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SYZYGIES OF ABELIAN VARIETIES

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Let A be an ample line bundle on an abelian variety X (over an algebraically closed field). A theorem of Koizumi ([Ko], [S]), developing Mumford's ideas and results ([M1]), states that if $m \geq 3$ the line bundle $L = A^{\otimes m}$ embeds X in projective space as a projectively normal variety. Moreover, a celebrated theorem of Mumford ([M2]), slightly refined by Kempf ([K4]), asserts that the homogeneous ideal of X is generated by quadrics as soon as $m \geq 4$. Such results turn out to be particular cases of a statement, conjectured by Rob Lazarsfeld, concerning the minimal resolution of the graded algebra $R_L = \bigoplus_{h=0}^{\infty} H^0(X, L^{\otimes h})$ over the polynomial ring $S_L = \bigoplus_{h=0}^{\infty} Sym^h H^0(X, L)$. The purpose of this paper is to prove Lazarsfeld's conjecture.

To put such matters into perspective, it is useful to review the case of projective curves. A classical theorem of Castelnuovo states that a curve X, embedded in projective space by a complete linear system $|L|$, is projectively normal as soon as $\deg L \geq 2g(X) + 1$, and a theorem of Mattuck, Fujita and Saint-Donat states that if $\deg L \geq 2g(X) + 2$, then the homogeneous ideal of X is generated by quadrics. Green ([G1]) unified, re-interpreted and generalized these results to a statement about syzygies. Specifically, given a (smooth) projective variety X and a very ample line bundle L on X, a minimal resolution of R_L as a graded S_L-module (notation as above) looks like

$$0 \to \cdots \to E_p \to \cdots \to E_1 \to E_0 \to R_L \to 0 \tag{1}$$

where $E_0 = S_L \oplus \bigoplus_j S_L(-a_{0j})$ (where, since X is embedded by a complete linear system, $a_{0j} \geq 2$ for any j), $E_1 = \bigoplus_j S_L(-a_{1j})$ (where, since the image of X in $\mathbf{P}(H^0(L)^\vee)$ is not contained in any hyperplane, $a_{1j} \geq 2$ for any j), and, in general, for $p \geq 1$, $E_p = \bigoplus_j S_L(-a_{pj})$ with $a_{pj} \geq p + 1$ for any j. Green introduced the following terminology: L is said to *satisfy property* N_0 if $E_0 = S_L$. This means that the map $S_L \to R_L$ is surjective, i.e. that the embedded variety X is projectively normal. Moreover L is said to *satisfy property* N_1 if it satisfies N_0 and $a_{1j} = 2$ for any j, i.e. the homogeneous ideal of the embedded variety X is generated by quadrics. Inductively, one says that L *satisfies condition* N_p if it satisfies condition N_{p-1} and $a_{pj} = p+1$ for any j. So N_2 means that the relations between the quadrics defining X are generated by linear ones and, for arbitrary $p \geq 2$, N_p means that the first $p - 1$ maps of the resolution of the homogeneous ideal are matrices with linear entries. In a word N_p means that, up to the p-th step, the resolution (1) is as "regular" as it could possibly be. The aforementioned theorem of Green states that *if X is a curve and* $\deg L \geq 2g(X) + 1 + p$, *then L satisfies* N_p. This result stimulated many interesting questions (see [G1], [G2], [L] and [EL]). One of these

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is how it extends to higher dimension. For arbitrary varieties there is a general conjecture of Mukai and results of Ein and Lazarsfeld ([EL]). In this article we will be concerned with the case of abelian varieties. Lazarsfeld's conjecture states that powers of ample line bundles on abelian varieties of arbitrary dimension behave exactly as in the case of elliptic curves:

Theorem ($char(k) = 0$)**.** *Let A be an ample line bundle on an abelian variety X. If $n \geq p+3$, then $A^{\otimes n}$ satisfies condition N_p.*

For elliptic curves this amounts to Green's theorem and, in arbitrary dimension, the cases $p = 0, 1$ are the aforementioned results of Koizumi and Mumford. It is also worthwhile to point out that, as it is easy to see, if A is a principal polarization, then the bound of Lazarsfeld's conjecture is the best one can hope for. We also prove a generalization of Lazarsfeld's conjecture (Theorem 4.3), about the first p steps of the resolution of $R_{A^{\otimes n}}$ when $n < p+3$ (we refer to Section 4 below for terminology, statement and proof).

It is worthwhile to recall that, shortly after the appearance of Lazarsfeld's conjecture, Kempf ([K5]) proved it "up to a factor two": his statement, in the present terminology, is that *$A^{\otimes n}$ satisfies condition N_p as soon as $n \geq$* $\max\{3, 2p+2\}$. Thus the above Theorem improves Kempf's result already for abelian surfaces and $p = 2$.

The main point of the proof will be a criterion – of independent interest – for the surjectivity of multiplication maps of global sections:

$$(*) \qquad H^0(X, A^{\otimes n}) \otimes H^0(X, E) \to H^0(X, A^{\otimes n} \otimes E),$$

where A and E are respectively a vector bundle and an ample line bundle on an abelian variety X. Roughly, the criterion (Theorem 3.1 and Corollary 3.3) states that the map $(*)$ is surjective as soon as certain vanishing conditions on the higher cohomology of the *Pontrjagin product* of $A^{\otimes n}$ and E are satisfied. The relation between maps as $(*)$ and syzygies is well known and goes as follows (in characteristic zero): let us denote by M_L the kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_X \to L$. If the multiplication map

$$(**) \qquad H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L^{\otimes h}) \to H^0(M_L^{\otimes p} \otimes L^{\otimes h+1})$$

is surjective for any $h \geq 1$, then L satisfies N_p. Thus the proof of Lazarsfeld's conjecture will consist in showing that for any $n \geq p+3$ and $h \geq 1$ the Pontrjagin product of $A^{\otimes n}$ and $E = (M_{A^{\otimes n}})^{\otimes p} \otimes A^{\otimes nh}$ satisfies the required cohomological conditions.

Let us describe more closely how the aforementioned criterion for the surjectivity of multiplication maps is obtained: in Section 1 it is shown that, given two vector bundles L and E and a point x on X, under mild hypotheses the multiplication map $H^0(T_x^*L) \otimes H^0(E) \to H^0((T_x^*L) \otimes E)$ is identified to the evaluation of global sections at x of another vector bundle, denoted $L\hat{*}E$, the (skew) Pontrjagin product of L and E. This suggests we look for ways of understanding the locus where $L\hat{*}E$ is not globally generated. More generally, it is natural to ask for a cohomological criterion for the global generation of vector bundles on abelian varieties. This is the theme of Section 2 (which is independent of the rest of the paper), where the required cohomological criterion (Theorem 2.1) is supplied. Theorem 2.1 follows from an important result (Lemma 2.2), implicitly contained in Kempf's work and pointed out by R. Lazarsfeld, whose proof we include. Putting together the results of Sections 1 and 2 one gets a criterion for the surjectivity of multiplication maps, in terms of the cohomology of the Pontrjagin products (Theorem 3.1). In Section 3 we supply some techniques about how to verify the hypotheses of such a criterion. If one of the two vector bundles L and E is a *line* bundle, everything can be put in a much more explicit form (Theorem 3.8), which, we hope, might have other applications. In Section 4, we apply the previous results to Lazarsfeld's conjecture and to its generalization (Theorem 4.3).

Although we work in the algebraic setting, we have made some simplifying assumptions on the characteristic of the ground field. Moreover one needs the characteristic zero for condition $(**)$ about syzygies.

This work owes much to the reading of George Kempf's penetrating works on these and related subjects, especially [K1]–[K6]. My understanding of [K5] in turn owes a lot to some UCLA lectures and seminar talks given by Rob Lazarsfeld on the subject, back in '88-89. More recently, I have also benefitted by some useful conversations with Lawrence Ein. Part of this work was done while the author was visiting Colorado State University at Fort Collins and U.N.A.M. at Morelia (Mexico), supported respectively by a C.N.R.-N.A.T.O. scholarship and by U.N.A.M. My thanks go to these institutions, and especially to Jeanne Duflot and Rick Miranda, at Fort Collins, and Alexis Garcia-Zamora and Sevin Recillas, at Morelia, for really great hospitality.

Throughout this paper X will denote an abelian variety over an algebraically closed field k. $T_x : X \to X$ will denote the translation by the point $x \in X$. Given two integers m and n, $mp_1+np_2 : X \times X \to X$ will mean the map $(x, y) \to mx+ny$. E.g. the group law $(x, y) \mapsto x + y$ will be denoted $p_1 + p_2$ and *not* m_X (which will denote the multiplication by the integer m).

1. Pontrjagin products and surjectivity of multiplication maps

As mentioned in the introduction, the main theme of the first part of the paper will be to find criteria ensuring the surjectivity of the multiplication map

$$H^0(L) \otimes H^0(E) \to H^0(L \otimes E),$$

where L and E are respectively a line bundle and a vector bundle on E. However, the (crucial) fact that L is a line bundle will be used only later (see Section 3(b)). So here we will assume that L and E are simply two vector bundles (i.e. locally free sheaves) on X. We will rather undertake a global study of the family of multiplication maps

$$m_x : H^0(T_x^* L) \otimes H^0(E) \to H^0((T_x^* L) \otimes E) \tag{1}$$

for x varying in X. Let us define the *"skew Pontrjagin product"* of L and E as

$$L \hat{*} E := p_{1*}((p_1 + p_2)^* L \otimes p_2^* E)$$

(see Remark 1.2 below for a justification of the terminology and some comments). For the sake of simplicity, we will assume throughout that

$$H^i((T_x^* L) \otimes E) = 0 \tag{2}$$

for any $x \in X$ and for any $i > 0$. Then Grauert's theorem yields that $L \hat{*} E$ is locally free and that, at the fibers level, the map $L \hat{*} E(x) \to H^0((T_x^* L) \otimes E)$ is an isomorphism. On the other hand, there is a natural isomorphism

$$\Phi : H^0(L) \otimes H^0(E) \xrightarrow{\sim} H^0(L \hat{*} E)$$

obtained as follows: via the automorphism $\phi = (p_1 + p_2, p_2)$ of $X \times X$ we have that $(p_1 + p_2)^* L \otimes p_2^* E = \phi^*(p_1^* L \otimes p_2^* E)$. Therefore we have the isomorphism

$$\phi^* : H^0(L) \otimes H^0(E) \to H^0((p_1 + p_2)^* L \otimes p_2^* E).$$

The isomorphism Φ follows by composing with the isomorphism

$$H^0((p_1 + p_2)^* L \otimes p_2^* E) \cong H^0(p_{1*}((p_1 + p_2)^* L \otimes p_2^* E)).$$

It follows that the evaluation map of $L \hat{*} E$ at x composed with the isomorphism Φ,

$$H^0(L) \otimes H^0(E) \xrightarrow{\Phi} H^0(L \hat{*} E) \xrightarrow{ev_x} L \hat{*} E(x),$$

coincides with the composed map

$$H^0(L) \otimes H^0(E) \xrightarrow{T_x^* \times id} H^0(T_x^* L) \otimes H^0(E) \xrightarrow{M_x} H^0((T_x^* L) \otimes E).$$

Therefore we have

Proposition 1.1. *Assuming hypothesis (2), the locus of points $x \in X$ such that the multiplication map m_x is not surjective coincides with the locus where $L \hat{*} E$ is not generated by its global sections.*

Of course the advantage of this is that the fact that the maps m_x are surjective *for any* $x \in X$ is equivalent to the global generation of the vector bundle $L \hat{*} E$ and this last condition can be investigated via global cohomological methods.

Remark 1.2. Given two sheaves F and E on X, the sheaf $F \hat{*} E$ is isomorphic to $F * (-1)_X^* E$, were "$*$" means the *Pontrjagin product* as defined in [Mu], p.160, i.e. $G * H = (p_1 + p_2)_*(p_1^* G \otimes p_2^* H)$. The isomorphism can be seen as above using the automorphism $(p_1 + p_2, -p_2)$. Note that $\hat{*}$ is "skew-symmetric", i.e. $F \hat{*} E \cong (-1)_X^*(E \hat{*} F)$.

2. A cohomological criterion for the global generation of vector bundles on abelian varieties

In view of Proposition 1.1, it is natural to look for criteria guaranteeing that a vector bundle on an abelian variety is generated by its global sections. E.g. a basic and elementary result on abelian varieties states that the product of two ample line bundles is base-point free. We generalize it to the following result, which can be seen as an analogue for vector bundles on abelian varieties of the classical Castelnuovo-Mumford criterion for sheaves on projective spaces.

Theorem 2.1. *Let A and F be respectively an ample line bundle and a vector bundle on an abelian variety X. If $h^i(F \otimes A^\vee \otimes \alpha) = 0$ for any $i > 0$ and for any line bundle $\alpha \in \mathrm{Pic}^0 X$, then F is generated by its global sections.*

(a) A lemma of Kempf (according to Lazarsfeld). Theorem 2.1 will be a consequence of a very sharp argument implicit in Kempf's work ([K5], [K6]). Since the matter is not easy to isolate in Kempf's paper, we include the proof for the reader's benefit. The following formulation is due to Lazarsfeld.

Lemma 2.2 (Kempf)**.** *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on an abelian variety X. If $h^i(\mathcal{E}\otimes\alpha)=h^i(\mathcal{F}\otimes\alpha)=0$ for any $i>0$ and for any $\alpha\in\mathrm{Pic}^0X$, then the map*

$$\bigoplus_{\alpha\in U} H^0(\mathcal{E}\otimes\alpha)\otimes H^0(\mathcal{F}\otimes\alpha^\vee)\stackrel{m_\alpha}{\to} H^0(\mathcal{E}\otimes\mathcal{F})$$

is surjective for any non-empty Zariski open set $U\subset\mathrm{Pic}^0X$.

Proof. STEP 1. Let us denote by p_i the three projections on $X\times X\times\mathrm{Pic}^0X$ and by p_{ij} the three intermediate projections. Moreover let Δ be the diagonal in $X\times X$, and let $\mathcal{P}$ be a Poincaré line bundle on $X\times\mathrm{Pic}^0X$. Finally, let

$$\mathcal{W}=p_{13}^*(p_1^*\mathcal{E}\otimes\mathcal{P})\otimes p_{23}^*(p_2^*\mathcal{F}\otimes\mathcal{P}^\vee),$$

and let $\Phi:\mathcal{W}\to\mathcal{W}_{|\Delta\times\mathrm{Pic}^0X}$ be the restriction map. Note that, since $p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee$ is trivial when restricted to $\Delta\times\mathrm{Pic}^0X$, the target is isomorphic to the sheaf $p_X^*(\mathcal{E}\otimes\mathcal{F})$ on $X\times\mathrm{Pic}^0X$ (via the isomorphism $X\stackrel{\sim}{\to}\Delta$). By the hypothesis on $\mathcal{E}$ and $\mathcal{F}$ and Grauert's theorem, the sheaf $p_{3*}(\mathcal{W})$ is locally free, with fiber at α isomorphic to $H^0(\mathcal{E}\otimes\alpha)\otimes H^0(\mathcal{F}\otimes\alpha^\vee)$. Moreover, denoting $p_{3*}\Phi:p_{3*}\mathcal{W}\to H^0(\mathcal{E}\otimes\mathcal{F})\otimes\mathcal{O}_{\mathrm{Pic}^0X}$, we have that $p_{3*}\Phi(\alpha)$ is identified to the multiplication map $m_\alpha:H^0(\mathcal{E}\otimes\alpha)\otimes H^0(\mathcal{F}\otimes\alpha^\vee)\to H^0(\mathcal{E}\otimes\mathcal{F})$.

STEP 2. *The surjectivity of the map $\bigoplus_{\alpha\in U}m_\alpha$ of the statement of Lemma 2.2 is equivalent to the surjectivity of the map $H^g(p_{3*}\Phi):H^g(p_{3*}\mathcal{W})\to H^0(\mathcal{E}\otimes\mathcal{F})\otimes H^g(\mathcal{O}_{\mathrm{Pic}^0X})$* (where $g=\dim X$; of course $H^g(\mathcal{O}_{\mathrm{Pic}^0X})\cong k$).

Proof. The surjectivity of the map $\bigoplus_{\alpha\in U}m_\alpha:\bigoplus_{\alpha\in U}p_{3*}\mathcal{W}(\alpha)\to H^0(\mathcal{E}\otimes\mathcal{F})$ is equivalent to the injectivity of the dual map $H^0(\mathcal{E}\otimes\mathcal{F})^\vee\to\prod_{\alpha\in U}p_{3*}\mathcal{W}(\alpha)^\vee$ which is in turn equivalent to the fact that the map $(p_{3*}\Phi)^\vee:H^0(\mathcal{E}\otimes\mathcal{F})\otimes\mathcal{O}_{\mathrm{Pic}^0X}\to p_{3*}\mathcal{W}^\vee$ is injective at the global sections level and this, by Serre duality, is equivalent to the surjectivity of $H^g(p_{3*}\Phi)$.

STEP 3. *The Leray spectral sequence $H^i(R^jp_{3*}\mathcal{W})\Rightarrow H^{i+j}(\mathcal{W})$ degenerates giving an isomorphism $H^g(p_{3*}\mathcal{W})\stackrel{\sim}{\to}H^g(\mathcal{W})$* (by hypothesis, $R^ip_{3*}(\mathcal{W})$ is zero for $i>0$).

STEP 4. *There is a canonical isomorphism $H^g(\mathcal{W})\stackrel{\sim}{\to}H^0(\mathcal{E}\otimes\mathcal{F})\otimes H^g(\mathcal{O}_{\mathrm{Pic}^0X})$ such that the map $H^g(p_{3*}\Phi)$ of Step 2 is the composition of this isomorphism with the one supplied by Step 3. Therefore $H^g(p_{3*}\Phi)$ is itself an isomorphism. This proves Lemma 2.2.*

Proof. In the first place let us note that, by the projection formula,

$$R^ip_{12*}(\mathcal{W})\cong p_1^*\mathcal{E}\otimes p_2^*\mathcal{F}\otimes R^ip_{12*}(p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee).\tag{1}$$

On the other hand, it is well known (see [K1] or [K6], p.53) that

$$R^ip_{12*}(p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee)\cong\begin{cases}0 & \text{if } i<g;\\ \mathcal{O}_\Delta\otimes H^g(\mathcal{O}_{\mathrm{Pic}^0X}) & \text{if } i=g.\end{cases}\tag{2}$$

Let us recall the proof of (2) for the sake of self-containedness: one considers the difference map $d=p_1-p_2:X\times X\to X$. By the see-saw principle it follows immediately that $p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee\cong(d,\mathrm{id}_{\mathrm{Pic}^0X})^*\mathcal{P}$. Therefore we are reduced to computing the R^ip_{12*}'s of the right hand side. By flat base extension $R^ip_{12*}((d,\mathrm{id}_{\mathrm{Pic}^0X})^*\mathcal{P})\cong d^*(R^ip_{X*}(\mathcal{P}))$. At this point one invokes Mumford's duality result ([M1], Ch.13, [K2], Th.3.15) which states that $R^ip_{X*}(\mathcal{P})=H^g(\mathcal{O}_{\mathrm{Pic}^0X})\otimes\mathcal{O}_e$ if $i=g$ and it is zero otherwise (e is the identity point of X). This proves (2). Then by (1) it follows that

$$R^i p_{12*}(\mathcal{W}) \cong \begin{cases} p_1^*\mathcal{E} \otimes p_2^*\mathcal{F} \otimes H^g(\mathcal{O}_{\mathrm{Pic}^0 X}) \otimes \mathcal{O}_\Delta & \text{if } i = g; \\ 0 & \text{if } i < g. \end{cases} \tag{3}$$

Finally, to get the isomorphism of Step 4 let us consider the other natural spectral sequence, that is, $H^i(R^j p_{12*}(\mathcal{W})) \Rightarrow H^{i+j}(\mathcal{W})$. By (3) it degenerates to the isomorphism $H^g(\mathcal{W}) \cong H^0(R^g p_{12*}(\mathcal{W})) \cong H^0(\mathcal{E} \otimes \mathcal{F}) \otimes H^g(\mathcal{O}_{\mathrm{Pic}^0 X})$. The second part of the statement of Step 4 follows easily. □

(b) Proof of Theorem 2.1. We start with a couple of easy corollaries of Lemma 2.2, whose proof is immediate.

Corollary 2.3. *If $\mathcal{E}$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 2.2, then there is a positive integer N such that given N general line bundles $(\alpha_1, \ldots, \alpha_N) \in (\mathrm{Pic}^0 X)^N$ the map $\bigoplus_{i=1}^N H^0(\mathcal{E} \otimes \alpha_i) \otimes H^0(\mathcal{F} \otimes \alpha_i^\vee) \stackrel{m_{\alpha_i}}{\to} H^0(\mathcal{E} \otimes \mathcal{F})$ is surjective.*

Corollary 2.4. *Let $\mathcal{E}$ be a vector bundle on an abelian variety X such that $H^i(\mathcal{E} \otimes \alpha) = 0$ for any $i > 0$ and for any $\alpha \in \mathrm{Pic}^0 X$. Then there is a positive integer N such that given N general line bundles $(\alpha_1, \ldots, \alpha_N) \in (\mathrm{Pic}^0 X)^N$ the map $\bigoplus_{i=1}^N H^0(\mathcal{E} \otimes \alpha_i) \otimes \alpha_i^\vee \stackrel{\phi_{\alpha_i}}{\to} \mathcal{E}$ is surjective (where ϕ_{α_i} denotes the evaluation map of global sections of $\mathcal{E} \otimes \alpha_i$ tensored with ${\alpha_i}^\vee$).*

Proof. Let A be an ample line bundle on X and $L = A^{\otimes n}$. If the map $\bigoplus(\phi_{\alpha_i} \otimes L) : \bigoplus_{i=1}^N H^0(\mathcal{E} \otimes \alpha_i) \otimes L \otimes \alpha_i^\vee \to \mathcal{E} \otimes L$ is not surjective and n is big enough, then it is not surjective at the H^0 level and this is in contrast with Corollary 2.3. □

Theorem 2.1 is now an easy consequence of Corollary 2.4:

Proof of Theorem 2.1. The hypothesis allows us to apply Corollary 2.4 to the vector bundle $\mathcal{E} = F \otimes A^\vee$: given a tuple $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N) \in (\mathrm{Pic}^0 X)^N$ we have the commutative diagram:

$$\begin{array}{ccc} \bigoplus\limits_{\alpha_i=1}^{N} H^0(F \otimes A^\vee \otimes \alpha_i) \otimes H^0(A \otimes \alpha_i^\vee) \otimes \mathcal{O}_X & \longrightarrow & H^0(F) \otimes \mathcal{O}_X \\ \Big\downarrow {\scriptstyle \bigoplus(\mathrm{id} \otimes ev_i)} & & \Big\downarrow {\scriptstyle ev} \\ \bigoplus\limits_{\alpha_i=1}^{N} H^0(F \otimes A^\vee \otimes \alpha_i) \otimes A \otimes \alpha_i^\vee & \xrightarrow{\bigoplus(\phi_{\alpha_i} \otimes A)} & F \end{array} \tag{4}$$

Let us denote: B^α is the base locus of the line bundle $A \otimes \alpha^\vee$, $B^{\bar{\alpha}} = B^{\alpha_1} \cup \cdots \cup B^{\alpha_N}$ and $B(F)$ is the support of the cokernel of the map ev (the evaluation map of global sections of F). The point is that $A \otimes \alpha^\vee$ is always effective (since A is ample), so that $B^{\bar{\alpha}}$ is always a proper or empty subvariety of X. By Corollary 2.4 we have that $\bigoplus_{i=1}^N (\phi_{\alpha_i} \otimes A)$ is surjective for $\bar{\alpha}$ general in $(\mathrm{Pic}^0 X)^N$ and therefore the diagram shows that $B(F)$ is contained in $B^{\bar{\alpha}}$ for $\bar{\alpha}$ varying in a certain non-empty Zariski-open set V of $(\mathrm{Pic}^0 X)^N$. (Actually it follows that $B(F)$ is contained in $B^{\bar{\alpha}}$ for any $\bar{\alpha} \in \mathrm{Pic}^0 X$.) But such intersection is clearly empty, since already $\bigcap_{\alpha \in U} B^\alpha$ is empty for any Zariski open set U of $\mathrm{Pic}^0 X$ (it is the intersection of translates of the base locus $B(A)$ by points varying in a Zariski open set of X). □

3. Calculus with Pontrjagin products

Going back to the surjectivity of the multiplication maps (1) of Section 1, we have, as a consequence of Proposition 1.1 and Theorem 2.1:

Theorem 3.1. *Let L and E be vector bundles on an abelian variety X such that $H^i((T_x^*L)\otimes E)=0$ for any $i>0$ and for any $x\in X$. If there is an ample line bundle A on X such that, for any $i>0$ and for any $\alpha\in \mathrm{Pic}^0X$,*

$$H^i((L\hat{*}E)\otimes A^\vee\otimes\alpha)=0,$$

then the multiplication maps $m_x: H^0(T_x^*L)\otimes H^0(E)\to H^0((T_x^*L)\otimes E)$ are surjective for any $x\in X$.

Given a sheaf $\mathcal{F}$, we will say that $\mathcal{F}$ *has the vanishing property* if $H^i(\mathcal{F}\otimes\alpha)=0$ for any $i>0$ and for any $\alpha\in\mathrm{Pic}^0X$. Theorem 3.1 leaves the problem of how to check in practice that a "mixed product" such as $(L\hat{*}E)\otimes A^\vee$ has the vanishing property. The goal of this section is to develop some tools to this purpose, under the further hypothesis, used in (b) below, that L is a *line bundle*.

(a) Exchanging tensor and Pontrjagin product under cohomology.

Lemma 3.2. *Let L, E and M be vector bundles on X such that $h^i((T_x^*L)\otimes E)=h^i((T_x^*L)\otimes M)=0$ for any $i>0$ and for any $x\in X$. Then, for any $i\geq 0$,*

$$h^i((L\hat{*}E)\otimes M)=h^i((L\hat{*}M)\otimes E).$$

Proof. By the projection formula we have that

$$(L\hat{*}E)\otimes M\cong p_{1*}(p_1^*(M)\otimes(p_1+p_2)^*(L)\otimes p_2^*(E)),$$
$$(L\hat{*}M)\otimes E\cong p_{1*}(p_1^*(E)\otimes(p_1+p_2)^*(L)\otimes p_2^*(M)).$$

To simplify the notation let us denote by respectively $\mathcal{F}$ and $\mathcal{G}$ the right hand sides of the first and of the second isomorphism above. Of course, since $\mathcal{F}$ and $\mathcal{G}$ are exchanged by the automorphism (p_2,p_1) of $X\times X$, $H^i(\mathcal{F})\cong H^i(\mathcal{G})$ for any i. The projection formula and the hypothesis ensure that $R^ip_{1*}\mathcal{F}$ and $R^ip_{1*}\mathcal{G}$ vanish for $i>0$. Therefore the Leray spectral sequence degenerates to isomorphisms $H^i((L\hat{*}E)\otimes M)\cong H^i(\mathcal{F})$ and $H^i((L\hat{*}M)\otimes E)\cong H^i(\mathcal{G})$. □

As a corollary of Theorem 3.1 and Lemma 3.2 we get

Corollary 3.3. *Let A and E be respectively an ample line bundle and a vector bundle on X, and let $n>1$ be an integer such that $E\otimes A^{\otimes n}$ has the vanishing property. If*

$$H^i((A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))\otimes E)=0 \tag{1}$$

for any $i>0$ and $\alpha\in\mathrm{Pic}^0X$, then the multiplication maps $m_x: H^0(T_x^*A^{\otimes n})\otimes H^0(E)\to H^0((T_x^*A^{\otimes n})\otimes E)$ are surjective for any $x\in X$.

(b) Pulling back via the multiplication by an integer. So far we have merely assumed that L and E, the factors of the Pontrjagin product, are vector bundles. From this point we will make the (essential) further assumption that one of them, namely L, *has rank one.*

Let n be a positive integer and let us denote by $n_X: X\to X$ the map $x\mapsto nx$. If L is a line bundle, we have the following standard formula, which will be a fundamental tool in the proof of Lazarsfeld's conjecture.

Proposition 3.4. $n_X^*(L\hat{*}E) \cong (L^{\otimes n}\hat{*}(E \otimes L^{\otimes -n+1})) \otimes n_X^* L \otimes L^{\otimes -n}$.

Let us point out why one should expect a formula like this. By the theorem of the square, $(T_x^*L)^{\otimes n} \cong T_x^*(L^{\otimes n}) \cong T_{nx}^*L \otimes L^{\otimes n-1}$. Therefore $T_x^*L \otimes E = T_x^*L \otimes L^{\otimes n-1} \otimes E \otimes L^{\otimes -n+1} \cong (T_{x/n}^* L^{\otimes n}) \otimes E \otimes L^{\otimes -n+1}$, where x/n means any $z \in X$ such that $nz = x$. It is therefore natural to expect that such an isomorphism at the H^0 level globalizes to an isomorphism $n_X^*(L\hat{*}E) \cong (L^{\otimes n}\hat{*}(E \otimes L^{\otimes -n+1})) \otimes$ (a line bundle).

Proof of Proposition 3.4. By flat base extension we have

$$\begin{aligned} n_X^* p_{1*}((p_1+p_2)^*L \otimes p_2^*E) &\cong p_{1*}((n_X, 1_X)^*(p_1+p_2)^*L \otimes p_2^*E) \\ &= p_{1*}((np_1+p_2)^*L \otimes p_2^*E). \end{aligned} \tag{2}$$

On the other hand,

$$(np_1+p_2)^*L \cong (p_1+p_2)^*L^{\otimes n} \otimes p_1^*(n_X^*L \otimes L^{\otimes -n}) \otimes p_2^* L^{\otimes -n+1}. \tag{3}$$

For $n = 2$, (3) follows by plugging $f = g = p_1$ and $h = p_2$ in [M3], Cor.2, p.58 (Theorem of the Cube). For any n, (3) follows by induction in the same way. Finally, plugging (3) into the last member of (2) and using the projection formula we get the statement. □

Let us remark that one could also deduce a (conceptually slightly more complicated) proof of Proposition 3.4 from Mukai's setting (see Remark 1.2). In fact, given a sheaf G, let us consider its "Fourier transform" $\hat{G} = q_{2*}(q_1^*G \otimes \mathcal{P})$ (where q_i are the projections on $X \times \mathrm{Pic}^0 X$ and $\mathcal{P}$ is a Poincaré sheaf). Then one can prove the following formula (which can be deduced e.g. from [Mu], 3.10): $L\hat{*}E \cong \phi_L^*(\widehat{L \otimes E}) \otimes L$ (where we denote by $\phi_L : X \to \mathrm{Pic}^0 X$ the polarization associated to L). Then, since $\phi_{L^{\otimes n}} = n\phi_L$, we have $n_X^*(L\hat{*}E) \cong \phi_{L^{\otimes n}}(\widehat{L \otimes E}) \otimes n_X^*L$. Thus Proposition 3.4 follows from the formula above applied to $L^{\otimes n}\hat{*}(E \otimes L^{\otimes -n+1})$.

The main reason why it is convenient to consider pullbacks of type $n_X^*(L\hat{*}E)$ rather than $L\hat{*}E$ itself is that if L and E are both line bundles algebraically equivalent to powers of the same ample line bundle A, then there is always a positive integer n such that the vector bundle $n_X^*(L\hat{*}E)$ is trivial up to twist by a line bundle algebraically equivalent to a power of A. To see this we need some basic remarks.

Remarks 3.5. (a) Let L, E and α be respectively two coherent sheaves on X and a line bundle in $\mathrm{Pic}^0 X$. Then

$$(F \otimes \alpha)\hat{*}E \cong (F\hat{*}(E \otimes \alpha)) \otimes \alpha. \tag{4}$$

Indeed if $\alpha \in \mathrm{Pic}^0 X$, then $(p_1+p_2)^*\alpha \cong p_1^*\alpha \otimes p_2^*\alpha$. Plugging this into the definition of $(F \otimes \alpha)\hat{*}E$ one gets (4) by the projection formula.

(b) *Let F be any coherent sheaf on X. There is a natural isomorphism*

$$\Psi : F\hat{*}\mathcal{O}_X \xrightarrow{\sim} H^0(F) \otimes \mathcal{O}_X.$$

Indeed let ψ be the automorphism $\psi = (p_1, p_2 - p_1)$ of $X \times X$. Then $\psi^*(p_1+p_2)^*F = p_2^*F$ and we have the isomorphism $\psi^* : p_{1*}((p_1+p_2)^*F) \to p_{1*}(p_2^*F)$. Then Ψ is obtained by composing ψ^* with the Künneth isomorphism $p_{1*}(p_2^*F) \cong H^0(F) \otimes \mathcal{O}_X$. (a) and (b) yield

(c) *Let F be a coherent sheaf on X and $\alpha \in \mathrm{Pic}^0 X$. Then $F\hat{*}\alpha \cong H^0(F \otimes \alpha) \otimes \alpha^\vee$.* □

Let us assume e.g. that E is a line bundle algebraically equivalent to a positive multiple of L. If $\alpha \in \mathrm{Pic}^0 X$ and $n \geq 2$, by Proposition 3.4 and Remark 3.5(c) we get that

$$n_X^*(L \hat{*} (L^{\otimes n-1} \otimes \alpha)) \cong H^0(L^{\otimes n} \otimes \alpha) \otimes n_X^*(L) \otimes L^{\otimes -n} \otimes \alpha^\vee. \tag{5}$$

Since $n_X^* L$ is algebraically equivalent to $L^{\otimes n^2}$, we get that $n_X^*(L \hat{*} (L^{\otimes n-1} \otimes \alpha))$ is isomorphic to a trivial bundle twisted by a line bundle algebraically equivalent to $L^{\otimes n(n-1)}$. More generally, we have the following

Proposition 3.6. *Let A be a line bundle, and let a and b two integers such that a and $a+b$ are positive. Then $(a+b)_X^*(A^{\otimes a} \hat{*} (A^{\otimes b} \otimes \alpha))$ is isomorphic to a trivial bundle times a line bundle algebraically equivalent to $A^{\otimes ab(a+b)}$. Specifically*

$$\begin{aligned}(a+b)_X^*(A^{\otimes a} \hat{*} (A^{\otimes b} \otimes \alpha)) \cong H^0(A^{\otimes a+b} \otimes \alpha) &\otimes (a+b)_X^*(A^{\otimes a}) \\ &\otimes a_X^*(A^{\otimes -a-b}) \otimes \alpha^\vee.\end{aligned}$$

Proof. By Proposition 3.4 we know that

$$\begin{aligned}(a+b)_X^*(A^{\otimes a} \hat{*} (A^{\otimes b} \otimes \alpha)) \cong &(A^{\otimes a(a+b)} \hat{*} (A^{\otimes b - a(a+b-1)} \otimes \alpha)) \\ &\otimes (a+b)_X^*(A^{\otimes a}) \otimes A^{\otimes -a(a+b)},\end{aligned} \tag{6}$$

$$\begin{aligned}a_X^*(A^{\otimes a+b} \hat{*} \alpha) \cong &(A^{\otimes a(a+b)} \hat{*} (\alpha \otimes A^{\otimes -(a+b)(a-1)})) \\ &\otimes a_X^*(A^{\otimes a+b}) \otimes A^{\otimes -a(a+b)}.\end{aligned} \tag{7}$$

The statement is obtained by plugging (7) into (6) (note that $b - a(a+b-1) = -(a+b)(a-1)$), and using Remark 3.5(c). □

(c) Examples and applications.

Example 3.7. Let us recover the following statement (see [K6]), which includes Koizumi's Theorem: *if $a, b \geq 2$ and $a+b \geq 5$, then the multiplication maps $H^0(T_x^* A^{\otimes a}) \otimes H^0(A^{\otimes b}) \to H^0((T_x^* A^{\otimes a}) \otimes A^{\otimes b})$ are surjective for any $x \in X$.* Proof: by Theorem 3.1 we need to show that $(A^{\otimes a} \hat{*} A^{\otimes b}) \otimes A^\vee$ has the vanishing property. Pulling back via $(a+b)_X^*$, by Proposition 3.6 we get a trivial bundle times a line bundle algebraically equivalent to $A^{(a+b)(ab-a-b)}$, which is positive precisely when $a, b \geq 2$ and $a+b \geq 5$. (By means of a finer analysis it is also possible to prove along these lines the stronger result that $m_x : H^0(T_x^* A^{\otimes 2}) \otimes H^0(A^{\otimes 2}) \to H^0((T_x^* A^{\otimes 2}) \otimes A^{\otimes 2})$ is surjective for general $x \in X$. One can also recover the explicit description of the locus where m_x is not surjective.)

The same result can be recovered, with a slightly different proof, as a particular case of the following more general statement.

Theorem 3.8. *Let X be an abelian variety over an algebraically closed field k, and let $n_0 \geq 2$ be an integer (such that $\mathrm{char}(k)$ does not divide $n_0 - 1$). Also let A and E be respectively an ample line bundle and a vector bundle on X such that $E \otimes A^{\otimes n}$ has the vanishing property for any $n \geq n_0$.*

If $(n_0-1)_X^*(E) \otimes A^{\otimes -n_0(n_0-1)}$ has the vanishing property, then the multiplication map

$$m_x : H^0(T_x^* A^n) \otimes H^0(E) \to H^0((T_x^* A^{\otimes n}) \otimes E) \tag{8}$$

is surjective for any $x \in X$ and for any $n \geq n_0$.

Proof of Theorem 3.8. By hypothesis and Corollary 3.3 what we need to show is that $h^i((A^{\otimes n_0} \hat{*} (A^\vee \otimes \alpha)) \otimes E) = 0$ for any $i > 0$, $\alpha \in \mathrm{Pic}^0 X$. By Proposition 3.6, pulling back via the map $(n_0 - 1)_X$ we get a trivial bundle times $(n_0 - 1)_X^*(E) \otimes A^{-n_0(n_0-1)} \otimes \alpha$. $\square$

Taking $n_0 = 2$ and $E = A^{\otimes m}$ with $m \geq 3$ one recovers Example 3.7. More generally, when $n_0 = 2$ one recovers the following criterion, which is (implicitly) the tool used by Kempf to prove his result on syzygies ([K5]): *if $E \otimes A^k = 0$ has the vanishing property for any $k \geq -2$, then for $n \geq 2$ the multiplication map $H^0(T_x^* A^{\otimes n}) \otimes H^0(E) \to H^0((T_x^* A^{\otimes n}) \otimes E)$ is surjective for any $x \in X$.* This result is clearly optimal but its weakness lies in the fact that in many applications (e.g. syzygies) one deals with higher powers of A, say $n \geq n_0$, and what is needed is a criterion for the surjectivity of multiplication maps such as (8) which is effective in function of n_0. Theorem 3.8 shows that, if $n_0 > 2$, to get such an optimal criterion one needs to pullback E via a suitable map (the multiplication by $n_0 - 1$). This of course makes the criterion more difficult to apply. As we will see in the sequel, for the bundles appearing in syzygy problems this difficulty can be easily circumvented.

4. Proof of Lazarsfeld's conjecture

Let us begin by rewiewing some background material – well-known and largely due to Green – about the relation between conditions about syzygies, like N_p, and the Koszul complex resolving the residue field k as a module over the symmetric algebra. The main fact (see [G1], [G2], Thm 1.2, [L], p.511, for details) is that condition N_p is equivalent to the exactness in the middle of the complex

$$\bigwedge^{p+1} H^0(L) \otimes H^0(L^{\otimes h}) \to \bigwedge^{p} H^0(L) \otimes H^0(L^{\otimes h+1}) \to \bigwedge^{p-1} H^0(L) \otimes H^0(L^{\otimes h+2}) \tag{1}$$

for any $h \geq 1$. Vector bundles come into play as follows: let M_L be the kernel of the evaluation map of L,

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0. \tag{2}$$

Taking wedge products of (2) one gets, for any p, exact sequences

$$0 \to \bigwedge^{p+1} M_L \to \bigwedge^{p+1} H^0(L) \otimes \mathcal{O}_X \to \bigwedge^{p} M_L \otimes L \to 0. \tag{3}$$

It follows (see e.g. [L]) that the exactness of the complex (1) is equivalent to the surjectivity of the map

$$\bigwedge^{p+1} H^0(L) \otimes H^0(L^{\otimes h}) \to H^0(\bigwedge^{p} M_L \otimes L^{\otimes h+1}) \tag{4}$$

obtained by twisting (3) with $L^{\otimes h}$ and taking H^0 of the third arrow. Therefore, if

$$H^1(\bigwedge^{p+1} M_L \otimes L^{\otimes h}) = 0 \tag{5}$$

for any $h \geq 1$, then condition N_p holds (the converse is also true as soon as $H^1(L^{\otimes h}) = 0$, as for abelian vareties). This leads to the following lemma.

Lemma 4.1. (a) *Assume* $\mathrm{char}(k)=0$. *If* $H^1(M_L^{\otimes p+1}\otimes L^{\otimes h})=0$ *for any* $h\geq 1$, *then* L *satisfies condition* N_p.

(b) *Assume that* $H^1(M_L^{\otimes p}\otimes L^{\otimes h})=0$. *Then* $H^1(M_L^{\otimes p+1}\otimes L^{\otimes h})=0$ *if and only if the multiplication map* $H^0(L)\otimes H^0(M_L^{\otimes p}\otimes L^{\otimes h})\to H^0(M_L^{\otimes p}\otimes L^{\otimes h+1})$ *is surjective.*

Proof. (a) follows immediately from (5) since, in characteristic zero, $\bigwedge^{p+1}M_L$ is a direct summand of $M_L^{\otimes p+1}$, while (b) follows from the exact sequence

$$0\to M_L^{\otimes p+1}\otimes L^{\otimes h}\to H^0(L)\otimes M_L^{\otimes p}\otimes L^{\otimes h}\to M_L^{\otimes p}\otimes L^{\otimes h+1}\to 0. \tag{6}$$

□

Proof of Lazarsfeld's conjecture. We will work over an algebraically closed field of characteristic zero. First, let us give a (rough) sketch of the argument: as announced, the strategy is to use Lemma 4.1(b) to reduce the problem to checking the surjectivity of the multiplication map $H^0(A^{\otimes n})\otimes H^0(M_{A^{\otimes n}}^{\otimes p}\otimes A^{\otimes n})\to H^0(M_{A^{\otimes n}}^{\otimes p}\otimes A^{\otimes 2n})$ for $n\geq p+3$ (to simplify the notation here we consider only the critical case, i.e. $h=1$). The point is to use Corollary 3.3: all we have to do is to check that its hypotheses are satisfied in this case. The serious condition is (1) of Corollary 3.3, namely the vanishing of the higher cohomology of all sheaves $(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))\otimes M_{A^{\otimes n}}^{\otimes p}\otimes A^{\otimes n}$ (however the same argument will work also for the other hypothesis of Corollary 3.3, namely that $M_{A^{\otimes n}}\otimes A^{\otimes 2n}$ has the vanishing property). Using a sequence like (2) above, twisted by $M_{A^{\otimes n}}^{\otimes p-1}\otimes A^{\otimes n}\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))$, one sees that the required vanishings follow from the vanishing of the higher cohomology of the sheaves $M_{A^{\otimes n}}^{\otimes p-1}\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))$ and from the surjectivity of the multiplication map

$$H^0(A^{\otimes n})\otimes H^0(M_{A^{\otimes n}}^{\otimes p-1}\otimes A^{\otimes n}\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha)))\\ \to H^0(M_{A^{\otimes n}}^{\otimes p-1}\otimes A^{\otimes 2n}\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))).$$

To verify this last condition we apply Corollary 3.3 another time, and we are reduced to proving the vanishing of the higher cohomology of

$$M_{A^{\otimes n}}^{\otimes p-1}\otimes A^{\otimes n}\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha))\otimes(A^{\otimes n}\hat{*}(A^\vee\otimes\beta)).$$

After repeating this procedure p times one eliminates all $M_{A^{\otimes n}}$'s, so that we are reduced to the vanishing of

$$A^{\otimes n}\otimes\bigotimes_{i=1}^{p+1}(A^{\otimes n}\hat{*}(A^\vee\otimes\alpha_i)). \tag{7}$$

Now, via the etale cover $(n-1)_X$, the pullback of a bundle like $A^{\otimes n}\hat{*}(A^\vee\otimes\alpha_i)$ is a trivial bundle times a line bundle equivalent to $A^{\otimes -n(n-1)}$ (Proposition 3.6) while the pullback of $A^{\otimes n}$ is equivalent to $A^{\otimes(n-1)^2n}$. Thus it is immediate to check that the required vanishing holds as soon as $n\geq p+3$.

Now let us give the formal proof. Throughout what follows, given $\alpha\in\mathrm{Pic}^0X$, we will denote

$$F_\alpha^{(n)}=A^{\otimes n}\hat{*}(A^\vee\otimes\alpha).$$

Proposition 4.2. *Let A be an ample line bundle on X, and let p, n, m be positive integers. Also let j and k be two integers such that $j, k \geq 0$ and $j + k \leq p + 1$. If $m(n-1) > n(p+1)$, then*

$$H^i(M_{A^{\otimes n}}^{\otimes j} \otimes A^{\otimes m} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}) = 0 \tag{8}$$

for any $\alpha_1, \dots, \alpha_l, \beta \in \mathrm{Pic}^0 X$ and for any $i > 0$.

Note that Lazarsfeld's conjecture follows as a special case: taking $m = hn$ (where $h \geq 1$), $k = 0$, $j = p+1$ and $\beta = \mathcal{O}_X$ one gets that $H^i(M_{A^{\otimes n}}^{\otimes p+1} \otimes A^{\otimes hn}) = 0$ as soon as $i > 0$ and $h(n-1) > p+1$ (note that in the critical case $h = 1$ this reduces to $n \geq p+3$). Then use Lemma 4.1(a). □

Proof. The proof will be by induction on j. For $j = 0$: to check the vanishings (8), it is enough to check them on the pullback via $(n-1)_X$. We have that $(n-1)_X^*(A^{\otimes m} \otimes \beta)$ is algebraically equivalent to $A^{\otimes (n-1)^2 m}$ while $(n-1)_X^* F_{\alpha_l}^{(n)}$ is a trivial bundle times a line bundle algebraically equivalent to $A^{\otimes -n(n-1)}$ (Proposition 3.6). Therefore the pullback via $(n-1)_X$ of our vector bundle is a trivial bundle times a line bundle algebraically equivalent to $A^{\otimes (n-1)((n-1)m - kn)}$, where $k \leq p+1$. This proves the case $j = 0$. Concerning the induction step, let us consider the exact sequence

$$0 \to (M_{A^{\otimes n}})^{\otimes j} \to H^0(A^{\otimes n}) \otimes (M_{A^{\otimes n}})^{\otimes j-1} \to A^{\otimes n} \otimes (M_{A^{\otimes n}})^{\otimes j-1} \to 0 \tag{9}$$

twisted by $A^{\otimes m} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}$. We have that the vanishings (8) are implied by
(i) *the vanishing of the higher cohomology of* $(M_{A^{\otimes n}})^{\otimes j-1} \otimes A^{\otimes m} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}$;
(ii) *the vanishing of the higher cohomology of* $(M_{A^{\otimes n}})^{\otimes j-1} \otimes A^{\otimes m+n} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}$;
(iii) *the surjectivity of the multiplication map*

$$\begin{aligned} H^0(A^{\otimes n}) \otimes H^0(M_{A^{\otimes n}}^{\otimes j-1} \otimes A^{\otimes m} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}) \\ \to H^0(M_{A^{\otimes n}}^{\otimes j-1} \otimes A^{\otimes m+n} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}). \end{aligned} \tag{10}$$

Now (i) and (ii) hold by induction. (iii): by Corollary 3.3 we have that the surjectivity of (10) follows from the following conditions:
(iv) *the vanishing of the higher cohomology of* $T_x^* A^{\otimes n} \otimes M_{A^{\otimes n}}^{\otimes j-1} \otimes A^{\otimes m} \otimes \beta \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}$ *for any* $x \in X$;
(v) *the vanishing of the higher cohomology of* $M_{A^{\otimes n}}^{\otimes j-1} \otimes A^{\otimes m} \otimes \beta \otimes F_\alpha \otimes \bigotimes_{l=1}^{k} F_{\alpha_l}^{(n)}$ *for any* $\alpha \in \mathrm{Pic}^0 X$ (this is condition (1) of Corollary 3.3).

Again, (iv) clearly holds by induction, so the only serious condition we have to check is (v) which holds by induction too since j decreases by one and k increases by one. □

A generalization of Lazarsfeld's conjecture. If $m < p+3$, property N_p may fail for $A^{\otimes m}$ (in fact this is what happens if A is a principal polarization). Our last result supplies an upper bound of "how much" it fails. For example one knows that $A^{\otimes 2}$ is h-normal for $h \geq 3$, i.e., going back to the notation of the introduction, $a_{0j} \leq 2$ for any j. Along the same lines a result of Kempf ([K4], [K6], Thm. 6.13(a)) states that the homogeneous ideal of the abelian variety X, embedded by the line bundle $A^{\otimes 3}$, is generated by quadrics or cubics , i.e. $a_{1j} \leq 3$ for any j. So in these cases the failure of N_p is of "at most one". More generally, given an integer $r \geq 0$, one may extend Green's condition as follows: L is said to *satisfy property* N_0^r if $a_{0j} \leq 1+r$ (i.e. the embedded variety X is h-normal for $h \geq 2+r$); L is said to *satisfy property* N_1^r if it satisfies N_0^r and $a_{1j} \leq 2+r$ for any j. Inductively one says that L *satisfies property* N_p^r if it satisfies N_{p-1}^r and $a_{pj} \leq p+1+r$ for any j. Roughly, this means that property N_p fails of at most r (note that N_p becomes N_p^0 in this terminology). It should be also said that Castelnuovo-Mumford's theorem yields that any very ample line bundle L satisfies N_p^{g+1}, where $g = \dim X$. The results of Kempf's work [K5] imply, in this terminology, that *if* $(r+1)n \geq \max(3, 2r+2, 2p+2)$*, then* $A^{\otimes n}$ *satisfies* N_p^r. Again, we improve such a result by a factor two:

Theorem 4.3 ($char(k) = 0$)**.** *If* $(r+1)(n-1) > p+1$*, then* $A^{\otimes n}$ *satisfies* N_p^r.

Proof. By Proposition 4.2 applied to the case $i = 1$, $m = hn$, $j = p+1$, $k = 0$ and $\beta = \mathcal{O}_X$ one gets that

$$H^1((M_{A^{\otimes n}})^{\otimes p+1} \otimes A^{\otimes hn}) = 0 \tag{11}$$

as soon as $h(n-1) > p+1$. The statement follows since, arguing exactly as for Lemma 4.1, the vanishing (11) for any $h \geq r+1$ yields property N_p^r. □

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