REAL RATIONAL CURVES IN GRASSMANNIANS

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INTRODUCTION

Fulton asked how many solutions to a problem of enumerative geometry can be real, when that problem is one of counting geometric figures of some kind having specified position with respect to some given general figures [5]. For the problem of plane conics tangent to five general (real) conics, the surprising answer is that all 3264 may be real [13]. Similarly, given any problem of enumerating \( p \)-planes incident on some given general subspaces, there are real subspaces such that each of the (finitely many) incident \( p \)-planes is real [17]. We show that the problem of enumerating parameterized rational curves in a Grassmannian satisfying simple (codimension 1) conditions may have all of its solutions real.

This problem of enumerating rational curves in a Grassmannian arose in at least two distinct areas of mathematics. The number of such curves was predicted by the formula of Vafa and Intriligator [20, 8] from mathematical physics. It is also the number of complex dynamic compensators which stabilize a given linear system, and the enumeration was solved in this context [12, 11]. The question of real solutions also arose in systems theory [3]. This application will be discussed in Section 4.

1. STATEMENT OF RESULTS

Fix integers \( m, p > 1 \) and \( q \geq 0 \). Set \( n := m + p \). Let \( G \) be the Grassmannian of \( p \)-planes in \( \mathbb{C}^n \). The space \( M_q \) of maps \( M : \mathbb{P}^1 \to G \) of degree \( q \) has dimension \( N := pm + qn \) [4, 19]. If \( L \) is an \( m \)-plane and \( s \in \mathbb{P}^1 \), then the collection of all maps \( M \in M_q \) satisfying \( M(s) \cap L \neq \{0\} \) is an irreducible subvariety of codimension 1. Consider the following enumerative problem.

Given general points \( s_1, \ldots, s_N \) in \( \mathbb{P}^1 \) and general \( m \)-planes \( L_1, \ldots, L_N \) in \( \mathbb{C}^n \), how many maps \( M \in M_q \) satisfy \( M(s_i) \cap L_i \neq \{0\} \) for \( i = 1, \ldots, N \)?

Since \( G \) is a homogeneous space, Kleiman’s Theorem [9] shows there are finitely many solutions and no multiplicities.

Rosenthal [14] interpreted the solutions as a linear section of a projective embedding of \( M_q \), and Ravi, Rosenthal, and Wang [12, 11] gave a formula for the degree \( d \) of its closure \( K_q \) in this embedding. Thus the number of solutions (counted with multiplicity) is at most \( d \). The difference between \( d \) and the number of solutions counts points common to both the linear section and to the boundary \( K_q - M_q \) of \( K_q \).

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Bertram [1] studied this and other intersection problems on $\mathcal{M}_q$ using a different compactification $Q_q$ of $\mathcal{M}_q$. He used an explicit moving lemma to show there are finitely many solutions to these problems on $Q_q$ with none in the boundary $Q_q - \mathcal{M}_q$ of $Q_q$. He also determined the small quantum cohomology ring of $G$, which gives formulas for these intersection numbers. For our problem [1], the formula of Bertram coincides with the formula of Ravi, Rosenthal, and Wang. This shows there are no points common to both the linear section and to the boundary of $K_q$, and so there are $d$ solutions to [1].

When the $s_i$ and $L_i$ are real, there may be fewer than $d$ real solutions. We show there are real $s_i$ and $L_i$ such that each of the $d$ solutions are real.

**Theorem 1.1.** There exist $m$-planes $L_1, \ldots, L_N$ in $\mathbb{R}^n$ and points $s_1, \ldots, s_N \in \mathbb{P}^1_R$ so that there are exactly $d$ maps $M: \mathbb{P}^1 \to G$ of degree $q$ which satisfy $M(s_i) \cap L_i \neq \{0\}$ for each $i = 1, \ldots, N$, and each of these are real.

Our proof is elementary in that it argues from the equations for the locus of maps $M$ which satisfy $M(s) \cap L \neq \{0\}$. A consequence is that we obtain fairly explicit choices of $s_i$ and $L_i$ that give only real maps, which we discuss in Section 4. Our proof uses neither Kleiman’s Theorem nor Bertram’s Moving Lemma, and thus it provides a new and elementary proof that there are $d$ solutions to the enumerative problem [1].

2. **The Quantum Grassmannian**

The space $\mathcal{M}_q$ of maps $\mathbb{P}^1 \to G$ of degree $q$ is a smooth quasi-projective algebraic variety. A smooth compactification is provided by a quotient scheme $Q_q$ [19]. By definition, there is a universal exact sequence

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1} \to T \to 0$$

of sheaves on $\mathbb{P}^1 \times Q_q$ where $S$ is a vector bundle of degree $-q$ and rank $p$. Twisting the determinant of $S$ by $\mathcal{O}_{\mathbb{P}^1}(q)$ and pushing forward to $Q_q$ induces a Plücker map

$$Q_q \to \mathbb{P}(\Lambda^p \mathbb{C}^n \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q))^\ast)$$

which is the analog of the Plücker embedding of $G$. The Plücker map is an embedding of $\mathcal{M}_q$, and so its image $K_q$ provides a different compactification of $\mathcal{M}_q$. We call $K_q$ the quantum Grassmannian. (This space is called the Uhlenbeck compactification in [2].) Our proof of Theorem [1] exploits some of its structures that were elucidated in systems theory.

The Plücker map fails to be injective on the boundary $Q_q - \mathcal{M}_q$ of $Q_q$. Indeed, Bertram [1] constructs a $\mathbb{P}^{n-1}$ bundle over $\mathbb{P}^1 \times Q_{q-1}$ that maps onto the boundary of $Q_q$, with its restriction over $\mathbb{P}^1 \times \mathcal{M}_{q-1}$ an embedding. On this projective bundle, the Plücker map factors through the base $\mathbb{P}^1 \times Q_{q-1}$ and the image of a point in the base is $s \cdot S$, where $s$ is the section of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing at $s \in \mathbb{P}^1$ and $S$ is the image of a point in $Q_{q-1}$ under its Plücker map. This identifies the image of the exceptional locus of the Plücker map with the image of $\mathbb{P}^1 \times K_{q-1}$ in $K_q$ under a map $\pi$ which is given in [2] below.

More concretely, a point in $Q_q$ may be (non-uniquely) represented by a $p \times n$-matrix $M$ of forms in $s, t$ with homogeneous rows and whose maximal minors have degree $q$ [10]. The image of such a point under the Plücker map is the collection
of maximal minors of $M$. The maps in $\mathcal{M}_q$ are represented by matrices whose maximal minors have no common factors: Given such a matrix $M$, the association

$$\mathbb{P}^1 \ni (s,t) \mapsto \text{row space } M(s,t)$$

defines a map of degree $q$.

The collection $\binom{[n]}{p}$ of $p$-subsets of $\{1, \ldots, n\}$ index the maximal minors of $M$. For $\alpha \in \binom{[n]}{p}$ and $0 \leq a \leq q$, let $z_{\alpha}^{(a)}$ be the coefficient of $s^a t^{q-a}$ in the $\alpha$th maximal minor of $M$. These $z_{\alpha}^{(a)}$ provide Plücker coordinates for the space $\mathbb{P} \left( \bigwedge^p \mathbb{C}^n \otimes H^0(\mathcal{O}_F(q)) \right)$. Let $\mathcal{C}_q := \{\alpha^{(a)} : \alpha \in \binom{[n]}{p}, \ 0 \leq a \leq q\}$ be the indices of these Plücker coordinates. Then the image of the exceptional locus in $\mathcal{K}_q$ is the image of the map $\pi : \mathbb{P}^1 \times \mathcal{K}_{q-1} \to \mathcal{K}_q$ defined by

$$(2.1) \quad \pi : \left( [A,B], (x_{\beta}^{(b)} : \beta^{(b)} \in \mathcal{C}_{q-1}) \right) \mapsto \left( Ax_{\alpha}^{(a)} - Bx_{\alpha}^{(a-1)} : \alpha^{(a)} \in \mathcal{C}_q \right),$$

where $x_{\alpha}^{(a)} = x_{\alpha}^{(a-1)} = 0$.

The relevance of the quantum Grassmannian $\mathcal{K}_q$ to the enumerative problem (1.1) is seen by considering the condition for a map $M \in \mathcal{M}_q$ to satisfy $M(s,t) \cap L \neq \{0\}$ where $L$ is an $m$-plane in $\mathcal{C}^n$ and $(s,t) \in \mathbb{P}^1$. If we represent $L$ as the row space of an $m \times n$ matrix, also written $L$, then this condition is

$$0 = \det \left[ \begin{array}{c} L \\ M(s,t) \end{array} \right] = \sum_{\alpha \in \binom{[n]}{p}} f_{\alpha}(s,t) l_{\alpha},$$

the second expression given by Laplace expansion of the determinant along the rows of $M$. Here, $l_{\alpha}$ is the appropriately signed maximal minor of $L$. If we expand the forms $f_{\alpha}(s,t)$ in this last expression, then we obtain

$$\sum_{\alpha^{(a)} \in \mathcal{C}_q} z_{\alpha^{(a)}} s^a t^{q-a} l_{\alpha} = 0,$$

a linear equation in the Plücker coordinates of $M$. Thus the solutions $M \in \mathcal{M}_q$ to the enumerative problem (1.1) are a linear section of $\mathcal{M}_q$ in its Plücker embedding, and so the degree $d$ of $\mathcal{K}_q$ provides an upper bound on the number of solutions.

The set $\mathcal{C}_q$ of Plücker coordinates has a natural partial order

$$\alpha^{(a)} \leq \beta^{(b)} \iff a \leq b \text{ and } \alpha_i \leq \beta_{b-a+i} \text{ for } i = 1, 2, \ldots, p - b + a.$$

The poset $\mathcal{C}_q$ is graded with the rank, $|\alpha^{(a)}|$, of $\alpha^{(a)}$ equal to $an + \sum_i \alpha_i - i$. It is also a distributive lattice. Figure 11 shows $\mathcal{C}_3$ when $p = 2$ and $m = 3$. Given $\alpha^{(a)} \in \mathcal{C}_q$, define the quantun Schubert variety

$$Z_{\alpha^{(a)}} := \{ (z_{\beta^{(b)}}) : \beta^{(b)} \in \mathcal{C}_q, z_{\beta^{(b)}} = 0 \text{ if } \beta^{(b)} \not\leq \alpha^{(a)} \}.$$

Let $\mathcal{H}_{\alpha^{(a)}}$ be the hyperplane defined by $z_{\alpha^{(a)}} = 0$. The main technical result we use is the following.

**Proposition 2.1** ([11] [12]). Let $\alpha^{(a)} \in \mathcal{C}_q$. Then

(i) $Z_{\alpha^{(a)}}$ is an irreducible subvariety of $\mathcal{K}_q$ of dimension $|\alpha^{(a)}|$. 
(ii) The intersection of $Z_{\alpha^{(a)}}$ and $\mathcal{H}_{\alpha^{(a)}}$ is generically transverse and we have

$$Z_{\alpha^{(a)}} \cap \mathcal{H}_{\alpha^{(a)}} = \bigcup_{\beta^{(b)} \leq \alpha^{(a)}} Z_{\beta^{(b)}}.$$
Another proof of (ii) is given in [18], which shows that (ii) is a scheme-theoretic equality. From (ii) and Bézout’s theorem, we obtain the following recursive formula for the degree of $Z_{\alpha(a)}$:

$$
\deg Z_{\alpha(a)} = \sum_{\beta(0) < \alpha(a)} \deg Z_{\beta(0)} .
$$

Since the minimal quantum Schubert variety is a point, we deduce the formula of [12].

**Corollary 2.2.** The degree $d$ of $K_q$ is the number of maximal chains in the poset $C_q$.

For example, when $p = 2$ and $m = 3$, the degree of $K_1$ is 55.

An alternative proof of Corollary 2.2 is given in [18] by explicitly deforming $K_q$ to the toric variety associated with the poset $C_q$.

## 3. Proof of Theorem 1.1

Let $L(s, t)$ be the $m$-plane osculating the parameterized rational normal curve

$$
\gamma : (s, t) \in \mathbb{P}^1 \mapsto (s^{n-1}, ts^{n-2}, \ldots, t^{n-2}s, t^{n-1}) \in \mathbb{P}^{n-1}
$$

at the point $\gamma(s, t)$. Then $L(s, t)$ is the row space of the $m \times n$ matrix of forms with rows $\gamma(s, t), \gamma'(s, t), \ldots, \gamma^{(m-1)}(s, t)$, the derivative taken with respect to the parameter $t$. Write $L(s, t)$ for this matrix. For $\alpha \in \binom{[n]}{p}$, the maximal minor of $L(s, t)$ complementary to $\alpha$ is $s^{(m)}(\alpha) \cdot (-1)^{|\alpha|} \cdot s^{(\alpha)} |^{m-p-|\alpha|}$, where $|\alpha| := \sum_i \alpha_i - i$ and $(-1)^{|\alpha|} \cdot l_{\alpha}$ is the corresponding maximal minor of $L(1, 1)$. For $(s, t) \in \mathbb{P}^1$, let $H(s, t)$ be the hyperplane given by the linear form

$$
\Lambda(s, t) := \sum_{\alpha(a) \in C_q} z_{\alpha(a)} l_{\alpha} s^{(\alpha)} |^{m-p-|\alpha|} .
$$
Let $M$ be a matrix representing a map in $M_q$. Then
\[
\det \begin{bmatrix} L(s, t) \\ M(s^n, t^n) \end{bmatrix} = s^{(2)} \sum_{\alpha^{(a)} \in C_q} z_{\alpha^{(a)}} s^{(q-a) n} l_{\alpha} s^{(a)} t^{m-p-a} = s^{(2)} \Lambda(s, t).
\]
Thus $M_q \cap \mathcal{H}(s, t)$ consists of all maps $M : \mathbb{P}^1 \to \mathbb{G}$ of degree $q$ which satisfy $M(s^n, t^n) \cap L(s, t) \neq \{0\}$. Theorem 1.1 is a consequence of the following two theorems.

**Theorem 3.1.** There exist real numbers $s_1, \ldots, s_N$ such that for any $\alpha^{(a)} \in C_q$ the intersection
\[
Z_{\alpha^{(a)}} \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_{|\alpha^{(a)}|}, 1)
\]
is transverse with all points of intersection real.

**Theorem 3.2.** If $s_1, \ldots, s_k \in \mathbb{C}$ are distinct, then for any $\alpha^{(a)} \in C_q$ the intersection
\[
Z_{\alpha^{(a)}} \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_k, 1)
\]
is proper in that it has dimension $|\alpha^{(a)}| - k$.

**Proof of Theorem 3.1.** By Theorem 3.1 there exist real numbers $s_1, \ldots, s_N$ (necessarily distinct) so that the intersection
\[
K_q \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_N, 1)
\]
is transverse and consists of exactly $d$ real points. To prove Theorem 1.1 we show that all these points lie in $M_q$. Thus each point in (3.3) represents a real map $\pi : \mathbb{P}^1 \times K_{q-1} \to K_q$ by the map (2.1) whose image is the complement of $M_q$ in $K_q$. Then
\[
\pi^* \Lambda(s, t) = \sum_{\alpha^{(a)} \in C_q} (Ax_{\alpha^{(a)}} - Bs_{\alpha^{(a-1)}}) l_{\alpha} s^{(a)} t^{N-|\alpha^{(a)}|} = (At^n - Bs^n) \sum_{\beta^{(b)} \in C_{q-1}} x_{\beta^{(b)}} l_{\beta} s^{(b)} t^{N-n-|\beta^{(b)}|} = (At^n - Bs^n) \Lambda(s, t),
\]
where $\mathcal{H}(s, t)$ is the linear form for $K_{q-1}$ analogous to $\Lambda(s, t)$. Let $\mathcal{H}'(s, t)$ be the hyperplane given by the linear form $\mathcal{H}'(s, t)$.

Any point in (3.3) but not in $M_q$ is the image of a point $([A, B], x)$ in $\mathbb{P}^1 \times K_{q-1}$ satisfying $\pi^* \Lambda(s_i, 1) = (A - Bs_i^n) \Lambda'(s_i, 1)$ for each $i = 1, \ldots, N$. As the $s_i$ are distinct and real, such a point can satisfy $A - Bs_i^n = 0$ for at most two $i$. Thus $x \in K_{q-1}$ lies in at least $N - 2$ of the hyperplanes $\mathcal{H}'(s_i, 1)$. Since $N - 2$ exceeds the dimension $N - n$ of $K_{q-1}$, there are no such points $x \in K_{q-1}$ by Theorem 3.2 for maps of degree $q - 1$.

**Proof of Theorem 3.2.** For any $s_1, \ldots, s_k$, the intersection (3.2) has dimension at least $|\alpha^{(a)}| - k$. We show it has at most this dimension, if $s_1, \ldots, s_k$ are distinct.

Suppose $k = |\alpha^{(a)}| + 1$ and let $z \in Z_{\alpha^{(a)}}$. Then $z_{\beta^{(b)}} = 0$ if $\beta^{(b)} \not\leq \alpha^{(a)}$ and so the form $\Lambda(s, 1)$ defining $\mathcal{H}(s, 1)$ evaluated at $z$ is
\[
\sum_{\beta^{(b)} \leq \alpha^{(a)}} z_{\beta^{(b)}} l_{\beta} s^{(b)}.
\]
This is a non-zero polynomial in $s$ of degree at most $|\alpha^{(\alpha)}|$ and thus it vanishes for at most $|\alpha^{(\alpha)}|$ distinct values of $s$. It follows that (3.2) is empty for $k > |\alpha^{(\alpha)}|$.

If $k \leq |\alpha^{(\alpha)}|$ and $s_1, \ldots, s_k$ are distinct, but (3.2) has dimension exceeding $|\alpha^{(\alpha)}| - k$, then completing $s_1, \ldots, s_k$ to a set of distinct numbers $s_1, \ldots, s_{|\alpha^{(\alpha)}| + 1}$ would give a non-empty intersection in (3.2), a contradiction. 

Proof of Theorem 3.1. We construct the sequence $s_i$ inductively. The unique element of rank 1 in $C_q$ is $\alpha^{(0)}$, where $\alpha$ is the sequence $1 < 2 < \cdots < p - 1 < p + 1$. The quantum Schubert variety $Z_{\alpha^{(0)}}$ is a line in Plücker space. Indeed, it is isomorphic to the set of $p$-planes containing a fixed $(p - 1)$-plane and lying in a fixed $(p + 1)$-plane. By Theorem 3.2, $Z_{\alpha^{(0)}} \cap \mathcal{H}(s, 1)$ is then a single, necessarily real, point, for any real number $s$. Let $s_1$ be any positive real number.

Suppose we have real numbers $s_1, \ldots, s_k$ with the property that for any $\beta^{(i)}$ with $|\beta^{(i)}| \leq k$,

$$Z_{\beta^{(i)}} \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_{|\beta^{(i)}|}, 1)$$

is transverse with all points of intersection real.

Let $\alpha^{(a)}$ be an index with $|\alpha^{(a)}| = k + 1$ and consider the 1-parameter family $Z(s)$ of schemes defined by $Z_{\alpha^{(a)}} \cap \mathcal{H}(s, 1)$. If we restrict the form $\Lambda(s, 1)$ to $z \in Z_{\alpha^{(a)}}$, then we obtain

$$\sum_{\beta^{(i)} \leq \alpha^{(a)}} z_{\beta^{(i)}} l_{\beta^{(i)}} s_{|\beta^{(i)}|},$$

a polynomial in $s$ with leading term $z_{\alpha^{(a)}} s_{|\alpha^{(a)}|}$. Thus $Z(\infty)$ is

$$Z_{\alpha^{(a)}} \cap \mathcal{H}_{\alpha^{(a)}} = \bigcup_{\beta^{(i)} \leq \alpha^{(a)}} Z_{\beta^{(i)}},$$

by Proposition 2.1(ii).

Claim: The cycle

$$Z(\infty) \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_k, 1)$$

is free of multiplicities.

If not, then there are two components $Z_{\beta^{(i)}}$ and $Z_{\gamma^{(c)}}$ of $Z(\infty)$ such that

$$Z_{\beta^{(i)}} \cap Z_{\gamma^{(c)}} \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_k, 1)$$

is non-empty. But this contradicts Theorem 3.2, as $Z_{\beta^{(i)}} \cap Z_{\gamma^{(c)}} = Z_{\delta^{(d)}}$, where $\delta^{(d)}$ is the greatest lower bound of $\beta^{(i)}$ and $\gamma^{(c)}$ in $C_q$, and so $\dim Z_{\delta^{(d)}} < \dim Z_{\beta^{(i)}} = k$.

From the claim, there is a real number $N_{\alpha^{(a)}} > 0$ such that if $s > N_{\alpha^{(a)}}$, then

$$Z(s) \cap \mathcal{H}(s_1, 1) \cap \cdots \cap \mathcal{H}(s_k, 1)$$

is transverse with all points of intersection real. Set

$$N_{k+1} := \max\{N_{\alpha^{(a)}} : |\alpha^{(a)}| = k + 1\}$$

and let $s_{k+1}$ be any real number satisfying $s_{k+1} > N_{k+1}$. 


4. Further remarks

The proof of Theorem 3.1 gives a rather precise choice of $s_i$ and $L_i$ in the enumerative problem (1.1) which gives only real maps. The positive real numbers $s_1, \ldots, s_N$ of Theorem 3.1 are constructed inductively, first choosing $s_1 > 0$. Then, having chosen $s_1, \ldots, s_k$, a number $N_{k+1} > 0$ is found with the property that for any $s_{k+1} > N_{k+1}$ and $\alpha^{(a)}$ with $|\alpha^{(a)}| = k + 1$, the intersection is transverse with all points real. By the quantifier $\forall s_1 \ll s_2 \ll \cdots \ll s_N$, we mean such a choice of real numbers $s_1, \ldots, s_N$. More precisely,

$$\forall s_1 > 0 \quad \exists N_2 > 0, \quad \text{such that } \forall s_2 > N_2 \quad \cdots \quad \exists N_N > 0, \quad \text{such that } \forall s_N > N_N.$$ We deduce a more precise form of Theorem 1.1, as the quantifiers $\forall s_1 \ll s_2 \ll \cdots \ll s_N$ and $\forall s_i^1 \ll s_i^2 \ll \cdots \ll s_i^N$ are equivalent.

**Corollary 4.1.** $\forall s_1 \ll s_2 \ll \cdots \ll s_N$, each of the $d$ maps $M : \mathbb{P}^1 \to G$ of degree $q$ which satisfy $M(s_1, 1) \cap L(s_i^{1/m}, 1) \neq \{0\}$ for $i = 1, \ldots, N$ are real.

When $q = 0$, there is substantial evidence that this choice of $s_1, \ldots, s_N$ is too restrictive. B. Shapiro and M. Shapiro have the following conjecture:

**Conjecture 4.2** (B. Shapiro and M. Shapiro). *Suppose $q = 0$. Then for distinct real numbers $s_1, \ldots, s_{mp}$ each of the finitely many $p$-planes $H$ which satisfy $H \cap L(s_1, 1) \neq \{0\}$ are real.*

In contrast, when $q > 0$ the restriction $\forall s_1 \ll s_2 \ll \cdots \ll s_N$ is necessary. We observe this in the case of $q = 1, p = m = 2$, so $N = 8$ and $d = 8$; that is, for parameterized curves of degree 1 in the Grassmannian of 2-planes in $\mathbb{C}^4$. Here, the choice of $s_i = i$ in (3.3) gives no real maps, while the choice $s_i = i^6$ gives 8 real maps.

We describe that calculation. There are 12 Plücker coordinates $z_{ij}(s)$ for $1 \leq i < j \leq 4$ and $a = 0, 1$. If we let $f_{ij} := t z_{ij}(0) + s z_{ij}(1)$, then

$$f_{14} f_{23} - f_{13} f_{24} + f_{12} f_{34} = 0,$$

as $f_{ij}(s, t) \in G$ for all $s, t$. The coefficients of $t^2$, $st$, and $s^2$ in this expression give three quadratic relations among the $z_{ij}(s)$:

$$z_{14}(0) z_{23}(0) - z_{13}(0) z_{24}(0) + z_{12}(0) z_{34}(0),$$

$$z_{12}(1) z_{34}(0) - z_{13}(1) z_{24}(0) + z_{14}(1) z_{23}(0) + z_{23}(1) z_{14}(0) - z_{24}(1) z_{13}(0) + z_{34}(1) z_{12}(0),$$

$$z_{14}(1) z_{23}(1) - z_{13}(1) z_{24}(1) + z_{12}(1) z_{34}(1),$$

and these constitute a Gröbner basis for the homogeneous ideal of $K_3$.

Here, the form $A(s, 1)$ is

$$A(s, 1) = z_{12}(1) - 2 s z_{13}(0) + s^2 z_{14}(0) + 3 s^2 z_{23}(0) - 2 s^3 z_{24}(0) + s^4 z_{34}(0) + s^4 z_{12}(1) - 2 s^5 z_{13}(1) + s^6 z_{14}(1) + 3 s^6 z_{23}(1) - 2 s^7 z_{24}(1) + s^8 z_{34}(1).$$

We set $z_{34}(1) = 1$ and work in local coordinates. Then the ideal generated by the 3 quadratic equations and 8 linear relations $A(s_i, 1)$ for $i = 1, \ldots, 8$ defines the 8 solutions to (3.3). We used Maple to generate these equations and compute the number of real solutions. There are no real solutions when $s_i = 1$, but all 8 are real when $s_i = i^6$. 

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We describe how the enumerative problem (1.1) arises in systems theory (see also [3]). A physical system (e.g. a mechanical linkage) with \( m \) inputs and \( p \) measured outputs whose evolution is governed by a system of linear differential equations is modeled by an \( m \times n \)-matrix \( L(s) \) of real univariate polynomials. The largest degree of a maximal minor of this matrix is the McMillan degree, \( r \), of the evolution equation. Consider now controlling this linear system by output feedback with a dynamic compensator. That is, use a \( p \)-input, \( m \)-output linear system \( M \) to couple the \( m \) inputs of the system \( L \) to its \( p \) outputs. The resulting closed system has characteristic polynomial

\[
\varphi(s) := \begin{bmatrix}
L(s) \\
M(s)
\end{bmatrix},
\]

and the roots of \( \varphi \) are the natural frequencies or poles of the closed system. The dynamic pole assignment problem asks, given a system \( L(s) \) and a desired characteristic polynomial \( \varphi \), can one find a (real) compensator \( M(s) \) of McMillan degree \( q \) so that the resulting closed system has characteristic polynomial \( \varphi \)? That is, if \( s_1, \ldots, s_{r+q} \) are the roots of \( \varphi \), then which \( M \in \mathcal{M}_q \) satisfy

\[
\det \begin{bmatrix}
L(s_i) \\
M(s_i)
\end{bmatrix} = 0, \quad \text{for } i = 1, 2, \ldots, r + q ?
\]

In the critical case when \( r + q = mp + qn \) (\( = \dim \mathcal{M}_q \)), this is an instance of the enumerative problem (1.1). When the degree \( d \) is odd, then for any real system \( L \) and a real characteristic polynomial \( \varphi \), there will be at least one real dynamic compensator. Part of the motivation for (1.1) was to obtain a closed formula for \( d \) from which its parity could be deduced for different values of \( q, m, \) and \( p \).

The choice of planes \( L_i \) that arise in the dynamic pole placement problem are \( N = mp + qn \) points on a rational curve of degree \( mp + (n - 1)q \) in the Grassmannian of \( m \)-planes in \( \mathbb{C}^n \). In contrast, the planes of Theorem 3.1 (and hence of Theorem 1.1) arise as \( N \) points on a rational curve of degree \( mp \). Only when \( q = 0 \) (the case of static compensators) is there overlap.

Our proof of Theorem 1.1 (like that in [17]) was inspired by the numerical Pieri homotopy algorithm of [7] for computing the solutions to (1.1) when \( q = 0 \). Likewise, our explicit degenerations of intersections of the \( \mathcal{H}(s, t) \), and more generally Proposition 2.1 (ii), suggested to Huber and Verschelde an optimal numerical homotopy algorithm for finding the solutions to (1.1) [6]. This is in exactly the same manner as the explicit degenerations of intersections of special Schubert varieties of [15] were used to construct the Pieri homotopy algorithm of [7] (see also [6]).

We close with one open problem concerning the enumeration of rational curves on a Grassmannian. For a point \( s \in \mathbb{P}^1 \) and a Schubert variety \( \Omega \) of \( G \), consider the quantum Schubert variety \( \Omega(s) \) of curves \( M \in \mathcal{M}_q \) satisfying \( M(s) \in \Omega \). Bertram’s quantum Schubert calculus gives formulas to compute the number of curves \( M \in \mathcal{M}_q \) which lie in the intersection of an appropriate number of these \( \Omega(s) \), and we ask when it is possible to have all solutions real. A modification of the proof of Theorem 3.1 shows that this is the case when all except possibly 2 are hypersurface Schubert varieties. In every case we have computed, all solutions may be real.

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REAL RATIONAL CURVES IN GRASSMANNIANS

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