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R-EQUIVALENCE IN SPINOR GROUPS

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The notion of R-equivalence in the set $X(F)$ of F-points of an algebraic variety X defined over a field F was introduced by Manin in [11] and studied for linear algebraic groups by Colliot-Thélène and Sansuc in [3]. For an algebraic group G defined over a field F, the subgroup $RG(F)$ of *R-trivial elements* in the group $G(F)$ of all F-points is defined as follows. An element g belongs to $RG(F)$ if there is a rational morphism $f : \mathbb{A}^1_F \to G$ over F, defined at the points 0 and 1 such that $f(0) = 1$ and $f(1) = g$. In other words, g can be connected with the identity of the group by the image of a rational curve. The subgroup $RG(F)$ is normal in $G(F)$ and the factor group $G(F)/RG(F) = G(F)/R$ is called the *group of R-equivalence classes.*

The group of R-equivalence classes is very useful while studying the rationality problem for algebraic groups, the problem to determine whether the variety of an algebraic group is rational or stably rational. We say that a group G is *R-trivial* if $G(E)/R = 1$ for any field extension E/F. If the variety of a group G is stably rational over F, then G is R-trivial. Thus, if one can establish non-triviality of the group of R-equivalence classes $G(E)/R$ just for one field extension E/F, the group G is not stably rational over F.

The group of R-equivalence classes for adjoint semisimple classical groups was computed in [15]. This computation was used to obtain examples of non-rational adjoint algebraic groups.

Consider simply connected algebraic groups of classical types. Let $G = \mathbf{GL}_1(A)$ be the algebraic group of invertible elements of a central simple F-algebra A of dimension n^2, and let $H = \mathbf{SL}_1(A)$ be the subgroup in G of the reduced norm 1 elements. The group H is a simply connected group of inner type A_{n-1}. V. Voskresenskiĭ [29] has shown that the natural homomorphism $A^\times \to K_1(A)$ induces an isomorphism

$$G(F)/RH(F) \xrightarrow{\sim} K_1(A).$$

Thus the group of R-equivalence classes $H(F)/R$ is the reduced Whitehead group $\mathrm{SK}_1(A)$. If index $\mathrm{ind}(A)$ is squarefree, then $\mathrm{SK}_1(A \otimes_F L) = 1$ for any field extension, i.e., the group H is R-trivial [5, §23]. If $\mathrm{ind}(A)$ is not squarefree, A. Suslin conjectured in [26] that H is not R-trivial (and therefore, is not stably rational).

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This conjecture is still open. The only known case [12] is when $\operatorname{ind}(A)$ is divisible by 4.

In the outer case A_{n-1}, a simply connected algebraic group is isomorphic to the *special unitary group* $\mathbf{SU}(B,\tau)$ of a central simple algebra B of dimension n^2 over a quadratic extension of F with involution τ of the second kind [9, §26]. The group of R-equivalence classes of $\mathbf{SU}(B,\tau)$ was computed in [2].

Simply connected groups of type C_n, the symplectic groups, are rational [30, 2, Prop. 2.4]. In the remaining classical cases B_n and D_n simply connected groups are the twisted forms of the spinor groups of non-degenerate quadratic forms. In the present paper we give a computation of the group of R-equivalence classes in these remaining cases.

To describe the main result of the paper, for an algebraic variety X over a field F, denote $A_0(X,K_1)$ to be the cokernel of the residue homomorphism

$$\coprod_{x\in X_{(1)}} K_2F(x)^\times \xrightarrow{\partial} \coprod_{x\in X_{(0)}} K_1F(x),$$

where $X_{(p)}$ is the set of all points in X of dimension p. Let (V,q) be a non-degenerate quadratic space over F. In [23], M. Rost defined a natural homomorphism

$$\Gamma^+(V,q)\to A_0(X,K_1),$$

where $\Gamma^+(V,q)$ is the special Clifford group of (V,q) and X is the projective quadric hypersurface given by q. We give another definition of Rost's homomorphism and prove that it induces isomorphisms (Theorem 6.2):

$$\Gamma^+(V,q)/R\operatorname{Spin}(V,q)\simeq A_0(X,K_1),$$

$$\operatorname{Spin}(V,q)/R\simeq \overline{A}_0(X,K_1),$$

where $\overline{A}_0(X,K_1)$ is the kernel of the norm homomorphism

$$N^1: A_0(X,K_1)\to K_1(F)=F^\times.$$

The result allows us to use the machinery of algebraic K-theory while dealing with the groups of R-equivalence classes.

The rationality of the group $\mathbf{Spin}(V,q)$ implies that the group $\overline{A}_0(X,K_1)$ is trivial. In particular, $\mathbf{Spin}(V,q)$ is rational if $q=f\perp g$, where f is a Pfister neighbor and $\dim g\le 2$ [16, Th. 6.4]. Note that triviality of $\overline{A}_0(X,K_1)$ in the case when q is a Pfister neighbor (M. Rost [23]) was used by V. Voevodsky in the proof of the Milnor Conjecture.

The main result of the paper (Theorem 5.10) can be applied for some other classical groups. In particular we recover the isomorphism (Theorem 6.1)

$$K_1(A)=\mathrm{GL}_1(A)/R\,\mathrm{SL}_1(A)\simeq A_0(X,K_1)$$

for a central simple algebra A and the Severi-Brauer variety X corresponding to A. This isomorphism was originally obtained in [18].

Another application deals with some twisted forms of spinor groups. Let A be a central simple algebra of even dimension, and let (σ,f) be a quadratic pair on A

[9, §5.B]. If $\operatorname{ind}(A) \leq 2$, there are isomorphisms (Theorem 6.5)

$$\Gamma(A,\sigma,f)/R\operatorname{Spin}(A,\sigma,f) \simeq A_0(X,K_1),$$
$$\operatorname{Spin}(A,\sigma,f)/R \simeq \overline{A}_0(X,K_1),$$

where $\Gamma(A,\sigma,f)$ is the Clifford group of the quadratic pair and X is the involution variety. The group $\mathbf{Spin}(A,\sigma,f)$ is a general twisted form of the classical spinor group of type D_n, $n \neq 4$. With the restriction on the index of A the result covers all simply connected groups of type D_n with odd n.

The paper is organized as follows.

In the first section we discuss the machinery we use in the paper: R-equivalence, cycle modules, invariants of algebraic groups, etc. In the second section the main objects of the paper are introduced: a reductive group G together with a character $\rho : G \to \mathbb{G}_m$ and a smooth projective variety X satisfying certain conditions. We will keep these assumptions and notation throughout the paper. We also consider three basic examples.

In the next section we define a homomorphism

$$\alpha_F : G(F)/RH(F) \to A_0(X,K_1),$$

where $H = \ker \rho$. The following two sections are devoted to the proof of the fact that α_F is an isomorphism (Theorem 5.10). First of all, we develop an evaluation technique (section 4) and then use it to construct the inverse of α_F, a homomorphism β_F.

In the last section we consider applications. First of all, we formulate the main result in the three special cases corresponding to examples given in section 2. At the end we consider spinor groups of "generic" quadratic forms. In particular we exhibit examples of non-rational spinor groups of quadratic forms of every dimension ≥ 6. Some examples of non-rational spinor groups in dimensions $\equiv 2$ (mod 4) were given by V. Platonov in [21]. Note that the spinor groups of quadratic forms of dimension < 6 are rational since they are the groups of rank ≤ 2.

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Notation. The letter F always denotes a perfect infinite field.

We denote the Milnor K-groups of F by $K_n(F)$ [19].

A *variety* is a separated scheme of finite type over a field.

For an algebraic variety X over a field F and a commutative F-algebra R, we denote the set of R-points $\operatorname{Mor}_F\big(\operatorname{Spec}(R),X\big)$ of X by $X(R)$. We identify the set $X(F)$ of *rational points* of X with a subset of X.

For a variety X over F, denote the set of all points in X of dimension p by $X_{(p)}$. If E/F is a field extension, X_E denotes $X \times_{\operatorname{Spec} F} \operatorname{Spec} E$. For any $x \in X$, $F(x)$ is the residue field of x. The degree $\deg x$ of a closed point x is $[F(x):F]$. The F-algebra of regular functions on X is $F[X]$. If X is irreducible and reduced, $F(X)$ is the function field of X.

An algebraic group G over F is a smooth affine group scheme over F. We consider G as a functor $E \mapsto G(E)$ from the category of field extensions of F to the category of groups.

In order to distinguish between algebraic groups and their groups of F-points, we use symbols in bold for certain algebraic groups. For example, $\mathbf{Spin}(V,q)$ is an algebraic group, but $\operatorname{Spin}(V,q)$ is the (abstract) group of F-points $\mathbf{Spin}(V,q)(F)$.

For an algebraic torus T denote the group of co-characters $\operatorname{Hom}(\mathbb{G}_m,T)$ by T_*.

1. Preliminaries

In this section we introduce necessary definitions and prove some auxiliary results.

1.1. **R-equivalence.** Let Y be an irreducible variety over a field F, and let $L = F(Y)$ be the function field of Y. For a point $y \in Y$ denote the local ring of y on Y by $\mathcal{O}_y$.

Let X be another algebraic variety defined over F. Clearly $X(\mathcal{O}_y)$ is a subset in $X(L)$. We say that an element $u \in X(L)$ is *defined at y* if $u \in X(\mathcal{O}_y)$. If u is defined at y, then the image $u(y)$ under the map $X(\mathcal{O}_y) \to X\big(F(y)\big)$, induced by the natural surjection $\mathcal{O}_y \to F(y)$, is called the *value of u at y*.

We consider the rational function field $F(t)$ as the function field of the affine line $\mathbb{A}^1_F$ or the projective line $\mathbb{P}^1_F$. We say that an element $u = u(t) \in X\big(F(t)\big)$ is *defined at a rational point $a \in \mathbb{P}^1_F(F)$* if it is defined at the corresponding point of $\mathbb{P}^1_F$. We denote the value at a by $u(a)$.

Let G be an algebraic group defined over F. An element $g \in G(F)$ is called *R-trivial* if there is $g(t) \in G\big(F(t)\big)$ defined at the points $t = 0$ and $t = 1$ such that $g(0) = 1$ and $g(1) = g$ [3]. In other words, there exists a rational morphism $f : \mathbb{A}^1_F \to X$ defined at $t = 0$ and $t = 1$ such that $f(0) = 1$ and $f(1) = g$.

The set of all R-trivial elements in $G(F)$ is denoted by $RG(F)$. It is a normal subgroup in $G(F)$. The factor group $G(F)/R = G(F)/RG(F)$ is called the *group of R-equivalence classes*. For a field extension E/F we define the group $G(E)/R$ as the group of R-equivalence classes of the group G_E over E. We say that a group G is *R-trivial* if $G(E)/R = 1$ for any field extension E/F.

The following properties of R-equivalence can be found in [3], [6, Lemme 2.1] and [7, Lemme II.1.1].

1. Let $g(t) \in G\big(F(t)\big)$ be defined at two rational points $t = a$ and $t = b$. Then the value $g(a) \cdot g(b)^{-1}$ belongs to $RG(F)$.
2. If G is stably rational, i.e., $G \times \mathbb{A}^n_F$ is birationally isomorphic to $\mathbb{A}^m_F$ for some n and m, then G is R-trivial.
3. The functor $E \mapsto G(E)/R$ is *rigid*, i.e., for a purely transcendental field extension L/F the natural homomorphism $G(F)/R \to G(L)/R$ is an isomorphism.

1.2. **Cycle modules.** Cycle modules were introduced by M. Rost in [22]. A *cycle module* M over a field F is an object function $E \mapsto M_*(E)$ from the category of field extensions of F to the category of $\mathbb{Z}$-graded abelian groups together with some data and rules [22, §2]. The data includes a graded module structure on M under the Milnor ring $K_*(F)$, a degree 0 homomorphism $\alpha_* : M(E) \to M(L)$ for any field homomorphism $\alpha : E \to L$, a degree 0 homomorphism (norm map) $\alpha^* : M(L) \to M(E)$ for any finite field homomorphism $\alpha : E \to L$ over F and also a degree -1 *residue homomorphism* $\partial_v : M(E) \to M\big(\kappa(v)\big)$ for a discrete, rank one, valuation v on E (here $\kappa(v)$ is the residue field). For example, the Milnor K-groups K_* for field extensions of F form a cycle module over F.

For a variety X over F and a cycle module M over F one can define a complex [22, §5]

$$\cdots \to C_p(X; M, n) \xrightarrow{\partial_p} C_{p-1}(X; M, n) \to \dots,$$

where

$$C_p(X; M, n) = \coprod_{x \in X_{(p)}} M_{n+p}\bigl(F(x)\bigr). \tag{1.1}$$

The p-th homology group of this complex is denoted $A_p(X, M_n)$.

In particular if $M_* = K_*$ is the cycle module of Milnor K-groups, we get the *K-homology groups* $A_p(X, K_n)$. In particular,

$$\mathrm{CH}_p(X) = A_p(X, K_{-p})$$

are the *Chow groups* of classes of cycles on X of dimension p.

Example 1.1. Let X be a variety over F. As shown in [22, §7], the function $E \mapsto A_0(X_E, K_*)$ is a cycle module over F denoted $A_0[X, K_*]$. We will be using this cycle module later to define invariants of certain algebraic groups.

Let M be a cycle module over F, and let Y be an irreducible variety over F of dimension d, with $y \in Y$ a point of codimension 1. We say that an element $u \in M_n\bigl(F(Y)\bigr)$ is *unramified at y* if u belongs to the kernel of the homomorphism induced by the differential ∂_d in the complex (1.1):

$$\partial_{d,y} : M_n\bigl(F(Y)\bigr) \to M_{n-1}\bigl(F(y)\bigr).$$

The subgroup of all elements in $M_n\bigl(F(Y)\bigr)$ unramified at all points of codimension 1 in Y is denoted $A^0(Y, M_n)$. Clearly,

$$A^0(Y, M_n) = A_d(Y, M_{n-d}).$$

For a morphism of varieties $f : Y' \to Y$ with smooth Y there is a well-defined *inverse image homomorphism* [22, §12]

$$f^* : A^0(Y, M_n) \to A^0(Y', M_n).$$

Let y be a smooth point of Y. The *evaluation homomorphism*

$$A^0(Y, M_d) \to M_d\bigl(F(y)\bigr), \quad v \mapsto v(y)$$

is the restriction to $A^0(Y, M_d)$ of the inverse image homomorphism

$$i^* : A^0(U, M_d) \to A^0\bigl(\operatorname{Spec} F(y), M_d\bigr) = M_d\bigl(F(y)\bigr),$$

where U is a smooth open neighborhood of y and $i : \operatorname{Spec} F(y) \to U$ is the canonical morphism.

We say that an element $v \in M_d\bigl(F(Y)\bigr)$ is *defined at $y \in Y$* if there is an open neighborhood U of y such that $v \in A^0(U, M_d)$. If v is defined at a smooth point y, then the value $v(y)$ is well defined.

1.3. **Invariants of algebraic groups.** Let G be a connected algebraic group over a field F, and let M be a cycle module over F. For any $d \in \mathbb{Z}$, we consider M_d as a functor from the category of field extensions of F to the category of abelian groups. An *invariant of G in M of dimension d* is a morphism (natural transformation) of functors $G \to M_d$ [17]. In other words, an invariant is a collection of compatible group homomorphisms

$$G(E) \to M_d(E)$$

for all field extensions E/F.

Consider two projections and multiplication morphisms $p_1, p_2, m : G \times G \to G$. An element $\alpha \in A^0(G, M_d)$ is called *multiplicative* if

$$p_1^*(\alpha) + p_2^*(\alpha) = m^*(\alpha) \in A^0(G \times G, M_d).$$

By [17, 2.1], a multiplicative element α of the group $A^0(G, M_d)$ defines an invariant u of G in M of dimension d as follows. For a point $g \in G(E)$, i.e., for a morphism $\operatorname{Spec}(E) \to G$, the value $u(g) \in M_d(E)$ is the image of α under the inverse image homomorphism

$$A^0(G, M_d) \to A^0(\operatorname{Spec} E, M_d) = M_d(E).$$

Conversely, any invariant u of G in M of dimension d can be obtained from a multiplicative element α of the group $A^0(G, M_d)$ this way.

Invariants are compatible with the evaluations at smooth points.

Proposition 1.2. *Let Y be an irreducible variety over F, and let $y \in Y$ be a smooth point. Then for any invariant u of an algebraic group G in a cycle module M of dimension d and any $g \in G\big(F(Y)\big)$ defined at y, the element $u_{F(Y)}(g) \in M_d\big(F(Y)\big)$ is defined at y and $\big(u_{F(Y)}(g)\big)(y) = u_{F(y)}\big(g(y)\big)$.*

Proof. Since g is defined at y, there is an open smooth neighborhood U of y contained in the set of definition of g. We consider g as a morphism $g : U \to G$. Let $i : \operatorname{Spec} F(y) \to U$ be the natural morphism. Denote by $\alpha \in A^0(G, M_d)$ the multiplicative element corresponding to the invariant u. Then $u_{F(Y)}(g) = g^*(\alpha)$ belongs to $A^0(U, M_d)$ and hence is defined at y. Moreover, the value $u_{F(Y)}(g)(y)$ is equal to $i^* \circ g^*(\alpha) = g(y)^*(\alpha) = u_{F(y)}\big(g(y)\big)$ since $g \circ i = g(y)$. □

1.4. **Index of a character.** Let $\rho : G \to \mathbb{G}_m$ be a non-trivial character of a reductive group G. The *index* $\operatorname{ind} \rho$ of ρ is the least positive integer in the image of the composition

$$G\big(F((t))\big) \xrightarrow{\rho} F((t))^\times \xrightarrow{v} \mathbb{Z},$$

where v is the discrete valuation of the field of formal Laurent series $F((t))$. By [14, Prop. 4.2], $\operatorname{ind} \rho$ is the smallest $n \in \mathbb{N}$ such that there exists a group homomorphism $\nu : \mathbb{G}_m \to G$ with the composition $\rho \circ \nu$ being the n-th power endomorphism of $\mathbb{G}_m$. In other words, $n\mathbb{Z}$ is the image of the induced homomorphism of co-character groups $T_* \to (\mathbb{G}_m)_* = \mathbb{Z}$ where T is a maximal split torus of G.

In the following proposition we collect some properties of the index.

Proposition 1.3. 1. *For a finite field extension L/F,* $\operatorname{ind} \rho$ *divides the product* $[L : F] \cdot \operatorname{ind} \rho_L$.

2. *The following three conditions are equivalent:*

(a) $\operatorname{ind} \rho = 1$*;*

(b) *the homomorphism ρ splits;*

(c) *for any field extension E/F, the homomorphism $\rho(E) : G(E) \to E^\times$ is surjective.*

3. *For any purely transcendental extension E/F,* $\operatorname{ind} \rho = \operatorname{ind} \rho_E$.

Proof. 1. Let $n = \operatorname{ind} \rho_L$. Then t^n belongs to the image of $\rho\big(L(t)\big) : G\big(L(t)\big) \to L(t)^\times$. By the norm principle for monic polynomials [14, Cor. 4.5], $t^{[L:F] \cdot n}$ belongs to the image of $\rho\big(F(t)\big)$, hence $\operatorname{ind} \rho$ divides $[L : F] \cdot n$.

2. Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To show (c) $\Rightarrow$ (a), we take $E = F((t))$.

3. The group T_* does not change under purely transcendental extensions. □

1.5. **Norms.** Let X be a complete variety over F. For any $m \geq 0$ there is a well-defined *norm homomorphism* [22]

$$N^m : A_0(X, K_m) \to K_m(F), \quad N\big(\sum (x, u_x)\big) = \sum N_{F(x)/F}(u_x).$$

In particular, if $m = 0$, the image of

$$N^0 : \mathrm{CH}_0(X) = A_0(X, K_0) \to K_0(F) = \mathbb{Z}$$

is equal to $n\mathbb{Z}$, where $n = \gcd(\deg x)$ for all closed points $x \in X$.

We denote the kernel of N^m by $\overline{A}_0(X, K_m)$.

2. Assumptions

In this section we introduce two objects: a character ρ of a (connected) reductive group G over F and a variety X over F satisfying certain properties. We will keep this notation and the assumptions throughout the paper.

Let $\rho : G \to \mathbb{G}_m$ be a non-trivial character of a reductive group G. We assume that G is a rational group and the group scheme $H = \mathrm{Ker}(\rho)$ is smooth. The subgroup $RH(F) \subset G(F)$ is normal by [2, Lemma 1.2]. Hence the homomorphism of the groups of F-points $\rho(F) : G(F) \to F^\times$ induces a homomorphism

$$\rho^F : G(F)/RH(F) \to F^\times$$

with the kernel $H(F)/R$.

If ρ satisfies the equivalent conditions of Proposition 1.3.2, then ρ^F is an isomorphism. Indeed, ρ splits, hence the variety of G is isomorphic to $H \times \mathbb{G}_m$. In particular, the group H is stably rational and $RH(F) = H(F)$. Therefore, ρ^F is injective and hence is an isomorphism.

Assume that there is a complete smooth variety X over F satisfying the following conditions:

1. For any field extension L/F, the norm homomorphism

$$N^0 = N^0_L : A_0(X_L, K_0) \to K_0(L) = \mathbb{Z}$$

is injective, i.e., $\overline{A}_0(X_L, K_0) = 0$.

2. For any field extension L/F such that $X(L) \neq \emptyset$, the norm homomorphism

$$N^1 = N^1_L : A_0(X_L, K_1) \to K_1(L) = L^\times$$

is an isomorphism.

3. For any field extension L/F,

$$\operatorname{ind} \rho_L = \gcd_{x \in (X_L)_{(0)}} (\deg x).$$

In particular, if $X(L) \neq \emptyset$, then $\operatorname{ind} \rho_L = 1$.

Proposition 2.1. *For any field extension L/F, the image of $\rho(L) : G(L) \to L^\times$ coincides with the image of the norm homomorphism*

$$N^1 = N^1_L : A_0(X_L, K_1) \to K_1(L) = L^\times.$$

Proof. Let $x \in (X_L)_{(0)}$ and $E = L(x)$. Since $X(E) \neq \emptyset$, condition 3 for X implies that $\rho(E) : G(E) \to E^\times$ is surjective. It follows from the rationality of G and the norm principle [14, Th. 3.9] for the field extension E/L that $N_{E/L}(E^\times) \subset \mathrm{Im}\big(\rho(L)\big)$ and hence $\mathrm{Im}(N^1_L) \subset \mathrm{Im}\big(\rho(L)\big)$.

Conversely, by [14, Th. 4.3], the image of $\rho(L)$ is the product of the subgroups $B(E) \overset{def}{=} N_{E/L}(E^\times)^{\operatorname{ind}\rho_E}$ for all finite field extensions E/L. By property 3 of X, $(E^\times)^{\operatorname{ind}\rho_E}$ is contained in the image of N^1_E, hence $B(E) \subset \operatorname{Im}(N^1_L)$ and $\operatorname{Im}\big(\rho(L)\big) \subset \operatorname{Im}(N^1_L)$. □

2.1. **Examples.** We consider three basic examples.

Example 2.2. Let A be a central simple algebra over F of dimension n^2, and let $G = \mathbf{GL}_1(A)$ be the group of invertible elements in A. More precisely, $G(R) = (A \otimes_F R)^\times$ for any commutative unitary F-algebra R. The group G is rational since it is an open subvariety of the affine space of A.

The reduced norm homomorphism [5, §22] gives rise to a character $\rho : G \to \mathbb{G}_m$. The kernel H of ρ is the special linear group $\mathbf{SL}_1(A)$. It is an absolutely simple simply connected algebraic group of type A_{n-1} [9, Th. 26.9].

Let X be the Severi-Brauer variety of right ideals in A of dimension n [9, §1.C]. Note that $X(E) \neq \emptyset$ for a field extension E/F if and only if the algebra $A_E = A \otimes_F E$ splits.

The variety X satisfies the following properties:

1. The injectivity of N^0 for X is proved in [20] (see also [18]).

2. If $X(L)$ is not empty, $X_L \simeq \mathbb{P}^{n-1}_L$. For projective spaces the norm homomorphism N^1_L is an isomorphism by [24].

3. The index $\operatorname{ind}\rho_L$ is equal to the index of the algebra A_L. On the other hand, for any closed point $x \in X_L$, the field $L(x)$ splits A_L, hence $\operatorname{ind}(A_L)$ divides $\deg(x)$. If E/L is a splitting field of A_L of degree $\operatorname{ind}(A_L)$, then $X(E) \neq \emptyset$; thus, $\operatorname{ind}(A_L) = \gcd_{x \in (X_L)_{(0)}}(\deg x)$.

Example 2.3. Let (V, q) be a non-degenerate quadratic space over F of dimension $n \geq 2$, and let G be the *special Clifford group* $\mathbf{\Gamma}^+(V, q)$ of the quadratic space (V, q) [9, 23.A]. The character ρ is the spinor norm homomorphism $G \to \mathbb{G}_m$. There is an exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathbf{O}^+(V, q) \to 1,$$

where $\mathbf{O}^+(V, q)$ is the special orthogonal group of (V, q). By Hilbert Theorem 90, this sequence splits rationally, hence G is birationally equivalent to $\mathbf{O}^+(V, q) \times \mathbb{G}_m$ and therefore is rational since the orthogonal group is rational ([30] if $\operatorname{char}(F) \neq 2$ and [2, Prop. 2.4] in general).

The kernel H of ρ is the spinor group $\mathbf{Spin}(V, q)$. It is an absolutely simple simply connected algebraic group of type B_m if $n = 2m + 1$ and D_m if $n = 2m > 4$ (semisimple if $n = 4$).

Let X be the projective quadric hypersurface given in the projective space $\mathbb{P}(V)$ by the equation $q = 0$. The variety X satisfies the following three properties:

1. The injectivity of N^0 for X is proved in [8] and [27].

2. We need to show that if q is isotropic, then $N^1 : A_0(X, K_1) \to F^\times$ is an isomorphism. As shown in [8], there is an open subset $U \subset X$ isomorphic to an affine space, a rational point $z \in Z = X \setminus U$ and a vector bundle $Z \setminus \{z\} \to X'$ where X' is a subquadric in X of codimension 2. The variety Z is a singular projective quadric corresponding to a degenerate quadratic form with one-dimensional radical. Since U is an affine space, by [25], $A_0(X, K_1) = A_0(Z, K_1)$. The statement follows from

Lemma 2.4. *The norm map $N^m : A_0(Z, K_m) \to K_m(F)$ is an isomorphism.*

Proof. Since Z has a rational point, N^m is surjective. In order to prove that N^m is an isomorphism, it suffices to show that $A_0(Z, K_m) = K_m(F) \cdot [z]$. By the vector bundle theorem [25],

$$A_0(Z \setminus \{z\}, K_m) = A_{-1}(X', K_{m+1}) = 0.$$

The desired equality follows from exactness of the localization sequence

$$K_m(F) = A_0(z, K_m) \to A_0(Z, K_m) \to A_0(Z \setminus \{z\}, K_m) = 0.$$

□

3. If q_L is anisotropic, then $\operatorname{ind} \rho_L = 2 = \gcd_{x \in (X_L)_{(0)}} (\deg x)$ since odd degree field extensions of L do not split q_L by Springer's Theorem. If q_L is isotropic, then the spinor norm is surjective and clearly, $\operatorname{ind} \rho_L = 1 = \gcd_{x \in (X_L)_{(0)}} (\deg x)$.

Example 2.5. Let A be a central simple algebra over F of dimension $n^2 = (2m)^2$, $n \geq 4$. Denote by $\operatorname{Sym}(A, \sigma)$ (resp. $\operatorname{Skew}(A, \sigma)$) the space of symmetric (resp. skew-symmetric) elements in A under σ. A *quadratic pair* on A is a couple (σ, f), where σ is an involution of the first kind on A and $f : \operatorname{Sym}(A, \sigma) \to F$ is a linear map such that:

1. $\dim_F \operatorname{Sym}(A, \sigma) = n(n+1)/2$ and $\operatorname{Trd}_A\bigl(\operatorname{Skew}(A, \sigma)\bigr) = 0$, where Trd is the reduced trace map;
2. $f\bigl(a + \sigma(a)\bigr) = \operatorname{Trd}_A(a)$ for all $a \in A$ [9, §5.B].

Let G be the *Clifford group* $\mathbf{\Gamma}(A, \sigma, f)$ of (A, σ, f), and let $\rho : G \to \mathbb{G}_m$ be the spinor norm homomorphism. Similarly to Example 2.3, it follows from the exactness of the sequence [9, 23.B]

$$1 \to \mathbb{G}_m \to G \to \mathbf{O}^+(A, \sigma, f) \to 1,$$

that the group G is rational.

The kernel H of ρ is the spinor group $\mathbf{Spin}(A, \sigma, f)$ [9, p. 351]. It is an absolutely simple simply connected algebraic group of type D_m if $m > 2$ (semisimple if $m = 2$). In fact, any simply connected group of type D_m, $m \neq 4$, is the spinor group of some quadratic pair by [9, Th. 26.15].

If A splits, i.e., $A = \operatorname{End}_F(V)$ for a vector space V over F, the quadratic pair is given by a non-degenerate quadratic form q on V [9, Prop. 5.11] and the groups G and H coincide with those in Example 2.3.

A right ideal $I \subset A$ is called *isotropic* with respect to the quadratic pair (σ, f) [9, Def. 6.5] if the following conditions hold:

1. $\sigma(I) \cdot I = 0$;
2. $f(a) = 0$ for all $a \in I \cap \operatorname{Sym}(A, \sigma)$.

We say that a quadratic pair (σ, f) is *isotropic* if A contains a non-zero isotropic ideal.

The variety $X = I(A, \sigma, f)$ of all isotropic ideals of dimension n is called the *involution variety* of the quadratic pair (σ, f). (In the case $\operatorname{char}(F) \neq 2$, the involution varieties have been introduced in [28].) If A splits, X coincides with the projective quadric hypersurface considered in Example 2.3.

Assume now that $\operatorname{ind}(A) \leq 2$. The variety X satisfies the following properties:

1. The injectivity of N^0 for X is proved in [13].
2. If $X(L) \neq \emptyset$ for a field extension L/F, the variety X_L is a projective isotropic quadric considered in Example 2.3, hence the condition holds.

3. If A_L splits and (σ_L, f_L) is isotropic, the spinor norm is surjective and therefore, $\operatorname{ind}\rho_L = 1 = \gcd_{x\in(X_L)_{(0)}}(\deg x)$. Otherwise, $\operatorname{ind}\rho_L = 2$ and for any closed point $x \in X_L$, the degree of x is even. Since $\operatorname{ind}(A) \le 2$, by [13], there is a closed point in X of degree 2 and therefore $\gcd_{x\in(X_L)_{(0)}}(\deg x) = 2$. Note that this condition does not hold if $\operatorname{ind}(A) > 2$.

Remark 2.6. The statements in [13] and [14] quoted in Example 2.5 are proved under assumption $\operatorname{char}(F) \neq 2$. But the proofs carry over in the general case too.

2.2. **More norms.** The group G is rational. As shown in [2, Sect. 4], for any finite field extension L/F there is a well-defined *norm homomorphism*

$$\mathcal{N}_{L/F} : G(L)/RH(L) \to G(F)/RH(F).$$

The homomorphism ρ commutes with the norms, i.e., the following diagram commutes:

$$\begin{tikzcd} G(L)/RH(L) \arrow[r, "\rho^L"] \arrow[d, "\mathcal{N}_{L/F}"'] & L^\times \arrow[d, "N_{L/F}"] \\ G(F)/RH(F) \arrow[r, "\rho^F"] & F^\times. \end{tikzcd}$$

The norm $\mathcal{N}$ is transitive [2, Sect. 4]: for a finite field extension E/L, $\mathcal{N}_{E/F} = \mathcal{N}_{L/F} \circ \mathcal{N}_{E/L}$.

3. Invariant α

3.1. **Definition of the invariant.** We would like to define a one-dimensional invariant α of G in the cycle module $A_0[X, K_*]$, i.e., to construct a collection of compatible group homomorphisms

$$\alpha_E : G(E) \to A_0(X_E, K_1)$$

for all field extensions E/F, so α is an invariant of dimension 1. By subsection 1.3, such an invariant can be determined by an unramified multiplicative element of the group $A_0(X_{F(G)}, K_1)$.

The character ρ can be considered as a regular function in $F(G)^\times$. Clearly, ρ is the image of the generic element ξ of G under

$$\rho\big(F(G)\big) : G\big(F(G)\big) \to F(G)^\times.$$

Hence, by Proposition 2.1, there is an $\alpha \in A_0(X_{F(G)}, K_1)$ such that $N^1_{F(G)}(\alpha) = \rho$.

Lemma 3.1. *Any such α is unramified with respect to all codimension 1 points of G, i.e., $\alpha \in A^0(G, A_0[X, K_1])$.*

Proof. Let x be a point of G of codimension 1. In the commutative diagram

$$\begin{tikzcd} A_0(X_{F(G)}, K_1) \arrow[r, "N^1_{F(G)}"] \arrow[d, "\partial_x"'] & F(G)^\times \arrow[d, "v_x"] \\ A_0(X_{F(x)}, K_0) \arrow[r, "N^0_{F(x)}"] & \mathbb{Z} \end{tikzcd}$$

the bottom homomorphism is injective by property 1 of X. Since $\rho \in F(G)$ is an invertible regular function, $v_x(\rho) = 0$, hence $\partial_x(\alpha) = 0$, i.e., α is unramified. $\square$

Thus, Lemma 3.1 implies that for any point $g \in G(F)$ and any α as above there is a well-defined value $\alpha(g) \in A_0(X, K_1)$.

The element α is not uniquely determined. To make a canonical choice of α we need the following

Lemma 3.2. *The functor $E \mapsto \overline{A}_0(X_E, K_1)$ is rigid, i.e., the natural homomorphism*

$$\overline{A}_0(X, K_1) \to \overline{A}_0(X_{F(t)}, K_1)$$

is an isomorphism.

Proof. Using the formalism of cycle modules [22, Prop. 2.2], we get the following commutative diagram with an exact sequence in the top row:

$$\begin{tikzcd} 0 \arrow[r] & A_0(X, K_1) \arrow[r] \arrow[d, "N_F^1"] & A_0(X_{F(t)}, K_1) \arrow[r] \arrow[d, "N_{F(t)}^1"] & \coprod_{p\in\mathbb{A}^1} A_0(X_{F(p)}, K_0) \arrow[r] \arrow[d, "\coprod N_{F(p)}^0"] & 0 \\ 0 \arrow[r] & F^\times \arrow[r] & F(t)^\times \arrow[r] & \coprod_{p\in\mathbb{A}^1} \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The result follows by the Snake Lemma and injectivity of the right vertical arrow (property 1 of X). $\square$

Corollary 3.3. *If Y is a rational irreducible variety, the natural homomorphism*

$$\overline{A}_0(X, K_1) \to \overline{A}_0(X_{F(Y)}, K_1)$$

is an isomorphism.

Since G is a rational group, it follows from the corollary that the element α is uniquely determined modulo $\overline{A}_0(X, K_1)$, hence it is uniquely determined by the value $\alpha(1) \in \overline{A}_0(X, K_1)$. Therefore, there exists a unique α such that $\alpha(1) = 0$. We will assume such a normalization.

Lemma 3.4. *The element α is multiplicative.*

Proof. We need to show that the element

$$\kappa \overset{def}{=} p_1^*(\alpha) + p_2^*(\alpha) - m^*(\alpha)$$

is trivial in $A_0(X_{F(G\times G)}, K_1)$. Since the function $\rho \in F(G)^\times$ is multiplicative, we have

$$\begin{aligned} N^1_{F(G\times G)}(\kappa) &= p_1^*\big(\rho(\xi)\big) \cdot p_2^*\big(\rho(\xi)\big) \cdot m^*\big(\rho(\xi)\big)^{-1} \\ &= \rho(\xi \times 1) \cdot \rho(1 \times \xi) \cdot \rho(\xi \times \xi)^{-1} = 1, \end{aligned}$$

i.e., $\kappa \in \overline{A}_0(X_{F(G\times G)}, K_1)$. But $G \times G$ is a rational group, hence by Corollary 3.3, $\kappa \in \overline{A}_0(X, K_1)$, i.e., κ is constant. Finally, the element κ is normalized, $\kappa(1) = 0$, hence $\kappa = 0$. $\square$

Thus, the element α defines an invariant of the group G which we also denote α, so that we have a collection of compatible homomorphisms

$$\alpha_E : G(E) \to A_0(X_E, K_1)$$

for any field extension E/F.

3.2. **Properties of α.**

Proposition 3.5. *The composition $G(F) \overset{\alpha_F}{\to} A_0(X, K_1) \overset{N_F}{\to} F^\times$ coincides with $\rho(F)$.*

Proof. Let $g \in G(F)$. Consider the commutative diagram

$$\begin{tikzcd} A_0(X_{F(G)}, K_1) \arrow[r, "N^1_{F(G)}"] \arrow[d, "i"'] & F(G)^\times \arrow[d, "j"] \\ A_0(X, K_1) \arrow[r, "N^1_F"] & F^\times \end{tikzcd}$$

where i and j are the evaluation homomorphisms at g. We have

$$N^1_F\big(\alpha_F(g)\big) = N^1_F\big(i(\alpha)\big) = j\big(N^1_{F(G)}(\alpha)\big) = j(\rho) = \rho(g).$$

□

Proposition 3.6. *The map α_F factors through a homomorphism (still denoted α_F)*

$$G(F)/RH(F) \to A_0(X, K_1).$$

Proof. Let $h(t)$ be an element of $H\big(F(t)\big)$ defined at the points $t = 0$ and $t = 1$ such that $h(0) = 1$. The image of $h(t)$ under $\alpha_{F(t)}$ belongs to $\overline{A}_0(X_{F(t)}, K_1) = \overline{A}_0(X, K_1)$, i.e., it is constant. By Proposition 1.2, the homomorphism α commutes with the evaluations, hence

$$\alpha_F\big(h(1)\big) = \alpha_{F(t)}(h)(1) = \alpha_{F(t)}(h)(0) = \alpha_F\big(h(0)\big) = 0.$$

□

In particular, α_F induces a homomorphism

$$H(F)/R \to \overline{A}_0(X, K_1).$$

Finally, we prove that α commutes with the norms.

Proposition 3.7. *Let L/F be a finite field extension. Then the following diagram commutes:*

$$\begin{tikzcd} G(L)/RH(L) \arrow[r, "\alpha_L"] \arrow[d, "\mathcal{N}_{L/F}"'] & A_0(X_L, K_1) \arrow[d, "N_{L/F}"] \\ G(F)/RH(F) \arrow[r, "\alpha_F"] & A_0(X, K_1) \end{tikzcd}$$

Proof. The difference of two compositions $N_{L/F} \circ \alpha_L - \alpha_F \circ \mathcal{N}_{L/F}$ can be considered as an invariant of the group $G' = R_{L/F}(G)$ and therefore is given by an element $\gamma \in A_0(X_{F(G')}, K_1)$. Since ρ commutes with the norms, the image $N^1(\gamma)$ in $F(G')^\times$ is trivial, hence $\gamma \in \overline{A}_0(X_{F(G')}, K_1)$. Since the group G' is rational, by Corollary 3.3, $\gamma \in \overline{A}_0(X, K_1)$, hence γ is constant and therefore $\gamma = \gamma(1) = 0$. □

3.3. **Homomorphism ν^F.** Let $\operatorname{ind}\rho = n$. By 1.4, there is a homomorphism $\nu : \mathbb{G}_m \to G$ such that the composition $\rho \circ \nu$ is the n-th power map. Then ν defines a homomorphism

$$\nu^F : F^\times = \mathbb{G}_m(F) \overset{\nu(F)}{\to} G(F) \to G(F)/RH(F).$$

Lemma 3.8. *The map ν^F does not depend on the choice of ν.*

Proof. Assume that $\rho \circ \nu = \rho \circ \nu'$ for another homomorphism ν'. Then $\nu' = \nu \cdot \varphi$ for some morphism $\varphi : \mathbb{G}_m \to H$ (not necessarily a homomorphism). Clearly, $\varphi(1) = 1$ and therefore $\varphi(a) \in RH(F)$ for all $a \in F^\times$. Hence $\nu'(a) \equiv \nu(a)$ modulo $RH(F)$. □

The composition $\rho^F \circ \nu^F$ is the n-th power endomorphism of $F^\times$.

If $\operatorname{ind}\rho = 1$, i.e., $\rho \circ \nu = \mathrm{id}$, then ν^F is the inverse isomorphism to ρ^F.

4. Evaluation

We elaborate the evaluation technique which we use in the next section.

4.1. **Value at a point.** Let Y be an irreducible variety over a field F, with $L = F(Y)$ the function field of Y, $y \in Y$. We say that an element $v \in G(L)/RH(L)$ is *defined at the point* y if v is represented by an element of $G(\mathcal{O}_y)$, where $\mathcal{O}_y$ is the local ring of y, i.e., if v is represented by an element of $G(L)$ defined at y.

We would like to show that the value $v(y)$ is well defined at least if $Y = \mathbb{A}^1_F$ and y is a rational point $t = b$, $b \in F$. Let $\mathcal{O}_b$ be the local ring of all functions in $F(\mathbb{A}^1_F) = F(t)$ defined at y.

Lemma 4.1. *For any $h(t) \in H(\mathcal{O}_b) \cap RH\big(F(t)\big)$ the value $h(b)$ belongs to $RH(F)$.*

Proof. Choose $h(s,t) \in H\big(F(s,t)\big)$ such that $h(0,t) = 1$ and $h(1,t) = h(t)$. The element $h(s,t)$ is given by an algebra homomorphism $f : F[H] \to F(s,t)$ with the property that the image of f is contained in the localization $F[s,t]_q$ where the polynomial $q \in F[s,t]$ is such that $q(0,t)$ and $q(1,t)$ are non-zero polynomials in $F[t]$. Since F is infinite, there is $c \in F$ such that $q(0,c) \neq 0$ and $q(1,c) \neq 0$. Then the elements $h(0,t)$ and $h(1,t)$ in $H\big(F(t)\big)$ are defined at the point $t = c$ and $h(0,c) = 1$, $h(1,c) = h(c)$. Hence the element $h(s,c) \in H\big(F(s)\big)$ is defined at $s = 0$, $s = 1$ and therefore $h(c) = h(1,c)/h(0,c) \in RH(F)$. On the other hand, $h(t) = h(1,t)$ is defined at $t = b$ and $t = c$, hence $h(b)/h(c) \in RH(F)$. Finally, $h(b) = h(c) \cdot h(b)/h(c) \in RH(F)$. □

Lemma 4.1 shows that if an element $v(t) \in G\big(F(t)\big)/RH\big(F(t)\big)$ is defined at a point $t = b$, then the *value* $v(b) = v|_{t=b} \in G(F)/RH(F)$ is well defined.

We show that the homomorphism ν^F commutes with the evaluations.

Proposition 4.2. *Let $g(s) \in F(s)^\times$ be defined at a point $s = a$, $a \in F$. Then $\nu^{F(s)}\big(g(s)\big)$ is defined at $s = a$ and*

$$\nu^{F(s)}\big(g(s)\big)(a) = \nu^F\big(g(a)\big).$$

Proof. By Proposition 1.3.3, $\operatorname{ind}(\rho) = \operatorname{ind}(\rho_{F(s)})$, hence $\nu \otimes_F F(s)$ can be taken in the definition of $\nu^{F(s)}$. □

4.2. **Properties of the evaluation.**

Lemma 4.3. *Let $\mathcal{O}$ be the local ring of a smooth point y of a rational irreducible variety Y, $L = F(Y)$. Assume that $\rho^L(w) \in \mathcal{O}^\times$ for an element $w \in G(L)/RH(L)$. Then w is defined at y.*

Proof. Let w be represented by $\tilde{w} \in G(L)$. By [4, Th. 3.2], the right vertical map in the commutative diagram with exact rows

$$\begin{tikzcd} G(\mathcal{O}) \arrow[r, "\rho"] \arrow[d, hook] & \mathcal{O}^\times \arrow[r] \arrow[d, hook] & H^1_{\acute{e}t}(\mathcal{O}, H) \arrow[d] \\ G(L) \arrow[r, "\rho"] & L^\times \arrow[r] & H^1_{\acute{e}t}(L, H) \end{tikzcd}$$

is injective. Since $\rho(\tilde{w}) \in \mathcal{O}^\times$, there exists $u \in G(\mathcal{O})$ such that $\rho(u) = \rho(\tilde{w})$. Replacing $\tilde{w}$ by $\tilde{w} \cdot u^{-1}$, we may assume that $\rho(\tilde{w}) = 1$, i.e., $\tilde{w} \in H(L)$. Therefore, since L/F is purely transcendental, $w \in H(L)/R = H(F)/R$ is constant and hence is defined at y. □

Let $\mathcal{O}$ be a local F-algebra. For any variety Y over F consider a map $j_Y : Y(\mathcal{O}) \to Y$ defined as follows. Let v be a point of $Y(\mathcal{O})$, i.e., v is a morphism $\operatorname{Spec} \mathcal{O} \to Y$. We set $j_Y(v) = v(x)$ where x is the closed point of $\operatorname{Spec} \mathcal{O}$.

Lemma 4.4. *The image of $j_G : G(\mathcal{O}) \to G$ is dense.*

Proof. Since G is rational, the group of rational points $G(F)$ is dense in G [1, Cor. 18.3]. Hence the image of j_G is also dense in G. □

Corollary 4.5. *Assume that the residue field of a local F-algebra $\mathcal{O}$ is F. Then for any non-empty open subset $U \subset G$, $G(\mathcal{O}) = U(\mathcal{O}) \cdot U(\mathcal{O})$.*

Proof. For a $g \in G(\mathcal{O})$ denote its value in $G(F)$ by $\bar{g}$. The group G is connected, hence the set $\bar{g}U^{-1} \cap U$ is non-empty. By Lemma 4.4, there is $g_1 \in G(\mathcal{O})$ such that $\bar{g}_1 \in \bar{g}U(F)^{-1} \cap U(F)$. Since $\bar{g}_1 \in U(F)$, the image of the closed point under $g_1 : \operatorname{Spec} \mathcal{O} \to G$ is contained in U, hence $\operatorname{Im}(g_1) \subset U$ and $g_1 \in U(\mathcal{O})$. Set $g_2 = g_1^{-1} g$. Again, $\bar{g}_2 \in U(F)$ implies that $g_2 \in U(\mathcal{O})$. Finally, $g = g_1 g_2$ and $g_1, g_2 \in U(\mathcal{O})$. □

Lemma 4.6. *Let $\mathcal{O}$ be the local ring of an F-point $p = (a, b)$ of the affine plane* $\operatorname{Spec} F[s,t]$. *Assume that an element $w = w(s,t) \in G\big(F(s,t)\big)/RH\big(F(s,t)\big)$ is defined at p, i.e., is represented by an element $\tilde{w} \in G(\mathcal{O})$. Then w is defined at $s = a$ over $F(t)$, the value $w(a,t) = w|_{s=a}$ is defined at $t = b$ and the value $(w|_{s=a})|_{t=b}$ in $G(F)/RH(F)$ is represented by $\tilde{w}(p)$.*

Proof. Let $\mathcal{O}_1$ be the local ring of the point $s = a$ over $F(t)$ and $\mathcal{O}_2$ the local ring of the point $t = b$ over F. The inclusion $\mathcal{O} \subset \mathcal{O}_1$ and evaluation homomorphisms

$$\mathcal{O} \overset{s=a}{\to} \mathcal{O}_2 \overset{t=b}{\to} F, \quad \mathcal{O}_1 \overset{s=a}{\to} F(t)$$

induce the following commutative diagram:

$$\begin{tikzcd} G(\mathcal{O}) \arrow[r] \arrow[d] & G(\mathcal{O}_2) \arrow[r] \arrow[d] & G(F) \\ G(\mathcal{O}_1) \arrow[r] & G\big(F(t)\big). \end{tikzcd}$$

The image of $\tilde{w}$ in $G(\mathcal{O}_1)$ represents w, hence w is defined in $s = a$ over $F(t)$. The value $w(a,t) = w|_{s=a}$ is represented by the image of $\tilde{w}$ in $G(\mathcal{O}_2)$, hence $w(a,t)$ is defined at $t = b$ and the value $(w|_{s=a})|_{t=b}$ coincides with the image of $\tilde{w}$ in $G(F)$, i.e., it is equal to $\tilde{w}(p)$. □

Corollary 4.7. $(w|_{t=b})|_{s=a} = (w|_{s=a})|_{t=b}$.

4.3. **Evaluation and norms.** We prove that the norms commute with the evaluation.

Lemma 4.8. *Let L/F be a finite field extension, and let*

$$v(s) \in G\big(L(s)\big)/RH\big(L(s)\big)$$

be an element defined at $s = a$, $a \in F$. Then $\mathcal{N}_{L(s)/F(s)}\big(v(s)\big)$ is defined at $s = a$ and

$$\big(\mathcal{N}_{L(s)/F(s)}v(s)\big)(a) = \mathcal{N}_{L/F}\big(v(a)\big).$$

Proof. By the definition of the norm map $\mathcal{N}$ given in [2], there is a non-empty open subset $U \subset R_{L/F}(G_L)$ and a morphism $i : U \to G$ such that for any field extension E/F the restriction to $U(E)$ of the composition

$$R_{L/F}(G_L)(E) = G(L \otimes_F E) \to G(L \otimes_F E)/RH(L \otimes_F E) \overset{\mathcal{N}_{L/F}}{\to} G(E)/RH(E)$$

is given by the composition

$$U(E) \overset{i(E)}{\to} G(E) \to G(E)/RH(E).$$

Let $\mathcal{O}$ be the local ring in $F(s)$ of the point $s = a$. If $v(s)$ is represented by an element $\tilde{v} \in U(\mathcal{O})$, then $\mathcal{N}_{L(s)/F(s)}\big(v(s)\big)$ is represented by $i(\tilde{v}) \in G(\mathcal{O})$ and hence is defined at $s = a$. It follows from the commutativity of the diagram

$$\begin{tikzcd} U(\mathcal{O}) \arrow[r, "i"] \arrow[d, "p"'] & G(\mathcal{O}) \arrow[d] \\ U(F) \arrow[r, "i"] & G(F) \end{tikzcd}$$

that $\big(\mathcal{N}_{L(s)/F(s)}v(s)\big)(a)$ is represented by $(ip)(\tilde{v})$ and hence is equal to $\mathcal{N}_{L/F}\big(v(a)\big)$. The general case follows from Corollary 4.5. □

Proposition 4.2 then implies

Corollary 4.9. *Let L/F be a finite field extension, and let $g(s) \in L(s)^\times$ be a function defined at $s = a$, $a \in F$. Then $\mathcal{N}_{L(s)/F(s)}\nu^{L(s)}\big(g(s)\big)$ is defined at the point $s = a$ and*

$$\big(\mathcal{N}_{L(s)/F(s)}\nu^{L(s)}g(s)\big)(a) = \mathcal{N}_{L/F}\nu^{L}\big(g(a)\big).$$

The following lemma is the first application of the evaluation technique. The idea of the proof of the lemma (and a series of statements in the next section) is as follows. Suppose we would like to prove an equality $v = 1$ in some "rigid object", where v is "R-trivial", i.e., there is a rational family $v(s)$ such that $v(0) = 1$ and $v(1) = v$. Since $v(s)$ takes values in a rigid object, $v(s)$ is a constant family, hence $v = v(1) = v(0) = 1$. Shortly: a map from "R-trivial" to "rigid" is constant.

Let L/F be a finite field extension such that $\operatorname{ind}\rho_L = 1$. Then by Proposition 1.3.1, $\operatorname{ind}\rho$ divides $[L : F]$.

Lemma 4.10. *Let L/F be a finite field extension such that* $\operatorname{ind}\rho_L = 1$*. Then for any $a \in F^\times$,*

$$\mathcal{N}_{L/F}\big(\nu^L(a)\big) = \nu^F(a)^{[L:F]/\operatorname{ind}(\rho)}.$$

Proof. Consider the following two elements in $G\big(F(s)\big)/RH(F(s))$:

$$v(s) = \mathcal{N}_{L(s)/F(s)}\big(\nu^{L(s)}(1 - s + sa)\big),$$

$$w(s) = \nu^{F(s)}(1 - s + sa)^{[L:F]/\operatorname{ind}\rho}.$$

Since ρ commutes with the norms and $\operatorname{ind}\rho_{F(s)} = \operatorname{ind}\rho$ by Proposition 1.3.3,

$$\rho\big(v(s)\big) = N_{L(s)/F(s)}(1 - s + sa) = (1 - s + sa)^{[L:F]} = \rho\big(w(s)\big),$$

we have $q(s) \stackrel{def}{=} v(s) \cdot w(s)^{-1} \in H\big(F(s)\big)/R = H(F)/R$, i.e., $q(s)$ is constant. By Corollary 4.9 and Proposition 4.2,

$$1 = q(0) = q(1) = \mathcal{N}_{L/F}\big(\nu^L(a)\big)/\nu^F(a)^{[L:F]/\operatorname{ind}\rho}.$$

□

5. Homomorphism β_F

In this section we prove that α_F is an isomorphism by constructing its inverse β_F.

5.1. **Definition of β_F.** For a closed point $x \in X$ we define a homomorphism

$$F(x)^\times \to G(F)/RH(F), \quad u \mapsto \mathcal{N}_{F(x)/F}\big(\nu^{F(x)}(u)\big),$$

and hence we have a homomorphism

$$\coprod_{x \in X_{(0)}} F(x)^\times \to G(F)/RH(F).$$

We will prove that this homomorphism is trivial on the image of the differential in the complex (1.1):

$$\coprod_{x \in X_{(1)}} K_2\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X_{(0)}} F(x)^\times \tag{5.1}$$

and hence factors through a well-defined homomorphism

$$\beta_F : A_0(X, K_1) \to G(F)/RH(F).$$

The closure of a point $y \in X_{(1)}$ is a projective curve in X. Let C be the normalization of this curve. By the definition of the complex (1.1), [22, 3.2], the image of $K_2\big(F(y)\big)$ under ∂ in (5.1) coincides with the image of the composition

$$K_2\big(F(C)\big) \xrightarrow{\partial} \coprod_{c \in C_{(0)}} F(c)^\times \to \coprod_{x \in X_{(0)}} F(x)^\times,$$

where the second homomorphism is induced by the norm maps $F(c)^\times \to F(x)^\times$ for all pairs $c|x$. Hence, by the transitivity of the norm map $\mathcal{N}$, it suffices to show that the composition

$$K_2\big(F(C)\big) \xrightarrow{\partial} \coprod_{c \in C_{(0)}} F(c)^\times \to G(F)/RH(F) \tag{5.2}$$

is zero. Note that the curve C is smooth projective and by property 3 of X, $\operatorname{ind}\rho_{F(c)} = 1$ for all points $c \in C$ since $X\big(F(c)\big) \neq \emptyset$.

For a function $f \in F(C)^\times$ we denote by $\operatorname{Sup}(f) \subset C$ the support of the principal divisor $\operatorname{div}(f)$. Clearly, $\operatorname{Sup}(f) = \emptyset$ if and only if $f \in F^\times$. For any closed point $x \in C$ we denote by v_x the discrete valuation of $F(C)$ associated to x.

Lemma 5.1. *The group $K_2\big(F(C)\big)$ is generated by the symbols $\{f, g\}$ for all $f, g \in F(C)^\times$ such that $\operatorname{Sup}(f) \cap \operatorname{Sup}(g) = \emptyset$.*

Proof. For any $n \geq 0$, denote by A_n the subgroup in $K_2\big(F(C)\big)$ generated by the symbols $\{f, g\}$ for all $f, g \in F(C)^\times$ such that $|\operatorname{Sup}(f) \cap \operatorname{Sup}(g)| \leq n$. Clearly $A_0 \subset A_1 \subset \cdots \subset K_2\big(F(C)\big)$ and we would like to show that $A_0 = K_2\big(F(C)\big)$.

We prove first that $A_1 = K_2\big(F(C)\big)$. Let $f, g \in F(C)^\times$ be such that $|\operatorname{Sup}(f) \cap \operatorname{Sup}(g)| = n > 1$. For a point $x \in \operatorname{Sup}(f) \cap \operatorname{Sup}(g)$ consider a function $h \in F(C)^\times$ such that $v_x(h) = v_x(f)$ and $v_y(h) = 0$ for all $y \in \operatorname{Sup}(g)$, $y \neq x$. Then $\{f, g\} = \{h, g\} + \{fh^{-1}, g\}$ and $\{h, g\} \in A_1$, $\{fh^{-1}, g\} \in A_{n-1}$, therefore $A_n = A_{n-1}$ for $n > 1$. The descending induction shows that $A_1 = K_2\big(F(C)\big)$.

It remains to show that $A_1 = A_0$. Let $f, g \in F(C)^\times$ be such that $\operatorname{Sup}(f) \cap \operatorname{Sup}(g) = \{x\}$ for a point $x \in C$. The curve $C' = C \setminus \{x\}$ is affine and the supports of the divisors of f and g on C' are disjoint. Choose a non-zero regular function $h \in F[C']$ such that $v_y(h) = -v_y(f)$ for all $y \in C'$ with $v_y(f) < 0$ and $v_y(h) = 0$ for all $y \in C'$ with $v_y(g) \neq 0$. Then $fh \in F[C']$ and $\{f, g\} = \{fh, g\} - \{h, g\}$ and functions in the pairs (fh, g) and (h, g) have disjoint supports on C'. Thus, we may assume that $f \in F[C']$. A similar argument shows that we may also assume that $g \in F[C']$. Let $n = -v_x(f) > 0$, $m = -v_x(g) > 0$ and $d = \gcd(n, m)$. Set $n' = n/d$ and $m' = m/d$. We have $m'\{f, g\} = \{f^{m'}, g\} = \{h, g\}$ where $h = \frac{f^{m'}}{(1-g)^{n'}}$. Since $v_x(h) = 0$ and the divisors of g and $1 - g$ are disjoint on C', the symbol $\{h, g\}$ belongs to A_0, i.e., $m'\{f, g\} \in A_0$. Similarly, $n'\{f, g\} \in A_0$. Finally, n' and m' are relatively prime, hence $\{f, g\} \in A_0$. $\square$

By Lemma 5.1, in order to prove that β_F is well defined, it suffices to show that the composition (5.2) is trivial on symbols $\{f, g\}$ with the functions $f, g \in F(C)^\times$ satisfying $\operatorname{Sup}(f) \cap \operatorname{Sup}(g) = \emptyset$. By the definition of the residue homomorphism ∂, we have to check that

$$\prod_{x \in \operatorname{Sup}(f)} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)} g(x)\big)^{v_x(f)} = \prod_{x \in \operatorname{Sup}(g)} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)} f(x)\big)^{v_x(g)}. \tag{5.3}$$

We may assume that one of the functions f or g, say f, is not constant. Thus f defines a finite morphism $f : C \to \mathbb{P}^1_F$. Denote by e_x the ramification index of a point $x \in C$. We identify the function field $F(t)$ of $\mathbb{P}^1_F$ with a subfield in $F(C)$. Under this identification, $t = f$. Thus, if $f(x) = 0$, then $e_x = v_x(t)$, and if $f(x) = \infty$, then $e_x = -v_x(t)$. Set

$$u(t) = \mathcal{N}_{F(C)/F(t)}\big(\nu^{F(C)}(g)\big) \in G\big(F(t)\big)/RH\big(F(t)\big),$$
$$h(t) = \rho\big(u(t)\big) = N_{F(C)/F(t)}(g) \in F(t)^\times.$$

The following lemma is standard.

Lemma 5.2. *Let $b \in \mathbb{P}^1(F)$, and let $l \in F(C)^\times$ be a function defined and non-zero at all points $x \in C$ such that $f(x) = b$. Then*

$$\big(N_{F(C)/F(t)}(l)\big)(b) = \prod_{f(x)=b} N_{F(x)/F}\big(l(x)\big)^{e_x}.$$

Lemma 5.3. *Let $b \in \mathbb{P}^1(F)$ be such that $b \notin f\big(\mathrm{Sup}(g)\big)$. Then $u(t)$ is defined at $t = b$ and*

$$u(b) = \prod_{f(x)=b} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)} g(x)\big)^{e_x} \in G(F)/RH(F).$$

Proof. Let $g(s) = 1 - s + sg \in F(C)(s)^\times$. Consider the following elements:

$$v(s) = \prod_{f(x)=b} \mathcal{N}_{F(x)(s)/F(s)}\big(\nu^{F(x)(s)} g(s)(x)\big)^{e_x} \in G\big(F(s)\big)/RH(F(s)),$$

$$w(s,t) = \mathcal{N}_{F(C)(s)/F(s,t)}\big(\nu^{F(C)(s)} g(s)\big) \in G\big(F(s,t)\big)/RH(F(s,t)).$$

The function $\rho\big(w(s,t)\big) = N_{F(C)(s)/F(s,t)}\big(g(s)\big)$ is defined at the points $s = 0, t = b$ and $s = 1, t = b$. By Lemma 4.3, $w(s,t)$ is defined at these points. Lemma 4.6 implies that $w(s,b) \in G\big(F(s)\big)/RH(F(s))$ is well defined. Since by Lemma 5.2, applied to the field $F(s)$ and function $l = g(s)$,

$$\begin{aligned}\rho\big(v(s)\big) = &\prod_{f(x)=b} N_{F(x)(s)/F(s)}\big(g(s)(x)\big)^{e_x} = N_{F(C)(s)/F(s,t)}\big(g(s)\big)(b)\\ =&\rho\big(w(s,t)\big)(b) = \rho\big(w(s,b)\big),\end{aligned}$$

we have $q(s) \overset{def}{=} v(s) \cdot w(s,b)^{-1} \in H\big(F(s)\big)/R = H(F)/R$, i.e., $q(s)$ is constant.

By Corollaries 4.9 and 4.7,

$$w(s,b)|_{s=1} = w(1,t)|_{t=b} = u(b) \quad \text{and} \quad w(s,b)|_{s=0} = w(0,t)|_{t=b} = 1.$$

Corollary 4.9 implies that

$$v(1) = \prod_{f(x)=b} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)} g(x)\big)^{e_x}, \quad v(0) = 1,$$

hence

$$1 = q(0) = q(1) = \prod_{f(x)=b} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)} g(x)\big)^{e_x} \cdot u(b)^{-1}.$$

□

Lemma 5.4. *Let $a, b \in F$, $a \neq 0$, and let $l(t) \in F(t)^\times$ be a rational function defined at $t = 0, \frac{b}{a}, \infty$. Then*

$$\prod_{p \in \mathbb{G}_m} N_{F(p)/F}\big(at(p) - b\big)^{v_p(l)} = l(\frac{b}{a}) \cdot l(\infty)^{-1}.$$

Proof. Since both sides of the equality are multiplicative in $l(t)$, we may assume that $l(t) = p(t)/(t-c)^n$ where $p(t) \neq t$ is an irreducible polynomial of degree n and

$c \in F$ is different from 0 and $\frac{b}{a}$. In this case $at(p) - b$ is a root of $p\left(\frac{t+b}{a}\right)$ and the product reduces to

$$\frac{N_{F(p)/F}\big(at(p)-b\big)}{(ac-b)^n} = \frac{(-a)^n p(\frac{b}{a})}{(ac-b)^n l(\infty)} = l(\frac{b}{a}) \cdot l(\infty)^{-1}.$$

$\square$

Since $\mathrm{Sup}(f)$ and $\mathrm{Sup}(g)$ are disjoint, $h(t)$ is defined at the points $t = 0, \infty$; hence by Lemma 4.3, $u(0)$ and $u(\infty)$ are well defined. By [14, Th. 4.4], $\operatorname{ind} \rho_{F(p)}$ divides $v_p(h)$.

Lemma 5.5.

$$\prod_{p \in \mathbb{G}_m} \mathcal{N}_{F(p)/F}\Big(\nu^{F(p)}\big(t(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}\Big) = u(0) \cdot u(\infty)^{-1} \in G(F)/RH(F).$$

Proof. Consider the following two elements in $G\big(F(s)\big)/RH(F(s))$:

$$v(s) = \prod_{p \in \mathbb{G}_m} \mathcal{N}_{F(p)(s)/F(s)}\Big(\nu^{F(p)(s)}\big(1 - s + st(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}\Big),$$

$$w(s) = u(\frac{s-1}{s}) \cdot u(\infty)^{-1}.$$

Since ρ commutes with the norms, by Lemma 5.4 applied to the field $F(s)$ and $a = s$, $b = s - 1$,

$$\begin{aligned} \rho\big(v(s)\big) &= \prod_{p \in \mathbb{G}_m} N_{F(p)(s)/F(s)}\big(1 - s + st(p)\big)^{v_p(h)} \\ &= h(\frac{s-1}{s}) \cdot h(\infty)^{-1} = \rho\big(w(s)\big), \end{aligned}$$

we have $q(s) \overset{def}{=} v(s) \cdot w(s)^{-1} \in H\big(F(s)\big)/R = H(F)/R$, i.e., $q(s)$ is constant. Corollary 4.9 implies that

$$1 = q(0) = q(1) = \prod_{p \in \mathbb{G}_m} \mathcal{N}_{F(p)/F}\Big(\nu^{F(p)}\big(t(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}\Big) \cdot u(0)^{-1} \cdot u(\infty).$$

$\square$

Lemma 5.6. *For any closed point $p \in \mathbb{G}_m \subset \mathbb{P}^1_F$,*

$$\prod_{f(x)=p} \mathcal{N}_{F(x)/F(p)}\big(\nu^{F(x)} f(x)\big)^{v_x(g)} = \nu^{F(p)}\big(t(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}$$

in $G\big(F(p)\big)/RH(F(p))$.

Proof. Since $f(x) = t(p) \in F(p)^\times$ and

$$\sum_{f(x)=p} v_x(g) \cdot [F(x) : F(p)] = v_p(h),$$

we have

$$\begin{aligned}
&\prod_{f(x)=p} \mathcal{N}_{F(x)/F(p)}\big(\nu^{F(x)}(f(x))\big)^{v_x(g)}\\
&\qquad= \prod_{f(x)=p} \nu^{F(p)}\big(t(p)\big)^{v_x(g)\cdot[F(x):F(p)]/\operatorname{ind}\rho_{F(p)}} \quad \text{(by Lemma 4.10)}\\
&\qquad= \nu^{F(p)}\big(t(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}.
\end{aligned}$$

□

Now we can prove (5.3):

$$\begin{aligned}
&\prod_{x\in \operatorname{Sup}(f)} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)}g(x)\big)^{v_x(f)}\\
&\quad= \prod_{f(x)=0,\infty} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)}(g(x))\big)^{v_x(t)}\\
&\quad= u(0)\cdot u(\infty)^{-1} \quad \text{(by Lemma 5.3)}\\
&\quad= \prod_{p\in\mathbb{G}_m} \mathcal{N}_{F(p)/F}\Big(\nu^{F(p)}\big(t(p)\big)^{v_p(h)/\operatorname{ind}\rho_{F(p)}}\Big) \quad \text{(by Lemma 5.5)}\\
&\quad= \prod_{p\in\mathbb{G}_m} \mathcal{N}_{F(p)/F} \prod_{f(x)=p} \mathcal{N}_{F(x)/F(p)}\big(\nu^{F(x)}f(x)\big)^{v_x(g)} \quad \text{(by Lemma 5.6)}\\
&\quad= \prod_{x\in \operatorname{Sup}(g)} \mathcal{N}_{F(x)/F}\big(\nu^{F(x)}f(x)\big)^{v_x(g)}.
\end{aligned}$$

Thus, the homomorphism β_F is well defined.

5.2. **Properties of β_F.** We prove first that β_F commutes with the norms.

Proposition 5.7. *For any finite field extension L/F the following diagram commutes:*

$$\begin{tikzcd}
A_0(X_L,K_1) \arrow[r, "\beta_L"] \arrow[d, "N_{L/F}"'] & G(L)/RH(L) \arrow[d, "\mathcal{N}_{L/F}"]\\
A_0(X,K_1) \arrow[r, "\beta_F"] & G(F)/RH(F)
\end{tikzcd}$$

Proof. Let $x\in X_L$ be a closed point, $u\in L(x)^\times$. Denote by y the image of x under the natural morphism $X_L\to X$ and by v the norm $N_{L(x)/F(y)}(u)\in F(y)^\times$. Since ρ commutes with the norms and by a property of the map ν, $\nu^E=(\rho^E)^{-1}$ for any field extension E/F such that $X(E)\neq\emptyset$, we have

$$\mathcal{N}_{L(x)/F(y)}\nu^{L(x)}(u)=\nu^{F(y)}\big(N_{L(x)/F(y)}(u)\big)=\nu^{F(y)}(v).$$

Then $N_{L/F}(x,u)=(y,v)$ and

$$\begin{aligned}
\beta_F\big(N_{L/F}(x,u)\big) &= \beta_F(y,v)=\mathcal{N}_{F(y)/F}\big(\nu^{F(y)}(v)\big)=\mathcal{N}_{F(y)/F}\big(\mathcal{N}_{L(x)/F(y)}\nu^{L(x)}(u)\big)\\
&=\mathcal{N}_{L/F}\big(\mathcal{N}_{L(x)/L}\nu^{L(x)}(u)\big)=\mathcal{N}_{L/F}\big(\beta_L(x,u)\big).
\end{aligned}$$

□

Corollary 5.8. *The composition*

$$A_0(X, K_1) \overset{\beta_F}{\to} G(F)/RH(F) \overset{\alpha_F}{\to} A_0(X, K_1)$$

is the identity.

Proof. Assume first that $X(F) \neq \emptyset$. Then for any $x \in X(F)$ and $a \in F^\times$,

$$N_F^1\big(\alpha_F(\beta_F(x, a))\big) = \rho^F\big(\beta_F(x, a)\big) = \rho^F\big(\nu^F(a)\big) = a = N_F^1(x, a);$$

hence $\alpha_F \circ \beta_F$ is the identity since $N_F^1 : A_0(X, K_1) \to F^\times$ is an isomorphism by property 2 of X.

In the general case, by Propositions 3.7 and 5.7, for any finite field extension L/F the following diagram is commutative:

$$\begin{tikzcd}
A_0(X_L, K_1) \arrow[r, "\beta_L"] \arrow[d, "N_{L/F}"] & G(L)/RH(L) \arrow[r, "\alpha_L"] \arrow[d, "\mathcal{N}_{L/F}"] & A_0(X_L, K_1) \arrow[d, "N_{L/F}"] \\
A_0(X, K_1) \arrow[r, "\beta_F"] & G(F)/RH(F) \arrow[r, "\alpha_F"] & A_0(X, K_1).
\end{tikzcd}$$

Since the composition in the top row is the identity if $X(L) \neq \emptyset$, the composition $\alpha_F \circ \beta_F$ is the identity on the image of $N_{L/F}$ for such field extensions. But these images generate the group $A_0(X, K_1)$. $\square$

Lemma 5.9. *Let $p(t)$ be a monic irreducible polynomial over F different from t and $t-1$, and let $L = F(p) \overset{def}{=} F[t]/p(t)F[t]$. Then there is an element $u(t) \in G\big(F(t)\big)/RH\big(F(t)\big)$ such that*

- $\rho\big(u(t)\big) = p(t)^{\operatorname{ind}\rho_L}$;
- *$u(t)$ is defined at $t = 0$, $t = 1$;*
- $u(0), u(1) \in \operatorname{Im}\beta_F$.

Proof. Let $\theta \in L$ be the canonical root of $p(t)$. For a closed point $x \in X_L$, consider the element

$$u_x(t) = \mathcal{N}_{L(x)(t)/F(t)}\big(\nu_{L(x)(t)}(t - \theta)\big) \in G\big(F(t)\big)/RH\big(F(t)\big).$$

Clearly,

$$\begin{aligned}
\rho\big(u_x(t)\big) &= N_{L(x)(t)/F(t)}(t - \theta) \\
&= N_{L(t)/F(t)}(t - \theta)^{\deg x} = p(t)^{\deg x}.
\end{aligned}$$

For $a = 0$ or 1, by Lemma 4.3 and Corollary 4.9, $u_x(t)$ is defined at $t = a$ and

$$u_x(a) = \mathcal{N}_{L(x)/F}\big(\nu_{L(x)}(a - \theta)\big) = \mathcal{N}_{L/F}(u'),$$

where $u' = \mathcal{N}_{L(x)/L}\big(\nu_{L(x)}(a - \theta)\big)$. By the definition of β, $u' \in \operatorname{Im}(\beta_L)$. Hence, Proposition 5.7 implies that $u_x(a) \in \operatorname{Im}(\beta_F)$.

Since by property 3 of X, $\gcd \deg(x) = \operatorname{ind}\rho_L$, there are finitely many closed points $x_i \in X_L$ and integers m_i such that $\sum m_i \deg x_i = \operatorname{ind}\rho_L$. Then the element $u(t) = \prod u_{x_i}(t)^{m_i}$ satisfies the necessary conditions. $\square$

Theorem 5.10. *The maps α_F and β_F are mutually inverse isomorphisms.*

Proof. In view of Corollary 5.8 it is sufficient to show that β_F is surjective. Let $g \in G(F)/RH(F)$. Since G is a rational group, there is

$$u(t) \in G\big(F(t)\big)/RH\big(F(t)\big)$$

defined at $t = 0, 1$ such that $u(0) = 1$ and $u(1) = g$. Set

$$\rho\big(u(t)\big) = a \cdot \prod p_i(t)^{k_i},$$

where $a \in F^\times$ and p_i are monic irreducible polynomials over F. By [14, Th. 4.4], $a = \rho(w)$ for some $w \in G(F)/RH(F)$ and $\operatorname{ind} \rho_{F(p_i)}$ divides k_i for all i. By Lemma 5.9, there exist elements $u_i(t) \in G\big(F(t)\big)/RH\big(F(t)\big)$ such that $\rho\big(u_i(t)\big) = p_i(t)^{k_i}$, $u(t)$ is defined at $t = 0, 1$ and $u_i(0), u(1) \in \operatorname{Im} \beta_F$. Set

$$v(t) = w \cdot \prod u_i(t), \quad h(t) = u(t) \cdot v(t)^{-1}.$$

Clearly, $\rho\big(h(t)\big) = 1$, hence $h(t) \in H\big(F(t)\big)/R = H(F)/R$ is constant. Thus,

$$g = \frac{u(1)}{u(0)} = \frac{v(1)}{v(0)} \in \operatorname{Im} \beta_F.$$

□

Corollary 5.11. *The map α_F induces an isomorphism $H(F)/R \simeq \overline{A}_0(X, K_1)$.*

6. Applications

We apply Theorem 5.10 and Corollary 5.11 in the situation considered in Examples 2.2, 2.3 and 2.5.

First of all, we get another proof of the following theorem [18].

Theorem 6.1. *Let A be a central simple algebra over F, and let X be the Severi-Brauer variety of A. Then there are canonical isomorphisms*

$$K_1(A) = \mathrm{GL}_1(A)/R\,\mathrm{SL}_1(A) \simeq A_0(X, K_1),$$

$$SK_1(A) = \mathrm{SL}_1(A)/R \simeq \overline{A}_0(X, K_1).$$

In the following two theorems we get a computation of the group of R-equivalence classes in spinor groups.

Theorem 6.2. *Let (V, q) be a non-degenerate quadratic space over F, and let X be the corresponding projective quadric hypersurface. Then there are canonical isomorphisms*

$$\Gamma^+(V, q)/R\,\mathrm{Spin}(V, q) \simeq A_0(X, K_1),$$

$$\mathrm{Spin}(V, q)/R \simeq \overline{A}_0(X, K_1).$$

Corollary 6.3. *If $q = f \perp g$, where f is a Pfister neighbor and* $\dim g \leq 2$, *then the group $\overline{A}_0(X, K_1)$ is trivial.*

Proof. By [16, Th. 6.4], the group $\mathbf{Spin}(V, q)$ is rational. □

Remark 6.4. One can show that the homomorphism

$$\alpha_F : \Gamma^+(V, q) \to A_0(X, K_1)$$

coincides with one defined by M. Rost in [23].

Theorem 6.5. *Let A be a central simple algebra over F of even dimension and index at most 2 with a quadratic pair (σ, f), and let X be the corresponding involution variety. Then there are canonical isomorphisms*

$$\Gamma(A,\sigma,f)/R\,\mathrm{Spin}(A,\sigma,f) \simeq A_0(X,K_1),$$

$$\mathrm{Spin}(A,\sigma,f)/R \simeq \overline{A}_0(X,K_1).$$

Remark 6.6. Theorem 6.5 covers all simply connected groups of type D_m with odd m.

6.1. **Generic quadric hypersurfaces.** Let F be a field of characteristic different from 2. Let q be a non-degenerate quadratic form over F of dimension n. Consider the quadratic form $q' = \langle t\rangle \perp q_{F(t)}$ over the rational function field $F(t)$. Since $q_{F(t)}$ is a subform of q', the group $\mathbf{Spin}(q)_{F(t)}$ is a subgroup of $\mathbf{Spin}(q')$.

Theorem 6.7. $\mathrm{Spin}(q)/R \simeq \mathrm{Spin}(q')/R$.

Proof. Consider the hypersurface Y in $\mathbb{A}^1_F \times \mathbb{P}^n_F$ given by the equation $tT_0^2 + q(T) = 0$ in the coordinates $t, T_0 : T_1 : \ldots : T_n$. The subvariety $Z \subset Y$ given by $T_0 = 0$ is isomorphic to $\mathbb{A}^1_F \times X$, where X is the projective quadric corresponding to q. The generic fiber of the projection $f : Y \to \mathbb{A}^1_F$ is the projective quadric X' corresponding to q' over the field $F(t)$. The top row of the following diagram is the exact sequence corresponding to the morphism f [22, §8]:

$$\begin{tikzcd}
\coprod_{p\in\mathbb{A}^1} A_1(Y_p,K_0) \arrow[r, "i"] & A_1(Y,K_0) \arrow[r] & A_0(X',K_1) \arrow[r, "\partial"] \arrow[d, "N^1"'] & \coprod_{p\in\mathbb{A}^1} A_0(Y_p,K_0) \arrow[d, "\coprod N^0"] \\
 & & F(t)^\times \arrow[r, "\partial"] & \coprod_{p\in\mathbb{A}^1} \mathbb{Z}
\end{tikzcd}$$

where Y_p is the fiber of f over $p \in \mathbb{A}^1$. Notice first that the right vertical homomorphism is injective. Indeed, if $p \neq 0$, the fiber Y_p is a smooth quadric and the norm homomorphism $N^0 : A_0(Y_p, K_0) \to \mathbb{Z}$ is injective (Example 2.3). If $p = 0$, the variety Y_p is a singular projective quadric corresponding to a degenerate quadric with one-dimensional radical, i.e., Y_p is of the form of the variety in Lemma 2.4. Hence the result follows from Lemma 2.4.

Let $U = Y \setminus Z$, i.e., U is an open subvariety in Y defined by $T_0 \neq 0$. Clearly, U is isomorphic to the affine space $\mathbb{A}^n_F$. The localization exact sequence for (Y, U) and the homotopy invariance then yield an isomorphism

$$A_1(Y, K_m) \simeq A_0(X, K_{m+1})$$

for any $m \geq 0$. Thus the diagram above gives the following exact sequence:

$$\coprod_{p\in\mathbb{A}^1} A_1(Y_p,K_0) \xrightarrow{i} \overline{A}_0(X,K_1) \to \overline{A}_0(X',K_1) \to 0.$$

In view of Theorem 5.10, it suffices to show that $i = 0$, i.e., for every closed point $p \in \mathbb{A}^1_F$, the direct image homomorphism

$$i_p : A_1(Y_p, K_0) \to A_1(Y, K_0)$$

is trivial. More generally, we will prove that for any $m \geq 0$, the direct image homomorphism

$$i_p^m : A_1(Y_p, K_m) \to A_1(Y, K_m)$$

is trivial.

Assume first that $p \neq 0$, i.e., the fiber Y_p is a smooth projective quadric. Consider the graph $\Delta \subset Y_p \times Y$ of the embedding of Y_p into Y. The idea to use the following lemma is due to M. Rost.

Lemma 6.8. *If Δ represents the trivial class in $\mathrm{CH}_{n-1}(Y_p \times Y)$, then i_p^m is the trivial homomorphism.*

Proof. By [10], the direct image i_p^m is the composition

$$A_1(Y_p, K_m) \xrightarrow{g} A_{n+1}(Y_p \times Y, K_{m-n}) \xrightarrow{k} A_1(Y_p \times Y, K_m) \xrightarrow{h} A_1(Y, K_m),$$

where g is the inverse image homomorphism with respect to the projection $Y_p \times Y \to Y_p$, k is the multiplication by the class of Δ in

$$\mathrm{CH}_{n-1}(Y_p \times Y) = A_{n-1}(Y_p \times Y, K_{1-n})$$

and h is the direct image under the projection $Y_p \times Y \to Y$. □

It remains to check that the class of Δ is trivial. The variety $Y_p \times Y$ is given in $\mathbb{A}_F^1 \times \mathbb{P}_F^n \times \mathbb{A}_F^1 \times \mathbb{P}_F^n$ with the coordinates $t, T_0 : \ldots : T_n, s, S_0 : \ldots : S_n$ by the equations

$$tT_0^2 + q(T) = 0, \quad sS_0^2 + q(S) = 0, \quad t = t(p).$$

Consider the closed subvariety V in $Y_p \times Y$ given by the equations $S_i T_j = S_j T_i$ for all $i, j = 1, 2, \ldots, n$. The restriction l on V of the function $\frac{S_i T_0}{S_0 T_i} - 1$ does not depend on the choice of $i = 1, 2, \ldots, n$. It is straightforward to check that Δ is the divisor of l on V so has trivial image in $\mathrm{CH}_{n-1}(Y_p \times Y)$.

Finally assume $p = 0$. The fiber Y_0 is a degenerate quadric with the singular point y given by $T_0 = 1$, $T_i = 0$ for $i \geq 1$. There is a natural vector bundle

$$Y_0 \setminus \{y\} \to X, \quad (t, T_0 : T_1 : \ldots : T_n) \mapsto (T_1 : \ldots : T_n).$$

The localization exact sequence and the homotopy invariance then yield isomorphisms

$$A_1(Y_0, K_m) \simeq A_1(Y_0 \setminus \{y\}, K_m) \simeq A_0(X, K_{m+1}).$$

The group $A_0(X, K_{m+1})$ is generated by the norms of the group

$$A_0(X_L, K_{m+1}) = K_{m+1}(L)$$

in all finite field extensions L/F such that $X(L) \neq \emptyset$. Since the homomorphisms i_0^m are K_*-linear and commute with the norms, it suffices to prove that $i_0^0 = 0$. But this is clear since the image of i_0^0 belongs to $\overline{A}_0(X, K_0) = 0$. □

Example 6.9. Let q be a non-degenerate quadratic form of dimension 6 over F, and $L = F(t_1, t_2, \ldots, t_n)$ a field of rational functions. Consider the quadratic form $q' = \langle t_1, t_2, \ldots, t_n \rangle \perp q_L$ over L. By Theorem 6.7, for any field extension E/F,

$$\mathrm{Spin}(q_E)/R \simeq \mathrm{Spin}(q'_{EL})/R.$$

Assume that q remains anisotropic over the discriminant quadratic extension $F(\sqrt{\mathrm{disc}(q)})$. Then by [16, Cor. 9.2], $\mathrm{Spin}(q_E)/R \neq 0$ for some field extension E/F. Hence $\mathrm{Spin}(q'_{EL})/R \neq 0$ and therefore, the group $\mathbf{Spin}(q')$ is not rational.

In particular we get examples of non-rational spinor groups for quadratic forms of any dimension ≥ 6. Note that some examples of non-rational spinor groups in dimensions $\equiv 2$ $(mod\ 4)$ were given in [21].

Example 6.10. Let q be the "generic" quadratic form $\langle t_1, t_2, \ldots, t_n \rangle$, $n \geq 2$, over the rational function field $L = F(t_1, t_2, \ldots, t_n)$. It follows from Example 6.9 that for any n, $\mathrm{Spin}(q)/R = 1$ but the group $\mathbf{Spin}(q)$ is not rational if and only if $n \geq 6$.

Remark 6.11. If $\mathrm{char}(F) = 2$, the form q' in Theorem 6.7 is degenerate if $\dim(q)$ is odd. Let q'' be the orthogonal sum of the form $q_{F(s,t)}$ with the non-degenerate binary form $sX^2 + XY + tY^2$ over the field $F(s,t)$. Then q'' is non-degenerate and one can prove that the natural map $\mathrm{Spin}(q)/R \to \mathrm{Spin}(q'')/R$ is an isomorphism.

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