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ON THE BRYLINSKI-KOSTANT FILTRATION

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Introduction

The base field k is assumed to be of characteristic zero. Let $\mathfrak{g}$ be a split semi-simple k-Lie algebra. Consider a finite-dimensional simple $\mathfrak{g}$ module V and fix a weight μ of V. This paper concerns the Brylinski-Kostant (or simply, BK) filtration defined on the μ weight space of V. In particular, the members of the n^{th} subspace in the filtration are those vectors of weight μ killed by the n^{th} power of a fixed regular nilpotent element. The q character corresponding to this filtration, referred to in [B1] as the jump polynomial, is the associated (finite) Poincaré series for the filtration in the variable q. A second q polynomial was introduced by Lusztig ([L1]); it is the coefficient of e^{μ} in a q version of the ordinary character formula for V defined using a q analog of Kostant's partition function (see Section 2.3 for a precise definition). The aim of this paper is to give a new proof of [B1, Theorem 3.4]: the jump polynomial of a dominant weight μ is equal to Lusztig's q polynomial at μ. We also obtain a natural extension of this result to non-dominant weights.

We briefly review some high points in the history of the BK filtration and its related jump polynomial. First, A. Shapiro and R. Steinberg independently found an empirical method of reading off the exponents in the Poincaré polynomial of the adjoint group from the root system for $\mathfrak{g}$. B. Kostant [K1] found the theoretical underpinnings of this procedure by studying the decomposition of $\mathfrak{g}$ into submodules for the action of the principal TDS. This computes the BK filtration for the adjoint representation. In a later paper [K2], Kostant considered generalized exponents associated to any V as described above. As a consequence it is possible to obtain a remarkable relation between the "harmonic degrees" of V and what we now call the BK filtration of the zero weight space V_0 of V, by combining [K2, Sect. 5, Cor. 4] and [B1, Lemma 2.5 and Prop. 2.6]. Central to the derivation of this observation is a difficult result of [K2] describing the ideal of definition for the nilpotent cone. Hesselink [H] and Peterson [P] then (independently) gave a purely combinatorial formula for these generalized exponents. Specifically, one can read the generalized exponents as Lusztig's q polynomial at weight zero, thus establishing the first connection between Lusztig's q formulas and jump polynomials. This combinatorial approach is useful, for example, in computing the so-called PRV determinants [J1].

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Inspired by Kostant's work, R.K. Brylinski defined the BK filtration for arbitrary weights of V and computed the jump polynomial for dominant weights (under mild restrictions). Her approach [B1, Theorem 3.4] involves a twisting process effected through invertible sheafs on the nilpotent cone (and on the flag variety). She reduced the equality of the jump polynomial and Lusztig's q polynomial to a vanishing of higher cohomology. By results of H.H. Andersen and J.C. Jantzen [AJ], and P. Griffiths [G], it followed that this vanishing condition held for most of the dominant weights. Later, B. Broer [B] extended this result (and thus established the equality of these two formulas) for all dominant weights.

The notion of a BK filtration was extended to the category $\mathcal{O}$-dual of a Verma module in [J1] and will be used here (see Section 4). It was noted in [J1] that the Brylinski-Broer result determined the associated jump polynomial. As might be expected, it is a q version of the character formula of the Verma module. This result had a simple proof in [J1]; unfortunately, it was not found possible to use similar techniques such as the BGG resolution of V to compute the jump polynomials associated to various BK filtrations of V.

The form of the jump polynomial associated to the BK filtration for non-dominant weights was conjectured in [Z, 3.2]. S. Zelikson [Z, Theorem 3.3.2] established this conjecture for weight spaces of multiplicity one in the simply-laced case using a positivity result of Lusztig [L2, 22.1.7]. We now prove the conjecture in general (Theorem 7.6). In particular, let μ be a dominant weight and let w be an element of the Weyl group of $\mathfrak{g}$. We show that the jump polynomial associated to $w\mu$ is equal to a power of q times Lusztig's q polynomial at the dominant weight μ. This power of q depends only on $w\mu$ and is equal to a natural upper bound, the height of $\mu - w\mu$ (see Section 7.2). This indicates that there are no accidental cancellations when applying powers of the regular nilpotent element.

We offer an alternative to the geometric approach in [B1] to prove the Brylinski-Broer result and Zelikson's conjecture. In the spirit of Gelfand-Kirillov, we twist the differential operators using maps of pairs of Verma modules. The resulting object is the Weyl algebra realized as a $\mathfrak{g}$ bimodule dependent on two parameters corresponding to the highest weights of the Verma modules. Our techniques are algebraic and representation theoretic. For the reader who would like to understand our interpretation from the geometric point of view, we note that these bimodules can be realized as twisted differential operators on the w_0 translate of the big open cell in the flag variety.

The operator filtration on the $\mathfrak{g}$ bimodules described above is defined by taking the degree of an element considered as a differential operator in the corresponding Weyl algebra. In Section 3, we relate (basically by Frobenius reciprocity) three filtrations: the BK filtration of V, the BK filtration on a dual Verma module and the operator filtration on differential operators. This correspondence is not precise for all values of the parameters. However our present method has the additional flexibility of a two-parameter theory.

An important tool in our argument is the graded injectivity of the $\mathfrak{g}$ bimodules of twisted differential operators, for certain values of the parameters, viewed as modules (in an extension of the $\mathcal{O}$ category) for the Lie algebra. Graded injectivity was first established in [J5, 3.6] when the parameters coincide; it was used in [JL] to obtain a parabolic generalization of Hesselink's exponent result that the exponents can be read off a parabolic version of Lusztig's q character formula. This in turn is used in [JLT] to compute the KPRV determinants (which are parabolic generalizations of the PRV determinants).

Graded injectivity in this paper replaces cohomological vanishing. Using translation principles, it is extended (Section 2.1) to the case when the parameter difference is dominant and both parameters are antidominant and regular. This gives the Brylinski-Broer result (Theorem 5.6). As might be expected, graded injectivity fails in the non-dominant case. However in Section 7, we use composition of differential operators (and the fact that their symbols form a domain), to reduce the computation of degrees to the case of what we call the unique minimal $\mathfrak{k}$-type, the lowest degree simple module for the filtration. In particular, we show by induction (hypothesis H_w in Section 6.6) that this module generates the filtration (in the sense of 6.5). By virtue of the above relations between filtrations, it is then enough to compute the BK filtration for extreme weights. As noted above, this obtains from Lusztig's deep positivity result for $\mathfrak{g}$ simply-laced. Except for E_8, there is also a much simpler positivity argument which we also present.

The theory we have described generalizes with no apparent difficulties to the case when a regular nilpotent element is replaced by a Richardson element, that is, when the Borel is replaced by a parabolic. However to avoid more complicated notation we shall stick to the Borel case. We remark that it might be interesting to compute the BK filtration for an arbitrary nilpotent element.

1. Preliminaries

1.1. For each Lie algebra $\mathfrak{a}$ let $U(\mathfrak{a})$ denote its enveloping algebra. Let $\mathfrak{g}$ be a split semisimple k-Lie algebra, and fix a Cartan subalgebra $\mathfrak{h}$ and a subset Δ^+ of positive roots in the set Δ of non-zero roots. Let $\pi \subset \Delta^+$ be the corresponding set of simple roots and $P(\pi)$ (resp. $P^+(\pi)$) the set of weights (resp. dominant weights). Let ρ be the half sum of the positive roots and for each $\alpha \in \Delta$, let s_α be the corresponding reflection with W the subgroup of $Aut\,\mathfrak{h}^*$ that they generate. Set $w.\lambda = w(\lambda+\rho)-\rho$, for all $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{b} \supset \mathfrak{h}$ be the Borel subgroup with nilradical $\mathfrak{n}^+$ having roots in Δ^+. Fix a Chevalley basis $e_\alpha, f_{-\alpha} : \alpha \in \Delta^+,\ h_\alpha : \alpha \in \pi$, and let κ be the corresponding Chevalley antiautomorphism. Let σ be the principal antiautomorphism and set $\iota = \sigma\kappa$, which is an involution. Set $\mathfrak{n}^- = \kappa(\mathfrak{n}^+)$. Given $\nu \in P(\pi)$, let $V(\nu)$ denote the simple finite-dimensional $U(\mathfrak{g})$ module with extreme weight ν. For any vector space R, we denote by $S(R)$ the symmetric algebra of R, or, more simply, the polynomial ring generated by the elements of R.

1.2. For all $\lambda \in \mathfrak{h}^*$ let k_λ denote the one-dimensional $\mathfrak{b}$ module of weight λ on which $\mathfrak{n}^+$ acts trivially. Then $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$ is the Verma module with highest weight λ and $v_\lambda := 1 \otimes 1$ is its canonical highest weight vector. Set $n = |\Delta^+|$ and let A_n (or simply, A) be the algebra on generators $q_{-\alpha}, p_\alpha := \partial/\partial q_{-\alpha}$ (or equivalently, $q_{-\alpha} = -\partial/\partial p_\alpha$) for all $\alpha \in \Delta^+$, where the $q_{-\alpha}$ (and hence the p_α) generate a polynomial subalgebra $\mathbf{Q}$ (resp. $\mathbf{P}$). The symmetrization map $s : S(\mathfrak{n}^-) \xrightarrow{\sim} U(\mathfrak{n}^-)$ identifies $\mathbf{Q}$ with any $M(\lambda)$. The natural action of A on $\mathbf{Q}$ then gives for any pair $\lambda, \mu \in \mathfrak{h}^*$, a linear embedding $A \to Hom_k(M(\lambda), M(\mu))$. If we note by r_λ the action of $U(\mathfrak{g})$ on $M(\lambda)$ (identified with $\mathbf{Q}$), then $r_\lambda(U(\mathfrak{g})) \subset A$ (see [C]). Hence, identifying A with its image in $Hom_k(M(\lambda), M(\mu))$, left composition with r_μ, and right composition with r_λ, give A the structure of a $U(\mathfrak{g})$ bimodule $A^{\lambda,\mu}$. We get then a $U(\mathfrak{g})$ module structure through diagonal action (for g in $\mathfrak{g}$, $g \circ \phi =$

$r_\mu(g)\circ\varphi-\varphi\circ r_\lambda(g))$. We recall below a number of properties of $A^{\lambda,\mu}$ summarizing the contents [J5, Sects. 1, 2].

1.3. Let $\mathcal{F}$ denote the filtration on A obtained by taking the degree filtration on $\mathbf{Q}$ and the trivial filtration on $\mathbf{P}$. In what follows, we refer to $\mathcal{F}$ as the operator filtration on A. [C, Section 5] gives us for any $g\in\mathfrak{g}$

$$r_\lambda(g)=\sum_{\gamma\in\Delta^+}P_\gamma^g q_\gamma+\sum_{i=1}^{l}\lambda_i P_i^g$$

where the P, P_i are polynomials in $\mathbf{P}$ and $\lambda_i=(\lambda,\alpha_i^\vee)$. We see that $\mathfrak{g}$ acts through elements of the first order in $q_{-\gamma}$. Now, as $[q_{-\gamma},p_\beta]=-\delta_{\gamma,\beta}$, it follows that $\mathcal{F}$ is an invariant filtration for the diagonal action of $U(\mathfrak{g})$.

In particular, the 0 degree of this filtration, $\mathbf{P}$, is a submodule of $A^{\lambda,\mu}$ of lowest weight $\mu-\lambda$. Its lowest weight vector is the $\mathfrak{b}^-$ module isomorphism of $M(\lambda)$ onto $M(\mu)$ sending v_λ to v_μ. Consider the $\mathcal{O}$-dual $\delta M(\lambda-\mu)$ of the Verma module $M(\lambda-\mu)$; then by [J6], $\mathbf{P}$ identifies with $(\delta M(\lambda-\mu))^\iota$.

1.4. Let M be a $U(\mathfrak{g})$ module and set $F_\mathfrak{h}(M):=\{m\in M\mid dim\ U(\mathfrak{h})m<\infty\}$. Then $F_\mathfrak{h}(M)$ is the direct sum of the generalized weight subspaces of M and is a $U(\mathfrak{g})$ submodule of M. In many cases, for example if M is the dual of a Verma module, $F_\mathfrak{h}(M)$ is a sum of $\mathfrak{h}$ weight spaces of M.

Set $\mathfrak{b}^-=\kappa(\mathfrak{b})$ and

$$F_{\mathfrak{b}^-}(M)=\{m\in F_\mathfrak{h}(M)\mid dim\ U(\mathfrak{n}^-)m<\infty\}.$$

Now consider $Hom(M(\lambda),M(\mu))$ under the diagonal action of $\mathfrak{g}$. By [J6, 3.5] one has

$$A^{\lambda,\mu}=F_{\mathfrak{b}^-}(Hom(M(\lambda),M(\mu))).$$

The main point of the proof is an argument along the lines of [J7, 2.6] to obtain

$$F_{\mathfrak{b}^-}(Hom(M(\lambda),M(\mu)))=Hom(M(\lambda),M(\mu))^{\mathfrak{n}^-}\,\mathbf{P}.$$

Now $Hom(M(\lambda),M(\mu))^{\mathfrak{n}^-}=End_{U(\mathfrak{n}^-)}(U(\mathfrak{n}^-))$ which is a subalgebra of A by [C].

1.5. We generalize slightly the construction of [J5, 2.1]. Take $\lambda,\mu,\nu\in\mathfrak{h}^*$. Composition of homomorphisms

$$Hom_k(M(\mu),M(\nu))\times Hom_k(M(\lambda),M(\mu))\to Hom_k(M(\lambda),M(\nu))$$

restricts to a map $A^{\mu,\nu}\times A^{\lambda,\mu}\to A^{\lambda,\nu}$ which corresponds to multiplication in A. Fix $\lambda,\mu\in\mathfrak{h}^*$ and set $I=\{a\in A^{\lambda,\mu}\mid av_\lambda=0\}$ which is an $A^{\mu,\mu}-U(\mathfrak{b})$ submodule of $A^{\lambda,\mu}$. Clearly $A^{\lambda,\mu}v_\lambda=M(\mu)$ which gives an isomorphism $a+I\mapsto av_\lambda$ of $A^{\lambda,\mu}/I$ onto $M(\mu)\mid_{U(\mathfrak{b})}\otimes k_{-\lambda}$ with respect to the *diagonal* action of $U(\mathfrak{b})$ on the target. Now assume μ is antidominant, that is, $(\mu+\rho,\alpha^\vee)\le 0$, for $\alpha\in\Delta^+$ for which the left hand side is an integer. One may recall [D, 7.6.23] that μ is antidominant if and only if $M(\mu)$ is simple and so isomorphic to its $\mathcal{O}$-dual $\delta M(\mu)$. Furthermore, [J5, 1.6] one has $\delta M(\mu)\mid_{U(\mathfrak{b})}\otimes k_{-\lambda}\cong\delta M(\mu-\lambda)\mid_{U(\mathfrak{b})}$. Hence one may identify $\delta M(\mu-\lambda)\mid_{U(\mathfrak{b})}$ with $M(\mu)\mid_{U(\mathfrak{b})}\otimes k_{-\lambda}$ when μ is antidominant. Using this identification and the above map, the operator filtration on A induces a filtration on $\delta M(\mu-\lambda)$. Let $\mathcal{F}^m(\delta M(\mu-\lambda))$ denote the image of $\mathcal{F}^m(A^{\lambda,\mu})$. It is finite dimensional and $U(\mathfrak{b})$ invariant.

With respect to the diagonal action of $U(\mathfrak{g})$ on $A^{\lambda,\mu}$, universality ([D, 5.5.3]) gives a $U(\mathfrak{g})$ module map ψ of $A^{\lambda,\mu}$ into $Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\mu-\lambda))$ which restricts to a map ψ_m of $\mathcal{F}^m(A^{\lambda,\mu})$ into $Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{F}^m(\delta M(\mu-\lambda)))$. As in [J5, 2.2 and 2.4] we obtain

Proposition. *Assume μ is antidominant. Then ψ_m (resp. ψ) is an isomorphism of $\mathcal{F}^m(A^{\lambda,\mu})$ (resp. $(A^{\lambda,\mu})$) onto $F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{F}^m(\delta M(\mu-\lambda))))$ (resp. $F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\mu-\lambda))))$.*

Let us recall the main ideas of the proof of the similar statement in [J5]. Injectivity of ψ [J5, Lemma 2.1] is rather easy. One shows by induction on n that $ker\,\psi \subset Ann_A U^n(\mathfrak{n}^-)v_{\mu-\lambda}$. (Here $U^n(\mathfrak{n}^-)$ is the n^{th} subspace corresponding to the canonical filtration of $U(\mathfrak{n}^-)$.) Hence one gets $ker\,\psi \subset Ann_A M(\mu-\lambda) = 0$. Surjectivity is more delicate. One has to compare weight multiplicities on both sides, so as to get [J5, Lemma 2.2] that ψ_m is an isomorphism of $\mathcal{F}^m(A^{\lambda,\mu})$ onto

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{F}^m(\delta M(\mu-\lambda)))).$$

The statement about ψ is obtained by showing [J5, Lemma 2.3] that any finitely generated submodule N of $F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\mu-\lambda)))$ actually lies inside $F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), V))$ for some finite $U(\mathfrak{b})$ module V of $\delta M(\mu-\lambda)$.

2. Graded injectivity and multiplicities

2.1. One of the main results of [J5] is that for μ dominant, $gr_{\mathcal{F}} A^{\mu,\mu}$ is injective as a module in a certain category of $U(\mathfrak{g})$ modules. A generalization of this result is needed here. First, we define the relevant categories of $U(\mathfrak{g})$ modules.

Let $\mathcal{O}$ denote the Bernstein-Gelfand-Gelfand category [BGG] of $U(\mathfrak{g})$ modules. By definition M is in $\mathcal{O}$ if and only if

(i) $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, *with* $M_\lambda = \{m \in M \mid hm = \lambda(h)m$, *for all* $h \in \mathfrak{h}\}$,
(ii) $dim\ M_\lambda < \infty$,
(iii) *the set* $\{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ *is contained in some cone* $\mu - \mathbb{N}\pi : \mu \in \mathfrak{h}^*$.

Let $\hat{\mathcal{O}}$ denote the $\mathfrak{g}$ module category consisting of all $U(\mathfrak{g})$ modules which are sums of objects in $\mathcal{O}$. Note that $Ob\hat{\mathcal{O}}$ consists of all weight modules with a locally finite action of $U(\mathfrak{b})$ and $\mathcal{O}$ is the full subcategory of objects in $\hat{\mathcal{O}}$ of finite length. Recall the involution ι defined in Section 1.1 and let $\mathcal{O}^-$ (resp. $\hat{\mathcal{O}}^-$) denote the $U(\mathfrak{g})$ module category obtained from $\mathcal{O}$ (resp. $\hat{\mathcal{O}}$) by transport under ι. For the diagonal action of $U(\mathfrak{g})$ one has by 1.4 that $A^{\lambda,\mu} \in \hat{\mathcal{O}}^-$.

The next proposition uses translation functors which we briefly review here. Consider $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P(\pi)$ and set χ_λ equal to the maximal ideal $Ann_{Z(\mathfrak{g})}M(\lambda)$ of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Let $\mathcal{M}_\lambda$ denote the category of $U(\mathfrak{g})$ modules annihilated by a power of χ_λ. Given a $U(\mathfrak{g})$ module M which admits a locally finite action of $Z(\mathfrak{g})$, define $\chi_\lambda(M)$ to be the $Z(\mathfrak{g})$ primary component of M with respect to the maximal ideal χ_λ. There is an exact functor L^μ_λ from $\mathcal{M}_\lambda$ to $\mathcal{M}_\mu$ defined by $L^\mu_\lambda : M \to \chi_\mu(V(\mu-\lambda)\otimes M)$ (see for example [J, Section 2.10]). Let R^μ_λ be the analogous functor on right modules defined using $V(\mu-\lambda)^*$.

A weight $\lambda \in \mathfrak{h}^*$ is called regular if $Stab_W\lambda$ is reduced to the neutral element.

Proposition. *Suppose $\lambda, \mu \in \mathfrak{h}^*$ are regular antidominant and $\lambda - \mu \in P^+(\pi)$. Then $gr_{\mathcal{F}} A^{\lambda,\mu}$ is injective in $\hat{\mathcal{O}}^-$.*

Proof. The assertion for $\lambda - \mu = 0$ is just [J5, 3.5, 3.6] and the main ideas are the following: The projectivity of $U(\mathfrak{g})\otimes_{U(\mathfrak{h})} k_0$ in the category of $\mathfrak{h}$-finite modules [GJ, 1.4.5] implies the injectivity of its dual $P^{*\sigma}$. By the restriction proposition for *Ext* groups [GJ, 1.5.7] this implies injectivity of $F_{\mathfrak{b}^-}(P^{*\sigma})$ in the category of $\mathfrak{b}^-$-finite modules. Now one shows that for any $N \in Ob\mathcal{O}^-$, one has a natural isomorphism

$$Hom_{U(\mathfrak{g})}(N, A^{\lambda,\lambda}) \xrightarrow{\sim} Hom_{U(\mathfrak{g})}(N, F_{\mathfrak{b}^-}(P^{*\sigma})).$$

Thus [J5, 2.7] $A^{\lambda,\lambda}$ is injective.

By [J5, 2.5], any indecomposable direct summand of

$$F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), \delta M(0)))$$

occurs as a submodule of $A^{\eta,\eta}$, for some $\eta \in P^+(\pi)$. As the filtrations of these $A^{\eta,\eta}$ are invariant under left and right translation [J5, 3.4], one then gets injectivity of $gr_{\mathcal{F}}A^{0,0}$ by splitting off injectives. Now for any $\lambda \in P$, $gr_{\mathcal{F}}A^{0,0}$ is isomorphic to $gr_{\mathcal{F}}A^{\lambda,\lambda}$.

For the general case we use translation functors. By hypothesis λ, μ are in the same facette (in the sense of Jantzen [J, 2.6]) and so $\chi_\lambda(V(\lambda-\mu)\otimes M(\mu)) \cong M(\lambda)$. This only needs that $\lambda - \mu \in P(\pi)$. Now

$$Hom_k(M(\mu), M(\mu)) \otimes Hom_k(V(\lambda-\mu), k) \cong Hom_k(M(\mu)\otimes V(\lambda-\mu), M(\mu))$$

admits a direct summand isomorphic to $Hom_k(M(\lambda), M(\mu))$. It follows as in [J5, 3.3] that the right translation functor R^λ_μ sends $A^{\mu,\mu}$ to $A^{\lambda,\mu}$. When $\lambda-\mu \in P^+(\pi)$, this is just right multiplication by the socle of $\mathbf{P}$ which is isomorphic to $V(\lambda-\mu)^*$. Now the elements of $\mathbf{P}$ are of filtration degree zero and so we conclude that $R^\lambda_\mu(\mathcal{F}^m(A^{\mu,\mu})) = \mathcal{F}^m(A^{\lambda,\mu})$. Moreover R^λ_μ is an equivalence of $U(\mathfrak{g})$ bimodule categories with inverse functor R^μ_λ. Viewed as a functor of $U(\mathfrak{g})$ modules (under diagonal action) it still consists of tensoring by finite-dimensional modules (which preserve injectives via Frobenius reciprocity) and taking direct summands (though not those corresponding to a fixed central character). Thus the injectivity of $\mathcal{F}^m(A^{\mu,\mu})$ (under diagonal action) implies the injectivity of $\mathcal{F}^m(A^{\lambda,\mu})$. Then $gr_{\mathcal{F}}(A^{\lambda,\mu})$ is injective by splitting off injectives. □

Remarks. Of course the conclusion still holds if just λ, μ are in the same facette; but fails if $\lambda - \mu \notin P^+(\pi)$, because there is no guarantee that degree is preserved. Indeed as we shall see in Section 6 the unique minimal $\mathfrak{k}$-type $V(\lambda-\mu)^*$ of $A^{\lambda,\mu}$ has strictly positive degree unless $\lambda - \mu \in P^+(\pi)$. Of course $R^\lambda_\mu(A^{\mu,\mu}) = A^{\lambda,\mu}$, so $A^{\lambda,\mu}$ is injective; but this already follows from 3.1 and [J5, 2.6] and is not so useful. Indeed from the analysis below it just implies what is already known from Frobenius reciprocity. Again from, say, [J5, 2.2] it is easy to see that $\mathcal{F}^m(A^{\lambda,\mu})$ has a dual Verma flag with quotients isomorphic to the $\delta M((\lambda-\mu)+\beta)^\iota$, where β is a sum of $\leq m$ positive roots. When $\lambda - \mu$ goes away from the walls, this becomes a direct sum of modules which are injective in $\mathcal{O}^-$ if and only if $\lambda - \mu \in P^+(\pi)$.

2.2. Define D_q to be an element of the group ring $\mathbb{Z}[q,q^{-1}]P(\pi)$ defined by

$$D_q = \prod_{\alpha\in\Delta^+}(1-qe^\alpha)^{-1},$$

and let D denote its value at $q=1$. Extend $e^\alpha \mapsto e^{-\alpha}$ linearly to an involution $a \mapsto \bar{a}$ of $\mathbb{Z}[q,q^{-1}]P(\pi)$. For any weight module M with an $\mathfrak{h}$ invariant filtration $\mathcal{G}$

define

$$ch_q^{\mathcal{G}} M = \sum_{\mu \in \mathfrak{h}^*} \sum_{m \in \mathbb{N}} q^m e^\mu dim_k(\mathcal{G}^m(M)_\mu / \mathcal{G}^{m-1}(M)_\mu),$$

where the subscript μ denotes the weight subspace $\mu \in \mathfrak{h}^*$ (assumed finite dimensional). When the invariant filtration on M is inherited from the operator filtration $\mathcal{F}$ on $A^{\lambda,\mu}$, then we simply write $ch_q(M)$ for the corresponding q character formula.

It is clear that for diagonal action

$$ch_q A^{\lambda,\mu} = D e^{\mu-\lambda} \bar{D}_q.$$

For any $U(\mathfrak{g})$ module M, let $F(M) := \{m \in M \mid dim\ U(\mathfrak{g})m < \infty\}$ denote its locally finite submodule. Consider now an injective indecomposable module $I := I(\lambda),\ \lambda \in P(\pi)$, of the category $\mathcal{O}$. It has a filtration with factors isomorphic to dual Verma modules $\delta M(w.\lambda)$ (whose multiplicities are known by the Kazhdan-Lusztig conjecture). Define

$$\chi_I = \sum_{w \in W} [I : \delta M(w.\lambda)] e^{w.\lambda}, \ \ J = \bar{D} \left(\sum_{w \in W} (-1)^{\ell(w)} w. \right).$$

Lemma ([J5, 4.2])**.** *Take $\lambda \in P^+(\pi)$. We then have*

$$J(\chi_I) = \begin{cases} ch\, V(\lambda), & w = e, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. For $w = e$ this is simply the Weyl character formula, $I(\lambda)$ being $\delta M(\lambda)$. Otherwise there is an $\alpha \in \pi$ for which $s_\alpha w < w$, and therefore $V(w.\lambda)$ is not α-locally finite. This implies, through B.G.G. duality, that for any couple $s_\alpha y > y$ of elements of W,

$$[I(w.\lambda) : \delta M(y.\lambda)] = [I(w.\lambda) : \delta M(s_\alpha w.\lambda)].$$

Thus the respective contributions to $J(\chi_I)$ cancel each other out. □

A remarkable property of injective modules in $\mathcal{O}$ (or $\mathcal{O}^-$) which follows from the above is that one can compute multiplicities of the finite part $F(I)$ simply by knowing the formal character of I. Thus by Proposition 2.1, we obtain

$$(*) \qquad \overline{ch_q F(A^{\lambda,\mu})} = J(e^{\lambda-\mu} D_q).$$

2.3. Let P_q be the Kostant q-partition function defined by

$$\sum_{\beta \in \mathbb{N}\pi} P_q(\beta) e^\beta = D_q.$$

Define Lusztig's q polynomial at weight μ as follows (see [L1, Section 9.4] or [B1, Section 3.3]):

$$m_\nu^\mu(q) = \sum_{w \in W} (-1)^{\ell(w)} P_q(w.\nu - \mu)$$

for all $\nu \in P^+(\pi),\ \mu \in P(\pi)$. One may remark that

$$m_\nu^\mu(1) = dim\ V(\nu)_\mu.$$

However unless $\mu \in P^+(\pi)$, the coefficients of $m_\nu^\mu(q)$ can be negative already in $\mathfrak{sl}(2)$.

Set $\mathcal{F}_m(F(A^{\lambda,\mu})) = \mathcal{F}^m(F(A^{\lambda,\mu}))/\mathcal{F}^{m-1}(F(A^{\lambda,\mu}))$. It is a direct sum of finite-dimensional highest weight modules. Let $a \mapsto [a]_0$ be the evaluation map on $\mathbb{Z}[q,q^{-1}]P(\pi)$ sending e^α to 1 if $\alpha = 0$ and to 0 otherwise. Let $\langle\ ,\ \rangle$ denote the Macdonald scalar product $\langle a, b\rangle := [a\bar{b}/D\bar{D}]_0$. It is well known that $\langle ch\, V(\mu), ch\, V(\nu)\rangle = \delta_{\mu,\nu}|W|$ (see [JL, 7.2] for example).

Proposition. *Assume λ, μ are antidominant and regular with $\lambda-\mu \in P^+(\pi)$. Then for all $\nu \in P^+(\pi)$ the multiplicity of $V(\nu)^*$ in $\mathcal{F}_m(F(A^{\lambda,\mu}))$ is just the coefficient of q^m in $m_\nu^{\lambda-\mu}(q)$.*

Proof. The calculation follows [JL, 7.4]. One has

$$\sum_m [\mathcal{F}_m(F(A^{\lambda,\mu})) : V(\nu)^*]q^m = \langle\overline{ch\ V(\nu)},\ \overline{J(e^{\lambda-\mu}D_q)}\rangle, \quad by \quad 2.2(*).$$

Writing $e^{\lambda-\mu}D_q$ as $\sum_\xi\ f(\xi)e^\xi$ and using the definition of $\langle\ ,\ \rangle$ this equals

$$\frac{1}{|W|}\left[\sum_{x,y\in W}(-1)^{\ell(x)}e^{x.\nu}(-1)^{\ell(y)}y.(\sum e^{-\xi}f(\xi))\right]_0$$
$$= \left[\sum_{w\in W}\ \sum_{\xi\in\mathbb{Z}\pi}(-1)^{\ell(w)}e^{w(\nu+\rho)-\xi-\rho}f(\xi)\right]_0$$
$$= \left[\sum_{w\in W}\ \sum_{\beta\in\mathbb{N}\pi}(-1)^{\ell(w)}e^{w.\nu-(\lambda-\mu)-\beta}P_q(\beta)\right]_0$$
$$= \sum_{w\in W}(-1)^{\ell(w)}P_q(w.\nu-(\lambda-\mu)) = m_\nu^{\lambda-\mu}(q),$$

as required. □

3. Extending Frobenius reciprocity

3.1. Given a left (resp. right) $U(\mathfrak{g})$ module M and an automorphism (resp. antiautomorphism) τ of $U(\mathfrak{g})$, define M^τ to be the left $U(\mathfrak{g})$ module which is M as a vector space and admits the action

$$(a, m) \mapsto \tau(a)m \qquad (\text{resp. } m\tau(a)).$$

In particular, for any $M \in Ob\mathcal{O}$, its $\mathcal{O}$-dual is defined as $\delta M := F_\mathfrak{h}(M^{*\kappa}) \in Ob\mathcal{O}$. Then for all $\nu \in h^*$ one has

$$\begin{aligned}\delta M(\nu) &= F_\mathfrak{h}(Hom_k(U(\mathfrak{g})\otimes_{U(\mathfrak{b})} k_\nu, k))^\kappa\\ &= F_\mathfrak{h}(Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), Hom_k(k_\nu, k)^\kappa)).\end{aligned}$$

It is convenient to designate the left $U(\mathfrak{b}^-)$ module $Hom_k(k_\nu, k)^\kappa$ simply by k_ν^*. It is trivial as a $U(\mathfrak{n}^-)$ module and has weight ν as a $U(\mathfrak{h})$ module because κ is the identity on $U(\mathfrak{h})$.

Lemma. *For all $\nu \in \mathfrak{h}^*$, Frobenius reciprocity gives an isomorphism*

$$F_\mathfrak{h}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\nu))) \xrightarrow{\sim} F_\mathfrak{h}(Hom_{U(\mathfrak{h})}(U(\mathfrak{g}), k_\nu^*)).$$

In particular, for all $\lambda, \mu \in \mathfrak{h}^*$ with μ antidominant, one has an isomorphism

$$A^{\lambda,\mu} \xrightarrow{\sim} F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{h})}(U(\mathfrak{g}), k_{\mu-\lambda}^*)).$$

Proof. Let N be a $U(\mathfrak{b})$ module and consider $\theta \in F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), N))$. Fix $a \in U(\mathfrak{g})$. For all $h \in \mathfrak{h}$ one has $(h.\theta)(a) = \theta(ah) = -\theta((ad\ h)a) + h\theta(a)$. Thus $U(\mathfrak{h})\theta(a)$ belongs to the subspace $(U(\mathfrak{h}).\theta)(a) + \theta(ad\ U(\mathfrak{h})a)$ of N which is finite dimensional. Consequently the natural injection

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), F_{\mathfrak{h}}(N))) \hookrightarrow F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), N))$$

is an isomorphism. In particular,

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\nu))) \xrightarrow{\sim} F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), k_\nu^*))).$$

Yet $Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), k_\nu^*)$ as a left $U(\mathfrak{b})$ module is just $Hom_{U(\mathfrak{h})}(U(\mathfrak{b}), k_\nu^*)$ and so the right hand side above is just

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), Hom_{U(\mathfrak{h})}(U(\mathfrak{b}), k_\nu^*))) \xrightarrow{\sim} F_{\mathfrak{h}}(Hom_{U(\mathfrak{h})}(U(\mathfrak{g}), k_\nu^*)).$$

The last assertion follows from 1.5. □

Remarks. A similar argument also works in the parabolic case. In particular, the notation of [J5, Section 2] gives

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{p})}(U(\mathfrak{g}), \delta M_{\pi'}(\nu))) \xrightarrow{\sim} F_{\mathfrak{h}}(Hom_{U(\mathfrak{r})}(U(\mathfrak{g}), k_\nu^*)),$$

which improves somewhat [J5, 2.6]. As in [J5, Section 2] it implies that $A^{\lambda,\mu}$ is injective in $\hat{\mathcal{O}}^-$ for all μ antidominant.

3.2. We recall here the definition of the filtration studied in [B1]. Take an element $x \in \mathfrak{n}^+$. By the Jacobson-Morosov theorem x lies in some TDS with basis (x, h, y). We assume that x and the TDS are chosen so that $h \in \mathfrak{h}$. (We remark that this restriction is already present in Brylinski's work [B1]. It is needed to translate Kostant's result [K2, Section 4, Corollary 3] to compute the BK filtration for a zero weight subspace.) Fix $\nu \in P^+(\pi)$ and let β be a weight of the finite-dimensional simple $U(\mathfrak{g})$ module $V(\nu)$. The BK filtration $\mathcal{J}_x$ on the β weight space $V(\nu)_\beta$ is defined by setting $\mathcal{J}_x^p(V(\nu)_\beta)$ equal to the subspace of elements in $V(\nu)_\beta$ annihilated by x^{p+1}. In what follows, we consider this filtration from a different perspective. In particular, the *left* $U(\mathfrak{g})$ module $V(\nu)$ is replaced with a corresponding *right* $U(\mathfrak{g})$ module. This approach allows us to connect the BK filtration of a finite-dimensional $U(\mathfrak{g})$ module with a similar filtration defined on $A^{\lambda,\mu}$.

Consider k as the trivial $U(\mathfrak{n}^+)$ module. One defines the generic Verma module $\mathcal{M} := U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^+)} k$. The surjection $\mathcal{M} \to M(\nu)$ gives rise to an embedding

$$\delta M(\nu) \hookrightarrow \mathcal{M}^{*\kappa} \cong Hom_{U(\mathfrak{n}^-)}(U(\mathfrak{g}), k^*),$$

where k^* denotes the (trivial) $U(\mathfrak{n}^-)$ module $Hom_k(k, k)^\kappa$. If we further define

$$\mathcal{A} = Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{M}^{*\kappa}),$$

then for μ antidominant we obtain an embedding

$$(*) \qquad A^{\lambda,\mu} \hookrightarrow \mathcal{A}.$$

As a $U(\mathfrak{b})$ module, $\mathcal{M}^{*\kappa}$ is isomorphic to $Hom_k(U(\mathfrak{b}), k^*)$. Thus Frobenius reciprocity gives an isomorphism

$$(**) \qquad \mathcal{A} \cong Hom_k(U(\mathfrak{g}), k^*).$$

Now fix $\nu \in P^+(\pi)$ and view $V(\nu)^*$ as a *left* $U(\mathfrak{g})$ module through the principal antiautomorphism σ, hence isomorphic to $V(-\nu)$. We write $V(\nu)^{**}$ below to emphasize that this is the dual of $V(\nu)^*$ and so considered as a *right* $U(\mathfrak{g})$ module.

In particular, $V(\nu)^{**}_{-(\lambda-\mu)} = V(\nu)_{\lambda-\mu}$, for $V(\nu)$ considered as a left $U(\mathfrak{g})$ module through σ. Frobenius reciprocity gives

$$V(\nu)^{**}_{-(\lambda-\mu)} \cong Hom_{U(\mathfrak{g})}(V(\nu)^*, (Hom_{U(\mathfrak{h})}(U(\mathfrak{g}), k^*_{\mu-\lambda}))).$$

Thus by 3.1 there is a vector space isomorphism

$$\varphi_{\lambda,\nu} : V(\nu)^{**}_{-(\lambda-\mu)} \xrightarrow{\sim} Hom_{U(\mathfrak{g})}(V(\nu)^*, A^{\lambda,\mu}).$$

By $(*)$, there is an embedding

$$Hom_{U(\mathfrak{g})}(V(\nu)^*, A^{\lambda,\mu}) \hookrightarrow Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{A}).$$

Furthermore $(**)$ yields

$$Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{A}) \cong Hom_{U(\mathfrak{g})}(V(\nu)^*, (Hom_k(U(\mathfrak{g}), k^*))).$$

This combined with Frobenius reciprocity, gives an isomorphism

$$\varphi : V(\nu)^{**} \xrightarrow{\sim} Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{A}).$$

Thus the embedding of $V(\nu)^{**}_{-(\lambda-\nu)}$ in $V(\nu)^{**}$ corresponds to the embedding of $Hom_{U(\mathfrak{g})}(V(\nu)^*, A^{\lambda,\mu})$ in $Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{A})$. Moreover the map $\varphi_{\lambda,\mu}$ can be thought of as restriction of φ to the $-(\lambda-\mu)$ weight space of $V(\nu)^{**}$.

The point of the above observation is that $\mathcal{A}$ (unlike $A^{\lambda,\mu}$), when viewed as $Hom_k(U(\mathfrak{g}), k^*)$, admits a right $U(\mathfrak{g})$ module structure coming from left multiplication in $U(\mathfrak{g})$. This gives $V(\nu)^{**}$ its natural right $U(\mathfrak{g})$ module structure. Now if we consider $\mathcal{A}$ as $Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{M}^{*\kappa})$, then the restriction of this right module structure to $U(\mathfrak{b})$ is exactly that which comes from the right $U(\mathfrak{b})$ module structure of $\mathcal{M}^{*\kappa} = Hom_k(U(\mathfrak{b}), k^*)$ coming from left multiplication in $U(\mathfrak{b})$. The above embedding allows us to consider an element $x \in U(\mathfrak{b})$ applied to $V(\nu)^{**}_{-(\lambda-\mu)} \hookrightarrow V(\nu)^{**}$. In particular, consider again an element $x \in \mathfrak{n}^+$ such that the corresponding TDS is chosen with $h \in \mathfrak{h}$. We now restate the BK filtration using this right module structure as follows. The BK filtration $\mathcal{F}_x$ of $V(\nu)^{**}_{-(\lambda-\mu)}$ relative to x is defined by

$$\mathcal{F}^n_x(V(\nu)^{**}_{-(\lambda-\mu)}) = \{v \in V(\nu)^{**}_{-(\lambda-\mu)} \mid vx^{n+1} = 0\}.$$

Recall that $V(\nu)^{**}$ is isomorphic to $V(\nu)$ considered as a left $U(\mathfrak{g})$ module through the antiautomorphism σ. Thus, given a weight γ, the filtration $\mathcal{F}_x$ induces a filtration on $V(\nu)_\gamma$ which we also denote by $\mathcal{F}_x$ and refer to as the BK filtration on $V(\nu)_\gamma$. It should be noted that this is the same filtration as $\mathcal{J}_{\sigma(x)}$, though we will not use this latter notation.

One may ask what is the image of $\mathcal{F}^n_x(V(\nu)^{**}_{-(\lambda-\mu)})$ under $\varphi_{\lambda,\mu}$? Now $\delta M(\mu-\lambda)$ is isomorphic to $F_{\mathfrak{h}}(Hom_{U(\mathfrak{h})}(U(\mathfrak{b}), k^*_{\mu-\lambda}))$ and thus embeds inside of $\mathcal{M}^{*\kappa}$. As a $U(\mathfrak{n}^+)$ module, $\delta M(\mu-\lambda)$ is just $F_{\mathfrak{h}}(Hom_k(U(\mathfrak{n}^+), k^*))$, that is, the graded dual of $U(\mathfrak{n}^+)$. Like $\mathcal{M}^{*\kappa}$, this has a right $U(\mathfrak{n}^+)$ module structure coming from left multiplication in $U(\mathfrak{n}^+)$ (resp. $U(\mathfrak{b})$) and the above embedding is a homomorphism of $U(\mathfrak{n}^+)$ bimodules. In $\delta M(\mu-\lambda)$ this right $U(\mathfrak{n}^+)$ action does not commute with its left $U(\mathfrak{g})$ action; but it does commute with its left $U(\mathfrak{b})$ action up to translation of weights (in the obvious manner). Thus

$$\mathcal{F}^n_x(\delta M(\nu)) := \bigoplus_{\gamma \in \nu - \mathbb{N}\pi} \{m \in \delta M(\nu)_\gamma \mid mx^{n+1} = 0\}$$

is a $U(\mathfrak{b})$ invariant filtration $\mathcal{F}_x$ of $\delta M(\nu)$. We call it the Brylinski-Kostant filtration of $\delta M(\nu)$ relative to x, more properly with respect to the pair $(x, h) : h \in \mathfrak{h}$.

Under the hypothesis that μ is antidominant so that 1.5 applies, set $\mathcal{F}_x^n(A^{\lambda,\mu}) = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{F}_x^n(\delta(M(\mu-\lambda)))))$.

Lemma. *Assume μ is antidominant. For all $n \in \mathbb{N}$ the Frobenius map $\varphi_{\lambda,\mu}$ restricts to an isomorphism of $\mathcal{F}_x^n(V(\nu)_{\lambda-\mu})$ onto $Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{F}_x^n(A^{\lambda,\mu}))$.*

Proof. Recall the isomorphism

$$\varphi : V(\nu)^{**} \xrightarrow{\sim} Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{A}).$$

Given $v \in V(\nu)^{**}$ and $\xi \in V(\nu)^*$, $\varphi(v)(\xi)$ is the element of $Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{M}^{*\kappa}) = \mathcal{A}$ such that $\varphi(v)(\xi)(a)(c) = v(ca\xi)$ for all $a \in U(\mathfrak{g})$ and $c \in U(\mathfrak{b})$. Note that $\varphi(v)(\xi)$ is a right $U(\mathfrak{b})$ module map and in particular $(\varphi(v)(\xi).b)(a)(c) = v(bca\xi)$ for all $v \in V(\nu)^{**}$, $\xi \in V(\nu)^*$, $b \in U(\mathfrak{b})$, $c \in U(\mathfrak{b})$, and $a \in U(\mathfrak{g})$. Fix $\xi \in V(\nu)^*$ and $a \in U(\mathfrak{g})$, and set $\theta(v) = \varphi(v)(\xi)(a) \in \mathcal{M}^{*\kappa}$. Recall that the right $U(\mathfrak{b})$ module structure of $\mathcal{A}$ comes from the right $U(\mathfrak{b})$ module structure of $\mathcal{M}^*$. Hence, the above formula shows that $\theta(v.b) = \theta(v).b$ with respect to the right action of $U(\mathfrak{b})$ on $V(\nu)^{**}$ and the previously defined right action of $U(\mathfrak{b})$ on $\mathcal{M}^{*\kappa}$. Now take $v \in V(\nu)^{**}_{-(\lambda-\mu)}$. Then $v.x^{n+1} = 0$ if and only if $v(x^{n+1}ba\xi) = 0$ for all $\xi \in V(\nu)^*, b \in U(\mathfrak{b})$, and $a \in U(\mathfrak{g})$. This latter condition is equivalent to $\varphi_{\lambda,\mu}(v)(\xi)(a).x^{n+1} = 0$ for all $\xi \in V(\nu)^*$ and $a \in U(\mathfrak{g})$. Thus $v.x^{n+1} = 0$ if and only if $\varphi_{\lambda,\mu}(v)(V(\nu)^*)(U(\mathfrak{g})).x^{n+1} = 0$. We conclude that $v \in \mathcal{F}_x^n(V(\nu)_{\lambda-\mu})$ if and only if $\varphi_{\lambda,\mu}(v)(V(\nu)^*)(U(\mathfrak{g})) \subset \mathcal{F}_x^n(\delta M(\mu-\lambda))$. Indeed the left-hand side of this inclusion is a $U(\mathfrak{b})$ submodule of $\mathcal{M}^{*\kappa}$ and from the definition of $\mathcal{F}_x$ the right-hand side is the largest $U(\mathfrak{b})$ submodule of $\delta M(\mu-\lambda)$ annihilated by x^{n+1}. Consequently $v \in \mathcal{F}_x^n(V(\nu)_{\lambda-\mu})$ if and only if $\varphi_{\lambda,\mu}(\nu)(V(\nu)^*) \subset \mathcal{F}_x^n(A^{\lambda,\mu})$ which is equivalent to $\varphi_{\lambda,\mu}(\nu) \in Hom_{U(\mathfrak{g})}(V(\nu)^*, \mathcal{F}^n(A^{\lambda,\mu}))$ as required. □

Remark. We see from the above that a Brylinski-Kostant filtration (which involves applying powers of x to weight vectors) is in fact rather natural since it leads to a $U(\mathfrak{b})$ invariant filtration on $\delta M(u-\lambda)$ and then to a $U(\mathfrak{g})$ invariant filtration on $A^{\lambda,\mu}$. This would have failed had we applied powers of x to arbitrary vectors.

3.3. The above result gives a q-version of Frobenius reciprocity. Recall the definition and notation of the q character associated to a filtration (see Section 2.2). Denote $ch_q^{\mathcal{F}_x}M$ by $ch_q^x M$ for those modules which admit a BK filtration $\mathcal{F}_x$. In particular,

$$ch_q^x V(\nu)_\gamma = \sum_{n=0}^{\infty} q^n dim(\mathcal{F}_x^n(V(\nu)_\gamma)/\mathcal{F}_x^{n-1}(V(\nu)_\gamma))$$

and

$$ch_q^x F(A^{\lambda,\mu}) = \sum_{n=0}^{\infty} q^n dim(\mathcal{F}_x^n F(A^{\lambda,\mu})/\mathcal{F}_x^{n-1} F(A^{\lambda,\mu})).$$

Corollary. *For all $\mu \in \mathfrak{h}^*$ antidominant and $\lambda \in \mathfrak{h}^*$ with $\lambda - \mu \in P(\pi)$ and $\nu \in P^+(\pi)$, one has*

$$ch_q^x V(\nu)_{\lambda-\mu} = \langle V(\nu)^*, ch_q^x F(A^{\lambda,\mu})\rangle.$$

Remark. Of course, for any $\gamma \in P(\pi)$, we can choose λ, μ antidominant and regular so that $\gamma = \lambda - \mu$. This is an additional flexibility not present in the Brylinski theory and allows us to analyze the case when γ is not dominant.

4. The principal BK filtration of a dual Verma module

4.1. Let $\mathbf{G}$ be the adjoint group of $\mathfrak{g}$ (generated by *exp ad* $x_\alpha : \alpha \in \Delta$), and let $\mathbf{B}$ (resp. $\mathbf{H}$) be the connected subgroup corresponding to $\mathfrak{b}$ (resp. $\mathfrak{h}$). Fix $x \in \mathfrak{n}^+$ embedded in some TDS with semisimple element $h \in \mathfrak{h}$ and recall the BK filtrations $\mathcal{F}_x$ on $V(\nu)_{\lambda-\nu}$, on $\delta M(\mu-\lambda)$ and on $A^{\lambda,\mu}$ defined in 3.2. We call a BK filtration *principal* if x is regular, equivalently if x belongs to a principal TDS. Recall that the regular elements of $\mathfrak{n}^+$ form a simple $\mathbf{B}$ orbit. Hence those satisfying the above condition on the (regular) semisimple element form a single $\mathbf{H}$ orbit. Obviously these filtrations depend on x, though should be independent of the $\{b \in \mathbf{B} \mid (Ad\ b)h \in \mathfrak{h}\}$ orbit to which x belongs. We shall show this for x regular. We may anticipate a similar result for x Richardson with $\mathbf{B}$ replaced by the corresponding parabolic and $\mathbf{H}$ by the corresponding Levi factor. The general situation is less clear. By Kostant's construction noted in the introduction, it follows that $ch_q^x V(\nu)_0$ is independent of the choice of x in its $\mathbf{G}$ orbit given that x is regular (nilpotent) and satisfies the above condition on h. We shall show exactly (Section 7) how this is modified for the remaining weight subspaces. The situation for x non-regular is less clear.

4.2. From now on we assume that $x \in \mathfrak{n}^+$ is embedded in a principal TDS, say (x, h, y), with $h \in \mathfrak{h}$ and consider the BK filtration $\mathcal{F}_x$ on $\delta M(\mu-\lambda)$. Here we can assume $\mu - \lambda = 0$ without loss of generality since up to a shift by $\mu-\lambda$ of weight spaces the $U(\mathfrak{b}) - U(\mathfrak{n}^+)$ bimodule structure of $\delta M(\mu-\lambda)$ is independent of $\mu-\lambda$.

By definition $\delta M(0) = F_{\mathfrak{h}}(Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), k_0^*))$ where k_0^* is the trivial $U(\mathfrak{b}^-)$ module using the notational conventions of 3.1, 3.2. One easily checks the well-known fact that $Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), k_0^*)$ and hence $\delta M(0)$ is a subalgebra of $U(\mathfrak{g})^*$ in which $\mathfrak{g}$ acts by derivations.

As a left $U(\mathfrak{b})$ module, $\delta M(0)$ identifies with

$$F_{\mathfrak{h}}(Hom_{U(\mathfrak{h})}(U(\mathfrak{b}), k_0^*)) \cong F_{\mathfrak{h}}(U(\mathfrak{n}^-)^{*\kappa}),$$

where the left multiplication of $U(\mathfrak{n}^-)$ on itself is extended to an action of $U(\mathfrak{b}^-)$ via the adjoint action of $U(\mathfrak{h})$. Thus $\delta M(0) \mid_{U(\mathfrak{b})}$ is just the graded dual of $U(\mathfrak{n}^-)$ with a left action of $U(\mathfrak{b})$ obtained through κ. In particular, $\delta M(0) \mid_{U(\mathfrak{b})}$ admits a right $U(\mathfrak{n}^+)$ action which commutes with its left $U(\mathfrak{b})$ action, up to (an obvious) translation of weights and in which the elements of $\mathfrak{n}^+$ also act by derivations. Set $x = \sum_{\alpha\in\pi} e_\alpha$, which we recall [D, 8.1.1] is regular and is embedded in a TDS with semisimple element $h \in \mathfrak{h}$.

Let V be the largest $U(\mathfrak{h})$ invariant subspace of $\delta M(0)$ satisfying $Vx^2 = 0$. The only weight vector in V annihilated by x is the highest weight vector of weight 0 which identifies with the identity 1 of the ring $\delta M(0)$. Thus V admits a unique weight space decomposition as $V = V^- \oplus k1$. The main result of [J1, Section 4] is the following.

Proposition. *As a $U(\mathfrak{h})$ module V^- is isomorphic to $\mathfrak{n}^-$. Furthermore $\delta M(0)$ is generated by V^- as a polynomial algebra and $\mathcal{F}_x^m(\delta M(0)) = k + V^- + \cdots + (V^-)^m$, for all $m \in \mathbb{N}$.*

Remarks. Of course this is really equivalent to the assertion that

$$ch_q^x \delta M(0) = \prod_{\alpha\in\Delta^+} (1 - qe^{-\alpha})^{-1}$$

which follows from the Brylinski result on $V(\nu)$ taking $\nu \to \infty$. However the proof in [J1] is elementary and in particular independent of Kostant's hard primeness result [K2] and Broer's vanishing theorem [B, Theorem 2.4] of the higher cohomology of a line bundle on $T^*(\mathbf{G}/\mathbf{B})$ corresponding to a dominant weight.

In the variables defined by the weight vectors of V^- the action of $\mathfrak{n}^+$ is by derivations with at most linear coefficients. Of course the existence of such an action was a key point in the proof. Here we add that the assertion in [J1, p. 406, lines 5, 6] is explicitly verified in [J3, Lemma 8.5]. This action comes from twisting slightly the (linear) coadjoint action of $\mathfrak{n}^+$ on $\mathfrak{n}^-$ through a suitable evaluation (depending on h) of the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^-$. For $\mathfrak{g}$ simple, the unique highest weight β of $\mathfrak{g}$ gives rise to a vector $p_{-\beta} \in V^-$ which through the embedding $V(\beta) \hookrightarrow \delta M(\beta)$ and shifting of weights by $-\beta$ identifies with the semisimple element h of the principal TDS containing x. (Observe that $V(\beta)$ is the adjoint representation and $(ad\ x)^2 h = 0$.) The remaining weight vectors $p_{-\gamma} : \gamma \in \Delta^+$ of V^- are obtained from $p_{-\beta}$ by the left action of $\mathfrak{n}^+$.

Consider as an example $\mathfrak{g}$ of type A_3, with its usual Chevalley basis. To avoid confusion, we shall note the generators of $\delta M(0) \cong S(\mathfrak{n}^-)$ by $a_{-\gamma}$. Note that $dim\, \mathcal{F}^1_x(\delta M(0)_{-\beta}) = 1$ and this vector space is spanned by $v = a_{-\beta} - 1/6\, a_{-\alpha_1} a_{-\alpha_2-\alpha_3} + 1/6\, a_{-\alpha_3} a_{-\alpha_1-\alpha_2}$. One may check that the $\mathfrak{n}^+$ space generated by v is

$$V^- = \{a_{-\alpha_1},\, a_{-\alpha_2},\, a_{-\alpha_3},\, a_{-\alpha_1-\alpha_2},\, a_{-\alpha_2-\alpha_3},\, v\}$$

which is isomorphic as an $\mathfrak{h}$ module to $\mathfrak{n}^-$ (as in [J1, p. 406 (*)] we get $e_{\alpha_2} v = 0$). Note also that the vector space $\mathcal{F}^1_x(\delta M(0))$ is different from $\mathcal{F}^1(\delta M(0))$.

4.3. The above result applies to any other principal TDS with $x \in \mathfrak{n}^+$ and $h \in \mathfrak{h}$. This is simply through conjugation by $\mathbf{H}$. Indeed the right action of x on $U(\mathfrak{n}^-)^{*\kappa}$ is $ad\ \mathfrak{h}$ equivariant. Thus if $\xi x^{n+1} = 0$, for some $\xi \in U(\mathfrak{n}^-)^{*\kappa}$, then $(Ad\ h)\xi((Ad\ h)x)^{n+1} = 0$, for all $Ad\ h \in \mathbf{H}$. Yet V is already $ad\ \mathfrak{h}$, hence $\mathbf{H}$ stable and so $\mathcal{F}_x(\delta M(0))$ is independent of x. A similar argument applies to $\mathcal{F}_x(V(\nu)_{\lambda-\mu})$.

5. The principal BK filtration of $A^{\lambda,\mu}$

5.1. Fix λ, μ regular and antidominant with $\lambda - \mu \in P(\pi)$. From now on we take $x = \sum_{\alpha\in\pi} e_\alpha$. By 4.3 this entails no loss of generality. Recall the operator filtration $\mathcal{F}$ and the BK filtration $\mathcal{F}_x$ defined on $A^{\lambda,\mu}$ and on $\delta M(\mu - \lambda)$ in Section 1 and Section 3, respectively. Unfortunately $\mathcal{F}$ and $\mathcal{F}_x$ cannot be expected to coincide (outside $\mathfrak{sl}(2)$). Indeed to construct $\mathcal{F}_x$ we made use of a choice of variables in which $\mathfrak{n}^+$ acts by derivations with at most linear coefficients. Then in order for $\mathcal{F}, \mathcal{F}_x$ to coincide we would need to know that for this same choice of variables $\mathfrak{n}^-$ acts by derivations (and possibly a further multiplicative term) with coefficients which are "on average" at most quadratic. This is a little too much to expect and indeed fails by the example in 4.2. Nevertheless we show that the filtrations $\mathcal{F}, \mathcal{F}_x$ on $A^{\lambda,\mu}$ are equivalent, more precisely coincide up to choice of the generating copy of $\mathfrak{g}$ (see below). One may further conclude that they induce equivalent filtrations on $F(A^{\lambda,\mu})$.

5.2. Let us write $A^{\mu,\mu}$ simply as A. Recall that by 1.4 and 1.5 one has

$$\mathcal{F}^m A = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathcal{F}^m(\delta M(0)))).$$

Now $\mathcal{F}^0(\delta M(0)) = k_0$ and so $\mathbf{P} = \mathcal{F}^0 A = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), k_0))$ which is $\delta M(0)^\iota$ as a $U(\mathfrak{g})$ module. Furthermore the latter has an algebra structure coming from multiplication in $U(\mathfrak{g})^*$. One may easily check (the well-known fact) that this coincides with the given polynomial algebra structure on $\mathbf{P}$.

The Conze embedding [C] gives an algebra homomorphism of $U(\mathfrak{g})$ into A which is injective when restricted to $U(\mathfrak{n}^-)$. Thus $U(\mathfrak{n}^-)$ can be thought of as a subalgebra of A. Furthermore there is a vector space isomorphism

$$(*) \qquad U(\mathfrak{n}^-) \otimes \mathbf{P} \xrightarrow{\sim} A$$

defined by multiplication in A (see [J5, 1.4(iii)]). As in [J5, Lemma 1.4], we have $\mathcal{F}^m A = Im(\mathcal{G}^m U(\mathfrak{n}^-) \otimes \mathbf{P})$, where $\mathcal{G}$ denotes the canonical filtration ([D, 2.1]) on $U(\mathfrak{g})$.

The commutator $[\mathfrak{g}, \mathbf{P}]$ in A is given by the action of $\mathfrak{g}$ on $\mathbf{P}$. In particular, the elements of $\mathfrak{g}$ can be viewed as first order differential operators on $\mathbf{P}$. Consider the matrix with entries $P_{\delta,\gamma} := [f_{-\gamma}, p_\delta] : \gamma, \delta \in \Delta^+$ in $\mathbf{P}$. By weight space considerations, $P_{\delta,\gamma}$ is triangular with respect to a lexicographic ordering on Δ^+, and furthermore

$$f_{-\gamma} = \sum_{\delta \in \Delta^+} P_{\delta,\gamma} \partial/\partial p_\delta \quad \text{for all } \gamma \in \Delta^+.$$

The injectivity assertion of $(*)$ implies that the $P_{\gamma,\gamma}$ are non-zero scalars and so this matrix is invertible. Recall (Section 1.2) the definition of the polynomial algebra $\mathbf{Q}$ generated by the elements $q_{-\gamma} = -\partial/\partial p_\gamma$. Note that multiplication in A gives a vector space isomorphism $\mathbf{Q} \otimes_k \mathbf{P} \xrightarrow{\sim} A$ and moreover by definition $\mathcal{F}^m A = Im(\mathcal{F}^m \mathbf{Q} \otimes \mathbf{P})$. In particular, $\mathcal{F}^m A$ is $\mathfrak{g}$ stable.

We remark that more generally each $g \in \mathfrak{g}$ viewed as an element of A takes the form

$$(*) \qquad g = \sum_{\gamma \in \Delta^+} P^g_\gamma \partial/\partial p_\gamma + P^g,$$

where $P^g_\gamma = [g, p_\gamma]$ and so are determined by the action of $\mathfrak{g}$ on A. If $g \in \mathfrak{h}$, then P^g is a scalar and the choice of these scalars determines, through the action of $\mathfrak{n}^-$, the remaining P^g. Comparison with [J5, 1.3, 1.4] shows that the possible solutions are exactly those given by the $A^{\mu,\mu} : \mu \in \mathfrak{h}^*$. We denote by $P^g(\mu)$ the solution corresponding to $\mu \in \mathfrak{h}^*$. The map $\mu \mapsto P^g(\mu)$ is easily seen to be linear.

Now consider $gr_{\mathcal{F}} A$. It is a commutative algebra isomorphic to $\mathbf{Q} \otimes_k \mathbf{P}$ and inherits a $\mathfrak{g}$ module structure via diagonal action. As already noted in [J5, 1.3] the isomorphism class of $gr_{\mathcal{F}} A$ is independent of the choice of μ (equivalently of the lower order terms P^g above) and so is completely determined by the action of $\mathfrak{g}$ on itself and on $\mathbf{P}$. Indeed we may write $gr_{\mathcal{F}} g = \sum_{\gamma \in \Delta^+} P^g_\gamma q_{-\gamma}$ and these expressions determine the kernel of the surjective map $S(gr_{\mathcal{F}} \mathfrak{g}) \otimes \mathbf{P} \twoheadrightarrow gr_{\mathcal{F}} A$.

5.3. By 3.1 we have an isomorphism $A \xrightarrow{\sim} F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{h})}(U(\mathfrak{g}), k_0^*))$ of $U(\mathfrak{g})$ modules. Now the right-hand side identifies with a subalgebra B of $U(\mathfrak{g})^*$. This is *not* isomorphic to A as an algebra since $U(\mathfrak{g})^*$ is commutative. Recall (4.2) that $\delta M(0) = F_{\mathfrak{h}}(Hom_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), k_0^*))$ also identifies with a subalgebra of $U(\mathfrak{g})^*$ and that $\mathcal{F}_x$ is a filtration of $\delta M(0)$ with this algebra structure. Define the coproduct $\Delta(a) = a_1 \otimes a_2$ on $U(\mathfrak{g})$, using the summation convention of [J2, 1.1.8]. One has $B = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(0)))$ with multiplication defined by $\theta\theta'(a) = \theta(a_1)\theta'(a_2)$. Then $(\mathcal{F}^m_x B)(\mathcal{F}^{m'}_x B)(a) \subset (\mathcal{F}^m_x B)(a_1)(\mathcal{F}^{m'}_x B)(a_2) \subset (\mathcal{F}^m_x(\delta M(0)))(\mathcal{F}^{m'}_x(\delta M(0)))$ $\subset \mathcal{F}^{m+m'}_x(\delta M(0))$ and so $\mathcal{F}^m_x$ is a filtration of B as an algebra. In what follows it is convenient to assume $\mathfrak{g}$ is simple. The general case is similar. Recall the definitions of V, V^-, and the $p_{-\beta}$ of Section 4.2. Define $\theta_{-\beta} \in Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(0))$ to vanish on the augmentation $U(\mathfrak{n}^-)_+$ of $U(\mathfrak{n}^-)$ and to take the value $p_{-\beta} \in \delta M(0)$ on 1.

Lemma. *The $U(\mathfrak{g})$ submodule generated by $\theta_{-\beta}$ is isomorphic to the adjoint representation $V(\beta)$.*

Proof. It is clear that $\theta_{-\beta}$ is $\mathfrak{n}^-$ invariant and of weight $-\beta$. Fix $\alpha \in \pi$ and set $n = (\alpha^\vee, \beta)$. A standard $\mathfrak{sl}(2)$ calculation shows that $x_{-\alpha}.(x_\alpha^{n+1}.\theta_{-\beta}) = 0$, whereas for any $\alpha' \in \pi \setminus \{\alpha\}$ one has $x_{-\alpha'}.(x_\alpha^{n+1}.\theta_{-\beta}) = 0$ trivially. It follows that $x_\alpha^{n+1}.\theta_{-\beta}$ vanishes on $U(\mathfrak{n}^-)_+$. On the other hand,

$$(x_\alpha^{n+1}.\theta_{-\beta})(1) = \theta_{-\beta}(x_\alpha^{n+1}) = x_\alpha^{n+1}(\theta_{-\beta}(1)) = x_\alpha^{n+1} p_{-\beta} \in V^-.$$

Yet $-\beta + (n+1)\alpha$ is not a root and hence cannot be a weight of V^-. This proves that $x_\alpha^{n+1}.\theta_{-\beta} = 0$ and hence the lemma. □

5.4. We now try to reconstruct

$$B = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(0)))$$

from $\mathbf{P} = F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), k_0)) = \mathcal{F}_x^0 B$ and $V(\beta) \subset \mathcal{F}_x^1 B$, following the observations in 5.1. Set $V_1 = \{\theta \in V(\beta) \mid \theta(1) \neq 0\}$. The subspace $\{\theta(1) \mid \theta \in V_1\}$ lies in $\mathcal{F}_x^1 \delta M(0) = V$ and contains $p_{-\beta}$. From the formulae in [J1, p. 406] describing the $\mathfrak{n}^+$ action on V, this subspace must equal V. Thus the set of weights of V_1 coincides with $-\Delta^+ \cup \{0\}$ and we let V_1^- denote the subspace spanned by vectors of non-zero weight. Let $\mathcal{K}$ denote the degree filtration on the polynomial ring $S(V(\beta))$ and its subring $S(V_1^-)$.

Lemma. *The map $S(V_1^-) \otimes \mathbf{P} \to B$ defined by multiplication in the commutative algebra B is an algebra isomorphism. Furthermore $\mathcal{F}_x^m B = Im(\mathcal{K}^m(S(V_1^-)) \otimes \mathbf{P})$.*

Proof. Observe that (paradoxically) V_1^- is $\mathfrak{n}^-$ invariant. Indeed it is just the subspace of the adjoint representation spanned by the weight vectors of weight $-\gamma : \gamma \in \Delta^+$. Recall further that $\mathfrak{g}$ acts by derivations on B.

Injectivity. Since $\mathfrak{n}^-$ acts locally nilpotent on $\mathbf{P}$ and on V_1^-, it is enough to show that there is no non-zero $\mathfrak{n}^-$ invariant element in the kernel. Recall that $\mathbf{P}^{\mathfrak{n}^-}$ reduces to scalars. Then a standard calculation shows that any non-zero $\mathfrak{n}^-$ invariant θ of $S(V_1^-) \otimes \mathbf{P}$, viewed as a sum of weight vectors of $\mathbf{P}$ with coefficients in $S(V_1^-)$, must have a *non-zero* coefficient, say θ', of 1.

For any weight vector $\theta_\gamma \in \mathbf{P}$ one has $\theta_\gamma(1) \in k1$, which hence vanishes if $\gamma \neq 0$. We conclude that $0 = \theta(1) = \theta'(1)$. On the other hand, the map $\theta' \mapsto \theta'(1)$ is an algebra homomorphism of $S(V_1^-)$ onto $\delta M(0) \cong S(V^-)$, which is the identity on V_1^- and so is an isomorphism. We conclude that $\theta' = 0$ and this contradiction establishes injectivity.

Surjectivity. Let M be a finite-dimensional submodule of $\delta M(0)$. Given $N, N' \in Ob\mathcal{O}$ we write $ch\ N \leq ch\ N'$ if $dim\ N_\nu \leq dim\ N'_\nu$ for all $\nu \in \mathfrak{h}^*$. Exactly as in [J5, 2.2] one obtains

$$(*) \qquad ch\ F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), M)) \leq D\ ch\ M.$$

Through the definition of $V(\beta)$ and $\mathbf{P}$ one has

$$(**) \qquad Im(\mathcal{K}^m S(V_1^-) \otimes \mathbf{P}) \subset \mathcal{F}_x^m B.$$

Consequently

$$ch\ Im(\mathcal{K}^m(S(V_1^-)\otimes \mathbf{P})) \leq ch\ F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}),\mathcal{F}_x^m(\delta M(0))))$$

$$\leq ch\ \mathcal{K}^m(S(V_1^-))D, \quad \text{by } (*).$$

Yet by the injectivity established above, the left-hand side equals

$$ch\ \mathcal{K}^m(S(V_1^-))ch\ \mathbf{P} = ch\ \mathcal{K}^m(S(V_1^-))D.$$

We conclude that equality holds in $(**)$. This proves surjectivity and the last part. ☐

5.5. We would now like to show that $\mathcal{F}^m A$ and $\mathcal{F}_x^m B$ are isomorphic as $U(\mathfrak{g})$ modules for all $m \in \mathbb{N}$. Now $\mathbf{P} = \mathcal{F}^0 A = \mathcal{F}_x^0 B$, while $\mathfrak{g} \subset \mathcal{F}^1 A$ and $V(\beta) \subset \mathcal{F}_x^1 B$. By 5.2 and 5.4 we have surjections

$$(*) \qquad \mathcal{G}^m(U(\mathfrak{g}))\otimes \mathbf{P} \twoheadrightarrow \mathcal{F}^m A, \qquad \mathcal{K}^m(S(V(\beta)))\otimes \mathbf{P} \twoheadrightarrow \mathcal{F}_x^m B$$

and the left-hand sides are isomorphic as $U(\mathfrak{g})$ modules. The trouble is that these maps have kernels. Already this can cause the image of some $a \in \mathcal{G}^m(U(\mathfrak{g}))$ to lie in some $\mathcal{G}^{m'}A$ with $m' < m$. Actually after Borho-Brylinski [BB] this only arises when $\mathfrak{b}$ is replaced by a parabolic which is not of confluent type in the terminology of [JL, Section 8]. (For an example, see [JLT, 12.3]). Nevertheless it serves as a warning.

To compare the above kernels, recall that the matrix with entries $P_{\delta,\gamma} : \delta,\gamma \in \Delta^+$ introduced in 5.2 is invertible. We may define elements $q'_{-\gamma} \in B : \gamma \in \Delta$, through

$$(**) \qquad f'_{-\gamma} = \sum_{\delta\in\Delta^+} q'_{-\delta}P_{\delta,\gamma},$$

with $f'_{-\gamma} \in V(\beta)_{-\gamma}$ corresponding to a Chevalley basis element through the isomorphism of 5.3. By 5.4, these are algebraically independent so they generate a polynomial subalgebra $\mathbf{Q}'$ of B. Furthermore the multiplication map $\mathbf{Q}'\otimes_k \mathbf{P} \xrightarrow{\sim} B$ is an algebra isomorphism. Define a Poisson bracket on B through $\{q'_{-\gamma}, p_\gamma\} = 1$ and all other Poisson brackets on generators equal to zero. It follows easily from 5.4 and the choice of $P_{\delta,\gamma}$ in $(**)$ that B is isomorphic to $gr_{\mathcal{F}}A$ as a Poisson algebra by sending $gr_{\mathcal{F}}\partial/\partial p_\gamma$ to $q'_{-\gamma}$. Furthermore $a \mapsto \{q'_{-\gamma}, a\}$ is just $\partial/\partial p_\gamma$ on $\mathbf{P}$. By this and the choice of $P_{\delta,\gamma}$ the action of $V_1^- \subset V(\beta)$ on $\mathbf{P}$ defined by this Poisson bracket coincides with the action of $\mathfrak{n}^-$. Moreover identifying V_1^- as a subspace of B, the Poisson bracket defines an action of V_1^- on itself which is just the adjoint action of $\mathfrak{n}^-$. This is a simple consequence of the expressions being first order in $q'_{-\gamma}$ (resp. $\partial/\partial p_\gamma$) and our embedding of $\mathfrak{g}$ into A being a Lie algebra homomorphism. Then by 5.4 the action on V_1^- on B as derivations can be recovered by its restriction to $\mathbf{P}$ and this adjoint action.

We can now describe how $V(\beta)$ lies in B in terms of the $q'_{-\gamma}, p_\delta$ variables. Since $V(\beta) \subset \mathcal{F}_x^1 B$, we can already write each $g \in V(\beta)$ in the form

$$g = \sum_{\gamma\in\Delta^+} q'_{-\gamma}P_\gamma^g + P^g \quad : \quad P_\gamma^g, P^g \in \mathbf{P}.$$

Now the action of $\mathfrak{n}^-$ on $V(\beta)$ brings each such element into V_1^- whose form we already know. Since in $\mathbf{P}$ only the constants Poisson-commute with all the $q'_{-\gamma} : \gamma \in \Delta^+$, it follows that the P_γ^g (which for g of positive weight are polynomials with no constant term) are uniquely determined. Again the P^g for $g \in V(\beta)$ of strictly positive weight have no constant terms; but the P^h, for $h \in V(\beta)$ of weight zero, are scalars. It follows that the P^g are uniquely determined by these scalars. (One may remark that their precise values are related to the choice of regular nilpotent x and are *not* zero.) Finally by comparison of commutation with Poisson bracket which coincide up to degree one we may further conclude the P^g_γ, P^g are *exactly* the same polynomials as given in 5.2, when these scalars match up.

It is not too obvious that the above observation means that we can also match up the kernels in $(*)$ which lie in non-isomorphic algebras. However we can immediately conclude that the action of $\mathfrak{g}$ on $gr_{\mathcal{F}}A$ and $gr_{\mathcal{F}_x}B$ coincide, since both are defined by using the Poisson bracket and the same expressions, namely $g = \sum_{\gamma\in\Delta^+} q_{-\gamma}P^g_\gamma$, for the elements of $\mathfrak{g}$. Then by 2.1 we conclude that B is graded injective for the principal BK filtration $\mathcal{F}_x$. This leads to the

Proposition. *Suppose $\lambda, \mu \in \mathfrak{h}^*$ are regular antidominant and $\lambda - \mu \in P^+(\pi)$. Then $gr_{\mathcal{F}_x}A^{\lambda,\mu}$ is injective in $\hat{\mathcal{O}}^-$.*

Proof. The case $\lambda = \mu$ is the above. For the general case we make a construction similar to 2.1 which has the advantage of being even more transparent. Indeed for the multiplication defined by the coproduct on $U(\mathfrak{g})$ we may observe that $A^{\lambda,\mu} = F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(\mu - \lambda)))$ becomes a module over the commutative algebra

$$F_{\mathfrak{b}^-}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(0))) = B.$$

Consider the unique up to scalars $\mathfrak{n}^-$ invariant element $\theta^{-(\lambda-\mu)} \in A^{\lambda,\mu}$ whose value on $1 \in U(\mathfrak{g})$ has weight $\mu - \lambda$. Forgetting the $\mathfrak{h}$ action this is just the identity in $B = A^{\lambda,\mu}$. Consequently $A^{\lambda,\mu} = B\theta^{-(\lambda-\mu)}$. Now $\theta^{-(\lambda-\mu)}$ generates the finite-dimensional simple $U(\mathfrak{g})$ submodule $V(-(\lambda - \mu))$ of $\delta M(\lambda - \mu)^\iota = \mathcal{F}^0_x A^{\lambda-\mu}$, and a fortiori $A^{\lambda,\mu} = BV(-(\lambda-\mu)) = A^{\mu,\mu}V(-(\lambda-\mu))$. Now we already know that $A^{\lambda,\mu}$ is a direct summand of $A^{\mu,\mu} \otimes V(-(\lambda - \mu))$, by the Jantzen translation principle (see Section 2.1, proof of Proposition 2.1, or [J5, 3.1, 3.2]) and because λ, μ are in the same facette. Consequently this direct summand is just obtained by the above multiplication which hence takes injectives to injectives. Finally $\mathcal{F}^m_x A^{\mu,\mu}$ is injective for each m and $V(-(\lambda - \mu)) \subset \mathcal{F}^0_x(A^{\lambda,\mu})$. Consequently $\mathcal{F}^m_x(A^{\lambda,\mu}) = \mathcal{F}^m_x(A^{\mu,\mu})V(-(\lambda - \mu))$ is injective. Splitting off injectives concludes the proof. □

5.6. Combining the above observations we now obtain the Brylinski-Broer result.

Theorem. *For all $\xi, \nu \in P^+(\pi)$ and $x = \sum_{\alpha\in\pi} c_\alpha e_\alpha : c_\alpha \in k \setminus \{0\}$, one has*

$$ch^x_q V(\xi)_\nu = m^\nu_\xi(q).$$

Proof. Take λ, μ sufficiently antidominant so we may write $\nu = \lambda - \mu$. Combine 5.4 and 2.3 to determine

$$[\mathcal{F}^m_x F(A^{\lambda,\mu}) : V(\xi)]$$

and conclude by 3.3. □

6. Minimal $\mathfrak{k}$-type

6.1. Take $g \in \mathfrak{g}$ and recall the definition of the linear map $\mu \mapsto P^g(\mu)$ of $\mathfrak{h}^*$ into $\mathbf{P}$ representing the possible solutions to 5.2$(*)$. They describe the zeroth order terms in the description of the algebra homomorphisms $U(\mathfrak{g}) \to A^{\mu,\mu}$. The diagonal action of $\mathfrak{g}$ on $A^{\lambda,\mu}$ results from these embeddings of $\mathfrak{g}$ and the composition of homomorphisms sending $A^{\mu,\mu} \times A^{\lambda,\mu} \times A^{\lambda,\lambda}$ to $A^{\lambda,\mu}$. The action of $x \in \mathfrak{g}$ in $gr_{\mathcal{F}}A^{\lambda,\mu}$

coming from the zeroth order terms is just multiplication by $P^g(\mu) - P^g(\lambda) = P^g(\mu - \lambda)$. In particular the identity $1 \in gr_{\mathcal{F}} A^{\lambda,\mu}$ has weight $-(\lambda - \mu)$. This analysis gives the following

Lemma. *The isomorphism class of* $gr_{\mathcal{F}} A^{\lambda,\mu} : \lambda, \mu \in \mathfrak{h}^*$ *depends only on* $\lambda - \mu$.

Remark. Of course the $gr_{\mathcal{F}} A^{\lambda,\mu}$ can all be identified with the polynomial ring $\mathbf{Q} \otimes \mathbf{P}$ and in this common space isomorphism can be replaced by equality.

6.2. Since we shall need to use $\mathcal{F}_x$ instead of $\mathcal{F}$ it is important to observe that the same results hold for $\mathcal{F}_x$.

Lemma. *For all* $\lambda, \mu \in \mathfrak{h}^*$, *one may identify* $gr_{\mathcal{F}} A^{\lambda,\mu}$ *and* $gr_{\mathcal{F}_x} A^{\lambda,\mu}$ *as* $\mathfrak{g}$ *modules by sending* $q_{-\gamma}$ *to* $q'_{-\gamma}$ *for all* $\gamma \in \Delta^+$.

Proof. When $\lambda = \mu$, this just summarizes a conclusion of 5.4. For the general case, observe that the identity $1 \in A$ becomes a vector of weight $-(\lambda - \mu)$ in $A^{\lambda,\mu}$. As in 5.4 one may use the action of $\mathfrak{n}^-$ to show that in $gr_{\mathcal{F}_x} A^{\lambda,\mu}$ the zeroth order term coming from the diagonal action of g is just multiplication by $P^g(-(\lambda - \mu))$, as required. □

Remark. Combined with 2.1 this gives a second proof of 5.5.

6.3. We already know by 3.1 that $A^{\lambda,\mu} : \lambda, \mu \in \mathfrak{h}^*$, μ antidominant, is injective in $\hat{\mathcal{O}}^-$. This allows us to calculate the multiplicities $[F(A^{\lambda,\mu}) : V(\nu)^*]$ and we remark that these may also be computed from Frobenius reciprocity using the simplicity of $M(\mu)$ (see [J4, 10.5], for example). Again if λ is dominant one may also calculate these multiplicities using (see [JLT, 10.7], for example) the projectivity of $M(\lambda)$. Finally if λ, μ are in the same facette, then assuming $\lambda - \mu \in P(\pi)$ one may use the injectivity of $A^{\mu,\mu}$ [J5, 3.6] combined with the translation principle of [J5, Section 3] to compute multiplicities. (Here and in what follows the reader can take $\lambda, \mu \in \mathfrak{h}^*$ in the same facette to just mean that λ, μ are regular and both dominant or both antidominant.) From the above one has the

Proposition. *Take* $\lambda, \mu \in \mathfrak{h}^*$, *with* $\lambda - \mu \in P(\pi)$ *and* $\nu \in P(\pi)$. *Assume one of the following holds:*
(i) μ *is antidominant.*
(ii) λ *is dominant.*
(iii) λ, μ *are in the same facette.*
Then $[F(A^{\lambda,\mu}) : V(\nu)^*] = dim\ V(\nu)_{\lambda-\mu}$.

Remarks. The same calculation goes through in the parabolic case. For arbitrary $\lambda, \mu \in \mathfrak{h}^*$ these multiplicities (in the present Borel case) were calculated in [GJ, 3.4] and are rather complicated in general. The latter shows that the isomorphism class of $A^{\lambda,\mu}$ does *not* only depend on $\lambda - \mu$, even if $\lambda - \mu \in P^+(\pi)$, when λ, μ are *not* in the same facette.

6.4. For all $\nu \in P(\pi)$, define the length of ν to be (ν, ν). If any one of the hypotheses of 6.3 holds it follows that $V(\lambda - \mu)^*$ occurs with multiplicity 1 in $F(A^{\lambda,\mu})$ and furthermore if any other $V(\nu)^*$ occurs in $F(A^{\lambda,\mu})$, then $(\nu, \nu) > (\lambda - \mu, \lambda - \mu)$. Following a convention introduced by D.A. Vogan (in the slightly different context of *simple* Harish-Chandra modules) we call $V(\lambda - \mu)^*$ the unique minimal $\mathfrak{k}$ type of $F(A^{\lambda,\mu})$. We say that $F(A^{\lambda,\mu})$ is generated by its minimal $\mathfrak{k}$-type if $F(A^{\lambda,\mu}) = F(A^{\mu,\mu})V(\lambda - \mu)^*$. The following is essentially well known. We give the proof for completion as it is rather important for our considerations.

Lemma. *Assume $\lambda, \mu \in \mathfrak{h}^*$ are in the same facette and $\lambda - \mu \in P(\pi)$. Then $F(A^{\lambda,\mu})$ is generated by its minimal $\mathfrak{k}$-type.*

Proof. View $V := V(\lambda-\mu)^*$ as the unique minimal $\mathfrak{k}$-type of $F(A^{\lambda,\mu})$. Then the composition $V \times M(\lambda) \to M(\mu)$ gives rise to a $U(\mathfrak{g})$ module map $\varphi : V \otimes M(\lambda) \to M(\mu)$. We claim that φ is just the projection onto the direct summand of $V \otimes M(\lambda)$ having the same central character as $M(\mu)$. Following [D, 7.6.14] let $\{u_{\xi_i}\}$ be a basis of V formed from weight vectors ordered so all the partial sums $\sum_{j=i+1}^{n} ku_{\xi_j}$ are $U(\mathfrak{b})$ submodules of V. Induction gives a Verma flag $V \otimes M(\lambda) = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{n+1} = 0$ with $u_{\xi_i}v_\lambda$ *mod* M_{i+1} the canonical generator of $M_i/M_{i+1} \cong M(\lambda+\xi_i)$. The hypothesis of the lemma implies after Jantzen [J, 2.6] that only one of these, namely $M(\lambda+\mu-\lambda)$, has the appropriate central character. Thus by universality φ factors through the projection onto $M(\mu)$ defined by central character decomposition and $Im\varphi$ is a quotient of this copy of $M(\mu)$. Yet $Vv_\lambda \neq 0$, so $Im\varphi$ is a non-zero submodule of $M(\mu)$. Since $M(\mu)$ is indecomposable, this proves the claim.

Since $dim\ V < \infty$, the natural injection is an isomorphism

$$\tilde{\varphi} : Hom_k(k, V) \otimes Hom_k(M(\lambda), M(\lambda)) \xrightarrow{\sim} Hom_k(M(\lambda), V \otimes M(\lambda))$$

with $Hom_k(M(\lambda), M(\mu))$ a direct summand of the right-hand side. Taking $U(\mathfrak{g})$ locally finite parts, we obtain an isomorphism

$$V \otimes F(A^{\lambda,\lambda}) \xrightarrow{\sim} F(Hom_k(M(\lambda), V \otimes M(\lambda)))$$

with $F(A^{\lambda,\mu})$ a direct summand of the right-hand side. From our claim and the definition of $\tilde{\varphi}$, it follows that the projection onto this direct summand is just that obtained by taking V to be the minimal $\mathfrak{k}$-type of $F(A^{\lambda,\mu})$ and through the composition of homomorphisms $V \times F(A^{\lambda,\lambda}) \to F(A^{\lambda,\mu})$ deduced from φ. Hence $VF(A^{\lambda,\lambda}) = F(A^{\lambda,\mu})$. Since the map $U(\mathfrak{g}) \to F(A^{\lambda,\lambda})$ defined by the action of $U(\mathfrak{g})$ on $M(\lambda)$ is surjective (see [J4, 10.5], for a proof independent of Kostant's primeness result [K2]), we may conclude that $F(A^{\mu,\mu})V \supset U(\mathfrak{g})V = VU(\mathfrak{g}) = VF(A^{\lambda,\lambda}) = F(A^{\lambda,\mu})$, as required. □

Remark. This argument also clarifies the argument in 2.1. A further point there was that in the dominant case $\lambda - \mu \in P^+(\pi)$, the minimal $\mathfrak{k}$-type has degree 0. We shall calculate this degree in general.

6.5. Now fix $\lambda, \mu \in \mathfrak{h}^*$ in the same facette and assume $\lambda - \mu \in P(\pi)$. Let r be the smallest integer ≥ 0 such that $V(\lambda-\mu)^* \subset \mathcal{F}_x^r(V(\lambda-\mu)^*)$. We say that $gr_{\mathcal{F}_x}(F(A^{\lambda,\mu}))$ is generated by its minimal $\mathfrak{k}$-type if one has

$$(*) \qquad \mathcal{F}_x^{m-r}(F(A^{\mu,\mu}))V(\lambda-\mu)^* = \mathcal{F}_x^m(F(A^{\lambda,\mu})), \quad \text{for all } m \in \mathbb{N},$$

or equivalently

$$(**) \qquad gr_{\mathcal{F}_x}(F(A^{\mu,\mu}))V(\lambda-\mu)^* = gr_{\mathcal{F}_x}(F(A^{\lambda,\mu})).$$

A priori, this is a much stronger property than the conclusion of 6.5 except if $\lambda \in P^+(\pi)$ and $\mu = 0$. Indeed when $\mu = 0$ (as noted already in [J5, Section 1]) the gradation on $A^{\mu,\mu} \cong \mathbf{Q} \otimes_k \mathbf{P}$ induced by the degree gradation on $\mathbf{Q}$ is a gradation of $U(\mathfrak{g})$ modules (for the diagonal action). Moreover the condition $\lambda - \mu \in P^+(\pi)$ implies that $V(\lambda-\mu)^*$ is a subspace of $\mathbf{P}$ (and hence of degree 0).

Right multiplication by $\mathbf{P}$ preserves this gradation and so by 6.4 we obtain for all $\lambda \in P^+(\pi)$ that

$$\mathcal{F}^m(F(A^{0,0}))V(\lambda)^* = \mathcal{F}^m(F(A^{\lambda,0})), \text{ for all } m \in \mathbb{N},$$

or equivalently

$$gr_{\mathcal{F}} F(A^{0,0})V(\lambda)^* = gr_{\mathcal{F}} F(A^{\lambda,0}).$$

More generally if $\lambda, \mu \in \mathfrak{h}^*$ are in the same facette and $\lambda - \mu \in P^+(\pi)$, by 6.1 and 6.3 we can identify $gr_{\mathcal{F}} F(A^{\lambda,\mu})$ with $gr_{\mathcal{F}} F(A^{\lambda-\mu,0})$ and $gr_{\mathcal{F}} F(A^{\mu,\mu})$ with $gr_{\mathcal{F}} F(A^{0,0})$. Taking account of 6.2, $gr_{\mathcal{F}_x} A^{\lambda,\mu}$ and $gr_{\mathcal{F}} A^{\lambda,\mu}$ can be identified as $\mathfrak{g}$ modules and as algebras. Therefore the argument above proves the

Proposition. *Assume $\lambda, \mu \in \mathfrak{h}^*$ are in the same facette and $\lambda - \mu \in P^+(\pi)$. Then $gr_{\mathcal{F}_x}(F(A^{\lambda,\mu}))$ is generated by its minimal $\mathfrak{k}$-type.*

6.6. Fix $\nu \in P^+(\pi)$ and $w \in W$. Choose $\lambda, \mu \in \mathfrak{h}^*$ sufficiently antidominant so that $\lambda - \mu = w\nu$. Recall 6.1 and 6.2. We denote by $H_w(\nu)$ the hypothesis that $gr_{\mathcal{F}_x}(F(A^{\lambda,\mu}))$ is generated by its minimal $\mathfrak{k}$ type (which is isomorphic to $V(\nu)^*$).

7. Non-dominant weights

7.1. Take x as in 5.1. Fix $\xi \in P^+(\pi)$ and $\nu \in P(\pi)$. We wish to calculate $ch_q^x V(\xi)_\nu$. Since we have determined this for ν dominant, it is enough to relate it with $ch_q^x V(\xi)_{s_\alpha \nu}$ for each $\alpha \in \pi$. Here we had the conjecture (which we now prove in Theorem 7.6)

$$\text{(C1)} \qquad ch_q^x V(\xi)_{s_\alpha \nu} = q^{(\alpha^\vee,\nu)} ch_q^x V(\xi)_\nu.$$

To motivate this, we first note that it is compatible with the action of the principal TDS denoted (x,h,y). Take $\nu \in P^+(\pi)$. Given $w \in W$, set $S(w) = \{\alpha \in \Delta^+ \mid w\alpha \in \Delta^-\}$. Then a standard calculation shows that (C1) implies

$$\text{(C1)}_w \qquad ch_q^x V(\xi)_{w\nu} = \left(\prod_{\beta \in S(w)} q^{(\beta^\vee,\nu)} \right) ch_q^x V(\xi)_\nu.$$

In particular, for the unique longest element $w_0 \in W$ one has

$$(*) \qquad ch_q^x V(\xi)_{w_0\nu} = q^{2(\rho^\vee,\nu)} ch_q^x V(\xi)_\nu,$$

where $\rho^\vee$ is the half-sum of the positive coroots. Yet for our choice of x, the semisimple element h is proportional to $2\rho^\vee$. Now suppose $v \in V(\xi)_\nu$ satisfies $x^{n+1}v = 0$ and $x^n v \neq 0$, $y^{n'+1}v = 0, y^{n'}v \neq 0$. With respect to h, the element $x^n v$ has weight $2n+2(\rho^\vee,\nu)$ and $y^{n'}v$ has weight $-2n'+2(\rho^\vee,\nu)$. From $\mathfrak{sl}(2)$ theory these sum to zero, that is, $2n + 2(\rho^\vee,\nu) = -(-2n' + 2(\rho^\vee,\nu))$. Hence $n' = n + 2(\rho^\vee,\nu)$, which is exactly the prediction of $(*)$.

7.2. For all $w \in W$ and $\nu \in P^+(\pi)$, let us define

$$\ell_\nu(w) = \sum_{\beta \in S(w)} (\beta^\vee, \nu).$$

Recall that $V(\nu)_{w\nu}$ is one-dimensional and fix $v_{w\nu} \in V(\nu)_{w\nu}$ non-zero. It is clear that $\nu - w\nu$ is a sum of exactly $\ell_\nu(w)$ simple roots. We conclude that $x^{n+1}v_{w\nu} = 0$ for all $n \geq \ell_\nu(w)$. On the other hand, (C1) implies

$$\text{(C2)} \qquad \max\{n \in \mathbb{N} \mid x^n v_{w\nu} \neq 0\} = \ell_\nu(w).$$

The truth of (C2) is that there are no "accidental" cancellations in applying x^n to $v_{w\nu}$.

7.3. From 6.2 and the remark in 2.1, it follows that $\mathcal{F}_x^m(A^{\lambda,\mu})/\mathcal{F}_x^{m-1}(A^{\lambda,\mu})$ admits a $U(\mathfrak{g})$ filtration with quotients isomorphic to the $(\delta M((\lambda-\mu)+\beta))^\iota$, where β is the sum of exactly m positive roots (their multiplicities being the coefficient of q^m in $P_q(\beta)$). Now assume $\lambda-\mu = w\nu$, for some $\nu \in P^+(\pi)$ and $w \in W$. Then the largest m for which $\delta M(\nu)^\iota$ can occur is exactly $\ell_\nu(w)$ and moreover it occurs with multiplicity 1. Consequently the largest possible degree in which the unique minimal $\mathfrak{k}$-type $V(\lambda-\mu)^*$ of $F(A^{\lambda,\mu})$ can occur is also $\ell_\nu(w)$. However it is not clear that it does occur in this degree because the corresponding copy of $\delta M(\nu)^\iota$ may not be a submodule of $\mathcal{F}_x^m(A^{\lambda,\mu})$. Thus we formulate the conjecture (which will also be proved)

$(\mathrm{C3})_w$ *Suppose $\lambda-\mu = w\nu$ with $\nu \in P^+(\pi)$ and $w \in W$. Then the unique minimal $\mathfrak{k}$-type of $F(A^{\lambda,\mu})$ occurs in degree $\ell_\nu(w)$.*

Denote by (C3) the common truth of the $(\mathrm{C3})_w : w \in W$.

Lemma. *Conjectures* (C2) *and* (C3) *are equivalent.*

Proof. This follows from 3.3. □

Remark. Again (C3), like (C2), is not so innocent. It means that the remaining $\delta M(\nu)^\iota$ occurring in lower degrees cannot be submodules.

Proposition. *Take $\nu \in P^+(\pi)$, $w \in W$ and $\lambda,\mu \in \mathfrak{h}^*$ sufficiently antidominant so that $w\nu = \lambda-\mu$. Given that* (C3) *holds, then so does* (C1) *and furthermore $gr_{\mathcal{F}_x}(F(A^{\lambda,\mu}))$ is generated by its unique minimal $\mathfrak{k}$-type (isomorphic to $V(\nu)^*$).*

Proof. The proof is by induction on the length of w. For the neutral element, the conjecture $(\mathrm{C1})_w$ is empty and the second assertion is just 6.5.

Now assume we have established $(\mathrm{C1})_w$ and $H_w(\nu)$ of 6.6. If $w = w_0$, there is nothing left to prove. If not, set $\nu' = w\nu$ and choose $\alpha \in \pi$ such that $t := (\alpha^\vee, \nu') \geq 0$. Set $\nu'' = s_\alpha\nu' = \nu' - t\alpha$. We may assume $t > 0$, without loss of generality. Now set $A' = A^{\mu+\nu',\mu}$, $A'' = A^{\mu+\nu'',\mu}$ and $F = F(A^{\mu,\mu}), F' = F(A'), F'' = F(A'')$. Composition of homomorphisms gives $A^{\mu+\nu',\mu}A^{\mu+\nu'',\mu+\nu'} \subset A^{\mu+\nu'',\mu}$. Recall that this can be viewed as multiplication in the Weyl algebra A. Thus, if we consider $A''' := A^{\mu+\nu'',\mu+\nu'}$ we have $A'A''' \subset A''$. We shall seek to use the term A''' to transfer $gr_{\mathcal{F}_x}(F')$, which we understand by the inductive hypothesis, onto $gr_{\mathcal{F}_x}(F'')$, which we want to determine. Recall also that by 6.2, for any $\eta_1, \eta_2 \in \mathfrak{h}^*$, $gr_{\mathcal{F}_x}(A^{\eta_1,\eta_2})$ can be identified as an algebra with $gr_{\mathcal{F}}(A^{\eta_1,\eta_2})$. Furthermore, the latter is just $gr_{\mathcal{F}}(A)$ as an algebra. (Only the $\mathfrak{g}$ module structure changes passing from A^{η_1,η_2} to the Weyl algebra A using the same filtration $\mathcal{F}$.) In particular, $gr_{\mathcal{F}_x}A''$ is a domain which is a key fact in the proof below.

As $(\mu+\nu'')-(\mu+\nu') = -t\alpha$, it follows that $X_t := (\mathcal{F}_x^tA''')/(\mathcal{F}_x^{t-1}A''')$ admits a dual Verma flag with quotients isomorphic to the $\delta M(-t\alpha+\beta)^\iota$, where β is a sum of exactly t positive roots. In particular $\delta M(0)^\iota$ occurs in X_t and is the only factor of the form $\delta M(w.0)$. (Indeed, it is the only factor annihilated by a power of the augmentation ideal of the centre of $U(\mathfrak{g})$.) Thus $\delta M(0)^\iota$ occurs as a submodule of X_t. Its unique simple submodule is a $\mathfrak{g}$ invariant element $\bar{z}$ of X_t. Choose a representative $z \in \mathcal{F}_x^t(A''')$. We remark that one cannot choose a representative which is $\mathfrak{g}$ invariant since $M(\mu+\nu')$ is simple and $\mu+\nu'' \neq \mu+\nu'$, which forces

$Hom_{\mathfrak{g}}(M(\mu+\nu''), M(\mu+\nu')) = 0$. (One may add that $s_\alpha \cdot 0 = -\alpha$ and that $\delta M(-\alpha)^\iota$ occurs in degree $t-1$. This permits $\delta M(0)^\iota$ to occur as a quotient of a non-trivial extension with $\delta M(-\alpha)^\iota$ as a submodule.) Nevertheless z is $\mathfrak{g}$ invariant *mod* $\mathcal{F}_x^{t-1}(A'')$.

Set $\ell' = \ell_w(\nu)$ and $\ell'' = \ell_{s_\alpha w}(\nu) = \ell_w(\nu) + (\alpha^\vee, \nu') = t + \ell'$. Multiplication in the Weyl algebra yields $\mathcal{F}_x^{\ell'}(A')z \subset \mathcal{F}_x^{\ell''}(A'')$. Admit $(\mathrm{C3})_w$. Then $V' = V(\nu')^*$ occurs as a submodule of $\mathcal{F}_x^{\ell'}(A')$. Now $V'z$ must have a non-zero component in degree ℓ'' since $gr_{\mathcal{F}_x}A''$ is a domain. Moreover its image in $\mathcal{F}_x^{\ell''}(A'')/\mathcal{F}_x^{\ell''-1}(A'')$ is isomorphic to $V(\nu)^*$. Again observe that $\delta M(\nu)^\iota$ occurs with multiplicity 1 in the dual Verma flag of this quotient. Hence $V(\nu)^\iota = V(\nu)^*$ occurs with multiplicity 1 as a Jordan-Holder composition factor. On the other hand, applying $(\mathrm{C3})_{s_\alpha w}$ to A'' we obtain $V(\nu)^*$ as a submodule V'' of $\mathcal{F}_x^{\ell''}(A'')$. We conclude that

$$(*) \qquad V'z = V'' \ mod \ \mathcal{F}_x^{\ell''-1}(A'').$$

By hypothesis $H_w(\nu)$, any simple module $\tilde{V}$ of F' occurring in degree r, that is, in $\mathcal{F}_x^r(F')/\mathcal{F}_x^{r-1}(F')$, must satisfy $\tilde{V} \subset \mathcal{F}_x^{r-\ell'}(F(A^{\mu,\mu}))V'$. Thus by $(*)$ we obtain $\tilde{V}z \subset \mathcal{F}_x^{r-\ell'}(F(A^{\mu,\mu}))V''$ *mod* $\mathcal{F}_x^{r-\ell'+\ell''-1}(A'')$. On the other hand, the image of $\tilde{V}z$ in $\mathcal{F}_x^{r-\ell'+\ell''}(A'')/\mathcal{F}^{r-\ell'+\ell''-1}(A'')$ is isomorphic to $\tilde{V}$. Now $gr_{\mathcal{F}_x}A''$ is a domain, so multiplication by z does not lower multiplicities. Hence, since $\mathcal{F}_x^{r-\ell'}(F(A^{\mu,\mu}))V'' \subset \mathcal{F}_x^{r-\ell'+\ell''}(F'')$ we conclude that

$$(**) \qquad \begin{aligned}[][\mathcal{F}_x^r(F')/\mathcal{F}_x^{r-1}(F') : \tilde{V}] &\leq [\mathcal{F}_x^{r-\ell'}(F)V''/\mathcal{F}_x^{r-\ell'+\ell''-1}(F'') : \tilde{V}] \\ &\leq [\mathcal{F}_x^{r-\ell'+\ell''}(F'')/\mathcal{F}_x^{r-\ell'+\ell''-1}(F'') : \tilde{V}].\end{aligned}$$

Define a partial order $\geq$ on the set of polynomials in q with integer coefficients ≥ 0 through $\sum a_i q^i \geq \sum b_i q^i$ if and only if $a_i \geq b_i$ for all i. Then by 3.3 we obtain

$$ch_q^x V(\xi)_{s_\alpha \nu'} \geq q^{(\alpha^\vee, \nu')} ch_q^x V(\xi)_{\nu'}.$$

Since equality holds at $q = 1$, this gives equality for all q, and so $(\mathrm{C1})_w$ implies $(\mathrm{C1})_{s_\alpha w}$. Again $[F' : \tilde{V}] = [F'' : \tilde{V}]$, by 6.3 which forces equality in $(**)$ and hence the truth of $H_{s_\alpha w}(\nu)$. □

7.5. Suppose $\mathfrak{g}$ is simply-laced. Then (C2) is a special case of [Z, Theorem 3.2.2] and follows from a result of Lusztig [L2, Theorem 22.1.7]. The latter asserts in particular that with respect to the global basis for $V(\xi)$ any monomial in $e_\alpha : \alpha \in \pi$ applied to $v_{w\xi}$ is a sum with positive (integer) coefficients. Thus there can be no "accidental" cancellations in applying a power of x to $v_{w\xi}$. One may therefore conclude

Proposition. *Suppose* $\mathfrak{g}$ *is simply-laced. Then* (C2) *holds.*

7.6. We now obtain our main result. Set $x = \sum_{\alpha\in\pi} e_\alpha$.

Theorem. *Take* $\xi, \nu \in P^+(\pi)$ *and* $w \in W$. *Then*

$$ch_q^x V(\xi)_{w\nu} = \left(\prod_{\beta\in S(w)} q^{(\beta^\vee, \nu)} \right) m_\xi^\nu(q).$$

Proof. Given (C2), 7.3–7.4 imply $(\mathrm{C1})_w$, and we may conclude by 5.6. By 7.5 this completes the proof for $\mathfrak{g}$ simply-laced. The non-simply-laced case is treated below. □

7.7. Let $(\mathrm{C2})_i$ denote (C2) when ν is the i^{th} fundamental weight and $(\mathrm{C2})_F$ is the common truth of the $(\mathrm{C2})_i$.

Lemma. $(\mathrm{C2})_F$ *implies* (C2).

Proof. With respect to multiplication in $\mathbf{P}$ the $V(\nu) : \nu \in P^+(\pi)$ satisfy Cartan multiplication, that is, $V(\nu)V(\nu') = V(\nu+\nu')$, for all $\nu, \nu' \in P^+(\pi)$. Furthermore $v_{w\nu}v_{w\nu'} = v_{w(\nu+\nu')}$, for all $w \in W$. Since $\mathbf{P}$ is a domain, it follows that (C2) for ν, ν' implies (C2) for $\nu + \nu'$ as required. □

7.8. Suppose $\mathfrak{g}$ is of type A_{n-1} and let $e_{i,j} : i, j = 1, 2, \cdots, n$, be the standard matrix elements in $End_k k^n$. Let V be the standard n-dimensional representation of $\mathfrak{g}$ with basis $\{v_1, v_2, \cdots, v_n\}$ chosen so that $e_{i,i+1}v_{i+1} = v_i$. Take $s \in \{1, 2, \cdots, n-1\}$. Then $\bigwedge^s V$ is a fundamental representation of $\mathfrak{g}$ and has basis $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_s}$: $1 \leq i_1 < i_2 < \cdots < i_s \leq n$. For this choice of basis, the matrix coefficients of the $e_{i,i+1} : i = 1, 2, \cdots, n-1$, are either 1 or 0. Thus $(\mathrm{C2})_s$ holds for all s. Combined with 7.7, this gives a simple proof of (C2) in type A_{n-1}.

7.9. Let (x, h, y) be an s-triple and V a finite-dimensional module for the corresponding $\mathfrak{sl}(2)$ algebra. Let $v \in V$ be an h-eigenvector having eigenvalue $j \in \mathbb{Z}$ and consider xyv. It is easy to show that there exists $c_i \in \mathbb{Q}$ such that

$$xyv = \sum c_i y^i x^i v$$

and moreover there may be several such expressions. We say that V admits positivity if it is always possible to take the $c_i \geq 0$. We remark that if V is simple, then xyv and the $y^i x^i v$ are integer multiples ≥ 0 of v.

Lemma. *Suppose every simple in* V *has dimension* ≤ 5. *Then* V *admits positivity.*

Proof. Since $xy = yx + h$ we can assume $j < 0$. We can assume $yv \neq 0$ and so by the hypothesis we are reduced to $j = -1, -2$. If $j = -1$, then yv belongs to the isotypical component of V corresponding to the simple with lowest weight -3 and moreover so does x^2v. Consequently y^2x^2v and xyv are non-zero and equal up to a positive rational number. If $j = -2$ a similar argument shows that the same holds for the pair y^3x^3v and xyv. □

7.10. Let V be a finite-dimensional $\mathfrak{g}$ module. We say that V admits positivity if for every $\alpha \in \pi$, V admits positivity with respect to the s-triple $(e_\alpha, h_\alpha, f_{-\alpha})$.

For each sequence $I = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ of simple roots let $F_I = f_{-\alpha_1} f_{-\alpha_2} \cdots f_{-\alpha_m}$ be the corresponding monomial in the negative root vectors of degree $|I| := m$.

Now take $V = V(\mu) : \mu \in P^+(\pi)$, with v_μ its highest weight vector.

Lemma. *Assume that* $V(\mu)$ *admits positivity. For every monomial* $F_I : |I| \geq 1$, *there exists* c_i *rational* ≥ 0 *and monomials* $F_{I_i} : |I_i| \leq |I| - 1$ *such that* $e_\alpha F_I v(\mu) = \sum_i c_i F_{I_i} v(\mu)$.

Proof. The proof is by induction on $|I|$. Choose $\alpha \in \pi$ such that $F_I = f_{-\alpha} F_{I'}$. By positivity

$$e_\alpha F_I v(\mu) = e_\alpha f_{-\alpha} F_{I'} v(\mu) = \sum c_i f_{-\alpha}^i e_\alpha^i F_{I'},$$

for some c_i rational ≥ 0, while if $\beta \in \pi \setminus \{\alpha\}$ one has

$$e_\beta F_I v(\mu) = f_{-\alpha} e_\beta F_{I'} v(\mu).$$

By the induction hypothesis the assertion follows easily. □

7.11. It is immediate from 7.10 that if $V(\mu)$ admits positivity it also satisfies (C2). Now the hypothesis of 7.9 is satisfied for the fundamental module $V(\omega_\alpha)$ with respect to every s-triple $(e_\beta, h_\beta, f_{-\beta}) : \beta \in \pi$, as long as the coefficient of $\alpha^\vee$ in any coroot does not exceed 4. This proves the

Corollary. (C2)$_F$ *and hence* (C2) *holds for every simple Lie algebra outside* E_8.

Remark. Of course E_8 is simply-laced, so (C2) holds by 7.5. This completes the proof of 7.6.

7.12. Take $\mathfrak{g}$ of type B_2 with α (resp. α') the short (resp. long) simple root. Let (x, h, y) (resp. (x', h', y')) be the corresponding s-triple. Consider $v \in V(3\omega_\alpha + \omega_{\alpha'})$ of highest weight and set $v' = y'y^3 v$. Then

$$xyv' = hy'y^3v + yy'xy^3v = -y'y^3v + 3yy'y^2v = -y^3y'v + 3y^2y'yv.$$

Since $y^3y'v \neq 0$, this does not have the required form. More generally if v' generates a direct sum V of the 6-, 4-, 2-dimensional simple $\mathfrak{s}_\alpha$ modules, then V does not admit positivity.

Index of notation

Symbols appearing frequently are given below in order of appearance.

Introduction k.
1.1 $\mathfrak{a}$, $U(\mathfrak{a})$, $S(\mathfrak{a})$, $\mathfrak{g}$, $\mathfrak{h}$, Δ^+, Δ, π, $P(\pi)$, $P^+(\pi)$, ρ, s_α, W, $w.\lambda$, $\mathfrak{b}$, $\mathfrak{n}^+$, e_α, $f_{-\alpha}$, h_α, κ, σ, ι, $V(\nu)$.
1.2 k_λ, $M(\lambda)$, v_λ, A, $q_{-\alpha}$, p_α, $\mathbf{Q}$, $\mathbf{P}$, $A^{\lambda,\mu}$.
1.3 $\mathcal{F}$.
1.4 $F_{\mathfrak{h}}(M)$, $\mathfrak{b}^-$, $F_{\mathfrak{b}^-}(M)$.
1.5 $\delta M(\mu)$.
2.1 $\mathcal{O}$, $\hat{\mathcal{O}}$, $\mathcal{O}^-$, $\hat{\mathcal{O}}^-$, R_μ^λ.
2.2 D_q, D, $F(M)$, S, J.
2.3 P_q, $m_\nu^\mu(q)$.
3.1 M^τ, δ.
3.2 $\mathcal{M}$, $\mathcal{A}$, $\mathcal{F}_x$.
3.3 ch_q^x.
4.1 $\mathbf{G}$, $\mathbf{B}$, $\mathbf{H}$.
4.2 V, V^-, $p_{-\gamma}$.
5.2 $P_{\delta,\gamma}$, P_γ^g, P^g, $P^g(\mu)$.
5.3 B.

5.4 V_1, V_1^-.
5.5 $q'_{-\gamma}$, $\mathbf{Q}'$.
6.6 $H_w(\nu)$.
7.1 $S(w)$, w_0, $\rho^\vee$.
7.2 $\ell_\nu(w)$.

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