L)) = d(L)$, since Ψ is symplectic and if ψ is volume preserving near p, we have that $\Psi : \{p\} \times \mathbb{R}^n \longmapsto \{\psi(p)\} \times \mathbb{R}^n$ is volume preserving (it is the adjoint of $d\psi(p)$) and we conclude that $\mathrm{vol}(\Psi(L); q) = \mathrm{vol}(L; q)$. Now given any set U, there is a diffeomorphism of $\mathbb{R}^n$, volume preserving near U, with $\psi(U) \subset [0, a]^n$ for any a such that $a^n > \mathrm{vol}(U)$. This is obviously true for U compact, using Moser's lemma, but we may always reduce ourselves to this case, since for each L, $\pi_p(L)$ is compact. Thus for $\mathrm{vol}(U) < 1$, $\tilde{\rho}(U) \leq \rho_n$ and by homogeneity we must have $\tilde{\rho}(U) \leq \rho_n \, \mathrm{vol}(U)$.

Finally we get

$$d(L)^n \leq \rho_n \, \mathrm{vol}(L; q) \, \mathrm{vol}(\pi_p(L)) \leq \frac{\rho_n}{2^n} \, \mathrm{vol}^2(L).$$

□

We actually proved a stronger statement, that is,

Proposition 3.3. *With the constant ρ_n defined above, we have:*

$$d(L)^n \leq \rho_n \, \mathrm{vol}(L; q) \cdot \mathrm{vol}(\pi_p(L)).$$

Using Chekanov's inequality between displacement energy and minimal area of a holomorphic disc ([Chek]), this implies:

Theorem 3.4. *Given a Lagrangian submanifold L in $\mathbb{R}^{2n}$ there is a holomorphic disc D with boundary in L and area less than $(\rho_n \cdot \mathrm{vol}(L))^{2/n}$.*

This is really the symplectization (in V.I. Arnold's terminology) of the usual isoperimetric inequality.

Remarks. 1. Note that one would like to replace $\mathrm{vol}(L; q)$ by $\mathrm{vol}(\pi_q(L))$ in the above inequality. However the above proof will not work. Even though the best constant in Proposition 3.2, $\rho(U)$, satisfies both $\rho(U) \leq \rho(V)$ for $U \subset V$ and $\rho(\lambda \cdot U) = \lambda^n \rho(U)$, it will not be invariant by volume preserving maps.

Indeed for ψ volume preserving, and Ψ the induced symplectic map, we do not have $\mathrm{vol}(\pi_p(\Psi(L))) = \mathrm{vol}(\pi_p(L))$, so we may not conclude as above. However, $\rho(U)$, the best constant in 3.2 is invariant by the action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n$. Thus $\rho(U)$ is a monotone affine invariant. There are many such invariants (beside the volume), for example the volume of the smallest ellipsoid containing U, or $\tilde{\delta}(U)^n$ where $\tilde{\delta}(U) = \inf\{\mathrm{diam}(AU) \mid A \in SL(n, \mathbb{R})\}$.

An important result would be to decide if $\rho(U)$ is bounded by a constant time the volume, or at least to identify affine invariants that are upper bounds for $\rho(U)$. Clearly $\tilde{\delta}(U)^n$ is such an invariant, but this is not a very good result.

2. From the definition of d it is clear that $d(L) \leq \pi \, \mathrm{diam}(L)^2$ since U is contained in a ball of radius $\mathrm{diam}(U)$. But $\mathrm{diam}(L)$ and $\mathrm{vol}(L)$ are independent quantities (one of them can be large and the other small). However since $\rho(\pi_q(L)) \leq \tilde{\delta}(\pi_q(L))^n \leq \mathrm{diam}(L)^n$ and $\mathrm{vol}(\pi_p(L)) \leq C \, \mathrm{diam}(L)^n$, Proposition 3.2 actually improves on both inequalities.

3. Let us set

$$v(L) = \inf\{\mathrm{vol}(\phi(L)) \mid \phi \text{ is the time one map of a Hamiltonian flow}\}.$$

Then the above proves that $v(L) > 0$. It thus makes sense to see whether the infimum $v(L)$ is achieved, at least by some submanifold with possible singularities. The question of volume minimization was raised by Y.G. Oh in [O2]. A Lagrange submanifold stationary for Hamiltonian deformations is called H-minimal. He proved in particular that the tori $T^n(r_1, \dots, r_n)$ product of n circles of radius r_i are H-minimal, and local minima of the volume.

In fact, it follows from the work of Chekanov that the tori $T^n(r_1, \dots, r_n)$ and $T^n(r'_1, \dots, r'_n)$ (we assume $r_1 \le r_2 \le \dots \le r_n$ and $r'_1 \le r'_2 \le \dots \le r'_n$) are Hamiltonianly equivalent if and only if:

- $r_1 = r'_1$,
- $\{j \mid r_j = r_1\} = \{j \mid r'_j = r'_1\}$,
- the lattices generated by $\{r_j - r_1 \mid j \in [1, n]\}$ and $\{r'_j - r'_1 \mid j \in [1, n]\}$ are isomorphic.

It follows for example that $T(1, 2, 2)$ and $T(1, 2, 3)$ are Hamiltonianly isotopic. Since the first has volume $4\pi^3$ and the second $6\pi^3$, the second one is certainly not an absolute minimum. As a result, using the mountain pass principle, there should be an H-minimal torus which is not a minimum of the volume (however this is meaningless unless some regularity is proved). It would be interesting to construct explicitly such tori.

Still one may conjecture that the torus $T(1, 2, \dots, 2)$ has minimal volume.

Our lower bound does not seem to be sharp enough to imply this. We refer to the work of Schoen and Wolfson for more on minimal Lagrange submanifolds (and currents).

4. The Lagrangian case is the only one for which this type of estimate may hold. Indeed David Hermann exhibited coisotropic submanifolds of any dimension k greater than n with positive capacity and arbitrarily small k-dimensional volume (see [H3]). However, it would be extremely interesting to get a bound on capacities from other metric quantities.

5. Clearly there can be no such inequality in the opposite direction. There are Lagrange submanifolds of arbitrarily large volume and small capacity (take a torus $S^1(\varepsilon) \times S^1(\frac{1}{\varepsilon}) \subset \mathbb{R}^4$). As far as symplectic isoperimetric inequalities are concerned, in the classical sense, there does not seem to be much room for other types of inequalities. One type of quantities one could hope to estimate are the "Quermass-integral". However, in the case of the torus, for example, all these integrals vanish, except for the volume.

3.2. **Deformations of the zero-section in cotangent bundles.** We may apply this theorem to the case of a Lagrange submanifold Hamiltonianly isotopic to the zero section, L. We shall work in $T^*\mathbb{R}^n$ or rather T^*S^n by adding a point at infinity. We consider a manifold L obtained by applying the flow of a compact supported Hamiltonian to the zero section, $0_{\mathbb{R}^n}$. By addition of a point at infinity, we may assume L to be in T^*S^n.

To any such Lagrangian submanifold, we may associate the numbers $c(1, L)$, $c(\mu, L)$, $\gamma(L)$, defined in [V1] (see Appendix A).

For $L_t = \phi_t(0_{S^n})$, we define the support of the deformation to be

$S = \{z \in 0_{\mathbb{R}^n} \mid \exists t \in [0, 1] \;\; \phi_t(z) \neq z\}$,

$\hat{d}(L) = \inf\{\gamma(N_1) \mid N_t = \psi_t(0_{S^n}) \text{ and } N_1 \cap L \cap S = \emptyset\}$ and

$\bar{d}(L) = \inf\{\gamma(\psi_1) \mid \psi_1(0_{S^n}) \cap L \cap S = \emptyset\}$.

In both definitions ψ_t is a compact supported Hamiltonian isotopy starting at the identity.

Note that for $L = \mathrm{graph}(df)$, we have $\bar{d}(L) = \hat{d}(L) = \gamma(L) = \max f - \min f$. This is easy to prove, using $N = graph(dg)$ with $|g - f| \leq \varepsilon$ small, and such that $g - f$ has no critical points in the support of f. Using such an N it follows from the definition of γ, and the fact that g is a generating function for N (see [V1]), that $\gamma(N) = \max g - \min g$. Using the map $\psi(q,p) = (q, p + dg(q))$ we get that $\bar{d}(L) \leq \max g - \min g \leq \max f - \min f + \varepsilon$.

Moreover, we shall see that for L different from the zero section, $\hat{d}(L)$ is nonzero (see Lemma 3.7).

Lemma 3.5. *The following inequalities hold:*

(1) $\gamma(\psi(O_{S^n})) \leq \gamma(\psi)$.
(2) $\bar{d}(L) \geq \hat{d}(L)$.

Proof. The argument is essentially due to David Hermann in his proof of Siburg's conjecture ([H2]).

The second inequality obviously follows from the first one. Let $\Gamma(\psi)$ be the graph of ψ: $\Gamma(\psi) = \{(z, \psi(z)) \mid z \in T^*\mathbb{R}^n\} \subset \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n$.

We shall consider $\psi(O_{S^n})$ as the reduction of $\Gamma(\psi) \times O_{S^n}$, by the coisotropic submanifold $\{(x, y, x) \in \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n\}$. Let C be a coisotropic linear subspace of the symplectic vector space $V \oplus V^*$.

Lemma 3.6. *Assume that C is transverse to V. Then after applying some linear symplectic transformation, preserving V, we have the decompositions $V = W \oplus S$ where $W = V \cap C$ and $C = W \oplus W^* \oplus \{0\} \oplus S^* \subset W \oplus W^* \oplus S \oplus S^*$. Moreover C/C^ω may be identified to $W \oplus W^*$.*

Proof. The proof is left to the reader, using the transitivity of the action of the symplectic group on pairs of transverse isotropic subspaces. This is applied to the isotropic subspaces $I = C^\omega$ and V, which by assumption satisfy $I \cap V = \emptyset$. $\square$

Now $\overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ may be identified to $T^*(\Delta) \times T^*\mathbb{R}^n$. Our coisotropic submanifold is then the conormal of a factor of $\Delta \times \mathbb{R}^n$. As a result, if $F : (\Delta \times \mathbb{R}^n) \times \mathbb{R}^l \mapsto \mathbb{R}$ is a generating function quadratic at infinity for a Lagrange submanifold $L \subset T^*(\Delta \times \mathbb{R}^n)$, then the reduction of L, L_C, has G as a generating function quadratic at infinity, where G is the restriction of F to $(C \cap (\Delta \times \mathbb{R}^n)) \times \mathbb{R}^l$.

According to [V1] (Proposition 5.1) we then have $\gamma(G) \leq \gamma(F)$. This concludes our proof. $\square$

Lemma 3.7. *We have $\gamma(L) \leq 2\hat{d}(L)$.*

Proof. Indeed, let L_t be a Hamiltonian isotopy from the zero section to $L = L_1$, let F_t be a generating function quadratic at infinity for L_t, and let R be a generating function quadratic at infinity for N, where $N = \psi_1(0_{S^n})$ and $N \cap L \cap S = \emptyset$. Then $c(\alpha, F_t - R)$ does not depend on t. Indeed it is associated to points in $L_t \cap N$ and this set is independent of t by assumption on N. The standard argument as in [V1] allows us to conclude that $c(\alpha, F_t - R)$ is constant. But we have, again from [V1], the inequality

$$c(\mu, R - F_1) \geq c(\mu, -F_1) + c(1, R).$$

Thus $\gamma(R) = c(\mu, R) - c(1, R) \geq c(\mu, -F_1)$.

Finally, since $c(\mu, -F_1) = -c(1, F_1)$, we have $\gamma(N) = \gamma(R) \geq -c(1, L)$. Similarly we prove $\gamma(N) \geq c(\mu, L)$, and the inequality $2 \cdot \gamma(N) \geq \gamma(L)$. $\square$

Theorem 3.8. *For K compact and L in T^*K, we have*

$$\gamma(L)^n \leq \rho_K(\mathrm{vol}(L)).$$

Proof. The proof is similar to that of Proposition 3.2. We first need to prove that

$$\bar{d}(L)^{k+1} \leq C_{k+1} \int \bar{d}(L_x)^k dx.$$

This is proved similarly to Proposition 2.2. The rest of the proof is then the same as that of the inequality

$$\bar{d}(L)^n \leq \rho_K(\mathrm{vol}(L))$$

in Proposition 3.2. We then conclude using Lemma 3.7. $\square$

Remark. One can easily prove that in fact $\gamma(L)^n \leq \rho_K(\mathrm{vol}(L;p))$, and as before, that $\rho_K \leq \rho_n \,\mathrm{vol}(K)$.

Note that already in dimension 2, we see that the dependence is on K, not on the support of the deformation, S.

Corollary 3.9. *We have for any C^2 function f with support on K*

$$\max_K |f| \leq (\rho_n)^{1/n} \,\mathrm{vol}(K)^{1/n} \cdot \left(\int_K |\det(\frac{\partial^2 f}{\partial x_i \partial x_j})| dx_1 \dots dx_n \right)^{1/n}.$$

Proof. Set $L = \{(x, df(x)) \mid x \in \mathbb{R}^n\}$, and use the fact that $\gamma(L) = \max_K f - \min_K f$ and $\mathrm{vol}(L) = \int_K \sqrt{\det\left(I + (\frac{\partial^2 f}{\partial^2 x_i x_j})^2\right)}$. By rescaling f to λf and letting λ go to infinity we get that $\mathrm{vol}(L)$ is equivalent to

$$\lambda^n \cdot \int_K |\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)| dx_1 \dots dx_n$$

while the left hand side is multiplied by λ. We then apply the above theorem to conclude our proof. $\square$

Remark. This is a kind of Alexandroff-Bakelman-Pucci estimate (see [CC], pp. 27-28). Such an estimate for the operator $\det(I + \partial_{i,j} u)^{1/n} = M(u)$ may be rewritten as

$$\max u \leq C(\mathrm{vol}(K))^{1/n} \|M(u)\|_{L^n}$$

for u convex. The fact that the constant is of the type $C(\mathrm{vol}(K))^{1/n}$ is due to Cabré in [Ca].

Note that here we replaced the convexity by the compact support assumption. We presumably can get rid of this last assumption by adding a boundary term to the right hand side.

Similarly we get estimates for $\gamma(S)$ as a function of the integral on Σ_S of the second partial derivatives of S.

Theorem 3.10. *Let $S(q, \xi)$ be a generating function quadratic at infinity on $N \times R^k$. Assume that $S(q, \xi) = Q(\xi)$ for q outside a compact set K. Set $\Sigma_S = \{(x, \xi) \mid \frac{\partial S}{\partial \xi} = 0\}$, $A(q, \xi) = \frac{\partial^2 S}{\partial q^2}$, $B(x, \xi) = \frac{\partial^2 S}{\partial q \partial \xi}$, and $C(q, \xi) = \frac{\partial^2 S}{\partial \xi^2}$.*

Then we have:

$$\gamma(S) \leq (\rho_n)^{1/n} \,\mathrm{vol}(K)^{1/n} \left(\int_{\Sigma_S} |\det\left(A(q,\xi) - B(q,\xi) C^{-1}(q,\xi) B^*(q,\xi)\right)| dq_1 \dots dq_n \right)^{1/n}.$$

In particular this means that S has at least one critical point with absolute value less than the right hand side term.

For a subset K in $\mathbb{R}^n$ we define $\widetilde{K}$ to be the set of midpoints of pairs of points of K, that is, $\widetilde{K} = \{\frac{x+y}{2} \mid (x,y) \in K^2\}$. Note that for K convex, $\widetilde{K} = K$.

Corollary 3.11. *There is a constant $C_K = C(\mathrm{vol}(\widetilde{K}))^{1/2n}$ such that any symplectic isotopy with support in K satisfies:*

$$\begin{aligned}\gamma(\psi) &\leq C_K \left(\int_{\mathbb{R}^{2n}} |\det(d\psi(z) - Id)| dq_1 \dots dq_n dp_1 \dots dp_n\right)^{1/2n} \\ &\leq C_K \|\psi - Id\|_{W^{1,2n}(K)}.\end{aligned}$$

Proof. This is left to the reader. □

Corollary 3.12. *Let K be a compact set in $\mathbb{R}^{2n}$. Then we have*

$$\gamma(K) \leq \rho(K)\,\mathrm{vol}(\widetilde{K})^{1/2n} \leq \rho_n \tilde{\delta}(K)\,\mathrm{vol}(\widetilde{K})^{1/2n}.$$

Proof. This is a consequence of the definition of $\gamma(K)$, and the fact that the projection of the graph L of a diffeomorphism of K on the diagonal is contained in $\widetilde{K}$ while its projection on the antidiagonal is contained in $\{J(\frac{x-y}{2}) \mid (x,y) \in K^2\}$ with affine diameter at most $\tilde{\delta}(K)$. □

Note that for K convex $\tilde{\delta}(K)$ and $\mathrm{vol}(\widetilde{K})$ are both equivalent to $\mathrm{vol}(K)$. Hence in this case the right-hand side may be replaced by $(\mathrm{vol}(K))^{1/n}$ (up to a constant). We refer to section 5 for a simpler approach in the convex case.

Remark. One may ask what happens for L not Hamiltonianly isotopic to the zero section. First of all we need a new definition of distance, for example we may set

$$\hat{\gamma}(L) = \sup\{\gamma(N_1) \mid N_t = \psi_t(O) \text{ and } \mathrm{supp}(\psi_t) \cap L = \emptyset\}.$$

We may prove as before that $\hat{\gamma}(L) \leq C\,\mathrm{vol}(L;p)^{1/n}$. We just point out that when L is Hamiltonianly isotopic to the zero section, $\hat{\gamma}(L) \leq 2\gamma(L)$, as we easily prove with the same argument as in Lemma 3.7. But the inequality for $\hat{\gamma}(L)$ may not be used to prove the estimate on $\gamma(L)$.

3.3. **Generalization to the case of $\mathbb{C}P^n$.** We now deal with the symplectic manifold $\mathbb{C}P^n$.

We denote by L a Lagrangian submanifold in $\mathbb{C}P^n$, and by $\tilde{L}$ its lift to S^{2n+1}. We denote by $d(L)$ the displacement energy in $\mathbb{C}P^n$, and by $\tilde{d}(L)$ the displacement energy of $\tilde{L}$ in $\mathbb{R}^{2n+2}$.

We clearly have

1. $\tilde{d}(L) \leq d(L)$,
2. $\tilde{d}(L) \leq \pi (= d(S^{2n+1}))$,

and it is tempting to conjecture:

1. $\tilde{d}(L) = \pi \Rightarrow d(L) = +\infty$ (example: $L = \mathbb{R}P^n$),
2. $\tilde{d}(L) \simeq 2\arctan(d(L))$, or at least $f(d(L)) \leq \tilde{d}(L)$ for some continuous function f with $f(0) = 0$.

The above would imply an estimate between $\mathrm{vol}(L)$ and $f(d(L))$ for some function f.

At present we have:

Proposition 3.13.

$$\tilde{d}(L)^n \leq \rho_n \operatorname{vol}^2(L).$$

Proof. Indeed, $\operatorname{vol}(\tilde{L}) = \operatorname{vol}(L)$ up to a constant factor (equal to the volume of S^1). The proof is then a consequence of Theorem 3.1. $\square$

Remarks. 1. Note that Chekanov's inequality relating $d(L)$ to $\sigma(P, L)$, the smallest area of a holomorphic curve with boundary in L, is far from optimal here, since for $\mathbb{R}P^n$, $d(L) = \infty$ while $\sigma(P, L) = \pi$. Here $\tilde{d}(L) = \pi$ and this is much better. Can one generalize this?

2. We refer to Reznikov's paper [Rez2] for an elementary but sometimes efficient approach to finding a lower bound for the volume of the images of L by Hamiltonian deformation. The idea is that if L bounds in (M, ω) we may write $L = \partial W^{n+1}$ and for $n = 2k-1$ the number $\rho(L) = \int_W \omega^k$ is—under some topological assumption—invariant in $\mathbb{R}/\Gamma$ where Γ is the image of $H_{2k}(M)$ under $c \to \int_c \omega^k$. Since W may be chosen with volume at most C times the volume of L, for some constant depending only on the ambient manifold M, we get that when $\rho(L) \neq 0$ the volume of L is bounded from below.

For example for n odd, if $M = \mathbb{C}P^n, L = \mathbb{R}P^n$, Reznikov proves $\rho(L) = 1/2$. Thus in this case, one recovers the Givental-Kleiner-Oh result by elementary means.

3. The case of cotangent bundles is also interesting. In a previous version we claimed to prove the case of T^*T^n as an easy consequence of the case of Euclidean space. We refer to current work of Oancea (in particular for T^*S^n) for recent progress on these questions.

4. An application to billiards

Theorem 4.1. *Let U be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then there exists a billiard trajectory on U of length ℓ with*

$$\ell^n \leq C_n \operatorname{vol}(U).$$

Remark. The existence of a periodic billiard trajectory is due to [BG]. They show that a trajectory with at most $n+1$ bounces exists, for any domain U and any metric on U. Most likely one can find a trajectory with at most $n+1$ bounces, satisfying the above length estimate.

Proof. Indeed the periodic trajectories on approximations of $W = U \times D^n$ (see [BG]) converge to billiard trajectories on U (we mean the lift to T^*U of such trajectories). And the action of such a trajectory corresponds to the length of the trajectory. So it is enough to show that $d(W) \leq C \cdot \operatorname{vol}(U)^{1/n}$.

We may use again Proposition 2.2. Indeed, we argue once more by induction to prove that for $W = U \times [0,1]^n$ and $W(x) = U(x) \times [0,1]^n$, where $U(x) = U \cap \mathbb{R}^{n-1} \times \{x\}$ and $W_x = U(x) \times [0,1]^{n-1}$, the following inequalities hold:

$$d(W)^n \leq C_n \int d(W_x)^{n-1} dx \leq C_n \rho_{n-1} \cdot \int \operatorname{vol}(U(x))^{n-1} dx = \rho_n \operatorname{vol}(U).$$

Now U has a billiard trajectory of area less than $d(U \times D^n) \leq d(U \times [0,1]^n) \leq \rho_n \operatorname{vol}(U)^{1/n}$. $\square$

Remarks. 1. This result is a kind of isosystolic inequality in a simply connected situation. We propose to call this an *isodiastolic inequality* since it is an estimate about the length of geodesic curves obtained by mountain pass, like the great circles on the two-sphere, and not by minimization as is the case for the usual isosystolic inequality.

A theorem by C. Croke ([Cr]) proves that for any metric on S^2 we have a geodesic of length ℓ with $\ell^2 \leq 961 \cdot \mathrm{vol}(S^2)$. The proof is rather subtle and beautiful, using a careful analysis of the Birkhoff curve shortening process. This is purely two-dimensional, while our results hold in any dimension. However there is also a result in Croke's paper about convex hypersurfaces in $\mathbb{R}^{n+1}$, of the same type (i.e. $\ell^n \leq C_n \,\mathrm{vol}(M)$). One should mention that our result only holds for some flat metric, while Croke's result holds for any metric.

2. Given some metric on M, and a submanifold N of M, let $D(M,g)$ be the unit disc bundle for the metric g in T^*M. Then, the reduction of $D(M,g)$ by the conormal $\nu(N) = \{(q,p) \mid p = 0 \text{ on } N\}$ is the disc bundle $D(N,k)$ for the metric k such that $k(q,p) = \inf\{g(q,\tilde{p}) \mid p = \tilde{p} \text{ on } T_qN\}$. Thus the unit disc bundle for T_q^*N is on each fiber the projection of the unit disc of T_q^*M on T_q^*N. Note that this is **not** the unit disc bundle for the induced metric, as this would be $h(q,p) = g(q,p_0)$ where p_0 is the unique cotangent vector on M such that $p_0 = p$ on T_qN and $p_0 = 0$ on the orthogonal of T_qN. Its unit disc bundle is the intersection of the unit disc bundle of T^*M with T^*N (the inclusion of T^*N into T^*M is obtained through the metric).

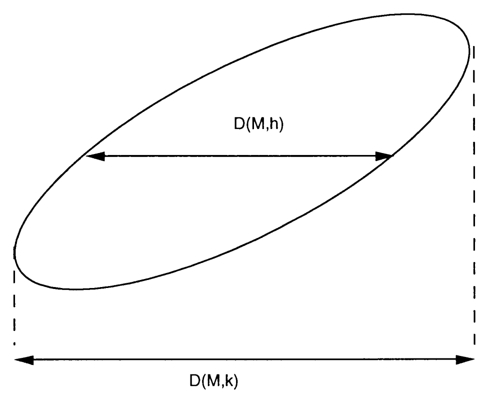

FIGURE 1. Illustration of Remark 2

From the last remark and the fact that any two metrics on the 2-disc are conformally equivalent, we get

Proposition 4.2. *Let U be a topological 2-disc. Then for any metric g on U, we have a billiard trajectory of length ℓ with*

$$\ell^2 \leq \rho_2 \,\mathrm{vol}_g(U)$$

where $\rho_2 \leq 32$.

Proof. Indeed, let us consider the metric $g(x,y)(dx^2+dy^2)$ on U. We shall in this proof identify the metric with the function. Let

$$W = D(U,g) = \{(x,y,\xi,\eta) \mid \frac{1}{g(x,y)}(\xi^2+\eta^2) \leq 1\}.$$

As above, then $W_x = D(U(x), g_x)$, where g_x is the function $g_x(y) = g(x,y)$. Indeed, here the projection of the unit disc for g on $\xi = \xi_0$ coincides with the intersection of the disc with $\xi = \xi_0$!

We use once more Proposition 2.1 as above:

$$d(W) \leq C_n \int d(W_x))dx \leq C_n \rho_{n-1} \cdot \int \mathrm{vol}(U(x), g_x)dx = C_n \rho_{n-1} \cdot \mathrm{vol}(U,g).$$

□

Remark. A similar inequality holds whenever the metric varies in a set such that the ratio between the projections and the sections of $D(z,g) \subset (\mathbb{R}^n)^*$ remain bounded by some constant k. In this case, we get a periodic orbit of the billiard problem of length ℓ with

$$\ell^n \leq \rho_n k^n \,\mathrm{vol}(U,g).$$

In particular, for a given metric on a compact manifold there is such an inequality holding for all metrics conformally equivalent to g_0, with the same constant.

Note that Theorem 4.1 corresponds to the case $D(z,g) = [0,1]^n$ and then $k = 1$.

5. Geometry of convex sets and periodic orbits

Theorem 5.1. *Let C be a convex set in $\mathbb{R}^{2n}$, let B be the unit ball, and denote by d any symplectic capacity. Then we have*

$$\frac{d(C)}{d(B)} \leq \gamma_n \left(\frac{\mathrm{vol}(C)}{\mathrm{vol}(B)}\right)^{1/n}$$

where $\gamma_n \leq 4n$ for a general convex set, and $\gamma_n \leq n$ if C is centrally symmetric.

Proof. Let E be the Loewner-Behrend-John ellipsoid, that is, the ellipsoid of minimal volume containing C. According to F. John (see [J]; a proof has been included for the reader's convenience in Appendix B) we have:

$$\gamma_n^n \,\mathrm{vol}(E) \leq \mathrm{vol}(C).$$

Let $r_1 \leq r_2 \leq \ldots \leq r_{n-1} \leq r_n$ be the symplectic axis of E. This means that in suitable (linear) symplectic coordinates, we have

$$E = \{(x_1,y_1,x_2,y_2,\ldots,x_n,y_n) \mid \sum_{i=1}^n \frac{1}{r_i^2}(x_i^2+y_i^2) \leq 1\}.$$

Then $\frac{d(E)}{d(B)} = r_1^2$ and $\frac{\mathrm{vol}(E)}{\mathrm{vol}(B)} = r_1^2 \cdot r_2^2 \cdot \ldots \cdot r_n^2$. Thus $\frac{d(E)}{d(B)} \leq \left(\frac{\mathrm{vol}(E)}{\mathrm{vol}(B)}\right)^{2/n}$.

Now

$$\frac{d(C)}{d(B)} \leq \frac{d(E)}{d(B)} \leq \left(\frac{\mathrm{vol}(E)}{\mathrm{vol}(B)}\right)^{2/n} \leq n^2 \cdot \left(\frac{\mathrm{vol}(C)}{\mathrm{vol}(B)}\right)^{2/n}.$$

Note that for a centrally symmetric set, $\frac{1}{\sqrt{n}}E \subset C$ holds, so n^2 may be replaced by n.

□

Remarks. 1. Of course, one would like to get an estimate independent of n. It is reasonable to conjecture that the constant equals one, and is only achieved by the ball. This sounds like a hard problem.

2. One proves similarly, using the Ekeland-Hofer capacities (see [EH2]), that $\gamma(C) = (c_1(C) \cdot \ldots \cdot c_n(C))^{1/n}$ satisfies

$$\frac{\gamma(C)}{\gamma(B)} \leq n^2 \cdot \left(\frac{\operatorname{vol}(C)}{\operatorname{vol}(B)}\right)^{1/n}.$$

Remark. As A. Weinstein mentioned long ago, convexity is not a symplectically invariant property, while our inequality is symplectically invariant. One would expect, as in the case of periodic orbits, that the result may be extended on any domain bounded by a hypersurface of contact type.

However this is not true, as the following example shows. But it leaves open the question of the correct generalisation of convexity. Restricted contact type could be a candidate.

Example 5.2. Let L be a Lagrangian submanifold. According to Chekanov ([Chek]; see also [V3] for the case of tori, and [Pol1] for the rational case) it has strictly positive displacement energy. But a tubular neighborhood of L has arbitrarily small volume, and positive capacity! However this neighborhood is of restricted contact type only if L is exact, and this is impossible[3] in $\mathbb{R}^{2n}$.

6. Compensated compactness and closure of the symplectic group

Let f, g be two C^1 functions on a symplectic manifold. We denote by $\{f, g\}$ their Poisson bracket. It is defined by $\{f, g\} = \omega(X_f, X_g)$.

In a Darboux chart, where $\omega = \sum_{j=1}^n dp_j \wedge dq^j$, we have

$$\{f, g\} = \sum_{j=1}^n \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j}.$$

Let k, p be positive numbers, and let $W^{k,p}$ be the Sobolev spaces of functions with k distributional derivatives belonging to L^p.

Theorem 6.1. *Let f_n, g_n be a sequence of functions such that $\{f_n, g_n\} = 1$. Then if $f_n \to f$ and $g_n \to g$ in $W^{1/2,2}$, we must have $\{f, g\} = 1$.*

Corollary 6.2. *The group of symplectic diffeomorphisms is closed for the $W^{1/2,2}$ norm in the set of smooth maps from $\mathbb{R}^{2n}$ into itself.*

Proof. Let ϕ be a compactly supported test function. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \{f_n, g_n\}\omega^n \phi \\
&= \int_{\mathbb{R}^{2n}} df_n \wedge dg_n \wedge \omega^{n-1}\phi \\
&= \int_{\mathbb{R}^{2n}} d(f_n dg_n) \wedge \omega^{n-1}\phi \\
&= -\int_{\mathbb{R}^{2n}} f_n dg_n \wedge \omega^{n-1} \wedge d\phi.
\end{aligned}$$

[3] David Hermann constructed starshaped Reinhardt domains of arbitrarily small volume and capacity equal to 1 (see [H3]).

By assumption, $f_n dg_n \to f dg$ weakly, hence doing the same computation backwards we have:

$$\lim_n \int_{\mathbb{R}^{2n}} \{f_n, g_n\}\omega^n \phi = \int_{\mathbb{R}^{2n}} \{f, g\}\omega^n \phi.$$

$\square$

Remarks. 1. What about C^0 convergence? There are several open questions, knowing that according to Gromov, if f_i^p, g_i^p is a sequence such that

$$\{f_i^p, g_j^p\} = \delta_{ij}; \quad \{f_i^p, f_j^p\} = \{g_i^p, g_j^p\} = 0$$

and if $f_i^p \to f_i; g_i^p \to g_j$ we have

$$\{f_i, g_j\} = \delta_{ij}; \quad \{f_i, f_j\} = \{g_i, g_j\} = 0.$$

2. Is there a counter-example to C^0 convergence if we only have one pair of functions f, g?

In other words, is it true that if $f^p \to f$ and $g^p \to g$ for the C^0 topology and $\{f^p, g^p\} = 1$, then $\{f, g\} = 1$?

3. Can one recover Gromov's theorem from compensated compactness techniques? Is the following statement true: Let f_i^p, g_j^p be such that

$$\{f_i^p, g_j^p\} = \alpha_{ij}^p; \quad \{f_i^p, f_j^p\} = \beta_{ij}^p; \quad \{g_i^p, g_j^p\} = \gamma_{ij}^p.$$

If $\alpha_{ij}^p \to \alpha_{ij}; \quad \beta_{ij}^p \to \beta_{ij}; \quad \gamma_{ij}^p \to \gamma_{ij}$ and $f_i^p \to f_i; g_i^p \to g_j$, then

$$\{f_i, g_j\} = \alpha_{ij}; \quad \{f_i, f_j\} = \beta_{ij}; \quad \{g_i, g_j\} = \gamma_{ij}.$$

This should hold (by Gromov's theorem) if the form $\sum_{i,j=1}^n \alpha_{ij} dq^i \wedge dp^j + \beta_{ij} dq^i \wedge dq^j + \gamma_{ij} dp^i \wedge dp^j$ is symplectic, that is, nondegenerate. Is one of these two assumptions sufficient? Are there counter-examples without these assumptions?

Appendix A. A summary of symplectic geometry through generating functions

We recall some facts about the symplectic invariants defined in [V1] for Lagrange submanifolds. Let $L \subset T^*N$ be a Lagrangian submanifold and let $S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$ be a smooth function. We say that S is a generating function of L if

$$L = \{(x, \frac{\partial S}{\partial x}(x, \xi)) \mid \frac{\partial S}{\partial \xi}(x, \xi) = 0\}.$$

We shall say that S is quadratic at infinity if it coincides with a nondegenerate quadratic form in ξ for ξ large enough.

According to [LauS], any Lagrangian submanifold Hamiltonianly isotopic to the zero section has such a generating function. The variational methods invented by Lusternik and Schnirelman allow us to define, for each cohomology class α on N, a critical value $c(\alpha, S)$. Indeed, for c large enough $H^*(S^c, S^{-c}) = H^*(Q^c, Q^{-c}) = H^{*-d}(N)$ where d is the dimension of the negative eigenspace of Q in the fiber variables, and the last isomorphism is Thom's. Thus each cohomology class in N may be identified to a class in $H^*(S^c, S^{-c})$, and we set:

$$c(\alpha, S) = \inf\{\lambda \mid \alpha \neq 0 \text{ in } H^*(S^\lambda, S^{-c})\}.$$

It is proved in [V1] and [Th] that for L Hamiltonianly isotopic to the zero section, $c(\alpha, S)$ only depends on L, and we denote it by $c(\alpha, L)$.

For $N = S^n$, we have only two invariants, $c(1, L)$ and $c(\mu, L)$ ($\mu \in H^n(S^n)$ is the generator), and we define an invariant $\gamma(L) = c(\mu, L) - c(1, L)$. We refer to [V1] for a detailed study of such invariants.

The main properties of c and γ are as follows (we refer to [V1] for proofs), where we use the notation $S_1 \oplus S_2(x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2)$.

Theorem A.1. 1. *$\gamma(L) = 0$ if and only if L coincides with the zero section.*
2. *$c(\mu, -S) = -c(1, S)$.*
3. *$c(\mu, S_1 \oplus S_2) \geq c(1, S_1) + c(\mu, S_1)$.*
4. *$c(\mu, S)$ is a critical value of S.*

One may associate to any open set U a number $c(U)$ by setting

$$c(U) = \sup\{c(L) \mid L \subset \Delta \cup U \times U\}$$

where Δ is the diagonal in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and c is measured by identifying this space with $T^*\Delta$.

This number is an analogue of the Ekeland-Hofer capacity. In fact we may use c instead of c_{EH}; it will still satisfy Proposition 1.2.

Appendix B. John's ellipsoid

It was pointed out by Albert Fathi that the following proof is to be found verbatim in Bollobas' book [Bollo]. It is nothing else than a modern formulation of John's original proof. We give here the proof of John's theorem, as Bollobas' book appears to be hard to find.

Theorem B.1 (John)**.** *Let C be a convex set in $\mathbb{R}^n$ with nonempty interior. Let E be the unique minimal volume ellipsoid containing C. Then we have*

1. *If C is symmetric,*
$$\frac{1}{\sqrt{n}} E \subset C \subset E.$$
2. *In general,*
$$\frac{1}{n} E \subset C \subset E$$
and
$$\frac{1}{4^n n^{n/2}} \operatorname{vol}(E) \leq \operatorname{vol}(C).$$

Proof. The existence and uniqueness of such an ellipsoid, due to Loewner and Behrend, follows from the following remarks:

1. The equation of an ellipsoid containing C is $\langle H(u-x), u-x\rangle \leq 1$, depending on the parameters (x, H) where x is the center of the ellipsoid and H is a positive definite symmetric matrix. The volume of the ellipsoid is given by $\det(H)^{-1}$, so we want $\det(H)$ to be maximal.
2. The existence follows from the fact that C contains a ball of positive volume. Now the set of all ellipsoids of bounded volume containing a given ball is compact.
3. If $\langle H(u - x), u - x\rangle \leq 1$ and $\langle K(u - y), u - y\rangle \leq 1$ on C, we have that for $L = \frac{H+K}{2}$, $\langle L(x - z), x - z\rangle \leq 1$, for z given by $Lz = \frac{1}{2}(Hx + Ky)$. Since $H \to \det(H)$ is concave, we have $\det(L) \geq \frac{1}{2}(\det(H) + \det(K))$, and the ellipsoid of minimal volume is unique.

By a linear change of coordinates, we may assume that the minimal volume ellipsoid is a sphere of radius 1. We shall provide a lower bound on the volume of C. For simplicity we may assume that C is a convex polytope, the convex envelope of a finite number of extremal points.[4]

The general result will follow from a limiting argument.

Assume the unit sphere to be the ellipsoid of minimal volume containing C; C must touch the sphere at (extremal) points $x_1, \dots, x_d$ such that for all (x_0, H) small enough, we must have that

$$\forall j \in [1,d],\ \langle (I+H)(x_j - x_0), x_j - x_0\rangle < 1 \text{ implies } \det(I+H) < 1$$

(this expresses the fact that if the ellipsoid $\langle (I+H)(x-x_0),(x-x_0)\rangle \leq 1$ contains C, its volume is larger than 1, and to check that C is contained in the ellipsoid it is enough, for (H, x_0) small enough, to check that the points x_j are in the ellipsoid).

Keeping only first order terms in (x_0, H), we get the following linear inequality on the set of pairs (x_0, H):

$$\forall j \in [1,d],\ \langle Hx_j, x_j\rangle - 2\langle x_j, x_0\rangle < 0 \text{ implies } \operatorname{trace} H < 0.$$

We must then have numbers $\lambda_1, \dots, \lambda_d$ such that $\lambda_j \geq 0$ and

$$\sum_{j=1}^{d} \lambda_j(\langle Hx_j, x_j\rangle - 2\langle x_j, x_0\rangle) = \operatorname{trace}(H),$$

that is,

$$\sum_{j=1}^{d} \lambda_j x_j = 0 \text{ (take } H = 0),$$

$$\sum_{j=1}^{d} \lambda_j \langle Hx_j, x_j\rangle = \operatorname{trace}(H) \text{ (take } x_0 = 0).$$

Applying this to $H = Id$ and then to $H(x) = \langle u, x\rangle u$ we get

$$\sum_{j=1}^{d} \lambda_j = n,$$

$$\sum_{j=1}^{d} \lambda_j \langle u, x_j\rangle^2 = 1.$$

These inequalities imply that for any u, there is a j such that

$$|\langle u, x_j\rangle| \geq \frac{1}{\sqrt{n}}.$$

If C is centrally symmetric, it contains a ball of radius $\frac{1}{\sqrt{n}}$ and $\frac{\operatorname{vol}(C)}{\operatorname{vol}(B)} \geq \frac{1}{n^{n/2}}$. Otherwise we have

$$\sum_{j=1}^{d} \lambda_j(\langle u, x_j\rangle^2 - t\langle u, x_j\rangle - \frac{1}{n}) = 0.$$

[4] An extremal point is a point that is not the convex combination of two distinct points of C.

This implies that for any t there is an i such that $\langle u, x_i\rangle^2 - t\langle u, x_i\rangle - \frac{1}{n} > 0$ and a k such that $\langle u, x_k\rangle^2 - t\langle u, x_k\rangle - \frac{1}{n} < 0$. In other words the $(\langle u, x_j\rangle)_{j\in 1,\dots,d}$ cannot all lie in an interval $[\alpha, \beta]$ with $\alpha \cdot \beta = \frac{1}{n}$. Since $|\langle u, x_j\rangle| \leq 1$ we have $\max_j\langle u, x_j\rangle \geq \frac{1}{n}$ and $\min_j\langle u, x_j\rangle \leq -\frac{1}{n}$. Thus C contains the ball of radius $\frac{1}{n}$.

Let us point out that if we only require a lower bound on the volume of C, we may improve on the nonsymmetric case as follows.

Set $K = C + -C = \{x - y \mid x, y \in C\}$, and assume E is the minimal ellipsoid containing K. Again, after applying a linear transformation we may assume that E is the unit ball. We have that $C \subset B$, and since $\mathrm{vol}(K) \geq \frac{1}{n^{n/2}}\mathrm{vol}(B)$ and $\mathrm{vol}(C) \geq 4^{-n}\mathrm{vol}(K)$ we get $\frac{\mathrm{vol}(C)}{\mathrm{vol}(B)} \geq \frac{1}{4^n n^{n/2}}$. $\square$

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