

SEMI-INVARIANTS OF QUIVERS AND SATURATION FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

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1. INTRODUCTION

Let Q be a quiver without oriented cycles. Let α be a dimension vector for Q . We denote by $\text{SI}(Q, \alpha)$ the ring of semi-invariants of the set of α -dimensional representations of Q over a fixed algebraically closed field K .

In this paper we prove some results about the set

$$\Sigma(Q, \alpha) = \{ \sigma \mid \text{SI}(Q, \alpha)_\sigma \neq 0 \}.$$

$\Sigma(Q, \alpha)$ is defined in the space of all weights by one homogeneous linear equation and by a finite set of homogeneous linear inequalities. In particular the set $\Sigma(Q, \alpha)$ is saturated, i.e., if $n\sigma \in \Sigma(Q, \alpha)$, then also $\sigma \in \Sigma(Q, \alpha)$.

These results, when applied to a special quiver $Q = T_{n,n,n}$ and to a special dimension vector, show that the GL_n -module V_λ appears in $V_\mu \otimes V_\nu$ if and only if the partitions λ , μ and ν satisfy an explicit set of inequalities. This gives new proofs of the results of Klyachko ([7, 3]) and Knutson and Tao ([8]).

The proof is based on another general result about semi-invariants of quivers (Theorem 1). In the paper [10], Schofield defined a semi-invariant c_W for each indecomposable representation W of Q . We show that the semi-invariants of this type span each weight space in $\text{SI}(Q, \alpha)$. This seems to be a fundamental fact, connecting semi-invariants and modules in a direct way. Given this fact, the results on sets of weights follow at once from the results in another paper of Schofield [11].

2. THE RESULTS

A quiver Q is a pair $Q = (Q_0, Q_1)$ consisting of the set of vertices Q_0 and the set of arrows Q_1 . Each arrow a has its head ha and tail ta , both in Q_0 :

$$ta \xrightarrow{a} ha.$$

We fix an algebraically closed field K . A representation (or a module) V of Q is a family of finite dimensional vector spaces $\{V(x) \mid x \in Q_0\}$ and of linear maps

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$V(a) : V(ta) \rightarrow V(ha)$. The dimension vector of a representation V is the function $\underline{d}(V) : Q_0 \rightarrow \mathbb{Z}_{\geq 0}$ defined by $\underline{d}(V)(x) := \dim V(x)$. The dimension vectors lie in the space Γ of integer-valued functions on Q_0 . A morphism $\phi : V \rightarrow V'$ of two representations is a collection of linear maps $\phi(x) : V(x) \rightarrow V'(x)$, $x \in Q_0$, such that for each $a \in Q_1$ we have $\phi(ha)V(a) = V'(a)\phi(ta)$. We denote the linear space of morphisms from V to V' by $\text{Hom}_Q(V, V')$.

A path p in Q is a sequence of arrows $p = a_1, \dots, a_n$ such that $ha_i = ta_{i+1}$ ($1 \leq i \leq n - 1$). We define $tp = ta_1, hp = ha_n$. We also have the trivial path $e(x)$ from x to x . If V is a representation and $p = a_1, \dots, a_n$, then we define $V(p) := V(a_n)V(a_{n-1}) \cdots V(a_1)$. We assume throughout the paper that Q has no oriented cycles, i.e., there are no paths $p = a_1, \dots, a_n$ such that $ta_1 = ha_n$.

For representations V and W of Q there is a canonical exact sequence ([9])

$$(1) \quad 0 \rightarrow \text{Hom}_Q(V, W) \xrightarrow{i} \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \xrightarrow{d_W^V} \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \xrightarrow{p} \text{Ext}_Q(V, W) \rightarrow 0.$$

The map i is the obvious inclusion, the map d_W^V is given by

$$\{f(x)\}_{x \in Q_0} \mapsto \{f(ha)V(a) - W(a)f(ta)\}_{a \in Q_1},$$

and the map p constructs an extension of the representations V and W by adding the maps $V(ta) \rightarrow W(ha)$ to the direct sum representation $V \oplus W$.

For $\alpha, \beta \in \Gamma$ we define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

It follows from (1) that $\langle \underline{d}(V), \underline{d}(W) \rangle = \dim_K \text{Hom}_Q(V, W) - \dim_K \text{Ext}_Q(V, W)$.

For a dimension vector α we denote by

$$\text{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \text{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$$

the vector space of α -dimensional representations of Q . The group

$$\text{GL}(Q, \alpha) := \prod_{x \in Q_0} \text{GL}(\alpha(x))$$

and its subgroup

$$\text{SL}(Q, \alpha) = \prod_{x \in Q_0} \text{SL}(\alpha(x))$$

act on $\text{Rep}(Q, \alpha)$ in an obvious way. We are interested in the ring of semi-invariants

$$\text{SI}(Q, \alpha) := K[\text{Rep}(Q, \alpha)]^{\text{SL}(Q, \alpha)}.$$

The ring $SI(Q, \alpha)$ has a weight space decomposition

$$SI(Q, \alpha) = \bigoplus_{\sigma} SI(Q, \alpha)_{\sigma}$$

where σ runs through the (one-dimensional irreducible) characters of $GL(Q, \alpha)$ and

$$SI(Q, \alpha)_{\sigma} = \{ f \in K[\text{Rep}(Q, \alpha)] \mid g(f) = \sigma(g)f \ \forall g \in GL(Q, \alpha) \}.$$

Suppose that σ lies in the dual space $\Gamma^* := \text{Hom}(\Gamma, \mathbb{Z})$. For each dimension vector α we can associate to σ a character of $GL(Q, \alpha)$ defined as

$$\prod_{x \in Q_0} d_x^{\sigma(e_x)}$$

where d_x is the determinant function on $GL(\alpha(x))$ and e_x is the dimension vector defined by

$$e_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

In this way we will identify characters with Γ^* . Sometimes, for convenience, we will write $\sigma(x)$ instead of $\sigma(e_x)$ (and treat σ as an element of Γ).

Let us choose the dimension vectors α and β in such way that $\langle \alpha, \beta \rangle = 0$. Then for every $V \in \text{Rep}(Q, \alpha)$ and $W \in \text{Rep}(Q, \beta)$ the matrix of d_W^V will be a square matrix. Following [10] we can therefore define the semi-invariant c of the action of $GL(Q, \alpha) \times GL(Q, \beta)$ on $\text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)$ by $c(V, W) := \det d_W^V$. The value of the determinant depends on the choices of bases, so c is well-defined up to a scalar. Notice that the semi-invariant c vanishes at the point (V, W) if and only if $\text{Hom}_Q(V, W) \neq 0$ which is equivalent to $\text{Ext}_Q(V, W) \neq 0$. For a fixed V the restriction of c to $\{V\} \times \text{Rep}(Q, \beta)$ defines a semi-invariant c^V in $SI(Q, \beta)$. Schofield proves ([10, Lemma 1.4]) that the weight of c^V equals $\langle \alpha, \cdot \rangle \in \Gamma^*$ which is defined as $\gamma \mapsto \langle \alpha, \gamma \rangle$. Similarly, for a fixed W the restriction of c to $\text{Rep}(Q, \alpha) \times \{W\}$ defines a semi-invariant c_W in $SI(Q, \alpha)$ of weight $-\langle \cdot, \beta \rangle$ ([10, Lemma 1.4]). If $V, V' \in \text{Rep}(Q, \alpha)$ and $V \cong V'$, then V and V' are in the same $GL(Q, \alpha)$ -orbit, and c^V and $c^{V'}$ are equal up to a constant scalar. Semi-invariants of the types c^V and c_W are well-defined up to a scalar. These semi-invariants have the following properties.

Lemma 1. *Suppose that V, V', V'' and W, W', W'' are representations of Q such that $\langle \underline{d}(V), \underline{d}(W) \rangle = 0$, and that there are exact sequences*

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0, \quad 0 \rightarrow W' \rightarrow W \rightarrow W'' \rightarrow 0.$$

- a) *If $\langle \underline{d}(V'), \underline{d}(W) \rangle < 0$, then $c^V(W) = 0$;*
- b) *If $\langle \underline{d}(V'), \underline{d}(W) \rangle = 0$, then $c^V(W) = c^{V'}(W)c^{V''}(W)$;*
- c) *If $\langle \underline{d}(V), \underline{d}(W') \rangle > 0$, then $c^V(W) = 0$;*
- d) *If $\langle \underline{d}(V), \underline{d}(W') \rangle = 0$, then $c^V(W) = c^V(W')c^V(W'')$.*

Proof. Consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \bigoplus_{x \in Q_0} \text{Hom}(V''(x), W(x)) & \xrightarrow{d_W^{V''}} & \bigoplus_{a \in Q_1} \text{Hom}(V''(ta), W(ha)) \\
 \downarrow & & \downarrow \\
 \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) & \xrightarrow{d_W^V} & \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \\
 \downarrow & & \downarrow \\
 \bigoplus_{x \in Q_0} \text{Hom}(V'(x), W(x)) & \xrightarrow{d_W^{V'}} & \bigoplus_{a \in Q_1} \text{Hom}(V'(ta), W(ha)) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

If $\langle \underline{d}(V'), \underline{d}(W) \rangle = 0$, then $d_W^{V'}$, d_W^V and $d_W^{V''}$ are all represented by square matrices. It follows that $c^V(W) = c^{V'}(W)c^{V''}(W)$. So b) follows and d) goes similarly. If $\langle \underline{d}(V'), \underline{d}(W) \rangle < 0$, then $d_W^{V'}$ cannot be surjective, hence d_W^V is not surjective. Now a) follows and c) goes similarly. \square

Our main result is that the semi-invariants of type c^V (resp. c_W) span all the weight spaces in the rings $\text{SI}(Q, \alpha)$.

Theorem 1. *Let Q be a quiver without oriented cycles and let β be a dimension vector. The ring of semi-invariants $\text{SI}(Q, \beta)$ is a K -linear span of semi-invariants c^V with $\langle \underline{d}(V), \beta \rangle = 0$. The analogous result is true for the semi-invariants c_W .*

After this paper was submitted we learned about the paper [12] where among other things the authors give another proof of Theorem 1 under the assumption that the characteristic of K is zero.

We will prove Theorem 1 in Section 4.

Remark 1. If $V = V_1 \oplus V_2$ is decomposable, then by Lemma 1 we have $c^V = 0$ if $\langle \underline{d}(V_1), \beta \rangle \neq 0$, and $c^V = c^{V_1}c^{V_2}$ if $\langle \underline{d}(V_1), \beta \rangle = 0$.

The algebra $\text{SI}(Q, \beta)$ is generated by all c^V where V is *indecomposable*. Generators of $\text{SI}(Q, \beta)$ therefore can be found in the degrees $\langle \alpha, \cdot \rangle$ such that a general representation of dimension α is indecomposable. By [5] this is equivalent to α being a Schur root.

Remark 2. If $\text{Rep}(Q, \beta)$ has a dense $\text{GL}(Q, \beta)$ -orbit, then Schofield showed in [10] that the invariants of type c^V with V indecomposable generate $\text{SI}(Q, \beta)$ (which is a polynomial ring in this case).

Theorem 1 has the following remarkable consequence.

Corollary 1 (Reciprocity Property). *Let α, β be two dimension vectors for the quiver Q . Assume that $\langle \alpha, \beta \rangle = 0$. Then*

$$\dim_K \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim_K \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Proof. Let V_1, \dots, V_s be the modules of dimension α such that c^{V_1}, \dots, c^{V_s} form a basis of $\text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$. These are linearly independent polynomials on $\text{Rep}(Q, \beta)$ so there exist s representations W_1, \dots, W_s in $\text{Rep}(Q, \beta)$ such that $\det(c^{V_i}(W_j))_{1 \leq i, j \leq s}$

is not zero. But $c^{V_i}(W_j) = c_{W_j}(V_i)$ and this means that the semi-invariants c_{W_1}, \dots, c_{W_s} are linearly independent. This proves that

$$\dim_K \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} \leq \dim_K \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

The other inequality is proven in exactly the same way. \square

In the remainder of this section we investigate the consequences of Theorem 1. First we recall the main results of [11]. They can be summarized as follows.

We say that for two dimension vectors α, β the space $\text{Hom}_Q(\alpha, \beta)$ (respectively $\text{Ext}_Q(\alpha, \beta)$) vanishes generically if and only if for general representations V, W of dimensions α, β respectively we have $\text{Hom}_Q(V, W) = 0$ (resp. $\text{Ext}_Q(V, W) = 0$). We also write $\alpha \hookrightarrow \beta$ if a general representation of dimension β has a subrepresentation of dimension α .

Theorem 2 (Schofield). *Let α and β be two dimension vectors for the quiver Q .*

- a) $\text{Ext}_Q(\alpha, \beta)$ vanishes generically if and only if $\alpha \hookrightarrow \alpha + \beta$,
- b) $\text{Ext}_Q(\alpha, \beta)$ does not vanish generically if and only if $\beta' \hookrightarrow \beta$ and $\langle \alpha, \beta - \beta' \rangle < 0$ for some dimension vector β' .

Part a) is proven in Section 3 of [11], and part b) is proven in Section 5.

Remark 3. Suppose that V and W are general modules of dimension α and β respectively, such that $\langle \alpha, \beta \rangle = 0$. The condition in b) is equivalent to $\exists \beta' \beta' \hookrightarrow \beta$ such that $\langle \alpha, \beta' \rangle > 0$. If $c^V(W) = 0$, then W must have a submodule W' such that $\langle \alpha, \underline{d}(W') \rangle > 0$. This means that the converse of Lemma 1.c) is true for general V and W .

Theorem 3. *Let Q be a quiver without oriented cycles and let β be a dimension vector. The semigroup $\Sigma(Q, \beta)$ is the set of all $\sigma \in \Gamma$ such that $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all β' such that $\beta' \hookrightarrow \beta$. Thus this condition is provided by one linear homogeneous equality and finitely many linear homogeneous inequalities. In particular the set $\Sigma(Q, \beta)$ is saturated in the lattice Γ .*

Proof. Suppose that $\sigma \in \Gamma^*$. We can write $\sigma = \langle \alpha, \cdot \rangle$ with $\alpha \in \Gamma$.

We will first assume that α is a dimension vector, i.e., $\alpha(x) \geq 0$ for all $x \in Q_0$. It follows from Theorem 1 that $\text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$ is non-zero if and only if there exists a representation V of dimension α such that c^V is not zero, which is equivalent to $\sigma(\beta) = \langle \alpha, \beta \rangle = 0$ and $\text{Ext}_Q(\alpha, \beta)$ vanishing generically. By part b) of Theorem 2, $\text{Ext}_Q(\alpha, \beta)$ vanishes generically if and only if for all β' such that $\beta' \hookrightarrow \beta$ we have $\langle \alpha, \beta - \beta' \rangle \geq 0$. This means that for all β' such that $\beta' \hookrightarrow \beta$ we have $\sigma(\beta') = \langle \alpha, \beta' \rangle \leq 0$. We conclude that $\text{SI}(Q, \beta)_\sigma \neq 0$ if and only if $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta' \hookrightarrow \beta$.

If α is not a dimension vector, then $\text{SI}(Q, \beta)_{n\sigma} = 0$ for all integers $n > 0$. Suppose that $W \in \text{Rep}(Q, \beta)$. From [6] it follows that either $\sigma(\underline{d}(W)) \neq 0$ or there exists a submodule W' of W such that $\sigma(\underline{d}(W')) > 0$. If W is in general position, then we obtain $\sigma(\beta) \neq 0$ or $\sigma(\beta') > 0$ for some $\beta' \hookrightarrow \beta$ (see also Remark 5). \square

Remark 4. Schofield in [11] gives an algorithm allowing one to determine the set of inequalities in Theorem 3 inductively. This algorithm is not very efficient.

Remark 5. A module $W \in \text{Rep}(Q, \beta)$ is called σ -stable if and only if there exist an $n > 0$ and an $f \in \text{SI}(Q, \beta)_{n\sigma}$ such that $f(W) \neq 0$. King proved in [6] that a module

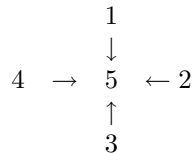
$W \in \text{Rep}(Q, \beta)$ is σ -stable if and only if $\sigma(W') \leq 0$ for all submodules W' of W . Applied to a general representation W of dimension β this gives us the equivalence:

$$\exists n > 0 \text{ SI}(Q, \beta)_{n\sigma} \neq 0 \Leftrightarrow \sigma(\beta) = 0 \text{ and } \forall \beta' \beta' \hookrightarrow \beta \text{ we have } \sigma(\beta') \leq 0.$$

This shows that the saturation of $\Sigma(Q, \beta)$ is given by linear inequalities but it does not show that $\Sigma(Q, \beta)$ is saturated.

Remark 6. In Theorem 3, instead of considering all β' with $\beta' \hookrightarrow \beta$ we only need to consider those β' such that the general representation of dimension β' is indecomposable, which is equivalent to β' being a Schur root. Still, the set of inequalities obtained in this way may not be a minimal set of inequalities as we will see in the next example.

Example 1. Let Q be the quiver



and let β be the dimension vector

$$\begin{array}{c}
 1 \\
 1 \ 2 \ 1 \ . \\
 1
 \end{array}$$

For a general representation V of Q with dimension vector β , the dimension vectors of indecomposable submodules are:

$$\begin{array}{cccc}
 0 & 1 & 1 & 1 \\
 1 \ 2 \ 1 & 1 \ 2 \ 0 & 1 \ 2 \ 1 & 0 \ 2 \ 1 \\
 1 & 1 & 0 & 1
 \end{array}$$

$$\begin{array}{cccc}
 1 & 0 & 0 & 0 \\
 0 \ 1 \ 0 & 0 \ 1 \ 1 & 0 \ 1 \ 0 & 1 \ 1 \ 0 \\
 0 & 0 & 1 & 0
 \end{array}$$

$$\begin{array}{c}
 0 \\
 0 \ 1 \ 0 \\
 0
 \end{array}$$

Let σ be the weight given by $\sigma(\alpha) = \sum_{i=1}^5 a_i \alpha(i)$, in other words

$$\sigma = \begin{array}{c} a_1 \\ a_4 \ a_5 \ a_2 \ . \\ a_3 \end{array}$$

We investigate when $\text{SI}(Q, \beta)_\sigma \neq 0$. First of all we must have $\sigma(\beta) = 0$, so $a_1 + a_2 + a_3 + a_4 + 2a_5 = 0$. In particular $a_1 + a_2 + a_3 + a_4$ must be even. The

indecomposable submodules listed above correspond to the inequalities (using $a_5 = -(a_1 + a_2 + a_3 + a_4)/2$):

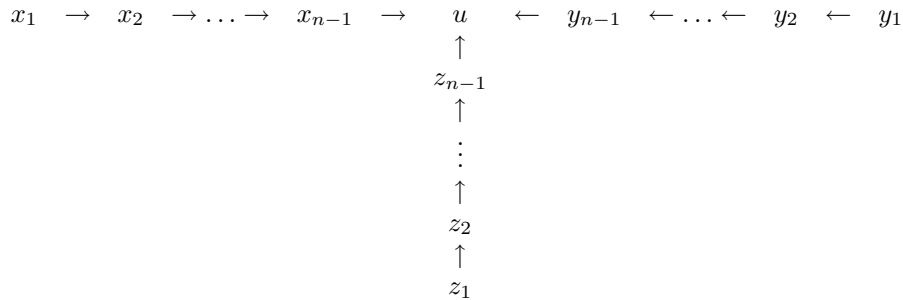
$$(2) \quad \begin{aligned} & a_1 \geq 0, a_2 \geq 0, a_3 \geq 0, a_4 \geq 0, \\ & a_1 \leq a_2 + a_3 + a_4, a_2 \leq a_1 + a_3 + a_4, a_3 \leq a_1 + a_2 + a_4, a_4 \leq a_1 + a_2 + a_3, \\ & a_1 + a_2 + a_3 + a_4 \geq 0. \end{aligned}$$

The last inequality is redundant.

In the next section we will see how semi-invariants can be interpreted in terms of tensor products of modules of the general linear group. This particular example shows that for a 2-dimensional vector space U , the tensor product of symmetric powers $S_{a_1}(U) \otimes S_{a_2}(U) \otimes S_{a_3}(U) \otimes S_{a_4}(U)$ contains a non-trivial $SL(U)$ -invariant subspace if and only if $a_1 + a_2 + a_3 + a_4$ is even and the inequalities (2) hold. In this case, the inequalities are obvious from the Clebsch-Gordan formula.

3. APPLICATION TO LITTLEWOOD-RICHARDSON COEFFICIENTS

Let us apply Theorem 3 in the following special case. Let us define the quiver $Q = T_{n,n,n}$ as follows:



Let us choose the dimension vector $\beta(x_i) = \beta(y_i) = \beta(z_i) = i$ for $i = 1, \dots, n - 1$, $\beta(u) = n$. The following proposition is a direct application of Cauchy’s formula and is a standard calculation in representation theory.

Proposition 1. *The weight space $SI(T_{n,n,n}, \beta)_\sigma$ is isomorphic to the space of $SL(U)$ -invariants in the triple tensor product $S_\lambda(U) \otimes S_\mu(U) \otimes S_\nu(U)$ of Schur functors on U , where U is the vector space of dimension n , and λ, μ, ν are partitions whose conjugate partitions are given as follows:*

$$(3) \quad \begin{aligned} \lambda' &= ((n - 1)^{\sigma(x_{n-1})}, (n - 2)^{\sigma(x_{n-2})}, \dots, 1^{\sigma(x_1)}), \\ \mu' &= ((n - 1)^{\sigma(y_{n-1})}, (n - 2)^{\sigma(y_{n-2})}, \dots, 1^{\sigma(y_1)}), \\ \nu' &= ((n - 1)^{\sigma(z_{n-1})}, (n - 2)^{\sigma(z_{n-2})}, \dots, 1^{\sigma(z_1)}). \end{aligned}$$

Here $\sigma(q)$ is defined as $\sigma(e_q)$ where the dimension vector e_q is given by $e_q(q) = 1$ and $e_q(p) = 0$ if $p \neq q$.

Proof. Let us denote by a_i (resp. b_i, c_i) the arrow in $T_{n,n,n}$ with $ta_i = x_i, ha_i = x_{i+1}$ (resp. $tb_i = y_i, hb_i = y_{i+1}, tc_i = z_i, hc_i = z_{i+1}$) for $1 \leq i \leq n - 1$. The space $\text{Rep}(T_{n,n,n}, \beta)$ can be identified with

$$\bigoplus_{1 \leq i \leq n-1} (\text{Hom}(V(x_i), V(x_{i+1})) \oplus \text{Hom}(V(y_i), V(y_{i+1})) \oplus \text{Hom}(V(z_i), V(z_{i+1})))$$

where we write $x_n = y_n = z_n = u$.

The Cauchy formula [4, §A.1] gives the decomposition of $K[\text{Rep}(T_{n,n,n}, \beta)]$ as a direct sum over the $3(n - 1)$ -tuples of partitions

$$((\alpha^i)_{1 \leq i \leq n-1}, (\beta^i)_{1 \leq i \leq n-1}, (\gamma^i)_{1 \leq i \leq n-1})$$

of the summands

$$\bigotimes_{1 \leq i \leq n-1} (S_{\alpha^i} V(x_i) \otimes S_{\alpha^i} V(x_{i+1})^* \otimes S_{\beta^i} V(y_i) \otimes S_{\beta^i} V(y_{i+1})^* \otimes S_{\gamma^i} V(z_i) \otimes S_{\gamma^i} V(z_{i+1})^*).$$

Let us denote $H = \prod_{1 \leq i \leq n-1} (\text{SL}(V(x_i)) \times \text{SL}(V(y_i)) \times \text{SL}(V(z_i)))$. Then it follows from the Littlewood-Richardson Rule [4, §A.1] that the summand corresponding to the $3(n - 1)$ -tuple

$$((\alpha^i)_{1 \leq i \leq n-1}, (\beta^i)_{1 \leq i \leq n-1}, (\gamma^i)_{1 \leq i \leq n-1})$$

contains an H -invariant if and only if we have for each i , $1 \leq i \leq n - 1$,

$$\begin{aligned} (\alpha^i)' &= ((i)^{\sigma(x_i)}, (i - 1)^{\sigma(x_{i-1})}, \dots, 1^{\sigma(x_1)}), \\ (\beta^i)' &= ((i)^{\sigma(y_i)}, (i - 1)^{\sigma(y_{i-1})}, \dots, 1^{\sigma(y_1)}), \\ (\gamma^i)' &= ((i)^{\sigma(z_i)}, (i - 1)^{\sigma(z_{i-1})}, \dots, 1^{\sigma(z_1)}) \end{aligned}$$

for some non-negative numbers $\sigma(x_i), \sigma(y_i), \sigma(z_i)$. Moreover, if these conditions are satisfied, then the space of H -invariants is isomorphic to

$$S_{\alpha^{n-1}} V(u)^* \otimes S_{\beta^{n-1}} V(u)^* \otimes S_{\gamma^{n-1}} V(u)^*.$$

Therefore the space of $\text{SL}(T_{n,n,n}, \beta)$ -semi-invariants can be identified with the space of $\text{SL}(V(u))$ -invariants in the above triple tensor product. \square

Corollary 2. *The set of triples of partitions (λ, μ, ν) such that the space of $\text{SL}(U)$ -invariants in $S_\lambda(U) \otimes S_\mu(U) \otimes S_\nu(U)$ is non-zero, in the space of triples of weights is given by a finite set of linear homogeneous inequalities in the parts of λ, μ, ν and the condition that $|\lambda| + |\mu| + |\nu|$ is divisible by $n := \dim U$.*

Proof. Let $\sigma \in \Gamma$ be given by (3) and let $\sigma(\beta) = 0$. All components of σ are integers only if $|\lambda| + |\mu| + |\nu|$ is divisible by n , because

$$0 = \sigma(\beta) = n\sigma(u) + \sum_{i=1}^{n-1} i(\sigma(x_i) + \sigma(y_i) + \sigma(z_i)) = n\sigma(u) + |\lambda| + |\mu| + |\nu|.$$

By Theorem 3 and Proposition 1, those (λ, μ, ν) for which $\text{SI}(T_{n,n,n}, \beta)_\sigma \neq 0$ are given by $\sigma(\beta) = 0$ and a finite set of homogeneous linear inequalities in $\sigma(x_i), \sigma(y_i), \sigma(z_i)$, $1 \leq i \leq n - 1$. These inequalities can be written as inequalities in the parts of λ, μ and ν . \square

4. THE PROOF OF THEOREM 1

We define $[x, y]$ to be the vector space with the basis formed by paths from x to y . We assumed that Q has no oriented cycles, so the spaces $[x, y]$ are finite dimensional.

The indecomposable projective representations are in a bijection with Q_0 . The indecomposable projective corresponding to x is defined by

$$P_x(y) = [x, y], \quad P_x(a) = a \circ \cdot : [x, ta] \rightarrow [x, ha],$$

where $P_x(a)$ is given by the composition $p \mapsto a \circ p$. We have $\text{Hom}_Q(P_x, V) = V(x)$. In particular $\text{Hom}_Q(P_x, P_y) = [y, x]$.

We choose a numbering $Q_0 = \{x_1, \dots, x_n\}$ of vertices of Q such that for every $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$, we have $i < j$. Let $b_{i,j}$ be the number of arrows $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$. Let $p_{i,j} = \dim[x_i, x_j]$ be the number of paths p in Q such that $tp = x_i, hp = x_j$.

The relations between the $\alpha(x_j)$ and $\sigma(x_i)$ are as follows:

$$(4) \quad \sigma(x_j) = \alpha(x_j) - \sum_{i < j} b_{i,j} \alpha(x_i),$$

$$(5) \quad \alpha(x_j) = \sigma(x_j) + \sum_{i < j} p_{i,j} \sigma(x_i).$$

We define the m -arrow quiver Θ_m as a quiver with two vertices x_+ and x_- , and m arrows a_1, \dots, a_m with $ta_i = x_-, ha_i = x_+$ for $i = 1, \dots, m$. We define the weight τ given by $\tau(x_+) = 1, \tau(x_-) = -1$. The dimension vector $\theta(n)$ is defined by $\theta(n)(x_+) = \theta(n)(x_-) = n$.

The idea of the proof of Theorem 1 is to reduce the calculation to the weight space $\text{SI}(\Theta_m, \theta(n))_\tau$. The method comes from Classical Invariant Theory with a slight adjustment to accomodate the definition of semi-invariants c^V .

Proof of Theorem 1. Let us fix Q, β and a weight σ . We proceed in three steps. In the first step, we reduce the theorem to the case that Q is a quiver with exactly one source x_- and one sink x_+ , and $\sigma(x_-) = 1, \sigma(x_+) = -1$ and σ is zero on all other vertices. In the second step we reduce to the case that there are no vertices x with $\sigma(x) = 0$. The only case left is the quiver Θ_m with weight τ . In Step 3 we will prove the theorem in this case.

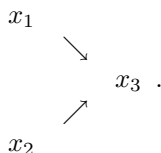
Step 1. Construct a quiver $Q(\sigma)$ as follows:

$$Q(\sigma)_0 = Q_0 \cup x_- \cup x_+,$$

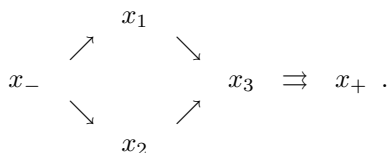
$$Q(\sigma)_1 = Q_1 \cup Q_- \cup Q_+$$

where Q_- consists of the set of arrows from x_- to x_i , with $\sigma(x_i)$ arrows going to the vertex x_i for which $\sigma(x_i) > 0$ and no arrows going to other vertices. The set Q_+ consists of the set of arrows from x_i to x_+ , with $-\sigma(x_i)$ arrows going from the vertex x_i for which $\sigma(x_i) < 0$ and no arrows going from other vertices to x_+ .

Example 2. Let Q be the quiver



Let $\sigma = (1, 1, -2)$. Then the quiver $Q(\sigma)$ is



We will write $\overline{Q} = Q(\sigma)$. Define the weight $\overline{\sigma}$ of \overline{Q} by $\overline{\sigma}(x_-) = 1, \overline{\sigma}(x_i) = 0, \overline{\sigma}(x_+) = -1$. The dimension vector $\overline{\beta} = \beta(\sigma)$ is defined by $\overline{\beta}(x_i) = \beta(x_i), \overline{\beta}(x_-) = \sum_{\{i|\sigma(x_i)>0\}} \sigma(x_i)\beta(x_i), \overline{\beta}(x_+) = \sum_{\{i|\sigma(x_i)<0\}} -\sigma(x_i)\beta(x_i)$. Suppose that $W \in \text{Rep}(\overline{Q}, \overline{\beta})$. The matrices of all maps $W(a)$ with $a \in Q_-$ form a square matrix. Let $D^-(W)$ be the determinant of this block matrix. Let $D^+(W)$ be the determinant of all $W(a)$ with $a \in Q_+$. Then the correspondence $c \rightarrow D^-cD^+$ gives the isomorphism of weight spaces $\text{SI}(Q, \beta)_\sigma \rightarrow \text{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$.

Let $\overline{\alpha}$ be the dimension vector of \overline{Q} such that $\overline{\sigma} = \langle \overline{\alpha}, \cdot \rangle$. Let \overline{V} be a representation of \overline{Q} with dimension vector $\overline{\alpha}$ and let $c^{\overline{V}}$ be the corresponding non-zero semi-invariant on $\text{SI}(\overline{Q}, \overline{\beta})$.

Proposition 2. *The factor c in the decomposition $c^{\overline{V}} = D^-cD^+$ is of the form c^V for some $V \in \text{Rep}(Q, \alpha)$.*

Proof. Notice that the weight of D^- is equal to $\langle \gamma_-, \cdot \rangle$ where

$$\gamma_-(x_-) = 1, \quad \gamma_-(x_j) = \gamma_-(x_+) = 0.$$

Similarly, by (5), the weight of D^+ equals $\langle \gamma_+, \cdot \rangle$ where

$$\begin{aligned} \gamma_+(x_-) &= 0, \quad \gamma_+(x_j) = - \sum_{\substack{i \leq j \\ \sigma(x_i) < 0}} p_{i,j} \sigma(x_i), \\ \gamma_+(x_+) &= -1 + \sum_{\substack{j \\ \sigma(x_j) < 0}} \sum_{\substack{i \leq j \\ \sigma(x_i) < 0}} p_{i,j} \sigma(x_i) \sigma(x_j). \end{aligned}$$

It is easy to see that $\langle \gamma_-, \overline{\beta} \rangle = \langle \gamma_+, \overline{\beta} \rangle = 0$.

Let $\overline{V} \in \text{Rep}(\overline{Q}, \overline{\alpha})$. Then \overline{V} has an obvious submodule $\overline{V}_1 = \overline{V}|_{\overline{Q}_0 \setminus \{x_-\}}$. We have an exact sequence

$$0 \rightarrow \overline{V}_1 \rightarrow \overline{V} \rightarrow \overline{V}_2 \rightarrow 0$$

with the dimension of \overline{V}_2 equal to γ_- .

Let M be the module defined by the exact sequence

$$0 \rightarrow P_{x_+} \xrightarrow{i} \bigoplus_{b, hb=x_+} P_{tb} \rightarrow M \rightarrow 0,$$

where the morphism i from P_{x_+} to a copy P_{tb} maps the trivial path $e(x_+)$ to the path b . The dimension vector of M is γ_+ , and c^M is the determinant D^+ . Consider the map

$$\sum_{\substack{b \\ hb=x_+}} \overline{V}_1(b) : \bigoplus_{b, hb=x_+} \overline{V}_1(tb) \rightarrow \overline{V}_1(x_+).$$

The dimension of the kernel is at least 1. Let $(s_b)_{b, hb=x_+}$ with $s_b \in \overline{V}_1(tb)$ be a non-trivial element in the kernel. We can now define a map $\bigoplus_{b, hb=x_+} P_{tb} \rightarrow \overline{V}_1$ by sending the generator $e(tb) \in P_{tb}(tb)$ to s_b for all b . Because $(s_b)_{b, hb=x_+}$ lies in the kernel, this actually defines a morphism $M \rightarrow \overline{V}_1$. Let \overline{V}_3 be the image of this morphism.

Now \overline{V}_3 is a submodule of \overline{V}_1 and $c^{\overline{V}_3} \neq 0$. By Lemma 1 a) we have $\langle \underline{d}(\overline{V}_3), \overline{\beta} \rangle \geq 0$. We also have $c^M = D^+ \neq 0$. If we apply Lemma 1 a) to the kernel N of

$M \rightarrow \bar{V}_3$, then we get $\langle \underline{d}(N), \bar{\beta} \rangle = \langle \gamma_+, -\underline{d}(\bar{V}_3) \rangle = -\langle \underline{d}(\bar{V}_3), \bar{\beta} \rangle \geq 0$. We conclude that $\langle \underline{d}(\bar{V}_3), \bar{\beta} \rangle = 0$. By Lemma 1 b) $c^{\bar{V}_3}$ divides the semi-invariant $c^M = D^+$. Because D^+ is an irreducible semi-invariant we must have $c^{\bar{V}_3} = D^+$, $\gamma_+ = \dim \bar{V}_3$ and \bar{V}_3 is isomorphic to M .

We have an exact sequence

$$0 \rightarrow \bar{V}_3 \rightarrow \bar{V}_1 \rightarrow \bar{V}_4 \rightarrow 0.$$

Now it is clear by the multiplicative property that $c^{\bar{V}} = c^{\bar{V}_2} c^{\bar{V}_4} c^{\bar{V}_3}$ with the first factor being proportional to D^- and the last one to D^+ . Let us also define a submodule $\bar{V}_5 = \bar{V}_4|_{\{x_+\}}$, so we have an exact sequence

$$0 \rightarrow \bar{V}_5 \rightarrow \bar{V}_4 \rightarrow \bar{V}_6 \rightarrow 0.$$

Note that \bar{V}_6 has support within Q . The restriction of \bar{V}_6 to Q will be denoted by V . We will prove that the restriction of $c^{\bar{V}}$ to $\text{Rep}(Q, \beta)$ is c^V .

Extend $W \in \text{Rep}(Q, \beta)$ to the module \bar{W} of dimension $\bar{\beta}$ by putting $\bar{W}(x_-) = \bigoplus_{a, ta=x_-} W(ha)$, $\bar{W}(x_+) = \bigoplus_{b, hb=x_+} W(tb)$, with the maps $\bar{W}(a)$ and $\bar{W}(b)$ being the components of the identity map. Define the canonical submodule $\bar{W}_1 = \bar{W}|_{\{x_+\}}$. We have an exact sequence

$$0 \rightarrow \bar{W}_1 \rightarrow \bar{W} \rightarrow \bar{W}_2 \rightarrow 0.$$

Define the submodule $\bar{W}_3 = \bar{W}_2|_{\hat{Q} \setminus \{x_-\}}$ of \bar{W}_2 . Now we have an exact sequence

$$0 \rightarrow \bar{W}_3 \rightarrow \bar{W}_2 \rightarrow \bar{W}_4 \rightarrow 0.$$

The representation \bar{W}_3 has support within Q and its restriction to Q is just W .

We now have

$$c^{\bar{V}}(\bar{W}) = c^{\bar{V}_4}(\bar{W}) = c^{\bar{V}_4}(\bar{W}_1) c^{\bar{V}_4}(\bar{W}_3) c^{\bar{V}_4}(\bar{W}_4) = c^{\bar{V}_4}(\bar{W}_3)$$

because $c^{\bar{V}_4}(\bar{W}_1)$ and $c^{\bar{V}_4}(\bar{W}_4)$ are constant. Moreover,

$$c^{\bar{V}_4}(\bar{W}_3) = c^{\bar{V}_5}(\bar{W}_3) c^{\bar{V}_6}(\bar{W}_3) = c^{\bar{V}_6}(\bar{W}_3) = c^V(W)$$

because $c^{\bar{V}_5}(\bar{W}_4)$ is constant. This concludes the proof of the proposition. □

Step 2. Let Q, β, σ be as above. Let $x \in Q_0$ be a vertex such that $\sigma(x) = 0$. Let a_1, \dots, a_s be the arrows in Q_1 with $ha_k = x$ ($k = 1, \dots, s$) and let b_1, \dots, b_t be the arrows in Q_1 with $tb_l = x$ ($l = 1, \dots, t$). Let \bar{Q} be the quiver such that $\bar{Q}_0 = Q_0 \setminus \{x\}$ and $\bar{Q}_1 = (Q_1 \setminus \{a_1, \dots, a_s, b_1, \dots, b_t\}) \cup \{ba_{k,l}\}_{1 \leq k \leq s, 1 \leq l \leq t}$, where $t(ba_{k,l}) = ta_k, h(ba_{k,l}) = hb_l$. Let $\bar{\beta}, \bar{\sigma}$ be the restrictions of β, σ to $Q_0 \setminus \{x\}$.

The Fundamental Theorem of Invariant Theory (see [2] for a characteristic free version) says that every semi-invariant from $\text{SI}(Q, \beta)_\sigma$ can be obtained from the semi-invariants from $\text{SI}(\bar{Q}, \bar{\beta})_{\bar{\sigma}}$ by substituting the actual compositions $b_l a_k$ for the arrows of type $ba_{k,l}$. Assuming Theorem 1 for $\text{SI}(\bar{Q}, \bar{\beta})_{\bar{\sigma}}$ to be true, we need to show that every semi-invariant $c^{\bar{V}}$ from $\text{SI}(\bar{Q}, \bar{\beta})_{\bar{\sigma}}$ pulls back to a semi-invariant of type c^V . For a given representation \bar{V} of \bar{Q} of dimension $\bar{\alpha}$ we define the representation $V = \text{ind } \bar{V}$ as follows. We notice that the condition $\sigma(x) = 0$ means that we expect $\dim V(x) = \sum_{k=1}^s \dim V(ta_k)$.

This means we put

$$V(y) = \begin{cases} \overline{V}(y) & \text{if } y \neq x, \\ \bigoplus_{k=1}^s \overline{V}(ta_k) & \text{if } y = x. \end{cases}$$

We define the linear maps $V(a)$ as follows:

$$V(a) = \begin{cases} \overline{V}(a) & \text{if } a \neq a_k, b_l, \\ i(a_k) & \text{if } a = a_k, \\ \sum_{k=1}^s \overline{V}(ba_{k,l}) & \text{if } b = b_l, \end{cases}$$

where $i(a_k) : V(ta_k) \rightarrow \bigoplus_{k=1}^s V(ta_k)$ is the injection on the k -th summand.

Then it is easy to check directly from the definition of semi-invariants c^V that if the representation $\overline{W} = \text{res } W$ of dimension $\overline{\beta}$ is a restriction of a representation W of Q of dimension β , then $c^{\overline{V}}(\overline{W}) = c^V(W)$.

Notice that the functor $\text{ind } \overline{V}$ is the left adjoint of the obvious restriction functor $\text{res} : \text{Rep}(Q) \rightarrow \text{Rep}(\overline{Q})$, i.e., we have the natural isomorphisms

$$\text{Hom}_Q(\text{ind } \overline{V}, W) = \text{Hom}_{\overline{Q}}(\overline{V}, \text{res } W)$$

which explains why $c^{\overline{V}}(\overline{W})$ and $c^V(W)$ vanish simultaneously.

Step 3. It remains to deal directly with the weight space $\text{SI}(\Theta_m, \theta(n))_\tau$. Writing the representation W of dimension $\theta(n)$ as an m -tuple of linear maps,

$$W(a_1), \dots, W(a_m) : W_- \rightarrow W_+,$$

we can introduce the additional action of the group $\text{GL}(m)$ acting on this space by taking linear combinations of the linear maps $W(a_1), \dots, W(a_m)$. Using the Cauchy formula (in its characteristic free version, say from [1]) we see that the space $\text{SI}(\Theta_m, \theta(n))_\tau$ of semi-invariants can be identified with $\bigwedge^n W_- \otimes \bigwedge^n W_+^* \otimes D_n(K^m)$. Here D_n denotes the n -th divided power. Since the divided power $D_n(K^m)$ is generated as a $\text{GL}(m)$ -module by its highest weight vector (which corresponds to the semi-invariant $\det W(a_1)$) and the set of semi-invariants of the form c^V is preserved by the action of $\text{GL}(m)$, it is enough to express $\det W(a_1)$ as the semi-invariant of the form c^V . Notice that $\tau = \langle \alpha, \cdot \rangle$ for the dimension vector $\alpha = (1, m - 1)$. Taking the module V to be the m -tuple of linear maps $V(a_1), \dots, V(a_m) : K \rightarrow K^{m-1}$ where $V(a_1) = 0$ and $V(a_i)$ is the embedding sending 1 to the $i - 1$ 'st basis vector, for $i = 2, \dots, m$, we check directly that $c^V = \det W(a_1)$. This concludes the proof of Theorem 1. □

We now will give another description for semi-invariants $\text{SI}(Q, \beta)_\sigma$. Let $\overline{Q} = Q(\sigma), \overline{\beta}$ and $\overline{\sigma}$ be as in Step 1 of the proof of Theorem 1. We know that $\text{SI}(Q, \beta)_\sigma \cong \text{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$. Let $\overline{\alpha}$ be a dimension vector of \overline{Q} such that $\langle \overline{\alpha}, \cdot \rangle = \overline{\sigma}$. Now $\text{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ is generated by semi-invariants $c^{\overline{V}}$ with $\underline{d}(\overline{V}) = \overline{\alpha}$. In fact we only need to take those $c^{\overline{V}}$ where \overline{V} lies in a Zariski dense set of $\text{Rep}(\overline{Q}, \overline{\alpha})$. A general representation \overline{V} of dimension $\overline{\alpha}$ has the following projective resolution:

$$0 \rightarrow P_{x_+} \xrightarrow{d_V} P_{x_-} \rightarrow \overline{V} \rightarrow 0$$

with $d_V \in \text{Hom}_Q(P_{x_+}, P_{x_-}) = [x_-, x_+]$. So d_V can be seen as some linear combination $\sum_{i=1}^r \lambda_i p_i$ where p_1, \dots, p_r are all paths from x_+ to x_- . For any $\overline{W} \in \text{Rep}(\overline{Q}, \overline{\beta})$ we have the following exact sequence:

$$0 \rightarrow \text{Hom}_{\overline{Q}}(\overline{V}, \overline{W}) \rightarrow \text{Hom}_{\overline{Q}}(P_{x_+}, \overline{W}) \xrightarrow{\tilde{d}_V} \text{Hom}_{\overline{Q}}(P_{x_-}, \overline{W}) \rightarrow \text{Ext}_{\overline{Q}}(\overline{V}, \overline{W}) \rightarrow 0.$$

It is easy to see that $\det(\tilde{d}_{\overline{V}}) = c^{\overline{V}}(\overline{W}) = c^V(W)$.

We have that

$$\begin{aligned}\mathrm{Hom}_{\overline{Q}}(P_{x_+}, \overline{W}) &\cong \overline{W}_{x_+} = \bigoplus_{\sigma(x_i) > 0} W(x_i)^{\sigma(x_i)}, \\ \mathrm{Hom}_{\overline{Q}}(P_{x_-}, \overline{W}) &\cong \overline{W}_{x_-} = \bigoplus_{\sigma(x_i) < 0} W(x_i)^{\sigma(x_i)}, \\ \tilde{d}_{\overline{V}} &= \sum_i \lambda_i \overline{V}(p_i).\end{aligned}$$

Let F be a function from the set of paths from x_+ to x_- to the set of non-negative integers. For each such F we can define the semi-invariant I_F as the coefficient of $\lambda_1^{F(p_1)} \lambda_2^{F(p_2)} \dots \lambda_r^{F(p_r)}$ in $\det(\tilde{d}_{\overline{V}})$.

Corollary 3. *The space of semi-invariants $\mathrm{SI}(Q, \beta)_\sigma$ is spanned by semi-invariants of the form I_F .*

A necessary condition for I_F to be non-zero is

$$\sum_i F(p_i) = \sum_{\sigma(x_i) > 0} \sigma(x_i) \beta(x_i) = \sum_{\sigma(x_i) < 0} -\sigma(x_i) \beta(x_i).$$

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