

THE SPECTRA OF NONNEGATIVE INTEGER MATRICES VIA FORMAL POWER SERIES

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1. INTRODUCTION

An old problem in matrix theory is to determine the n -tuples of complex numbers which can occur as the spectrum of a matrix with nonnegative entries (see [BP94, Chapter 4] or [Min88, Chapter VII]). Authors have studied the case where the n -tuple is comprised of real numbers [Bor95, Cia68, Fri78, Kel71, Per53, Sal72, Sou83, Sul49], the case where the matrices under consideration are symmetric [Fie74, JLL96], and the general problem [Joh81, LM99, LL79, Rea94, Rea96, Wuw97]. Various necessary conditions and sufficient conditions have been provided, but a complete characterization is known for real n -tuples only for $n \leq 4$ [Kel71, Sul49] and for complex n -tuples only for $n \leq 3$ [LL79].

Motivated by symbolic dynamics, Boyle and Handelman refocused attention on the *nonzero part* of the spectrum by making the following “Spectral Conjecture” [BH91, BH93] (see also [Boy93, §8] and [LM95, Chapter 11]).

Below, a matrix A is primitive if all entries of A are nonnegative and for some n , all entries of A^n are strictly positive. Also,

$$\operatorname{tr}(\Lambda^n) = \sum_{i=1}^d (\lambda_i)^n \quad \text{and} \quad \operatorname{tr}_n(\Lambda) = \sum_{k|n} \mu\left(\frac{n}{k}\right) \operatorname{tr}(\Lambda^k)$$

where μ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^e & \text{if } n \text{ is the product of } e \text{ distinct primes,} \\ 0 & \text{if } n \text{ is divisible by } k^2 \text{ for some } k \geq 2. \end{cases}$$

Spectral Conjecture (Boyle, Handelman). *Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a d -tuple of nonzero complex numbers and let S be a unital subring of \mathbb{R} . There is a primitive matrix over S with characteristic polynomial $t^m \prod_{i=1}^d (t - \lambda_i)$ for some $m \geq 0$ if and only if*

- (1) *the coefficients of the polynomial $\prod_{i=1}^d (t - \lambda_i)$ all belong to S ,*
- (2) *there exists λ_j in Λ such that $\lambda_j > |\lambda_i|$ for all $i \neq j$,*
- (3) *if $S = \mathbb{Z}$, then $\operatorname{tr}_n(\Lambda) \geq 0$ for all $n \geq 1$,*
- (3') *if $S \neq \mathbb{Z}$, then for all $n \geq 1$, $\operatorname{tr}(\Lambda^n) \geq 0$ and for all $k \geq 1$, $\operatorname{tr}(\Lambda^n) > 0$ implies $\operatorname{tr}(\Lambda^{nk}) > 0$.*

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It is not hard to show that these conditions are necessary. Moreover, the conditions can be easily verified for any example (condition (2) implies (3) or (3') for all but finitely many n, k).

Boyle and Handelmann proved the conditions of the Spectral Conjecture to be sufficient in many cases, including the case $S = \mathbb{R}$ [BH91, BH93]. In this paper we resolve an important remaining case by proving the Spectral Conjecture for $S = \mathbb{Z}$ (Theorem 2.2). The Spectral Conjecture for $S = \mathbb{Q}$ (Corollary 2.3) follows from this result. Characterizations of the possible nonzero spectra of irreducible and general nonnegative matrices follow from the primitive case [BH91].

We begin by restating the argument for the necessity of the Boyle-Handelman conditions in the case $S = \mathbb{Z}$. Suppose A is a primitive integer matrix with $\det(tI - A) = t^m \prod_{i=1}^d (t - \lambda_i)$. Then condition (1) is clearly satisfied. Condition (2) follows from Perron-Frobenius Theory, the spectral radius of primitive matrix is always an eigenvalue of multiplicity one for that matrix. Condition (3) follows when we interpret A as the adjacency matrix for a directed graph. In other words, if A is a $k \times k$ matrix we can construct a directed graph G_A with k vertices such that the number of distinct edges from vertex i to vertex j is given by $A_{(i,j)}$. Here, the trace of A^n , given by $\text{tr}(A^n)$, represents the number of loops of length n in G_A , i.e., labelling each edge distinctly, $\text{tr}(A^n)$ is the number of distinct edge sequences $e_1 e_2 \cdots e_n$ such that the terminal vertex of e_i is equal to the initial vertex of e_{i+1} and the terminal vertex of e_n is the initial vertex of e_1 . The n th net trace of A , given by $\text{tr}_n(A)$, represents the number of loops of length n in G_A which are not formed by concatenating a single loop with itself.

Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfy the conditions of the Spectral Conjecture for \mathbb{Z} . We outline our construction of a matrix A over \mathbb{Z}_+ with nonzero spectrum Λ . First, we note that there is a primitive matrix $A(t)$ over $t\mathbb{Z}_+[t]$ (the ring of polynomials with nonnegative integer entries and no constant term) such that $\det(I - A(t)) = \prod_{i=1}^d (1 - \lambda_i t)$ if and only if there is a primitive integer matrix A with nonzero spectrum Λ (see [Boy93, §5]). In Lemma 3.1, we use this fact to essentially reduce the problem of constructing a polynomial matrix $A(t)$ as above to finding polynomials $q_1, q_2, \dots, q_n \in t\mathbb{Z}_+[t]$ and a power series $r \in t\mathbb{Z}_+[[t]]$ such that

$$\prod_{i=1}^d (1 - \lambda_i t) = (1 - r(t)) \prod_{i=1}^n (1 - q_i(t)).$$

In particular, we show that there are integers $o(n) \geq 0$, $n_0 \geq 1$ and a polynomial $q \in t\mathbb{Z}_+[t]$ such that for each nonzero degree term in the power series

$$\frac{\prod_{i=1}^d (1 - \lambda_i t)}{(1 - q(t)) \prod_{n=1}^{n_0} (1 - t^n)^{o(n)}}$$

the coefficient is nonpositive. We prove this by giving estimates of coefficients of the power series $\prod_{n=1}^{n_0} (1 - t^n)^{-o(n)}$. We use different estimates for terms with degrees in linear, polynomial, subexponential, and exponential ranges of n_0 . For sufficiently large n_0 , these estimates imply the nonpositivity of coefficients of the above quotient up to a degree where the additional factor $1 - q(t)$ takes over.

With the results here and in [BH91], a complete characterization of the spectra of primitive matrices over \mathbb{R} or \mathbb{Z} would follow from sharp bounds on the size of the realizing matrix. However, it seems that bounds of this type will be quite difficult

to pin down. In particular, it follows from an inequality in [Joh81, LL79] that given any N one can construct a 3-tuple satisfying the Boyle-Handelman conditions for \mathbb{R} such that the size of a realizing matrix must be at least $N \times N$ (see [JLL96, LM98]). This is in dramatic contrast to the symmetric case where Johnson, Laffey and Loewy showed that if a d -tuple $(\lambda_1, \lambda_2, \dots, \lambda_d)$ of real numbers is the nonzero spectrum of a symmetric matrix, then it is the nonzero spectrum of a symmetric matrix of size no more than $d(d+1)/2$ [JLL96].

We propose that a variant of the size bound problem may be more approachable: Given $(\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfying the Boyle-Handelman conditions, produce sharp bounds on the size of a polynomial matrix $A(t)$ with $\det(I - A(t)) = \prod_{i=1}^d (1 - \lambda_i t)$. In some respects, bounds of this type would be more useful than bounds on ordinary matrices since one can exhibit a much wider range of behavior with polynomial matrices of a fixed size. Moreover, it is conceivable that this line of study could lead to a solution to the original problem. A result of Perrin underscores the freedom afforded by polynomial matrices: For every λ which occurs as the spectral radius of a primitive integer matrix, there is a 2×2 matrix $A(t)$ over $t\mathbb{Z}_+[t]$ with $\det(I - A(t)) = (1 - \lambda t) \prod_{i=1}^d (1 - \lambda_i t)$ and $\lambda > |\lambda_i|$ for all i [Per92] (see also [Boy93, §5]).

Our characterization of the possible nonzero spectra of nonnegative matrices over \mathbb{Z} translates into a characterization of the possible zeta functions for shifts of finite type in symbolic dynamics. In this setting, our work follows Lind's classification of entropies of shifts of finite type [Lin84], Boyle and Handelman's classification of zeta functions for finitely presented systems [BH91] and various authors' development of polynomial matrices as tools in symbolic dynamics [BGM80, KRW97, MT91]. In general, there is a deep connection between symbolic dynamics and the asymptotic algebra of nonnegative matrices (see [Boy93, BH93, LM95]). Boyle and Handelman's Generalized Spectral Conjecture (see [Boy93, §8]) concerns the matrix relation, strong shift equivalence over R , which in the case $R = \mathbb{Z}_+$ corresponds to conjugacy between associated shifts of finite type. Strong shift equivalence over a semiring R is defined as the transitive closure of the elementary relation \sim_R where $A \sim_R B$ if there exist matrices U, V over R such that

$$A = UV \quad \text{and} \quad VU = B.$$

Strong shift equivalence seems to be the strongest asymptotic equivalence relation for matrices over a given semiring [Boy93, BH93]. Boyle and Handelman conjecture that a matrix A over a unital ring $S \subseteq \mathbb{R}$ is strong shift equivalent over S_+ to a primitive matrix if and only if the nonzero spectrum of A satisfies the conditions of the Spectral Conjecture.

In Section 2 of this paper we provide statements of our results. In Section 3 we prove a reduction lemma and give a more detailed outline of the proof of the main theorem. The proof itself is contained in Sections 4–6. We thank Mike Boyle for many helpful discussions and for his detailed examination of portions of this manuscript. We also thank Doug Lind for his comments and for bringing the example from [LM95, Chapter 11] to our attention (see Appendix B).

2. STATEMENT OF THE MAIN RESULTS

For a unital ring $S \subseteq \mathbb{R}$ we will often refer to the conditions of the Spectral Conjecture for S as the *Boyle-Handelman conditions for S* . We will also clean

up notation a bit by noting that for a matrix A , there is an $m \geq 0$ such that $\det(tI - A) = t^m \prod_{i=1}^d (t - \lambda_i)$ if and only if $\det(I - At) = \prod_{i=1}^d (1 - \lambda_i t)$.

Definition 2.1. A matrix A over \mathbb{R}_+ is *primitive* if there is an $n > 0$ such that all of the entries of A^n are positive.

Main Theorem 2.2. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a d -tuple of nonzero complex numbers with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$. There exists a primitive integer matrix A such that $\det(I - At) = \prod_{i=1}^d (1 - \lambda_i t)$ if and only if

- (1) the polynomial $\prod_{i=1}^d (1 - \lambda_i t)$ has integer coefficients,
- (2) $\lambda_1 > |\lambda_i|$ for $i = 2, 3, \dots, d$,
- (3) $\text{tr}_n(\Lambda) \geq 0$ for all $n \geq 1$.

As shown in [BH91], the Spectral Conjecture for \mathbb{Q} follows from our main result.

Corollary 2.3. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a d -tuple of nonzero complex numbers with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$. There is a primitive rational matrix A with $\det(I - At) = \prod_{i=1}^d (1 - \lambda_i t)$ if and only if

- (1) the polynomial $\prod_{i=1}^d (1 - \lambda_i t)$ has rational coefficients,
- (2) $\lambda_1 > |\lambda_i|$ for $i = 2, 3, \dots, d$,
- (3) for all $n \geq 1$, $\text{tr}(\Lambda^n) \geq 0$ and for all $k \geq 1$, $\text{tr}(\Lambda^n) > 0$ implies $\text{tr}(\Lambda^{nk}) > 0$.

We give a sketch of how the characterization of the spectra of irreducible and nonnegative matrices follows from the primitive case (see [BH91] for more details).

Definition 2.4. A matrix A over \mathbb{R}_+ is *irreducible* if for all (i, j) there is an $n \geq 0$ such that the (i, j) entry of A^n is positive.

If a matrix A is irreducible, then there is a $p \geq 1$ and a primitive matrix B such that $\det(I - At) = \det(I - Bt^p)$. This leads to the following.

Corollary 2.5. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a d -tuple of nonzero complex numbers. There exists an irreducible matrix A over \mathbb{Z} (\mathbb{Q}) such that $\det(I - At) = \prod_{i=1}^d (1 - \lambda_i t)$ if and only if there exist an integer $p > 0$ and a partition of Λ into subtuples $\{\Lambda_k : 0 \leq k < p\}$ such that

- (1) $\Lambda_{k+1} = e^{2\pi i/p} \Lambda_k$ for $0 \leq k < (p - 1)$,
- (2) $(\Lambda_0)^p$ satisfies the Boyle-Handelman conditions for \mathbb{Z} (\mathbb{Q}).

If A is a nonnegative matrix, then there are irreducible matrices A_1, A_2, \dots, A_m such that $\det(I - At) = \prod_{j=1}^m \det(I - A_j t)$.

Corollary 2.6. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a d -tuple of nonzero complex numbers. There exists a nonnegative matrix A over \mathbb{Z} (\mathbb{Q}) such that $\det(I - At) = \prod_{i=1}^d (1 - \lambda_i t)$ if and only if there exist an integer $n > 0$, integers $p(1), p(2), \dots, p(n) > 0$ and a partition of Λ into subtuples $\{\Lambda_{(j,k)} : 1 \leq j \leq n, 0 \leq k < p(j)\}$ such that

- (1) $\Lambda_{(j,k+1)} = e^{2\pi i/p(j)} \Lambda_{(j,k)}$ for all $0 \leq k < (p(j) - 1)$ and $1 \leq j \leq n$,
- (2) $(\Lambda_{(j,0)})^{p(j)}$ satisfies the Boyle-Handelman conditions for \mathbb{Z} (\mathbb{Q}) for all $1 \leq j \leq n$.

Remark 2.7. A characterization of zeta functions for shifts of finite type follows from Corollary 2.6. Suppose A is a nonnegative matrix A and σ_A is the corresponding shift of finite type. If $\text{Fix}((\sigma_A)^n)$ is the set of fixed points of the map

$(\sigma_A)^n$, then we have the following definition and result:

$$\zeta_{(\sigma_A)}(t) := \exp \left(\sum_{n=1}^{\infty} \frac{\#\text{Fix}((\sigma_A)^n)}{n} t^n \right) = \frac{1}{\det(\mathbf{I} - At)}.$$

Therefore, a power series in $\mathbb{Z}[[t]]$ is a zeta function if and only if it is equal to $\prod_{i=1}^d (1 - \lambda_i t)^{-1}$ where $(\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfies the conditions of Corollary 2.6. Moreover, since irreducible matrices give rise to irreducible shifts of finite type and primitive matrices give rise to mixing shifts of finite type, Theorem 2.2 and Corollary 2.5 give finer characterizations for zeta functions associated to shifts of finite type in those classes.

3. PROOF SCHEME FOR THE MAIN THEOREM

For the remainder of the paper, suppose $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfies the Boyle-Handelman conditions for \mathbb{Z} and $1 - p(t) = \prod_{i=1}^d (1 - \lambda_i t)$. Let $\lambda = \lambda_1$.

We begin this section by showing that the problem of creating a primitive matrix A over \mathbb{Z} with $\det(\mathbf{I} - At) = 1 - p(t)$ reduces to writing

$$1 - p(t) = (1 - r(t)) \prod_{i=1}^n (1 - q_i(t))$$

where $q(t) \in t\mathbb{Z}_+[t]$ and $r(t) \in t\mathbb{Z}_+[[t]]$. The reduction uses in a critical way the fact that there is a primitive matrix A with entries in \mathbb{Z}_+ such that $\det(\mathbf{I} - At) = 1 - p(t)$ if and only if there is a primitive polynomial matrix $A(t)$ with entries in $t\mathbb{Z}_+[t]$ such that $\det(\mathbf{I} - A(t)) = 1 - p(t)$ (see [Boy93] for details).

3.1. Main reduction. Before we justify the reduction to the factorization problem in formal power series, we prove a lemma about coefficients of large degree in a rational power series.

Lemma 3.1. *Suppose $f(t) \in \mathbb{Z}[t]$, $u(t) \in \mathbb{R}[t]$ and $\alpha > 0$ such that all roots of $u(t)$ have modulus greater than $1/\alpha$. Let $r(t) = \sum r_n t^n$ be the power series*

$$r(t) = \frac{f(t)}{(1 - \alpha t)u(t)}.$$

Then there exist constants $K, k > 0$ such that

$$\left| \frac{r_n}{\alpha^n} - \frac{f(1/\alpha)}{u(1/\alpha)} \right| < K e^{-kn} \quad \text{for } n \geq 0.$$

Proof. Let $a = f(1/\alpha)/u(1/\alpha)$. Then $1/\alpha$ is a root of $f(t) - au(t)$ and $b(t) = (f(t) - au(t))/(1 - \alpha t)$ is a polynomial. We may rewrite $r(t)$ as

$$r(t) = \frac{a}{1 - \alpha t} + \frac{b(t)}{u(t)} = a \sum_{n=0}^{\infty} \alpha^n t^n + \frac{b(t)}{u(t)}.$$

Let $b(t)/u(t) = \sum c_n t^n$. By the condition on the roots of $u(t)$, the radius of convergence of the power series $\sum c_n t^n$ is larger than $1/\alpha$. Therefore there exist constants $K, k > 0$ such that $|c_n| \alpha^{-n} < K e^{-kn}$. Since $r_n = a \alpha^n + c_n$, the result follows. □

If, as in the previous lemma, a polynomial has a factorization of the form $(1 - \alpha t)u(t)$ where the modulus of all roots of $u(t)$ are larger than $1/\alpha$, we will call α the *Perron value* of the polynomial $(1 - \alpha t)u(t)$. If $q \in t\mathbb{Z}_+[t]$ and the degrees of terms in $q(t)$ with nonzero coefficients have greatest common divisor one, then $1 - q(t)$ has such a factorization.

We now show that producing a polynomial matrix $A(t)$ over $t\mathbb{Z}_+[t]$ with $\det(I - A(t)) = 1 - p(t)$ reduces to producing a certain factorization of $1 - p(t)$ in $\mathbb{Z}[[t]]$.

Main Reduction Lemma. *Let $p \in t\mathbb{Z}[t]$. If there exist polynomials q_1, q_2, \dots, q_m in $t\mathbb{Z}_+[t]$ and a power series r in $t\mathbb{Z}_+[[t]]$ such that*

- (1) *there is a Perron value α for the polynomial $\prod_{j=1}^m (1 - q_j(t))$,*
- (2) $1 - p(1/\alpha) < 0$,
- (3) $1 - p(t) = (1 - r(t)) \prod_{j=1}^m (1 - q_j(t))$ in $\mathbb{Z}[[t]]$,

then there exists a matrix $A(t)$ with entries in $t\mathbb{Z}_+[t]$ such that

$$\det(I - A(t)) = 1 - p(t).$$

Proof. Assume that the polynomials q_i are numbered so that $1/\alpha$ is a root of the first polynomial $1 - q_1(t)$. Then $1 - q_1(t)$ factors as $(1 - \alpha t)u(t)$ where $u(1/\alpha) = q_1'(1/\alpha)/\alpha > 0$. Note that for $j \geq 2$ we have $1 - q_j(1/\alpha) > 0$. This follows from $1 - q_j(0) = 1$ and assumption (1) which implies that the positive number $1/\alpha$ is strictly smaller than the unique positive root of $1 - q_j(t)$.

Since $1 - p(1/\alpha) < 0$ and $1 - q_j(1/\alpha) > 0$ for all $j \geq 2$, it follows from Lemma 3.1 that there is an $N > 0$ such that for all $1 \leq k \leq m$, the n th coefficient of $(1 - p(t)) \prod_{j=1}^k (1 - q_j(t))^{-1}$ is negative for all $n > N$. Select this N to exceed the degree of $p(t)$, the degree of all polynomials $q_j(t)$ and m .

Let $a_0(t) = p(t)$ and for $1 \leq k \leq m$ let $1 - a_k(t)$ be the sum of terms in the power series $(1 - p(t)) \prod_{j=1}^k (1 - q_j(t))^{-1}$ up to degree $2N$. Let $b_k(t)$ be the polynomial defined by the relation

$$(3.1) \quad 1 - a_{k-1}(t) = (1 - a_k(t))(1 - q_k(t)) - b_k(t).$$

Since the first $2N$ terms of $1 - a_{k-1}(t)$ and $(1 - a_k(t))(1 - q_k(t))$ agree, all coefficients of $b_k(t)$ in degrees $2N$ and less are zero. In degrees greater than $2N$, the coefficients of $b_k(t)$ equal the coefficients of the product $a_k(t)q_k(t)$. By the choice of N , $q_k(t) \in \mathbb{Z}_+[t]$ has degree at most N and $a_k(t)$ has nonnegative coefficients in degrees $(N + 1)$ and higher. Therefore, all coefficients of $b_k(t)$ are nonnegative.

Beginning with $1 - p(t) = 1 - a_0(t)$ and successively using equation (3.1), we have

$$(3.2) \quad 1 - p(t) = (1 - a_m(t)) \prod_{j=1}^m (1 - q_j(t)) - \sum_{i=1}^m \left[b_i(t) \prod_{j=1}^{i-1} (1 - q_j(t)) \right].$$

It follows from the hypotheses of the lemma that the polynomial $a_m(t)$ has all nonnegative coefficients since it is equal to the first $2N$ terms of the power series $r(t)$.

Now let $M(t)$ be the following polynomial matrix over $\mathbb{Z}_+[t]$:

$$M(t) = \begin{pmatrix} a_m(t) & 1 & 0 & \dots & 0 \\ b_m(t) & q_m(t) & 1 & \ddots & \vdots \\ b_{m-1}(t) & 0 & q_{m-1}(t) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ b_1(t) & 0 & \dots & 0 & q_1(t) \end{pmatrix}.$$

Using equation (3.2), we find $\det(I - M(t)) = 1 - p(t)$.

Finally, we replace each occurrence of 1 in $M(t)$ with a t and each $b_i(t)$ with $b_i(t)/t^{m+1-i}$. The result is a matrix $A(t)$ over $t\mathbb{Z}_+[t]$ with $\det(I - A(t)) = 1 - p(t)$. □

We now turn our attention to producing a factorization of our candidate polynomial $1 - p(t)$ as in the Main Reduction Lemma. One can recursively define a sequence of integers $o(i)$ such that $1 - p(t)$ has a formal factorization as

$$1 - p(t) = \prod_{i=1}^{\infty} (1 - t^i)^{o(i)}.$$

We claim $o(i) = \text{tr}_i(\Lambda)/i \geq 0$ for all i . To see this, first note the relation

$$1 - p(t) = \exp\left(\sum_{j=1}^d \log(1 - \lambda_j t)\right) = \exp\left(-\sum_{i=1}^{\infty} \frac{\text{tr}(\Lambda^i)}{i} t^i\right).$$

Taking logarithms and comparing coefficients in the resulting power series, we see $\text{tr}(\Lambda^i) = \sum_{k|i} ko(k)$. Since $\text{tr}(\Lambda^i) = \sum_{k|i} \text{tr}_k(\Lambda)$, we obtain $o(i) = \text{tr}_i(\Lambda)/i$.

Thus we use the Net Trace Condition to obtain an infinite factorization as above with nonnegative exponents $o(i)$. Moreover, we will know from the Perron condition that, for large i , $io(i)\lambda^{-i} \approx 1$.

It seems natural to attempt to truncate the infinite factorization and write

$$1 - p(t) = (1 - r(t)) \prod_{i=1}^{n_0} (1 - t^i)^{o(i)}$$

where $r \in t^{n_0+1}\mathbb{Z}_+[[t]]$. With such a factorization, we automatically have that the first n_0 coefficients of $r(t)$ are zero, and coefficients in degrees n_0 through $2n_0$ are positive. In fact we can use this truncated product approach to factor many examples, including $1 - 4t + 6t^2 - 6t^3$ from [LM95] (see Appendix B). This example was presented as one which satisfies the Boyle/Handelman conditions, but one for which there was no known matrix with the corresponding nonzero spectrum.

However, we are not able to factor general polynomials as above. In particular, to apply the Main Reduction Lemma, we need the denominator polynomial to have a Perron value in an appropriate region to control coefficients of large degree for the quotient power series.

The final factorization we seek will be one of the form

$$1 - p(t) = (1 - r(t))(1 - q(t)) \prod_{i=1}^{n_0} (1 - t^i)^{o(i)}$$

where $q(t) \in t\mathbb{Z}_+[t]$ and $r(t) \in t^{n_0+1}\mathbb{Z}_+[[t]]$. We are assuming $1 - p(t)$ has a Perron value λ . If we select $1 - q(t)$ so that its Perron value α is smaller than λ but larger

than $|\lambda_i|$ for $i > 1$, then we can guarantee $1 - p(1/\alpha) < 0$. Thus by Lemma 3.1, all coefficients of large degree terms $r(t)$ have the correct sign.

We know the n th coefficient of $(1 - p(t))/(1 - q(t)) \prod_{i=1}^{n_0} (1 - t^i)^{o(i)}$ has the appropriate sign when n is small (less than $2n_0$) and when n is sufficiently large. The difficulty lies in bridging the gap between these two regions. The proof we present relies on careful estimates of the coefficients of rational power series in various ranges. For example, we use different arguments for $n \in (2n_0 + \log^2 n_0, n_0^{20})$, $n \in [n_0^{20}, e^{rn_0})$ and $n \in [e^{rn_0}, \infty)$. The sequence of lemmas is somewhat long, and we present now an overview which we hope will help the reader digest the proof.

3.2. Overview of the estimates. We wish to show that the n th coefficient of the product of the polynomial $1 - p(t)$ with the power series $(1 - q(t))^{-1} \prod_{i=1}^{n_0} (1 - t^i)^{-o(i)}$ is nonpositive for $n > 0$. To do so we will follow one of two arguments.

Let $c(t) = \sum_{i=0}^{\infty} c_n t^n$ be any power series with $c_0 = 1$ and $c_n \geq 0$ for all n . Suppose we would like to show that the n th coefficient of $(1 - p(t))c(t)$ is negative. We may write the n th coefficient of the product of $(1 - p(t))c(t)$ as $c_n(1 - \sum_{i=1}^d p_i c_{n-i}/c_n)$.

To apply the *Difference Argument*, we show that there is a $D > 0$ such that $c_{n-i}/c_n \approx \lambda^{-i}(1 + iD)$ for $i = 1, 2, \dots, d$. Then

$$1 - \sum_{i=1}^d p_i c_{n-i}/c_n \approx 1 - \underbrace{\sum_{i=1}^d p_i \lambda^{-i}}_0 - D\lambda^{-1} \underbrace{\sum_{i=1}^d i p_i \lambda^{-i+1}}_{p'(1/\lambda)}.$$

Since $p'(1/\lambda) > 0$, the expression on the right-hand side is negative. In order to make this argument precise, we will have to show that the error in the approximation is small compared to the number D .

To apply the *Ratio Argument*, we show that there is a number $\alpha < \lambda$ such that $c_{n-i}/c_n \approx \alpha^{-i}$ for $i = 1, 2, \dots, d$ and $1 - p(1/\alpha) < 0$. Then

$$1 - \sum_{i=1}^d p_i c_{n-i}/c_n \approx 1 - \sum_{i=1}^d p_i \alpha^{-i} = 1 - p(1/\alpha).$$

To make this argument precise, we will have to show that the error in the approximation is small compared to $1 - p(1/\alpha)$. By the Mean Value Theorem, this will reduce to showing that the error in the approximation is small compared to $|\lambda - \alpha|$.

In Section 4, we examine the product of the power series

$$S(t) = \sum_{n=0}^{\infty} S_n t^n = \prod_{i=1}^{n_0} (1 - t^i)^{-o(i)}$$

and the polynomial $1 - p(t)$. We apply the Difference Argument to show that the n th coefficient of this product is negative for $n \in (n_0, n_0^{20})$. To do so, we first give bounds on S_n/S_{n+1} for n in an initial range. Then we introduce a different power series $E(t)$, the coefficients of which approximate the coefficients of $S(t)$. Let

$$E(t) = \sum_{n=0}^{\infty} E_n t^n = \exp \left(\sum_{i=1}^{n_0} \frac{\text{tr}(\Lambda^i)}{i} t^i \right).$$

By differentiating $E(t)$, we find that the coefficients of $E(t)$ satisfy a recursion relation:

$$nE_n = \sum_{i=1}^{n_0} \text{tr}(\Lambda^i)E_{n-i}.$$

With the estimates on ratios of S_n/S_{n+1} as the base case, we use the recursion formula to give estimates on E_n/E_{n+1} for n up to some exponential function of n_0 . We show that for large n_0 , all approximations introduce errors which are small compared to $D = \lambda S_{n-1}/S_n - 1 > 0$ for $n \in (n_0, n_0^{20})$. Then by the Difference Argument the series $(1 - p(t))S(t)$ has nonnegative coefficients in this polynomial range.

In Section 5 we make further use of the recursion formula for E_n and apply the Ratio Argument to the n th coefficient of $S(t)(1 - p(t))$ for $n \in [n_0^{20}, e^{rn_0})$ (r is a number chosen based on other constants). Here we think of the recursion relation on the coefficients of $E(t)$ as being given by a matrix. In other words, there is a matrix A_n such that

$$(E_n, E_{n-1}, \dots, E_{n-n_0+1})^T = A_n(E_{n-1}, E_{n-2}, \dots, E_{n-n_0})^T.$$

Thus, beginning with an initial string of n_0 consecutive coefficients, we obtain a string of coefficients of larger degree by applying the matrix $A_n A_{n-1} \cdots A_{n-N+1}$. As we will show for $n > n_0^4$ and $N = n_0^6$, applying this product of matrices to a vector $\mathbf{v} \in \mathbb{R}_+^{n_0}$ is approximately the same as multiplying by the N th power of the single matrix A_n to \mathbf{v} . The matrix A_n has a Perron eigenvalue α_n . A fairly general argument involving estimates of moduli of other eigenvalues of A_n implies that when a large power of A_n is applied to \mathbf{v} the result is a vector which is very close to an eigenvector for A_n corresponding to α_n . Since eigenvectors corresponding to α_n are exactly the vectors with ratios of successive entries equal to α_n , we see that $E_n/E_{n-1} \approx 1/\alpha_n$. Applying the Ratio Argument, all of these approximations taken with a lower bound on $\lambda - \alpha_n$ imply that the n th coefficient of $S(t)(1 - p(t))$ is negative for $n \in [n_0^{20}, e^{rn_0})$.

In Section 6, we incorporate an additional factor $(1 - q(t))^{-1}$ into the factorization scheme. We first select a real number β such that $\beta < \lambda$ and $1 - p(1/\beta) < 0$. Then let $1 - q(t)$ be an integer polynomial approximation of

$$1 - \frac{1}{n_0^3 - n_0^2} \left(\beta^{n_0^2+1} t^{n_0^2+1} + \beta^{n_0^2+2} t^{n_0^2+2} + \dots + \beta^{n_0^3} t^{n_0^3} \right).$$

The Perron value for $1 - q(t)$ is approximately β . Using the matrix arguments from Section 5 on the recursion formula for coefficients of $(1 - q(t))^{-1}$, we show that ratios of successive coefficients of $(1 - q(t))^{-1}$ are approximately β for $n > n_0^{20}$. Again we can apply the Ratios Argument to show that the n th coefficient of $(1 - p(t))/(1 - q(t))$ is negative for $n > n_0^{20}$.

Finally, we combine all of the estimates to show that the n th coefficient of the product $(1 - p(t))S(t)(1 - q(t))^{-1}$ is nonpositive for all $n > 0$. We think of the n th coefficient of $(1 - p(t))S(t)(1 - q(t))^{-1}$ either as the n th coefficient of the product of $(1 - p(t))S(t)$ and $(1 - q(t))^{-1}$ or as the n th coefficient of the product of $S(t)$ and $(1 - p(t))(1 - q(t))^{-1}$. In either case, we will write the n th coefficient as the sum of a large number of negative terms and a relatively small number of positive terms. We are able to conclude that the overall sum is negative.

4. POLYNOMIAL RANGE

In this section we follow the Difference Argument to show that if n_0 is sufficiently large and $n \in (n_0, n_0^{20})$, then the n th coefficient of the power series $S(t)(1 - p(t))$ is bounded above by S_n times a negative constant over n_0 (Polynomial Range Lemma). We must show that for each $n \in (n_0, n_0^{20})$ there is a number $D > 0$ such that $S_{n-i}/S_n \approx \lambda^{-i}(1 + iD)$ for $i = 1, 2, \dots, d$. We will let $D = \lambda S_{n-1}/S_n - 1$.

It will be convenient for us to write many of our estimates in terms of coefficients scaled by λ . Our convention will be to write coefficients of power series with capital letters and scaled coefficients with lower case letters. For example, let $s_n = S_n \lambda^{-n}$. Let $Z(t)$ denote the candidate zeta function $Z(t) = \sum_{i=1}^{\infty} z_n (\lambda t)^n = (1 - p(t))^{-1}$.

Throughout the remainder of the paper, we will use the fact that there exist constants $a, K_0, k_0 > 0$ such that

$$(4.1) \quad |z_n - a| < K_0 e^{-k_0 n} \quad \text{and} \quad |n o(n) \lambda^{-n} - 1| < K_0 e^{-k_0 n}$$

for all n .

In the following lemmas we use the term ‘‘constant’’ or ‘‘universal constant’’ to mean a number which is chosen based solely on $1 - p(t)$. In particular, it is important that constants do not depend upon the choice of n_0 . Many constants used in the statements of lemmas will be denoted by K or k and will be numbered so as to distinguish them.

4.1. Initial bounds. In the following two lemmas, our goal is to show that $s_n/s_{n+1} - 1$ is bounded between two positive constants times $1/n_0$ for $n \in [n_0 + \log^2 n_0, 2n_0 + \log^2 n_0]$. In Section 4.2 we will use these initial bounds along with a recursive formula to produce estimates on $s_n/s_{n+1} - 1$ for larger n .

We begin by proving a uniform lower bound on the coefficients s_n for n in this initial range.

Lemma 4.1. *There exists a constant $K_1 > 0$ such that if n_0 is sufficiently large, then*

$$s_n > K_1 \quad \text{for} \quad n \in (n_0, 2n_0 + \log^2 n_0].$$

Proof. We have the following expressions for $S(t)$:

$$S(t) = \prod_{i=1}^{n_0} (1 - t^i)^{-o(i)} = \prod_{i=n_0+1}^{\infty} (1 - t^i)^{o(i)} / (1 - p(t)).$$

Thus for $n \in (n_0, 2n_0 + \log^2 n_0]$, we have

$$s_n = z_n - \sum_{i=n_0+1}^n \frac{o(i)}{\lambda^i} z_{n-i} + \sum_{\substack{i,j > n_0 \\ i+j \leq n}} \frac{o(i)o(j)}{\lambda^{i+j}} z_{n-i-j} + \sum_{i=n_0+1}^{n/2} \lambda^{-2i} \binom{o(i)}{2} z_{n-2i}.$$

(The last two summations only occur when $n \geq 2n_0 + 2$.) Since the last two terms summations above are nonnegative,

$$s_n \geq z_n - \sum_{i=n_0+1}^n \frac{o(i)}{\lambda^i} z_{n-i} = z_n \left(1 - \sum_{i=n_0+1}^n \frac{o(i)z_{n-i}}{\lambda^i z_n} \right).$$

From estimates on z_n and $o(n)$, the difference between the summation of the terms $\sum_{i=n_0+1}^n o(i)z_{n-i}/\lambda^i z_n$ and $\sum_{i=n_0+1}^n 1/i$ is at most a constant times $1/n_0$.

Choose n_0 large enough so that

$$\sum_{i=n_0+1}^n \frac{1}{i} < \log\left(\frac{n+1}{n_0+1}\right) \leq \log\left(\frac{2n_0 + \log^2 n_0 + 1}{n_0 + 1}\right) < \log(2.1)$$

and

$$\sum_{i=n_0+1}^n \frac{o(i)z_{n-i}}{\lambda^i z_n} < \log(2.2).$$

Then $s_n > z_n(1 - \log(2.2))$.

The result follows since we may choose n_0 so that all scaled coefficients z_n are uniformly bounded away from zero for $n > n_0$. □

We are now ready to give bounds on $s_n/s_{n+1} - 1$ for n in this initial range. Recall that this quantity will represent D when we apply the Difference Argument.

Lemma 4.2. *There exist constants $0 < k_1 < k_2$ such that if n_0 is sufficiently large, then*

$$1 + \frac{k_1}{n_0} < \frac{s_n}{s_{n+1}} < 1 + \frac{k_2}{n_0} \quad \text{for } n \in [n_0 + \log^2 n_0, 2n_0 + \log^2 n_0].$$

Proof. We will rewrite the ratio s_n/s_{n+1} as $1 + (s_n - s_{n+1})/s_{n+1}$ and examine the difference $s_n - s_{n+1}$.

First, assume $n \in [n_0 + \log^2 n_0, 2n_0]$. Then

$$s_n - s_{n+1} = z_n - z_{n+1} - \sum_{i=n_0+1}^n \frac{o(i)}{\lambda^i} z_{n-i} + \sum_{i=n_0+1}^{n+1} \frac{o(i)}{\lambda^i} z_{n+1-i}.$$

Let $n_1 = \lfloor \log^2 n_0 \rfloor$ and rewrite the two summations above as the sum of the following three terms:

$$- \sum_{i=n_0+1}^{n-n_1} \frac{o(i)}{\lambda^i} (z_{n-i} - z_{n+1-i}) - \sum_{i=n-n_1+1}^n \left(\frac{o(i)}{\lambda^i} - \frac{o(i+1)}{\lambda^{i+1}} \right) z_{n-i} + \frac{o(n-n_1)}{\lambda^{n-n_1}} z_{n_1}.$$

By estimating each term, we will show that the last term dominates.

In the first summation, all terms of the form $o(i)\lambda^{-i}$ are less than a universal constant over n_0 . The differences between consecutive scaled coefficients $|z_k - z_{k+1}|$ form a summable series in k . Since the indices which occur in $|z_{n-i} - z_{n+1-i}|$ are greater than n_1 , by selecting n_0 sufficiently large, we can make the sum as small as we like compared to $1/n_0$.

In the second summation, z_{n-i} is bounded above by a constant and the difference $|o(i)\lambda^{-i} - o(i+1)\lambda^{-(i+1)}|$ is less than a constant over n_0^2 . Since there are at most $\log^2 n_0$ terms in the sum, the sum is as small as we like compared to $1/n_0$.

In the last term, for n_0 sufficiently large, z_{n_1} is close to a positive constant, and $o(n-n_1)\lambda^{n_1-n}$ is approximately $1/(n - \log^2 n_0)$. We may select constants $0 < c_1 < c_2$ such that for $n \in [n_0 + \log^2 n_0, 2n_0]$,

$$\frac{c_1}{n_0} < \frac{o(n-n_1)}{\lambda^{n-n_1}} z_{n_1} < \frac{c_2}{n_0}.$$

All other terms in $s_n - s_{n+1}$ have upper bounds which are small compared to $1/n_0$. Therefore if n_0 is sufficiently large,

$$\frac{c_1}{2n_0} < s_n - s_{n+1} < \frac{2c_2}{n_0}.$$

Finally, by Lemma 4.1 we may assume that s_{n+1} is bounded away from zero. Thus we may choose constants $0 < k_1 < k_2$ such that

$$1 + \frac{k_1}{n_0} < 1 + \frac{s_n - s_{n+1}}{s_{n+1}} < 1 + \frac{k_2}{n_0}.$$

This completes the proof in the case $n \in [n_0 + \log^2 n_0, 2n_0]$.

When $n \in (2n_0, 2n_0 + \log^2 n_0]$, we have

$$s_n = z_n - \sum_{i=n_0+1}^n \frac{o(i)}{\lambda^i} z_{n-i} + \sum_{\substack{i,j>n_0 \\ i+j \leq n}} \frac{o(i)o(j)}{\lambda^{i+j}} z_{n-i-j} + \sum_{i=n_0+1}^{n/2} \lambda^{-2i} \binom{o(i)}{2} z_{n-2i}.$$

In the summation $\sum \lambda^{-(i+j)} o(i)o(j) z_{n-i-j}$ there are at most $\log^4 n_0$ terms. In $\sum \lambda^{-2i} \binom{o(i)}{2} z_{n-2i}$, there are at most $\log^2 n_0$ terms. In both, each summand is at most a constant over n_0^2 . For sufficiently large n_0 , these terms are small compared to our dominating term. By adjusting our constants a bit, the result still follows. \square

4.2. An approximation of $S(t)$. To get further results for ratios s_n/s_{n+1} , we work with a power series $E(t)$ which approximates $S(t)$ and has coefficients which satisfy a useful recursion relation.

First notice

$$\begin{aligned} \log S(t) &= \log \left(\prod_{i=1}^{n_0} (1 - t^i)^{-o(i)} \right) \\ &= - \sum_{i=1}^{n_0} o(i) \log(1 - t^i) \\ &= \sum_{i=1}^{n_0} \frac{\text{tr}(\Lambda^i)}{i} t^i + \sum_{n=n_0+1}^{\infty} \sum_{\substack{i|n \\ i \leq n_0}} \frac{io(i)}{n} t^n. \end{aligned}$$

We will define $E(t)$ as the exponential of the first of the two summations above. As we will show, the exponential of the second summation has relatively small coefficients.

Letting $\tau_n = \lambda^{-n} \text{tr}(\Lambda^n)$,

$$E(t) = \sum_{n=0}^{\infty} e_n (\lambda t)^n = \exp \left(\sum_{i=1}^{n_0} \frac{\tau_i}{i} (\lambda t)^i \right).$$

The recursion formula below follows from $\frac{d}{dt} E(t) = E(t) \sum_{i=1}^{n_0} \tau_i \lambda^i t^{i-1}$:

$$(4.2) \quad ne_n = \sum_{i=1}^{n_0} \tau_i e_{n-i} \quad \text{for } n > n_0.$$

Throughout the rest of the paper, we will use the fact that $|\tau_i - 1|$ is less than an exponentially small function of i .

In the following lemma, we show that e_n approximates s_n in an initial range. We will then use the recursion formula to produce upper and lower bounds on ratios

e_n/e_{n+1} . This allows us to show that e_n approximates s_n for values of n up to some exponential function of n_0 .

Lemma 4.3. *There exists a constant $K_2 > 0$ such that if n_0 is sufficiently large, then*

$$\left| \frac{s_n}{e_n} - 1 \right| < K_2 \lambda^{-n_0/3} \quad \text{for } n \in (n_0, 2n_0 + \log^2 n_0].$$

Proof. Let

$$L(t) = \log S(t) - \log E(t) = \sum_{n=n_0+1}^{\infty} \sum_{\substack{i|n \\ i \leq n_0}} \frac{io(i)}{n} t^n.$$

The coefficients of $L(t)$ are all nonnegative and are zero in degrees one through n_0 . For n_0 sufficiently large and $n < n_0$,

$$L_n = \sum_{\substack{i|n \\ i \leq n_0}} \frac{io(i)}{n} < \lambda^{n/2}.$$

This follows since the indices i in the above summation are less than $n/2$ so we may assume $io(i) < 2\lambda^{n/2}$. Thus we have at most $n/2$ terms in the summation all less than $2\lambda^{n/2}/n$.

Let $E'(t) = \sum E'_n t^n = \exp L(t)$. By the above inequality, E'_n is always less than the n th coefficient of the series $\exp \sum_{i=1}^{\infty} \lambda^{i/2} t^i$. This series has radius of convergence $\lambda^{-1/2}$. Therefore, there exists a constant $C > 0$ such that $E'_n < C\lambda^{2n/3}$ for all n .

Since $E(t) = S(t) \exp(-L(t))$, we have

$$e_n = s_n \left(1 + \sum_{i=n_0+1}^n \frac{s_{n-i}}{s_n} \frac{[\exp(-L(t))]_i}{\lambda^i} \right).$$

The absolute value of the i th coefficient of $\exp(-L(t))$ is at most E'_i which is bounded above by $C\lambda^{2i/3}$. The ratios s_{n-i}/s_n are uniformly bounded in this region by Lemma 4.1 and the fact that the scaled coefficients have a uniform upper bound (the power series $S(t)$ has radius of convergence $1 > 1/\lambda$). Altogether this gives a constant $K_2 > 0$ such that $|e_n/s_n - 1| < K_2 \lambda^{-n_0/3}$ for $n \in (n_0, 2n_0 + \log^2 n_0]$. \square

The next step is to obtain upper and lower bounds on ratios of consecutive coefficients in $E(t)$. The key will be to use the recursion formula to consider the ratio e_n/e_{n+1} as $(n+1)/n$ times a weighted sum of the previous n_0 ratios. In other words, we write

$$\begin{aligned} \frac{e_n}{e_{n+1}} &= \left(\frac{n+1}{n} \right) \frac{\sum_{i=1}^{n_0} \tau_i e_{n-i}}{(n+1)e_{n+1}} \\ &= \left(\frac{n+1}{n} \right) \sum_{i=1}^{n_0} \left(\frac{\tau_i e_{n+1-i}}{(n+1)e_{n+1}} \right) \frac{e_{n-i}}{e_{n+1-i}}. \end{aligned}$$

For fixed n , if we let $w_i = \tau_i e_{n+1-i}/(n+1)e_{n+1}$, we have $\sum_{i=1}^{n_0} w_i = 1$.

We will use Lemma 4.3 and results of Section 4.1 to establish bounds on ratios of the form e_n/e_{n+1} for $n \in [n_0 + \log^2 n_0, 2n_0 + \log^2 n_0]$. Then working by induction, we will use the recursion formula to establish upper and lower bounds for e_n/e_{n+1} for $n \in [n_0 + \log^2 n_0, e^{rn_0}]$ where $r > 0$ is some constant.

Lemma 4.4. *There exists a constant $k_3 > 0$ such that if n_0 is sufficiently large, then*

$$\frac{e_n}{e_{n+1}} > 1 + \frac{k_3}{n_0} \quad \text{for } n \in [n_0 + \log^2 n_0, \infty).$$

Proof. Suppose n_0 is large enough for all previous lemmas and assume $n \in [n_0 + \log^2 n_0, 2n_0 + \log^2 n_0]$. By Lemma 4.2, $s_n/s_{n+1} < 1 + k_1/n_0$. By Lemma 4.3, the difference between e_n/e_{n+1} and s_n/s_{n+1} is small compared to $1/n_0$. Therefore, there is a constant $k_3 > 0$ such that for n_0 sufficiently large, $e_n/e_{n+1} > 1 + k_3/n_0$ for n in this initial range.

Now fix $n > 2n_0 + \log^2 n_0$ and assume $e_{n-i}/e_{n+1-i} > 1 + k_3/n_0$ for $1 \leq i \leq n_0$. Then by the recursion formula,

$$\frac{e_n}{e_{n+1}} = \left(\frac{n+1}{n}\right) \sum_{i=1}^{n_0} w_i \frac{e_{n-i}}{e_{n+1-i}} > \sum_{i=1}^{n_0} w_i \left(1 + \frac{k_3}{n_0}\right) = 1 + \frac{k_3}{n_0}.$$

□

It is somewhat more complicated to obtain an upper bound on the ratios e_n/e_{n+1} because of the factor $(n+1)/n$ in front of the weighted sum. Our upper bound will have to grow with n .

The following lemma is the first which contains the choice of the value of r . It is important to note that in this and subsequent lemmas, r is always specified before n_0 and any smaller value of r would also work.

Lemma 4.5. *For some $r > 0$, if n_0 is sufficiently large, then*

$$\frac{e_n}{e_{n+1}} < 1 + \frac{4 \log n}{n_0} \quad \text{for } n \in [n_0 + \log^2 n_0, e^{rn_0}].$$

Proof. Select $r \in (0, 1/12)$.

Reasoning in a manner similar to the previous proof, we may assume that there is a constant $k_4 > 0$ such that $e_n/e_{n+1} < 1 + k_4/n_0$ for $n \in [n_0 + \log^2 n_0, 2n_0 + \log^2 n_0]$. Therefore, for n_0 sufficiently large, we can insure that e_n/e_{n+1} is no more than $1 + 4 \log n/n_0$ for n in this initial range.

Now assume that $n \in (2n_0 + \log^2 n_0, e^{rn_0})$ and for $1 \leq i \leq n_0$, $e_{n-i}/e_{n+1-i} < 1 + 4 \log(n-i)/n_0$. Then by the recursion formula,

$$\begin{aligned} \frac{e_n}{e_{n+1}} &= \left(\frac{n+1}{n}\right) \sum_{i=1}^{n_0} w_i \frac{e_{n-i}}{e_{n+1-i}} \\ &< \left(\frac{n+1}{n}\right) \sum_{i=1}^{n_0} w_i \left(1 + \frac{4 \log(n-i)}{n_0}\right) \\ &< \left(\frac{n+1}{n}\right) \left(1 + \frac{4}{n_0} \sum_{i=1}^{n_0} w_i \log(n-i)\right). \end{aligned}$$

Using facts about logarithms, we may write

$$\sum_{i=1}^{n_0} w_i \log(n-i) < \log \left(\sum_{i=1}^{n_0} w_i (n-i) \right) = \log \left(n - \sum_{i=1}^{n_0} w_i i \right).$$

We will later show that $\sum_{i=1}^{n_0} w_i i > n_0/3$. Assuming this, the result is proven as follows. First we observe

$$\begin{aligned} \frac{e_n}{e_{n+1}} &< \left(\frac{n+1}{n}\right) \left(1 + \frac{4}{n_0} \log\left(n - \frac{n_0}{3}\right)\right) \\ &= \left(1 + \frac{1}{n}\right) \left(1 + \frac{4 \log n}{n_0} + \frac{4}{n_0} \log\left(1 - \frac{n_0}{3n}\right)\right). \end{aligned}$$

We want to show that the above estimate is less than $1 + 4 \log n/n_0$, which means we want to show

$$\frac{1}{n} + \frac{4 \log n}{nn_0} + \left(1 + \frac{1}{n}\right) \frac{4}{n_0} \log\left(1 - \frac{n_0}{3n}\right) < 0.$$

The above inequality follows since

$$\left(1 + \frac{1}{n}\right) \frac{4}{n_0} \log\left(1 - \frac{n_0}{3n}\right) < -\frac{4}{3n}$$

and $n < e^{rn_0} < e^{n_0/12}$ implies

$$\frac{4 \log n}{nn_0} < \frac{1}{3n}.$$

It remains to show $\sum w_i i > n_0/3$. We first note that if n_0 is sufficiently large, the weights w_i increase for $i > \log^2 n_0$. This follows since we have a lower bound on τ_{i+1}/τ_i of the form $1 - Ce^{-ci}$ for some constants $C, c > 0$ and a lower bound on $e_{n-i}/e_{n+1-i} > 1 + k_3/n_0$ from Lemma 4.4.

Let $n_1 = \lfloor \log^2 n_0 \rfloor$ and let W denote the sum of the first n_1 weights $W = \sum_{i=1}^{n_1} w_i$. By choosing n_0 sufficiently large, we can make W as small as we like. This follows since W is the sum of n_1 terms, all of which are no more than some universal constant times the last term w_{n_1} (recall $w_i = \tau_i e_{n-i}$, e_{n-i} increases with i , τ_i is uniformly bounded above). On the other hand, $1 - W$ is the sum of $n_0 - n_1$ terms all of which are greater than the weight w_{n-1} .

To continue, we note

$$\frac{1}{1 - W} \sum_{i=n_1+1}^{n_0} w_i i > \frac{1}{n_0 - n_1} \sum_{i=n_1+1}^{n_0} i.$$

This follows since in the summation on the left we have a weighted sum where increasing weights are multiplied by increasing terms, whereas the summation on the right has an even amount of weight on each term (this idea is made precise in Lemma A.1).

Now we have

$$\sum_{i=1}^{n_0} w_i i > \frac{1 - W}{n_0 - n_1} \sum_{i=n_1+1}^{n_0} i = \frac{1 - W}{n_0 - n_1} \left(\frac{n_0^2 + n_0}{2} - \frac{n_1^2 + n_1}{2}\right).$$

By selecting n_0 sufficiently large, we can force the right-hand side to be as close as we like to $n_0/2$. In particular, we can force it to be at least $n_0/3$, our desired lower bound. □

Our estimates on e_n now allow us to show that the e_n 's approximate s_n for $n \in (2n_0 + \log^2 n_0, e^{rn_0})$.

Lemma 4.6. *There exists a constant $K_3 > 0$ such that if n_0 is sufficiently large, then*

$$\left| \frac{s_n}{e_n} - 1 \right| < K_3 \lambda^{-n_0/6} \quad \text{for } n \in [n_0 + \log^2 n_0, e^{rn_0}].$$

Proof. Assume $(1 + 4r) < \lambda^{1/6}$ and r is small enough for Lemma 4.5.

Recall from the proof of Lemma 4.3 that the power series $E'(t)$ was defined by the relation $S(t) = E(t)E'(t)$ and there is a constant $C > 0$ such that $E'_i < C\lambda^{2i/3}$ for all i . We have the following:

$$(4.3) \quad s_n = e_n \left(1 + \sum_{i=n_0+1}^n \frac{e_{n-i} E'_i}{e_n \lambda^i} \right) < e_n \left(1 + C \sum_{i=n_0+1}^n \frac{e_{n-i} \lambda^{-i/3}}{e_n} \right).$$

Assume that n_0 is large enough to give bounds from all previous lemmas and assume $n \in [n_0 + \log^2 n_0, e^{rn_0}]$. For estimates on the terms e_{n-i}/e_n , we have two cases.

If $n_0 + \log^2 n_0 \leq n - i < n$, then by Lemma 4.5,

$$\frac{e_{n-i}}{e_n} < \left(1 + \frac{4 \log n}{n_0} \right)^i < (1 + 4r)^i < \lambda^{i/6}.$$

Now assume $n - i < n_0 + \log^2 n_0$. Then either $n - i \leq n_0$ in which case $e_{n-i} = s_{n-i}$ or $n - i \in (n_0, n_0 + \log^2 n_0]$ in which case Lemma 4.3 tells us that e_{n-i} is approximately equal to s_{n-i} . Lemma 4.1 implies that there is a constant $C' > 0$ such that $s_k < C' s_{[n_0 + \log^2 n_0]}$ for all $k < n_0 + \log^2 n_0$. Therefore, $e_{n-i}/e_n < 2C' \lambda^{i/6}$ for $n - i < n_0 + \log^2 n_0$.

Substituting our estimates in equation (4.3), we can find a constant $K_3 > 0$ such that

$$\left| \frac{s_n}{e_n} - 1 \right| < K_3 \lambda^{-n_0/6}.$$

□

We have now established bounds on ratios e_n/e_{n+1} and shown that we introduce a small error by replacing s_n/s_{n+1} with e_n/e_{n+1} . We are ready to prove the main lemma of this section. Define

$$R(t) = \sum_{n=0}^{\infty} R_n t^n = (1 - p(t))S(t).$$

Polynomial Range Lemma. *There exists a constant $K_4 > 0$ such that if n_0 is sufficiently large, then*

$$R_n < -K_4 S_n/n_0 \quad \text{for } n \in (n_0, n_0^{20}).$$

Proof. First assume that $n \in (n_0, 2n_0 + 2 \log^2 n_0]$. From the equation

$$R(t) = \prod_{i=n_0+1}^{\infty} (1 - t^i)^{o(i)}$$

we have

$$R_n = -o(n) + \sum_{\substack{i+j=n \\ i,j > n_0}} o(i)o(j) + \binom{o(n/2)}{2}.$$

(The last two terms occur only if $n \geq 2n_0 + 2$, the last term occurs only if n is even.)

We use estimates similar to those in Lemma 4.1 to show this expression is negative. For sufficiently large n_0 , $o(n) > \lambda^n/2n > \lambda^n/6n_0$. The second term $\sum o(i)o(j)$ is the sum of no more than $\log^2 n_0$ terms, each of which is less than a constant times λ^n/n_0^2 . The final term $\binom{o(n/2)}{2}$ is less than λ^n/n_0^2 .

It also follows from the above argument that for n_0 sufficiently large, R_n is less than a negative constant times λ^n/n_0 in this range. Since the coefficients $s_n = \lambda^{-n}S_n$ are uniformly bounded, we obtain the desired upper bound of a negative constant times S_n/n_0 .

Now fix $n \in (2n_0 + 2 \log^2 n_0, n_0^{20})$. It follows from Lemma 4.6 that the absolute value of the difference between $1 - \sum_{i=1}^d p_i \lambda^{-i} s_{n-i}/s_n$ and $1 - \sum_{i=1}^d p_i \lambda^{-i} e_{n-i}/e_n$ is less than a constant times $\lambda^{-n_0/6}$. In particular, this difference is small compared to $1/n_0$ for large n_0 . It remains to show the expression $1 - \sum_{i=1}^d p_i \lambda^{-i} e_{n-i}/e_n$ is less than a negative constant over n_0 .

Let D_i denote the difference $e_{n-i}/e_n - e_{n+1-i}/e_n$ for $1 \leq i \leq n_0$ and notice

$$\frac{e_{n-i}}{e_n} = 1 + \sum_{j=1}^i D_j = 1 + iD_1 + \underbrace{\sum_{j=1}^i (i-j)(D_{j+1} - D_j)}_{\text{ith error term}}$$

Lemma 4.4 implies $D_1 > k_3/n_0$. Therefore, following the Difference Argument, it suffices to show that the i th error term is small compared to $1/n_0$ for $i = 1, 2, \dots, d$.

We will work with the second order difference $D_2 - D_1$. An upper bound on second order differences $D_{j+1} - D_j$ for $j = 1, 2, \dots, d$ follows similarly. Note that we may rewrite $D_2 - D_1$ in the following way:

$$\begin{aligned} D_2 - D_1 &= \left(\frac{e_{n-2}}{e_n} - \frac{e_{n-1}}{e_n} \right) - \left(\frac{e_{n-1}}{e_n} - 1 \right) \\ &= \left(\frac{(n-2)e_{n-2}}{ne_n} - \frac{(n-1)e_{n-1}}{ne_n} \right) - \left(\frac{(n-1)e_{n-1}}{ne_n} - \frac{ne_n}{ne_n} \right) \\ &\quad + \frac{2}{n} \left(\frac{e_{n-2}}{e_n} - \frac{e_{n-1}}{e_n} \right). \end{aligned}$$

Lemma 4.5 implies the last term above has the following upper bound:

$$\begin{aligned} \frac{2}{n} \left(\frac{e_{n-2}}{e_n} - \frac{e_{n-1}}{e_n} \right) &= \frac{2e_{n-1}}{ne_n} \left(\frac{e_{n-2}}{e_{n-1}} - 1 \right) \\ &< \frac{2}{n} \left(1 + \frac{4 \log n}{n_0} \right) \left(\frac{4 \log n}{n_0} \right). \end{aligned}$$

Since $n \in (n_0, n_0^{20})$, for sufficiently large n_0 this upper bound is less than a constant times $\log n_0/n_0^2$.

We use the recursion formula for coefficients e_n to rewrite the remaining terms as follows:

$$\begin{aligned} \frac{1}{ne_n} \sum_{i=1}^{n_0} \tau_i (e_{n-2-i} - e_{n-1-i}) - \tau_i (e_{n-1-i} - e_n) \\ = \frac{1}{n} \sum_{i=1}^{n_0} \tau_i (D_{i+2} - D_{i+1}) \\ = \frac{\tau_{n_0} D_{n_0+2}}{n} + \frac{1}{n} \sum_{i=2}^{n_0} (\tau_{i-1} - \tau_i) D_{i+1} - \frac{\tau_1 D_2}{n}. \end{aligned}$$

The first term $\tau_{n_0} D_{n_0+2}/n$ is the most difficult to get an upper bound on. It is a first order difference $(e_{n-n_0-2}/e_{n-n_0-1} - 1)$ times a factor $\tau_{n_0} e_{n-n_0-1}/ne_n$ for which we do not yet have a sufficient upper bound. To show that this factor is small, we consider the ratio $2e_{n-n_0}/ne_n$, which is greater than $\tau_{n_0} e_{n-n_0-1}/ne_n$ for n_0 sufficiently large. Using the recursion formula,

$$\frac{e_{n-n_0}}{ne_n} = \frac{e_{n-n_0}}{\sum_{i=1}^{n_0} \tau_i e_{n-i}} = \frac{1}{\sum_{i=1}^{n_0} \tau_i e_{n-i}/e_{n-n_0}}.$$

By Lemma 4.5, we have $e_{n-i}/e_{n-n_0} > (1 + 4 \log n/n_0)^{i-n_0}$. Therefore,

$$\begin{aligned} \sum_{i=1}^{n_0} \tau_i e_{n-i}/e_{n-n_0} &> \sum_{i=n_0/2}^{n_0} \tau_i (1 + 4 \log n/n_0)^{i-n_0} \\ &> \frac{1}{2} \sum_{i=n_0/2}^{n_0} (1 + 4 \log n/n_0)^{i-n_0} \\ &\geq \frac{1 - (1 + 4 \log n/n_0)^{-n_0/2}}{8 \log n/n_0}. \end{aligned}$$

For n_0 sufficiently large and for $n > n_0$, the numerator of this last expression is at least $1/2$ so the entire expression is at least $n_0/16 \log n$. Combining all of the bounds, we have

$$\frac{\tau_{n_0} D_{n_0+2}}{n} < \frac{32 \log n}{n_0} \left(\frac{e_{n-n_0-2}}{e_{n-n_0-1}} - 1 \right) < \frac{128 \log^2 n}{n_0^2}.$$

For $n \in (n_0, n_0^{20})$, this is less than a constant times $\log^2 n_0/n_0^2$.

For the second term, we have $(1/n) \sum_{i=2}^{n_0} (\tau_{i+1} - \tau_i) D_{i+1}$. Recall the differences $|\tau_{i+1} - \tau_i|$ are less than a Ce^{-ci} for some constants $C, c > 0$. For D_{i+1} , we have an upper bound of the form

$$D_{i+1} = \frac{e_{n-i}}{e_n} \left(\frac{e_{n+1-i}}{e_{n-i}} - 1 \right) < \left(1 + \frac{4 \log n}{n_0} \right)^i \left(\frac{4 \log n}{n_0} \right).$$

Since $n < n_0^{20}$, by selecting n_0 sufficiently large, we can guarantee $1 + 4 \log n/n_0 < e^{-c/2}$. This implies $(1/n) \sum_{i=2}^{n_0} (\tau_{i+1} - \tau_i) D_{i+1}$ has an upper bound of the form constant times $\log n/n n_0 < 20 \log n_0/n_0^2$.

For the third term, we have

$$\frac{\tau_1 D_2}{n} < \tau_1 \left(1 + \frac{4 \log n}{n_0} \right) \frac{4 \log n}{n n_0}.$$

Again, for $n \in (n_0, n_0^{20})$, this term has an upper bound of the form constant times $\log n_0/n_0^2$.

Therefore, $D_2 - D_1$ can be written as the sum of terms, each of which is small compared to $1/n_0$ for large n_0 . Therefore the result follows by the Difference Argument. \square

5. POLYNOMIAL TO EXPONENTIAL RANGE

In this section, we follow the Ratio Argument to show that for some $r > 0$, if n_0 is sufficiently large and $n \in [n_0^{20}, e^{rn_0})$, then the n th coefficient of the power series $S(t)(1-p(t))$ is bounded above by S_n times a negative constant over n_0 (Polynomial to Exponential Range Lemma). Specifically, by Lemma 4.6, we have $s_{n-i}/s_n \approx e_{n-i}/e_n$. In this section we will show that $e_{n-i}/e_n \approx \alpha_n^{-i}$ where $1 - p(1/\alpha_n\lambda) < 0$. Then the n th coefficient of $S(t)(1 - p(t))$ is approximately S_n times

$$1 - \sum_{i=1}^d \frac{p_i}{\lambda^{-i}} \frac{e_{n-i}}{e_n} \approx 1 - \sum_{i=1}^d p_i(\alpha_n\lambda)^{-i} = 1 - p(1/\alpha_n\lambda) < 0.$$

In order to prove that ratios e_{n-i}/e_n approximate powers of α_n , we make further use of the recursion formula for e_n . However, in this section we think of this relation in the following matrix form:

$$\begin{pmatrix} e_n \\ e_{n-1} \\ e_{n-2} \\ \vdots \\ e_{n-n_0+1} \end{pmatrix} = \begin{pmatrix} \tau_1/n & \tau_2/n & \tau_3/n & \cdots & \tau_{n_0}/n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_{n-1} \\ e_{n-2} \\ e_{n-3} \\ \vdots \\ e_{n-n_0} \end{pmatrix}.$$

Letting A_n be the above matrix, we will get estimates on ratios e_{n-1}/e_n for $n \geq n_0^{20}$ by considering ratios of entries of $A_n A_{n-1} \cdots A_{n-N+1} \mathbf{v}$ where \mathbf{v} is an arbitrary vector with positive entries. Since

$$\chi_n(t) = \det(It - A_n) = t^{n_0} - \tau_1 t^{n_0-1}/n - \cdots - \tau_{n_0}/n,$$

there is a unique positive real root α_n of $\chi_n(t)$ and the corresponding eigenvector of A_n is $(\alpha_n^{n_0-1}, \alpha_n^{n_0-2}, \dots, 1)^T$. Our goal is to show that e_{n-1}/e_n is approximately equal to this eigenvalue α_n .

We begin by obtaining bounds on α_n and upper bounds on the other eigenvalues of A_n . Using these, we will see that for $N = n_0^6$, the vector $(A_n)^N \mathbf{v}$ is very close to a vector in the the eigenspace for α_n , so ratios of consecutive entries approximate α_n . Then we will need to see that applying the sequence of matrices $A_n A_{n-1} \cdots A_{n-N+1}$ to a vector is approximately the same as applying the N th power of the single matrix A_n .

Lemma 5.1. *If n_0 is sufficiently large and $n > 2n_0$, then*

$$\left(\frac{1}{2n}\right)^{1/n_0} < \alpha_n < \left(\frac{2n_0}{n}\right)^{1/n_0}.$$

Proof. For the lower bound on α_n , we only need to select n_0 large enough so that $\tau_{n_0} > 1/2$. Then

$$\alpha_n^{n_0} = \frac{1}{n} \sum_{i=1}^{n_0} \tau_i \alpha_n^{n_0-i} > \frac{\tau_{n_0}}{n} > \frac{1}{2n}.$$

For the upper bound on α_n , we need n_0 to be large enough so that $\sum_{i=1}^{n_0} \tau_i/2n_0 < 1$. Then since $\sum_{i=1}^{n_0} \tau_i \alpha_n^{-i}/n = 1$ and $\sum_{i=1}^{n_0} \tau_i/n < 1$, we know $\alpha_n < 1$. But since $\alpha_n < 1$,

$$\alpha_n^{n_0} = \frac{1}{n} \sum_{i=1}^{n_0} \tau_i \alpha_n^{n_0-i} < \frac{1}{n} \sum_{i=1}^{n_0} \tau_i < \frac{2n_0}{n}.$$

□

Next, we would like an upper bound on the modulus of the other roots of the characteristic polynomial for A_n . We will prove that if β_n is the root of $\chi_n(t)$ of the second largest modulus, then $|\beta_n|/\alpha_n < 1 - 1/n_0^4$. To begin we give a lower bound on the distance between the Perron root and any other root of a polynomial which has the same sign pattern as $\chi_n(t)$.

Lemma 5.2. *Let $f(t) = t^d - \sum_{i=1}^d f_i t^{d-i}$ be a polynomial with $d > 2$ and $f_i > 0$ for all i . Let $\alpha > 0$ be the unique positive real root of $f(t)$. If $\beta \neq \alpha$ is a complex number such that $f(\beta) = 0$, then*

$$|\alpha - \beta| > \alpha/d.$$

Proof. Let $g(t) = \sum_{i=1}^d g_i t^{d-i}$ be the polynomial $g(t) = f(t)/(t - \alpha)$. Equating coefficients from $(t - \alpha)g(t)$ and $f(t)$, we see $g_d \alpha = f_d > 0$ and $\alpha g_i = f_i + g_{i+1} > 0$ for $1 \leq i < d$.

Now let $\beta \neq \alpha$ be a complex number with $f(\beta) = 0$. Then

$$(5.1) \quad \left| \frac{g(\alpha) - g(\beta)}{\alpha - \beta} \right| \leq \frac{g(\alpha) - g(|\beta|)}{\alpha - |\beta|}.$$

The above follows since all of the coefficients of $g(t)$ are positive and $|\alpha^i - \beta^i|/|\alpha - \beta| \leq (\alpha^i - |\beta|^i)/(\alpha - |\beta|)$. By the Mean Value Theorem in \mathbb{R} ,

$$(5.2) \quad \frac{g(\alpha) - g(|\beta|)}{\alpha - |\beta|} = g'(x)$$

for some $x \in (|\beta|, \alpha)$. Since $g(t)$ has all positive coefficients and $d > 2$, the function $g'(t)$ is strictly increasing and $g'(x) < g'(\alpha)$.

Combining equations (5.1) and (5.2), and using $g(\beta) = 0$, $g(\alpha) = f'(\alpha)$ and $g'(\alpha) = f''(\alpha)/2$, we have the following inequality:

$$|\alpha - \beta| > \frac{2f'(\alpha)}{f''(\alpha)}.$$

To prove the claim, we need a lower bound for $2f'(\alpha)/f''(\alpha)$. Since $f(\alpha) = 0$,

$$f'(\alpha) = d\alpha^{d-1} - \sum_{i=1}^{d-1} (d-i)f_i \alpha^{d-i-1} = \sum_{i=1}^d i f_i \alpha^{d-i-1}.$$

Similarly,

$$\begin{aligned} f''(\alpha) &= d(d-1)\alpha^{d-2} - \sum_{i=1}^d (d-i)(d-i-1)f_i \alpha^{d-i-2} \\ &= \sum_{i=1}^d [2id - i^2 - i] f_i \alpha^{d-i-2}. \end{aligned}$$

Multiplying our expression for $f'(\alpha)$ by $2d$ and our expression for $f''(\alpha)$ by α , we see $2df'(\alpha) > \alpha f''(\alpha)$. Therefore,

$$|\alpha - \beta| > 2f'(\alpha)/f''(\alpha) > \alpha/d.$$

□

The above lemma gives a lower bound on the distance between α_n and a second root β_n of $\chi_n(t)$. To get an upper bound on the modulus of β_n , we write

$$\left| \frac{\beta_n}{\alpha_n} \right|^{n_0} = \left| \frac{\sum_{i=1}^d \tau_i \beta_n^{d-i}}{\sum_{i=1}^d \tau_i \alpha_n^{d-i}} \right|$$

and get an upper bound on the right-hand side of the form one minus a constant over n_0^2 . This bound follows since if we assume $|\beta_n|$ is close to α_n , then by the previous lemma the angle β_n makes with the positive real axis cannot be too small. Since $\tau_i \approx 1$ for large i , this forces enough cancellation in the sum $\sum_{i=1}^d \tau_i \beta_n^{d-i}$ to obtain the upper bound. We give an argument in a more general setting since it will be used again later in another context.

Lemma 5.3. *Let f_1, f_2, \dots, f_d be a finite sequence with $d > 2$ an even number and $f_i > 0$ for all i . Let $\alpha > 0$ be a real number and β a complex number with $|\beta| < \alpha$. Then*

$$\left| \frac{\sum_{i=1}^d f_i \beta^{d-i}}{\sum_{i=1}^d f_i \alpha^{d-i}} \right| < 1 - \frac{m|\alpha - \beta|^2}{2\alpha(\alpha + M)^2}$$

where $m = \min f_{i+1}/f_i$ and $M = \max f_{i+1}/f_i$.

Proof. First, let $x > 0$ be a real number. We claim

$$|\alpha + x|^2 - x|\alpha - \beta|^2/\alpha < |\beta + x|^2.$$

Let θ be the angle that β makes with the positive real axis. Then

$$|\beta + x|^2 = |\beta|^2 + x^2 + 2x|\beta| \cos \theta.$$

Now consider

$$\begin{aligned} (\alpha + x)^2 - x|\alpha - \beta|^2/\alpha &= \alpha^2 + x^2 + 2x\alpha - x\alpha - x|\beta|^2/\alpha + 2x|\beta| \cos \theta \\ &> \alpha^2 + x^2 + 2x|\beta| \cos \theta \\ &> |\beta + x|^2. \end{aligned}$$

This proves the claim.

Now we have for any $x > 0$,

$$(5.3) \quad \left| \frac{\beta + x}{\alpha + x} \right| < \sqrt{1 - \frac{x|\alpha - \beta|^2}{\alpha(\alpha + x)^2}} < 1 - \frac{x|\alpha - \beta|^2}{2\alpha(\alpha + x)^2}.$$

To finish the proof we write

$$\begin{aligned} \left| \sum_{i=1}^d f_i \beta^{d-i} \right| &\leq \sum_{i=1}^{d/2} |f_{2i-1} \beta^{d-2i+1} + f_{2i} \beta^{d-2i}| \\ &< \sum_{i=1}^{d/2} f_{2i-1} \alpha^{d-2i} \left| \beta + \frac{f_{2i}}{f_{2i-1}} \right|. \end{aligned}$$

We can also write $\left| \sum_{i=1}^d f_i \alpha^{d-i} \right| = \sum_{i=1}^{d/2} f_{2i-1} \alpha^{d-2i} |\alpha + f_{2i}/f_{2i-1}|$. Using inequality (5.3) with $r = f_{2i}/f_{2i-1}$, our upper bound follows. \square

We cannot directly apply the previous lemma to $\chi_n(t)$ since the coefficients τ_i may be zero when i is small. However, we will be able to use the approximation $\tau_i \approx 1$ and continue.

Lemma 5.4. *If n_0 is sufficiently large and $\beta_n \neq \alpha_n$ is a root of $\chi_n(t) = t^{n_0} - \sum_{i=1}^{n_0} \tau_i t^{n_0-i}/n$, then*

$$|\beta_n| < (1 - 1/n_0^4)\alpha_n \quad \text{for } n \in (n_0^4, e^{rn_0}).$$

Proof. Recall, there exist constants $C, c > 0$ such that $|\tau_n - 1| < Ce^{-cn}$ for all n . Fix $r \in (0, c/2)$ small enough for all previous lemmas. Fix $n \in (n_0^4, e^{rn_0})$. Let $\alpha = \alpha_n$ and $\beta = \beta_n$.

First, we write

$$(5.4) \quad \left| \frac{\beta}{\alpha} \right|^n = \left| \frac{\sum_{i=1}^{n_0} \tau_i \beta^{n_0-i}}{\sum_{i=1}^{n_0} \tau_i \alpha^{n_0-i}} \right|$$

and examine the effect of replacing τ_i by one on the right-hand side.

By Lemma 5.1, $1/\alpha < (2n)^{1/n_0} < 2^{1/n_0} e^r$. By the way we chose r , for sufficiently large n_0 , the sum $|\sum_{i=1}^{n_0} (\tau_i - 1)\alpha^{n_0-i}|$ is less than a universal constant times α^{n_0} . Also by Lemma 5.1, $\alpha^{n_0} < 2n_0/n < 2/n_0^3$.

If $|t| < \alpha$, it follows that the absolute value of $1 - (\sum_{i=1}^{n_0} \tau_i t^{n_0-i} / \sum_{i=1}^{n_0} t^{n_0-i})$ is less than a constant over n_0^3 . Therefore, we have a constant $C' > 0$ such that

$$\left| \frac{\beta}{\alpha} \right|^{n_0} < \left| \frac{\sum_{i=1}^{n_0} \beta^{n_0-i}}{\sum_{i=1}^{n_0} \alpha^{n_0-i}} \right| \left(1 + \frac{C'}{n_0^3} \right).$$

Combining Lemmas 5.2 and 5.3,

$$\left| \frac{\sum_{i=1}^{n_0} \beta^{n_0-i}}{\sum_{i=1}^{n_0} \alpha^{n_0-i}} \right| < 1 - \frac{\alpha}{2n_0^2(\alpha + 1)^2} < 1 - \frac{\alpha}{8n_0^2}.$$

By Lemma 5.1, $\alpha > 2^{-1/n_0} e^{-r}$; we may assume that α is bounded below by $1/2$. Altogether, we have a constant $C'' > 0$ such that if n_0 is sufficiently large, then

$$\left| \frac{\beta}{\alpha} \right|^{n_0} < \left(1 - \frac{C''}{n_0^2} \right).$$

For sufficiently large n_0 , by taking the n_0 root of both sides and noting $(1 - x)^{1/n} < 1 - x/n$ we have

$$\left| \frac{\beta}{\alpha} \right| < 1 - 1/n_0^4.$$

\square

Now we have estimates from Lemma 5.1 on the Perron eigenvalue of A_n and from Lemma 5.4 on the second largest modulus of a root. We would like to use these to show that if \mathbf{v} is a vector with nonnegative entries and $N \geq n_0^6$, then $(A_n)^N \mathbf{v}$ is close to the eigenvector for A_n . It will be convenient for us to use the matrix B_n

where B_n is A_n scaled by the Perron eigenvalue α_n , i.e.,

$$B_n = \begin{pmatrix} \tau_1/n\alpha_n & \tau_2/n\alpha_n^2 & \tau_3/n\alpha_n^3 & \cdots & \tau_{n_0}/n\alpha_n^{n_0} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

For the matrix B_n the maximal eigenvalue is 1 and the column eigenvector is $(1, 1, \dots, 1)^T$. If the matrix B_n is multiplied by a vector which has the scaled coefficients $e_{n-i}\alpha_n^{n-i}$ in the i th position for $i = 1, 2, \dots, n_0$ the result is a vector of the scaled coefficients $e_{n-j}\alpha_n^{n-j}$ for $j = 0, 1, \dots, n_0 - 1$.

It will also be convenient for us to use the ∞ -norm in the following lemmas, i.e., for $\mathbf{u} = (u_1, u_2, \dots, u_{n_0})$, $\|\mathbf{u}\| = \max |u_j|$.

Lemma 5.5. *There exist constants $K_5, k_5 > 0$ such that if n_0 is sufficiently large and $n \in (n_0^4, e^{rn_0})$, then the following holds. Let $\mathbf{v} \in (\mathbb{R}_+)^{n_0}$. For $j = 1, 2, \dots, n_0$,*

$$\left[(B_n)^{n_0} \mathbf{v} \right]_j = a + \epsilon_j$$

where $a > \|\mathbf{v}\|/n_0^2$ and $|\epsilon_j| < \|\mathbf{v}\|K_5e^{-k_5n_0}$.

Proof. Assume $r > 0$ is small enough for all previous lemmas. Fix $n \in (n_0^4, e^{rn_0})$. Let $B = B_n$, $\alpha = \alpha_n$. Let $\mathbf{v} \in (\mathbb{R}_+)^{n_0}$.

Let $\mathbf{c} = (1, 1, \dots, 1)^T$, the eigenvector for the eigenvalue one. By Lemma 5.4, all other eigenvalues have modulus less than $1 - 1/n_0^4$. Let W denote the span of these complementary eigenspaces. We wish to write the vector \mathbf{v} as $a\mathbf{c} + \mathbf{w}$ where $\mathbf{w} \in W$. Then we will have

$$B^N \mathbf{v} = a\mathbf{c} + B^N \mathbf{w}$$

and our result will follow from a lower bound on a and an upper bound on $\|B^N \mathbf{w}\|$.

To produce the lower bound for a , let $\mathbf{r} = (r_1, r_2, \dots, r_{n_0})$ be the row eigenvector for 1. Here $r_j = \sum_{i=j}^{n_0} \tau_i \alpha^{-i} / n$. The column vector \mathbf{r}^T is orthogonal to all of W . To see this suppose $\beta \neq 1$ and $(I\beta - B)^m \mathbf{x} = 0$. Since $\mathbf{r}(I\beta - B) = (\beta - 1)\mathbf{r}$,

$$(\beta - 1)^m \mathbf{r}\mathbf{x} = \mathbf{r}(I\beta - B)^m \mathbf{x} = 0.$$

Since \mathbf{r}^T is orthogonal to W we can write

$$\mathbf{v} = \left(\frac{\mathbf{r}\mathbf{v}}{\mathbf{r}\mathbf{r}} \right) \mathbf{r}^T + \mathbf{w}_1 \quad \text{and} \quad \mathbf{c} = \left(\frac{\mathbf{r}\mathbf{c}}{\mathbf{r}\mathbf{r}} \right) \mathbf{r}^T + \mathbf{w}_2 \quad \text{where } \mathbf{w}_1, \mathbf{w}_2 \in W.$$

Eliminating \mathbf{r}^T and solving for \mathbf{v} ,

$$\mathbf{v} = \left(\frac{\mathbf{r}\mathbf{v}}{\mathbf{r}\mathbf{c}} \right) \mathbf{c} + \mathbf{w}_1 - \left(\frac{\mathbf{r}\mathbf{v}}{\mathbf{r}\mathbf{c}} \right) \mathbf{w}_2.$$

Our lower bound for $a = \left(\frac{\mathbf{r}\mathbf{v}}{\mathbf{r}\mathbf{c}} \right)$ follows from Lemma 5.1. For n_0 sufficiently large,

$$\mathbf{r}\mathbf{v} \geq \|\mathbf{v}\| \min r_j \geq \|\mathbf{v}\| \tau_{n_0} \alpha^{-n_0} / n > \|\mathbf{v}\| / 4n_0$$

and

$$\mathbf{r}\mathbf{c} = \sum_{i=1}^{n_0} i\tau_i \alpha^{-i} / n < n_0 \sum_{i=1}^{n_0} \tau_i \alpha^{-i} / n = n_0.$$

Therefore $a > \|\mathbf{v}\| / 4n_0^2$.

Let $\mathbf{w} = \mathbf{w}_1 - \left(\frac{\mathbf{r}\mathbf{v}}{\mathbf{r}\mathbf{c}} \right) \mathbf{w}_2$. Since $\mathbf{r}\mathbf{v} \leq \mathbf{r}\mathbf{c}\|\mathbf{v}\|$, we have $a \leq \|\mathbf{v}\|$ and $\|\mathbf{w}\| \leq \|\mathbf{v}\| + a\|\mathbf{c}\| \leq 2\|\mathbf{v}\|$.

Now we would like an upper bound for $\|B^{n_0^6}\mathbf{w}\|$. Using Lemma A.2, there is an orthonormal matrix C over \mathbb{C} such that C^TBC is upper triangular. The diagonal entries of this matrix are the eigenvalues of B and the off-diagonal entries are at most n_0 in modulus. Without loss of generality, assume the eigenvalue one is in the $(1, 1)$ entry of C^TBC . Then to estimate $\|B^{n_0^6}\mathbf{w}\|$, we consider the submatrix of C^TBC which results from deleting the first row and first column. By Lemma A.3 the entries of this matrix are all less than the following expression:

$$n_0^{n_0} \binom{n_0^6 + n_0 - 1}{n_0} (1 - 1/n_0^4)^{n_0^6 - n_0}.$$

By Lemma A.4,

$$\binom{n_0^6 + n_0 - 1}{n_0} < n_0(1 + n_0^{-5})^{n_0^6} (1 + n_0^5)^{n_0}$$

and there is a constant $c > 0$ such that $n_0^{n_0} \binom{n_0^6 + n_0 - 1}{n_0} < e^{cn_0 \log n_0}$. We know $(1 - 1/n_0^4)^{n_0^6} < e^{-n_0^2}$, therefore the last term $(1 - 1/n_0^4)^{n_0^6 - n_0}$ is less than $e^{-n_0^2 + 1/n_0^3}$. The term $e^{-n_0^2}$ makes the entire expression less than an exponentially small function of n_0 .

Therefore, we can conclude that there exist constants $K_5, k_5 > 0$ such that for n_0 sufficiently large,

$$\|B^{n_0^6}\mathbf{w}\| < K_5 \|\mathbf{v}\| e^{-k_5 n_0}.$$

□

It remains to show that the difference between using the matrix $(A_n)^{n_0^6}$ and the product $A_n A_{n-1} \cdots A_{n-n_0^6+1}$ is negligible. This follows fairly easily since one obtains A_{n-k} from A_n by scaling some entries by a factor of $n/(n-k)$. For large n_0 , since $n \geq n_0^{20}$ and $k < n_0^6$, this factor is very close to one.

The estimates we need are a little more delicate than this so again it will be convenient to use the scaled matrices, but we would like to scale all matrices by the same factor. Fix $n \in [n_0^{20}, e^{rn_0})$ and let B_{n-i} denote the matrix A_{n-i} scaled by α_n , the eigenvalue of A_n . The maximal eigenvalue of B_{n-i} will then be α_n/α_{n-i} .

Lemma 5.6. *If n_0 is sufficiently large and $n \geq n_0^{20}$, then for $\mathbf{v} \in (\mathbb{R})^{n_0}$,*

$$\|(B_n)^{n_0^6}\mathbf{v} - B_n B_{n-1} \cdots B_{n-n_0^6+1}\mathbf{v}\| < n_0^{-4} \|\mathbf{v}\|.$$

Proof. Note for any $1 \leq i \leq n_0^6$, the only nonzero row of $B_{n-i} - B_n$ has a sum of $\sum_{i=1}^{n_0} i\tau_i \alpha_n^{-i}/n(n-i) = i/(n-i)$. If $n_0^{20} - n_0^6 > n_0^{16}$, this sum is less than n_0^{-10} . Therefore, $\|B_n\mathbf{v} - B_{n-i}\mathbf{v}\| \leq n_0^{-10} \|\mathbf{v}\|$.

Noting also that $\|B_{n-i}\mathbf{v}\| \leq \|\mathbf{v}\|$ for all i , we have

$$\|(B_n)^{n_0^6}\mathbf{v} - B_n B_{n-1} \cdots B_{n-n_0^6+1}\mathbf{v}\| \leq n_0^6 n_0^{-10} \|\mathbf{v}\| \leq n_0^{-4} \|\mathbf{v}\|.$$

□

Recall our definition

$$R(t) = (1 - p(t))S(t).$$

Polynomial to Exponential Range Lemma. *For some $r > 0$, there exists a constant $K_6 > 0$ such that if n_0 is sufficiently large, then*

$$R_n < -K_6 S_n/n_0 \quad \text{for } n \in [n_0^{20}, e^{rn_0}).$$

Proof. Recall $p'(1/\lambda) > 0$. Fix $\delta > 0$ such that $|x - 1| < \delta$ implies $p'(1/x\lambda) > p'(1/\lambda)/2$. Fix $r > 0$ small enough so that $e^{-r} > 1 - \delta/2$ and so that all previous lemmas hold.

Fix $n \in [n_0^{20}, e^{rn_0})$. We have the following formula for R_n :

$$R_n = \lambda^n s_n \left(1 - \sum_{i=1}^d p_i \lambda^{-i} \frac{s_{n-i}}{s_n} \right).$$

Lemma 4.6 implies that the difference between the above expression in parentheses and that expression with s_{n-i}/s_n replaced by e_{n-i}/e_n is no more than a constant times $\lambda^{-n_0/6}$. We will now use the lemmas of this section to show that $e_{n-i}/e_n \approx \alpha_n^{-i}$, so

$$(5.5) \quad 1 - \sum_{i=1}^d p_i \lambda^{-i} \frac{e_{n-i}}{e_n} \approx 1 - \sum_{i=1}^d p_i (\alpha_n \lambda)^{-i} = 1 - p(1/\alpha_n \lambda).$$

Lemma 5.1 provides the lower bound $(2n)^{-1/n_0}$ for the Perron eigenvalue α_n of the matrix A_n . Since $n < e^{rn_0}$ we have $\alpha_n > 2^{-1/n_0} e^{-r}$. By choice of r , for sufficiently large n_0 ,

$$1 - p(1/\alpha_n \lambda) < -\frac{p'(1/\lambda)}{2\lambda} |\alpha_n - 1|.$$

Lemma 5.1 also provides an upper bound $\alpha_n < (2n_0/n)^{1/n_0}$. Since $n \geq n_0^{20}$, for sufficiently large n_0 , we have $\alpha_n < 1 - 1/n_0$. Therefore, $1 - p(1/\alpha_n \lambda)$ is less than a negative constant over n_0 .

Following the Ratio Argument, it remains to show that the approximation in equation (5.5) has an error which is small compared to $1/n_0$. Let $m = n - n_0^6$, and let $\alpha = \alpha_n$. Let \mathbf{v} denote the vector $(e_{m-1} \alpha^{-m+1}, e_{m-2} \alpha^{-m+2}, \dots, e_{m-n_0} \alpha^{-m+n_0})^T$. Then there exist constants $C, c > 0$ such that for $1 \leq i \leq n_0$,

$$e_{n-i} \alpha^{-n+i} = a + \epsilon_i + \delta_i$$

where $a > \|\mathbf{v}\|/4n_0^2$, $|\epsilon_i| < C\|\mathbf{v}\|e^{-cn_0}$ (Lemma 5.5) and $|\delta_i| < \|\mathbf{v}\|/n_0^4$ (Lemma 5.6).

For n_0 sufficiently large, both $|\epsilon_i|/a$ and $|\delta_i|/a$ are less than a constant over n_0^2 . Therefore the difference between $\alpha^i e_{n-i}/e_n$ and 1 is less than a constant over n_0^2 . Since this error is smaller than $1/n_0$, the result follows from the Ratio Argument. \square

6. FINAL FACTORIZATION

In order to complete the proof, we multiply $(1 - p(t))S(t)$ by a power series of the form $(1 - q(t))^{-1}$ where $q_i > 0$ for all i . We wish to have the n th coefficient of $(1 - q(t))^{-1}$ closely approximate β^n for some β .

Fix $r > 0$ small enough to insure the Polynomial to Exponential Range Lemma holds, $1 - p(e^{r/2}/\lambda) < 0$ and $\lambda e^{-r/2} > 1$. Let $\beta = \lambda e^{-r/2}$ and let

$$1 - q(t) = 1 - \sum_{i=n_0^2+1}^{n_0^3} \left\lfloor \frac{\beta^i}{n_0^3 - n_0^2} \right\rfloor t^i.$$

We will show that the Perron value for $1 - q(t)$ is approximately β . Then using techniques similar to those in Section 5 (Polynomial to Exponential Range), we will show that consecutive coefficients of $(1 - q(t))^{-1}$ have ratios approximately equal to β . Then, letting $U(t) = (1 - q(t))^{-1}$ and $V(t) = (1 - p(t))U(t)$, we will show that for $n > n_0^{20}$,

$$V_n = U_n \left(1 - \sum_{i=1}^d p_i U_{n-i} / U_n \right) \approx U_n (1 - p(1/\beta)) < 0.$$

It will remain then to combine all of the lemmas to show that the n th coefficient of $(1 - p(t))S(t)U(t)$ is negative for all n .

Lemma 6.1. *Let γ denote the Perron value of $1 - q(t)$. There is a constant $K_7 > 0$ such that if n_0 is sufficiently large, then*

$$|\gamma - \beta| < K_7 \beta^{-n_0^2}.$$

Proof. First notice that $1 - q(1/\beta) > 0$ which implies $\gamma < \beta$. Therefore, by the Mean Value Theorem there is a $t \in (1/\beta, 1/\gamma)$ such that

$$|1/\beta - 1/\gamma| = (1 - q(1/\beta)) / |q'(t)|.$$

We obtain an upper bound on $1 - q(1/\beta)$ from

$$1 - q(1/\beta) = 1 - \sum_{i=n_0^2+1}^{n_0^3} \left\lfloor \frac{\beta^i}{n_0^3 - n_0^2} \right\rfloor \beta^{-i} \leq \sum_{i=n_0^2+1}^{n_0^3} \beta^{-i}.$$

Therefore, $1 - q(1/\beta) < \beta^{-n_0^2} / (\beta - 1)$.

Now since $t > 1/\beta$ and $q(t)$ has all nonnegative coefficients,

$$|q'(t)| > q'(1/\beta) = \sum_{i=n_0^2+1}^{n_0^3} i \left\lfloor \frac{\beta^i}{n_0^3 - n_0^2} \right\rfloor \beta^{-i+1}.$$

Taking only the last term in the sum, for n_0 sufficiently large, we see $|g'(t)| > \beta$. The desired inequality follows. □

Now in order to estimate the coefficients of $U(t) = (1 - q(t))^{-1}$, we will use the recursion relation

$$U_n = \sum_{i=n_0^2+1}^{n_0^3} q_i U_{n-i}.$$

As in Section 5, we can think of this relation as being given by a matrix. This matrix has Perron eigenvalue $\gamma \approx \beta$, so we will be able to use the same proofs we used in Section 5 to show $U_n / U_{n+1} \approx 1/\beta$. Letting $V(t) = (1 - p(t))U(t)$, this will give $V_n \approx U_n (1 - p(1/\beta))$.

Lemma 6.2. *There exist constants $K_8, K_9, k_8 > 0$ such that if n_0 is sufficiently large and $n > n_0^{20}$, then*

$$\left| \frac{U_{n+1}}{U_n} - \gamma \right| < K_8 e^{-k_8 n_0^3}$$

and

$$V_n < -K_9 U_n.$$

Proof. Let $1/\zeta$ be any other root of $1-q(t)$. By Lemma 5.2 we know $|\gamma-\zeta| > \gamma/n_0^3$. We have bounds on the ratio of consecutive coefficients

$$\beta \left(1 - \frac{n_0^3 - n_0^2}{\beta n_0^2}\right) < \frac{q_{i+1}}{q_i} < \beta \left(1 - \frac{n_0^3 - n_0^2}{\beta n_0^2}\right)^{-1}.$$

Therefore using Lemma 5.3, there is a constant $C > 0$ such that

$$\left|\frac{\zeta}{\gamma}\right|^{n_0^3} = \left|\frac{\sum_{i=n_0^2+1}^{n_0^3} q_i \zeta^{n_0^3-i}}{\sum_{i=n_0^2+1}^{n_0^3} q_i \gamma^{n_0^3-i}}\right| < 1 - C/n_0^6.$$

Taking the n_0^3 root of both sides, for sufficiently large n_0 ,

$$|\zeta| < (1 - 1/n_0^{12})\gamma.$$

Following the argument in Lemma 5.5, there exist constants $K_8, k_8 > 0$ such that if we multiply an arbitrary vector \mathbf{v} with nonnegative entries by the recursion matrix raised to the n_0^{19} power, the result will be a vector \mathbf{u} whose entries satisfy $|\mathbf{u}_{j+1}/\mathbf{u}_j - \gamma| < K_8 e^{-k_8 n_0^3}$. Since the recursion formula applies for all $n > n_0^3$, this gives the desired estimate on ratios of coefficients U_{n+1}/U_n for $n > n_0^{20} > n_0^{19} + n_0^3$.

Therefore by the previous lemma, when n_0 is sufficiently large, the ratio of coefficients U_{n-i}/U_n is as close as we like to β^{-i} for $i \leq d$. Therefore the quantity $1 - \sum_{i=1}^d p_i U_{n-i}/U_n$ is as close as we like to $1 - p(1/\beta)$. Since we chose β to be a fixed amount less than λ , for some $K_9 > 0$, $1 - \sum_{i=1}^d p_i U_{n-i}/U_n < -K_9$ and $V_n < -K_9 U_n$. □

The final step is to combine all of these estimates and show that the n th coefficient of $(1-p(t))S(t)U(t)$ is always negative. For this, we consider these coefficients in two ranges.

For $n < e^{rn_0}$, we have $R(t) = (1-p(t))S(t)$ and the n th coefficient of $R(t)U(t)$ can be written as

$$R_n + U_{n_0^2} R_{n-n_0^2} + U_{n_0^2+1} R_{n-n_0^2-1} + \dots + U_{n-n_0-1} R_{n_0+1} + U_n.$$

By the lemmas from Section 4, the R_n coefficients are negative. All coefficients of $U(t)$ are nonnegative so all coefficients in the sum above are nonpositive except U_n . To show that the entire sum is nonpositive, all we need to show is that U_n is smaller than the absolute value of just one other term in the sum.

Then for $n > e^{3rn_0/4}$ we have $V(t) = (1-p(t))U(t)$. The n th coefficient of $S(t)V(t)$ can be written as

$$S_n + V_{n_0^2} S_{n-n_0^2} + V_{n_0^2+1} S_{n-n_0^2-1} + \dots + V_{n-1} S_1 + V_n.$$

We know that all but the first n_0^{20} terms of $V(t)$ are negative and all terms of $S(t)$ are positive. We will show that the first n_0^{20} terms are small compared to the entire sum.

In order to show that the positive terms in the above summation do not affect the overall sign, we will need upper bounds for coefficients in $U(t)$ and in large degree terms in $S(t)$.

Lemma 6.3. *The coefficient of t^n in $U(t) = (1-q(t))^{-1}$ is less than $n\beta^n$ for all n .*

Proof. We consider the n th coefficient of $(1 - q(t))^{-1} = \sum_{i=0}^{\infty} (q(t))^i$. The result follows from the fact that the coefficient of t^n from $(q(t))^i$ is always less than β^n . This holds since the number of ways to select i coefficients of $q(t)$ whose indices sum to n is less than or equal to $(n_0^3 - n_0^2)^i$. \square

Lemma 6.4. *There exists a $\delta > 0$ such that if n_0 is sufficiently large and $n > e^{3rn_0/4}$, then $S_n < (\beta - \delta)^n$.*

Proof. Recall $S(t) = \prod_{i=1}^{n_0} (1 - t^i)^{o(i)}$. For i dividing n , we find estimates on $\binom{o(i)+n/i-1}{n/i}$, the n th coefficient of the series $(1 - t^i)^{-o(i)}$. We use the formula

$$\binom{o(i) + n/i - 1}{n/i} < n \left(1 + \frac{n}{io(i)}\right)^{o(i)} \left(1 + \frac{io(i)}{n}\right)^{n/i}$$

from Lemma A.4. We would like to show that the logarithm of the right-hand side is less than $n \log(\beta - 2\delta)$ for $n > e^{3rn_0/4}$.

The logarithm of the right-hand side is the sum of three terms. The first, $\log n$, is clearly small compared to n .

For the second term, $o(i) \log(1 + n/io(i))$, there are two cases. If $i < \log n_0$, then

$$o(i) < \lambda^{\log n_0} = n_0^{\log \lambda} < \left(\frac{4 \log n}{3r}\right)^{\log \lambda}.$$

In the case $i \geq \log n_0$, we use $\log(1 + x) < x$ to show

$$o(i) \log \left(1 + \frac{n}{io(i)}\right) < \frac{n}{i} < \frac{n}{\log n_0}.$$

In either case, by selecting n_0 large enough the ratio of this term with n is as small as we like.

For the third term, we have three cases. If $i < \log n_0$, then

$$\frac{n}{i} \log \left(1 + \frac{io(i)}{n}\right) < o(i) < \left(\frac{4 \log n}{3r}\right)^{\log \lambda}.$$

If $i \geq \log n_0$ and $io(i) < n$, then

$$\frac{n}{i} \log \left(1 + \frac{io(i)}{n}\right) < \frac{n}{i} \log 2 \leq \frac{n \log 2}{\log n_0}.$$

In both of these cases, for n_0 large, the term is small compared to n . If $i \geq \log n_0$ and $io(i) \geq n$, then we may assume $io(i) < 2\lambda^i$. Therefore

$$\frac{n}{i} \log \left(1 + \frac{io(i)}{n}\right) < \frac{n}{i} \log \left(\frac{4\lambda^i}{n}\right) = n \left(\log \lambda + \frac{\log 4}{i} - \frac{\log n}{i}\right).$$

Recalling $\log n > 3rn_0/4$, this term has an upper bound of the form

$$\frac{n}{i} \log \left(1 + \frac{io(i)}{n}\right) < n \left(\log \lambda + \frac{\log 4}{\log n_0} - \frac{3r}{4}\right).$$

Since $\log \beta = \log \lambda - r/2$, in this last case, we can choose a $\delta > 0$ such that for sufficiently large n_0 , the term is less than $n \log(\beta - 3\delta)$.

Altogether, we may assume that the n th coefficient of $(1 - t^i)^{-o(i)}$ is at most $(\beta - 2\delta)^n$ for $n > e^{3rn_0/4}$. The n th coefficient in the product $\prod_{i=1}^{n_0} (1 - t^i)^{-o(i)}$ is made up of at most $\binom{n+n_0-1}{n}$ terms each of which is less than $(\beta - 2\delta)^n$. Using

the upper bound $\binom{n+n_0-1}{n} < (2n)^{n_0}$, this tells us that if n_0 is sufficiently large and $n > e^{3rn_0/4}$, then

$$S_n < (2n)^{n_0}(\beta - 2\delta)^n < (\beta - \delta)^n.$$

□

We are now ready to combine the previous lemmas to show that the desired factorization of $1 - p(t)$ over the ring of formal power series exists.

Final Factorization Lemma. *There is a choice of $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for*

$$q(t) = \sum_{i=n_0^2+1}^{n_0^3} \left\lfloor \frac{\beta^i}{n_0^3 - n_0^2} \right\rfloor t^i$$

the n th coefficient of the product

$$\frac{1 - p(t)}{(1 - q(t)) \prod_{i=1}^{n_0} (1 - t^i)^{o(i)}}$$

is nonpositive for all $n \geq 1$.

Proof. Select $r > 0$ small enough for all previous lemmas, then select n_0 large enough for all previous lemmas. For $n \in (0, n_0]$, the n th coefficient of the product $1 - p(t)(1 - q(t)) \prod_{i=1}^{n_0} (1 - t^i)^{-o(i)}$ is zero.

For $n > n_0$ we follow the aforementioned argument. Recall the notation $R(t) = (1 - p(t))S(t)$ and $U(t) = (1 - q(t))^{-1}$. By the lemmas of Sections 4 and 5, for $n \in (n_0, e^{rn_0})$ the n th coefficient $R(t)$ is nonpositive. All coefficients of $U(t)$ are nonnegative. Therefore, for $n \in (n_0, e^{rn_0})$ it suffices to show that U_n is smaller than the absolute value of just one term of the form $R_i U_{n-i}$ where $i > 0$.

Region I: Fix $n \in (n_0, n_0^3]$. Pick $\epsilon > 0$ such that $\lambda - 2\epsilon > \beta$. By the Polynomial Range Lemma, there is a constant $C_1 > 0$ such that $R_n < -C_1 S_n/n_0$. Also, it follows from Lemmas 4.5 and 4.6 that there is a constant $C_2 > 0$ such that $S_n > C_2(\lambda - \epsilon)^n$. Therefore,

$$\left| \frac{U_n}{R_n} \right| < \frac{nn_0\beta^n}{C_1 C_2(\lambda - \epsilon)^n}.$$

For n_0 sufficiently large, this is less than one.

Region II: Fix $n \in (n_0^3, e^{rn_0})$. Here we compare U_n to $U_{n-n_0-1}R_{n_0+1}$. We have $|R_{n_0+1}| > C_2(\lambda - \epsilon)^{n_0+1}/n_0$ and $U_n < (\beta + \epsilon)^{n_0+1}U_{n-n_0-1}$ by the results of this section. Therefore,

$$\left| \frac{U_n}{R_{n_0+1}U_{n-n_0-1}} \right| < \frac{n_0(\beta + \epsilon)^{n_0+1}}{C_2(\lambda - \epsilon)^{n_0+1}}.$$

For n_0 sufficiently large, this is less than one.

Region III: Now for $n > e^{rn_0}$ we change the argument. Recall the notation $V(t) = (1 - p(t))U(t)$. By Lemma 6.2 the n th coefficient of $V(t)$ is nonpositive for $n > n_0^{20}$. Since all coefficients of $S(t)$ are nonnegative it suffices to show that V_n is larger in absolute value than $\sum_{i=0}^{n_0^{20}} S_i V_{n-i}$ where $i < n$.

Fix $n \in [e^{rn_0}, \infty)$. Assuming $e^{rn_0} > e^{3rn_0/4} + n_0^{20}$, by Lemma 6.4, $S_{n-i} < (\beta - \delta)^{n-i}$ for $0 < i < n_0^{20}$. Also, for $i < n_0^{20}$, by the results of this section,

there exist constants $C_3, C_4 > 0$ such that $|V_n| > C_3 U_n > C_4 (\beta - \delta/2)^{n-n_0^{20}} |V_i|$. Therefore,

$$\left| \frac{S_n + \sum_{i=1}^{n_0^{20}} S_{n-i} V_i}{V_n} \right| < \frac{n_0^{20}}{C_4} \left(\frac{\beta - \delta}{\beta - \delta/2} \right)^{n-n_0^{20}} (\beta - \delta)^{n_0^{20}}.$$

For n_0 sufficiently large, this is less than one. □

The above shows that we may factor $1 - p(t)$ in the form appearing in the Main Reduction Lemma. This proves the remaining direction of the theorem.

APPENDIX A. LEMMAS ON BINOMIAL COEFFICIENTS AND MATRIX THEORY

These results are probably known to experts in the appropriate areas but we do not have convenient references.

Lemma A.1. *Suppose $w_1 \leq w_2 \leq \dots \leq w_n$ are real numbers with $\sum_{i=1}^n w_i = 1$. If $r_1 \leq r_2 \leq \dots \leq r_n$, then*

$$\sum_{i=1}^n w_i r_i \geq (1/n) \sum_{i=1}^n r_i.$$

Proof. Since the weights w_i are increasing there is an index j such that $w_i < 1/n$ for all $1 \leq i < j$ and $w_i \geq 1/n$ for all $j \leq i \leq n$.

Notice that

$$\sum_{i=j}^n (w_i - 1/n) = \sum_{i=1}^{j-1} (1/n - w_i).$$

Therefore

$$\sum_{i=j}^n (w_i - 1/n) r_i \geq \sum_{i=1}^{j-1} (1/n - w_i) r_i.$$

Rearranging we obtain the desired inequality. □

Lemma A.2. *Let A be an $n \times n$ matrix over \mathbb{R} . Then there is a unitary, orthonormal matrix C over \mathbb{C} such that $C^T A C$ is upper triangular.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{C}^n$ be a set of \mathbb{C} -linearly independent column eigenvectors for A . Apply Gram-Schmidt to these vectors to create a set of orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n_0-1} \in \mathbb{C}^{n_0}$ such that $A \mathbf{u}_k \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ for all k .

Let C be the $n \times n$ matrix with columns \mathbf{u}_j . Then $C^T A C$ is a triangular matrix since $\mathbf{u}_i^T A \mathbf{u}_j = 0$ for $i > j$. □

Lemma A.3. *If A is an $n_0 \times n_0$ triangular matrix over \mathbb{R}_+ with diagonal entries less than β and off-diagonal entries less than $K > \beta$, then the entries of A^N are all less than $K^{n_0} \binom{N+n_0-1}{N} \beta^{N-n_0}$.*

Proof. The entry $A^N(i, j)$ is the sum of all possible products of the form

$$A(i, k_1) A(k_1, k_2) \cdots A(k_{N-1}, j).$$

Since A is triangular, only products where $i \leq k_1 \leq k_2 \leq \dots \leq k_{N-1} \leq j$ are nonzero. Therefore there are at most $\binom{N+n_0-1}{N}$ nonzero products in this sum. We have the bounds $A(k_n, k_{n+1}) < K$ for $k_n < k_{n+1}$ and $A(k_n, k_{n+1}) < \beta$ for

$k_n = k_{n+1}$. There are at most n_0 terms $A(k_n, k_{n+1})$ in each product with $k_n < k_{n+1}$. Therefore each term $A^N(i, j)$ is less than $K^{n_0} \binom{N+n_0-1}{N} \beta^{N-n_0}$. \square

Lemma A.4. For all $n, k \geq 1$,

$$\binom{n+k-1}{k} < n(1+k/n)^n(1+n/k)^k.$$

Proof. We have the following estimates on $\log(n!) = \sum_{i=1}^n \log i$:

$$\int_1^n \log x dx < \log(n!) < \int_1^{n+1} \log x dx.$$

Thus we obtain

$$\begin{aligned} \log \binom{n+k-1}{k} &< (n+k) \log(n+k) - k \log k \\ &\quad - (n-1) \log(n-1) - 2. \end{aligned}$$

Using the inequality $(n-1) \log(n-1) + 2 > (n-1) \log n$ and combining terms,

$$\log \binom{n+k-1}{k} < n \log(1+k/n) + k \log(1+n/k) + \log n.$$

This gives the inequality. \square

APPENDIX B. LIND/MARCUS EXAMPLE

The polynomial $1 - p(t) = 1 - 4t + 6t^2 - 6t^3$ has a factorization $\prod_{i=1}^3 (1 - \lambda_i t)$ where $\lambda_1 \approx 2.57474$ and $\lambda_2 = \overline{\lambda_3} \approx .71263 + 1.35000i$. Lind and Marcus [LM95, Chapter 11] give $(\lambda_1, \lambda_2, \lambda_3)$ as an example of an n -tuple satisfying the Boyle-Handelman conditions, but one for which there was no known primitive matrix with the corresponding nonzero spectrum. We will need a polynomial matrix $A(t)$ of at least size 4×4 to realize $1 - p(t)$. To see this, suppose $A(t)$ is a 3×3 matrix with entries in $t\mathbb{Z}_+[t]$,

$$A(t) = \begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix}$$

such that $\det(I - A(t)) = 1 - p(t)$. Equating first and second degree coefficients, we see $a_1 + e_1 + i_1 = 4$ and $a_1 e_1 + a_1 i_1 + e_1 i_1 \geq 6$. This is impossible since all coefficients are nonnegative integers.

Following our factorization scheme, we begin by dividing $1 - p(t)$ by powers of $1 - t^i$. A calculation shows

$$\begin{aligned} \frac{1-p(t)}{1-t} &= 1 - 3t + 3t^2 - 3t^3 - 3t^4 - 3t^5 - 3t^6 - 3t^7 - 3t^8 - \dots, \\ \frac{1-p(t)}{(1-t)^2} &= 1 - 2t + t^2 - 2t^3 - 5t^4 - 8t^5 - 11t^6 - 14t^7 - 17t^8 - \dots, \\ \frac{1-p(t)}{(1-t)^3} &= 1 - t - 2t^3 - 7t^4 - 15t^5 - 26t^6 - 40t^7 - 57t^8 - \dots. \end{aligned}$$

The last rational power series seems to be of the correct form. In fact, we notice from the Main Reduction Lemma that it suffices to have an $N > 3$ such that the coefficients of the power series $(1 - p(t))(1 - t)^{-1}$ and $(1 - p(t))(1 - t)^{-2}$ are nonpositive in degrees N to $2N$ and the coefficients of $(1 - p(t))(1 - t)^{-3}$ are nonpositive in degrees 1 to N . The value $N = 4$ suffices here.

Following the scheme in the Main Reduction Lemma,

$$\begin{aligned} a_0(t) &= 4t - 6t^2 + 6t^3, \\ a_1(t) &= 3t - 3t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + 3t^7 + 3t^8, \\ a_2(t) &= 2t - t^2 + 2t^3 + 5t^4 + 8t^5 + 11t^6 + 14t^7 + 17t^8, \\ a_3(t) &= t + 2t^3 + 7t^4 + 15t^5 + 26t^6 + 40t^7 + 57t^8 \end{aligned}$$

and

$$b_k(t) = (1 - a_k(t))(1 - t) - 1 - a_{k-1}(t)$$

which gives

$$b_1(t) = 3t^9, \quad b_2(t) = 17t^9, \quad b_3(t) = 57t^9.$$

Thus we construct the matrix

$$A(t) = \begin{pmatrix} t + 2t^3 + 7t^4 + 15t^5 + 26t^6 + 40t^7 + 57t^8 & t & 0 & 0 \\ & 57t^8 & t & t & 0 \\ & 17t^7 & 0 & t & t \\ & 3t^6 & 0 & 0 & t \end{pmatrix}$$

with $\det(\mathbf{I} - A(t)) = 1 - 4t + 6t^2 - 6t^3$. Notice that $A(t)^3$ has no nonzero entries, i.e., $A(t)$ is primitive.

We note that in this case, one can actually use the noninteger value of $N = 5/2$ to truncate the power series. This method results in the matrix

$$B(t) = \begin{pmatrix} t + 2t^3 + 7t^4 + 15t^5 & t & 0 & 0 \\ & 15t^5 & t & t & 0 \\ & 8t^4 & 0 & t & t \\ & 3t^3 & 0 & 0 & t \end{pmatrix}$$

with $\det(\mathbf{I} - B(t)) = 1 - 4t + 6t^2 - 6t^3$. If one writes down a primitive integer matrix which corresponds to $B(t)$ in the most straightforward way, the result is a 179×179 matrix. There are certainly smaller size matrices with the same nonzero spectrum. Determining the minimal size of a nonnegative integer matrix in this case remains an interesting open problem.

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