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THE "HOT SPOTS" CONJECTURE FOR DOMAINS WITH TWO AXES OF SYMMETRY

DAVID JERISON AND NIKOLAI NADIRASHVILI

§1. Introduction

Consider a convex planar domain with two axes of symmetry. We show that the maximum and minimum of a Neumann eigenfunction with lowest nonzero eigenvalue occur at points on the boundary only. We deduce J. Rauch's "hot spots" conjecture in the following form. If the initial temperature distribution is not orthogonal to the first nonzero eigenspace, then the point at which the temperature achieves its maximum tends to the boundary. In fact the maximum point reaches the boundary in finite time if the boundary has positive curvature.

Results of this type have already been proved by Bañuelos and Burdzy [BB] using the heat equation and probabilistic methods to deform initial conditions to eigenfunctions. We introduce here a new technique based on deformation of the domain. An advantage of our method is that it works even in the case of multiple eigenvalues. On the way toward our results, we prove monotonicity properties for Neumann eigenfunctions for symmetric domains that need not be convex and deduce a sharp comparison of eigenvalues with the Dirichlet problem of independent interest.

Theorem 1.1. *Let Ω be a bounded domain in the plane. Suppose that Ω is symmetric with respect to both coordinate axes and that all vertical and horizontal cross sections of Ω are intervals. Let u be the Neumann eigenfunction with lowest eigenvalue among functions that are odd with respect to the reflection $(x_1, x_2) \mapsto (x_1, -x_2)$. After multiplication by ± 1 we may assume that $u > 0$ in $x_2 > 0$. Then*

$$\partial u/\partial x_2 > 0 \text{ in } \Omega.$$

Except in the case of a rectangle,

$$\partial u/\partial x_1 < 0 \text{ for } x_1 x_2 > 0 \text{ in } \Omega,$$
$$\partial u/\partial x_1 > 0 \text{ for } x_1 x_2 < 0 \text{ in } \Omega.$$

The maximum and minimum of u on $\bar{\Omega}$ are achieved[1] *at the points where the x_2 axis meets $\partial\Omega$ and, except in the case of the rectangle, at no other points.*

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[1] We show in Section 7 that u is continuous in $\bar{\Omega}$. This is only at issue at the points where the axes meet the boundary at which the domain may fail to be Lipschitz.

In the case of the rectangle, the eigenfunction is $u = \sin(ax_2)$. The maximum and minimum are achieved on the whole top and bottom sides, $x_2 = \pm\pi/2a$, and $\partial u/\partial x_1 \equiv 0$.

Corollary 1.2. *Let* λ *be the eigenvalue of* u *in Theorem* 1.1. *Then*

$$\lambda \le \mu(\Omega),$$

where $\mu(\Omega)$ *is the lowest Dirichlet eigenvalue of* Ω.

Corollary 1.2 should be compared to the following well-known theorem.

Theorem 1.3. *The lowest Neumann nonzero eigenvalue of a planar domain is strictly less than the lowest Dirichlet eigenvalue for any domain with the same area.*

Theorem 1.3 follows from the fact that the Neumann eigenvalue is increased and the Dirichlet eigenvalue is decreased under radial symmetrization of the domain preserving area. One can then compare the eigenvalues on the disk explicitly. The rearrangement theorems are due to Szëgo and Faber-Krahn, respectively. (See [P, (3.21)].) Thus Corollary 1.2 is a very modest special case of Theorem 1.3 if λ is the lowest nonzero eigenvalue for the Neumann problem. But by symmetry the inequality of Corollary 1.2 also applies to the eigenvalue corresponding to the Neumann eigenfunction with lowest nonzero eigenvalue among functions that are odd in the x_1 variable. Thus unless the eigenvalue is multiple, Corollary 1.2 is a stronger comparison among eigenvalues of a single domain. On the other hand Theorem 1.3 applies to different domains with the same area.

Theorem 1.4. *Let* Ω *be a bounded convex domain in the plane that is symmetric with respect to both coordinate axes. Let* u *be any Neumann eigenfunction with lowest nonzero eigenvalue. Then, except in the case of a rectangle,* u *achieves its maximum over* $\bar{\Omega}$ *on the boundary at exactly one point, and likewise for its minimum. Furthermore, if* $x^0 \in \partial\Omega$ *and* $-x^0$ *denote the places where* u *achieves its maximum and minimum, then* u *is monotone along the two arcs of the boundary from* $-x^0$ *to* x^0. *Let* ν *be any outer normal to* $\partial\Omega$ *at* x^0, *that is,* $\nu \cdot (x - x^0) < 0$ *for all* $x \in \Omega$. *Then* $\nu \cdot \nabla u(x) > 0$ *for all* $x \in \Omega$.

One axis of symmetry is not enough. A straightforward argument using reflection shows that an equilateral triangle has an eigenfunction with a maximum at two vertices and a local minimum at the midpoint of the side in between. The absolute minimum is at the third vertex. (There are perturbations of this example that are smooth domains and that have a simple eigenvalue.)

Corollary 1.5 (hot spots). *Let* Ω *be a convex, bounded domain in the plane that is symmetric with respect to both coordinate axes. Let* $h(x,t)$ *be any solution to the heat equation in* $C(\bar{\Omega} \times [0,\infty))$ *with Neumann boundary conditions and such that*

$$\int_\Omega h(x,0)u(x)dx \neq 0$$

for some Neumann eigenfunction u *with lowest nonzero eigenvalue. Let* x_t *denote any point of* $\bar{\Omega}$ *at which* $x \mapsto h(x,t)$ *achieves its maximum. Then* x_t *tends to* $\partial\Omega$ *as* $t \to \infty$. *If* $\partial\Omega$ *is smooth and positively curved, then there is* T *such that* $x_t \in \partial\Omega$ *for* $t \ge T$.

The main idea of the proof of Theorems 1.1 and 1.4 is to keep track of the zero set of the directional derivatives of u as the domain varies. This method has been used very effectively by A. Melas [M] to prove that the nodal set of the second Dirichlet eigenfunction touches the boundary. Theorem 1.3 plays an important role. As pointed out by Pleijel [Pl], Theorem 1.3 implies that the nodal line of the least nonconstant Neumann eigenfunction cannot enclose a subdomain. We note that Pleijel's proof applies to directional derivatives of u, since these satisfy the same eigenvalue equation — the Neumann boundary conditions are not relevant.

Here is a sketch of the proofs of Theorems 1.1 and 1.4. Following the method of continuity, consider a continuously varying family of domains Ω_t such that Ω_0 is a domain for which one knows some monotonicity properties, and Ω_1 is the domain for which one aims to prove them. (For technical convenience, one approximates the domains by piecewise smooth domains or polygons.) Suppose, by contradiction, that monotonicity fails for Ω_1, and let t_0 be the infimum of times t for which monotonicity fails. Observe that the functions $w_i = \partial u/\partial x_i$ satisfy the eigenfunction equation. It follows that when w_i is zero at a point, either it is identically zero or its zero set has at least one branch through the point and it takes on values of both signs in a neighborhood of the point. But by continuity from earlier times the functions w_i at time t_0 cannot change sign in the interior of a quadrant, so what goes wrong at t_0 must occur on the boundary. If monotonicity fails on Ω_{τ_j} for $\tau_j \searrow t_0$ at points tending to the boundary, then the eigenfunction on Ω_{t_0} has a degenerate critical point on the boundary at time t_0. But the degeneracy of the critical point implies that the zero set of w_1 at that point has more than one branch pointing into the domain Ω_{t_0}. It then follows that every possible topological configuration of this zero set leads to a nodal domain associated to the eigenvalue λ. A comparison of Dirichlet with Neumann eigenvalues shows that this is impossible. (It is also necessary to prove certain nondegeneracy conditions at the vertices of polygons.)

In Theorem 1.1, to prove monotonicity of odd eigenfunctions, one starts from a diamond domain Ω_0, that is, the square with sides of slope ± 1. This domain has an explicit odd eigenfunction, written as a sum of trigonometric functions. In Theorem 1.4, to prove monotonicity in convex domains with a multiple eigenvalue one starts from the equilateral polygon, whose eigenfunctions are far from explicit. One must first prove monotonicity for this special case. The monotonicity property satisfied by the full family of eigenfunctions is monotonicity along the boundary between the minimum and the maximum, or, equivalently, monotonicity in the direction of the segment from the minimum to the maximum. The proof of monotonicity for the equilateral polygon again follows the method of continuity applied to linear combinations of eigenfunctions starting from the the odd eigenfunction whose monotonicity was already proved in Theorem 1.1. The contradiction is obtained using the full strength of Theorem 1.3: in one topological configuration of the zero set the comparison domain is not a subset of the equilateral polygon, but rather a domain obtained by reflection across a side. (The comparisons in the earlier parts of the proof are elementary and do not depend on this theorem at all.)

The organization of the paper is as follows. In the second section we prove continuous dependence of the eigenvalue and odd eigenfunction under certain variations of the domain. In the third section we prove the main monotonicity results for the odd eigenfunctions. This is proved first for a special class of piecewise smooth domains. We then pass to the limit to obtain a similar result for all Lipschitz domains. (The passage to the limit needed to handle the case of cusps at the points where the domain meets the axes is deferred to the seventh section because the result is not needed in the rest of the paper.) In the fourth section we prove strict monotonicity, nondegeneracy at the maximum and minimum and a strict eigenvalue comparison with the Dirichlet eigenvalue in the case of polygons. In the fifth section we deduce that all the eigenfunctions of equilateral polygons have monotonicity. In the sixth section we prove monotonicity in the case of multiple eigenvalues. This is done first in the case of convex polygons and then more generally for convex domains by passing to the limit. In Section 7 we prove Theorem 1.1 and Corollary 1.2 in full generality including the case of domains with cusps. Finally, in Section 8 we add some remarks about multiplicity and make some conjectures.

We thank Richard Laugesen and Krzysztof Burdzy for describing the “hot spots” conjecture to us. The monograph by B. Kawohl [K] also played an important role in calling attention to the problem. In 1974 Jeff Rauch [R, p. 359] made the hypothesis that the first nonzero Neumann eigenfunction achieves its maximum and minimum at isolated points on the boundary and suggested that it should be valid in “unexceptional cases”. He deduced from it that the point where the temperature attains its maximum tends to one of those boundary points. In [K], B. Kawohl made the more explicit conjecture that for every convex domain in $\mathbf{R}^n$, a first nonzero Neumann eigenfunction achieves its maximum and minimum on the boundary only. Positive results on this problem can be found in [BB] and [K], and there are a number of counterexamples in multiply-connected planar domains. The present authors have an argument (unpublished) proving the existence of a domain with many holes for which a lowest Neumann eigenfunction has an interior maximum. The argument is based on a homogenization method known as the Neumann sieve. A domain with three holes was constructed by Burdzy and Werner [BW], and a domain with many holes for which both the maximum and the minimum are in the interior was constructed by Bass and Burdzy [BaB].

§2. Continuous dependence under variation of the domain

Our theorems concern domains Ω given in polar coordinates by

$$\Omega = \{(r\cos\theta, r\sin\theta) : r < \phi(\theta)\}, \tag{2.1}$$

where ϕ is a continuous periodic function of period 2π. Symmetry of Ω with respect to the axes can be written as

$$\phi(\theta) = \phi(-\theta),\ 0 \le \theta \le \pi; \quad \phi(\theta) = \phi(\pi - \theta),\ 0 \le \theta \le \pi/2. \tag{2.2}$$

In particular ϕ on $[0, \pi/2]$ determines Ω. The assumption that the horizontal and vertical cross sections are intervals can be written as

$$-\cot\theta \le (\log\phi(\theta))' \le \tan\theta, \text{ for a.e. } \theta,\ 0 \le \theta \le \pi/2. \tag{2.3}$$

We begin by imposing an extra Lipschitz condition on the domains to make it easier to prove continuous dependence of the eigenvalue and eigenfunction when the domain varies. Denote by L_M the collection of domains Ω satisfying (2.2), (2.3) and in addition the Lipschitz condition

$$|\log\phi| + |(\log\phi)'| \le M. \tag{2.4}$$

Let u be the Neumann eigenfunction on Ω with lowest eigenvalue among functions that are odd with respect to the reflection $(x_1, x_2) \to (x_1, -x_2)$. This eigenfunction u can also be characterized as the lowest eigenvalue for the mixed boundary value problem on $\Omega^+ = \Omega \cap \{x : x_2 > 0\}$ with Neumann conditions on $\partial\Omega \cap \{x : x_2 > 0\}$

and Dirichlet conditions on $\Omega \cap \{x : x_2 = 0\}$. It is well known that u is unique up to a multiple and that one can choose the multiple so that $u > 0$ in Ω^+ and so that

$$\int_\Omega u^2 = 1.$$

We refer to this eigenfunction as normalized.

For two domains Ω_i given in polar coordinates by functions f_i we define the distance between the domains by

$$d(\Omega_1, \Omega_2) = \|\phi_1 - \phi_2\|_{L^\infty}.$$

Lemma 2.5. *There is a constant C depending only on M such that if Ω_i belong to L_M and λ_i are the corresponding lowest Neumann eigenvalues for functions odd with respect to x_2, then*

$$|\lambda_1 - \lambda_2| \le C d(\Omega_1, \Omega_2)^{1/2}.$$

Proof. Let u_i denote the lowest odd normalized eigenfunction for Ω_i. By the regularity theorem of [JK], u_i belongs uniformly to the Sobolev space $H^{3/2}(\Omega_i)$ with bounds depending only on M. In particular, $u_i \in L^\infty$ and $\nabla u_i \in L^4$. By Calderón's extension theorem, u_i can be extended to a function $\tilde{u}_i$ defined on a domain Ω containing both Ω_1 and Ω_2 with

$$\|\tilde{u}_i\|_{L^\infty(\Omega)} + \|\nabla \tilde{u}_i\|_{L^4(\Omega)} \le C$$

and preserving oddness with respect to the x_2 variable. It follows that

$$\int_{\Omega_1} \tilde{u}_2^2 \ge 1 - O(|\Omega_1 \backslash \Omega_2|); \quad \int_{\Omega_1} |\nabla \tilde{u}_2|^2 \le \lambda_2 + O(|\Omega_2 \backslash \Omega_1|^{1/2}).$$

Hence, $\lambda_1 \le \lambda_2 + O(d(\Omega_1, \Omega_2)^{1/2})$, and similarly for the inequality with 1 and 2 exchanged.

Lemma 2.6. *Suppose that the domains Ω_t belong to the class L_M. Let u_t denote the normalized odd eigenfunction for Ω_t with lowest eigenvalue. If $d(\Omega_t, \Omega_0) \to 0$ as $t \to 0$, then u_t tends to u_0 in C^k on compact subsets of Ω_0.*

Proof. Denote by λ_t the eigenvalue of u_t. The estimates of Lemma 2.5 imply

$$\int_{\Omega_0} |\nabla \tilde{u}_t|^2 \le \lambda_t + O(d(\Omega_t, \Omega_0)^{1/2}) \le \lambda_0 + O(d(\Omega_t, \Omega_0)^{1/2}),$$

$$\int_{\Omega_0} \tilde{u}_t^2 \ge 1 - O(d(\Omega_t, \Omega_0)).$$

The uniqueness of the lowest odd eigenfunction u_0 on Ω_0 implies that the second eigenvalue associated to odd eigenfunctions on Ω_0 is strictly greater. Therefore, for big-O constants depending on this spectral gap,

$$\left| \int_{\Omega_0} \tilde{u}_t u_0 \right| \ge 1 - O(d(\Omega_t, \Omega_0)^{1/2}).$$

One also has

$$\int_{\Omega_0} \tilde{u}_t u_0 = \int_{\Omega_0 \cap \Omega^t} \tilde{u}_t u_0 + O(d(\Omega_t, \Omega_t)).$$

Hence the normalization implies

$$\int_{\Omega_0 \cap \Omega^t} \tilde{u}_t u_0 > 0.$$

Consequently

$$\int_{\Omega_0} \tilde{u}_t u_0 \geq 1 - O(d(\Omega_t, \Omega_0)^{1/2})$$

and

$$\int_{\Omega_0} |\tilde{u}_t - u_0|^2 \leq O(d(\Omega_t, \Omega_0)^{1/2}). \tag{2.7}$$

Now suppose that there is a compact subset $K \subset \Omega$ on which u_t does not tend to u in $C^k(K)$ norm. By interior regularity, u_t is uniformly bounded in $C^{k+1}(K)$ norm, so there must be a subsequence u_{t_j} converging in $C^k(K)$ to a function v, but $v \neq u_0$. But this contradicts (2.7).

Next define an even more special class of domains $S_{M,\epsilon}$ as the collection of domains $\Omega = \{(r\cos\theta, r\sin\theta) : r < \phi(\theta)\}$ belonging to L_M, such that $\partial\Omega$ forms a right angle in an ϵ-neighborhood of each of the four points where it meets the axes and such that $\phi \in C^\infty([0, \pi/2])$.

Lemma 2.8. *Suppose that $\Omega_t = \{(r\cos\theta, r\sin\theta) : r < \phi_t(\theta)\}$ belongs to $S_{M,\epsilon}$ and $t \mapsto f_t$ is a continuous mapping into $C^\infty([0, \pi/2])$. Let u_t denote the normalized odd Neumann eigenfunction with least eigenvalue. There is an extension $\tilde{u}_t$ of u_t to a C^∞ function in a neighborhood $\tilde{\Omega}_t$ of Ω_t such that $\bar{\Omega}_{t_0} \subset \tilde{\Omega}_t$ for t near t_0 and $\tilde{u}_t$ tends to u_{t_0} in $C^k(\bar{\Omega}_{t_0})$ as $t \to t_0$ for any k.*

Proof. To extend u_t near the right angle corners, use reflection, which gives a real-analytic extension. Away from the corners use the fact that for every k, u_t is uniformly in C^k up to the boundary. It follows that for each k, $\tilde{u}_t$ belongs to $C^k(\tilde{\Omega}_t)$ uniformly in t. Hence by compactness of $C^k(\bar{\Omega}_{t_0})$ in $C^{k-1}(\bar{\Omega}_{t_0})$ and (2.7), $\tilde{u}_t$ converges in $C^{k-1}(\bar{\Omega}_{t_0})$ to u_{t_0}.

§3. Monotonicity of the odd eigenfunction

Denote by Ω_0 the diamond region formed by $|x_1 + x_2| < 1$ and $|x_1 - x_2| < 1$. Thus $\Omega_0 = \{(r\cos\theta, r\sin\theta) : r < \phi_0(\theta)\}$, where

$$\phi_0(\theta) = \frac{1}{\cos\theta + \sin\theta}. \tag{3.1}$$

Here is the main step towards proving monotonicity of the lowest odd eigenfunction.

Proposition 3.2. *Let $\Omega_1 = \{(r\cos\theta, r\sin\theta) : r < \phi_1(\theta)\}$ belong to $S_{M,\epsilon}$ and satisfy*

$$-\cot\theta < (\log\phi_1(\theta))' < \tan\theta$$

for all $0 \leq \theta \leq \pi/2$. Then the normalized odd eigenfunction u_1 associated to Ω_1 satisfies

$$\partial u_1/\partial x_2 \geq 0 \quad \text{and} \quad x_1 x_2 \partial u_1/\partial x_1 \leq 0.$$

Proof. Denote

$$\phi_t(\theta) = \phi_1(\theta)^t \phi_0(\theta)^{1-t}; \quad \Omega_t = \{(r\cos\theta, r\sin\theta) : r < \phi_t(\theta)\}.$$

Each of the domains

$$\Omega_t = \{(r\cos\theta, r\sin\theta) : r < \phi_t(\theta)\}, \quad 0 \leq t \leq 1,$$

belongs to $S_{M,\epsilon}$. In particular, if $\partial\Omega_1$ meets the positive x_1 and x_2 axes at $Q = (q_1, 0)$ and $P = (0, p_2)$, then Ω_t has a right angle at the corners $Q_t = (q_1^t, 0)$ and $P_t = (0, p_2^t)$. Furthermore, ϕ_t inherits from ϕ_1 and ϕ_0 the property

$$-\cot\theta < (\log\phi_t(\theta))' < \tan\theta, \quad 0 \le t \le 1. \tag{3.3}$$

Denote by u_t the lowest normalized Neumann eigenfunction of Ω_t, let λ_t be the corresponding eigenvalue and let $\tilde{u}_t$ be the extension as defined in Lemma 2.8. Denote the positive quadrant and upper half of Ω_t by

$$\Omega_t^{++} = \{x \in \Omega_t : x_1 > 0,\, x_2 > 0\}, \quad \Omega_t^{+} = \{x \in \Omega_t : x_2 > 0\},$$

respectively.

Assume, by contradiction, that either $\partial u_1/\partial x_1 > 0$ or $\partial u_1/\partial x_2 < 0$ at some point of Ω_1^{++}. Let t_0 be the infimum of all values of t for which this is the case. The formula for u_0 is

$$u_0 = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{\pi}{2}(x_1 + x_2)\right) - \sin\left(\frac{\pi}{2}(x_1 - x_2)\right)\right].$$

Since initially $\partial u_0/\partial x_1 < 0$ and $\partial u_0/\partial x_2 > 0$ in Ω_0^{++}, it follows from continuous dependence on t for $t \le t_0$ that

$$\partial u_{t_0}/\partial x_1 \le 0 \text{ and } \partial u_{t_0}/\partial x_2 \ge 0 \text{ on } \Omega_{t_0}^{++}. \tag{3.4}$$

Claim 1. *For t sufficiently close to t_0, $\partial u_t/\partial x_1 \le 0$ and $\partial u_t/\partial x_2 \ge 0$ in Ω_t^{++} in a neighborhood of fixed size of each of the two corners Q_t and P_t.*

Proof. Consider coordinates

$$y_1 = x_1 - q_1^t, \quad y_2 = x_2$$

centered at the corner Q_t. Recall that $\tilde{u}_t$ is real analytic in a neighborhood of $y = 0$. It satisfies $\tilde{u}_t = 0$ on $y_2 = x_2 = 0$ and the equation $\Delta u_t = -\lambda_t u_t$. Moreover $\tilde{u}_t$ is even with respect to reflection across the axes $y_1 = \pm y_2$. Hence

$$\operatorname{sign}\tilde{u}_t = -\operatorname{sign} y_1 y_2. \tag{3.5}$$

In particular, $\tilde{u}_t$ vanishes on $y_1 = 0$ and $y_2 = 0$. Finally, the coefficients of the power series of $\tilde{u}_t$ are continuous with respect to t. Thus the Taylor series in y is

$$\tilde{u}_t(y) = c_t y_1 y_2 + \text{ terms of degree at least 3}$$

and c_t tends to c_{t_0} as t tends to t_0. The main point is to show that $c_t < 0$. Indeed, the Taylor expansion of u_t must begin with a nonzero harmonic homogeneous polynomial of some degree k. But a harmonic polynomial of degree k has a zero set consisting of k equally spaced lines through $y = 0$ and the sign alternates on successive sectors. The only way this is consistent with (3.5) is if the harmonic polynomial is a negative multiple of $y_1 y_2$.

By continuity, $c_t \le -c < 0$ uniformly for t sufficiently close to t_0. It follows that for y in a fixed neighborhood of $y = 0$, depending on c and uniform bounds on higher derivatives,

$$\operatorname{sign}\frac{\partial u_t}{\partial y_1} = \operatorname{sign} c_t y_2 = -\operatorname{sign} y_2,$$
$$\operatorname{sign}\frac{\partial u_t}{\partial y_2} = \operatorname{sign} c_t y_1 = -\operatorname{sign} y_1.$$

Thus Claim 1 is proved near Q_t.

Next, near the corner P_t, consider coordinates

$$y_1 = x_1, \quad y_2 = x_2 - p_2^t.$$

In a neighborhood of $y = 0$, u_t is an analytic function that is even with respect to reflection across the lines $y_2 = \pm y_1$. The symmetry and the equation imply that the Taylor series takes the form

$$u_t = a_t - \frac{1}{2} a_t \lambda_t (y_1^2 + y_2^2) + b_t y_1 y_2 + \text{ terms of degree at least 3.}$$

The coefficients are continuous in t as $t \to t_0$. Since $a_t > 0$, $a_t \geq a > 0$ uniformly for all t near t_0. The additional fact that u_t is even with respect to x_1 (or y_1) implies that $b_t = 0$. It follows that for y in a fixed neighborhood of $y = 0$, depending on a and uniform bounds on higher derivatives,

$$\operatorname{sign} \frac{\partial u_t}{\partial y_1} = -\operatorname{sign} \lambda_t a_t y_1 = -\operatorname{sign} y_1,$$
$$\operatorname{sign} \frac{\partial u_t}{\partial y_2} = -\operatorname{sign} \lambda_t a_t y_2 = -\operatorname{sign} y_2.$$

This concludes Claim 1.

Claim 2. *There exists $x^0 \in (\partial\Omega_{t_0}) \cap (\partial\Omega_{t_0}^{++})$, $x^0 \neq P_{t_0}$ and $x^0 \neq Q_{t_0}$ such that $\nabla u_{t_0}(x^0) = 0$.*

Proof. Let $\mu(\Omega)$ denote the lowest Dirichlet eigenvalue of a region. Because λ_t is the lowest eigenvalue for the mixed problem on Ω_t^+ with Dirichlet data on the flat bottom and Neumann data on the top,

$$\lambda_t < \mu(\Omega_t^+). \tag{3.6}$$

Case 1. Suppose that the set in Ω_t where $w_t = \partial u_t / \partial x_2 < 0$ is nonempty for a sequence of values of t tending to t_0.

Consider a connected component U_t of $w_t < 0$. Since $u_t > 0$ in $x_2 > 0$, and $u_t = 0$ on $x_2 = 0$, $w_t \geq 0$ on $x_2 = 0$. It follows that U_t does not meet $x_2 = 0$ and we may assume by symmetry that U_t is a subset of Ω_t^+. If $(\partial U_t) \cap (\partial\Omega_t) = \emptyset$, then $w_t = 0$ on ∂U_t. But $\Delta w_t = -\lambda_t w_t$ on U_t, so $\mu(\Omega_t^+) \leq \mu(U_t) \leq \lambda_t$. This contradicts (3.6). In conclusion

$$(\partial U_t) \cap (\partial\Omega_t) \neq \emptyset. \tag{3.7}$$

Furthermore, Claim 1 implies that for t sufficiently close to t_0, U_t does not meet a fixed neighborhood of the corners. Thus if there is a sequence of values of $t \searrow t_0$ for which the set $w_t < 0$ is nonempty that set has limit points on $\partial\Omega_t$, and by the continuity of w_t with respect to t, we find in the limit $x^0 \in \partial\Omega_{t_0}$, not at a corner such that

$$\frac{\partial u_{t_0}(x^0)}{\partial x_2} \leq 0.$$

By symmetry we may as well assume that x^0 is in the first quadrant. By (3.4) we have the opposite inequality, so the derivative of u_{t_0} with respect to x_2 vanishes at x^0. Since by (3.3) the normal derivative as x^0 is not the x_2 direction, we conclude that the full gradient of u_{t_0} is zero at x^0.

Case 2. $\Omega_t^{++} \cap \{x : (\partial u_t / \partial x_1)(x) > 0\}$ is nonempty for a sequence of $t \nearrow t_0$.

This case is similar and simpler than Case 1 because $\partial u_t / \partial x_1 = 0$ on both of the axes, so the domain on which a Dirichlet eigenfunction is produced is a subset of an even smaller domain Ω_t^{++}. The same argument as in Case 1 shows that there is a point of $(\partial\Omega_{t_0}) \cap \partial\Omega_{t_0}^{++}$ that is not a corner but at which

$$\frac{\partial u_{t_0}(x^0)}{\partial x_1} \geq 0.$$

Then (3.4) implies that $\partial u_{t_0}(x^0)\partial x_1 = 0$, and (3.3) says that the normal is not in the x_1 direction. Hence the Neumann condition implies that the full gradient of u_t vanishes at x^0.

Claim 3. *There exist points of Ω_{t_0} near x^0, the point defined in Claim 2, at which $\partial u_{t_0}/\partial x_1 > 0$. This is a contradiction since $\partial u_{t_0}/\partial x_1 \leq 0$ for all points of $\Omega_{t_0}^{++}$.*

Proof. Consider a rectangular coordinate system (y_1, y_2) with origin at x^0 and so that $\partial/\partial y_2$ is the outer normal at x^0 and $\partial/\partial y_1$ is the tangential derivative in the clockwise direction. Write

$$\frac{\partial}{\partial x_1} = A\frac{\partial}{\partial y_1} + B\frac{\partial}{\partial y_2}.$$

It follows from assumption (3.3) that $A > 0$ and $B > 0$.

Denote $v = u_{t_0}$. The function v has the Taylor expansion

$$v = a - \frac{\lambda_{t_0}}{2}ay_2^2 + \alpha y_1 y_2 + \beta(y_1^2 - y_2^2) + \text{ (terms of order} \geq 3).$$

The boundary is given by a Taylor series $y_2 = py_1^2 + \cdots$. Substituting py_1^2 for y_2 one sees that, along the boundary,

$$v = a + \beta y_1^2 + O(y_1^3).$$

But v is monotone decreasing along the boundary as y_1 increases, so it must be that $\beta = 0$. Next, the Neumann condition implies that, along the boundary,

$$\partial v/\partial y_2 - 2py_1\partial v/\partial y_1 = O(y_1^2).$$

But $y_1\partial v/\partial y_1 = O(y_1^2)$ and $\partial v/\partial y_2 = \alpha y_1 + O(y_1^2)$ and hence $\alpha = 0$. In all,

$$v = a - \frac{\lambda_{t_0}}{2}ay_2^2 + \text{ (terms of order} \geq 3).$$

Note also that $a > 0$ and $\lambda_{t_0} > 0$. It follows that in a neighborhood of $y = 0$,

$$\frac{\partial v}{\partial y_2} = -\lambda_{t_0}ay_2 + O(|y|^2); \quad \frac{\partial v}{\partial y_1} = O(|y|^2).$$

Therefore, for $y_2 < 0$, $y_1 = 0$, and $|y|$ sufficiently small,

$$\frac{\partial v}{\partial x_1} = \text{sign}\left(A\frac{\partial v}{\partial y_1} + B\frac{\partial v}{\partial y_2}\right) = -\text{sign}\, B\lambda_{t_0}ay_2 > 0.$$

This concludes the proof of Claim 3 and of Proposition 3.2.

Corollary 3.8. *Let Ω belong to L_M. Then its normalized odd eigenfunction u satisfies*

$$\partial u_1/\partial x_2 \geq 0 \quad \textit{and} \quad x_1x_2\partial u_1/\partial x_1 \leq 0.$$

Proof. By Lemma 2.6, it suffices to approximate Ω in L_M by domains for which the hypotheses of Proposition 3.2 are satisfied. Take ϕ corresponding to Ω and let $a_\epsilon = \phi(\epsilon)/\phi_0(\epsilon)$ and $b_\epsilon = \phi(\pi/2-\epsilon)/\phi_0(\pi/2-\epsilon)$. Define

$$\begin{aligned}&\eta_\epsilon(\theta) = \phi(\theta),\ \epsilon \le \theta \le \pi/2-\epsilon,\\ &\eta_\epsilon(\theta) = a_\epsilon\phi_0(\theta),\ 0 \le \theta \le \epsilon,\\ &\eta_\epsilon(\theta) = b_\epsilon\phi_0(\theta),\ \pi/2-\epsilon \le \theta \le \pi/2,\end{aligned}$$

and extend η_ϵ by symmetry to the other three quadrants. The function η_ϵ is continuous and belongs to $L_{M'}$ with $M' = \max(M, 2a_\epsilon, 2b_\epsilon)$. Furthermore, η_ϵ tends uniformly to ϕ as $\epsilon \to 0$ because ϕ_0 and ϕ are continuous. Next, let $0 < \delta < 1$. Since ϕ_0 satisfies the strict inequality (3.3), $\eta_\epsilon^{1-\delta}\phi_0^\delta$ satisfies the corresponding strict inequality uniformly in $[0,\pi/2]$ with a bound depending on $\delta > 0$. Such functions tend uniformly to ϕ as $\epsilon \to 0$ and $\delta \to 0$. Moreover $\eta_\epsilon^{1-\delta}\phi_0^\delta$ is equal to a multiple of ϕ_0 near the corners. Finally, for each fixed $\delta > 0$, take $\log(\eta_\epsilon^{1-\delta}\phi_0^\delta)$, and use convolution to approximate by a $C^\infty([0,\pi/2])$ function (but leave the function unchanged in a neighborhood of 0 and $\pi/2$). The constraints of Proposition 3.2 are linear in the logarithm, so they are preserved. In this way one approximates the original function ϕ uniformly by functions satisfying the hypotheses of Proposition 3.2, and which all belong to $L_{M'}$.

§4. Strict monotonicity for polygons

We begin by stating the nondegeneracy conditions valid on polygons except at the vertices.

Theorem 4.1. *Suppose that Ω is a polygon that is symmetric with respect to both axes and for which vertical and horizontal cross sections are intervals. Let u be the lowest Neumann eigenfunction odd with respect to x_2 as in Theorem* 1.1. *Unless Ω is a rectangle, u has a unique maximum and minimum at the two points of $\partial\Omega$ on the x_2 axis and*

(a) *$\partial u/\partial x_2 > 0$ at every point of Ω.*

(b) *The tangential derivative along the boundary is nonzero except on the x_2 axis and at vertices (where it is zero or undefined).*

(c) *The second tangential derivative at the maximum (and minimum) is nonzero if the maximum does not occur at a vertex.*

In the exceptional case of the rectangle, the symmetry implies that the rectangle has sides parallel to the axes. The domain has the form $-L_1 < x_1 < L_1$, $-L_2 < x_2 < L_2$, and $u = \sin(\pi x_2/2L_2)$. The maximum and minimum are achieved on the whole upper and lower sides, $x_2 = \pm L_2$.

Proof. Corollary 3.8 implies that $w = \partial u/\partial x_2 \ge 0$ in Ω. It follows from the generalized mean value property (for small circles) that $w > 0$ in Ω or else $w \equiv 0$. But $w \equiv 0$ and $u = 0$ on $x_2 = 0$ implies $u \equiv 0$. This proves (a). In particular, the maximum and minimum cannot be achieved in the interior.

To prove (b), consider coordinates centered at a boundary point on a side of the polygon, not a vertex and not on the x_2 axis. Choose a rectangular coordinate system y_1, y_2 so that $y = 0$ is the boundary point and the y_1 axis contains the side of the polygon. We may assume, without loss of generality, that $y = 0$ is a boundary point in the second quadrant $x_2 \geq 0$, $x_1 < 0$ and that

$$\frac{\partial}{\partial y_1} = A\frac{\partial}{\partial x_1} + B\frac{\partial}{\partial x_2},$$

where $A \geq 0$ and $B \geq 0$. Corollary 3.8 says that in the second quadrant $\partial u/\partial x_i \geq 0$, $i = 1, 2$. Hence

$$v = \frac{\partial u}{\partial y_1} \geq 0$$

near $y = 0$. Note that by the Neumann condition we may reflect u evenly across $y_2 = 0$, and v can thus also be considered an even analytic function in a full neighborhood of $y = 0$.

We first show that if $v \equiv 0$, then Ω is a rectangle. By analytic continuation $\partial u/\partial y_1 = 0$ on all of Ω and $u = f(y_2)$. Let m be the smallest value of y_2 for points of $\bar{\Omega}$ and let M be the largest. Then for $m < y_2 < M$, there exists an interior point (of Ω) with that coordinate. Since $\partial u/\partial x_2$ is nonzero on the interior, $f'(y_2) \neq 0$ for all such y_2. The Neumann condition now implies that the sides of the polygon in the range $m < y_2 < M$ are normal to the y_1 direction. It follows that the region is a rectangle of the form $m < y_2 < M$, $c < y_1 < C$.

Thus, if the domain is not a rectangle, then v is not identically zero. We can now show that $a_1 > 0$, where

$$u = a_0 + a_1 y_1 + O(|y|^2); \quad v = a_1 + O(|y|).$$

Both u and v are even functions of y_2 and $\Delta v = -\lambda v$. But $v \geq 0$ implies $a_1 \geq 0$. If $a_1 = 0$, then $v = O(|y|)$. Therefore, since v is not identically zero, there is a nonzero harmonic polynomial, homogeneous of degree $k \geq 1$ such that $v = h_k + O(|y|^{k+1})$. But this contradicts the fact that $v \geq 0$ near $y = 0$. It follows that $a_1 > 0$. This concludes the proof of (b).

The function u is continuous up to the boundary.[2] We have just shown that if Ω is not a rectangle, then u is strictly increasing on each side, so it is strictly increasing along the boundary on each arc starting from the bottom point on the x_2 axis to the top point. Thus the maximum and minimum are unique and attained on the boundary intersected with the x_2 axis.

To prove (c), consider coordinates at the maximum point $P = (0, p_2)$, given by $y_1 = x_1$ and $y_2 = x_2 - p_2$. Then u is even with respect to y_1, and the Neumann condition implies that it can be extended across the top side to be even with respect to y_2. If u has zero second derivative in the $x_1 = y_1$ direction, then it has an expansion of the form

$$u = a_0 - \frac{1}{2}\lambda a_0 y_2^2 + w(y),$$

where $w = O(|y|^4)$ and w is even with respect to both y_1 and y_2 and satisfies $(\Delta + \lambda)w = \lambda^2 a_0 y_2^2/2$. But then

$$w_1 = \frac{\partial u}{\partial x_1} = \frac{\partial w}{\partial y_1}$$

satisfies $(\Delta + \lambda)w_1 = 0$. If $w_1 \equiv 0$, then as we showed above, the region is a rectangle. If not, then $w_1 = h_k(y) + O(|y|^{k+1})$ for some nonzero harmonic polynomial h_k that is homogeneous of degree k for some $k \geq 3$. But the pattern of signs in sectors of

[2]In two dimensions the Sobolev space $H^{3/2}(\bar{\Omega})$ is a subset of a Hölder class.

h_k is inconsistent with the fact (from Corollary 3.8) that w_1 is nonnegative in the subset of Ω where $x_1x_2 < 0$ and nonpositive in the subset of Ω where $x_1x_2 > 0$.

Next we prove nondegeneracy at the vertices of the polygon. To formulate the theorem, we describe the asymptotic expansion of u at vertices. Let Q be a vertex of a polygon Ω with interior angle σ. Let $\gamma = \gamma(x)$ be the angle that the segment from x to Q makes with an edge with endpoint Q, $\rho = |x - Q|$, and $\alpha = \pi/\sigma$. One can expand u on the circular arc $\rho = \rho_0$ and $0 < \gamma < \sigma$ in a Fourier cosine series

$$u(x) = \sum_{n=0}^{\infty} c_n \cos(n\alpha\gamma), \quad |x - Q| = \rho_0. \tag{4.2}$$

The series converges and the coefficients c_n are rapidly decreasing. It follows that for ρ_0 sufficiently small,

$$u(x) = \sum_{n=0}^{\infty} c_n (J_{n\alpha}(\sqrt{\lambda}\rho)/J_{n\alpha}(\sqrt{\lambda}\rho_0)) \cos(n\alpha\gamma) \tag{4.3}$$

for all x, $|x - Q| = \rho \le \rho_0$. Recall that

$$J_s(r) = \sum_{m=0}^{\infty} (-1)^m (r/2)^{s+2m}/[m!\Gamma(m + s + 1)].$$

It follows in particular that if $0 < \sigma < \pi$ (i.e., $\alpha > 1$), then

$$u(x) = a_0 + a_1\rho^\alpha \cos(\alpha\gamma) - \frac{1}{2}a_0\lambda\rho^2 + O(\rho^{\min(2\alpha,4)}). \tag{4.4}$$

On the other hand, if $\pi < \sigma \le 3\pi/2$ (i.e., $2/3 \le \alpha < 1$), then

(4.5)

$$u(x) = a_0 + a_1\rho^\alpha \cos(\alpha\gamma) + a_2\rho^{2\alpha} \cos(2\alpha\gamma) + a_3\rho^{3\alpha} \cos(3\alpha\gamma) - \frac{1}{2}a_0\lambda\rho^2 + O(\rho^{\alpha+2}).$$

Theorem 4.6. *With the hypothesis of Theorem* 4.1*,*

(a) *Let $Q \in \partial\Omega$ be a vertex of the polygon on the x_2 axis. Then the expansion of u has the form*

$$u(x) = a_0(1 - \frac{1}{2}\lambda\rho^2) + o(\rho^2)$$

with $a_0 \neq 0$.

(b) *If $Q \in \partial\Omega$ is a vertex not on the x_2 axis, then u has an expansion of the form*

$$u(x) = a_0 + a_1\rho^\alpha \cos(\alpha\gamma) + o(\rho^\alpha)$$

with $a_1 \neq 0$.

Proof. For part (a), note that the interior angle at Q is convex, $\sigma < \pi$. The expansion has the form (4.4), but because u is even, $a_1 = 0$. Moreover, $a_0 = u(Q) \neq 0$.

To prove (b) consider first the case of a vertex Q on the x_1 axis. Then $0 < \sigma < \pi$ ($\alpha > 1$). Furthermore, $a_0 = 0$. Let a_k denote the first nonzero coefficient. Then the expansion takes the form

$$u(x) = a_k\rho^{k\alpha} \cos(k\alpha\gamma) + o(\rho^{k\alpha}).$$

But $u > 0$ for $x_2 > 0$ and $u < 0$ for $x_2 < 0$ in the domain, whereas for every $k > 1$, the function $a_k \rho^{k\alpha} \cos(k\alpha\gamma)$ changes sign more times in the sector in Ω. Therefore $a_1 \neq 0$.

Consider next the case $\sigma = 3\pi/2$, $\alpha = 2/3$. This only occurs if the sides of Ω that meet at Q are horizontal and vertical. Because Q is not on the x_1 axis, $a_0 \neq 0$ in expansion (4.5). Assume, by contradiction, that $a_1 = 0$. Then monotonicity along $\partial\Omega$ near Q implies $a_2 = 0$. Since $\rho^2 = \rho^{3\alpha}$, the sum of the a_3 term and the ρ^2 term is a quadratic polynomial. The Neumann boundary conditions on the horizontal and vertical sides imply that this quadratic polynomial has the form $\alpha(x_1 - q_1)^2 + \beta(x_2 - q_2)^2$ $(Q = (q_1, q_2))$. It follows that $\partial u/\partial x_1 \approx 2\alpha(x_1 - q_1)$ and $\partial u/\partial x_2 \approx 2\beta(x_2 - q_2)$. The coefficients α and β cannot both be zero because $a_0 \neq 0$. Hence one of these linear approximations changes sign in the interior of the domain. This contradicts the fact that $\partial u/\partial x_i$ does not change sign in Ω near Q.

Next assume that Q is a vertex with $\sigma < 3\pi/2$, but not on either axis. Then $a_0 \neq 0$. Assume by contradiction that $a_1 = 0$ in (4.4) or (4.5). Either the $\rho^{2\alpha}$ term or the ρ^2 term determines the behavior of u near Q, depending on whether $\alpha > 1$ or $\alpha < 1$. They do not cancel each other because $\alpha \neq 1$. But $\cos(2\alpha\gamma) = 1$ along both sides that meet at Q. Thus both the functions ρ^2 and $\rho^{2\alpha}\cos(2\alpha\gamma)$ are decreasing as ρ tends to zero along each side. But this contradicts monotonicity of u along the boundary on both of the paths joining the minimum to the maximum.

Finally, we deduce Corollary 1.2 in the special case of polygons.

Proposition 4.7. *Let λ be the eigenvalue of u in Theorem* 1.1. *Assume in addition that Ω is a polygon. Then*

$$\lambda < \mu(\Omega),$$

where $\mu(\Omega)$ is the lowest Dirichlet eigenvalue of Ω.

Proof. Denote $w = \partial u/\partial x_2$. By Theorem 4.1 (a) and (b), $w > 0$ in Ω, $w \geq 0$ on $\partial\Omega$ and $w > 0$ on any side of the polygon that is not horizontal. Let $v > 0$ be the Dirichlet eigenfunction of Ω with eigenvalue $\mu = \mu(\Omega)$. By Green's formula,

$$\begin{aligned} \int_{\partial\Omega} w \frac{\partial v}{\partial \nu} d\sigma &= \int_{\partial\Omega} \left(w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right) d\sigma \\ &= \int_{\Omega} (w\Delta v - v\Delta w) \\ &= (\lambda - \mu) \int_{\Omega} vw. \end{aligned}$$

(Green's formula can be proved by passing to the limit from parallel boundaries using the asymptotics at vertices of w and v: $\nabla^k v = O(\rho^{2/3-k})$ and $\nabla^k w = O(\rho^{2/3-1-k})$.) Since $\partial v/\partial \nu < 0$ on $\partial\Omega$ except at the vertices, one has $\lambda - \mu < 0$.

§5. Equilateral polygons

We begin now to deal with the case of a multiple eigenvalue.

Proposition 5.1. *Let Ω be the equilateral polygon with $4k$ sides, $k > 1$. Then every second eigenfunction u for the Neumann problem satisfies the following.*

(a) *u is strictly monotone increasing on both arcs of the boundary from the minimum to the maximum.*

(b) *The tangential derivative of* u *is nonzero except at the minimum and maximum and the vertices.*

(c) *The second tangential derivative of* u *is nonzero at the maximum and minimum, assuming these occur on a side, not a vertex.*

(d) *If the maximum and minimum occur at vertices, then the expansion is nondegenerate in the sense of Theorem* 4.6 (a). *At vertices other than the maximum and minimum* u *has an expansion of the form* (4.4) *with* $a_1 \neq 0$.

(e) u *has no critical points on the interior of* Ω.

Proof. The main point is to prove a weak form of (a), namely that u is monotone along the two arcs of the boundary from the minimum to the maximum.

Orient the polygon so that there are horizontal and vertical sides. Let u^0 be the eigenfunction that is odd with respect to x_2, positive in $x_2 > 0$ and normalized to have $L^2(\Omega)$ norm 1. Denote rotation counterclockwise by the angle θ by

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $v(x) = u^0(A_{\pi/2}x)$. Thus $v > 0$ in $x_1 < 0$ and v is odd with respect to x_1. Since the eigenspace is two dimensional [N], the family of all normalized eigenfunctions can be written as

$$u^\theta(x) = (\cos\theta)u^0(x) + (\sin\theta)v(x).$$

Observe that

$$\nabla u^\theta(0) = (\cos\theta)\nabla u^0(0) + (\sin\theta)\nabla v(0) = (-\sin\theta, \cos\theta)|\nabla u(0)|$$

and u^θ is the unique normalized eigenfunction with gradient at 0 in this direction. Let

$$Y_1 = \cos(\pi/4k)x_1 + \sin(\pi/4k)x_2, \quad Y_2 = -\sin(\pi/4k)x_1 + \cos(\pi/4k)x_2.$$

Then Ω is also symmetric with respect to the Y_1 and Y_2 axes. In these coordinates Ω has a corner at the maximum and minimum value of Y_2. If w is positive in $Y_2 > 0$ and odd with respect to Y_2, then ∇w at 0 is parallel to ∇Y_2 and nonzero and hence

$$w = u^{\pi/4k}. \tag{5.2}$$

By similar reasoning, $u^{\pi/2k}(x) = u(A_{\pi/2k}x)$ is the unique normalized eigenfunction with gradient at 0 in the direction $A_{\pi/2k}e_2$ and $u^{\pi/2+\pi/2k}(x) = v(A_{\pi/2k}x)$ is the unique normalized eigenfunction with gradient at 0 in the direction $A_{\pi/2+\pi/2k}e_2$. Hence, taking linear combinations,

$$u^{\theta+\pi/2k}(x) = u^\theta(A_{\pi/2k}x) \tag{5.3}$$

for all θ. Let R denote the reflection $(x_1, x_2) \to (-x_1, x_2)$ across the x_2 axis. Then because u is even with respect to R and v is odd,

$$u^{-\theta}(x) = u^\theta(Rx). \tag{5.4}$$

It follows from (5.3) that it suffices to prove monotonicity for u^θ for $-\pi/4k\theta < \pi/4k$. From (5.4) it follows that it suffices to consider only $0 < \theta < \pi/4k$.

In the range $0 < \theta < \pi/4k$, u^θ is a linear combination of u^0 and w with positive coefficients. We focus our attention on the segment S defined as the left half of the top horizontal side. The left endpoint of S is the point $Q \in \partial\Omega$ at which $Y_1 = 0$ and $Y_2 > 0$. The right endpoint of S is the point $P \in \partial\Omega$ at which $x_1 = 0$, $x_2 > 0$. Since w is strictly monotone along the boundary between the two vertices on the Y_2 axis and u is strictly monotone along the boundary from the two vertices on the x_2 axis, u^θ is strictly monotone except possibly on S and its reflection through the origin $A_\pi(S)$ (the right half of the bottom horizontal side). Since all the functions u^θ have odd symmetry under the rotation by π, $u^\theta(x) = -u^\theta(A_\pi x)$, our monotonicity property will be valid if we can prove that u^θ has a unique maximum in S and is monotone on both sides.

Assume (by contradiction) that for some θ, $0 < \theta < \pi/4k$, u^θ is not monotone along the boundary from its minimum to its maximum. Let ϕ be the infimum of $\theta > 0$ for which monotonicity fails. We show first that $\phi > 0$ and that on the top side of the boundary u^ϕ has either an additional critical point or else the maximum is degenerate. We will then rule out these two cases.

The expansion of u^0 at Q is

$$u^0(x) = a_0 + a_1\rho^\alpha \cos(\alpha\gamma) - \frac{1}{2}a_0\lambda\rho^2 + O(\rho^{2\alpha})$$

with $a_0 > 0$, $\alpha = 2k/(2k-1)$, $\rho = |x - Q|$, and γ is the angle $x - Q$ makes with the horizonal side S. Moreover, Theorem 4.6 implies that $a_1 > 0$. On the other hand, w attains its maximum at Q and so its expansion at the vertex Q begins (w is even with respect to Y_1) with a term of size $\rho^{2\alpha}$. It follows that a linear combination of u and w with nonzero factor on u is strictly monotone in a neighborhood of Q. In other words, for $0 < \theta < \pi/4k$, $\partial u^\theta/\partial x_1 > 0$ in a set including a neighborhood of Q along S. This set only increases as θ decreases to 0.

For any $\epsilon > 0$ the exists $\tau > 0$ such that for $0 < \theta < \tau$,

$$\partial u^\theta/\partial x_1 > 0$$

on S except for an ϵ-neighborhood of P. This follows from the fact that $\partial u/\partial x_1 > 0$ on S except at the endpoints Q and P and the fact that we have already proved this strict monotonicity near Q. But by Theorem 4.1 (c), $\partial^2 u/\partial x_1^2(P) = -c < 0$. Therefore, for $\theta > 0$ sufficiently small, $\partial^2 u^\theta/\partial x_1^2 = -c/2 < 0$ in a fixed neighborhood of P. It follows that for θ sufficiently small u^θ has exactly one critical point, a nondegenerate maximum. In particular, $\phi > 0$.

Let Q_1 be a limit point of the maximum points of u^θ as $\theta \nearrow \phi$. Because u^ϕ is the uniform limit on S of u^θ, u^ϕ is strictly increasing on $\partial\Omega$ from Q to Q_1 and strictly decreasing from Q_1 to P. (The strict increasing and decreasing properties follow from the fact that u^ϕ is real analytic on the side of Ω containing S and because it is strictly increasing on the right half of the side, it cannot be constant on S.) But the perturbation of an analytic function with one nondegenerate critical point (a maximum) has exactly one nondegenerate critical point (also a maximum). Since u^ϕ is the limit from $\theta > \phi$ of functions that are not monotone on each side of a maximum, it must be either that u^ϕ has a degenerate maximum at Q_1 or a critical point at a point $Q_2 \neq Q_1$. (Note also that as θ increases the proportion of u in u^θ decreases and the proportion of w increases, so that the maximum moves to the left and a new critical point Q_2 can only appear to the left of Q_1.)

We will derive a contradiction using the behavior of the zero set of

$$V = \frac{\partial u^\phi}{\partial x_1}.$$

Observe that

$$\frac{\partial V}{\partial x_2} = 0 \text{ on } S.$$

Thus we can extend V and u^ϕ by reflection across S as even functions.

Claim. *The function V is nonnegative along the open segment QQ_1 and negative along the segment Q_1P. It is strictly positive in QQ_1 except at a finite number of points of type Q_2 on this segment. Each of the points of type Q_2 is a degenerate critical point (i.e., the tangential first and second derivatives of u^ϕ vanish there) and there are at least two branches of $V = 0$ (or more generally an even number of branches) pointing into Ω with the sign of V alternating in the sectors in between consistent with the sign $V \geq 0$ on the boundary near Q_2. At Q_1, the maximum point for u^ϕ, there are an odd number of branches of $V = 0$ pointing into the region, consistent with the sign of V, namely, $V > 0$ on the left to $V < 0$ on the right of Q_1. The presence of only one branch at Q_1 means that Q_1 is a nondegenerate maximum.*

Proof. Choose coordinates y_1 and y_2 translated so that Q_1 is the origin. Then $\partial u^\phi/\partial y_1 = 0$ at Q_1 ($y = 0$). Consider the case in which Q_1 is a degenerate maximum. Then $\partial^2 u^\phi/\partial^2 y_1 = 0$ and the maximum property implies in addition that $\partial^3 u^\phi/\partial^3 y_1 = 0$. The Taylor expansion of u^ϕ has only even powers of y_2. Together with the eigenfunction equation these equations at Q_1 imply an expansion of the form

$$u^\phi = a_0 - \frac{1}{2}\lambda a_0 y_2^2 + O(|y|^4).$$

Thus $V = O(|y|^3)$ and $(\Delta + \lambda)V = 0$. Moreover, since u^ϕ is not constant on S, V is not identically zero. Hence $V = h_k + O(|y|^{k+1})$ for some nonzero harmonic polynomial homogeneous of degree $k \geq 3$. Since V is even in y_2, h_k is even in y_2, which specifies it uniquely in the two-dimensional space of homogeneous harmonic polynomials of degree k. In particular, its zero set has k branches pointing into Ω. The sign of V alternates between positive and negative between the branches. Since $V > 0$ at all but finitely many points on the part of S to the left of Q_1 and $V < 0$ on the part of S to the right of Q_1, it follows that k is odd. (This can also be seen by other means directly from the expansion.) In the case of a nondegenerate maximum at Q_1, u^ϕ has a term with y_1^2, and V has a linear term in y_1 and there is exactly one branch in Ω.

Next, suppose there is a critical point $Q_2 \neq Q_1$. Translate the coordinates x_1 and x_2 to coordinates z_1 and z_2 centered at Q_2. We have $\partial u^\phi/\partial z_1 = 0$ at Q_2 ($z = 0$). Since u^ϕ is monotone increasing at Q_2, we also have $\partial^2 u^\phi/\partial z_1^2 = 0$ at $z = 0$. In other words, the new critical point must be degenerate. The Neumann condition and eigenfunction equation imply

$$u^\phi = a_0 - \frac{1}{2}\lambda a_0 z_2^2 + \alpha(z_1^3 - 3z_1 z_2^2) + O(|z|^4).$$

Thus $V = 3\alpha(z_1^2 - z_2^2) + O(|z|^3)$, and if $\alpha \neq 0$, then the zero set of V has two branches pointing into Ω. In the two sectors containing S, $V > 0$, so that there is a sector strictly inside on which $V < 0$. (Equivalently, since u^ϕ is monotone increasing in z_1, $\alpha \geq 0$. This determines the sign of V in each sector.) On the other hand, if $\alpha = 0$, then $V = h_k + O(|z|^{k+1})$ where h_k is a nonzero harmonic polynomial, homogeneous of degree k, even in z_2. Thus there are $k > 2$ branches of the zero set of V pointing into Ω. Furthermore, because $V \geq 0$ on S on both sides of Q_2, k is even.

Having proved the claim, our task now is to show that this pattern of the zero set of V leads to a contradiction in all cases in which there is at least one degenerate critical point.

The line $Y_2 = 0$ has positive slope in x coordinates. Except at $x = 0$ it belongs to the two quadrants satisfying $x_1 x_2 > 0$ and hence $\partial u/\partial x_1 \le 0$ on this line. On the other hand, $\partial w/\partial Y_2 > 0$ and $\partial w/\partial Y_1 = 0$ on this line, which implies $\partial w/\partial x_1 < 0$. In all,

$$\frac{\partial u^\theta}{\partial x_1} < 0 \tag{5.5}$$

in Ω on the line $Y_2 = 0$ for all θ, $0 < \theta < \pi/4k$. It is also true that this inequality is true in the whole sector interior to Ω in a neighborhood of the points where $Y_2 = 0$ meets $\partial\Omega$. (This is because $\partial(\rho^\alpha \cos(\alpha\gamma))/\partial x_2 > 0$ in the sector $0 < \theta < \sigma$ with $\alpha = \pi/\sigma$ and $0 < \sigma < \pi$. And this is the main term in the expansion – sum of u and w contributions which have the same sign here.)

Consider the region $R = \{x \in \Omega : Y_2 > 0\}$. Define A_1 and A_2 to be the left and right endpoints, respectively, of the open interval $\{Y_2 = 0\} \cap \Omega$, that is, A_1 is in the region $x_1 < 0$ and A_2 is in $x_1 > 0$. Then $V \ge 0$ on the arc of ∂R from A_1 to Q_1 traveling clockwise. (The places where $V = 0$ include the vertical side adjacent to A_1 and the degenerate critical points of type Q_2.) Furthermore, $V < 0$ on the arc of ∂R from Q_1 to A_1 traveling clockwise. (The arc from Q_1 to A_2 is on $\partial\Omega$ where we already have this sign condition from monotonicity properties of u and w. The sign on the segment $Y_2 = 0$ from A_2 to A_1 is handled by (5.5).)

The most important special case is the one in which there is exactly one extra degenerate critical point of type Q_2 and a nondegenerate maximum Q_1. Then, in the simplest case, there are exactly two branches of $V = 0$ from Q_2 and along any segment S' formed by the intersection with $\bar{\Omega}$ of a line parallel to S and slightly below S, V changes sign exactly 3 times. The segment S' will be fixed at some sufficiently small distance from S for the remainder of the argument. Thus S' is the union of four closed intervals S_k, $k = 1, \ldots, 4$, ordered from left to right, overlapping only at endpoints and such that $V > 0$ on the interior of S_1 and S_3 and $V < 0$ on the interior of S_2 and S_4.

Case 1. The component of $R \backslash \{V = 0\}$ that contains S_1 also contains S_3.

In this case, the component R_1 of $\{x \in R : V(x) < 0\}$ containing S_2 cannot reach the negative portion of the boundary (the arc of the boundary going clockwise from Q_1 to A_1). Thus $\partial R_1 \cap \partial R = \{Q_2\}$. Therefore, in particular, $V = 0$ on ∂R_1. Since we also have $(\Delta + \lambda)V = 0$ it follows that the Dirichlet eigenvalue $\mu(R_1) \le \lambda$. But R_1 is a subset of Ω, so $\mu(\Omega) \le \lambda$. This contradicts Payne's theorem (Theorem 1.3).

Case 2. The component of $R \backslash \{V = 0\}$ that contains S_2 also contains S_4.

In this case, the component of R_2 of $\{x \in R : V(x) > 0\}$ that contains S_3 does not meet ∂R except on a subset of the top side (the interval from Q_2 to Q_1). In particular, $V = 0$ on ∂R_2 except at the points of ∂R_2 on the top side. But V satisfies the Neumann condition $\partial V/\partial x_2 = 0$ on the top side. Define a region R_2^* as the double of R_2 reflected across the top side. Extend V to an even function V^* with respect to this reflection. Then $V^* = 0$ on ∂R_2^* and $(\Delta + \lambda)V^* = 0$ in R_2^*. Hence $\mu(R_2^*) \le \lambda$. On the other hand, since $R_2 \subset R$, the area of R_2^* is less than the area of Ω. This contradicts Payne's theorem.

The other degenerate cases are similar, but easier. For example, if Q_1 is a degenerate maximum, then there are at least three branches of $V = 0$ from Q_1 and a contradiction of the type of Case 1 can be obtained. In general, V changes sign k times for some odd number $k \geq 3$. Thus we have shown by contradiction that u^θ is monotone from its minimum to its maximum.

To see that u^θ is strictly monotone along each arc of the boundary note that the proof showed that u^ϕ does not have any critical points other than a nondegenerate maximum and minimum, using only the fact that u^ϕ was monotone. The same now applies to every u^θ. Thus we have confirmed (a), (b), and (c) of Proposition 5.1.

To prove (d) note that the proof that $a_1 \neq 0$ is similar to the proof of Theorem 4.6 (b). If $a_0 \neq 0$ the proof is exactly the same. If it were true that $a_1 = 0$, then the nonzero second order terms in the expansion of u^θ at the vertex violate the monotonicity of u^θ along the boundary near the vertex. If both $a_0 = 0$ and $a_1 = 0$, then following the argument of Theorem 4.6 (b), monotonicity implies that the expansion has a leading term of the form $\rho^{3\alpha}\cos(3\alpha\gamma)$ (or even higher order) which gives (at least) three branches of the zero set entering Ω. This contradicts the fact that u^θ has only two nodal domains [CH] combined with the fact that no nodal domain can be closed, which is a consequence of Payne's theorem.

Finally, to prove part (e), observe that for all θ, $0 \leq \theta \leq \pi/4k$, the monotonicity of u^θ in the tangential direction along the boundary implies that

$$\frac{\partial u^\theta}{\partial x_2} \geq 0 \quad \text{on } \partial\Omega.$$

But Proposition 4.7 implies that the lowest Dirichlet eigenvalue for Ω is strictly larger than λ. Therefore, the generalized maximum principle implies that

$$\frac{\partial u^\theta}{\partial x_2} > 0 \quad \text{on } \Omega.$$

§6. Monotonicity for multiple eigenfunctions

Proposition 6.1. *Let $m > 1$. Let Ω_1 be a convex polygon with $4m$ sides, symmetric with respect to both axes. Assume also that there is no vertex on an axis. Let u be any Neumann eigenfunction with lowest nonzero eigenvalue. Then u achieves its maximum (minimum) at exactly one point, that point is on the boundary and u is monotone increasing along both arcs of the boundary from the minimum to the maximum.*

Proof. The only case that has not been covered by Theorem 4.1 is the case in which Ω_1 has a multiple eigenvalue.

Lemma 6.2. *There is a family of convex polygons Ω_t, $0 \leq t \leq 1$, such that*

(a) *Ω_t has two axes of symmetry.*
(b) *Ω_t has $4m$ vertices, $m > 1$, such that not one of them is on an axis.*
(c) *Each vertex of Ω_t depends continuously on t.*
(d) *Ω_t has a multiple eigenvalue for all t.*
(e) *Ω_0 is an equilateral polygon.*

Proof. To begin with we ignore property (d), which we will enforce later. Deform Ω_1 by way of convex, symmetric polygons to a polygon with all its vertices on the unit circle as follows. Consider

$$F_t(x) = \frac{x}{(1-t)|x| + t}.$$

Then $F_1(x) = x$ and $F_0(x) = x/|x|$. If x^k are the vertices of Ω_1, define a new polygon $\tilde{\Omega}_t$ with vertices $F_t(x^k)$. We must confirm that these vertices define a convex polygon. Let $x = x^1$, $y = x^2$ and $z = x^3$. Then convexity and the fact that the angles between successive rays to the vertices are less than right angles implies that x and y are in the same half-plane. In particular they are independent and there exists α, $0 < \alpha < 1$, and $\beta > 1$ such that

$$z = \beta((1-\alpha)x + \alpha y).$$

There exist α_t such that $0 < \alpha_t < 1$ and $\beta_t > 0$ such that

$$F_t(z) = \beta_t((1-\alpha_t)F_t(x) + \alpha_t F_t(y)).$$

We want to show that $\beta_t > 1$ (strict convexity at the vertex $F_t(z)$). This is a routine computation following from the triangle inequality. Denote $f_t(x) = (1-t)|x| + t$. Then $\alpha_t\beta_t/f_t(y) = \alpha\beta/f_t(z)$ and $(1-\alpha_t)\beta_t/f_t(x) = (1-\alpha)\beta/f_t(z)$. Thus, eliminating α_t,

$$\beta_t f_t(z) = (1-\alpha)\beta f_t(x) + \alpha\beta f_t(y);$$

in other words,

$$(1-t)\beta_t|z| + \beta_t t = (1-t)[(1-\alpha)\beta|x| + \alpha\beta|y|] + \beta t$$

and the triangle inequality implies $|z| < (1-\alpha)\beta|x| + \alpha\beta|y|$. So it must be that $\beta_t > 1$.

After deforming to a convex region with vertices on the circle, one can move the vertices on the circle until they are equidistant. (Do this deformation in each quadrant symmetrically.) Reparametrizing t to the range $[0,1]$, one obtains a continuous family $\tilde{\Omega}_t$ with $\tilde{\Omega}_1 = \Omega_1$ and $\tilde{\Omega}_0 = \Omega_0$, an equilateral polygon with vertices not on the axes.

It remains to modify the construction so that the multiplicity property (d) is satisfied. Given a convex polygon Ω, symmetric with respect to both axes with boundary points $(a,0)$ and $(0,b)$, and given $s \geq 0$, denote[3]

$$\Omega(s) = \{x \in \Omega : |x_1| < a - s\}.$$

Let $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ denote the lowest Neumann eigenvalue of Ω among functions that are odd with respect to x_1 and x_2, respectively. The case of multiplicity is achieved whenever $\lambda_1(\Omega) = \lambda_2(\Omega)$. The key observation is for $s > s'$, $\Omega(s) \subset \Omega(s')$,

$$\lambda_1(\Omega(s)) > \lambda_1(\Omega(s')); \quad \lambda_2(\Omega(s')) \leq \lambda_2(\Omega(s)). \tag{6.3}$$

To prove this, denote by u_1 and u_2 the odd eigenfunctions with respect to x_1 and x_2, respectively, on $\Omega(s)$, with eigenvalues $\lambda_i = \lambda_i(\Omega(s))$. Define v_1 and v_2 similarly on $\Omega(s')$, with eigenvalues $\lambda_i' = \lambda_i(\Omega(s'))$. After multiplication of u_i and v_i by ± 1

[3]We attribute the idea of cutting or extending a body to fit a constraint to Procrustes [Bl].

we may assume that $\operatorname{sign} x_i = \operatorname{sign} u_i = \operatorname{sign} v_i$. Let $\Gamma = \Omega(s') \cap \partial\Omega(s)$. Then

$$\begin{aligned}
\int_{\Omega(s)} (-\lambda_i + \lambda_i')u_i v_i &= \int_{\Omega(s)} (-\lambda_i + \lambda_i')u_i v_i \\
&= \int_{\Omega(s)} (v_i \Delta u_i - u_i \Delta v_i) \\
&= \int_{\partial\Omega(s)} \left(v_i \frac{\partial u_i}{\partial \nu} - u_i \frac{\partial v_i}{\partial \nu} \right) d\sigma \\
&= -\int_{\Gamma} u_i \frac{\partial v_i}{\partial \nu} d\sigma \\
&= -\int_{\Gamma} u_i (\operatorname{sign}(x_1)) \frac{\partial v_i}{\partial x_1} d\sigma.
\end{aligned}$$

For $i = 1$, $(\operatorname{sign}(x_1))u_1 \geq 0$ and $\partial v_1/\partial x_1 > 0$ implies that the last integral is positive. With the minus sign in front we deduce $-\lambda_1 + \lambda_1' < 0$. On the other hand, for $i = 2$, $\operatorname{sign} x_2 u_2 \geq 0$ and $\operatorname{sign}(x_1)\operatorname{sign}(x_2)\partial v_2/\partial x_1 \leq 0$ implies that the last integral is negative. With the minus sign in front we obtain $-\lambda_2 + \lambda_2' \geq 0$ (with strict inequality except in the case of the rectangle, for which $\partial v_2/\partial x_1 \equiv 0$).

Because of this monotonicity, there is a unique choice of $\tilde{\Omega}_t$ truncated either in the x_1 variable (as above) or in the x_2 variable in such a way that $\lambda_1(\tilde{\Omega}_t) = \lambda_2(\tilde{\Omega}_t)$. There is a uniform lower bound on the height and width of $\tilde{\Omega}_t$. Indeed, it is easy to show that Ω is as above; then there is an absolute constant C such that

$$C^{-1}a^{-2} \leq \lambda_1(\Omega) \leq Ca^{-2}; \quad C^{-1}b^{-2} \leq \lambda_2(\Omega) \leq Cb^{-2}.$$

By continuity the vertical and horizontal sides of the family Ω_t are bounded below uniformly by a positive number. Therefore the same is true for $\tilde{\Omega}_t$. (Furthermore the diameter of $\tilde{\Omega}_t$ is bounded above uniformly in t.)

The difficulty with the family $\tilde{\Omega}_t$ is that the number of vertices may change. The vertices depend continuously on t, but two vertices can coalesce into one and then split apart again. To avoid this we perturb the construction slightly. In order to show that we can perturb without violating monotonicity, we must prove the uniform version of (6.3),

$$\lambda_1(\Omega(s)) - \lambda_1(\Omega(s')) \geq c(s - s') \tag{6.4}$$

for a constant $c > 0$ that depends only on an upper bound for the diameter of Ω and a positive lower bound for length of the vertical side and horizontal sides.

Let $S = \{x_1 > 0\} \cap \Gamma$. On the middle half of S, we observe that

$$u_1 \frac{\partial v_1}{\partial x_1} \geq c(\max u_1)(\max v_1)(s - s').$$

Indeed, u_1 can be extended by reflection across Γ. It satisfies the Harnack inequality in a neighborhood of the maximum point at $(a, 0)$, so u_1 is comparable to the maximum of u_1 on half of the vertical side. Similarly one can find a lower bound for v_1. But v_1 is zero on $x_1 = 0$ and so $\partial v_1/\partial x_1$ is comparable to the maximum of v_1 in a region of unit size at unit distance from the side S. Finally, $\partial v_1/\partial x_1 \geq 0$ on all of $\Omega(s')$ and the function vanishes on a segment $x_1 = a - s'$ which is at a distance $s - s'$ from S. It follows by a barrier argument that $\partial v_1/\partial x_1 \geq c(s - s') \max v_1$ on

S. Finally since $u_1 \frac{\partial v_1}{\partial x_1} \geq 0$ on all of S,

$$\int_S u_1 \frac{\partial v_1}{\partial x_1} d\sigma \geq c(\max u_1)(\max v_1)(s - s')$$

with a constant c depending on a lower bound for the length of S. Since the maximum majorizes the L^2 norm for domains of bounded area, (6.4) follows.

We will fix a number $\epsilon > 0$ sufficiently small in a way specified later depending on a lower bound for the length of the vertical and horizontal sides of $\tilde{\Omega}_t$ and on the constant c in (6.4) on m and on the maximum diameter of the domains Ω_t. To modify a domain $\Omega = \Omega_t$, assume that $\lambda_1(\Omega) < \lambda_2(\Omega)$. Then define a family of domains as

$$\Omega'(s) = \Omega(s)$$

for $0 < s \leq s_1$ where s_1 is the least value of s for which there are sides (four of them by symmetry) of length ϵ. Let us number the sides in the positive quadrant going counterclockwise. S_0 is the vertical side, S_1 the first side above it, S_2 the next side in counterclockwise order tending toward the line $x_1 = 0$, etc. Let us refer to the endpoints of a side as first if it is reached first in order going around the boundary counterclockwise in the first quadrant, and second if it is encountered second. For $s > s_1$, $\Omega'(s)$ is defined so that there is a side S_0 on the line $x_1 = (a-s)$ of increasing length. The side S_1 will have fixed length ϵ for all $s \geq s_1$ and fixed orientation (fixed normal). It will be translated in the plane so that the second endpoint follows the side S_2. Thus the side S_2 will be getting shorter as this vertex moves along it and the side S_0 will be getting longer in the way that is forced by the fact that S_1 has fixed orientation and length. We stop at $s = s_2$ when the length of S_2 is ϵ. Now continue to move with the pair S_1 and S_2 being a rigid set at the same orientation, with the second endpoint of S_2 following S_3 and S_0 lengthening as it is forced to do. As always, S_0 is a segment of the line $x_1 = a - s$.

The uniform strict lower bound (6.4) and monotonicity of $\lambda_2(\Omega(s))$ in s shows that for sufficiently small ϵ, $\lambda_1(\Omega'(s)) - \lambda_2(\Omega'(s))$ is a strictly increasing function of s. This is because the difference between this domain variation from the one of the form $\Omega(s)$ involves variation of sides of total length at most $4m\epsilon$. The variational formula for Neumann eigenvalues is given by

$$\dot{\lambda} = \int_{\partial\Omega} (u_T^2 - \lambda u^2) v d\sigma,$$

where u is the normalized eigenfunction, u_T is the tangential derivative, λ is the eigenvalue and v is the normal variation (see [Z]). For a Lipschitz domain (with uniform bounds depending on the Lipschitz constant), u_T belongs uniformly to $L^p(d\sigma)$ for some $p > 2$; u also belongs to this class (and even to L^∞). It follows that the square integral over a set of size $4m\epsilon$ tends to zero as some power of ϵ. The variation v is bounded, so for ϵ sufficiently small the lower bound of (6.4) dominates. Thus for each there is a unique value of s for which $\Omega'(s)$ satisfies $\lambda_1(\Omega'(s)) = \lambda_2(\Omega'(s))$. The vertices of $\Omega(s)$ depend continously on s. Therefore, given Ω_t we can define the unique s_t such that $\Omega'_t(s_t)$ has multiplicity. This family satisfies assumption (d) as well as the others of Lemma 6.2.

Remark 6.5. The assumption in Proposition 6.1 that there is no vertex on the axes already played a role in the proof of Lemma 6.2. But there is another easy consequence, namely that Ω_t has only obtuse angles. On the other hand, with the help of Proposition 6.1 we will show in Proposition 8.1 that if the eigenvalue is multiple, then no angle can be acute.

Dilate also so that the multiple Neumann eigenvalue of Ω_t is $\lambda \equiv 1$ for all t. Let t_0 be the infimum of t such that monotonicity along both arcs of the boundary from the minimum to the maximum fails for some second eigenfunction u_t on Ω_t. We may assume that u_t has an extension $\tilde{u}_t$ that tends to an eigenfunction u_{t_0} of Ω_{t_0} uniformly on $\bar{\Omega}_{t_0}$ along some sequence $t \searrow t_0$. In fact, because these domains belong uniformly to L_M for some fixed M, $\nabla u_t \in L^p$ for some $p > 2$ uniformly for all t and we may choose an extension $\tilde{u}_t$ such that $\tilde{u}_t$ converges uniformly on $\bar{\Omega}_{t_0}$ and in $W^{1,2}(\Omega_{t_0})$ norm to a Neumann eigenfunction u_{t_0} with eigenvalue 1. (First observe by Lemmas 2.5 and 2.6 that there is convergence in Dirichlet integral and that the $W^{1,2}$ norm distance to the family of all normalized eigenfunctions with eigenvalue 1 on Ω_{t_0} tends to zero. But this collection of eigenfunctions is two dimensional, so the normalized ones form a compact set — parametrized by a circle. So one can take a subsequence of u_t that is convergent.)

In a neighborhood of $\bar{\Omega}_t$ away from the vertices u_t is real analytic and can be extended by reflection so that u_t tends to u_{t_0} in C^k norm on compact subsets of $\bar{\Omega}_{t_0}$ that do not contain the vertices. But then the Fourier series representation of the solution in a neighborhood of the vertices shows that those coefficients also depend continuously on t. See (4.2) and (4.3). (The angle α at each vertex depends continuously on t.)

Step 1. Let Q be a point $\partial\Omega_{t_0}$ at which u_{t_0} attains its maximum. Then $\nu \cdot \nabla u_{t_0} > 0$ on Ω_{t_0}, where ν is the outer normal to Ω_{t_0} at Q.

Since u_{t_0} is the uniform limit from $t \le t_0$ of monotone eigenfunctions, it is monotone along the boundary from its minimum to its maximum. Thus $\nu \cdot \nabla u_{t_0} \ge 0$ on $\partial\Omega_{t_0}$. Since the Dirichlet eigenvalue is strictly larger than the Neumann eigenvalue and this directional derivative satisfies the Neumann eigenvalue equation, $\nu \cdot \nabla u_{t_0} \ge 0$ on Ω_{t_0}. By the generalized mean value property the inequality must be strict unless the function is identically zero. But it cannot be identically zero since the maximum and minimum values of u_{t_0} are not equal.

Step 2. The coefficient $a_1 \ne 0$ in the expansion of u_{t_0} near any vertex that is not a maximum or minimum.

For every eigenfunction for Ω_t one gets nondegeneracy of the a_1 coefficient by a similar argument to the one in the proof of Theorem 4.6. In fact if $u(Q) \ne 0$, then $a_1 = 0$ violates monotonicity on the boundary. If $u(Q) = 0$, then monotonicity plus $a_1 = 0$ implies that

$$u(x) = a_k \rho^{k\alpha} \cos(k\alpha\gamma) + o(\rho^{k\alpha})$$

with $a_k \ne 0$ for some odd $k > 1$. This implies that there are at least three branches of the nodal set that enter Ω_{t_0} at Q. But this contradicts the fact that u_{t_0} has only two nodal domains.

Step 3. u_{t_0} cannot have a degenerate maximum (or minimum) at a vertex or on a side.[4]

[4]This is where we take advantage of the fact that we are working with polygons. On a flat side a degenerate maximum creates three branches for the tangential derivative of u_{t_0}. This is not the case for a curved boundary. In that case, if we consider analytic boundaries, the zero set is a pair of curves meeting at right angles. One branch is tangent to the region, but outside, and a second branch is perpendicular to the boundary. We suspect that this kind of degeneracy never happens, but cannot rule it out in advance.

Since the vertices are not on an axis and the domain is not a rectangle, the angle is necessarily obtuse. Thus $\alpha < 2$ and the $\rho^\alpha \cos(\alpha\gamma)$ term is more important than the ρ^2 term. Thus at a maximum $a_1 = 0$ and $u_{t_0} = a_0 - (a_0/2)\rho^2 + o(\rho^2)$. This argument is different from the one in Theorem 4.6 (a). There we allowed interior right angles and acute angles, but used the fact that the eigenfunction was even across the axis at the vertex where the maximum occurs.

Suppose now that u_{t_0} has a degenerate maximum on a side, i.e., the first and second tangential derivatives vanish at the maximum. Let e be the tangential direction at the maximum and $V = e \cdot \nabla u$. Recall that we showed, using an even reflection across the side, that the set $V = 0$ has at least three branches entering Ω_{t_0}. By A_π symmetry the tangential direction at the minimum is also e and $V = 0$ has three branches at the minimum, diametrically opposite the maximum. There are four components of $\{V \neq 0\}$ of alternating signs near the minimum and the maximum. By convexity, the boundary is divided into quadrants where the dot product of the tangential vector alternates in sign. By weak monotonicity of u along the boundary the sign of V alternates. But the configuration we have described in all cases yields a component of either $V > 0$ or $V < 0$ that meets $\partial\Omega$ in at most a points where $V = 0$. Thus V is an eigenfunction vanishing on the boundary of a component that is a proper subset of Ω, and this violates the property that the Dirichlet eigenvalue is greater than the Neumann eigenvalue.

Step 4. u_{t_0} has a critical point on a side (tangential derivative zero) that is neither the maximum nor the minimum.

Suppose that u_{t_0} has no extra critical point. In other words its tangential derivative along the boundary vanishes only at the maximum and the minimum and the vertices. We have shown in Step 3 that if the maximum and minimum of u_{t_0} occur on sides, then they are nondegenerate critical points. So by standard perturbation theory, for t sufficiently close to t_0, u_t has exactly one nondegenerate maximum/minimum nearby and no other critical points.

Suppose that the maximum of u_{t_0} is at a vertex. The first terms in the expansion centered at that point are

$$a_0(t)J_0(\rho) + a_1(t)J_{\alpha(t)}(\rho)\cos(\alpha(t)\gamma)$$

with $a_0(0) > 0$ and $a_1(0) = 0$. If $a_1(t) = 0$, then we are done; the $\rho^{2\alpha}$ term is negligible and the function is monotone increasing in a neighborhood going toward the maximum which stays at the vertex.

If $a_1(t) \neq 0$, we may as well assume $a_1(t) > 0$. The other case is symmetric because the cosine is just ± 1 on the two edges. Divide by $a_0(t)$, which is nonzero. Then the renormalized perturbation u_t has the form

$$u_t = (1 - z^2/2) + \eta|z|^\alpha \mathrm{sign}\,(z) + f(z,t),$$

where α, η and f are continuous functions of t, $\eta(t_0) = 0$ and $1 < \alpha < 2$. (Recall that an obtuse angle implies $\alpha < 2$. Convexity implies $\alpha > 1$.) The letter z represents the tangential length along the side of the polygon Ω_t with $z > 0$ in one direction and $z < 0$ in the other, i.e., $z = \pm\rho$.

$$|\partial_z f| \le Cz^{2\alpha-1}; \quad |\partial_{zz} f| \le Cz^{2\alpha-2}.$$

By symmetry, as noted earlier, we may as well assume $\eta(t) > 0$. Then

$$\partial_z u_t = -z + \alpha\eta|z|^{\alpha-1} + \partial_z f > 0$$

for $z < 0$ in a sufficiently small neighborhood. (The term from $\partial_z f$ is negligible compared to z because $\alpha > 1$.)

We want to show that $\partial_z u_t$ changes sign exactly once in $0 < z < A$, where A is independent of the size of η (as η tends to zero). Let $v(z) = -z + \alpha\eta z^{\alpha-1}$. Let

$$z_0 = (\alpha\eta)^{1/(2-\alpha)}$$

(the unique positive root solution to $v(z) = 0$). For any β, $0 < \beta < 1$, there exists $c > 0$, depending only on β such that

$$v(z) > cz$$

for all z, $0 < z \le \beta^{1/(2-\alpha)} z_0$. Since f_z is negligible compared to z,

$$\partial_z u_t > 0$$

for $t - t_0$ sufficiently small depending on β, but not on z_0. Now fix β sufficiently close to 1 such that $(\alpha - 1)/\beta < 1$. If $z \ge \beta^{1/(2-\alpha)} z_0$,

$$\partial_z v = -1 + \alpha(\alpha - 1)\eta z^{\alpha-2} \le -1 + (\alpha - 1)/\beta < 0.$$

Moreover, $|\partial_{zz} f| = O(z^{2\alpha-2})$ is negligible compared to 1, so

$$\partial_{zz} u_t \le -c < 0$$

for all $A \ge z \ge \beta^{1/(2-\alpha)} z_0$. Thus, $\partial_z u_t$ is decreasing and crosses zero exactly once in this range.

We have now shown in all cases that the tangential derivative of u_t changes sign exactly once in a fixed neighborhood of the maximum of u_{t_0} (and this point is a local maximum). On the other hand, at each vertex for which $a_1 \ne 0$, every u_t for t near t_0 is strictly monotone in a neighborhood. Similarly, in a neighborhood of any point at which the tangential derivative of u_{t_0} is nonzero on a side, u_t has the same property for t near t_0. In all, u_t is monotone increasing along the boundary from the minimum to the maximum, contradicting the definition of t_0.

Step 5. If u_{t_0} has a critical point on a side other than the maximum and minimum, then there is another second eigenfunction v_{t_0} for Ω_{t_0} that violates monotonicity.

Consider the odd and even eigenfunctions restricted to the side and call them f and g as functions on the interval $[0, a]$ so that

$$Cu_{t_0} = (1 - s_0)f(x) + s_0 g(x)$$

without loss of generality, $f' > 0$ and $g' < 0$ on this side. We also have monotonicity of u_{t_0} in a neighborhood of the critical point x_0, so that

$$(1 - s_0)f'(x) + s_0 g'(x) \ge 0$$

for x near x_0 and

$$(1 - s_0)f'(x_0) + s_0 g'(x_0) = 0.$$

But $f'(x_0) > 0$ and $g'(x_0) < 0$. Also the analytic function $(1 - s_0)f'(x) + s_0 g'(x)$ is not identically zero because that would say that both the normal and tangential derivatives of u_{t_0} vanish on an open subset of the boundary, which would imply u_{t_0} is a function of one variable (the case of the rectangle). For any $\epsilon > 0$, there exists $\delta > 0$ such that if $s_0 + \delta > s > s_0$, then

$$(1 - s)f'(x) + sg'(x)$$

changes sign twice in $[x_0-\epsilon, x_0+\epsilon]$. Therefore, Ω_{t_0} has an eigenfunction that strictly violates monotonicity and one can perturb to $t < t_0$ with the same violation. This contradicts the definition of t_0 as the infimum of times for which monotonicity fails. This concludes the proof of Proposition 6.1.

Proof of Theorem 1.4. Having proved monotonicity for polygons, we now deduce monotonicity along the boundary from the minimum to the maximum for any convex domain with two axes of symmetry by taking limits (uniformly in L_M). This follows as above in Step 1 of the proof of Proposition 6.1. If the maximum is achieved at x^0, then the only other places where it can be achieved are boundary points on a segment of the boundary of the tangent line at x^0, $x^0\cdot(x-x^0) = 0$. Thus, if the maximum is achieved at more than one point, then $\partial\Omega$ contains a segment of its tangent line at x^0 and the tangential derivative of u is zero on an open subset of that line. By reflection across this interval and analytic continuation, the derivative of u with respect to that direction is identically zero and u is a function of a single variable. As we have seen earlier in the proof of Theorem 4.1 (b), this only happens in the case of a rectangle.

Proof of Corollary 1.5. Because $h(\cdot, 0)$ is not orthogonal to the first nonzero eigenspace, there are $c_2 > 0$ and u a Neumann eigenfunction with lowest nonzero eigenvalue λ such that $h(x,t) = c_1 + c_2e^{-\lambda t}u(x) + o(e^{-\lambda t})$ uniformly in $x \in \bar{\Omega}$ as $t \to \infty$. It follows from Theorem 1.4 that the location of any maximum point tends to the boundary.

To show that the maximum hits the boundary in finite time in the smooth, positively curved case, we need to prove nondegeneracy in the normal direction.

Lemma 6.6. *If u is a Neumann eigenfunction with lowest nonzero eigenvalue in a smoothly bounded domain Ω with positively curved boundary, and $x^0 \in \partial\Omega$ is the place where u attains its maximum and ν is the normal at x_0, then*

$$(\nu \cdot \nabla)^2 u(x^0) < 0.$$

Proof. Choose rectangular coordinates y_1 and y_2 with $y = 0$ at x^0, the tangent at x^0 is the y_1 axis, and the y_2 axis points outside the domain for $y_2 > 0$. The boundary is given by

$$y_2 = -py_1^2 + O(|y_1|^3)$$

for some $p > 0$. The expansion of u is

$$u = a_0 + ay_1^2 + by_2^2 + \alpha(y_1^3 - 3y_1y_2^2) + \beta(y_2^3 - 3y_2y_1^2) + O(|y|^4)$$

with $a+b = a_0\lambda/2$. (The term y_1y_2 has the coefficient zero because of the Neumann condition.) To prove the lemma we need to show that $b > 0$.

Suppose not. If $b < 0$, then $u > u(x^0)$ on the axis $y_1 = 0$ for small $y_2 < 0$. So the only case that remains to rule out is the case $b = 0$. If $b = 0$, then $u = \beta y_2^3 + O(|y|^4)$ on $y_1 = 0$. But this leads to $u > u(x^0)$ for small $y_2 < 0$ unless $\beta \geq 0$. Now consider the Neumann condition,

$$\left(\frac{\partial}{\partial y_2} + (2py_1 + O(y_1^2))\frac{\partial}{\partial y_1}\right) u = 0$$

on $y_2 = -py_1^2$. Recalling $b = 0$ and $a = -a_0\lambda/2$, this leads to

$$-6\alpha y_1y_2 + 3\beta(y_2^2 - y_1^2) - 2pa_0\lambda y_1^2 = O(|y|^3)$$

for $y_2 = -py_1^2$. In other words,

$$-3\beta y_1^2 - 2pa_0\lambda y_1^2 = O(|y_1|^3).$$

But this contradicts $a_0 > 0$, $p > 0$ and $\beta \geq 0$. This concludes the proof of the lemma.

Let $h(x,t)$ be the solution to the heat equation with $c_2 > 0$ as above. In the smoothly bounded case,

$$\|h(\cdot,t) - c_1 + c_2 e^{-\lambda t} u\|_{C^2(\bar{\Omega})} = o(e^{-\lambda t}).$$

Hence by Lemma 6.6 there is a neighborhood N of x^0 in $\bar{\Omega}$ such that for all $t > T$

$$(\nu' \cdot \nabla)^2 h(x,t) < 0 \tag{6.7}$$

for all $x \in N$ and all ν' normal to $\partial\Omega$ at any point of $N \cap \partial\Omega$. Suppose that a point x_t where $h(\cdot,t)$ attains its maximum is not on the boundary. Let D_t be the largest disk in Ω centered at x_t. Choose $z_t \in (\partial\Omega) \cap \partial D_t$. Since x_t tends to x^0, for T sufficiently large the segment from x_t to z_t is contained in N for all $t > T$. Let ν' be the normal to ∂D_t at z_t. Then ν' is normal to $\partial\Omega$ and the Neumann condition says $\nu' \cdot \nabla h(z_t,t) = 0$. By (6.7), $(\nu' \cdot \nabla)^2 h(x,t) < 0$ for all x on the segment from z_t to x_t. Hence $h(z_t,t) > h(x_t,t)$, a contradiction. It follows that x_t cannot be on the interior.

§7. Limiting cases

We have proved Theorem 1.1 for Ω in L_M. Our next task is to pass to the limit to handle cusps on the axes. Consider the general case of a domain Ω satisfying the hypothesis of Theorem 1.1. Let $a > 0$ and $b > 0$ be such that $(a,0) \in \partial\Omega$ and $(0,b) \in \partial\Omega$. Denote

$$\Omega_\epsilon = \{x \in \Omega : |x_1| < a - \epsilon,\ |x_2| < b - \epsilon\}, \quad \epsilon > 0. \tag{7.1}$$

Let u_ϵ denote the lowest Neumann eigenfunction for Ω_ϵ that is odd with respect to x_2 and let λ_ϵ be the eigenvalue. Assume that u_ϵ is normalized to have $L^2(\Omega_\epsilon)$ norm 1 and so that $u_\epsilon > 0$ in $x_2 > 0$. Note that Ω_ϵ belongs to L_M for some M depending on ϵ.

Lemma 7.2. *$|u_\epsilon| \leq C$ with C independent of ϵ.*

Proof. Fix $\Lambda > \lambda_\epsilon$ for all small ϵ. Denote

$$B(x) = \cos(\sqrt{\Lambda}(x_2 - b)).$$

Then $(\Delta + \Lambda)B = 0$ and

$$\frac{\partial B}{\partial x_2} > 0, \ \text{ for } b - \pi/2\sqrt{\Lambda} < x_2 < b.$$

Fix $h = \min(a/4, b/4, \pi/4\sqrt{\Lambda})$. For each fixed $\epsilon > 0$ approximate Ω_ϵ by domains $\Omega_{\epsilon,\delta}$ so that $d(\Omega_\epsilon, \Omega_{\epsilon,\delta}) \to 0$ and $\Omega_{\epsilon,\delta}$ is a C^∞ domain with two axes of symmetry whose normal satisfies

$$\nu \cdot e_2 > 0$$

for all $x \in \partial\Omega_{\epsilon,\delta}$ such that $x_2 \geq b - h$. (The cross section hypothesis of Theorem 1.1 implies that $\nu \cdot e_2 \geq 0$, and so to find this smooth approximation, one perturbs by convolution of the logarithm of the function representing the graph of the boundary in polar coordinates as in the approximation in Corollary 3.8.) Since

$\partial\Omega_\epsilon$ is uniformly Lipschitz, independent of ϵ in the range $b-2h \le |x_2| \le b-h/2$, we may assume that $\partial\Omega_{\epsilon,\delta}$ is uniformly Lipschitz, independent of both $\delta > 0$ and $\epsilon > 0$, in the same range of x_2. Let $u_{\epsilon,\delta}$ be the normalized odd eigenfunction for $\Omega_{\epsilon,\delta}$. Let $U = \Omega_{\epsilon,\delta} \cap \{x : x_2 > b-h\}$. Denote $\Gamma_1 = \partial U \cap \{x_2 > b-h\}$ and $\Gamma_2 = \partial U \cap \{x_2 = b-h\}$. It follows from uniform Hölder continuity estimates up to the boundary in regions where the boundary is uniformly Lipschitz that there is a constant independent of δ and ϵ such that

$$\max_{\Gamma_2} u_{\epsilon,\delta} \le C.$$

Since $\nu \cdot e_2 > 0$ on Γ_1, $\partial B/\partial\nu > 0$ on Γ_1. Furthermore, $B > 0$ on $\bar{U}$ and $(\Delta + \lambda_{\epsilon,\delta})B < 0$ where $\lambda_{\epsilon,\delta}$ is the eigenvalue of $u_{\epsilon,\delta}$, provided δ is sufficiently small so that $\lambda_{\epsilon,\delta} < \Lambda$. In addition, $\partial u_{\epsilon,\delta}/\partial n = 0$ and $u_{\epsilon,\delta} > 0$ on $\bar{U}$, so $(\partial/\partial\nu)(u_{\epsilon,\delta}/B) < 0$ on Γ_1, and so the maximum of $u_{\epsilon,\delta}/B$ over $\bar{U}$ cannot be attained on Γ_1. By the generalized maximum principle,

$$\max_{\bar{U}} u_{\epsilon,\delta}/B \le \max_{\Gamma_2} u_{\epsilon,\delta}/B.$$

This proves a uniform upper bound for $u_{\epsilon,\delta}$ independent of ϵ and δ. For fixed ϵ, we may extend $u_{\epsilon,\delta}$ to a function $\tilde{u}_{\epsilon,\delta}$ defined in a neighborhood of Ω_ϵ so that $\tilde{u}_{\epsilon,\delta}$ tends uniformly to u_ϵ as $\delta \to 0$. This proves the uniform upper bound for u_ϵ.

Corollary 3.8 implies that u_ϵ is monotone with respect to each variable in each quadrant. Extend u_ϵ to a function $\tilde{u}_\epsilon$ that is even across the vertical and horizontal segments of the boundary coinciding with the lines $|x_1| = a-\epsilon$ and $|x_2| = b-\epsilon$. Take a subsequence, converging weakly in $W^{1,2}(\Omega)$ to a function u and also converging in C^k on compact subsets of Ω and converging uniformly on compact subsets of $\bar{\Omega}$ that are disjoint from the four points where $\partial\Omega$ meets the axes. Then $\Delta u = \lambda u$ for $\lambda = \lim \lambda_\epsilon$. Moreover, u is monotone with respect to both variables in each quadrant in Ω. Furthermore, u is continuous up to the boundary except possibly at the points where the boundary meets the axes. Since u_ϵ tends uniformly to u on Ω_{ϵ_0} for each fixed $\epsilon_0 > 0$ and the area of $\Omega \backslash \Omega_{\epsilon_0}$ tends to zero as $\epsilon_0 \to 0$, Lemma 7.2 implies that u_ϵ tends to u in L^2 and the $L^2(\Omega)$ norm of u is 1.

Next we confirm that u is an eigenfunction. For every function $\psi \in C^\infty(\bar{O})$ such that ψ vanishes in a neighborhood of the four corners $(\pm a, 0)$, $(0, \pm b)$,

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\Omega \nabla \tilde{u}_\epsilon \cdot \nabla \psi &= \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla \psi \\ &= \lim_{\epsilon \to 0} \lambda_\epsilon \int_{\Omega_\epsilon} u_\epsilon \psi \\ &= \lambda \int_\Omega u\psi. \end{aligned}$$

Since the set of ψ is dense in $W^{1,2}(\Omega)$, it follows that u is the unique odd Neumann eigenfunction for Ω. Thus we have proved that the function u of Theorem 1.1 is monotone with respect to each variable in each quadrant of Ω. It is also continuous in $\bar{\Omega}$ except possibly at the boundary points on the axes.

Proposition 7.3. *The odd Neumann eigenfunction u is continuous on $\bar{\Omega}$.*

Proof. Continuity at the boundary points on the x_1 axis follows routinely from monotonicity and continuity inside. In fact, if $x_2 > 0$, then by monotonicity in x_1,

$$\max_{x_1} u(x_1, x_2) = u(0, x_2).$$

But u is continuous on the interior, so $u(0, x_2) \to u(0, 0) = 0$ as $x_2 \searrow 0$. Therefore,

$$\max_{x_1} u(x_1, x_2) \to 0$$

as $x_2 \searrow 0$, uniformly in x_1.

Note that by monotonicity and boundedness, u is continuous when restricted to the x_2 axis. We need to show that along the boundary to the topmost point u tends to the same limit instead of something smaller. We introduce an integral operator Γ due to Hans Lewy that is analogous to harmonic conjugation for eigenfunctions. Let $J(x) = J_0(\sqrt{\lambda}|x|)$, the Bessel function satisfying $(\Delta + \lambda)J = 0$. For a rectifiable curve γ in Ω with $\gamma(0) = 0$ and $\gamma(T) = x$, define

$$\begin{aligned} v(x) &= (\Gamma u)(x) \\ &= \int_0^T [u(\gamma(t))\dot{\gamma}(t)^{\perp} \cdot \nabla J(x - \gamma(t)) - J(x - \gamma(t))\dot{\gamma}(t)^{\perp} \cdot \nabla u(\gamma(t))]dt. \end{aligned} \tag{7.4}$$

Lewy [L] has shown that this definition is path-independent, that the operator Γ is analogous to the harmonic conjugation in the sense that $(\Delta + \lambda)v = 0$, and that $\Gamma v = -u$.

Recall from [JK] that on Lipschitz domains ∇u has nontangential maximal function in $L^2(\partial\Omega)$. It follows that the same is true here on compact subsets of the boundary away from the cusps. Thus the integral formula (7.4) can be extended to the boundary. In particular, by a limiting argument with parallel boundaries, one can show that the formula is well defined for paths that follow the boundary, provided one does not cross a cusp. Since the tangential and normal directions are orthogonal, the Neumann condition implies that $\dot{\gamma}^{\perp} \cdot \nabla u = 0$ for curves along the boundary. Hence v is bounded. Moreover, since u is bounded, v is Lipschitz continuous when restricted to $\partial\Omega$.

Since v is Lipschitz continuous on $\partial\Omega$, a standard Dirichlet barrier argument implies that

$$|v(x) - v(x')| \le C|x - x'|^{1/2}.$$

(In fact, Ω has exterior right angle sectors so there is Hölder continuity up to order 2/3.) Standard interior regularity, suitably scaled, then implies

$$|\nabla v| \le C\delta(x)^{-1/2}, \tag{7.5}$$

where $\delta(x) = \mathrm{dist}\,(x, \partial\Omega)$.

Now we make use of the represention $u = \Gamma v$ to prove continuity at the top boundary point on the x_2 axis, $(0, a) \in \partial\Omega$; consider any $Q = (z_1, z_2) \in \partial\Omega$ in the first quadrant with $z_2 > z_1$. Let γ be the path from 0 to Q that follows the x_2 axis from 0 to $P = (0, z_2 - z_1)$ and then follows the segment of slope 1 from P to Q. The formula $u = -\Gamma v$ along γ shows that

$$|u(P) - u(Q')| \le C \int_0^{z_1} s^{-1/2} ds \le C z_1^{1/2}$$

for every Q' on the segment between P and Q. Since $u(P)$ tends to $u(0, a)$ as P tends to $(0, a)$, it follows that $u(x)$ tends to $u(0, a)$ as x_2 tends to a.

Having proved continuity of u, let us finish the remaining parts of the proof of Theorem 1.1. By following piecewise horizontal and vertical paths we see that the maximum and minimum of u are achieved at $(0, \pm b)$. The strict inequality

$$\partial u/\partial x_2 > 0 \text{ in } \Omega$$

follows from the fact that $\partial u/\partial x_2$ satisfies the eigenvalue equation and is nonnegative on Ω. If it were zero, then the generalized mean value property would force it to be strictly negative at some point nearby, unless it were identically zero. But it is not identically zero. Similarly,

$$x_1 x_2 \partial u/\partial x_1 < 0 \text{ in } \Omega \text{ for } x_1 x_2 \neq 0$$

unless $\partial u/\partial x_1 \equiv 0$. In this last case one has $u(x_1, x_2) = f(x_2)$ with $f'(x_2) > 0$ for all $-b < x_2 < b$. The only way that this can satisfy the Neumann condition is if the boundary is vertical for all $-b < x_2 < b$. In other words, this only happens if Ω is a rectangle.

All that remains to prove is the last assertion of Theorem 1.1 that unless Ω is a rectangle, the maximum and minimum are achieved at $(0, \pm b)$ only. The strict monotonicity already proved shows that the maximum can only be achieved at more than one point if the boundary contains a horizontal segment of the line $x_2 = b$ and that u is constant on that segment. One also has $\partial u \partial x_2 = 0$ on the segment because that is the Neumann condition there. Thus the power series of u is uniquely determined and by analytic continuation $u = c\cos(\sqrt{\lambda}(x_2 - b))$ in all of Ω. Therefore, Ω is a rectangle. This concludes the proof of Theorem 1.1.

To deduce Corollary 1.2, consider the odd Neumann eigenfunction u with eigenvalue λ. Denote

$$w = \frac{\partial u}{\partial x_2}.$$

Then $w > 0$ in Ω and satisfies $\Delta w = -\lambda w$. Let v be the lowest Dirichlet eigenfunction for any smooth domain $\Omega' \subset\subset \Omega$ with eigenvalue μ'. After multiplication by ± 1, we may assume $v > 0$. Integrating by parts (see Proposition 4.7)

$$(\mu' - \lambda) \int_{\Omega'} vw = -\int_{\partial\Omega'} w \frac{\partial v}{\partial \nu} d\sigma > 0$$

since $(\partial/\partial\nu)v < 0$ almost everywhere on $\partial\Omega'$. Therefore, $\mu' > \lambda$. Furthermore, $\mu' = \mu(\Omega') \to \mu = \mu(\Omega)$ as Ω' tends to Ω. Therefore, $\lambda \leq \mu(\Omega)$.

§8. Remarks and open problems

We observe that monotonicity of all eigenfunctions in the lowest eigenspace implies in certain cases that the space is one dimensional.

Proposition 8.1. *Let Ω be a convex polygon with two axes of symmetry. If Ω has an acute angle, then the lowest nonzero Neumann eigenvalue is simple.*

Proof. Suppose that Ω has an acute angle $\sigma < \pi/2$. Vertices not on the axes can only have obtuse angles, so without loss of generality we may assume that the vertex is $(0, b) \in \partial\Omega$. If Ω has a two-dimensional family of eigenfunctions, then it has an eigenfunction u_0 that is odd with respect to the x_2 axis and even with respect to the x_1 axis and another eigenfunction u_1 that is odd with respect to the x_1 axis and even with respect to the x_2 axis. After multiplication by ± 1 we may assume that u_0 attains its maximum at $(0,b)$ and u_1 attains its maximum at $(a,0) \in \partial\Omega$. The convex combinations, $0 \le t \le 1$,

$$u_t = (1-t)u_0 + tu_1$$

achieve their maximum at points x^t in the first quadrant. Since Ω is not a rectangle, Theorem 1.4 implies that the maximum point is unique. The first observation is that the point x^t depends continuously on t. (Indeed, every limit point of the points x^t as $t \to t_0$ is a maximum point of u_{t_0}, which, by uniqueness, must be x^{t_0}.) The expansion of u_0 at $(0,b)$ has the form

$$u_0 = a_0(1 - \lambda\rho^2/2) + o(\rho^2)$$

with ρ the distance to $(0,b)$. Let ρ be the distance to $(0,b)$. Then

$$u_1 = O(\rho^\alpha)$$

with $\alpha = \pi/\sigma > 2$. It follows that $u_t(x)$ increases to $u_t(0,b) = (1-t)a_0$ as $x \in \partial\Omega$ tends to $(0,b)$ from both sides. Since by Theorem 1.4 u_t is monotone along both arcs of $\partial\Omega$ from the minimum to the maximum, it must be that u_t attains its maximum at $(0,b)$. But then $x^t = (0,b)$ for all $0 \le t < 1$, whereas $x^1 = (a,0)$, contradicting continuous dependence. Thus Ω cannot have an acute angle.

Note that the main ingredient of Theorem 1.4 is Proposition 6.1, which was proved using polygons with multiplicity and obtuse angles. The obtuse angles were technically important in the proof. On the other hand, Theorem 1.4 applies to all convex polygons and permits us to deduce Proposition 8.1, which shows, post facto, that polygons with multiplicity cannot have acute angles. (See Remark 6.5.)

Conjecture 8.2. *If Ω is a convex polygon with two axes of symmetry and a right angle, then the only case in which the eigenvalue is multiple is the case when Ω is a square (with sides of slopes ± 1).*

We can analyze the case of equality $\lambda = \mu(\Omega)$ in Corollary 1.2 in the case of smoothly bounded domains as follows. The proof of the corollary shows that if equality holds, then $w = \partial u/\partial x_2$ vanishes on the boundary. This and the Neumann condition show that ∇u vanishes on an open subset of the boundary (all points at which the boundary is not horizontal). After reflection u is the solution of an elliptic divergence form equation with Lipschitz coefficients whose gradient vanishes to infinite order at a point and hence u is constant. Note that $\lambda = \mu(\Omega)$ is achieved in the limit of rectangles as the length of the horizontal side tends to infinity, that is, it is achieved in the case of an infinite horizontal strip. This is consistent with the proof just given, which depends on the presence of a portion of the boundary that is not horizontal. We conjecture that the inequality is strict for every domain satisfying the hypotheses of Theorem 1.1. A proof may depend on some suitable regularity of $\partial u/\partial x_2$ in the nonsmooth case.

We propose that the eigenvalue comparison of Corollary 1.2 is valid in much greater generality.

Conjecture 8.3. *Let Ω be any bounded, C^∞ domain in $\mathbf{R}^n$ that is symmetric with respect to the plane $x_1 = 0$. Let λ be the least Neumann eigenvalue λ with an odd eigenfunction with respect to the reflection $x_1 \to -x_1$. Let μ be the least Dirichlet eigenvalue. Then $\lambda < \mu$.*

Note that, by uniqueness, the Dirichlet eigenfunction is even with respect to reflection. Thus, another way of phrasing this problem is as a comparison between the lowest eigenvalues of two mixed boundary problems on $\Omega^+ = \Omega \cap \{x_1 > 0\}$. The number λ is the least eigenvalue for the problem with Neumann conditions on $\{x_1 > 0\} \cap \partial\Omega^+$ and Dirichlet conditions on $\{x_1 = 0\} \cap \partial\Omega^+$. The number μ is the least eigenvalue for the problem with Dirichlet conditions on $\{x_1 > 0\} \cap \partial\Omega^+$ and Neumann conditions on $\{x_1 = 0\} \cap \partial\Omega^+$.

To address the problem of extending the proof given here of the hot spots conjecture to general convex domains, it is important to note that the first nonconstant Neumann eigenfunction on a convex domain need not be increasing along the boundary from a minimum to a maximum. In the case of a narrow circular sector the maximum is attained along the entire circular boundary and the minimum at the vertex. It seems natural to expect that in the case of an isosceles triangle the maximum occurs at the two symmetric vertices and the minimum at the third vertex with a fourth critical point, a local minimum at the midpoint of the side between the two global maxima. This can be confirmed for isosceles triangles that are small perturbations of the equilateral triangle. (The eigenvalue is simple and even with respect to the axis of symmetry provided the angle at the vertex on the axis of symmetry is less than $\pi/3$.) The circular sector example alone tells us that the eigenfunction need not be a Morse function on the boundary. Nevertheless we can still hope that in general convex domains no eigenfunction has a critical point on the interior.

While for general convex domains one must abandon monotonicity, a natural hypothesis under which it may still hold is that the domain be convex and centrally symmetric, that is, symmetric under $(x_1, x_2) \mapsto (-x_1, -x_2)$.

Conjecture 8.4. *Let Ω be a centrally symmetric convex domain. Any Neumann eigenfunction with the lowest nonzero eigenvalue is monotone increasing on the boundary from the minimum to the maximum. Also the directional derivative in the direction from the minimum to the maximum is strictly positive in Ω.*

Conjecture 8.5. *Let Ω be a convex, centrally symmetric domain contained in a circular sector with vertex P and acute angle. If $P \in \partial\Omega$, then the maximum or minimum of the lowest Neumann eigenfunction is achieved at P. In particular, the eigenfunction is simple.*

We mention one final conjecture concerning higher eigenvalues.

Conjecture 8.6. *Let Ω be a centrally symmetric convex domain. Let λ_2 be the lowest nonzero Neumann eigenvalue for an eigenfunction that is even with respect to the central symmetry $x \mapsto -x$ and let μ_2 be the lowest Dirichlet eigenvalue that is odd with respect to this symmetry. Then $\lambda_2 < \mu_2$.*

Postscriptum. L. Friedlander has settled Conjecture 8.6 in a recent manuscript [F]. He proved that $\lambda_2 < \mu_2$ for C^2 domains in $\mathbf{R}^n$ with nonnegative mean curvature.

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[Z] D. Zanger, Thesis MIT, June 1997.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: `jerison@math.mit.edu`

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

E-mail address: `nicholas@math.uchicago.edu`