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# SEMI-INVARIANTS OF QUIVERS AND SATURATION FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

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## 1. Introduction

Let $Q$ be a quiver without oriented cycles. Let $\alpha$ be a dimension vector for $Q$. We denote by $\mathrm{SI}(Q,\alpha)$ the ring of semi-invariants of the set of $\alpha$-dimensional representations of $Q$ over a fixed algebraically closed field $K$.

In this paper we prove some results about the set

$$\Sigma(Q,\alpha) = \{\, \sigma \mid \mathrm{SI}(Q,\alpha)_\sigma \neq 0 \,\}.$$

$\Sigma(Q,\alpha)$ is defined in the space of all weights by one homogeneous linear equation and by a finite set of homogeneous linear inequalities. In particular the set $\Sigma(Q,\alpha)$ is saturated, i.e., if $n\sigma \in \Sigma(Q,\alpha)$, then also $\sigma \in \Sigma(Q,\alpha)$.

These results, when applied to a special quiver $Q = T_{n,n,n}$ and to a special dimension vector, show that the $\mathrm{GL}_n$-module $V_\lambda$ appears in $V_\mu \otimes V_\nu$ if and only if the partitions $\lambda$, $\mu$ and $\nu$ satisfy an explicit set of inequalities. This gives new proofs of the results of Klyachko ([7, 3]) and Knutson and Tao ([8]).

The proof is based on another general result about semi-invariants of quivers (Theorem 1). In the paper [10], Schofield defined a semi-invariant $c_W$ for each indecomposable representation $W$ of $Q$. We show that the semi-invariants of this type span each weight space in $\mathrm{SI}(Q,\alpha)$. This seems to be a fundamental fact, connecting semi-invariants and modules in a direct way. Given this fact, the results on sets of weights follow at once from the results in another paper of Schofield [11].

## 2. The results

A quiver $Q$ is a pair $Q = (Q_0, Q_1)$ consisting of the set of vertices $Q_0$ and the set of arrows $Q_1$. Each arrow $a$ has its head $ha$ and tail $ta$, both in $Q_0$:

$$ta \xrightarrow{a} ha.$$

We fix an algebraically closed field $K$. A representation (or a module) $V$ of $Q$ is a family of finite dimensional vector spaces $\{\, V(x) \,|\, x \in Q_0 \,\}$ and of linear maps

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$V(a) : V(ta) \to V(ha)$. The dimension vector of a representation $V$ is the function $\underline{d}(V) : Q_0 \to \mathbb{Z}_{\geq 0}$ defined by $\underline{d}(V)(x) := \dim V(x)$. The dimension vectors lie in the space $\Gamma$ of integer-valued functions on $Q_0$. A morphism $\phi : V \to V'$ of two representations is a collection of linear maps $\phi(x) : V(x) \to V'(x)$, $x \in Q_0$, such that for each $a \in Q_1$ we have $\phi(ha)V(a) = V'(a)\phi(ta)$. We denote the linear space of morphisms from $V$ to $V'$ by $\mathrm{Hom}_Q(V, V')$.

A path $p$ in $Q$ is a sequence of arrows $p = a_1, \dots, a_n$ such that $ha_i = ta_{i+1}$ $(1 \leq i \leq n-1)$. We define $tp = ta_1, hp = ha_n$. We also have the trivial path $e(x)$ from $x$ to $x$. If $V$ is a representation and $p = a_1, \dots, a_n$, then we define $V(p) := V(a_n)V(a_{n-1}) \cdots V(a_1)$. We assume throughout the paper that $Q$ has no oriented cycles, i.e., there are no paths $p = a_1, \dots, a_n$ such that $ta_1 = ha_n$.

For representations $V$ and $W$ of $Q$ there is a canonical exact sequence ([9])

$$(1) \quad 0 \to \mathrm{Hom}_Q(V, W) \xrightarrow{i} \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \xrightarrow{p} \mathrm{Ext}_Q(V, W) \to 0.$$

The map $i$ is the obvious inclusion, the map $d^V_W$ is given by

$$\{f(x)\}_{x \in Q_0} \mapsto \{f(ha)V(a) - W(a)f(ta)\}_{a \in Q_1},$$

and the map $p$ constructs an extension of the representations $V$ and $W$ by adding the maps $V(ta) \to W(ha)$ to the direct sum representation $V \oplus W$.

For $\alpha, \beta \in \Gamma$ we define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

It follows from (1) that $\langle \underline{d}(V), \underline{d}(W) \rangle = \dim_K \mathrm{Hom}_Q(V, W) - \dim_K \mathrm{Ext}_Q(V, W)$.

For a dimension vector $\alpha$ we denote by

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$$

the vector space of $\alpha$-dimensional representations of $Q$. The group

$$\mathrm{GL}(Q, \alpha) := \prod_{x \in Q_0} \mathrm{GL}(\alpha(x))$$

and its subgroup

$$\mathrm{SL}(Q, \alpha) = \prod_{x \in Q_0} \mathrm{SL}(\alpha(x))$$

act on $\mathrm{Rep}(Q, \alpha)$ in an obvious way. We are interested in the ring of semi-invariants

$$\mathrm{SI}(Q, \alpha) := K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(Q, \alpha)}.$$

The ring $\mathrm{SI}(Q,\alpha)$ has a weight space decomposition

$$\mathrm{SI}(Q,\alpha)=\bigoplus_{\sigma}\mathrm{SI}(Q,\alpha)_{\sigma}$$

where $\sigma$ runs through the (one-dimensional irreducible) characters of $\mathrm{GL}(Q,\alpha)$ and

$$\mathrm{SI}(Q,\alpha)_{\sigma}=\{\, f\in K[\mathrm{Rep}(Q,\alpha)] \mid g(f)=\sigma(g)f \ \forall g\in \mathrm{GL}(Q,\alpha)\,\}.$$

Suppose that $\sigma$ lies in the dual space $\Gamma^{\star}:=\mathrm{Hom}(\Gamma,\mathbb{Z})$. For each dimension vector $\alpha$ we can associate to $\sigma$ a character of $\mathrm{GL}(Q,\alpha)$ defined as

$$\prod_{x\in Q_0} d_x^{\sigma(e_x)}$$

where $d_x$ is the determinant function on $\mathrm{GL}(\alpha(x))$ and $e_x$ is the dimension vector defined by

$$e_x(y)=\begin{cases} 1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

In this way we will identify characters with $\Gamma^{\star}$. Sometimes, for convenience, we will write $\sigma(x)$ instead of $\sigma(e_x)$ (and treat $\sigma$ as an element of $\Gamma$).

Let us choose the dimension vectors $\alpha$ and $\beta$ in such way that $\langle\alpha,\beta\rangle=0$. Then for every $V\in\mathrm{Rep}(Q,\alpha)$ and $W\in\mathrm{Rep}(Q,\beta)$ the matrix of $d^V_W$ will be a square matrix. Following [10] we can therefore define the semi-invariant $c$ of the action of $\mathrm{GL}(Q,\alpha)\times\mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\alpha)\times\mathrm{Rep}(Q,\beta)$ by $c(V,W):=\det d^V_W$. The value of the determinant depends on the choices of bases, so $c$ is well-defined up to a scalar. Notice that the semi-invariant $c$ vanishes at the point $(V,W)$ if and only if $\mathrm{Hom}_Q(V,W)\neq 0$ which is equivalent to $\mathrm{Ext}_Q(V,W)\neq 0$. For a fixed $V$ the restriction of $c$ to $\{V\}\times\mathrm{Rep}(Q,\beta)$ defines a semi-invariant $c^V$ in $\mathrm{SI}(Q,\beta)$. Schofield proves ([10, Lemma 1.4]) that the weight of $c^V$ equals $\langle\alpha,\cdot\rangle\in\Gamma^{\star}$ which is defined as $\gamma\mapsto\langle\alpha,\gamma\rangle$. Similarly, for a fixed $W$ the restriction of $c$ to $\mathrm{Rep}(Q,\alpha)\times\{W\}$ defines a semi-invariant $c_W$ in $\mathrm{SI}(Q,\alpha)$ of weight $-\langle\cdot,\beta\rangle$ ([10, Lemma 1.4]). If $V,V'\in\mathrm{Rep}(Q,\alpha)$ and $V\cong V'$, then $V$ and $V'$ are in the same $\mathrm{GL}(Q,\alpha)$-orbit, and $c^V$ and $c^{V'}$ are equal up to a constant scalar. Semi-invariants of the types $c^V$ and $c_W$ are well-defined up to a scalar. These semi-invariants have the following properties.

**Lemma 1.** *Suppose that $V,V',V''$ and $W,W',W''$ are representations of $Q$ such that $\langle\underline{d}(V),\underline{d}(W)\rangle=0$, and that there are exact sequences*

$$0\to V'\to V\to V''\to 0,\qquad 0\to W'\to W\to W''\to 0.$$

a) *If $\langle\underline{d}(V'),\underline{d}(W)\rangle<0$, then $c^V(W)=0$;*
b) *If $\langle\underline{d}(V'),\underline{d}(W)\rangle=0$, then $c^V(W)=c^{V'}(W)c^{V''}(W)$;*
c) *If $\langle\underline{d}(V),\underline{d}(W')\rangle>0$, then $c^V(W)=0$;*
d) *If $\langle\underline{d}(V),\underline{d}(W')\rangle=0$, then $c^V(W)=c^V(W')c^V(W'')$.*

*Proof.* Consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \mathrm{Hom}(V''(x), W(x)) & \xrightarrow{d_W^{V''}} & \bigoplus_{a\in Q_1} \mathrm{Hom}(V''(ta), W(ha)) \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \mathrm{Hom}(V(x), W(x)) & \xrightarrow{d_W^{V}} & \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ta), W(ha)) \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \mathrm{Hom}(V'(x), W(x)) & \xrightarrow{d_W^{V'}} & \bigoplus_{a\in Q_1} \mathrm{Hom}(V'(ta), W(ha)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

If $\langle \underline{d}(V'), \underline{d}(W)\rangle = 0$, then $d_W^{V'}$, $d_W^V$ and $d_W^{V''}$ are all represented by square matrices. It follows that $c^V(W) = c^{V'}(W)c^{V''}(W)$. So b) follows and d) goes similarly. If $\langle \underline{d}(V'), \underline{d}(W)\rangle < 0$, then $d_W^{V'}$ cannot be surjective, hence $d_W^V$ is not surjective. Now a) follows and c) goes similarly. ◻

Our main result is that the semi-invariants of type $c^V$ (resp. $c_W$) span all the weight spaces in the rings $\mathrm{SI}(Q, \alpha)$.

**Theorem 1.** *Let $Q$ be a quiver without oriented cycles and let $\beta$ be a dimension vector. The ring of semi-invariants $\mathrm{SI}(Q,\beta)$ is a $K$-linear span of semi-invariants $c^V$ with $\langle \underline{d}(V), \beta\rangle = 0$. The analogous result is true for the semi-invariants $c_W$.*

After this paper was submitted we learned about the paper [12] where among other things the authors give another proof of Theorem 1 under the assumption that the characteristic of $K$ is zero.

We will prove Theorem 1 in Section 4.

*Remark* 1. If $V = V_1 \oplus V_2$ is decomposable, then by Lemma 1 we have $c^V = 0$ if $\langle \underline{d}(V_1), \beta\rangle \neq 0$, and $c^V = c^{V_1}c^{V_2}$ if $\langle \underline{d}(V_1), \beta\rangle = 0$.

The algebra $\mathrm{SI}(Q, \beta)$ is generated by all $c^V$ where $V$ is *indecomposable*. Generators of $\mathrm{SI}(Q, \beta)$ therefore can be found in the degrees $\langle \alpha, \cdot\rangle$ such that a general representation of dimension $\alpha$ is indecomposable. By [5] this is equivalent to $\alpha$ being a Schur root.

*Remark* 2. If $\mathrm{Rep}(Q, \beta)$ has a dense $\mathrm{GL}(Q, \beta)$-orbit, then Schofield showed in [10] that the invariants of type $c^V$ with $V$ indecomposable generate $\mathrm{SI}(Q, \beta)$ (which is a polynomial ring in this case).

Theorem 1 has the following remarkable consequence.

**Corollary 1** (Reciprocity Property)**.** *Let $\alpha, \beta$ be two dimension vectors for the quiver $Q$. Assume that $\langle \alpha, \beta\rangle = 0$. Then*

$$\dim_K \mathrm{SI}(Q, \beta)_{\langle \alpha,\cdot\rangle} = \dim_K \mathrm{SI}(Q, \alpha)_{-\langle \cdot,\beta\rangle}.$$

*Proof.* Let $V_1, \dots, V_s$ be the modules of dimension $\alpha$ such that $c^{V_1}, \dots, c^{V_s}$ form a basis of $\mathrm{SI}(Q, \beta)_{\langle \alpha,\cdot\rangle}$. These are linearly independent polynomials on $\mathrm{Rep}(Q, \beta)$ so there exist $s$ representations $W_1, \dots, W_s$ in $\mathrm{Rep}(Q, \beta)$ such that $\det(c^{V_i}(W_j))_{1\le i,j\le s}$

is not zero. But $c^{V_i}(W_j) = c_{W_j}(V_i)$ and this means that the semi-invariants $c_{W_1}, \dots, c_{W_s}$ are linearly independent. This proves that

$$\dim_K \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle} \leq \dim_K \mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

The other inequality is proven in exactly the same way. □

In the remainder of this section we investigate the consequences of Theorem 1. First we recall the main results of [11]. They can be summarized as follows.

We say that for two dimension vectors $\alpha, \beta$ the space $\mathrm{Hom}_Q(\alpha,\beta)$ (respectively $\mathrm{Ext}_Q(\alpha,\beta)$) vanishes generically if and only if for general representations $V, W$ of dimensions $\alpha, \beta$ respectively we have $\mathrm{Hom}_Q(V,W) = 0$ (resp. $\mathrm{Ext}_Q(V,W) = 0$). We also write $\alpha \hookrightarrow \beta$ if a general representation of dimension $\beta$ has a subrepresentation of dimension $\alpha$.

**Theorem 2** (Schofield). *Let $\alpha$ and $\beta$ be two dimension vectors for the quiver $Q$.*

a) $\mathrm{Ext}_Q(\alpha,\beta)$ *vanishes generically if and only if* $\alpha \hookrightarrow \alpha+\beta$,

b) $\mathrm{Ext}_Q(\alpha,\beta)$ *does not vanish generically if and only if* $\beta' \hookrightarrow \beta$ *and* $\langle\alpha, \beta-\beta'\rangle < 0$ *for some dimension vector* $\beta'$.

Part a) is proven in Section 3 of [11], and part b) is proven in Section 5.

*Remark* 3. Suppose that $V$ and $W$ are general modules of dimension $\alpha$ and $\beta$ respectively, such that $\langle\alpha,\beta\rangle = 0$. The condition in b) is equivalent to $\exists\beta'\beta' \hookrightarrow \beta$ such that $\langle\alpha,\beta'\rangle > 0$. If $c^V(W) = 0$, then $W$ must have a submodule $W'$ such that $\langle\alpha, \underline{d}(W')\rangle > 0$. This means that the converse of Lemma 1.c) is true for general $V$ and $W$.

**Theorem 3.** *Let $Q$ be a quiver without oriented cycles and let $\beta$ be a dimension vector. The semigroup $\Sigma(Q,\beta)$ is the set of all $\sigma \in \Gamma$ such that $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta'$ such that $\beta' \hookrightarrow \beta$. Thus this condition is provided by one linear homogeneous equality and finitely many linear homogeneous inequalities. In particular the set $\Sigma(Q,\beta)$ is saturated in the lattice $\Gamma$.*

*Proof.* Suppose that $\sigma \in \Gamma^\star$. We can write $\sigma = \langle\alpha,\cdot\rangle$ with $\alpha \in \Gamma$.

We will first assume that $\alpha$ is a dimension vector, i.e., $\alpha(x) \geq 0$ for all $x \in Q_0$. It follows from Theorem 1 that $\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}$ is non-zero if and only if there exists a representation $V$ of dimension $\alpha$ such that $c^V$ is not zero, which is equivalent to $\sigma(\beta) = \langle\alpha,\beta\rangle = 0$ and $\mathrm{Ext}_Q(\alpha,\beta)$ vanishing generically. By part b) of Theorem 2, $\mathrm{Ext}_Q(\alpha,\beta)$ vanishes generically if and only if for all $\beta'$ such that $\beta' \hookrightarrow \beta$ we have $\langle\alpha,\beta-\beta'\rangle \geq 0$. This means that for all $\beta'$ such that $\beta' \hookrightarrow \beta$ we have $\sigma(\beta') = \langle\alpha,\beta'\rangle \leq 0$. We conclude that $\mathrm{SI}(Q,\beta)_\sigma \neq 0$ if and only if $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta' \hookrightarrow \beta$.

If $\alpha$ is not a dimension vector, then $\mathrm{SI}(Q,\beta)_{n\sigma} = 0$ for all integers $n > 0$. Suppose that $W \in \mathrm{Rep}(Q,\beta)$. From [6] it follows that either $\sigma(\underline{d}(W)) \neq 0$ or there exists a submodule $W'$ of $W$ such that $\sigma(\underline{d}(W')) > 0$. If $W$ is in general position, then we obtain $\sigma(\beta) \neq 0$ or $\sigma(\beta') > 0$ for some $\beta' \hookrightarrow \beta$ (see also Remark 5). □

*Remark* 4. Schofield in [11] gives an algorithm allowing one to determine the set of inequalities in Theorem 3 inductively. This algorithm is not very efficient.

*Remark* 5. A module $W \in \mathrm{Rep}(Q,\beta)$ is called $\sigma$-stable if and only if there exist an $n > 0$ and an $f \in \mathrm{SI}(Q,\beta)_{n\sigma}$ such that $f(W) \neq 0$. King proved in [6] that a module

$W \in \mathrm{Rep}(Q,\beta)$ is $\sigma$-stable if and only if $\sigma(W') \leq 0$ for all submodules $W'$ of $W$. Applied to a general representation $W$ of dimension $\beta$ this gives us the equivalence:

$$\exists n > 0 \ \ \mathrm{SI}(Q,\beta)_{n\sigma} \neq 0 \Leftrightarrow \sigma(\beta) = 0 \text{ and } \forall \beta' \ \beta' \hookrightarrow \beta \text{ we have } \sigma(\beta') \leq 0.$$

This shows that the saturation of $\Sigma(Q,\beta)$ is given by linear inequalities but it does not show that $\Sigma(Q,\beta)$ is saturated.

*Remark* 6. In Theorem 3, instead of considering *all* $\beta'$ with $\beta' \hookrightarrow \beta$ we only need to consider those $\beta'$ such that the general representation of dimension $\beta'$ is indecomposable, which is equivalent to $\beta'$ being a Schur root. Still, the set of inequalities obtained in this way may not be a minimal set of inequalities as we will see in the next example.

**Example 1.** Let $Q$ be the quiver

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 4 & \rightarrow & 5 & \leftarrow & 2 \\ & & \uparrow & & \\ & & 3 & & \end{array}$$

and let $\beta$ be the dimension vector

$$\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array}.$$

For a general representation $V$ of $Q$ with dimension vector $\beta$, the dimension vectors of indecomposable submodules are:

$$\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 1 & \end{array}$$

$$\begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{array}$$

$$\begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array}$$

Let $\sigma$ be the weight given by $\sigma(\alpha) = \sum_{i=1}^{5} a_i \alpha(i)$, in other words

$$\sigma = \begin{array}{ccc} & a_1 & \\ a_4 & a_5 & a_2 \\ & a_3 & \end{array}.$$

We investigate when $\mathrm{SI}(Q,\beta)_\sigma \neq 0$. First of all we must have $\sigma(\beta) = 0$, so $a_1 + a_2 + a_3 + a_4 + 2a_5 = 0$. In particular $a_1 + a_2 + a_3 + a_4$ must be even. The indecomposable submodules listed above correspond to the inequalities (using $a_5 = -(a_1 + a_2 + a_3 + a_4)/2$):

(2)
$$\begin{gathered} a_1 \geq 0,\ a_2 \geq 0, a_3 \geq 0, a_4 \geq 0, \\ a_1 \leq a_2 + a_3 + a_4,\ a_2 \leq a_1 + a_3 + a_4,\ a_3 \leq a_1 + a_2 + a_4,\ a_4 \leq a_1 + a_2 + a_3, \\ a_1 + a_2 + a_3 + a_4 \geq 0. \end{gathered}$$

The last inequality is redundant.

In the next section we will see how semi-invariants can be interpreted in terms of tensor products of modules of the general linear group. This particular example shows that for a 2-dimensional vector space $U$, the tensor product of symmetric powers $S_{a_1}(U) \otimes S_{a_2}(U) \otimes S_{a_3}(U) \otimes S_{a_4}(U)$ contains a non-trivial $\mathrm{SL}(U)$-invariant subspace if and only if $a_1 + a_2 + a_3 + a_4$ is even and the inequalities (2) hold. In this case, the inequalities are obvious from the Clebsch-Gordan formula.

## 3. Application to Littlewood-Richardson coefficients

Let us apply Theorem 3 in the following special case. Let us define the quiver $Q = T_{n,n,n}$ as follows:

$$\begin{array}{ccccccccccccc} x_1 & \to & x_2 & \to \dots \to & x_{n-1} & \to & u & \leftarrow & y_{n-1} & \leftarrow \dots \leftarrow & y_2 & \leftarrow & y_1 \\ & & & & & & \uparrow & & & & & & \\ & & & & & & z_{n-1} & & & & & & \\ & & & & & & \uparrow & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & \uparrow & & & & & & \\ & & & & & & z_2 & & & & & & \\ & & & & & & \uparrow & & & & & & \\ & & & & & & z_1 & & & & & & \end{array}$$

Let us choose the dimension vector $\beta(x_i) = \beta(y_i) = \beta(z_i) = i$ for $i = 1, \dots, n-1$, $\beta(u) = n$. The following proposition is a direct application of Cauchy's formula and is a standard calculation in representation theory.

**Proposition 1.** *The weight space* $\mathrm{SI}(T_{n,n,n}, \beta)_\sigma$ *is isomorphic to the space of* $\mathrm{SL}(U)$*-invariants in the triple tensor product* $S_\lambda(U) \otimes S_\mu(U) \otimes S_\nu(U)$ *of Schur functors on* $U$*, where* $U$ *is the vector space of dimension* $n$*, and* $\lambda, \mu, \nu$ *are partitions whose conjugate partitions are given as follows:*

$$\begin{array}{rcl} \lambda' & = & ((n-1)^{\sigma(x_{n-1})}, (n-2)^{\sigma(x_{n-2})}, \dots, 1^{\sigma(x_1)}), \\ \mu' & = & ((n-1)^{\sigma(y_{n-1})}, (n-2)^{\sigma(y_{n-2})}, \dots, 1^{\sigma(y_1)}), \\ \nu' & = & ((n-1)^{\sigma(z_{n-1})}, (n-2)^{\sigma(z_{n-2})}, \dots, 1^{\sigma(z_1)}). \end{array} \quad (3)$$

*Here* $\sigma(q)$ *is defined as* $\sigma(e_q)$ *where the dimension vector* $e_q$ *is given by* $e_q(q) = 1$ *and* $e_q(p) = 0$ *if* $p \neq q$.

*Proof.* Let us denote by $a_i$ (resp. $b_i, c_i$) the arrow in $T_{n,n,n}$ with $ta_i = x_i, ha_i = x_{i+1}$ (resp. $tb_i = y_i, hb_i = y_{i+1}, tc_i = z_i, hc_i = z_{i+1}$) for $1 \leq i \leq n-1$. The space $\mathrm{Rep}(T_{n,n,n}, \beta)$ can be identified with

$$\bigoplus_{1 \leq i \leq n-1} \big( \mathrm{Hom}(V(x_i), V(x_{i+1})) \oplus \mathrm{Hom}(V(y_i), V(y_{i+1})) \oplus \mathrm{Hom}(V(z_i), V(z_{i+1})) \big)$$

where we write $x_n = y_n = z_n = u$.

The Cauchy formula [4, §A.1] gives the decomposition of $K[\mathrm{Rep}(T_{n,n,n},\beta)]$ as a direct sum over the $3(n-1)$-tuples of partitions

$$((\alpha^i)_{1\le i\le n-1},(\beta^i)_{1\le i\le n-1},(\gamma^i)_{1\le i\le n-1})$$

of the summands

$$\bigotimes_{1\le i\le n-1}\big(S_{\alpha^i}V(x_i)\otimes S_{\alpha^i}V(x_{i+1})^*\otimes S_{\beta^i}V(y_i)\otimes S_{\beta^i}V(y_{i+1})^* \\ \otimes S_{\gamma^i}V(z_i)\otimes S_{\gamma^i}V(z_{i+1})^*\big).$$

Let us denote $H=\prod_{1\le i\le n-1}\big(\mathrm{SL}(V(x_i))\times\mathrm{SL}(V(y_i))\times\mathrm{SL}(V(z_i))\big)$. Then it follows from the Littlewood-Richardson Rule [4, §A.1] that the summand corresponding to the $3(n-1)$-tuple

$$((\alpha^i)_{1\le i\le n-1},(\beta^i)_{1\le i\le n-1},(\gamma^i)_{1\le i\le n-1})$$

contains an $H$-invariant if and only if we have for each $i$, $1\le i\le n-1$,

$$(\alpha^i)'=((i)^{\sigma(x_i)},(i-1)^{\sigma(x_{i-1})},\dots,1^{\sigma(x_1)}),$$
$$(\beta^i)'=((i)^{\sigma(y_i)},(i-1)^{\sigma(y_{i-1})},\dots,1^{\sigma(y_1)}),$$
$$(\gamma^i)'=((i)^{\sigma(z_i)},(i-1)^{\sigma(z_{i-1})},\dots,1^{\sigma(z_1)})$$

for some non-negative numbers $\sigma(x_i),\sigma(y_i),\sigma(z_i)$. Moreover, if these conditions are satisfied, then the space of $H$-invariants is isomorphic to

$$S_{\alpha^{n-1}}V(u)^*\otimes S_{\beta^{n-1}}V(u)^*\otimes S_{\gamma^{n-1}}V(u)^*.$$

Therefore the space of $\mathrm{SL}(T_{n,n,n},\beta)$-semi-invariants can be identified with the space of $\mathrm{SL}(V(u))$-invariants in the above triple tensor product. □

**Corollary 2.** *The set of triples of partitions $(\lambda,\mu,\nu)$ such that the space of $\mathrm{SL}(U)$-invariants in $S_\lambda(U)\otimes S_\mu(U)\otimes S_\nu(U)$ is non-zero, in the space of triples of weights is given by a finite set of linear homogeneous inequalities in the parts of $\lambda,\mu,\nu$ and the condition that $|\lambda|+|\mu|+|\nu|$ is divisible by $n:=\dim U$.*

*Proof.* Let $\sigma\in\Gamma$ be given by (3) and let $\sigma(\beta)=0$. All components of $\sigma$ are integers only if $|\lambda|+|\mu|+|\nu|$ is divisible by $n$, because

$$0=\sigma(\beta)=n\sigma(u)+\sum_{i=1}^{n-1}i\,(\sigma(x_i)+\sigma(y_i)+\sigma(z_i))=n\sigma(u)+|\lambda|+|\mu|+|\nu|.$$

By Theorem 3 and Proposition 1, those $(\lambda,\mu,\nu)$ for which $\mathrm{SI}(T_{n,n,n},\beta)_\sigma\neq 0$ are given by $\sigma(\beta)=0$ and a finite set of homogeneous linear inequalities in $\sigma(x_i),\sigma(y_i),\sigma(z_i)$, $1\le i\le n-1$. These inequalities can be written as inequalities in the parts of $\lambda$, $\mu$ and $\nu$. □

## 4. The proof of Theorem 1

We define $[x,y]$ to be the vector space with the basis formed by paths from $x$ to $y$. We assumed that $Q$ has no oriented cycles, so the spaces $[x,y]$ are finite dimensional.

The indecomposable projective representations are in a bijection with $Q_0$. The indecomposable projective corresponding to $x$ is defined by

$$P_x(y)=[x,y],\quad P_x(a)=a\circ\cdot:[x,ta]\to[x,ha],$$

where $P_x(a)$ is given by the composition $p \mapsto a \circ p$. We have $\mathrm{Hom}_Q(P_x, V) = V(x)$. In particular $\mathrm{Hom}_Q(P_x, P_y) = [y, x]$.

We choose a numbering $Q_0 = \{x_1, \ldots, x_n\}$ of vertices of $Q$ such that for every $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$, we have $i < j$. Let $b_{i,j}$ be the number of arrows $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$. Let $p_{i,j} = \dim[x_i, x_j]$ be the number of paths $p$ in $Q$ such that $tp = x_i, hp = x_j$.

The relations between the $\alpha(x_j)$ and $\sigma(x_i)$ are as follows:

$$\sigma(x_j) = \alpha(x_j) - \sum_{i<j} b_{i,j}\alpha(x_i), \tag{4}$$

$$\alpha(x_j) = \sigma(x_j) + \sum_{i<j} p_{i,j}\sigma(x_i). \tag{5}$$

We define the $m$-arrow quiver $\Theta_m$ as a quiver with two vertices $x_+$ and $x_-$, and $m$ arrows $a_1, \ldots, a_m$ with $ta_i = x_-$, $ha_i = x_+$ for $i = 1, \ldots, m$. We define the weight $\tau$ given by $\tau(x_+) = 1, \tau(x_-) = -1$. The dimension vector $\theta(n)$ is defined by $\theta(n)(x_+) = \theta(n)(x_-) = n$.

The idea of the proof of Theorem 1 is to reduce the calculation to the weight space $\mathrm{SI}(\Theta_m, \theta(n))_\tau$. The method comes from Classical Invariant Theory with a slight adjustment to accomodate the definition of semi-invariants $c^V$.

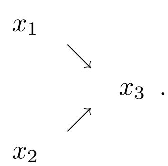
*Proof of Theorem 1.* Let us fix $Q$, $\beta$ and a weight $\sigma$. We proceed in three steps. In the first step, we reduce the theorem to the case that $Q$ is a quiver with exactly one source $x_-$ and one sink $x_+$, and $\sigma(x_-) = 1$, $\sigma(x_+) = -1$ and $\sigma$ is zero on all other vertices. In the second step we reduce to the case that there are no vertices $x$ with $\sigma(x) = 0$. The only case left is the quiver $\Theta_m$ with weight $\tau$. In Step 3 we will prove the theorem in this case.

**Step 1.** Construct a quiver $Q(\sigma)$ as follows:

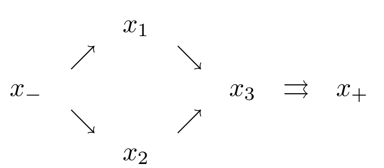
$$Q(\sigma)_0 = Q_0 \cup x_- \cup x_+,$$
$$Q(\sigma)_1 = Q_1 \cup Q_- \cup Q_+$$

where $Q_-$ consists of the set of arrows from $x_-$ to $x_i$, with $\sigma(x_i)$ arrows going to the vertex $x_i$ for which $\sigma(x_i) > 0$ and no arrows going to other vertices. The set $Q_+$ consists of the set of arrows from $x_i$ to $x_+$, with $-\sigma(x_i)$ arrows going from the vertex $x_i$ for which $\sigma(x_i) < 0$ and no arrows going from other vertices to $x_+$.

**Example 2.** Let $Q$ be the quiver




Let $\sigma = (1, 1, -2)$. Then the quiver $Q(\sigma)$ is



We will write $\overline{Q} = Q(\sigma)$. Define the weight $\overline{\sigma}$ of $\overline{Q}$ by $\overline{\sigma}(x_-) = 1$, $\overline{\sigma}(x_i) = 0$, $\overline{\sigma}(x_+) = -1$. The dimension vector $\overline{\beta} = \beta(\sigma)$ is defined by $\overline{\beta}(x_i) = \beta(x_i)$, $\overline{\beta}(x_-) = \sum_{\{i|\sigma(x_i)>0\}} \sigma(x_i)\beta(x_i)$, $\overline{\beta}(x_+) = \sum_{\{i|\sigma(x_i)<0\}} -\sigma(x_i)\beta(x_i)$. Suppose that $W \in \mathrm{Rep}(\overline{Q}, \overline{\beta})$. The matrices of all maps $W(a)$ with $a \in Q_-$ form a square matrix. Let $D^-(W)$ be the determinant of this block matrix. Let $D^+(W)$ be the determinant of all $W(a)$ with $a \in Q_+$. Then the correspondence $c \to D^-cD^+$ gives the isomorphism of weight spaces $\mathrm{SI}(Q,\beta)_\sigma \to \mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$.

Let $\overline{\alpha}$ be the dimension vector of $\overline{Q}$ such that $\overline{\sigma} = \langle \overline{\alpha}, \cdot \rangle$. Let $\overline{V}$ be a representation of $\overline{Q}$ with dimension vector $\overline{\alpha}$ and let $c^{\overline{V}}$ be the corresponding non-zero semi-invariant on $\mathrm{SI}(\overline{Q}, \overline{\beta})$.

**Proposition 2.** *The factor $c$ in the decomposition $c^{\overline{V}} = D^-cD^+$ is of the form $c^V$ for some $V \in \mathrm{Rep}(Q,\alpha)$.*

*Proof.* Notice that the weight of $D^-$ is equal to $\langle \gamma_-, \cdot \rangle$ where

$$\gamma_-(x_-) = 1, \quad \gamma_-(x_j) = \gamma_-(x_+) = 0.$$

Similarly, by (5), the weight of $D^+$ equals $\langle \gamma_+, \cdot \rangle$ where

$$\gamma_+(x_-) = 0, \quad \gamma_+(x_j) = -\sum_{\substack{i \le j \\ \sigma(x_i)<0}} p_{i,j}\sigma(x_i),$$

$$\gamma_+(x_+) = -1 + \sum_{\substack{j \\ \sigma(x_j)<0}} \sum_{\substack{i \le j \\ \sigma(x_i)<0}} p_{i,j}\sigma(x_i)\sigma(x_j).$$

It is easy to see that $\langle \gamma_-, \overline{\beta} \rangle = \langle \gamma_+, \overline{\beta} \rangle = 0$.

Let $\overline{V} \in \mathrm{Rep}(\overline{Q}, \overline{\alpha})$. Then $\overline{V}$ has an obvious submodule $\overline{V}_1 = \overline{V} \mid_{\overline{Q}_0 \setminus \{x_-\}}$. We have an exact sequence

$$0 \to \overline{V}_1 \to \overline{V} \to \overline{V}_2 \to 0$$

with the dimension of $\overline{V}_2$ equal to $\gamma_-$.

Let $M$ be the module defined by the exact sequence

$$0 \to P_{x_+} \xrightarrow{i} \bigoplus_{b, hb=x_+} P_{tb} \to M \to 0,$$

where the morphism $i$ from $P_{x_+}$ to a copy $P_{tb}$ maps the trivial path $e(x_+)$ to the path $b$. The dimension vector of $M$ is $\gamma_+$, and $c^M$ is the determinant $D^+$. Consider the map

$$\sum_{\substack{b \\ hb=x_+}} \overline{V}_1(b) : \bigoplus_{b, hb=x_+} \overline{V}_1(tb) \to \overline{V}_1(x_+).$$

The dimension of the kernel is at least 1. Let $(s_b)_{b,hb=x_+}$ with $s_b \in \overline{V}_1(tb)$ be a non-trivial element in the kernel. We can now define a map $\bigoplus_{b,hb=x_+} P_{tb} \to \overline{V}_1$ by sending the generator $e(tb) \in P_{tb}(tb)$ to $s_b$ for all $b$. Because $(s_b)_{b,hb=x_+}$ lies in the kernel, this actually defines a morphism $M \to \overline{V}_1$. Let $\overline{V}_3$ be the image of this morphism.

Now $\overline{V}_3$ is a submodule of $\overline{V}_1$ and $c^{\overline{V}_1} \neq 0$. By Lemma 1 a) we have $\langle \underline{d}(\overline{V}_3), \overline{\beta} \rangle \geq 0$. We also have $c^M = D^+ \neq 0$. If we apply Lemma 1 a) to the kernel $N$ of

$M \to \overline{V}_3$, then we get $\langle \underline{d}(N), \overline{\beta} \rangle = \langle \gamma_+, -\underline{d}(\overline{V}_3) \rangle = -\langle \underline{d}(\overline{V}_3), \overline{\beta} \rangle \geq 0$. We conclude that $\langle \underline{d}(\overline{V}_3), \overline{\beta} \rangle = 0$. By Lemma 1 b) $c^{\overline{V}_3}$ divides the semi-invariant $c^M = D^+$. Because $D^+$ is an irreducible semi-invariant we must have $c^{\overline{V}_3} = D^+$, $\gamma_+ = \dim \overline{V}_3$ and $\overline{V}_3$ is isomorphic to $M$.

We have an exact sequence

$$0 \to \overline{V}_3 \to \overline{V}_1 \to \overline{V}_4 \to 0.$$

Now it is clear by the multiplicative property that $c^{\overline{V}} = c^{\overline{V}_2} c^{\overline{V}_4} c^{\overline{V}_3}$ with the first factor being proportional to $D^-$ and the last one to $D^+$. Let us also define a submodule $\overline{V}_5 = \overline{V}_4 \mid_{\{x_+\}}$, so we have an exact sequence

$$0 \to \overline{V}_5 \to \overline{V}_4 \to \overline{V}_6 \to 0.$$

Note that $\overline{V}_6$ has support within $Q$. The restriction of $\overline{V}_6$ to $Q$ will be denoted by $V$. We will prove that the restriction of $c^{\hat{V}}$ to $\mathrm{Rep}(Q, \beta)$ is $c^V$.

Extend $W \in \mathrm{Rep}(Q, \beta)$ to the module $\overline{W}$ of dimension $\overline{\beta}$ by putting $\overline{W}(x_-) = \bigoplus_{a, ta = x_-} W(ha)$, $\overline{W}(x_+) = \bigoplus_{b, hb = x_+} W(tb)$, with the maps $\overline{W}(a)$ and $\overline{W}(b)$ being the components of the identity map. Define the canonical submodule $\overline{W}_1 = \overline{W} \mid_{\{x_+\}}$. We have an exact sequence

$$0 \to \overline{W}_1 \to \overline{W} \to \overline{W}_2 \to 0.$$

Define the submodule $\overline{W}_3 = \overline{W}_2 \mid_{\hat{Q} \setminus \{x_-\}}$ of $\overline{W}_2$. Now we have an exact sequence

$$0 \to \overline{W}_3 \to \overline{W}_2 \to \overline{W}_4 \to 0.$$

The representation $\overline{W}_3$ has support within $Q$ and its restriction to $Q$ is just $W$.

We now have

$$c^{\overline{V}}(\overline{W}) = c^{\overline{V}_4}(\overline{W}) = c^{\overline{V}_4}(\overline{W}_1) c^{\overline{V}_4}(\overline{W}_3) c^{\overline{V}_4}(\overline{W}_4) = c^{\overline{V}_4}(\overline{W}_3)$$

because $c^{\overline{V}_4}(\overline{W}_1)$ and $c^{\overline{V}_4}(\overline{W}_4)$ are constant. Moreover,

$$c^{\overline{V}_4}(\overline{W}_3) = c^{\overline{V}_5}(\overline{W}_3) c^{\overline{V}_6}(\overline{W}_3) = c^{\overline{V}_6}(\overline{W}_3) = c^V(W)$$

because $c^{\overline{V}_5}(\overline{W}_4)$ is constant. This concludes the proof of the proposition. □

**Step 2.** Let $Q$, $\beta$, $\sigma$ be as above. Let $x \in Q_0$ be a vertex such that $\sigma(x) = 0$. Let $a_1, \ldots, a_s$ be the arrows in $Q_1$ with $ha_k = x$ $(k = 1, \ldots, s)$ and let $b_1, \ldots, b_t$ be the arrows in $Q_1$ with $tb_l = x$ $(l = 1, \ldots, t)$. Let $\overline{Q}$ be the quiver such that $\overline{Q}_0 = Q_0 \setminus \{x\}$ and $\overline{Q}_1 = (Q_1 \setminus \{a_1, \ldots, a_s, b_1, \ldots, b_t\}) \cup \{ba_{k,l}\}_{1 \leq k \leq s, 1 \leq l \leq t}$, where $t(ba_{k,l}) = ta_k, h(ba_{k,l}) = hb_l$. Let $\overline{\beta}$, $\overline{\sigma}$ be the restrictions of $\beta, \sigma$ to $Q_0 \setminus \{x\}$.

The Fundamental Theorem of Invariant Theory (see [2] for a characteristic free version) says that every semi-invariant from $\mathrm{SI}(Q, \beta)_\sigma$ can be obtained from the semi-invariants from $\mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ by substituting the actual compositions $b_l a_k$ for the arrows of type $ba_{k,l}$. Assuming Theorem 1 for $\mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ to be true, we need to show that every semi-invariant $c^{\overline{V}}$ from $\mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ pulls back to a semi-invariant of type $c^V$. For a given representation $\overline{V}$ of $\overline{Q}$ of dimension $\overline{\alpha}$ we define the representation $V = \mathrm{ind}\, \overline{V}$ as follows. We notice that the condition $\sigma(x) = 0$ means that we expect $\dim V(x) = \sum_{k=1}^s \dim V(ta_k)$.

This means we put

$$V(y) = \begin{cases} \overline{V}(y) & \text{if } y \neq x, \\ \bigoplus_{k=1}^{s} \overline{V}(ta_k) & \text{if } y = x. \end{cases}$$

We define the linear maps $V(a)$ as follows:

$$V(a) = \begin{cases} \overline{V}(a) & \text{if } a \neq a_k, b_l, \\ i(a_k) & \text{if } a = a_k, \\ \sum_{k=1}^{s} \overline{V}(ba_{k,l}) & \text{if } b = b_l, \end{cases}$$

where $i(a_k) : V(ta_k) \to \bigoplus_{k=1}^{s} V(ta_k)$ is the injection on the $k$-th summand.

Then it is easy to check directly from the definition of semi-invariants $c^V$ that if the representation $\overline{W} = \text{res } W$ of dimension $\overline{\beta}$ is a restriction of a representation $W$ of $Q$ of dimension $\beta$, then $c^{\overline{V}}(\overline{W}) = c^V(W)$.

Notice that the functor ind $\overline{V}$ is the left adjoint of the obvious restriction functor res : $\text{Rep}(Q) \to \text{Rep}(\overline{Q})$, i.e., we have the natural isomorphisms

$$\text{Hom}_Q(\text{ind } \overline{V}, W) = \text{Hom}_{\overline{Q}}(\overline{V}, \text{res } W)$$

which explains why $c^{\overline{V}}(\overline{W})$ and $c^V(W)$ vanish simultaneously.

**Step 3.** It remains to deal directly with the weight space $\text{SI}(\Theta_m, \theta(n))_\tau$. Writing the representation $W$ of dimension $\theta(n)$ as an $m$-tuple of linear maps,

$$W(a_1), \dots, W(a_m) : W_- \to W_+,$$

we can introduce the additional action of the group $\text{GL}(m)$ acting on this space by taking linear combinations of the linear maps $W(a_1), \dots, W(a_m)$. Using the Cauchy formula (in its characteristic free version, say from [1]) we see that the space $\text{SI}(\Theta_m, \theta(n))_\tau$ of semi-invariants can be identified with $\bigwedge^n W_- \otimes \bigwedge^n W_+^* \otimes D_n(K^m)$. Here $D_n$ denotes the $n$-th divided power. Since the divided power $D_n(K^m)$ is generated as a $\text{GL}(m)$-module by its highest weight vector (which corresponds to the semi-invariant $\det W(a_1)$) and the set of semi-invariants of the form $c^V$ is preserved by the action of $\text{GL}(m)$, it is enough to express $\det W(a_1)$ as the semi-invariant of the form $c^V$. Notice that $\tau = \langle \alpha, \cdot \rangle$ for the dimension vector $\alpha = (1, m-1)$. Taking the module $V$ to be the $m$-tuple of linear maps $V(a_1), \dots, V(a_m) : K \to K^{m-1}$ where $V(a_1) = 0$ and $V(a_i)$ is the embedding sending 1 to the $i-1$'st basis vector, for $i = 2, \dots, m$, we check directly that $c^V = \det\ W(a_1)$. This concludes the proof of Theorem 1. □

We now will give another description for semi-invariants $\text{SI}(Q, \beta)_\sigma$. Let $\overline{Q} = Q(\sigma)$, $\overline{\beta}$ and $\overline{\sigma}$ be as in Step 1 of the proof of Theorem 1. We know that $\text{SI}(Q, \beta)_\sigma \cong \text{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$. Let $\overline{\alpha}$ be a dimension vector of $\overline{Q}$ such that $\langle \overline{\alpha}, \cdot \rangle = \overline{\sigma}$. Now $\text{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ is generated by semi-invariants $c^{\overline{V}}$ with $\underline{d}(\overline{V}) = \overline{\alpha}$. In fact we only need to take those $c^{\overline{V}}$ where $\overline{V}$ lies in a Zariski dense set of $\text{Rep}(\overline{Q}, \overline{\alpha})$. A general representation $\overline{V}$ of dimension $\overline{\alpha}$ has the following projective resolution:

$$0 \to P_{x_+} \xrightarrow{d_V} P_{x_-} \to \overline{V} \to 0$$

with $d_V \in \text{Hom}_Q(P_{x_+}, P_{x_-}) = [x_-, x_+]$. So $d_V$ can be seen as some linear combination $\sum_{i=1}^{r} \lambda_i p_i$ where $p_1, \dots, p_r$ are all paths from $x_+$ to $x_-$. For any $\overline{W} \in \text{Rep}(\overline{Q}, \overline{\beta})$ we have the following exact sequence:

$$0 \to \text{Hom}_{\overline{Q}}(\overline{V}, \overline{W}) \to \text{Hom}_{\overline{Q}}(P_{x_+}, \overline{W}) \xrightarrow{\tilde{d}_{\overline{V}}} \text{Hom}_{\overline{Q}}(P_{x_-}, \overline{W}) \to \text{Ext}_{\overline{Q}}(\overline{V}, \overline{W}) \to 0.$$

It is easy to see that $\det(\tilde{d}_{\overline{V}}) = c^{\overline{V}}(\overline{W}) = c^V(W)$.

We have that

$$\begin{aligned} \mathrm{Hom}_{\overline{Q}}(P_{x_+}, \overline{W}) &\cong \overline{W}_{x_+} = \textstyle\bigoplus_{\sigma(x_i)>0} W(x_i)^{\sigma(x_i)}, \\ \mathrm{Hom}_{\overline{Q}}(P_{x_-}, \overline{W}) &\cong \overline{W}_{x_-} = \textstyle\bigoplus_{\sigma(x_i)<0} W(x_i)^{\sigma(x_i)}, \\ \tilde{d}_{\overline{V}} &= \textstyle\sum_i \lambda_i \overline{V}(p_i). \end{aligned}$$

Let $F$ be a function from the set of paths from $x_+$ to $x_-$ to the set of non-negative integers. For each such $F$ we can define the semi-invariant $I_F$ as the coefficient of $\lambda_1^{F(p_1)}\lambda_2^{F(p_2)}\dots\lambda_r^{F(p_r)}$ in $\det(\tilde{d}_{\overline{V}})$.

**Corollary 3.** *The space of semi-invariants* $\mathrm{SI}(Q,\beta)_\sigma$ *is spanned by semi-invariants of the form* $I_F$.

A necessary condition for $I_F$ to be non-zero is

$$\sum_i F(p_i) = \sum_{\sigma(x_i)>0} \sigma(x_i)\beta(x_i) = \sum_{\sigma(x_i)<0} -\sigma(x_i)\beta(x_i).$$

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