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VAUGHT'S CONJECTURE ON ANALYTIC SETS

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0. Prehistory

The Vaught conjecture stands as a major problem in mathematical logic and may be stated as follows:

0.1. Conjecture (Original Vaught conjecture)**.** *Let T be a first order theory in a countable language. Then either T has $\leq \aleph_0$ many non-isomorphic countable models or it has $2^{\aleph_0}$ many non-isomorphic countable models.*

The study of this conjecture and the proof or refutation of its specializations and generalizations – found in papers such as [1], [5], [10], [13], [18], [19], [20] – has almost become a research area in its own right.

One natural extension of the Vaught conjecture is to classes of models defined by countably infinitary sentences in the sense of [16].

0.2. Conjecture (Vaught conjecture for infinitary logic)**.** *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then either σ has $\leq \aleph_0$ many non-isomorphic countable models or it has $2^{\aleph_0}$ many non-isomorphic countable models.*

Conjecture 0.2, like 0.1, remains open. Under the assumption of the continuum hypothesis both these conjectures are trivially true, and for this reason it is often customary to consider a strengthened version which has the same spirit as the original. To phrase this stronger version we need to equip the collection of all models whose underlying set is $\mathbb{N}$ with a topology, whose basic open sets consist of all $\mathcal{M}$ for which

$$\mathcal{M} \models \varphi(a_1, a_2, ..., a_n),$$

for some first order formula φ and $\vec{a} = a_1, ..., a_n \in \mathbb{N}$. It is known from [9] that the space of all structures or models on $\mathbb{N}$ in that given topology forms a Polish space. The collection of models satisfying a given theory T is then closed inside this space, and thus again a Polish space.

0.3. Conjecture. *Let T be a first order theory in a countable language $\mathcal{L}$. Then either T has $\leq \aleph_0$ many non-isomorphic countable models or there is a perfect set P included in the above described space of $\mathcal{L}$-structures modeling T for which any two models in P are non-isomorphic.*

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0.4. Conjecture. *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then either σ has $\leq \aleph_0$ many non-isomorphic countable models or there is a perfect set P included in the space of $\mathcal{L}$-structures which model σ for which any two models in P are non-isomorphic.*

Since any perfect set has size $2^{\aleph_0}$, 0.3 and 0.4 clearly imply 0.1 and 0.2, respectively. More subtle is the fact that 0.3 and 0.4 are *equiprovable* over ZFC with their counterparts 0.1 and 0.2. A proof of this further fact can be found in [20].

It was demonstrated in [22] that the isomorphism invariant Borel subsets of the space of $\mathcal{L}$-structures on $\mathbb{N}$ are exactly those defined by some infinitary sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Thus 0.4 asserts that any invariant Borel set in our space of countable structures has either only countably many orbits or it has *perfectly many* orbits, in the sense of there being a perfect set of pairwise non-isomorphic models. Thus we might be led to asking whether the Vaught conjecture can hold even on *analytic sets*.

0.5. Definition. Let X be a Polish space. $A \subset X$ is *analytic*, or $\underset{\sim}{\Sigma}^1_1$, if there is some other Polish space Y, some Borel function

$$f : Y \to X$$

and some Borel $B \subset Y$ with

$$f[B] = A.$$

0.6. Fact. There are countable languages $\mathcal{L} \subset \mathcal{L}^*$ and a first order $\mathcal{L}^*$ theory S such that the class

$$A_S$$

of $\mathcal{L}$-structures on $\mathbb{N}$ admitting an expansion to a model of S has uncountably many non-isomorphic countable models but not perfectly many non-isomorphic models. (See [2] for a discussion of this and related results.)

Since the process of assigning to each $\mathcal{L}^*$-structure on $\mathbb{N}$ its reduction to an $\mathcal{L}$-structure is Borel, we obtain the failure of Vaught's conjecture on $\underset{\sim}{\Sigma}^1_1$ sets: We *can* have an analytic subset of the space of countable $\mathcal{L}$-structures having uncountably many but not perfectly many non-isomorphic models.

A second direction in which we might try to extend the Vaught conjecture is to consider general actions of Polish groups. Here it is important to note that the Polish group S_∞, consisting of all permutations of the natural numbers in the topology of pointwise convergence, acts continuously on any such space of $\mathcal{L}$-structures with underlying set $\mathbb{N}$, and moreover its orbit equivalence relation is exactly the isomorphism relation on these countable structures.

0.7. Conjecture (Topological Vaught conjecture)**.** *Let G be a Polish group acting continuously on a Polish space X. Then either X has only countably many orbits under this action or it has perfectly many orbits.*

This conjecture just for the group S_∞ implies 0.3, and hence 0.1, since the subspace of models for T is a closed subset of the space of structures on $\mathbb{N}$. By a clever choice of the topology on our space of countable structures it can also be shown to imply 0.4. More recently Becker and Kechris in [4] have shown that 0.4 is in fact *equivalent* to the topological Vaught conjecture for the Polish group S_∞.

Conjecture 0.7, like 0.1–0.4, remains open. It has however been proved for various classes of Polish groups.

0.8. Theorem (Folklore)**.** *All locally compact Polish groups satisfy Vaught's conjecture, in the sense that if G is a locally compact Polish group acting continuously on a Polish space X, then either $|X/G| \leq \aleph_0$ or there is a perfect set of points with different orbits (and hence $|X/G| \geq 2^{\aleph_0}$).*

0.9. Theorem (Sami; see [18])**.** *Abelian Polish groups satisfy Vaught's conjecture.*

0.10. Theorem (Hjorth, Solecki [13])**.** *Invariantly metrizable and nilpotent Polish groups satisfy Vaught's conjecture.*

0.11. Theorem (Becker [2])**.** *Complete left invariant metric groups satisfy Vaught's conjecture.*

As discussed in [2], this implies the Vaught conjecture for solvable groups, as well as the version obtained in 0.10.

In each of these cases the result was shortly or immediately after extended to analytic sets.

0.12. Definition. For G a Polish group let $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ be the assertion that whenever G acts continuously on a Polish space X and $A \subset X$ is analytic, then either $|A/G| \leq \aleph_0$ or there is a perfect set of orbit inequivalent points in A.

Thus we have $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ for G in each of the collections groups mentioned in 0.8–0.11 above. But it is known that $\mathrm{TVC}(S_\infty, \underset{\sim}{\Sigma}^1_1)$ *fails*; A_S from 0.6 provides a counterexample.

In this paper we show that the presence of S_∞ is a necessary condition for $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ to fail:

0.13. Theorem. *If G is a Polish group for which the Vaught conjecture fails on analytic sets, then there is a closed subgroup of G that has S_∞ as a continuous homomorphic image.*

From the failure of $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ we may obtain the homomorphism onto S_∞. The converse of 0.13 is known and follows by 2.3.5 of [4]. Thus we have an exact characterization of $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$. If, as widely suspected, the Vaught conjecture should fail for S_∞, then this would as well characterize the groups for which the topological Vaught conjecture holds.

Theorem 0.13 implies the earlier results of 0.8–0.11. For instance one can use [8] to show that no group having S_∞ as its continuous homomorphic image can be given a compatible complete left invariant metric.

1. Preliminaries

This paper uses much the same techniques as [11]. The notation is also similar to [11], though in this regard [4] probably provides a better reference to the notation relating to Polish group actions.

In other respects we follow the usual set-theoretic conventions, as can be found in [14]. Perhaps the most alarming of these is to use ω for

$$\mathbb{N} = \{0, 1, 2, 3, ...\}.$$

A basic tool in the descriptive set theory of Polish group actions is provided by the Effros lemma.

1.1. Lemma (Effros; see [7] or 2.2.2 in [4])**.** *Let G be a Polish group acting continuously on a Polish space X (in other words, let X be a Polish G-space). For $x \in X$ we have $[x]_G \in \underset{\sim}{\Pi}^0_2$ if and only if*

$$G \to [x]_G,$$

$$g \mapsto g \cdot x$$

is open.

1.2. Corollary. *Let G be a Polish group and X a Polish G-space. Suppose that $[x]_G$ is $\underset{\sim}{\Pi}^0_2$. Then for all V containing the identity we may find open U containing x such that for all $x' \in U \cap [x]_G$ and $U' \subset X$ open*

$$[x]_G \cap U' \cap U \neq \emptyset$$

implies that there exists $g \in V$ such that

$$g \cdot x' \in U'.$$

Proof. Choose an open neighborhood W of the identity with $W^{-1} = W$ and $W^2 \subset V$. Then by 1.1 let U be an open set with $U \cap [x]_G = W \cdot x$.

For all $x' \in [x]_G \cap U$ and open U' with $[x]_G \cap U' \cap U \neq \emptyset$ we choose some $\hat{x} \in [x]_G \cap U' \cap U$. Since $x', \hat{x} \in U \cap [x]_G$, the assumption on U provides $g_0, g_1 \in W$ with

$$g_0 \cdot x = x',$$

$$g_1 \cdot x = \hat{x}.$$

Then $g_1 g_0^{-1}$ is as required. □

1.3. Definition. Let X be a Polish space and $\mathcal{B}$ a basis. Let $\mathcal{L}(\mathcal{B})$ be the propositional language formed from the atomic propositions $\dot{x} \in U$, for $U \in \mathcal{B}$. Let $\mathcal{L}_{\infty,0}(\mathcal{B})$ be the infinitary version, obtained by closing under negation and arbitrary disjunction and conjunction. $F \subset \mathcal{L}_{\infty,0}(\mathcal{B})$ is a *fragment* if it is closed under subformulas and the finitary Boolean operations of negation and finite disjunction and finite conjunction and it includes all atomic propositions.

For a point $x \in X$ and $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$, we can then define $x \models \varphi$ by induction in the usual fashion, starting with

$$x \models \dot{x} \in U$$

if in fact $x \in U$, then $x \models \bigvee_{i \in \Lambda}(\varphi_i)$ if there is some $i \in \Lambda$ with $x \models \varphi_i$, and $x \models \neg\varphi$ if x does not model φ. For $\mathbb{V}[H]$ a generic extension of $\mathbb{V}$ and $x, \varphi \in \mathbb{V}$ we have $\mathbb{V} \models (x \models \varphi)$ if and only if $\mathbb{V}[H] \models (x \models \varphi)$.

In the case that X is a Polish G-space and $V \subset G$ is open we may also define the Vaught transform $\varphi^{\Delta V}$ by induction on the logical complexity of φ, mimicking the definition of the usual Vaught transforms for Borel sets. For this purpose let $(V_i)_{i \in \mathbb{N}}$ be an enumeration of the non-empty basic open subsets of G. $(\dot{x} \in U)^{\Delta V}$ is just

$$\bigvee_{U' \in \mathcal{B}'} \dot{x} \in U',$$

where $\mathcal{B}'$ is the set of basic open sets U' for which there is some i with $V_i \cdot U' \subset U$ and $V_i \subset V$.

$$(\bigvee_{i \in \Lambda} \varphi_i)^{\Delta V}$$

is

$$\bigvee_{i\in\Lambda}(\varphi_i^{\Delta V}).$$

$$(\neg\varphi)^{\Delta V}$$

is

$$\bigvee\{\neg(\varphi^{\Delta V_n}) : V_n \subset V\}.$$

In any generic extension in which φ is hereditarily countable

$$x \models \varphi^{\Delta V}$$

if and only if

$$\exists^* g \in V(g \cdot x \models \varphi)$$

(where $\exists^*$ is the categoricity quantifier "there exists non-meagerly many").

In general $\varphi^{\Delta V}$ depends on the choice of $\mathcal{B}_0$ and $\mathcal{B}$. Our notation suppresses this dependence.

Note that if $\varphi \in \mathcal{L}_{\infty,0}(\mathcal{B})$ and $g \in G$ is such that $\mathcal{B}$ is fixed set-wise under g translation, then we can canonically define $g\varphi$ with the property that

$$x \models g\varphi$$

if and only if $g^{-1} \cdot x \models \varphi$. We define $g\varphi$ by induction on the complexity of φ, the base case given by $g(\dot{x} \in U)$ being the sentence $\dot{x} \in g \cdot U$, and the inductive steps through the infinitary Boolean operations carried out in the usual way.

In what follows there is frequent peril that we may confuse the usual name for the generic object – $\dot{G}$ – with the customary name for the group – G. For this reason I will generally resist explicit mention of the generic. If σ is a term, existing in our ground model, then $\dot{\sigma}$ will denote the effect of applying the term to the generic object in the generic extension. At times we will wish to consider product forcing, and then I will use $\dot{G}_l \times \dot{G}_r$ to name the generic object on a product forcing $\mathbb{P} \times \mathbb{Q}$. Similarly to avoid confusion with an open set, we use the embellished font $\mathbb{V}$, rather than the plain V, to indicate the universe of all sets.

d_X will always denote a complete metric on X and d_G a complete metric on the group G. G_x indicates the stabilizer of x, that is,

$$G_x = \{g \in G : g \cdot x = x\}.$$

1.4. Lemma. *Let X be a Polish G-space, $\mathbb{P}$ a forcing notion, $p \in \mathbb{P}$ a condition, and σ a $\mathbb{P}$-term. Suppose that $\mathcal{B}$ is a countable basis for X and $\mathcal{B}_0$ a countable basis for G. Suppose that G_0 is a countable dense subgroup of G and $\mathcal{B}$ is closed under G_0 translation and that $\mathcal{B}_0$ is closed under left and right G_0 translation. Then there is a formula φ_0 and a fragment F_0 containing φ_0 so that:*

(i)

$$\{\{x \in X : x \models \psi^{\Delta V}\} : \psi \in F_0, V \in \mathcal{B}_0\}$$

provides a basis for a topology $\tau_0(F_0)$, and in any generic extension in which F_0 becomes countable $(X, \tau_0(F_0))$ is a Polish G-space;

(ii) *φ_0 absolutely describes the equivalence class (if any) indicated by the triple $(\mathbb{P}, p, \sigma)$, in the sense that in any generic extension $\mathbb{V}[H]$ we find*

$$\mathbb{V}[H] \models \forall x \in X(p \Vdash_{\mathbb{P}} xE_G\dot{\sigma} \Leftrightarrow x \models \varphi_0).$$

Proof. Note that if F is any fragment and $\tau(F)$ is the topology with basis $\{\{x \in X : x \models \psi\} : \psi \in F\}$, then $\tau(F)$ includes the original topology by virtue of F including all the atomic propositions.

Claim (1). If F is a fragment, then in any generic extension in which F becomes countable, $\tau(F)$ generates a Polish topology.

Proof of claim. Recall from 13.2 in [15] that whenever we have a Polish topology τ_Y on a Polish space Y and $A \subset Y$ a τ_Y-open set, then the minimal topology obtained from τ_Y by adding A as a clopen set gives a topology on Y equivalent to taking the disjoint union of (A, τ_Y) and $(Y \setminus A, \tau_Y)$, and is therefore again Polish. Recall also that if τ_i is an increasing sequence of Polish topologies on Y, then the minimal topology including the union is again Polish, since it corresponds to the diagonal in the resulting space

$$\prod_{i\in\omega}(Y, \tau_i).$$

Iterating these two operations one shows by induction on the logical complexity of the infinitary sentences in F that $\tau(F)$ is Polish in any generic extension in which F becomes countable. (Claim (1) $\square$)

Let F_0 be a fragment closed under the Vaught transforms ($\psi \mapsto \psi^{\Delta V}$ for $V \subset G$ basic open) and such that for each $\psi \in F_0$ and $g \in G_0$ we have a corresponding $g\psi \in F_0$ with the property that through all generic extensions

$$(x \models g\psi) \Leftrightarrow (g^{-1} \cdot x \models \psi).$$

By the Becker-Kechris theorem on changing topologies, as found at [3] or as in §5.2 and the proof of 7.1.3 of [4], we have that

$$\{\{x \in X : x \models \psi^{\Delta V}\} : \psi \in F_0, V \in \mathcal{B}_0\}$$

generates a topology $\tau_0(F_0)$, with $(X, \tau_0(F_0))$ a Polish G-space whenever F_0 becomes countable. Since the sets of the form

$$\{x \in X : x \models \psi\},$$

$\psi \in F_0$, form an algebra closed under G_0 translation, it is easily seen that the collection $\{\{x \in X : x \models \psi^{\Delta V}\} : \psi \in F_0, V \in \mathcal{B}_0\}$ provides a basis for the topology $\tau_0(F_0)$.

In what follows it is crucial that the properties of most interest are absolute, and thus we can be ambiguous about the generic extension in which we make an evaluation. In particular, for any x appearing in any generic extension, the assertion $p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$ depends solely on x, and not the generic extension in which we consider this question:

Claim (2). If $\mathbb{V}[H_1] \subset \mathbb{V}[H_1][H_2]$ are two generic extensions of $\mathbb{V}$ and $x \in \mathbb{V}[H_1]$, then

$$\mathbb{V}[H_1] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$$

if and only if

$$\mathbb{V}[H_1][H_2] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}.$$

Proof of claim. Otherwise we may find $H \subset \mathbb{P}$ that is generic for both models, and such that the generic extensions $\mathbb{V}[H_1][H]$ and $\mathbb{V}[H_1][H_2][H]$ disagree on $xE_G\sigma[H]$, with a violation of absoluteness of $\underset{\sim}{\Sigma}^1_1$. (Claim (2) $\square$)

Claim (3). There is a generic extension of $\mathbb{V}[H]$ and $x \in \mathbb{V}[H]$ with

$$\mathbb{V}[H] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$$

if and only if

$$\mathbb{V} \models (p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l]E_G\sigma[\dot{G}_r].$$

Proof of claim. Clearly if $(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l]E_G\sigma[\dot{G}_r]$, then we can let $H \subset \mathbb{P}$ be generic below p, and obtain some $x = \sigma[H]$ in $\mathbb{V}[H]$ with

$$\mathbb{V}[H] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}.$$

Conversely, if we have $\mathbb{V}[H] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$ in some generic extension $\mathbb{V}[H]$ of $\mathbb{V}$ and $q_1, q_2 \leq p$ with

$$\mathbb{V} \models (q_1,q_2) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l]\neg E_G\sigma[\dot{G}_r],$$

then we may choose $H_1 \times H_2$ as a $\mathbb{V}[H]$-generic filter on $\mathbb{P}\times\mathbb{P}$ below (q_1,q_2). Then

$$\sigma[H_1]\neg E_G\sigma[H_2],$$

by assumption on (q_1,q_2), and hence for some $i \in \{1,2\}$ we have

$$\sigma[H_i]\neg E_G x,$$

contradicting the assumption on p. (Claim (3) $\Box$)

Claim (4). The set of x for which

$$p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$$

is $\underset{\sim}{\Sigma}^1_1$ in any generic extension in which $(\mathcal{P}(\mathbb{P}))^{\mathbb{V}}$ ($=_{df}$ Power set of $\mathbb{P}$ as calculated in $\mathbb{V}$) becomes countable.

Proof of claim. It suffices by Claim (3) to show that if

$$\mathbb{V} \models (p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l]E_G\sigma[\dot{G}_r],$$

then for any two $H_1, H_2 \subset \mathbb{P}$ meeting all the dense open sets in $\mathbb{V}$ we have

$$\sigma[H_1]E_G\sigma[H_2].$$

But given H_1, H_2 we can go to a third generic extension $\mathbb{V}[H_3]$ where $H_3 \subset \mathbb{P}$ is $\mathbb{V}[H_1]$ and $\mathbb{V}[H_2]$ generic below p, and thus $\sigma[H_3]E_G\sigma[H_i]$ for *both* $i=1$ and $i=2$. (Claim (4) $\Box$)

Thus we may find a transfinite sequence $(\psi_\alpha)_{\alpha\in\mathrm{Ord}}$ of infinitary propositions in $\mathcal{L}_{\infty 0}(\mathcal{B})$ such that in any extension $\mathbb{V}[H]$ in which $|\mathcal{P}(\mathbb{P})^{\mathbb{V}}| \leq \aleph_0$ and for any $x \in \mathbb{V}[H]$ we have

$$(p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}) \Leftrightarrow \exists\alpha < \omega_1^{\mathbb{V}[H]}(x \models \psi_\alpha).$$

This is obtained either by appeal to the usual decomposition of $\underset{\sim}{\Sigma}^1_1$ into Borel sets or by letting ψ_α be chosen canonically so that for all x in any generic extension

$$(x \models \psi_\alpha) \Leftrightarrow (L_\alpha(G, X, \sigma, (\mathcal{P}(\mathbb{P}))^{\mathbb{V}}, x) \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}).$$

Claim (5). In any generic extension $\mathbb{V}[H]$ in which $(\mathcal{P}(\mathbb{P}))^{\mathbb{V}}$ becomes countable and for any $x \in \mathbb{V}[H]$ with $p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$ we have some $\alpha < \omega_1^{\mathbb{V}[H]}$ with

$$\exists^* g \in G(g\cdot x \models \psi_\alpha).$$

Proof of claim. Since for any particular $\alpha < \omega_1^{\mathbb{V}[H]}$ the displayed sentence is absolute, we may as well assume $\mathrm{MA}_{\aleph_1}$. By invariance of the set in question we have that for all $g \in G$ there will be some $\alpha \in \omega_1^{\mathbb{V}[H]}$ with

$$g \cdot x \models \psi_\alpha.$$

By $\mathrm{MA}_{\aleph_1}$, for some α the set of g for which $g \cdot x \models \psi_\alpha$ must be non-meager. (Claim (5) $\square$)

So far there is no guarantee that *any* ψ_α is non-trivial; nothing yet has ruled against us *never* having the situation that $p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}$.

Claim (6). If there is some generic extension in which

$$\mathbb{V}[H] \models p \Vdash_{\mathbb{P}} xE_G\dot{\sigma},$$

then there is some α for which $\varphi_0 = (\psi_\alpha)^{\Delta G}$ satisfies (ii) from the statement of the lemma.

Proof of claim. Following Claim (4) and applying the absoluteness of $\underset{\sim}{\Sigma}^1_1$ we can assume that $\mathbb{V}[H]$ arises by simply collapsing $(\mathcal{P}(\mathbb{P}))^{\mathbb{V}}$ to $\aleph_0$. Following Claim (5) let us choose some $q \in \mathrm{Coll}(\omega, (\mathcal{P}(\mathbb{P}))^{\mathbb{V}})$ and $\alpha < (|\mathcal{P}(\mathbb{P})|^+)^{\mathbb{V}}$ such that

$$q \Vdash_{\mathrm{Coll}(\omega,(\mathcal{P}(\mathbb{P}))^{\mathbb{V}})} \exists x \exists^* g (g \cdot x \models \psi_\alpha).$$

Now suppose that $\mathbb{V}[H_1]$ is any generic extension with some $x_1 \in \mathbb{V}[H_1]$ such that

$$\mathbb{V}[H_1] \models (p \Vdash_{\mathbb{P}} x_1 E_G \dot{\sigma}).$$

Then we may choose $H_2 \subset \mathrm{Coll}(\omega, (\mathcal{P}(\mathbb{P}))^{\mathbb{V}})$ to be $\mathbb{V}[H_1]$-generic below q. Thus by assumption on q we may find some $x_2 \in \mathbb{V}[H_2]$ with

$$\mathbb{V}[H_2] \models \exists^* g \in G(x_2 \models \psi_\alpha),$$

and thus in particular

$$\mathbb{V}[H_2] \models (p \Vdash_{\mathbb{P}} x_2 E_G \dot{\sigma}).$$

Then by Claim (2)

$$\mathbb{V}[H_1][H_2] \models (p \Vdash_{\mathbb{P}} x_2 E_G \dot{\sigma}),$$
$$\mathbb{V}[H_1][H_2] \models (p \Vdash_{\mathbb{P}} x_1 E_G \dot{\sigma}).$$

So if we let $H \subset \mathbb{P}$ be $\mathbb{V}[H_1][H_2]$-generic below p we obtain

$$\mathbb{V}[H_1][H_2][H] \models x_2 E_G \sigma[H],$$

$$\mathbb{V}[H_1][H_2][H] \models x_1 E_G \sigma[H].$$

Thus $x_1 E_G x_2$.

But $(\psi_\alpha)^{\Delta G}$ is absolute between forcing extensions, and thus

$$\mathbb{V}[H_1][H_2][H] \models \exists^* g \in G(g \cdot x_2 \models \psi_\alpha).$$

Then by invariance

$$\mathbb{V}[H_1][H_2][H] \models \exists^* g \in G(g \cdot x_1 \models \psi_\alpha).$$

One final application of absoluteness gives

$$\mathbb{V}[H_1] \models (x_1 \models \psi_\alpha^{\Delta G}).$$

(Claim (6) $\square$)

Nothing in the statement of the lemma rules out (ii) holding trivially. It could just be the case that in every extension $\mathbb{V}[H_0]$ we have $\mathbb{V}[H_0] \models \forall x(p \Vdash_{\mathbb{P}} x \neg E_G \dot{\sigma})$. However in the non-trivial case where there is a generic extension $\mathbb{V}[H_0]$ satisfying $\exists x(p \Vdash_{\mathbb{P}} x E_G \dot{\sigma})$ it will follow by absoluteness for $\underset{\sim}{\Sigma}^1_1$ that for any generic extension $\mathbb{V}[H_1]$ in which F_0 becomes countable we must have $\mathbb{V}[H_1] \models \exists x(x \models \varphi_0)$, and therefore $\mathbb{V}[H_1] \models \exists x(p \Vdash_{\mathbb{P}} x E_G \dot{\sigma})$.

1.5. Lemma. *Let G be a Polish group, X a Polish G-space, and $A \subset X$ a $\underset{\sim}{\Sigma}^1_1$ set displaying a counterexample to $TVC(G, \underset{\sim}{\Sigma}^1_1)$ – so that A/G has uncountably many orbits, but no perfect set of E_G-inequivalent points. Then for each ordinal δ there is a sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)_{\alpha<\delta}$ so that:*

(0) *for each $\alpha < \delta$*

$$p_\alpha \Vdash_{\mathbb{P}_\alpha} \dot{\sigma}_\alpha \in A;$$

(i) *for each $\alpha < \delta$*

$$(p_\alpha, p_\alpha) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\dot{G}_l] E_G \sigma_\alpha[\dot{G}_r];$$

(ii) *for each $\alpha < \beta < \delta$*

$$(p_\alpha, p_\beta) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} \neg(\sigma_\alpha[\dot{G}_l] E_G \sigma_\beta[\dot{G}_r]).$$

Proof. Claim (1). If $\mathbb{P}$ is any forcing notion and σ a $\mathbb{P}$-term for an element of A, then there is $p \in \mathbb{P}$ such that $(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l] E_G \sigma[\dot{G}_r]$.

Proof of claim. It is known from [6], [21], and recorded in many places, that whenever we have a $\underset{\sim}{\Sigma}^1_1$ equivalence relation F on a Polish space Y for which there is no perfect set of inequivalent reals, then for all $\mathbb{P} \Vdash \dot{\tau} \in Y$ we may find some p with

$$(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau[\dot{G}_l] F \tau[\dot{G}_r].$$

Taking $\pi : Y \twoheadrightarrow A$ to be the continuous map witnessing $A \in \underset{\sim}{\Sigma}^1_1$ we may let F be the pullback of E_G. Since A has no perfect set of E_G-inequivalent reals, Y has no perfect set of F-inequivalent reals. Thus if σ is a term for an element of A, then we let τ be a term for an element of Y with

$$\mathbb{P} \Vdash \pi(\dot{\tau}) = \dot{\sigma},$$

and we obtain as above some $p \in \mathbb{P}$ with

$$(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau[\dot{G}_l] F \tau[\dot{G}_r],$$

$$\therefore (p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l] E_G \sigma[\dot{G}_r].$$

(Claim (1) $\Box$)

Claim (2). In every generic extension of $\mathbb{V}$ we have $|A/G| \geq \aleph_1$.

Proof of claim. (Compare also [18].) We use the fact that every orbit in X is Borel in any code for its stabilizer. (See 7.1.2 of [4].)

Let $\mathcal{F}(G)$ be the standard Borel space of closed subsets of G in the Effros Borel structure. Let $S \subset X \times \mathcal{F}(G)$ be $\underset{\sim}{\Pi}^1_1$ with $S = \{(x, G_x) : x \in X\}$. Let

$$E \subset X \times X \times \mathcal{F}(G)$$

be $\underset{\sim}{\Pi}^1_1$ such that $(x, y, G_x) \in E$ if and only if $x E_G y$. Then the uncountability of A/G becomes the $\underset{\sim}{\Pi}^1_2$ statement

$$\forall (x_i, F_i)_{i\in\omega} \in (X \times \mathcal{F}(X))^\omega \exists y \in A(\exists i(\neg S(x_i, F_i)) \vee \forall i(\neg E(x_i, y, F_i))),$$

and is therefore absolute. (Claim (2) $\Box$)

Thus we may successively choose $\mathbb{P}_\alpha$ and σ_α such that

$$\mathbb{P}_\alpha \Vdash \dot{\sigma}_\alpha \in A$$

and for all $\beta < \alpha$ and $q \in \mathbb{P}_\alpha$

$$(q, p_\beta) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} \neg(\sigma_\alpha[\dot{G}_l] E_G \sigma_\beta[\dot{G}_r]).$$

Refining to some $p_\alpha \in \mathbb{P}_\alpha$ with

$$(p_\alpha, p_\alpha) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\dot{G}_l] E_G \sigma_\alpha[\dot{G}_r]$$

completes the proof. □

Note that in the situation of 1.5 we may find for each $\alpha < \delta$ some corresponding $\varphi_{0,\alpha}$ for the forcing notion $\mathbb{P}_\alpha$ below p_α as in 1.4. We are guaranteed that this is non-trivial, in the sense that if $H \subset \mathbb{P}_\alpha$ is $\mathbb{V}$-generic below p_α and $x = \sigma_\alpha[H]$, then

$$p_\alpha \Vdash_{\mathbb{P}} x E_G \dot{\sigma}_\alpha,$$

$$\therefore x \models \varphi_{0,\alpha}.$$

On the other hand, if $x_0 \in \mathbb{V}[H_0]$ is an element of some generic extension $\mathbb{V}[H_0]$, and if $x \models \varphi_{0,\alpha}$ and $H_1 \subset \mathbb{P}_\alpha$ is any $\mathbb{V}$-generic with $x_1 = \sigma_\alpha[H_1]$, then we may choose $H_2 \subset \mathbb{P}_\alpha$ that is generic for both $\mathbb{V}[H_1]$ and $\mathbb{V}[H_2]$. Thus

$$x_0 E_G \sigma_\alpha[H_2]$$

by assumption on $\varphi_{0,\alpha}$. Moreover

$$x_1 E_G \sigma_\alpha[H_2]$$

by the assumption $\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha$. Hence $x_0 E_G x_1$.

In this sense we can associate to $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)_{\alpha<\delta}$ a corresponding sequence $(\varphi_{0,\alpha})_{\alpha\in\delta}$, with each $\varphi_{0,\alpha}$ providing an ∞-Borel code for the indicated equivalence class $[\sigma_\alpha[H]]_G$ whenever $H \subset \mathbb{P}_\alpha$ is $\mathbb{V}$-generic below p_α.

2. Proof

2.1. Definition. U is a *regular open* set if U equals the interior of its closure:

$$(\overline{U})^o = U.$$

For a set A let $RO(A) = (\overline{A})^o$.

Note that $RO(A)$ is always regular open.

2.2. Lemma. *Let G be a Polish group. For regular open sets $V_0, V_1 \subset G$,*

$$\{g \in G : V_0 \cdot g = V_1\}$$

is a closed subset of G.

Proof. The set in question arises as the intersection of the closed sets

$$\{g \in G : V_0 \cdot g \subset \overline{V_1}\}$$

and

$$\{g \in G : \overline{V_0} \cdot g \supset V_1\}.$$

□

I ask that the reader be willing to allow us to speak of an ω-model of set theory containing a Polish space, group, action, Borel set, and so on, provided suitable codes exist in the well-founded part. I will lean on the usual set-theoretical identifications, and speak of open sets and Borel sets being "in" an ω-model, when really it is only the case that the model has some suitable subset of the natural numbers which codes them.

Ill-founded ω-models are essential to the arguments below. The structure of the argument is to use the large sequence of forcing notions layed out in 1.5 to obtain a suitable ω-model with generating indiscernibles of order type $(\mathbb{Q}, <)$. Thus we can inject $\mathrm{Aut}(\mathbb{Q}, <)$ into the automorphisms of the ω-model, and then using a kind of "back-and-forth" argument at 2.3 obtain for each automorphism some corresponding element of the group G. Since well-founded models are rigid and since we want automorphisms in great number, the passage to an ω-model occupies a necessary step in the proofs.

Let ZFC* be some large fragment of ZFC; at the very least strong enough to prove all the lemmas of §1, but weak enough to admit a finite axiomatization.

The next lemma gives the connection between automorphisms of an ω-model and the corresponding elements of G. Here it should be understood that we do everything honestly: We are supposing that the M-generic H exists in $\mathbb{V}$, our ground model, and conclude with $\hat{g}$ again in the ground model. We are working inside a single class model of ZFC which sees not only M and π but also X, G, G_0, $\mathbb{P}$, F_0, $\mathcal{B}$, $\mathcal{B}_0$, φ_0, and H.

2.3. Lemma. *Let M be an ω-model of ZFC*. Let X, G, G_0, $\mathbb{P}$, F_0, $\mathcal{B}$, $\mathcal{B}_0$, φ_0 satisfy 1.4 inside M. Suppose*

$$\pi : M \cong M$$

is an automorphism of M fixing X, G, G_0, $\mathbb{P}$, F_0, φ_0, and all elements of $\mathcal{B}$ and $\mathcal{B}_0$. Suppose $H \subset \mathrm{Coll}(\omega, F_0)$ is M-generic and $x \in X^{M[H]}$ with

$$M[H] \models (x \models \varphi_0).$$

Then there exists $\hat{g} \in G$ so that for all $\psi \in F_0$ and $V \in \mathcal{B}_0$

$$RO(\{g \in G_0 : M[H] \models (g \cdot x \models \psi^{\Delta V})\})\hat{g}^{-1}$$
$$= RO(\{g \in G_0 : M[H] \models (g \cdot x \models \pi(\psi)^{\Delta V})\}).$$

Proof. It suffices to find $g_0, g_1 \in G$ so that

$$RO(\{g \in G_0 : M[H] \models (g \cdot x \models \psi^{\Delta V})\})g_0^{-1}$$
$$= RO(\{g \in G_0 : M[H] \models (g \cdot x \models \pi(\psi)^{\Delta V})\})g_1^{-1}$$

for all ψ and V.

Let $\mathbb{P}_0$ be the forcing notion $\mathrm{Coll}(\omega, F_0)$ in M. Fixing a complete metric d_G on G we build $h_i, h_i' \in G_0$, $\psi_i, \psi_i' \in F_0$, $W_i, W_i', V_i \in \mathcal{B}_0$ so that

(i) W_i, W_i' are open neighborhoods of the identity, with

$$W_{i+1} \subset W_i,$$
$$W_{i+1}' \subset W_i',$$
$$d_G(W_i) < 2^{-i},$$
$$d_G(W_i') < 2^{-i};$$

(ii) $\pi(\psi_i) = \psi_i'$;
(iii) $h_{2i} = h_{2i+1}$; for all $g \in W_{2i+1}h_{2i}$

$$d_G(g, h_{2i}) < 2^{-i};$$

(iv) $h'_{2i+1} = h'_{2i+2}$; for all $g \in W_{2i+2}h'_{2i+1}$

$$d_G(g, h'_{2i+1}) < 2^{-i};$$

(v) $h_{i+1} \in W_i h_i$ and $h'_{i+1} \in W_i h'_i$;
(vi) $M[H] \models (h_i \cdot x \models (\psi_i)^{\Delta V_i})$;
(vii) $M[H] \models (h'_i \cdot x \models (\psi'_i)^{\Delta V_i})$;
(viii) $M^{\mathbb{P}_0}$ satisfies that for all $y_0, y_1 \in X$, all $\psi \in F_0$, and all $V \in \mathcal{B}_0$, if

$$y_0 \models \varphi_0 \wedge (\psi_i)^{\Delta V_i} \quad \text{and} \quad y_1 \models \varphi_0 \wedge (\psi_i)^{\Delta V_i} \wedge \psi^{\Delta V},$$

then

$$y_0 \models ((\psi_i)^{\Delta V_i} \wedge \psi^{\Delta V})^{\Delta W_i};$$

(ix) conversely $M^{\mathbb{P}_0}$ satisfies that for all $y_0, y_1 \in X$, $\psi \in F_0$, and $V \in \mathcal{B}_0$, if

$$y_0 \models \varphi_0 \wedge (\psi'_i)^{\Delta V_i} \quad \text{and} \quad y_1 \models \varphi_0 \wedge (\psi'_i)^{\Delta V_i} \wedge \psi^{\Delta V},$$

then

$$y_0 \models ((\psi'_i)^{\Delta V_i} \wedge \psi^{\Delta V})^{\Delta W_i}.$$

Actually (ix) follows from (viii), (ii), and the elementarity of π.

Before verifying that we may produce $h_i, h'_i \in G_0$, $\psi_i, \psi'_i \in F_0$, and $W_i, V_i \in \mathcal{B}_0$ as above, let us imagine that it is already completed and see how to finish. Using (iii), (iv), and (v) we may obtain

$$g_0 = \lim_{i \to \infty} h_i$$

and

$$g_1 = \lim_{i \to \infty} h'_i.$$

It suffices to check that for all $\psi \in F_0$, $V \in \mathcal{B}_0$, and

$$g \in RO(\{h \in G_0 : M[H] \models (h \cdot x \models \psi^{\Delta V})\})g_0^{-1},$$

we have

$$g \in \overline{(\{h \in G_0 : M[H] \models (h \cdot x \models \pi(\psi)^{\Delta V})\})}g_1^{-1}$$

(the converse implication will be exactly symmetric). Fix g in

$$RO(\{h \in G_0 : M[H] \models (h \cdot x \models \psi^{\Delta V})\})g_0^{-1}.$$

By assumption on g there are arbitrarily large i and $h \in G_0$ with $h(h_i)^{-1}$ arbitrarily close to g and $M[H] \models (h \cdot x \models \psi^{\Delta V})$.

Thus replacing $h(h_i)^{-1}$ with $\hat{g}$ for sufficiently large i we may choose a sufficiently small open neighborhood $W \in \mathcal{B}_0$ of the identity and $\hat{g} \in G_0$ sufficiently close to g so that $W\hat{g}W_i$ is an arbitrarily small neighborhood of g and

$$M[H] \models (\hat{g}h_i \cdot x \models \psi^{\Delta V})$$

$$\therefore M[H] \models (h_i \cdot x \models (\psi^{\Delta V})^{\Delta W\hat{g}}),$$

hence, as witnessed by $y = h_i \cdot x$,

$$M^{\mathbb{P}_0} \models \exists y (y \models \varphi_0 \wedge (\psi_i)^{\Delta V_i} \wedge (\psi^{\Delta V})^{\Delta W\hat{g}}),$$

$$\therefore M^{\mathbb{P}_0} \models \exists y (y \models \varphi_0 \wedge (\psi'_i)^{\Delta V_i} \wedge (\pi(\psi)^{\Delta V})^{\Delta W\hat{g}})$$

by (ii) and elementarity of π,

$$\therefore M[H] \models (h'_i \cdot x \models ((\pi(\psi)^{\Delta V})^{\Delta W \hat{g}})^{\Delta W_i})$$

by (ix). So there exists some $\bar{g} \in W\hat{g}W_i$ such that

$$M[H] \models (\bar{g}h'_i \cdot x \models \pi(\psi)^{\Delta V}).$$

By letting $d_G(W\hat{g}W_i) \to 0$ and $h'_i \to g_1$ we get

$$g \in \overline{\{h \in G_0 : M[H] \models (h \cdot x \models \pi(\psi)^{\Delta V})\}}g_1^{-1},$$

as required.

We are left to hammer out the sequences.

Suppose that we have $\psi_j, \psi'_j, W_j, V_j, h_j, h'_j$ for $j \leq 2i$. Immediately we may set $h_{2i+1} = h_{2i}$ and find $W_{2i+1} \subset W_{2i}$ giving (iii), and then by 1.2 and 1.4(i) we can produce ψ_{2i+1} and V_{2i+1} satisfying (viii) and such that

$$M[H] \models h_{2i} \cdot x =_{df} h_{2i+1} \cdot x \models (\psi_{2i+1})^{V_{2i+1}}.$$

Then, by considering that π is elementary,

$$M^{\mathbb{P}_0} \models \exists y(y \models \varphi_0 \wedge \pi(\psi_{2i})^{\Delta V_{2i}} \wedge \pi(\psi_{2i+1})^{\Delta V_{2i+1}}).$$

Thus by (ix) we may find $h' \in G_0 \cap W_{2i}$ so that

$$M[H] \models (h'h'_{2i} \cdot x \models \pi(\psi_{2i})^{\Delta V_{2i}} \wedge \pi(\psi_{2i+1})^{\Delta V_{2i+1}}).$$

In other words, by (ii), if we let $\psi'_{2i+1} = \pi(\psi_{2i+1})$, then

$$M[H] \models (h'h'_{2i} \cdot x \models (\psi'_{2i})^{\Delta V_{2i}} \wedge (\psi'_{2i+1})^{\Delta V_{2i+1}}).$$

Taking $h'_{2i+1} = h'h'_{2i}$ we complete the transition from $2i$ to $2i+1$.

The further step of producing $\psi_{2i+2}, \psi'_{2i+2}, W_{2i+2}, h_{2i+2}$, V_{2i+2} and h'_{2i+2} is completely symmetrical. □

2.4. Definition. S_∞ *divides* a Polish group G if there is a closed subgroup $H < G$ and a continuous onto homomorphism

$$\pi : H \twoheadrightarrow S_\infty.$$

By Pettis' lemma, as at 1.2.6 of [4], any Borel homomorphism between Polish groups must be continuous.

2.5. Lemma. S_∞ *divides* $\mathrm{Aut}(\mathbb{Q}, <)$*, the automorphism group of the rationals equipped with the usual linear ordering.*

Proof. Let $\{D_i : i \in \omega\}$ be a partition of $\mathbb{Q}$ into countably many dense subsets.

Now let H be the group of all $\sigma \in \mathrm{Aut}(\mathbb{Q}, <)$ such that, for all $i, j \in \omega$ and all $q_1, q_2 \in D_i$,

$$\sigma(q_1) \in D_j \Leftrightarrow \sigma(q_2) \in D_j$$

and

$$\sigma^{-1}(q_1) \in D_j \Leftrightarrow \sigma^{-1}(q_2) \in D_j.$$

For any $\sigma \in H$ we define $\pi_\sigma \in S_\infty$ by $\pi_\sigma(i) = j$ if and only if $\sigma[D_i] = D_j$. The usual back-and-forth arguments show that for any permutation σ of ω we may find $\hat{\sigma} \in \mathrm{Aut}(\mathbb{Q}, <)$ such that, for all i, j,

$$\hat{\sigma}[D_i] = D_j$$

if and only if $\sigma(i) = j$. □

2.6. Definition. For X, G, F_0, etc., as in 1.4, $\mathbb{P}_0$= Coll(ω, F_0), $\psi_0, \psi_1 \in F_0$, and $V_0, V_1 \in \mathcal{B}_0$, set

$$(\psi_0, V_0)R(\psi_1, V_1)$$

if in $\mathbb{V}^{\mathbb{P}_0}$ for all $x \models \varphi_0$

$$RO(\{g \in G_0 : g \cdot x \models (\psi_0)^{\Delta V_0}\}) \cap RO(\{g \in G_0 : g \cdot x \models (\psi_1)^{\Delta V_1}\}) \neq \emptyset.$$

By the density of G_0 and the continuity of the action with respect to $\tau_0(F_0)$, the issue of whether this intersection is non-empty does not depend on the choice of $x \models \varphi_0$. For $V \in \mathcal{B}_0$ let $\mathcal{B}(V)$ be the set of pairs (φ, W) such that for all $\psi \in F_0$ and $W' \in \mathcal{B}_0$

$$\mathbb{V}^{\mathbb{P}_0} \models \forall x_0((x_0 \models \varphi_0 \wedge \varphi^{\Delta W} \wedge (\exists x_1 \models \varphi_0 \wedge \varphi^{\Delta W} \wedge \psi^{\Delta W'})) \Rightarrow x_0 \models (\varphi^{\Delta W} \wedge \psi^{\Delta W'})^{\Delta V}).$$

In other words, $\mathcal{B}(V)$ corresponds to the basic open sets witnessing 1.2 for V in the topology $\tau_0(F_0)$. The relation R corresponds to whether relatively open subsets of the orbit overlap.

The next lemma states that if the equivalence class corresponding to φ_0 requires large forcing to be introduced, then the formulas $\{\psi^{\Delta V} : \psi \in F_0, V \in \mathcal{B}_0\}$ have large R-discrete sets.

2.7. Lemma. *Let X, G, F_0, $\mathbb{P}$, p, σ, φ_0, etc., be as in 1.4. Assume that φ_0 is non-trivial, in the sense that $p \Vdash_{\mathbb{P}} (\dot\sigma \models \varphi_0)$. Let R be as in 2.6. Let κ be a cardinal. Suppose no forcing notion of size less than κ introduces a point in X satisfying φ_0. Then there is no infinite $\delta < \kappa$ such that each $\mathcal{B}(V)$ for $V \in \mathcal{B}_0$ has a maximal R-discrete set of size $\leq \delta$.*

Proof. Suppose otherwise and choose a large $\theta > \kappa$ so that $\mathbb{V}_\theta \models$ ZFC*, and choose an elementary substructure

$$A \prec \mathbb{V}_\theta$$

so that

$$|A| = \delta,$$

$$\delta + 1 \subset A,$$

and X, G, F_0, φ_0, etc., are in A. Let N be the transitive collapse of A and

$$\pi : N \to \mathbb{V}_\theta$$

the inverse of the collapsing map. Set $\hat{\mathbb{P}} = \pi^{-1}(\mathbb{P}_0)$ (where $\mathbb{P}_0$= Coll(ω, F_0)), $\hat\varphi_0 = \pi^{-1}(\varphi_0)$, and $\hat F_0 = \pi^{-1}(F_0)$, choose

$$\hat H \subset \hat{\mathbb{P}},$$

$$H \subset \mathbb{P}_0$$

to be $\mathbb{V}$-generic, and choose $\hat x \in N[\hat H]$ and $x \in \mathbb{V}[H]$ so that

$$N[\hat H] \models (\hat x \models \hat\varphi_0),$$

$$\mathbb{V}[H] \models (x \models \varphi_0)$$

(x and $\hat x$ exist for the reasons mentioned following 1.4). Note that for each $V \in \mathcal{B}_0$ there will be a maximal R-discrete subset of $\mathcal{B}(V)$ included in N. It suffices to show

$$\hat x E_G x.$$

As in the proof of 2.3 find $h_i, h'_i \in G_0$, $\psi_i \in F_0$, $\psi'_i \in \hat{F}_0$, $V_i, V'_i \in \mathcal{B}_0$, $W_i \in \mathcal{B}_0$ and $U_i \subset X$ basic open so that:

(i) $W_{i+1} \subset W_i$, $W_i = (W_i)^{-1}$, $d_G(W_i) < 2^{-i}$, $1_G \in W_i$; $U_{i+1} \subset U_i$, $d_X(U_i) < 2^{-i}$;

(ii) $h_{2i+1} = h_{2i}$; for all $g \in (W_{2i+1})^3 h_{2i}$

$$d_G(g, h_{2i}) < 2^{-i};$$

(iii) $h'_{2i+2} = h'_{2i+1}$; for all $g \in (W_{2i+2})^3 h'_{2i+1}$

$$d_G(g, h'_{2i+1}) < 2^{-i};$$

(iv) $h_{i+1} \in (W_i)^3 h_i$ and $h'_{i+1} \in (W_i)^3 h'_i$;

(v) $\mathbb{V}[H] \models (h_i \cdot x \models (\psi_i)^{\Delta V_i})$;

(vi) $N[\hat{H}] \models (h'_i \cdot \hat{x} \models (\psi'_i)^{\Delta V'_i})$;

(vii) $\mathbb{V} \models (\psi_i, V_i) \in \mathcal{B}(W_i)$;

(viii) $N \models (\psi'_i, V'_i) \in \mathcal{B}(W_i)$;

(ix) $(\pi(\psi'_i), V'_i) R (\psi_i, V_i)$;

(x) $h_i \cdot x, h'_i \cdot \hat{x} \in U_i$.

Granting that all this may be found, we finish quickly. By (ii), (iii), and (iv) we get $g_0 = \lim h_i$ and $g_1 = \lim h'_i$, whence

$$g_0 \cdot x = g_1 \cdot \hat{x}$$

by (x) and (i). This would contradict $\hat{\mathbb{P}}$ being too small to introduce a representative of $[x]_G$.

So instead suppose we have built V_j, V'_j, ψ_j and so on for $j \leq 2i$ and concentrate on trying to show that we may continue the construction up to $2i+2$.

First choose $W_{2i+1} \subset W_{2i}$ in accordance with (i) and (ii) and then for (x) and (i) choose $U_{2i+1} \subset U_{2i}$ containing $h_{2i} \cdot x (=_{df} h_{2i+1} \cdot x)$ with $d_X(U_{2i+1}) < 2^{-2i-1}$. Then by 1.2 we may choose $(V_{2i+1}, \psi_{2i+1}) \in \mathcal{B}(W_{2i+1})$ with

$$h_{2i} \cdot x \models (\psi_{2i+1})^{\Delta V_{2i+1}}.$$

On the N side we use the assumption on R to find V'_{2i+1} and ψ'_{2i+1} in N so that

$$N^{\hat{\mathbb{P}}} \models (\psi'_{2i+1}, V'_{2i+1}) \in \mathcal{B}(W_{2i+1})$$

and

$$(\pi(\psi'_{2i+1}), V'_{2i+1}) R (\psi_{2i+1}, V_{2i+1}).$$

Unwinding the definitions of $\mathcal{B}(W_{2i})$ and $\mathcal{B}(W_{2i+1})$ gives

$$\mathbb{V}^{\mathbb{P}_0} \models (y \models \varphi_0 \wedge \pi(\psi'_{2i})^{\Delta V'_{2i}}) \Rightarrow y \models ((\psi_{2i})^{\Delta V_{2i}} \wedge \pi(\psi'_{2i})^{\Delta V'_{2i}})^{\Delta W_{2i}},$$

$$\mathbb{V}^{\mathbb{P}_0} \models (y \models \varphi_0 \wedge (\psi_{2i})^{\Delta V_{2i}}) \Rightarrow y \models ((\psi_{2i})^{\Delta V_{2i}} \wedge (\psi_{2i+1})^{\Delta V_{2i+1}})^{\Delta W_{2i}},$$

$$\mathbb{V}^{\mathbb{P}_0} \models (y \models \varphi_0 \wedge (\psi_{2i+1})^{\Delta V_{2i+1}}) \Rightarrow y \models ((\psi_{2i+1})^{\Delta V_{2i+1}} \wedge \pi(\psi'_{2i+1})^{\Delta V'_{2i+1}})^{\Delta W_{2i+1}}.$$

After possibly tightening the $\tau_0(F_0)$-open set corresponding to $(\psi_{2i+1})^{\Delta V_{2i+1}}$ we can assume

$$(\psi_{2i+1})^{\Delta V_{2i+1}} \Rightarrow \dot{x} \in U_{2i+1},$$

and thus we have

$$\mathbb{V}^{\mathbb{P}_0} \models (y \models \varphi_0 \wedge \pi(\psi'_{2i})^{\Delta V'_{2i}}) \Rightarrow y \models ((\pi(\psi'_{2i+1}))^{\Delta V'_{2i+1}} \wedge \dot{x} \in U_{2i+1})^{\Delta (W_{2i})^3}.$$

Thus by elementarity of π we may find $h' \in (W_{2i})^3 \cap G_0$ so that $h' h'_{2i} \cdot \hat{x} \in U_{2i+1}$ and

$$N[\hat{H}] \models (h' h'_{2i} \cdot \hat{x} \models (\psi'_{2i+1})^{\Delta V'_{2i+1}}).$$

Then setting $h'_{2i+1} = h' h'_{2i}$ completes the transition from $2i$ to $2i+1$.

The step from $2i+1$ to $2i+2$ is similar, though easier since it will be a trivial task to find for $\mathbb{V}$ some $(\psi_{2i+2}, V_{2i+2}) \in \mathcal{B}(W_{2i+2})$ with $(\pi(\psi'_{2i+2}), V'_{2i+2})R(\psi_{2i+2}, V_{2i+2})$. □

We need a fact from infinitary model theory.

2.8. Theorem. *Let $\varphi \in \mathcal{L}_{\omega_1,\omega}$. Suppose*

$$N \models \varphi$$

and P is a predicate in the language of N with

$$|(P)^N| \geq \beth_{\aleph_1}.$$

Then φ has a model with generating indiscernibles in P.

More precisely there is a model M with language $\mathcal{L}^* \supset \mathcal{L}$, $\mathcal{L}^*$ having a new symbol $<$, along with new function symbols of the form $f_{\hat{\varphi}}$ for $\hat{\varphi}$ in the fragment of $\mathcal{L}_{\omega_1,\omega}$ generated by φ, and distinguished elements $(c_i)_{i\in\mathbb{N}}$, so that:

(i) *$(<)^M$ linearly orders $(P)^M$;*

(ii) *each $f_{\hat{\varphi}}$ is a Skolem function for $\hat{\varphi}$;*

(iii) *M is the Skolem hull of $\{c_i : i \in \mathbb{N}\}$ (under the functions of the form $f_{\hat{\varphi}}$);*

(iv) *each $c_i \in (P)^M$;*

(v) *for all ψ in the fragment of $\mathcal{L}^*_{\omega_1,\omega}$ generated by φ, for all $i_1 < i_2 < ... < i_n$ and $j_1 < ... < j_n$ in $\mathbb{N}$*

$$M \models \psi(c_{i_1}, c_{i_2}, ..., c_{i_n}) \Leftrightarrow \psi(c_{j_1}, c_{j_2}, ..., c_{j_n});$$

(vi) *$M \models \varphi$.*

(See the proof that $\beth_{\aleph_1}$ is the Hanf number of sentences in $\mathcal{L}_{\omega_1,\omega}$ from [16].) □

2.9. Theorem. *Let G be a Polish group for which $TVC(G, \undertilde{\Sigma}^1_1)$ fails. Then S_∞ divides G.*

Proof. Choose some Polish G-space X witnessing the failure of TVC$(G, \undertilde{\Sigma}^1_1)$. Following 1.5 we may find some $(\mathbb{P}, p, \sigma)$ introducing an equivalence class as in 1.4 that may not be produced by a forcing notion of size less than $\beth_{\aleph_1}$. Fix φ_0, $\mathcal{B}$, $\mathcal{B}_0$, F_0, G_0, etc., as in 1.4, so that in all generic extensions $\mathbb{V}[H]$ of $\mathbb{V}$

$$\mathbb{V}[H] \models \forall x \in X((p \Vdash_{\mathbb{P}} xE_G\dot{\sigma}) \Leftrightarrow y \models \varphi_0)$$

and

$$\mathbb{V} \models p \Vdash_{\mathbb{P}} (\dot{\sigma} \models \varphi_0).$$

Let $\mathbb{V}_\theta$ be large enough to contain X, G, φ_0, etc., and to satisfy ZFC*. By 2.7 choose $W \in \mathcal{B}_0$ and $P \subset \mathbb{V}_\theta \cap \mathcal{B}(W)$ of size $\beth_{\aleph_1}$ so that for all

$$(\psi, V), (\psi', V') \in P$$

we have

$$(\psi, V) \neq (\psi', V') \Rightarrow ((\psi, V)\neg R(\psi', V')).$$

By countability of $\mathcal{B}_0$ we may assume there is a single $V \in \mathcal{B}_0$ so that every element of P has the form

$$(\psi, V),$$

for some $\psi \in F_0$. Applying 2.8 to

$$N = (\mathbb{V}_\theta; \in, P, X, G, G_0, \varphi_0, ...)$$

we may obtain an ω-model M with indiscernibles $(\psi_q, V)_{q\in\mathbb{Q}}$ in P^M. Let $H \subset \mathrm{Coll}(\omega, (F_0)^M)$ be M-generic. Choose $x \in M[H]$ so that

$$M[H] \models (x \models \varphi_0).$$

All this granted, we may define G_1 to be the set of $\bar{g} \in G$ so that for all $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ with

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g}^{-1} \\ &\quad = RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}), \end{aligned}$$

and for $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ with

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g} \\ &\quad = RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}). \end{aligned}$$

G_1 is $\underset{\sim}{\Pi}^0_2$ in G, by 2.2 and since $\bar{g}$ is in G_1 if and only if the following four conditions hold:

(i) for all $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ with

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g}^{-1} \\ &\quad \cap RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}) \neq \emptyset; \end{aligned}$$

(ii) for all $q, r \in \mathbb{Q}$

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g}^{-1} \\ &\quad \cap RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}) \neq \emptyset \end{aligned}$$

implies

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g}^{-1} \\ &\quad = RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}); \end{aligned}$$

(iii) for all $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ with

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g} \\ &\quad \cap RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}) \neq \emptyset; \end{aligned}$$

(iv) for all $q, r \in \mathbb{Q}$

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_q)^{\Delta V})\})\bar{g} \\ &\quad \cap RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}) \neq \emptyset \end{aligned}$$

implies

$$\begin{aligned} &RO(\{g \in G_0 : M[H] \models (g\cdot x \models \psi_q^{\Delta V})\})\bar{g}^{-1} \\ &\quad = RO(\{g \in G_0 : M[H] \models (g\cdot x \models (\psi_r)^{\Delta V})\}). \end{aligned}$$

Since G_1 is a $\underset{\sim}{\Pi}^0_2$ subgroup of G, it must be closed.

For $g \in G_1$ we may define the permutation $\hat{\pi}(g)$ of $\mathbb{Q}$ by the specification that for all $q \in \mathbb{Q}$

$$(\hat{\pi}(g))(q) = r$$

if and only if r is as in part (i) of the definition of G_1.

Now let G_2 be the set of $g \in G_1$ such that $\hat{\pi}(g)$ defines an automorphism of the structure $(\mathbb{Q}, <)$. G_2 is a closed subgroup of G_1 and hence G. Noting that every order preserving permutation of the indiscernibles induces an automorphism of M and using the R-discreteness of the set $(P)^M$, the map

$$\hat{\pi} : G_2 \to \mathrm{Aut}(\mathbb{Q}, <)$$

is seen to be onto by 2.3. Then by 2.5 S_∞ divides G. □

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