JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 3, Pages 623–715
S 0894-0347(01)00366-6
Article electronically published on March 20, 2001

BOCHNER-KÄHLER METRICS

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Contents

Received by the editors July 6, 2000 and, in revised form, December 19, 2000.
2000 *Mathematics Subject Classification.* Primary 53B35; Secondary 53C55.
Key words and phrases. Kähler metrics, Bochner tensor, momentum map, polytope.
The research for this article was made possible by support from the National Science Foundation through grant DMS-9870164 and from Duke University.

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1. Introduction

1.1. **Historical background.** In Riemannian geometry, the decomposition of the curvature tensor into its irreducible summands under the orthogonal group is regarded as fundamental. There are three such summands, the scalar curvature, the traceless Ricci curvature, and the Weyl curvature.[1] The metrics for which one or more of these irreducible tensors vanishes have been the subject of much research and a great deal is now known about restrictions on the topology of the complete or compact examples. For example, consult [3], where the bulk of the work is devoted to studying the metrics for which the traceless Ricci curvature vanishes, i.e., the Einstein metrics. The metrics in dimensions 4 or higher for which the Weyl curvature vanishes are the conformally flat metrics. While such metrics are trivial to describe locally, their global geometry is rather delicate, so that classifying the complete or compact examples remains a challenge.

In Kähler geometry, the corresponding decomposition of the curvature tensor into its irreducible summands under the unitary group is not quite as familiar, although it has been known since the 1949 work of Bochner [4]. (For a more recent treatment, see [3, 2.63].) The Kähler decomposition bears some resemblance to the Riemannian one, there being three irreducible summands, the scalar curvature, the traceless Ricci curvature, and what has become known as the *Bochner* curvature.[2] Bochner's interest in this latter tensor was due to its appearance in certain Weitzenbock-type formulae. In [4], he proved some cohomological vanishing theorems for compact Kähler manifolds with vanishing Bochner tensor or, more generally, for manifolds for which the pointwise norm of the Bochner tensor was sufficiently small relative to the smallest eigenvalue of the Ricci tensor.

While Kähler metrics with vanishing scalar curvature or vanishing traceless Ricci curvature (i.e., Kähler-Einstein metrics) have been much studied, those with vanishing Bochner tensor, now known as *Bochner-Kähler* metrics, have received considerably less attention. For surveys of what has been known up to now about these metrics, the reader might consult [8], [15], [16], [29], or [30] in addition to §2 of the present article. One will be struck by the paucity of examples. For example, up

[1]The Weyl curvature exists as a nontrivial summand only when the dimension n of the underlying manifold is 4 or more. When $n = 4$, the Weyl curvature is further reducible under the special orthogonal group, but not the full orthogonal group.

[2]N.B.: The Bochner curvature is one component of the Weyl curvature, but not the only component. For example, in complex dimension 2 the Bochner curvature is the anti-self-dual part of the Weyl curvature. See §2.1.3.

until now, every known complete Bochner-Kähler metric was also locally symmetric. (The symmetric examples are the products of the form $M^p_c \times M^{n-p}_{-c}$ where M^p_c denotes the p-dimensional complex space form of constant holomorphic sectional curvature c.)

1.2. **New results.** At first glance, one might expect the theory of Bochner-Kähler manifolds to parallel the theory of conformally flat manifolds. However, this expectation is quickly abandoned. Unlike the local description of conformally flat metrics, a local description of Bochner-Kähler metrics is far from trivial. In fact, no such description was known until now.

Theorem 3.1 and Corollary 3.4 show that the space of isometry classes of germs of C^5 Bochner-Kähler metrics in complex dimension n can be naturally regarded as a closed semi-algebraic subset $F_n \subset \mathbb{R}^{2n+1}$ (with a nonempty interior). More precisely, if M is a complex n-manifold endowed with a C^5 Bochner-Kähler metric g, there is a mapping $f : M \to F_n \subset \mathbb{R}^{2n+1}$ (which is a polynomial function of the curvature tensor of g and its first two covariant derivatives) with the property that $f(x) = f(y)$ for $x, y \in M$ if and only if the germ of g at x is holomorphically isometric to the germ of g at y. Moreover, we show that for every $v \in F_n$, there is a Bochner-Kähler metric g on a neighborhood U of $0 \in \mathbb{C}^n$ so that the associated classifying map $f : U \to F_n$ satisfies $f(0) = v$. (This existence theorem relies on some old results of Élie Cartan that are not readily available in the current literature, so we have included an appendix that exposes these results in a form convenient for the applications in this article.) A by-product of this analysis is that any C^5 Bochner-Kähler metric is necessarily real-analytic.[3] Accordingly, for the rest of the article, we will assume that the Bochner-Kähler metrics under consideration are real-analytic.

Theorem 3.1 suggests that a notion of 'analytic continuation' of Bochner-Kähler metrics might be useful. Elements $v_1, v_2 \in F_n$ are said to be *analytically connected* if there is a connected Bochner-Kähler manifold (M^n, g) for which $f(M)$ contains both v_1 and v_2. This is an equivalence relation, so denote the analytically connected equivalence class of $v \in F_n$ by $[v] \subset F_n$. In Theorem 3.8, we construct a polynomial submersion $C : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$ and show that it is constant on each $[v]$. Eventually, Theorem 4.10 will show that each fiber $C^{-1}(c) \cap F_n$ consists of a finite number of analytically connected equivalence classes and explicitly identify each one as a (not necessarily closed) semi-algebraic set of (real) dimension at most n. Thus, the components of C furnish a set of 'coarse moduli' for Bochner-Kähler metrics. The image $C(F_n) \subset \mathbb{R}^{n+1}$ (which will be explicitly identified below) has nonempty interior, so it makes sense to say that, roughly speaking, the moduli space of Bochner-Kähler metrics in complex dimension n has real dimension $n+1$.

Since each equivalence class $[v] \subset F_n$ has real dimension $m \leq n$ at its smooth points, this suggests that a connected Bochner-Kähler manifold of complex dimension n must always have a nontrivial local isometry 'group', acting with some cohomogeneity $m \leq n$. In Theorem 3.6 and Proposition 3.7, we show that when M^n is simply-connected, the Lie algebra $\mathfrak{g}$ of Killing fields for a Bochner-Kähler structure on M does indeed have dimension at least n and we compute its precise dimension for each analytically connected equivalence class $[v] \subset F_n$. Moreover, for each $v \in F_n$, we compute the dimension of the orbit of the local isometry pseudogroup through an $x \in M$ with $f(x) = v$. In particular, we show in §3.3.3 how to

[3]Presumably, any C^2 Bochner-Kähler metric is real-analytic, but this is not shown here.

compute the cohomogeneity m for each $v \in F_n$. (Interestingly enough, it turns out that m cannot, in general, be computed from the coarse moduli $C(v)$ alone. This is a reflection of the fact that not all of the equivalence classes $[v]$ are closed sets in F_n.) The ultimate conclusion is that a Bochner-Kähler metric always possesses a rather high degree of infinitesimal symmetry.

Perhaps the greatest surprise and what, ultimately, turns out to be the key to understanding the geometry of Bochner-Kähler metrics is that the Lie algebra $\mathfrak{g}$ contains a canonical central subalgebra $\mathfrak{z}$ whose dimension m is the same as that of $\bigl[f(x)\bigr] \subset F_n$ for some (and hence any) $x \in M$. This infinitesimal torus action can be described explicitly as follows: Let Ω be the Kähler form and let $\rho = \mathrm{Ric}(\Omega)$ be its associated Ricci form [3, 2.44]. Define a 'renormalized' Ricci 2-form η by

$$\eta = \frac{1}{2(n{+}1)(n{+}2)}\,(\mathrm{tr}_\Omega\,\rho)\;\;\Omega - \frac{1}{2(n{+}2)}\,\rho$$

and define $p_h(t)$ by the formula $\bigl(t\Omega - \eta\bigr)^n = p_h(t)\,\Omega^n$. Thus,

$$p_h(t) = t^n - h_1\,t^{n-1} + \cdots + (-1)^n h_n$$

where $h_j : M \to \mathbb{R}$ is a certain symmetric polynomial of degree j in the eigenvalues of the Ricci tensor. Then Theorem 3.11 asserts that the Ω-Hamiltonian vector fields X_j defined by $X_j \lrcorner \Omega = -dh_j$ for $1 \le j \le n$ are Killing fields for the metric g and that they Lie commute, i.e., span a torus $\mathfrak{z} \subset \mathfrak{g}$. Of course, this infinitesimal action is Poisson since $h = (h_1, \ldots, h_n) : M \to \mathbb{R}^n$ is a momentum mapping by definition.

As is shown in §3.4.2, the map $h : M \to \mathbb{R}^n$ can be written as $\Psi \circ f$ where Ψ is a weighted homogeneous polynomial mapping from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^n$. When M is connected, the maps h and f have the same fibers. The image of h is m-dimensional and lies in an affine subspace $\mathfrak{a} \subset \mathbb{R}^n$ of dimension m (the same $m \le n$ as defined above). This number m is defined to be the *cohomogeneity* of the Bochner-Kähler structure. Theorem 3.13 shows that, in fact, $p_h(t)$ has a polynomial factor $p_{h''}(t)$ with constant coefficients and of degree $n{-}m$. Thus, $p_h(t) = p_{h''}(t)\,p_{h'}(t)$ where

$$p_{h'}(t) = t^m - h'_1\,t^{m-1} + \cdots + (-1)^m h'_m$$

and the functions $h'_j : M \to \mathbb{R}^m$ for $1 \le j \le m$ are smooth. Theorem 3.13 also shows that, outside a (possibly singular) complex submanifold $N \subset M$ (called the *exceptional locus*), the *reduced momentum mapping* $h' = (h'_1, \ldots, h'_m) : M \to \mathbb{R}^m$ is a submersion. This exceptional locus N is the union of some number of totally geodesic complex submanifolds of M. Let $M^\circ = M \setminus N$ be its complement, the *regular locus*.

Theorem 4.10 gives a polynomial embedding $\iota_v : [v] \to \mathbb{R}^m$ of each m-dimensional analytically connected equivalence class $[v]$ into $\mathbb{R}^m$ as a convex polytope, i.e., an intersection of half-spaces (which can be open or closed). The embedding ι_v satisfies $h' = \iota_v \circ f$ when $f(M)$ lies in $[v]$. Moreover, h' maps M° into the interior of the polytope. Theorem 4.13 shows that the interior of $\iota_v\bigl([v]\bigr)$ carries a canonical Riemannian metric so that $h' : M^\circ \to \iota_v\bigl([v]\bigr)^\circ$ is a Riemannian submersion. In fact, this metric on $\iota_v\bigl([v]\bigr)^\circ$ has rational polynomial coefficients when expressed in terms of linear coordinates on $\mathbb{R}^m$. These metrics are related to certain metrics considered by Guillemin in his study of Kähler structures on toric varieties [12], as will be explained.

Since the metric on the polytope is very explicitly computed, this allows conclusions to be drawn about the existence of complete Bochner-Kähler metrics based on the geometry of the polytopes. In Proposition 4.14, we show that if there is a complete Bochner-Kähler metric whose moduli image lies in $[v]$, then $[v]$ must be bounded (which turns out to be the same as saying that its corresponding polytope is bounded). Essentially, it turns out that when $[v]$ is unbounded, any attempt to 'analytically continue' the metric to a maximal domain will run into curvature blow-up at finite distance. Since there are very few $[v]$ that are bounded, this considerably narrows the search for complete examples.

On the other hand, Proposition 4.16 shows that if $[v]$ is compact but is not a single point, then there is no complete Bochner-Kähler manifold whose moduli image lies in $[v]$. In this case, the problem is not curvature blow-up but is, instead, the presence of essential orbifold singularities in any attempted completion.

A corollary of Proposition 4.16 is Kamishima's result [14] that the only compact Bochner-Kähler manifolds are the compact quotients of the known symmetric ones. This result renders vacuous or trivial many of the results in the literature about Bochner-Kähler metrics. For example, the only Kähler n-manifold satisfying the conditions of [5, Theorems 8.25 and 8.26] is $\mathbb{CP}^n$ endowed with a constant multiple of the Fubini-Study metric. The conclusions of these theorems (which concern the vanishing of various cohomology groups) are trivial for these manifolds.

Theorem 4.24 provides explicit models for the Bochner-Kähler metrics in dimension n that are of cohomogeneity n (i.e., the *least* symmetric ones) on the regular locus. It constructs, for each n-dimensional class $[v] \subset F_n$, a Bochner-Kähler metric on $\iota_v\big([v]\big)^\circ \times \mathbb{R}^n$ with the following universal embedding property: If (M, g) is a Bochner-Kähler n-manifold with $f(M) \subset [v]$, then the universal cover $\widetilde{M^\circ}$ can be isometrically immersed into $\iota_v\big([v]\big)^\circ \times \mathbb{R}^n$, lifting the momentum submersion $h : M^\circ \to \iota_v\big([v]\big)^\circ$. Completeness issues can then be addressed by studying the model metric on $\iota_v\big([v]\big)^\circ \times \mathbb{R}^n$. These metrics are closely related to the metrics studied in [12] and [1]. In particular, Abreu's results in [1] can be generalized to show that the above metrics are actually extremal in the sense of Calabi.

Theorem 4.27 provides a contractible n-parameter family of complete Bochner-Kähler metrics on $\mathbb{C}^n$ and proves that every simply-connected, complete Bochner-Kähler manifold that is not homogeneous is isometric to a unique member of this family.

Thus, the set of complete Bochner-Kähler manifolds is very restricted. However, if one is willing to consider orbifolds, it turns out that there are many nontrivial complete Bochner-Kähler metrics on orbifolds. We include some discussion of these at the end of the article. In fact, by Theorem 4.29, every weighted projective space carries a Bochner-Kähler metric,[4] presumably unique up to constant multiples, though we have not shown this. For example, the Fubini-Study metric is, up to isometry and constant multiples, the unique Bochner-Kähler metric on $\mathbb{CP}^n$. For more detail on this, see §4.3.2 and §4.4.6.

Finally, in §5, we collect some miscellaneous and incidental remarks about generalizations and related problems. In particular, we comment on how this work in the dimension 2 case is related to the recent work of Apostolov and Gauduchon [2]

[4] A natural guess would be that this metric is the one that comes by symplectic reduction from the standard metric on $\mathbb{C}^{n+1}$ via the weighted S^1-action that defines the weighted projective space. However, this 'reduced' metric is never Bochner-Kähler except in the case of equal weights.

that classifies the self-dual Hermitian Einstein metrics in (real) dimension 4 and use the normal forms constructed in this article to produce the first known complete examples of such metrics that are of cohomogeneity 2 (the maximum possible, as it turns out).

1.3. **Glossary.** This article is rather long, so a summary of the terms and standard notations may be of use to the reader. A list of the most important of such terms is included below:

- (M, g, Ω)**:** A complex n-manifold with Kähler metric g and Kähler form Ω, usually assumed to be Bochner-Kähler.
- $\pi : P \to M$**:** The unitary coframe bundle of (M, g, Ω).
- $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$**:** The structure function of (M, g, Ω).
- $\mathfrak{g}$**:** The Lie algebra of Killing fields of (M, g, Ω), i.e., the Killing fields of g that preserve Ω.
- $\mathfrak{z}$**:** The central subalgebra of $\mathfrak{g}$.
- $h : M \to \mathbb{R}^n$**:** The momentum mapping associated to the canonical central elements of $\mathfrak{g}$.
- m**:** The cohomogeneity of (M, g, Ω).
- $h' : M \to \mathbb{R}^m$**:** The reduced momentum mapping.
- M°**:** The regular locus in M (i.e., the complement of the critical locus of h').
- $p_h(t)$**:** The momentum polynomial.
- $p_{h'}(t)$**:** The reduced momentum polynomial.
- $p_C(t)$**:** The characteristic polynomial.
- $p_D(t)$**:** The reduced characteristic polynomial.
- $C(p_D, \mu)$**:** The momentum cell in $\mathbb{R}^m$ associated to p_D and μ.
- R_D**:** The metric on $C(p_D, \mu)$ for which $h' : M^\circ \to C(p_D, \mu)^\circ \subset \mathbb{R}^m$ is a Riemannian submersion.

2. The structure equations of Bochner-Kähler metrics

First, some standard notation. Let $\mathbb{C}^n$ (thought of as columns of height n whose entries are complex numbers) be endowed with its usual Hermitian inner product, in which $\langle z, w\rangle = {}^t\bar{z}w$ for all $w, z \in \mathbb{C}^n$. Let $\mathrm{U}(n) \subset M_n(\mathbb{C})$ denote the group of unitary matrices and let $\mathfrak{u}(n) \subset M_n(\mathbb{C})$ denote its Lie algebra, i.e., the space of skew-Hermitian n-by-n matrices. As is customary, the conjugate transpose operation will be denoted by a superscript asterisk. Thus, $\langle z, w\rangle = z^*w$, and $a \in M_n(\mathbb{C})$ lies in $\mathfrak{u}(n)$ if and only if $a^* = -a$.

2.1. **The unitary coframe bundle.** Let (M, g, Ω) be a Kähler manifold, i.e., M is an n-dimensional complex manifold and g is an Hermitian metric on M whose associated Kähler 2-form Ω is closed. As is customary, let $J : TM \to TM$ be the associated almost complex structure endomorphism.

For $x \in M$, let P_x be the set of unitary isomorphisms $u : T_xM \to \mathbb{C}^n$. Then $P = \bigcup_{x\in M} P_x$ is a principal right $\mathrm{U}(n)$-bundle over M, with the basepoint projection $\pi : P \to M$ given by $\pi\bigl(P_x\bigr) = x$ and $\mathrm{U}(n)$-action given by $u \cdot a = a^{-1}\circ u$ for $a \in \mathrm{U}(n)$.

2.1.1. *The first and second structure equations.* Let ω be the $\mathbb{C}^n$-valued 1-form on P defined by the rule $\omega(v) = u\bigl(\pi'(v)\bigr)$ for all $v \in T_uP$. Then $\pi^*\Omega = -\frac{i}{2}\,\omega^*\wedge\omega$.

Because the structure (M, g, Ω) is Kählerian, there exists a unique $\mathfrak{u}(n)$-valued 1-form ϕ on P satisfying the *first structure equation* of É. Cartan,

$$d\omega = -\phi \wedge \omega. \tag{2.1}$$

The *second structure equation* of É. Cartan takes the form

$$d\phi = -\phi \wedge \phi + \tfrac{1}{2} R(\omega \wedge \omega^*) \tag{2.2}$$

where $R : P \to \mathrm{Hom}\big(\mathfrak{u}(n), \mathfrak{u}(n)\big)$ is the *Kähler curvature function.* The adjoint representation of $\mathrm{U}(n)$ on $\mathfrak{u}(n)$ induces a representation ρ of $\mathrm{U}(n)$ on $\mathrm{Hom}\big(\mathfrak{u}(n), \mathfrak{u}(n)\big)$. The curvature function R is equivariant with respect to this action, i.e., $R(u \cdot a) = \rho(a^{-1})\big(R(u)\big)$ for $a \in \mathrm{U}(n)$.

The first Bianchi identity is $0 = d(d\omega) = -R(\omega\wedge\omega^*)\wedge\omega$. Thus, R takes values in the subspace $\mathcal{K}\big(\mathfrak{u}(n)\big)$ consisting of those elements $r \in \mathrm{Hom}\big(\mathfrak{u}(n), \mathfrak{u}(n)\big)$ that satisfy

$$r(xy^* - yx^*)z + r(yz^* - zy^*)x + r(zx^* - xz^*)y = 0, \quad \forall\, x, y, z \in \mathbb{C}^n.$$

2.1.2. *Tensors, vector fields, and symmetries.* The reader will recall that any (real or complex) representation $\chi : \mathrm{U}(n) \to \mathrm{Aut}(V)$ defines a (tensor) vector bundle $P_\chi = P \times_\chi V$ over M. A section σ of P_χ is then uniquely defined by a function $s : P \to V$ that satisfies the equivariance condition $s(u\cdot a) = \chi(a^{-1})\big(s(u)\big)$ for all $u \in P$ and $a \in \mathrm{U}(n)$ and $\sigma(x) = \big[u, s(u)\big]_\chi$ for some (and hence any) $u \in P_x$. The function s is said to *represent* σ. For example, R represents the Kähler curvature tensor.

For notational simplicity, we will use χ also to denote the induced map on Lie algebras; thus, $\chi : \mathfrak{u}(n) \to \mathrm{End}(V)$. The $\mathrm{U}(n)$-equivariance of a representative function $s : P \to V$ implies that the 1-form $ds + \chi(\phi)\, s$ is π-semibasic. Thus, there exists a linear mapping $Ds : P \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, V)$ satisfying

$$ds + \chi(\phi)\, s = Ds(\omega).$$

Naturally, Ds represents the covariant derivative of the section σ represented by s.

For example, the standard inclusion $\iota : \mathrm{U}(n) \hookrightarrow \mathrm{Aut}(\mathbb{C}^n)$ yields $P_\iota \simeq TM$. A vector field Z on M is represented by the function $z : P \to \mathbb{C}^n$ defined by $z(u) = u\big(Z_{\pi(u)}\big)$. Now, $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}^n) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)\, C$ where $C : \mathbb{C}^n \to \mathbb{C}^n$ is conjugation. Thus, since $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) = M_n(\mathbb{C})$, there are functions z' and z'' on P with values in $M_n(\mathbb{C})$ so that

$$dz + \phi\, z = z'\,\omega + z''\,\bar\omega\,.$$

These functions have the $\mathrm{U}(n)$-equivariance

$$z'(u\cdot a) = a^{-1} z'(u) a, \qquad z''(u\cdot a) = a^{-1} z''(u) \bar a$$

and thus represent tensors on M. In fact, z' represents $\nabla^{1,0}(Z - iJZ)$ while z'' represents $\nabla^{0,1}(Z - iJZ) = \bar\partial(Z - iJZ)$.

In particular, Z is the real part of a holomorphic vector field, namely $Z - iJZ$, if and only if $z'' = 0$. Moreover, computation shows that

$$\pi^*\big(Z \lrcorner \Omega\big) = -\tfrac{i}{2}(z^*\,\omega - \omega^*\, z),$$

implying, in particular, that

$$\pi^*\big(\mathfrak{L}_Z\, \Omega\big) = -\tfrac{i}{2}\,\omega^*\big(z' + (z')^*\big)\,\omega - \tfrac{i}{2}\,\omega^*\, z''\,\bar\omega - \tfrac{i}{2}\,\bar\omega^*\,(z'')^*\,\omega.$$

Thus, the flow of Z is both holomorphic and symplectic (and hence an infinitesimal symmetry of the Kähler structure) if and only if $z'' = 0$ and $z' + (z')^* = 0$. In such a case, $Z = \pi'(Z')$ where Z' is the vector field on P that satisfies

$$\omega(Z') = z, \qquad \phi(Z') = z'.$$

The flow of Z' preserves both ω and ϕ. In fact,

$$\mathfrak{L}_{Z'}\,\omega = d\big(\omega(Z')\big) + Z' \lrcorner \big(-\phi\wedge\omega\big) = dz + \phi\, z - z'\,\omega = 0,$$

so the flow of Z' does indeed preserve ω. Moreover, since ϕ is the unique $\mathfrak{u}(n)$-valued 1-form that satisfies $d\omega = -\phi{\wedge}\omega$, the flow of Z' must preserve ϕ as well.

Conversely, any vector field on P whose flow preserves both ω and ϕ is of the form Z' where Z is a symmetry vector field of the Kähler structure.

If Z is a symmetry vector field of the Kähler structure and Z vanishes at $x \in M$, then $\nabla Z(x) \in T_xM \otimes T^*_xM$ is both skew-symmetric and commutes with the complex structure J_x. Moreover, the flow Φ_Z of Z is complete on the open geodesic ball $B_\delta(x)$ for all sufficiently small $\delta > 0$ and is isometric there. Let $z : P \to \mathbb{C}^n$ represent Z. Then $z(u) = 0$ for all $u \in P_x$ and, by the above discussion, $z'(u)$ belongs to $\mathfrak{u}(n)$. In particular, the linear transformation $a = u^{-1} \circ z'(u) \circ u : T_xM \to T_xM$ is a well-defined skew-Hermitian transformation of T_xM.

Then, for all $v \in T_xM$ with $|v| < \delta$,

$$\Phi_Z\big(t, \exp_x(v)\big) = \exp_x\big(e^{-a\,t}v\big),$$

i.e., the map $\exp_x : B_\delta(0_x) \to B_\delta(x)$ intertwines the linear 1-parameter subgroup action on T_xM generated by exponentiating $-a$ with the flow of Z.

This has two consequences that will be needed in this article (see §4.3.3). First, exponentiating the kernel of a gives the component of the fixed locus of the flow of Z that passes through x, which is therefore a totally geodesic complex submanifold of M. Second, when M is connected, the flow of Z will be periodic of period T if and only if the eigenvalues of $z'(u)$ generate the discrete subgroup of $i\mathbb{R} \subset \mathbb{C}$ that consists of the integral multiples of $2\pi i/T$.

A 'micro-local' version of symmetry will be useful. Two coframes $u, v \in P$ are said to be *equivalent* if there is a connected u-neighborhood U, a connected v-neighborhood V, and a diffeomorphism $\psi : U \to V$ that satisfies $p(u) = v$ and $p^*(\omega_V) = \omega_U$. (It follows, as a consequence, that $p^*(\phi_V) = \phi_U$.) Such a p, when it exists, is unique once U is specified and is locally of the form $p(w) = w \circ (\bar p')^{-1}$ for some local isomorphism $\bar p : \pi(U) \to \pi(V)$ of the Kähler structure on M.

If u and v are equivalent, then $R(u) = R(v)$; in fact, $D^kR(u) = D^kR(v)$ for all $k \geq 0$.[5] Let $\Gamma \subset P \times P$ consist of the equivalent pairs. Then the set $\bar\Gamma = \Gamma/\,\mathrm{U}(n)$ (where the $\mathrm{U}(n)$-action is the diagonal one on $P \times P$) can be identified with the set of pointed local isomorphisms of the Kähler structure on M. For want of a better name, we will refer to $\bar\Gamma$ as the *symmetry pseudo-groupoid* of the Kähler structure.

For any x, the set $\bar\Gamma \cdot x$ is defined to consist of the points $\pi(v)$ where $\pi(u) = x$ and (u, v) lies in Γ. Thus, $\bar\Gamma \cdot x \subset M$ consists of the points $y \in M$ about which the Kähler structure is locally isomorphic to the Kähler structure about x. Even though $\bar\Gamma$ is not a group, we will, by an extension of the usual language, refer to $\bar\Gamma \cdot x$ as the *x-orbit* of the symmetry pseudo-groupoid of the Kähler structure. For any $x \in M$, the x-orbit is a smooth (but not necessarily closed) submanifold of M.

[5] The converse is not generally true, although it is when the Kähler structure is real-analytic.

The subset $\bar{\Gamma}_x = \big(\Gamma \cap (P_x \times P_x)\big)/\,\mathrm{U}(n)$ actually is a group in a natural way, canonically represented as a closed subgroup of $\mathrm{U}(T_xM)$ as the (local) rotations about x that preserve the metric and complex structure. This group will be known as the *stabilizer* of x.

2.1.3. *Curvature decomposition.* Now, the curvature representation $\mathcal{K}\big(\mathfrak{u}(n)\big)$ is a $\mathrm{U}(n)$-invariant subspace of $\mathrm{Hom}\big(\mathfrak{u}(n), \mathfrak{u}(n)\big)$. It is known [17] that $\mathcal{K}\big(\mathfrak{u}(n)\big)$ is isomorphic as a $\mathrm{U}(n)$-module to $S^{2,2}_{\mathbb{R}}(\mathbb{C}^n) = \big(S^{2,0}(\mathbb{C}^n) \otimes_{\mathbb{C}} S^{0,2}(\mathbb{C}^n)\big)_{\mathbb{R}}$, the real-valued quartic functions on $\mathbb{C}^n$ that are complex quadratic and complex conjugate quadratic.

Now, for each $p > 0$, the $\mathrm{U}(n)$-invariant Hermitian inner product on $\mathbb{C}^n$ induces a surjective $\mathrm{U}(n)$-equivariant 'trace' (also called a 'contraction' or 'Laplacian')

$$\mathrm{tr} : S^{p,p}_{\mathbb{R}}(\mathbb{C}^n) \to S^{p-1,p-1}_{\mathbb{R}}(\mathbb{C}^n).$$

Its kernel $S^{p,p}_{\mathbb{R},0}(\mathbb{C}^n) \subset S^{p,p}_{\mathbb{R}}(\mathbb{C}^n)$ is an irreducible $\mathrm{U}(n)$-module [11].

It follows that there is an isomorphism of $\mathrm{U}(n)$-modules

$$\mathcal{K}\big(\mathfrak{u}(n)\big) \simeq S^{2,2}_{\mathbb{R}}(\mathbb{C}^n) \simeq \mathbb{R} \oplus S^{1,1}_{\mathbb{R},0}(\mathbb{C}^n) \oplus S^{2,2}_{\mathbb{R},0}(\mathbb{C}^n) \tag{2.3}$$

where the $\mathrm{U}(n)$-irreducible modules on the right hand side have (real) dimensions 1, n^2-1, and $\frac{1}{4}n^2(n-1)(n+3)$, respectively. Thus, there are unique $\mathrm{U}(n)$-invariant subspaces $\mathcal{K}_i \subset \mathcal{K}\big(\mathfrak{u}(n)\big)$ satisfying $\mathcal{K}_i \simeq S^{i,i}_{\mathbb{R}}(\mathbb{C}^n)$ for $i = 0$, 1, and 2.

The Kähler curvature function R can therefore be written as a sum

$$R = R_0 + R_1 + R_2$$

where R_i takes values in $\mathcal{K}_i$ and represents a section of the bundle $S^{i,i}_{\mathbb{R}}(TM)$, i.e., a tensor associated to the Kähler structure Ω.

The function R_0 represents the scalar curvature, R_1 represents the traceless Ricci tensor, and R_2 represents the *Bochner tensor*, identified in 1949 by S. Bochner [4]. When $n = 1$, both R_1 and R_2 are zero by definition, but when $n \geq 2$, all three tensors are nonzero for the generic Kähler metric.

The Kähler structures for which R_0 vanishes are the scalar-flat Kähler structures. When $n \geq 2$, those for which R_1 vanishes are the Kähler-Einstein structures and those for which R_2 vanishes are known as *Bochner-Kähler* structures.

Remark 2.1 (The Riemannian analogy). Bochner's decomposition of the Kähler curvature bears a resemblance to the more familiar decomposition of the Riemann curvature tensor of a Riemannian metric into the scalar curvature, the traceless Ricci tensor, and the Weyl curvature tensor. However, this resemblance is somewhat misleading.

While the scalar curvature and the Ricci curvature in the two cases do correspond, the Weyl curvature tensor of a Kähler metric is not simply the Bochner curvature tensor. For example, when $n = 2$, so that the underlying manifold has dimension 4 and is canonically oriented, the Bochner tensor turns out to be W^-, the anti-self-dual part of the Weyl curvature. Thus, in complex dimension 2, the Bochner-Kähler metrics are the same as the self-dual Kähler metrics.[6]

Bochner observed [4] that the Weyl curvature of a Kähler metric breaks up into two or three irreducible components under the action of $\mathrm{U}(n) \subset \mathrm{O}(2n)$, one of

[6]These metrics have been studied from this point of view. For example, see [9] and the forthcoming [2]. For further comments on this relationship, see §5.3.

which is the Bochner curvature tensor. One of the other components is equivalent to the scalar curvature while, when $n > 2$, another is equivalent to the traceless Ricci curvature. Thus, when $n > 2$, the vanishing of the Weyl curvature of a Kähler metric implies that the metric is flat. In particular, when $n > 2$, a conformally flat Kähler metric is flat. When $n = 2$, the conformal flatness of a Kähler metric implies only that the structure is Bochner-Kähler, with vanishing scalar curvature.[7]

2.2. **Explicit Bochner-Kähler structures.** Few explicit examples of Bochner-Kähler structures have been found up to now. The main strategy for constructing examples so far has been to look for examples that satisfy conditions sufficiently stringent to reduce the construction to an ODE problem.

Example 2.2 (Locally symmetric). The simplest Bochner-Kähler metric is the complex n-dimensional space M^n_c of constant holomorphic sectional curvature $c \in \mathbb{R}$. (In fact, $R_1 = R_2 = 0$ characterizes these metrics.)

Tachibana and Liu [28, §2] showed that the products $M^p_c \times M^{n-p}_{-c}$ are Bochner-Kähler for any n, p, and c. Moreover, they showed that any Bochner-Kähler structure that is a product in a nontrivial way is locally isomorphic to $M^p_c \times M^{n-p}_{-c}$.

Matsumoto [19, Theorem 2] proved that a Bochner-Kähler structure with constant scalar curvature is locally symmetric. Matsumoto and Tanno [20] then proved that any locally symmetric Bochner-Kähler structure is locally isomorphic to one of the above examples. (For a simple proof, see Proposition 2.5 below.)

Note that their results, combined with the preceding remark, imply the well-known result that the only conformally flat Kähler structures in dimension $n = 2$ are those that are locally isometric to $M^1_c \times M^1_{-c}$ for some $c \geq 0$.

Example 2.3 (Rotationally symmetric). The first examples with nonconstant scalar curvature appear to be due to Tachibana and Liu [28], who considered Kähler structures of the form

$$\Omega = \tfrac{i}{2}\,\partial\bar\partial f\big(|z|^2\big) = -\tfrac{i}{2}\;dz^* \wedge \big[f'\big(|z|^2\big)\;\mathrm{I}_n + f''\big(|z|^2\big)\,z\,z^*\big]\;dz \tag{2.4}$$

where f is smooth and real-valued on some interval $I \subset \mathbb{R}$. The $(1,1)$-form Ω is positive on $D = \{\, z \in \mathbb{C}^n \mid |z|^2 \in I \,\}$ if and only if $f'(t) + t f''(t) > 0$ and $f'(t) > 0$ (when $n > 1$) for all $t \in I \cap [0, \infty)$.

For $n \geq 2$, they showed that Ω is Bochner-Kähler on D if and only if f' satisfies

$$f''(t) = \big(a\,t f'(t) + k\big) f'(t)^2 \tag{2.5}$$

for some constants a and k.[8]

For such an Ω, the eigenvalues of $\mathrm{Ric}(\Omega)$ with respect to Ω are

$$\begin{aligned}\rho_1 &= -2(n{+}1)k - 2(n{+}2)a\,|z|^2 f'\big(|z|^2\big)\,,\\ \rho_2 &= -2(n{+}1)k - 4(n{+}2)a\,|z|^2 f'\big(|z|^2\big)\,,\end{aligned}$$

with ρ_1 having multiplicity $n{-}1$, representing the $(n{-}1)$-plane orthogonal to the radial direction, and ρ_2 having multiplicity 1, representing the radial direction. Thus, the solutions of (2.5) for which $a \neq 0$ yield Bochner-Kähler structures that are not homogeneous.

[7] However, as will be seen in Example 2.2, when $n = 2$ there are essentially only two conformally flat Kähler structures up to local isomorphism and homothety.

[8] While equation (2.5) makes sense even when $n = 1$, 'Bochner-Kähler' has not yet been defined for $n = 1$. This will be remedied in §2.3.6 in such a way that the present discussion extends without change to the case $n = 1$.

Tachibana and Liu integrated the above equation when $k = 0$, thereby giving explicit examples of Bochner-Kähler structures that are not homogeneous. They do not discuss completeness issues, but it is evident from their formulae that none of their explicit examples are complete.

2.2.1. *Further analysis.* Now, (2.5) can be integrated even when $k \neq 0$. Set $x(t) = tf'(t)$, so that (2.5) becomes

$$t\,x'(t) = x(t)\big(1 + k\,x(t) + a\,x(t)^2\big). \tag{2.6}$$

Admissible solutions must satisfy $x > 0$ when $t > 0$ and $x' = f' + tf'' > 0$. Now, (2.6) can be integrated by separation of variables

$$\frac{dx}{x\,(1{+}k\,x{+}a\,x^2)} = \frac{dt}{t}\,. \tag{2.7}$$

Scaling equivalences. Relation (2.7) is invariant under scaling t, which corresponds geometrically to homothety in $\mathbb{C}^n$. Thus, solutions of (2.7) that differ by constant scaling in t represent isomorphic Kähler structures and can be regarded as equivalent. Similarly, multiplying x by a positive constant corresponds to multiplying the Kähler form Ω by that constant, so solutions for a given pair of constants (k, a) can be regarded as equivalent to the solutions for any other pair $(\lambda k, \lambda^2 a)$ with $\lambda \in \mathbb{R}^+$.

The two types of solutions. For any fixed $(k, a) \in \mathbb{R}^2$, let $J_{k,a} \subset \mathbb{R}$ be the maximal x-interval containing 0 on which $(1{+}k\,x{+}a\,x^2)$ is positive. Define a positive function F on $J_{k,a}$ by the formula

$$\log F(x) = -\int_0^x \frac{k{+}a\,\xi}{(1{+}k\,\xi{+}a\,\xi^2)}\,d\xi\,.$$

One solution to (2.7) can then be written implicitly in the form

$$xF(x) = t.$$

The expression on the left hand side of this equation defines a function on $J_{k,a}$ that has positive derivative and that vanishes at $x = 0$. Let $I_{k,a} \subset \mathbb{R}$ denote the range of this function. (More will be said about this range below.) The above relation can then be solved for x, yielding a real-analytic solution $x : I_{k,a} \to J_{k,a}$ to (2.6).

This solution satisfies $x'(0) = 1$. Any other solution to (2.6) whose range lies in $J_{k,a}$ differs from this one by scaling in t. These solutions will be said to be of *type one*.

When $a > 0$ and $k \le -2\sqrt{a}$, there is a second, geometrically distinct, admissible solution to (2.6). Under these assumptions, let $J^*_{k,a}$ be the interval (p, ∞) where

$$p = \frac{-k + \sqrt{k^2 - 4a}}{2a} > 0$$

is the larger root of $(1{+}k\,p{+}a\,p^2) = 0$. (When the two roots are equal, p is simply the root.) Define a function F^* on $J^*_{k,a}$ by the formula

$$\log F^*(x) = -\int_x^\infty \frac{1}{\xi\,(1{+}k\,\xi{+}a\,\xi^2)}\,d\xi.$$

Then (2.7) can be integrated in the form $F^*(x) = t$. Since the integral diverges to infinity as x approaches p from above, the function F^* maps (p, ∞) diffeomorphically onto $(0, 1)$. Thus, the equation $F^*(x) = t$ can be solved for x, yielding a real analytic solution $x^* : (0, 1) \to (p, \infty)$ to (2.6). Any other solution to (2.6) whose range lies in $J^*_{k,a}$ differs from this one by scaling in t. These solutions will be said to be of *type two*.

Completeness. For any admissible solution $x : I \to J$ to (2.6), consider the Bochner-Kähler structure Ω obtained by setting $f'(t) = x(t)/t$. The differential of arc length σ along a radial curve $\{s\,v \mid s^2 \in I\}$ for any fixed $v \in \mathbb{C}^n$ with $|v| = 1$ can be calculated to be

$$d\sigma = \frac{d\big(x(s^2)\big)}{2\sqrt{x(s^2)\big(1 + k\,x(s^2) + a\,x(s^2)^2\big)}}\,. \tag{2.8}$$

This formula permits an analysis of the completeness properties of Ω without having to write down an explicit formula for x.

When $a = 0$, the solution of type one is $x(t) = t/(1-kt)$ and Ω has constant holomorphic sectional curvature. This metric is complete on $\mathbb{C}^n$ when $k = 0$. When $k > 0$, it is complete on the ball $|z|^2 < k^{-1}$. When $k < 0$ it is not complete, since the radial arc length

$$\int_0^{-k^{-1}} \frac{d\xi}{2\sqrt{\xi(1 + k\,\xi)}}$$

is finite. However, in this case, the metric on $\mathbb{C}^n$ extends smoothly to (a multiple of) the Fubini-Study metric on $\mathbb{CP}^n$.

When $a > 0$ and $1 + k\,p + a\,p^2 = 0$ has no positive root p, the interval $J_{k,a}$ contains some interval of the form (α, ∞) for $\alpha < 0$. Because the integral

$$\int_1^{\infty} \frac{d\xi}{\xi(1 + k\,\xi + a\,\xi^2)}$$

converges, the type one solution to (2.6) is defined on an interval $I_{k,a} = (-\delta, R^2)$ for δ and R^2 positive, with $x(t)$ tending to infinity as t approaches R^2. Thus, Ω is defined and nondegenerate on a ball $|z| < R$. Since

$$\int_0^{\infty} \frac{d\xi}{2\sqrt{\xi(1 + k\,\xi + a\,\xi^2)}} < \infty,$$

the metric is not complete. Yet, Ω cannot be extended beyond $|z| < R$ because the two curvatures ρ_1 and ρ_2 tend to $-\infty$ as $|z|$ approaches R.

Suppose now that $1 + k\,p + a\,p^2 = 0$ does have at least one positive root. By the x-scaling argument, it can be assumed that $p = 1$ is a root and that there is no root in the interval $(0, 1)$. Thus, $k = -(1+a)$, so that $(1 + k\,p + a\,p^2) = (1-p)(1-a\,p)$, and $a \leq 1$.

Suppose first that $a = 1$ (the extreme value), so that $(1+k\,p+a\,p^2) = (1-p)^2$. Since the integral

$$\int_0^1 \frac{d\xi}{\xi(1-\xi)^2}$$

diverges at both endpoints, the type one solution h to (2.6) is defined on all of $\mathbb{R}$ and maps $[0, \infty)$ to $[0, 1)$. Thus, Ω is defined and nondegenerate on all of $\mathbb{C}^n$. Since

$$\int_0^1 \frac{d\xi}{2\sqrt{\xi(1-\xi)^2}} = \infty,$$

this metric is complete on $\mathbb{C}^n$. As $|z|^2$ goes to infinity, the curvatures ρ_1 and ρ_2 approach $2n$ and -4, respectively.

Still assuming $(1+k\,p+a\,p^2) = (1-p)^2$, consider the type two solution to (2.6). The form Ω is defined and nondegenerate on the punctured ball $0 < |z| < 1$. The arc length integral shows that this metric is complete on a neighborhood of the puncture but not complete near the boundary $|z| = 1$. Since x goes to infinity as $|z|$ tends to 1, the curvatures ρ_1 and ρ_2 tend to $-\infty$ near this boundary. Thus, Ω cannot be extended beyond the punctured ball $0 < |z| < 1$.

Now suppose $a < 1$. The integral

$$\int_0^1 \frac{d\xi}{\xi(1-\xi)(1-a\xi)}$$

still diverges at both endpoints, so the type one solution to (2.6) is defined on an open interval in $\mathbb{R}$ that contains $[0,\infty)$. Moreover, x maps $[0,\infty)$ to $[0,1)$. Again, Ω is defined and positive definite on all of $\mathbb{C}^n$. However, now, the elliptic integral

$$\int_0^1 \frac{d\xi}{2\sqrt{\xi(1-\xi)(1-a\xi)}}$$

is finite, so the metric is not complete. The curvatures ρ_1 and ρ_2 approach the limits $2(n+1)-2a$ and $2(n+1)(1-a)$, respectively, as $|z|^2$ goes to infinity. It can be shown that this Bochner-Kähler structure extends to an 'orbifold' Bochner-Kähler structure on $\mathbb{CP}^n$ even when $a \neq 0$. We will not discuss this extension here since its nature will be more clear after the considerations to be taken up in the next section. Unless $a = 0$ (the Fubini-Study case), this is not a homogeneous metric.

Finally, when $0 < a < 1$, consider the type two solution to (2.6), whose range is (a^{-1},∞). Since the integral

$$\int_{a^{-1}+1}^{\infty} \frac{d\xi}{\xi(1-\xi)(1-a\xi)}$$

converges, the domain of this solution is $(0,1)$. Then Ω is defined and nondegenerate on the punctured unit ball $0 < |z| < 1$. When $0 < a < 1$, the elliptic integral

$$\int_{a^{-1}}^{\infty} \frac{d\xi}{2\sqrt{\xi(1-\xi)(1-a\xi)}}$$

is finite, so the metric is not complete at either the puncture or the boundary. The curvatures ρ_1 and ρ_2 approach $-\infty$ as $|z|$ approaches 1. However, these curvatures remain bounded and approach a limit when z approaches 0. The nature of the singularity at $|z| = 0$ and whether or not it can be removed will be discussed in §4.

Conclusion. Up to constant multiples and scaling, the Ansatz of Tachibana and Liu provides exactly one example of a complete Bochner-Kähler metric (on $\mathbb{C}^n$) that is not locally symmetric.

Example 2.4 (Ejiri metrics)**.** Ejiri [10] considered a more general Ansatz, seeking Bochner-Kähler metrics for which the Ricci tensor has at most two distinct eigenvalues, an evident property of the Tachibana-Liu examples and the locally symmetric examples. He showed that when $n \geq 3$, such examples that are not locally symmetric have cohomogeneity one and that the isometry stabilizer of the general point is $\mathrm{U}(n-1) \subset \mathrm{U}(n)$. Thus, the problem of describing these examples reduces to an ODE problem, which Ejiri integrated up to a Weierstraß-type equation, thereby producing the desired examples.

In [10, §4], Ejiri remarked that none of his examples (aside from the locally symmetric ones) were known to be complete. However, since the Tachibana-Liu examples are special cases of his examples, at least one of his examples is complete. In fact, Ejiri's example in [10, §4] of a complete, C^2 Bochner-Kähler metric on $\mathbb{R}^{2n}$ turns out to be the complete example of Tachibana and Liu on $\mathbb{C}^n$, but presented in unusual coordinates in which it is not fully regular at the origin. This will become apparent in §4.4, when all of the complete Bochner-Kähler metrics in dimension n will be classified.

2.3. **The differential analysis.** Now suppose that M is a complex manifold of complex dimension $n \geq 2$ endowed with a Bochner-Kähler structure Ω. As before, let $\pi : P \to M$ be the unitary coframe bundle of Ω and denote its canonical forms by ω, with values in $\mathbb{C}^n$, and ϕ, with values in $\mathfrak{u}(n)$. Let $R : P \to \mathcal{K}(\mathfrak{u}(n))$ be the Kähler curvature function.

2.3.1. *Simplification of the curvature.* By definition, Ω is Bochner-Kähler if and only if R_2 vanishes identically. The curvature decomposition of §2.1.3 shows that the remaining part of R takes values in a representation isomorphic to $S^{1,1}_{\mathbb{R}}(\mathbb{C}^n)$, the Hermitian symmetric quadratic forms. Now, for any function $S = S^* : P \to i\,\mathfrak{u}(n) \subset M_n(\mathbb{C})$, the 2-form

$$\Phi = S\,\omega^* \wedge \omega - S\,\omega \wedge \omega^* - \omega \wedge \omega^* S + \omega^* \wedge S\omega\ \mathrm{I}_n$$

takes values in $\mathfrak{u}(n)$ and satisfies $\Phi{\wedge}\omega = 0$ (which is the first Bianchi identity). Moreover, Φ vanishes if and only if S vanishes.

It follows that the assumption that Ω be Bochner-Kähler is equivalent to the existence of a function $S : P \to i\,\mathfrak{u}(n) \subset M_n(\mathbb{C})$ for which

$$d\phi + \phi \wedge \phi = S\,\omega^* \wedge \omega - S\,\omega \wedge \omega^* - \omega \wedge \omega^* S + \omega^* \wedge S\omega\ \mathrm{I}_n\ . \tag{2.9}$$

Now, S does not represent the Ricci tensor *per se*. However, the identity

$$\pi^*\bigl(\mathrm{Ric}(\Omega)\bigr) = i\ \mathrm{tr}\bigl(d\phi{+}\phi \wedge \phi\bigr) = i\,\bigl(\mathrm{tr}(S)\omega^* \wedge \omega + (n{+}2)\,\omega^* \wedge S\omega\bigr)$$

shows how S is related to the Ricci form. In particular, the scalar curvature of the underlying metric is $2\,\mathrm{tr}_\Omega\bigl(\mathrm{Ric}(\Omega)\bigr) = -8(n{+}1)\,\mathrm{tr}\,S$.

2.3.2. *Higher Bianchi identities.* Now, consider the consequences of differentiating (2.9). Setting $\sigma = dS + \phi\,S - S\,\phi$ and taking the exterior derivative of (2.9) leads to the identity

$$\sigma \wedge \omega^* \wedge \omega - \sigma \wedge \omega \wedge \omega^* - \omega \wedge \omega^* \wedge \sigma - \omega^* \wedge \sigma \wedge \omega\ \mathrm{I}_n = 0.$$

This, coupled with the evident identity $\sigma = \sigma^*$ implies, by a straightforward variant of Cartan's Lemma, that there must exist a function $T : P \to \mathbb{C}^n$ so that

$$dS + \phi\,S - S\,\phi = \sigma = T\,\omega^* + \omega\,T^* + \tfrac{1}{2}(T^*\omega + \omega^* T)\ \mathrm{I}_n\ . \tag{2.10}$$

(Equation (2.10) is the second Bianchi identity for Bochner-Kähler structures.)

Setting $\tau = dT + \phi\,T - S^2\,\omega$ and computing the exterior derivative of (2.10) yields

$$\tau \wedge \omega^* - \omega \wedge \tau^* + \tfrac{1}{2}(\tau^* \wedge \omega - \omega^* \wedge \tau)\ \mathrm{I}_n = 0.$$

By another variant of Cartan's Lemma, there is a function $U : P \to \mathbb{R}$ so that

$$dT + \phi\,T - S^2\,\omega = \tau = U\omega. \tag{2.11}$$

(This might be thought of as a sort of *third* Bianchi identity.)

Finally, setting $\upsilon = dU - (T^*S\omega + \omega^*ST)$ and differentiating (2.11) yields $\upsilon\wedge\omega = 0$, implying that $\upsilon = 0$, i.e., that

$$dU = T^*S\omega + \omega^*ST. \tag{2.12}$$

(This is a *fourth* Bianchi identity.) The exterior derivative of (2.12) is an identity.

The collection of formulae

$$\begin{aligned}
d\omega &= -\phi\wedge\omega,\\
d\phi &= -\phi\wedge\phi + S\,\omega^*\wedge\omega - S\,\omega\wedge\omega^* - \omega\wedge\omega^* S + \omega^*\wedge S\omega\ \mathrm{I}_n\,,\\
dS &= -\phi\, S + S\,\phi + T\,\omega^* + \omega\, T^* + \tfrac{1}{2}(T^*\omega + \omega^* T)\ \mathrm{I}_n\,,\\
dT &= -\phi\, T + (U\ \mathrm{I}_n + S^2)\,\omega,\\
dU &= T^*S\omega + \omega^*ST
\end{aligned} \tag{2.13}$$

will be referred to as the *structure equations* of a Bochner-Kähler structure.

2.3.3. *First consequences.* The equations (2.13) allow simple proofs of some known results about Bochner-Kähler structures.

The first part of the following result is due to Matsumoto [19] and the second part is due to Matsumoto and Tanno [20].

Proposition 2.5. *If a Bochner-Kähler structure has constant scalar curvature, then it is a locally symmetric space. Any locally symmetric Bochner-Kähler structure is locally isometric to* $M^p_c \times M^{n-p}_{-c}$ *for some* n, p, *and* c.

Proof. Since the pullback of the scalar curvature to P is $-8(n{+}1)\,\mathrm{tr}\,S$, the hypothesis of constant scalar curvature is equivalent to $d(\mathrm{tr}\,S) = 0$. Now, by the structure equations

$$d(\mathrm{tr}\,S) = \tfrac{1}{2}(n{+}2)\big(T^*\omega + \omega^*T\big),$$

so $d(\mathrm{tr}\,S) = 0$ implies that T vanishes identically. However, if T vanishes identically, then $dS = -\phi S + S\phi$, so that the curvature tensor is parallel. Thus, the structure is locally symmetric. In particular, the eigenvalues of S are all constant.

Also, $T = 0$ implies that $S^2 = -U\,\mathrm{I}_n$. This, combined with the constancy of the eigenvalues of S, implies that U is constant and equal to $-s^2$ for some real number $s \geq 0$. This, in turn, implies that $(S - s\,\mathrm{I}_n)(S + s\ \mathrm{I}_n) = 0$. Consequently, S has at most two distinct eigenvalues. It follows that either $S = \pm s\,\mathrm{I}_n$, in which case the structure has constant holomorphic sectional curvature $\mp 4s$, or else that there is a symmetric frame reduction of P to a $\big(\mathrm{U}(p)\times \mathrm{U}(n{-}p)\big)$-subbundle $P' \subset P$ on which

$$S = \begin{pmatrix} -s\ \mathrm{I}_p & 0 \\ 0 & s\ \mathrm{I}_{n-p} \end{pmatrix}.$$

Thus, the structure is locally isomorphic to $M^p_c \times M^{n-p}_{-c}$ where $c = 4s$. □

The structure equations also yield a simple proof of the following result of Tachibana and Liu.

Proposition 2.6. *If a Bochner-Kähler structure is locally a nontrivial product, then it is locally isometric to* $M^p_c \times M^{n-p}_{-c}$ *for some* n, p, *and* c.

Proof. Assume that the Bochner-Kähler structure is locally a nontrivial product. Then for some $1 \le p \le n/2$, there is a $\bigl(\mathrm{U}(p)\times \mathrm{U}(n{-}p)\bigr)$-subbundle $P' \subset P$ on which ϕ is blocked in the form

$$\phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$$

where ϕ_1 takes values in $\mathfrak{u}(p)$ and ϕ_2 takes values in $\mathfrak{u}(n{-}p)$. This forces S to be blocked in the corresponding form

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

The vanishing of the off-diagonal blocks of the structure equation for dS then shows that T must be zero, thus implying that the structure is locally symmetric. Now apply Proposition 2.5. □

2.3.4. *The structure function.* It turns out[9] to be more convenient to work with $H = S - \frac{1}{n+2}(\operatorname{tr} S)\,\mathrm{I}_n$ than to work with S directly. Thus, $S = H + \frac{1}{2}(\operatorname{tr} H)\,\mathrm{I}_n$, and the structure equations (2.13) assume the form

$$\begin{aligned}
d\omega &= -\phi\wedge\omega,\\
d\phi &= -\phi\wedge\phi + H\ \omega^*\wedge\omega - H\,\omega\wedge\omega^* - \omega\wedge\omega^* H + \omega^*\wedge H\omega\ \mathrm{I}_n\\
&\qquad\qquad + (\operatorname{tr} H)\bigl(\omega^*\wedge\omega\ \mathrm{I}_n - \omega\wedge\omega^*\bigr),\\
dH &= -\phi\, H + H\,\phi + T\,\omega^* + \omega\, T^*,\\
dT &= -\phi\, T + \bigl(H^2 + (\operatorname{tr} H)\, H + V\ \mathrm{I}_n\bigr)\,\omega,\\
dV &= (\operatorname{tr} H)\bigl(T^*\omega + \omega^* T\bigr) + \bigl(T^* H\omega + \omega^* H T\bigr)
\end{aligned} \tag{2.14}$$

where we have also set $V = U + \frac{1}{4}(\operatorname{tr} H)^2$. The map $(H,T,V) : P \to i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$ will be known as the *structure function.*

While several of these equations seem more complicated than their counterparts in (2.13), the decisive simplification is the formula for dH versus the formula for dS, as will be seen. For later use, we record the identity

$$\pi^*\bigl(\mathrm{Ric}(\Omega)\bigr) = (n{+}2)\, i\,\omega^*\wedge\bigl(H + (\operatorname{tr} H)\ \mathrm{I}_n\bigr)\omega \tag{2.15}$$

which follows from the earlier formula for the Ricci form in terms of S.

2.3.5. *Scaling weights.* If Ω is a Bochner-Kähler structure on a complex manifold M, then so is $c\,\Omega$ for any constant $c > 0$. The unitary coframe bundle of this scaled structure is

$$\sqrt{c}\,P = \bigl\{\, \sqrt{c}\,u \mid u \in P \,\bigr\}.$$

The structure functions on the two bundles P and $\sqrt{c}\,P$ then satisfy

$$H\bigl(\sqrt{c}\,u\bigr) = c^{-1} H(u), \quad T\bigl(\sqrt{c}\,u\bigr) = c^{-3/2}\, T(u), \quad V\bigl(\sqrt{c}\,u\bigr) = c^{-2}\, V(u). \tag{2.16}$$

This motivates assigning ‘scaling weights’ to the components of the structure function as follows: H has scaling weight 1, T has scaling weight $\frac{3}{2}$, and V has scaling weight 2. (Taking positive, rather than negative, scaling weights is a simplifying convention.)

[9] This was only noticed in hindsight, after the momentum mapping construction of §3.5.

2.3.6. *Dimension* 1. Equations (2.14) still make sense when $n = 1$, i.e., when M is a complex curve endowed with a positive 2-form Ω and $\pi : P \to M$ is its U(1)-coframe bundle. In this case, H is an $\mathbb{R}$-valued function while T is $\mathbb{C}$-valued. The equations (2.14) then simplify to the scalar equations

$$\begin{aligned} d\omega &= -\phi \wedge \omega\,, \\ d\phi &= -6H\,\omega \wedge \bar\omega\,, \\ dH &= \bar T\,\omega + T\,\bar\omega\,, \\ dT &= -\phi\, T + \big(2H^2 + V\big)\,\omega\,, \\ dV &= 2H\,\big(\bar T\,\omega + T\,\bar\omega\big) = 2H\,dH. \end{aligned} \tag{2.17}$$

Accordingly, when $n = 1$, the satisfaction of these structure equations can be taken to be the *definition* of the Bochner-Kähler property. Throughout this article, this will be done. It is not difficult to check that the rotationally symmetric analysis of Example 2.3 extends to the case $n = 1$ when one takes this as the definition of Bochner-Kähler.

The Gaussian curvature of the associated metric g is $K = -12H$. In fact, the geometric interpretation of the equations (2.17) is just that the Ω-Hamiltonian flow associated to K should be g-isometric. (Compare §2.1.2.) Thus, any constant curvature metric in (complex) dimension 1 is Bochner-Kähler. Moreover, any nonconstant curvature metric in dimension 1 that is Bochner-Kähler has a canonically defined nontrivial Killing field.

Assume that M is connected, which implies that P is also connected. The last structure equation of (2.17) implies that $V - H^2$ is a constant C_2 (the index denotes the scaling weight), and the next-to-last equation of (2.17) then implies that there is a constant C_3 so that $|T|^2 = H^3 + C_2\,H + C_3$, or equivalently, that $|T|^2 - VH = C_3$.

These two 'constants of the structure' will be generalized considerably in higher dimensions, as will the existence of nontrivial symmetry vector fields.

3. Existence and moduli

3.1. **Existence.** In [6], Élie Cartan proved a powerful existence and uniqueness theorem that generalizes Lie's Third Fundamental Theorem from the case of a transitive group action to the case of an intransitive group action.

For the convenience of the reader and because Cartan's rather sketchy treatment needs amplification on some minor points, a discussion of his theorem is included in the Appendix.

Cartan's conditions for the existence of a (local) coframing and system of functional invariants satisfying a given set of structure equations are satisfied by the system (2.14). The following result is then an immediate consequence of his general theorem.

Theorem 3.1. *For any $(H_0, T_0, V_0) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$, there exists a Bochner-Kähler structure Ω on a neighborhood V of $0 \in \mathbb{C}^n$ whose unitary coframe bundle $\pi : P \to V$ contains a $u_0 \in P_0 = \pi^{-1}(0)$ for which $H(u_0) = H_0$, $T(u_0) = T_0$, and $V(u_0) = V_0$. Any two real-analytic Bochner-Kähler structures with this property are isomorphic on a neighborhood of $0 \in \mathbb{C}^n$. Finally, any Bochner-Kähler structure that is C^5 is real-analytic.*

Proof. Since the exterior derivatives of the equations (2.14) are identities, Cartan's conditions (i.e., his generalization of the Jacobi conditions) are satisfied for these equations as structure equations of a coframing.

Thus, by Theorem A.1 (see the Appendix), for any $(H_0, T_0, V_0) \in i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$, there exists a real-analytic manifold N of dimension $n^2{+}2n$ on which there are two real-analytic 1-forms ω and ϕ, taking values in $\mathbb{C}^n$ and $\mathfrak{u}(n)$, respectively, and a real-analytic function $(H, T, V) : P \to i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$ with the properties that (ω, ϕ) is a $\mathbb{C}^n\oplus\mathfrak{u}(n)$-valued coframing on N, that the equations (2.14) are satisfied on N, and that there exists a $u_0 \in N$ for which $\big(H(u_0), T(u_0), V(u_0)\big) = (H_0, T_0, V_0)$.

Since $d\omega = -\phi{\wedge}\omega$, the equation $\omega = 0$ defines an integrable plane field of codimension $2n$ on N. After shrinking N to an open neighborhood of u_0 if necessary, an application of the complex Frobenius theorem shows that there is a submersion $z : N \to \mathbb{C}^n$ with $z(u_0) = 0$ so that the leaves of this integrable plane field are the fibers of π and, moreover, that $dz = p\,\omega$ for some function $p : N \to \mathrm{GL}(n, \mathbb{C})$ that satisfies $p(u_0) = \mathrm{I}_n$.

Since $\phi = -\phi^*$, the 2-form

$$\Omega = -\tfrac{i}{2}\,\omega^* \wedge \omega = -\tfrac{i}{2}\,dz^* \wedge (pp^*)^{-1}\,dz$$

is closed. Since Ω is z-semibasic and since, by definition, the fibers of z are connected, it follows that Ω is actually the pullback to N of a closed, positive (1,1)-form on the open set $V = z(N) \subset \mathbb{C}^n$, i.e., a Kähler structure on V.

Let $\pi : P \to V$ be the unitary coframe bundle of this Kähler structure. Define a mapping $\tau : N \to P$ as follows: If $z(u) = x \in V$, then $dz_u : T_uN \to T_x\mathbb{C}^n \simeq \mathbb{C}^n$ is surjective and, by construction, has the same kernel as $\omega_u : T_uN \to \mathbb{C}^n$. Thus, there is a unique linear isomorphism $\tau(u) : T_x\mathbb{C}^n \to \mathbb{C}^n$ so that $\omega_u = \tau(u) \circ dz_u$. In fact, $\tau(u)$ is complex linear; using the standard identification $T_x\mathbb{C}^n \simeq \mathbb{C}^n$, one sees that $\tau(u)$ becomes $p(u)^{-1} \in M_n(\mathbb{C})$.

The equation $\Omega = -\frac{i}{2}\,\omega^*{\wedge}\omega$ implies that $\tau(u)$ is a unitary coframe for all $u \in N$. Since (ω, ϕ) is a coframing, it follows that $\tau : N \to P$ is an open immersion of N into P. Shrinking N again if necessary, it can be assumed that τ embeds N as an open subset of P. Thus, nothing is lost by identifying N with this open subset of P.

The structure equations (2.14) now become identified with the structure equations of the unitary coframe bundle P, implying that the underlying Kähler structure on V is, in fact, Bochner-Kähler, and that the structure function (H, T, V) takes on the value (H_0, T_0, V_0) at $u_0 \in P$, as desired. Further details are left to the reader. This completes the existence proof.

Uniqueness in the real-analytic category now follows directly from Theorem A.1. Now, while Cartan states the uniqueness part of Theorem A.1 only in the real-analytic category, uniqueness can actually be proved using only ordinary differential equations (i.e., the Frobenius theorem); the Cauchy-Kowalewski or Cartan-Kähler Theorems are not needed. Thus, his uniqueness result is valid as long as the form Ω is sufficiently differentiable for P to exist as a differentiable bundle and for H, T, and V to be defined and differentiable. For this to be true, it certainly suffices for Ω to be C^5.

Since Cartan's existence proof produces a real-analytic example, uniqueness then implies that any C^5 Bochner-Kähler structure is real-analytic. □

Remark 3.2 (Minimal regularity). With some work, one can show that if H and T are differentiable, then V (which, by (2.11), must exist) must be differentiable as well, thus reducing the regularity needed to apply Cartan's Theorem to C^4. However, this is almost certainly not optimal since, presumably, when $n \geq 2$, any C^2 Kähler structure that is Bochner-Kähler is real-analytic. However, the above proof does not show this.

From now on, we will assume that the Bochner-Kähler structures under consideration are real-analytic.

3.2. **Local moduli.** The group $\mathrm{U}(n)$ acts on the space $i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ in the usual way:

$$a \cdot (h, t, v) = (aha^*,\, at,\, v) \tag{3.1}$$

for $a \in \mathrm{U}(n)$. This action makes the structure function of a Bochner-Kähler structure $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ equivariant with respect to the right bundle action, i.e.,

$$\big(H(u{\cdot}a), T(u{\cdot}a), V(u{\cdot}a)\big) = a^{-1} \cdot \big(H(u), T(u), V(u)\big). \tag{3.2}$$

Consequently, it will be useful to have an understanding of the orbits of $\mathrm{U}(n)$ acting on this space.

3.2.1. *Orbits.* Let $W \subset i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ be the linear subspace consisting of the triples (h, t, v) for which h is diagonal and t is real. Then W is a linear subspace of (real) dimension $2n{+}1$. Let $C \subset W$ be the 'chamber' defined by the inequalities $h_{1\bar1} \geq h_{2\bar2} \geq \cdots \geq h_{n\bar n}$ augmented by the conditions that $t_j \geq 0$, with equality if $h_{j\bar j} = h_{i\bar i}$ for any $i < j$. N.B.: The set C has nonempty interior in W. Note, however, that C is not closed when $n \geq 2$.

Proposition 3.3. *Each* $\mathrm{U}(n)$*-orbit in* $i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ *meets* C *in exactly one point.*

Proof. Consider any $(h, t, v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$. Act by an element $a \in \mathrm{U}(n)$ so as to reduce to the case where h is diagonal and its (real) eigenvalues are arranged in decreasing order down the diagonal. If there are integers $i \leq j$ so that $h_{j\bar j} = h_{i\bar i}$, suppose that $i, i{+}1, \ldots, j$ is a maximal unbroken string with this property. Then the stabilizer of h in $\mathrm{U}(n)$ will contain a subgroup isomorphic to $\mathrm{U}(j{-}i{+}1)$ that will act as unitary rotations on the subvector $(t_i, \ldots, t_j)$. Acting by an element of the stabilizer of h, one can then reduce to the case where t_i is real and nonnegative while $t_{i+1} = \cdots = t_j = 0$. By definition, the resulting new (h, t, v) is an element of C. It is clear from the construction that this element is unique. □

Corollary 3.4. *The set of isomorphism classes of germs of Bochner-Kähler structures in dimension* n *is in one-to-one correspondence with the elements of* C*.*

3.2.2. *Invariant polynomials.* It is not difficult to exhibit enough $\mathrm{U}(n)$-invariant polynomials on $i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ to separate the $\mathrm{U}(n)$-orbits. For $k \geq 0$, define the $\mathrm{U}(n)$-invariant polynomials

$$a_k(h, t, v) = \mathrm{tr}(h^k), \qquad b_{k+3}(h, t, v) = t^* h^k\, t\,, \tag{3.3}$$

and set $b_2(h, t, v) = v$. (The indexing is chosen so as to indicate the scaling weight as defined in §2.3.5. The anomalous definition of b_2 will be explained below.) Then an easy argument using Proposition 3.3 shows that the collection of $2n{+}1$ functions

$$\varphi = (a_1, \dots, a_n, b_2, b_3, \dots, b_{n+2}) \tag{3.4}$$

separates the $\mathrm{U}(n)$-orbits in $i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$.[10]

When $n = 1$, the function $a_1^2 + {b_2}^2 + b_3 \geq 0$ is evidently a proper function on $i\,\mathfrak{u}(1)\oplus\mathbb{C}\oplus\mathbb{R}$ while, for $n \geq 2$, the function $a_2 + {b_2}^2 + b_3 \geq 0$ is a proper function on $i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$.

It follows that φ is a proper mapping, implying that $F_n = \varphi\bigl(i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}\bigr)$ is closed in $\mathbb{R}^{2n+1}$. The set $F_n \subset \mathbb{R}^{2n+1}$ is thus the proper moduli space of orbits.

If $(h_0, t_0, v_0) \in i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$ is such that h_0 has n distinct eigenvalues and t_0 is not orthogonal to any of the eigenvectors of h_0, then an elementary computation shows that $\varphi'(h_0, t_0, v_0) : i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R} \to \mathbb{R}^{2n+1}$ is surjective. It follows from this that F_n is the closure of its interior. Of course, $\varphi : C \to F_n$ is a bijection.

3.2.3. *The moduli mapping.* This description of F_n can be interpreted as saying that the germs of Bochner-Kähler structures in dimension n form a singular space of real dimension $2n{+}1$. It is F_n that is the natural moduli space for germs of Bochner-Kähler structures in the following sense: For any Bochner-Kähler structure (M, g, Ω), there is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{(H,T,V)} & i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ M & \xrightarrow{\ f\ } & F_n \subset \mathbb{R}^{2n+1} \end{array} \tag{3.5}$$

where $f : M \to F_n$ is a real-analytic map each of whose fibers is an orbit of the symmetry pseudo-groupoid of the Bochner-Kähler structure on M. This function will be known as the *moduli mapping* of the Bochner-Kähler structure.

3.2.4. *Analytic connectedness.* However, this description does not really say 'how many' Bochner-Kähler structures there are locally since, for a given Bochner-Kähler structure, the map $f : M \to F_n$ might have rather large image in F_n. A priori, the image could even have dimension as large as $2n$, in which case one would be tempted to say that the 'generic' Bochner-Kähler structures depend on only one parameter, the parameter that distinguishes the 'hypersurfaces' in F_n that are the images of generic Bochner-Kähler structure maps. However, as will be shown in the next subsection, this is not the case. Instead, the dimension of the image $f(M)$ turns out to be no more than n for any Bochner-Kähler structure.

One of the difficulties that arises in discussing this 'how many' question is that it turns out that not every connected Bochner-Kähler structure can be regarded as an open subset of a unique 'maximal' connected Bochner-Kähler structure (cf. the discussion of the dimension $n = 1$ at the end of §3.2.5). Even when one restricts attention to the simply-connected, connected Bochner-Kähler structures, this difficulty persists. Compare this situation with that of locally symmetric spaces: Every simply-connected, connected locally symmetric space has an isometric open immersion (sometimes called a developing map) into a unique (complete) simply-connected symmetric space and this immersion is unique up to ambient isometry.

[10]In fact, by [23, Theorem 12.1], the components of φ generate the ring of $\mathrm{U}(n)$-invariant polynomials on $i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$.

The discussion carried out in Example 2.3 and in §3.2.5 below shows that no such result could hold for Bochner-Kähler structures.

Two elements $v_1, v_2 \in F_n$ will be said to be *analytically connected* if there exists a connected Bochner-Kähler manifold (M, Ω) so that both v_1 and v_2 lie in $f(M)$. An elementary argument shows that this is an equivalence relation. One of the tasks of this article is to describe these equivalence classes explicitly.

3.2.5. *Dimension* 1. Now, $i\,\mathfrak{u}(1) = \mathbb{R}$ and the map $\varphi : \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{R} \to \mathbb{R}^3$ takes the form

$$\varphi(h, t, v) = \big(h, v, |t|^2\big).$$

Thus $F_1 \subset \mathbb{R}^3$ is the closed upper half-space. The fiber $\varphi^{-1}(x, y, 0)$ is a single point for each $(x, y, 0)$ on the boundary of F_1 while the fiber $\varphi^{-1}(x, y, z)$ is a circle when (x, y, z) lies in the interior F_1°, i.e., when $z > 0$.

By Theorem 3.1, every point of F_1 lies in the f-image of some Bochner-Kähler structure in dimension 1.

Let (M, g, Ω) be a connected Bochner-Kähler manifold of dimension 1, so that $\varphi \circ (H, T, V) = (H, V, |T|^2)$. As was pointed out in §2.3.6, there are constants C_2 and C_3 so that

$$V - H^2 = C_2 \qquad \text{and} \qquad |T|^2 - HV = C_3\,.$$

In other words, $\varphi \circ (H, T, V) = (H,\ H^2{+}C_2,\ H^3{+}C_2H{+}C_3)$, implying that the map $f : M \to \mathbb{R}^3$ has its image either a point (if H is constant) or a curve.

For any $C = (C_2, C_3)$, let $p_C(t) = t^3{+}C_2\,t{+}C_3$ and set

$$\Gamma_C = \big\{(t,\ t^2{+}C_2,\ t^3{+}C_2\,t{+}C_3) \,\big|\, p_C(t) \ge 0\,\big\}.$$

Since $dH = \bar{T}\,\omega + T\,\bar{\omega}$, it follows that df vanishes only at those $x \in M$ where $|T|^2 = 0$. In other words, if $M^\circ = f^{-1}(F_1^\circ)$ is the locus where $|T|^2$ is nonzero, then $f : M^\circ \to F_1^\circ$ is a submersion onto an open subset of $\Gamma_C^\circ = \Gamma_C \cap F_1^\circ$.

Since $4|T|^2$ is the squared norm of the gradient of dH and the Ω-Hamiltonian of H is a Killing field on the surface, it follows that either $|T|^2$ vanishes identically or else it vanishes only at isolated points in M and then only to second order.

In the former case, H is constant on M. By the structure equations (2.17), since T vanishes identically it follows that $V \equiv -2H^2$. Thus, each of the points $v = (r, -2r^2, 0) \in F_1$ constitutes a single analytically connected equivalence class that is the f-image of any surface endowed with a metric of constant curvature $K = -12r$. Note that, in this case, the constants C_2 and C_3 assume the values $C_2 = -3r^2$ and $C_3 = 2r^3$, so that $p_C(t) = (t - r)^2(t + 2r)$ has either a double or triple root (if $r = 0$). Let $\Pi = \big\{(r, -2r^2, 0) \,\big|\, r \in \mathbb{R}\big\}$ be the parabola of 'isolated' classes. These are the only points that can be the value of a constant f.

Now suppose that $|T|^2$ is not identically zero, so that M° is simply M minus a set of isolated points.

When $p_C(t)$ has only one real simple root, say r_0, then Γ_C is connected and homeomorphic to a closed half-line. Call this Case 1. In this case $\Gamma_C \cap \Pi = \emptyset$, so that it is not possible for M to satisfy $f(M) \subset \Gamma_C$ and have $f(M)$ be a point. Since $f : M^\circ \to \Gamma_C^\circ$ is a submersion, it follows that if $f(M)$ lies in Γ_C, then $f(M)$ is an open subset of Γ_C. Since Γ_C is connected, it follows that Γ_C must constitute a single analytically connected equivalence class.

When $p_C(t)$ has only one real root, but this root is multiple, the only possibility is that this root is $t = 0$ and, in fact, $p_C(t) = t^3$. Call this Case 2. In this case,

$\Gamma_C = \{(0,0,0)\} \cup \Gamma_C^\circ$ where $\Gamma_C^\circ = \big\{(t,\, t^2,\, t^3)\big| t > 0\,\big\}$. In this case, $f(M)$ can lie in Γ_C only if either $f(M) = \{(0,0,0)\}$ or $f(M)$ is an open subset of Γ_C°. Since Γ_C° is connected, it follows that Γ_C° constitutes a single analytically connected equivalence class.

When $p_C(t)$ has two real distinct roots, say $r_1 > r_2$, one must be double, so there are two possibilities. Case 3-i will be that in which r_i is the double root.

In Case 3-1, $p_C(t) = (t-r)^2(t+2r)$ where $r > 0$. Since $\Gamma_C \cap \Pi = \{(r, -2r^2, 0)\}$, define

$$\begin{aligned}\Gamma_C^a &= \big\{\big(t,\, t^2{-}3r^2,\, (t{-}r)^2(t{+}2r)\big) \,\big|\, t > r\,\big\},\\ \Gamma_C^b &= \big\{\big(t,\, t^2{-}3r^2,\, (t{-}r)^2(t{+}2r)\big) \,\big|\, -2r \le t < r\,\big\}.\end{aligned}$$

Then $\Gamma_C = \Gamma_C^b \cup \{(r,-2r^2,0)\} \cup \Gamma_C^a$, and each of Γ_C^a, $\{(r,-2r^2,0)\}$, and Γ_C^b is evidently a single analytically connected equivalence class.

In Case 3-2, $p_C(t) = (t-r)^2(t+2r)$ where $r < 0$. Still, $\Gamma_C \cap \Pi = \{(r, -2r^2, 0)\}$, but now $\Gamma_C = \{(r,-2r^2,0)\} \cup \Gamma_C^a$ where

$$\Gamma_C^a = \big\{\big(t,\, t^2{-}3r^2,\, (t{-}r)^2(t{+}2r)\big) \,\big|\, -2r \le t\,\big\},$$

and each of $\{(r,-2r^2,0)\}$ and Γ_C^a is evidently a single analytically connected equivalence class.

When $p_C(t)$ has three distinct real roots, say $r_0 > r_1 > r_2$, then $r_0 + r_1 + r_2 = 0$ and again $\Gamma_C \cap \Pi = \emptyset$. Call this Case 4. In this case, Γ_C has two components

$$\begin{aligned}\Gamma_C^0 &= \big\{\big(t,\, t^2 + (r_0r_1{+}r_0r_2{+}r_1r_2),\, (t{-}r_0)(t{-}r_1)(t{-}r_2)\big) \,\big|\, r_0 \le t\,\big\},\\ \Gamma_C^1 &= \big\{\big(t,\, t^2 + (r_0r_1{+}r_0r_2{+}r_1r_2),\, (t{-}r_0)(t{-}r_1)(t{-}r_2)\big) \,\big|\, r_2 \le t \le r_1\,\big\},\end{aligned}$$

each of which is a single analytically connected equivalence class.

Now, in all these cases, the metric $g = \omega \circ \bar\omega$ can be expressed directly in terms of the invariants. Restrict attention to $M^\circ \subset M$ and note that, by the structure equations, the complex-valued 1-form ω/T is closed and therefore a nowhere vanishing holomorphic 1-form on M°. Since $|T|^2$ vanishes only to second order at each of its zeroes, ω/T extends to all of M as a meromorphic 1-form with simple poles at the places where ω/T vanishes.

Also, since $|T|^2 = H^3 + C_2\,H + C_3$, it follows that

$$\frac{dH}{H^3 + C_2\,H + C_3} = \frac{dH}{|T|^2} = \frac{\omega}{T} + \frac{\bar\omega}{\bar T}.$$

Thus,

$$\frac{\omega}{T} = \frac{dH}{2(H^3 + C_2\,H + C_3)} + 2i\,d\theta$$

where θ is locally well-defined on M° up to a (real) additive constant (the factor of 2 in front of the $d\theta$ term provides for consistency with later notation). Consequently, one has the formula

$$g = \omega \circ \bar\omega = \frac{dH^2}{4(H^3 + C_2\,H + C_3)} + 4(H^3 + C_2\,H + C_3)\,d\theta^2\,.$$

More precisely, the simply-connected cover $\widetilde{M^\circ}$ admits a developing map $(H,\theta) : \widetilde{M^\circ} \to \mathbb{R}^2$ that isometrically embeds $\widetilde{M^\circ}$ into the region R_C in the $H\theta$-plane defined by the inequality $H^3 + C_2\,H + C_3 > 0$, endowed with the above metric.

Using this representation, one can determine which of the Cases above can allow complete Bochner-Kähler metrics in dimension 1.

For example, let R be the largest real root of $p_C(t)$. Then because the integral

$$\int_{R+1}^{\infty} \frac{dH}{2\sqrt{H^3 + C_2\,H + C_3}}$$

converges, the metric g defined above is not complete at the 'edge' $H = \infty$ of the half-plane $H > R$. This implies that if (M, g, Ω) is a Bochner-Kähler metric with characteristic polynomial p_C, satisfying $H \geq R$, and having H nonconstant, then the length of the gradient lines of H would be finite in the increasing direction and so could not be complete.

Consequently, a complete Bochner-Kähler metric must have its image lie in a bounded region of F_1. In particular, the analytically connected component that contains $f(M)$ must be bounded. The only bounded analytically connected equivalence classes are

(1) Case 3 with $f(M) = \{(r, -2r^2, 0)\}$;
(2) Case 3-1 with $f(M) = \Gamma^b_C$; and
(3) Case 4 with $f(M) = \Gamma^1_C$.

The case of a single point has already been discussed: There is a unique connected and simply-connected complete example for each r.

In Case 3-1, with $f(M) = \Gamma^b_C$ with $r > 0$, the metric g on the region $-2r < H < r$ in the $H\theta$-plane is of the form

$$g = \frac{dH^2}{4(H{-}r)^2(H{+}2r)} + 4(H{-}r)^2(H{+}2r)\,d\theta^2\,.$$

Because

$$\int_0^r \frac{dH}{2\sqrt{(H{-}r)^2(H{+}2r)}} = \infty,$$

this metric is complete near the 'edge' $H = r$. However, since

$$\int_{-2r}^0 \frac{dH}{2\sqrt{(H{-}r)^2(H{+}2r)}} < \infty,$$

the metric is not complete near the 'edge' $H = -2r$. In fact, making the substitution $H + 2r = 3r\rho^2$, the metric takes the form

$$g = \frac{d\rho^2 + \rho^2(1-\rho^2)^4\,(18r^2\,d\theta)^2}{(1-\rho^2)^2}\,,$$

and one recognizes that g will extend to a smooth metric at $\rho = 0$ in polar coordinates (ρ, θ) on the disk $\rho < 1$ if and only if θ is taken to be periodic with period $\pi/(9r^2)$. This disk endowed with this complete metric is conformally equivalent to $\mathbb{C}$. The Gaussian curvature decreases monotonically from $24r$ at $\rho = 0$ to a limiting value of $-12r$ as ρ approaches 1.

Finally, consider Case 4 with image in Γ^1_C. Let $r_0 > r_1 > r_2$ be the three roots satisfying $r_0 = -(r_1 + r_2)$, so that $H^3 + C_2\,H + C_3 = (H{-}r_0)(H{-}r_1)(H{-}r_2)$. Consider the metric on the strip $r_2 < H < r_1$ in the $H\theta$-plane given by

$$g = \frac{dH^2}{4(H-r_0)(H-r_1)(H-r_2)} + 4(H-r_0)(H-r_1)(H-r_2)\,d\theta^2\,.$$

Since

$$\int_{r_2}^{r_1} \frac{dH}{2\sqrt{(H-r_0)(H-r_1)(H-r_2)}} < \infty,$$

this metric is not complete at either edge $H = r_i$ for $i = 1, 2$.

Letting $H = r_2 + v^2$ and computing as above, one finds that the metric will extend to a smooth metric on a disk about $v = 0$ in (v, θ) polar coordinates if and only if θ is taken to be periodic with period

$$\tau_2 = \frac{\pi}{3{r_2}^2 + C_2} > 0\,.$$

Similarly, setting $H = r_1 - w^2$, and computing as above, one finds that the metric will extend to a smooth metric on a disk about $w = 0$ in (w, θ) polar coordinates if and only if θ is taken to be periodic with period

$$\tau_1 = \frac{-\pi}{3{r_1}^2 + C_2} > 0\,.$$

Now, computation shows that $\tau_1 = \tau_2$ has no solutions with $r_1 > r_2$.

Consequently, there is no complete Bochner-Kähler metric on a surface whose moduli image is Γ^1_C.

However, complete Bochner-Kähler metrics on orbifolds do exist: Taking $r_1 = r(q-2p)$ and $r_2 = r(p-2q)$ where $0 < p < q$ are relatively prime integers and r is a positive real number, one can choose a period for θ so that the resulting quotient completes to an orbifold metric on S^2 with one conical point of order $1/q$ and the other of order $1/p$.

This orbifold is the weighted projective line $\mathbb{CP}^{(p,q)}$, i.e., $\mathbb{C}^2$ minus the origin modulo the $\mathbb{C}^*$-action $\lambda \cdot (z, w) = \bigl(\lambda^p z, \lambda^q w\bigr)$. This compact Riemannian orbifold could reasonably be regarded as the natural complete model for this case. Note that the Gaussian curvature of this metric will be strictly positive if and only if $q < 2p$.

3.3. **Infinitesimal symmetries.** It turns out that any Bochner-Kähler structure has a nontrivial symmetry pseudo-groupoid $\bar\Gamma$. In this subsection, some useful information about the 'dimension' and orbits of $\bar\Gamma$ will be collected.

For $(h, t, v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$, let $G^0_{(h,t,v)} \subset \mathrm{U}(n)$ be the stabilizer of (h, t, v) under the action defined in §3.2. Since $a \in \mathrm{U}(n)$ lies in $G^0_{(h,t,v)}$ if and only if $aha^* = h$ and $at = t$, it follows that $G^0_{(h,t,v)}$ is a closed, connected subgroup of $\mathrm{U}(n)$.

In fact, $G^0_{(h,t,v)}$ is a product of unitary groups and can be described as follows: Let $h_1 > h_2 > \cdots > h_\delta$ be the distinct eigenvalues of h and, for $1 \le \alpha \le n$, let $L_\alpha \subseteq \mathbb{C}^n$ be the h_α-eigenspace of h. Since h is Hermitian symmetric, there is an orthogonal direct sum decomposition

$$\mathbb{C}^n = L_1 \oplus L_2 \oplus \cdots \oplus L_\delta$$

with $\dim L_\alpha = n_\alpha \ge 1$. Write $t = t_1 + \cdots + t_\delta$ where t_α lies in L_α and let $t_\alpha^\perp \subseteq L_\alpha$ be the subspace of L_α that is perpendicular to t_α. Then, using obvious notation,

$$G^0_{(h,t,v)} = \mathrm{U}(t_1^\perp) \times \mathrm{U}(t_2^\perp) \times \cdots \times \mathrm{U}(t_\delta^\perp).$$

The uniqueness part of Cartan's Theorem A.1 then has the following useful corollary.

Corollary 3.5. *Let $P \to M$ be a Bochner-Kähler structure. Then for any $u \in P_x$, the unitary isomorphism $u : T_xM \to \mathbb{C}^n$ induces an isomorphism*

$$\bar{\Gamma}_x \simeq G^0_{(H(u),T(u),V(u))}.$$

Thus, $\bar{\Gamma}_x$ is isomorphic to a product of unitary groups and, in particular, is connected.

3.3.1. *Existence and lower bounds.* Roughly speaking, a Bochner-Kähler structure has at least an n-dimensional 'infinitesimal symmetry group'. As will be seen below, this lower bound is reached for the 'generic' Bochner-Kähler structure.

Theorem 3.6. *Let M be a simply-connected complex n-manifold endowed with a Bochner-Kähler structure Ω. Let $\mathfrak{g} \subset \mathfrak{X}(M)$ denote the Lie algebra of vector fields on M whose flows preserve the complex structure and Ω. Then $\dim_{\mathbb{R}} \mathfrak{g} \geq n$.*

Proof. Let (M, Ω) satisfy the assumptions of the theorem, let $\pi : P \to M$ be the unitary coframe bundle, with canonical forms ω and ϕ, and let $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ be the structure function.

Because M is simply-connected and the Bochner-Kähler structure Ω is real-analytic, any symmetry vector field of the structure defined on a connected open subset of M can be uniquely analytically continued to a symmetry vector field on all of M. Moreover if $Z \in \mathfrak{X}(M)$ is such a symmetry vector field, then, as discussed in §2.1, there is a unique vector field Z' on P satisfying $\pi'(Z') = Z$ and $\mathfrak{L}_{Z'}\,\omega = \mathfrak{L}_{Z'}\,\phi = 0$. Conversely, if Y is a vector field on P satisfying $\mathfrak{L}_Y\,\omega = \mathfrak{L}_Y\,\phi = 0$, then $Y = Z'$ where $Z = \pi'(Y)$ is a symmetry vector field on M.

In other words, the mapping $Z \mapsto Z'$ defines an embedding $\mathfrak{g} \hookrightarrow \mathfrak{X}(P)$ that realizes $\mathfrak{g}$ as the Lie algebra of vector fields on P whose flows preserve the coframing $\eta = (\omega, \phi)$. By the structure equations (2.14) the flow of such a vector field must necessarily preserve the structure function $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$, which is a submersion onto its (connected) image.

Applying Cartan's Theorem A.2 (see the Appendix), for any $u \in P$ the evaluation map $e_u : \mathfrak{g} \to T_uP$ defined by $e_u(Z) = Z'(u) \in T_uP$ is a vector space isomorphism between $\mathfrak{g}$ and the kernel of $(H, T, V)'(u) : T_uP \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$. Let $K_u \subset T_uP$ denote this kernel. Then, by (2.14), the image $(\omega, \phi)(K_u) \subset \mathbb{C}^n \oplus i\,\mathfrak{u}(n)$ consists of the pairs $(w, f) \in \mathbb{C}^n \oplus i\,\mathfrak{u}(n)$ that satisfy

$$\begin{aligned}
0 &= H(u)\,f - f\,H(u) + T(u)\,w^* + w\,T(u)^*\,,\\
0 &= -f\,T(u) + \big(H(u)^2 + (\operatorname{tr} H(u))\,H(u) + V(u)\ \mathrm{I}_n\big)\,w\,,\\
0 &= (\operatorname{tr} H(u))\big(T(u)^*\,w + w^*T(u)\big) + T(u)^*H(u)\,w + w^*H(u)T(u).
\end{aligned}$$

By the first of these equations,

$$T(u)^*\,w + w^*T(u) = \operatorname{tr}\big([f, H(u)]\big) = 0$$

and

$$2(T(u)^*H(u)\,w + w^*H(u)T(u)) = 2\operatorname{tr}\big([f, H(u)]H(u)\big) = \operatorname{tr}\big([f, H(u)^2]\big) = 0.$$

Thus, the third equation is a consequence of the first and so can be ignored for the rest of this discussion.

Let $(H, T, V)(u_0) = (H_0, T_0, V_0) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$. By $\mathrm{U}(n)$-equivariance, it is enough to show that the dimension of K_{u_0} is at least n at any point u_0 where H_0 and T_0 are both real, so assume this for the rest of the argument.

By the structure equations (2.14), the dimension of K_{u_0} is equal to the dimension of the space of solutions of the linear equations

$$(3.6)\qquad \begin{aligned} 0 &= H_0\, f - f\, H_0 + T_0\, w^* + w\, T_0^*, \\ 0 &= -f\, T_0 + \bigl({H_0}^2 + (\operatorname{tr} H_0)\, H_0 + V_0\ \mathrm{I}_n\bigr)\, w \end{aligned}$$

for $w \in \mathbb{C}^n$ and $f \in \mathfrak{u}(n)$.

Consider the solutions of (3.6) for which f and w are purely imaginary, i.e., where $f = is$ and $w = iy$ for some symmetric (real) matrix s and some $y \in \mathbb{R}^n$. Then the equations in (3.6) reduce to

$$(3.7)\qquad \begin{aligned} 0 &= H_0\, s - s\, H_0 - T_0\, {}^t y + y\, {}^t T_0\,, \\ 0 &= -s\, T_0 + \bigl({H_0}^2 + (\operatorname{tr} H_0)\, H_0 + V_0\ \mathrm{I}_n\bigr)\, y\,. \end{aligned}$$

The right hand side of the first equation of (3.7) takes values in $\mathfrak{so}(n)$ and the right hand side of the second equation of (3.7) takes values in $\mathbb{R}^n$. Thus, this is $\frac{1}{2}n(n{-}1) + n$ equations for the $\frac{1}{2}n(n{+}1) + n$ components of $s = {}^t s$ and y. Consequently, the space of solutions is at least of dimension n. □

3.3.2. *The symmetry algebra.* For $(h, t, v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$, let $\mathfrak{g}_{(h,t,v)} \subset \mathbb{C}^n \oplus \mathfrak{u}(n)$ be the space of solutions $(w, f) \in \mathbb{C}^n \oplus \mathfrak{u}(n)$ of the linear equations

$$(3.8)\qquad \begin{aligned} 0 &= h\, f - f\, h + t\, w^* + w\, t^*, \\ 0 &= -f\, t + \bigl(h^2 + (\operatorname{tr} h)\, h + v\ \mathrm{I}_n\bigr)\, w. \end{aligned}$$

As was established in the course of the above proof, $\mathfrak{g}_{(h,t,v)}$ is isomorphic as a vector space to the symmetry algebra $\mathfrak{g}$ of any simply-connected Bochner-Kähler manifold whose structure function assumes the value (h, t, v). In fact, the structure equations (2.14) show that, if $\bigl(H(u), T(u), V(u)\bigr) = (h, t, v)$, then the vector space isomorphism $\mathfrak{g} \to \mathfrak{g}_{(h,t,v)}$ defined by $X \mapsto \bigl(\omega_u(X'), \phi_u(X')\bigr)$ induces a Lie algebra structure on $\mathfrak{g}_{(h,t,v)}$ that is given by the formula

$$\bigl[(x, x'), (y, y')\bigr] = \bigl(-x'y + y'x,\ -[x', y'] + \{x, y\}_h\bigr)$$

where, for $x, y \in \mathbb{C}^n$, the element $\{x, y\}_h$ in $\mathfrak{u}(n)$ is defined by

$$\begin{aligned} \{x, y\}_h = &- h\,(xy^* - yx^*) - (xy^* - yx^*)\, h + (x^*y - y^*x)\, h \\ &+ (x^*hy - y^*hx)\ \mathrm{I}_n + (\operatorname{tr} h)\bigl((x^*y - y^*x)\ \mathrm{I}_n - xy^* + yx^*\bigr). \end{aligned}$$

For $x \in M$, let $\mathfrak{g}_x \subset \mathfrak{g}$ denote the subalgebra that consists of the vector fields in $\mathfrak{g}$ that vanish at x. Under the vector space isomorphism $\mathfrak{g} \to \mathfrak{g}_{(h,t,v)}$ defined above, $\mathfrak{g}_x$ maps into the subalgebra $\mathfrak{g}^0_{(h,t,v)} \subset \mathfrak{g}_{(h,t,v)}$ defined by $w = 0$.

Since information about $\mathfrak{g}^0_{(h,t,v)}$ and $\mathfrak{g}_{(h,t,v)}$ will be needed later, these spaces will now be described more fully.

Fix $(h, t, v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$. Suppose that $h_1 > h_2 > \cdots > h_\delta$ are the distinct eigenvalues of h, that h has $L_\alpha \subset \mathbb{C}^n$ as its h_α-eigenspace, and that $n_\alpha \ge 1$ is the (complex) dimension of L_α. Write

$$t = t_1 + \cdots + t_\delta$$

where t_α lies in L_α. Define the quantities

$$v_\alpha = {h_\alpha}^2 + (\operatorname{tr} h)\, h_\alpha + v + \sum_{\beta \ne \alpha} \frac{|t_\beta|^2}{(h_\alpha - h_\beta)}\,.$$

Now define

$$\tau_\alpha = \begin{cases} 1 \\ 0 \\ 2n_\alpha \end{cases} \qquad \text{and} \qquad \rho_\alpha = \begin{cases} (n_\alpha{-}1)^2 & \text{if } t_\alpha \neq 0; \\ (n_\alpha)^2 & \text{if } t_\alpha = 0 \text{ but } v_\alpha \neq 0; \\ (n_\alpha)^2 & \text{if } t_\alpha = 0 \text{ and } v_\alpha = 0. \end{cases}$$

Proposition 3.7. *For any* $(h,t,v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$*,*

$$\dim \mathfrak{g}^0_{(h,t,v)} = \rho_1 + \cdots + \rho_\delta$$

and

$$\dim \mathfrak{g}_{(h,t,v)} = \dim \mathfrak{g}^0_{(h,t,v)} \; + \tau_1 + \cdots + \tau_\delta \,.$$

Proof. Because all the integers involved are invariant under the action of $\mathrm{U}(n)$, it suffices to prove this formula in the case that (h,t,v) lies in C. Maintaining the notation introduced above, this means that

$$h = \begin{pmatrix} h_1\,\mathrm{I}_{n_1} & 0 & \cdots & 0 \\ 0 & h_2\,\mathrm{I}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_\delta\,\mathrm{I}_{n_\delta} \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_\delta \end{pmatrix}$$

where t_α takes values in $\mathbb{R}^{n_\alpha}$ for $1 \le \alpha \le \delta$ and has all of its entries equal to zero except possibly the top one, which is nonnegative.

For $f \in \mathfrak{u}(n)$, write f in 'block' form as $f = (f_{\alpha\bar\beta})$ where $f_{\alpha\bar\beta} = -{f_{\beta\bar\alpha}}^*$ takes values in n_α-by-n_β complex matrices for $1 \le \alpha, \beta \le \delta$. Correspondingly, write $w \in \mathbb{C}^n$ in 'block' form as $w = (w_\alpha)$ where w_α takes values in $\mathbb{C}^{n_\alpha}$. Then the first equation of (3.8) breaks into blocks as

$$0 = (h_\alpha{-}h_\beta) f_{\alpha\bar\beta} + t_\alpha\, {w_\beta}^* + w_\alpha\, {t_\beta}^*.$$

When $\alpha = \beta$, this forces $t_\alpha\, {w_\alpha}^* + w_\alpha\, {t_\alpha}^* = 0$, so that either $t_\alpha = 0$, in which case this places no restriction on w_α, or else $t_\alpha \neq 0$, in which case w_α must be a purely imaginary multiple of t_α, say $w_\alpha = i\, r_\alpha\, t_\alpha$ for some $r_\alpha \in \mathbb{R}$. In either case, ${w_\alpha}^*\, t_\alpha$ is purely imaginary.

When $\alpha \neq \beta$, the above equation can be written as

$$f_{\alpha\bar\beta} = \frac{t_\alpha\, {w_\beta}^* + w_\alpha\, {t_\beta}^*}{(h_\beta - h_\alpha)}, \qquad \alpha \neq \beta.$$

Substituting this equation into the second equation of (3.8) yields

$$0 = -f_{\alpha\bar\alpha}\, t_\alpha - \sum_{\beta\neq\alpha} \frac{t_\alpha\, {w_\beta}^* + w_\alpha\, {t_\beta}^*}{(h_\beta - h_\alpha)}\, t_\beta + ({h_\alpha}^2 + (\operatorname{tr} h)\, h_\alpha + v)\, w_\alpha\,,$$

which, by the definition of v_α and the purely imaginary nature of ${w_\beta}^*\, t_\beta$, can be written in the form

$$0 = -f_{\alpha\bar\alpha}\, t_\alpha + \left(\sum_{\beta\neq\alpha} \frac{{t_\beta}^*\, w_\beta}{(h_\beta{-}h_\alpha)} \right)\, t_\alpha + v_\alpha\, w_\alpha\,.$$

Now, if $t_\alpha \neq 0$, then this equation can be written in the form

$$0 = \left(-f_{\alpha\bar\alpha} + \left(iv_\alpha\, r_\alpha + \sum_{\beta\neq\alpha} \frac{t_\beta{}^*\, w_\beta}{(h_\beta - h_\alpha)} \right) \frac{t_\alpha\, t_\alpha{}^*}{|t_\alpha|^2} \right) t_\alpha\,,$$

which is $2n_\alpha-1$ real equations for the $n_\alpha{}^2$ entries of $f_{\alpha\bar\alpha}$. In fact, the solutions of this equation can be written in the form

$$f_{\alpha\bar\alpha} = f^\perp_{\alpha\bar\alpha} + \left(iv_\alpha\, r_\alpha + \sum_{\beta\neq\alpha} \frac{t_\beta{}^*\, w_\beta}{(h_\beta - h_\alpha)} \right) \frac{t_\alpha\, t_\alpha{}^*}{|t_\alpha|^2}$$

where $f^\perp_{\alpha\bar\alpha} \in \mathfrak{u}(n_\alpha)$ is any solution to $f^\perp_{\alpha\bar\alpha}\, t_\alpha = 0$, an equation that defines the stabilizer subalgebra of t_α in $\mathfrak{u}(n_\alpha)$ and so has a solution space of dimension $(n_\alpha-1)^2$.

If $t_\alpha = 0$, then the equation above simplifies to $v_\alpha\, w_\alpha = 0$. If $v_\alpha \neq 0$, then this implies that $w_\alpha = 0$, while if $v_\alpha = 0$, the equation degenerates to an identity.

In particular, it follows that the equations (3.8) impose no interrelations among the w_α, just the condition $w_\alpha = ir_\alpha\, t_\alpha$ if $t_\alpha \neq 0$, the condition $w_\alpha = 0$ if $t_\alpha = 0$ but $v_\alpha \neq 0$, and no condition on w_α if $t_\alpha = v_\alpha = 0$.

Moreover, once the w_α have been chosen subject to these conditions, the $f_{\alpha\bar\beta}$ for $\alpha \neq \beta$ are completely determined while the $f_{\alpha\bar\alpha} \in \mathfrak{u}(n_\alpha)$ are determined up to a choice of $f^\perp_{\alpha\bar\alpha}$ if $t_\alpha \neq 0$ or are freely specifiable if $t_\alpha = 0$.

The desired dimension formulae follow immediately. □

3.3.3. *Orbit dimension and slices.* The proof of Proposition 3.7 shows how to compute the dimension of the x-orbit for any $x \in M$ for which there is a coframe $u \in P_x$ with $\big(H(u), T(u), V(u)\big) = (h, t, v)$. Maintain the notation introduced above for the invariants of (h, t, v).

Let $O_x \subset T_xM$ be the tangent to the orbit through x. Then $u(O_x) \subset \mathbb{C}^n$ is the direct sum of the lines $\mathbb{R}{\cdot}t_\alpha \subset L_\alpha$ for those α with $t_\alpha \neq 0$ and the subspaces L_α for those α with $t_\alpha = 0$ and $v_\alpha = 0$. Thus, the dimension of the orbit through x is equal to $\tau_1 + \cdots + \tau_\delta$.

A more interesting result is the calculation of a near-slice to the orbits near x. For each α, let $t^\perp_\alpha \subset L_\alpha$ be the (complex) subspace $\{\, w \in L_\alpha \mid t^*_\alpha w = 0 \,\}$.

If $O^\perp_x \subset T_xM$ is the perpendicular to O_x, then $u(O^\perp_x) \subset \mathbb{C}^n$ is the direct sum of the subspaces $i\mathbb{R}{\cdot}t_\alpha \oplus t^\perp_\alpha$ for those α with $t_\alpha \neq 0$ together with the subspaces L_α for those α with $t_\alpha = 0$ and $v_\alpha \neq 0$.

Now, from the description of $\mathfrak{g}^0_{(h,t,v)}$, it follows that the flows of the vector fields in $\mathfrak{g}_x$ generate a group of rotations about x that, via the unitary identification $u : T_xM \to \mathbb{C}^n$, is carried isomorphically into the product of the unitary groups $\mathrm{U}(t^\perp_\alpha)$ for all α. This is a closed subgroup of $\mathrm{U}(n)$ that evidently preserves the subspace $u(O^\perp_x)$.

A near-slice to this action can be constructed as follows: For each α with $t_\alpha \neq 0$ and $t^\perp_\alpha = 0$ (i.e., $n_\alpha = 1$), let $S_\alpha = i\mathbb{R}{\cdot}t_\alpha$. If $t_\alpha \neq 0$ and $t^\perp_\alpha \neq 0$ (i.e., $n_\alpha > 1$), choose a unit vector $s_\alpha \in t^\perp_\alpha$ and let $S_\alpha = i\mathbb{R}{\cdot}t_\alpha \oplus \mathbb{R}{\cdot}s_\alpha$. If $t_\alpha = 0$ and $v_\alpha \neq 0$, choose a unit vector $s_\alpha \in t^\perp_\alpha = L_\alpha$ and let $S_\alpha = \mathbb{R}{\cdot}s_\alpha$. Finally, if $t_\alpha = 0$ and $v_\alpha = 0$, then set $S_\alpha = 0$.

Then the direct sum $S_1 \oplus \cdots \oplus S_\delta \subset u(O^\perp_x)$ is of the form $u(S_x)$ for a subspace $S_x \subset O^\perp_x$ that is a near-slice to the action of the isometric rotations about x. Consequently, the submanifold $\exp_x(S_x)$ near x meets each orbit in a finite number of points and meets the generic orbit transversely. Let $m_\alpha = \dim S_\alpha$, so that

$$m_\alpha = \begin{cases} 2 & \text{if } t_\alpha \neq 0 \text{ and } n_\alpha > 1; \\ 1 & \text{if } t_\alpha \neq 0 \text{ and } n_\alpha = 1; \\ 1 & \text{if } t_\alpha = 0 \text{ but } v_\alpha \neq 0; \\ 0 & \text{if } t_\alpha = 0 \text{ and } v_\alpha = 0. \end{cases}$$

Then the 'generic' orbit in M has codimension $m = m_1 + \cdots + m_\delta$. Since $m_\alpha \leq n_\alpha$ for all α, it follows that $m \leq n$.

3.3.4. *Minimal symmetry.* By Proposition 3.7, if $(h, t, v) \in C$ satisfies $h_{i\bar{\imath}} > h_{j\bar{\jmath}}$ for $i < j$ and $t_i > 0$ for all i, then $\dim \mathfrak{g}_{(h,t,v)} = n$. Thus, any simply-connected Bochner-Kähler manifold whose structure function assumes such an (h, t, v) must have its symmetry algebra $\mathfrak{g}$ be of dimension n exactly. Moreover, from the above discussion, it follows that the generic orbits of such a Bochner-Kähler structure have codimension n, the maximum possible.

Thus, a 'generic' Bochner-Kähler structure has its infinitesimal symmetry algebra of dimension n as well as cohomogeneity equal to n.

3.4. **Constants of the structure.** In the previous subsection, it was shown that the structure function $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ has rank at most $n^2 + n$. Since the image of P under this map is $\mathrm{U}(n)$-invariant, it is natural to look for a set of $\mathrm{U}(n)$-invariant polynomials whose simultaneous level sets will contain the images of structure functions. In this section, we will exhibit $n+1$ such polynomials and show that they are independent.

3.4.1. *Conserved polynomials.* Let Ω be any Bochner-Kähler structure on a connected complex manifold M and let $\pi : P \to M$ be the unitary coframe bundle, with canonical forms ω and ϕ and structure functions S, T, and V as above.

By (2.14), there are identities

$$\begin{aligned} d(\operatorname{tr} H) &= (T^*\,\omega + \omega^*\,T), \\ d(\operatorname{tr} H^2) &= 2\,(T^*\,H\,\omega + \omega^*\,H\,T). \end{aligned}$$

Thus, by the last equation of (2.14)

$$dV = (\operatorname{tr} H)\, d(\operatorname{tr} H) + \tfrac{1}{2}\, d(\operatorname{tr} H^2).$$

Since P is connected, there is a constant C_2 for which

$$V - \tfrac{1}{2}\,\operatorname{tr}(H^2) - \tfrac{1}{2}\,(\operatorname{tr} H)^2 = C_2.$$

We will now show that this example can be generalized by constructing n additional polynomials on $i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ that have this constancy property.

Define A_k and B_k for $k \geq 0$ by the formulae

$$\begin{aligned} A_k &= \operatorname{tr}(H^k), \\ B_0 &= 1, \quad B_1 = \operatorname{tr} H, \quad B_2 = V, \\ B_k &= T^* H^{k-3} T, \quad k \geq 3. \end{aligned} \tag{3.9}$$

Because these functions are constant on the fibers of $\pi : P \to M$, they can be regarded as the pullbacks to P of well-defined smooth functions on M. In what follows, we will usually treat them as functions on M. For convenience, define $A_k = B_k = 0$ when $k < 0$.

Also, for $0 \le k \le n$, let h_k denote the k-th elementary symmetric function of the eigenvalues of H. These functions can be expressed as polynomials in the A_k and hence are smooth functions on M. For example, $h_0 = 1$, $h_1 = A_1$, $h_2 = \frac{1}{2}({A_1}^2 - A_2)$, etc. For convenience, set $h_k = 0$ for $k < 0$ or $k > n$.

Theorem 3.8. *For any connected Bochner-Kähler n-manifold* (M, Ω)*, the functions*

$$C_k = B_k - h_1\, B_{k-1} + h_2\, B_{k-2} - \cdots + (-1)^{k-1} h_{k-1}\, B_1 + (-1)^k h_k\, B_0 \tag{3.10}$$

are locally constant for $2 \le k \le n+2$*.*

Example 3.9 (Lowest constants)**.** For example, in addition to the evident constancy of the function

$$C_2 = B_2 - h_1\, B_1 + h_2\, B_0 = B_2 - \tfrac{1}{2}\, A_2 - \tfrac{1}{2}\, {A_1}^2,$$

one has the constancy of

$$C_3 = B_3 - h_1\, B_2 + h_2\, B_1 - h_3\, B_0 = B_3 - A_1\, B_2 - \tfrac{1}{3}\left(A_3 - {A_1}^3\right).$$

The reader may notice that the above formula for C_k makes sense for $k = 1$ and for $k > n{+}2$. Now, the expression C_1 is just $B_1 - h_1 = B_1 - A_1$, which vanishes by definition. When $k \ge n+3$, applying the Cayley-Hamilton theorem to the definition of C_k yields

$$C_k = T^* H^{k-n-3}\,(H^n - h_1\, H^{n-1} + h_2\, H^{n-2} - \cdots + (-1)^n h_n\ \mathrm{I}_n)T = 0.$$

However, when $2 \le k \le n{+}2$, the expression C_k is a nontrivial polynomial of weighted degree k in the variables A_j and B_j. In fact, the above expressions for $(C_2, \ldots, C_{n+2})$ can obviously be solved for $(B_2, \ldots, B_{n+2})$.

Proof. Define 1-forms $\alpha_0 = 0$ and

$$\alpha_{k+1} = T^* H^k \omega + \omega^* H^k T, \qquad \text{for } k \ge 0. \tag{3.11}$$

(The indexing is determined by 'scaling weight' considerations.) The α_k are visibly π-semibasic, but they are also invariant under the $\mathrm{U}(n)$-action on P. Thus, they are the π-pullbacks of well-defined 1-forms on M. Consequently, they will, by abuse of language, be treated as 1-forms on M.

The first step will be to prove the following identities for all $k \ge 0$:

$$\begin{aligned} &dA_k = k\, \alpha_k\,, \qquad d\alpha_k = 0\,,\\ &dB_k = B_0\, \alpha_k + B_1\, \alpha_{k-1} + B_2\, \alpha_{k-2} + \cdots + B_{k-1}\, \alpha_1\,. \end{aligned} \tag{3.12}$$

Now, the first set of identities is just a calculation. The case $k = 0$ is obvious, so assume $k > 0$. Using $\mathrm{tr}(PQ) = \mathrm{tr}(QP)$ and the structure equation, one has

$$\begin{aligned} dA_k &= d\big(\mathrm{tr}(H^k)\big) = k\, \mathrm{tr}(H^{k-1}\, dH)\\ &= k\, \mathrm{tr}\big(H^{k-1}(T\, \omega^* + \omega\, T^*)\big) = k\, \alpha_k\,. \end{aligned}$$

(The terms in dH involving ϕ cancel since A_k is constant on the fibers of π.) Taking the exterior derivative of this relation and dividing by k yields

$$0 = d\alpha_k\,.$$

The second set is a little more complicated, but still just a calculation. The case $k = 0$ is trivial, and the case $k = 1$ follows from the fact that $B_1 = A_1$, so $dB_1 = dA_1 = \alpha_1 = B_0\, \alpha_1$.

Because $B_2 = V$, the second identity for $k = 2$ is just the structure equation for dV. Also,

$$\begin{aligned} dB_3 &= dT^*\,T + T^* dT = \omega^*\big(H^2{+}(\operatorname{tr} H)\,H{+}V\ \mathrm{I}_n\big)T + T^*\big(H^2{+}(\operatorname{tr} H)\,H{+}V\ \mathrm{I}_n\big)\,\omega \\ &= B_0\,\alpha_3 + B_1\,\alpha_2 + B_2\,\alpha_1\,, \end{aligned}$$

verifying the formula when $k = 3$. Thus, suppose from now on that $k > 3$ and compute (again ignoring terms involving ϕ, which must cancel)

$$\begin{aligned} dB_k &= dT^*\,H^{k-3}\,T + T^*\,H^{k-3}\,dT + \sum_{l=0}^{k-4} T^*\,H^l\,dH\,H^{k-l-4}\,T \\ &= \omega^*\big(H^2{+}(\operatorname{tr} H)\,H{+}V\ \mathrm{I}_n\big)H^{k-3}\,T + T^*\,H^{k-3}\big(H^2{+}(\operatorname{tr} H)\,H{+}V\ \mathrm{I}_n\big)\omega \\ &\qquad + \sum_{l=0}^{k-4} T^*\,H^l\,(T\,\omega^* + \omega\,T^*)\,H^{k-l-4}\,T \\ &= \alpha_k + (\operatorname{tr} H)\,\alpha_{k-1} + V\,\alpha_{k-2} + \sum_{l=0}^{k-4} B_{l+3}\,\omega^*\,H^{k-l-4}\,T + T^*\,H^l\,\omega\,B_{k-l-1} \\ &= B_0\,\alpha_k + B_1\,\alpha_{k-1} + B_2\,\alpha_{k-2} + \sum_{l=0}^{k-4} B_{l+3}\,\alpha_{k-l-3} \\ &= B_0\,\alpha_{k+2} + B_1\,\alpha_{k-1} + B_2\,\alpha_k + B_3\,\alpha_{k-3} + \cdots + B_{k-1}\,\alpha_1\,. \end{aligned}$$

Thus, the formulae (3.12) are established.

Now, we claim that the functions h_k satisfy the differential equations

$$dh_k = h_{k-1}\,\alpha_1 - h_{k-2}\,\alpha_2 + \cdots + (-1)^{k+1} h_0\,\alpha_k\,. \tag{3.13}$$

Granting (3.13) for the moment, computation gives

$$\begin{aligned} dC_k &= d\left(\sum_{j=0}^{k}(-1)^j h_j B_{k-j}\right) = \sum_{j=0}^{k}(-1)^j\big(dh_j\,B_{k-j} + h_j\,dB_{k-j}\big) \\ &= \sum_{j=0}^{k}(-1)^j\left(\sum_{l=0}^{j}(-1)^{j-l+1}h_l\alpha_{j-l}\,B_{k-j} + \sum_{l=0}^{k-j} h_j\,B_l\alpha_{k-j-l}\right) \\ &= \sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^{-l+1}h_l B_{k-j}\,\alpha_{j-l} + \sum_{j=0}^{k}\sum_{l=0}^{k-j}(-1)^j h_j B_l\,\alpha_{k-j-l} \\ &= \sum_{l=0}^{k}\sum_{j=l}^{k}(-1)^{-l+1}h_l B_{k-j}\,\alpha_{j-l} + \sum_{l=0}^{k}\sum_{j=0}^{k-l}(-1)^l h_l B_j\,\alpha_{k-j-l} \\ &= \sum_{l=0}^{k}\sum_{j=l}^{k}(-1)^{-l+1}h_l B_{k-j}\,\alpha_{j-l} + \sum_{l=0}^{k}\sum_{j=l}^{k}(-1)^l h_l B_{k-j}\,\alpha_{j-l} \\ &= 0. \end{aligned} \tag{3.14}$$

It remains to verify (3.13). This is a classical identity: Let $\lambda_1, \ldots, \lambda_n$ be free variables, let s_k be the k-th elementary function of the λ_i, and let p_k be the k-th power function of the λ_i, i.e., $p_k = {\lambda_1}^k + \cdots + {\lambda_n}^k$. For any constant t, one has

$$(1 + s_1\,t + s_2\,t^2 + \cdots + s_n\,t^n) = (1 + \lambda_1\,t)\cdots(1 + \lambda_n\,t).$$

Taking the logarithm and then computing the differential of both sides yields

$$\frac{(ds_1\,t + ds_2\,t^2 + \cdots + ds_n\,t^n)}{(1 + s_1\,t + s_2\,t^2 + \cdots + s_n\,t^n)} = \sum_{i=0}^{n} \frac{t\,d\lambda_i}{1+\lambda_i\,t}.$$

Expanding the right hand side out as a formal geometric power series in t and collecting like powers of t yields

$$\frac{(ds_1\,t + ds_2\,t^2 + \cdots + ds_n\,t^n)}{(1 + s_1\,t + s_2\,t^2 + \cdots + s_n\,t^n)} = \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{k} dp_k \right) t^k\,.$$

It follows that

$$\frac{(dh_1\,t + dh_2\,t^2 + \cdots + dh_n\,t^n)}{(1 + h_1\,t + h_2\,t^2 + \cdots + h_n\,t^n)} = \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{k} dA_k \right) t^k = \sum_{k=1}^{\infty} (-1)^{k+1} \alpha_k\,t^k\,.$$

Thus,

$$dh_1\,t + \cdots + dh_n\,t^n = (1 + h_1\,t + \cdots + h_n\,t^n)(\alpha_1\,t - \alpha_2\,t^2 + \alpha_3\,t^3 - \cdots), \tag{3.15}$$

which, after equating coefficients of like powers of t on each side, is (3.13). □

3.4.2. *The moduli map.* The map $f : M \to F_n \subset \mathbb{R}^{2n+1}$ defined in §3.2 satisfies

$$f = (A_1, \ldots, A_n, B_2, B_3, \ldots, B_{n+2}).$$

As is well known,[11] there is a unique, invertible weighted-homogeneous polynomial mapping $\Sigma : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies

$$\Sigma(A_1, \ldots, A_n) = (h_1, \ldots, h_n).$$

From the definition of the C_k as polynomials in B_j and h_j, it follows easily that Σ can be extended to an invertible, weighted-homogeneous polynomial mapping $\Delta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ so that

$$\Delta\circ f = (h_1, \ldots, h_n, C_2, C_3, \ldots, C_{n+2}).$$

Thus, the fibers and the rank of the map $\Delta\circ f : M \to \mathbb{R}^{2n+1}$ are the same as for the map $f : M \to \mathbb{R}^{2n+1}$. In particular, the fibers of $\Delta\circ f$ are the orbits of the symmetry pseudo-groupoid of the Bochner-Kähler structure.

By Theorem 3.8, when M is connected, the functions C_k are constants, so the fibers and rank of f are the same as the fibers and rank of the map $h : M \to \mathbb{R}^n$ defined by

$$h = (h_1, \ldots, h_n).$$

3.5. **Central symmetries.** The symmetry algebra $\mathfrak{g}$ of a Bochner-Kähler structure Ω on a connected M turns out to contain a canonical central subalgebra $\mathfrak{z}$, whose dimension is equal to the infinitesimal cohomogeneity of the structure.

[11] In fact, this mapping can be computed by comparing like powers of t in the formal series expansions of the identity

$$1 + h_1\,t + \cdots + h_n\,t^n = \exp(A_1\,t - \tfrac{1}{2}A_2\,t^2 + \tfrac{1}{3}A_3\,t^3 - \cdots).$$

The mapping in the other direction can be computed by taking the logarithm of both sides, expanding the left side as a series in t, and then comparing like powers.

3.5.1. *An isometry vector field.* Consider the vector field Z_2' on P that satisfies

$$\omega(Z_2') = 2i\,T, \qquad \phi(Z_2') = 2i(H^2 + \operatorname{tr}(H)\,H + V\ \mathrm{I}_n). \tag{3.16}$$

(The indexing is dictated by scaling weight considerations.) Since $\omega\oplus\phi : T_uP \to \mathbb{C}^n\oplus\mathfrak{u}(n)$ is an isomorphism for all $u \in P$ and since H is Hermitian symmetric while V is real, this does indeed define a unique vector field Z_2'. Because of the $\mathrm{U}(n)$-equivariance of the structure functions, the vector field Z_2' is invariant under the right $\mathrm{U}(n)$-action. Consequently, there is a unique vector field Z_2 on M that is π-related to Z_2', i.e., that satisfies $\pi'\big(Z_2'(u)\big) = Z_2\big(\pi(u)\big)$. The structure equation $dT + \phi\,T = (H^2 + \operatorname{tr}(H)\,H + V\ \mathrm{I}_n)\,\omega$ coupled with the discussion in §2.1.2 shows that Z_2 is the real part of a holomorphic vector field on M.

Since

$$\pi^*(Z_2 \lrcorner\, \Omega) = Z_2' \lrcorner \big(-\tfrac{i}{2}\,\omega^*\wedge\omega\big) = -\big(T^*\omega + \omega^*T\big) = -\,d\big(\operatorname{tr}(H)\big), \tag{3.17}$$

the flow of Z_2 is the Ω-Hamiltonian flow associated to h_1.

Alternatively, one can see directly that the (local) flow of Z_2 preserves the Bochner-Kähler structure. It has already been observed that Z_2 is π-related to the $\mathrm{U}(n)$-invariant vector field Z_2' on P. The defining formulae for Z_2' yields

$$\begin{aligned}
\mathfrak{L}_{Z_2'}\,\omega &= d\big(\omega(Z_2')\big) + Z_2' \lrcorner\, d\omega = d\big(\omega(Z_2')\big) + Z_2' \lrcorner\,(-\phi\wedge\omega)\\
&= d(2i\,T) - \phi(Z_2')\,\omega + \phi\,\omega(Z_2')\\
&= 2i\,\big(dT - (H^2 + \operatorname{tr}(H)\,H + V\ \mathrm{I}_n)\,\omega + \phi\,T\big) = 0.
\end{aligned}$$

Thus, the flow of Z_2' preserves ω. In turn, this implies that the flow of Z_2' preserves ϕ (since ϕ is the unique $\mathfrak{u}(n)$-valued 1-form that satisfies $d\omega = -\phi\wedge\omega$). Thus, the vector field Z_2' is π-related to the symmetry vector field Z_2 of the $\mathrm{U}(n)$-structure that P defines, i.e., the original Bochner-Kähler structure.

Remark 3.10 (Matsumoto's observation). To my knowledge it was Matsumoto [19, Theorem 2] who first observed that one could construct a holomorphic vector field on M by Ω-dualizing the exterior derivative of the scalar curvature, at least when M is compact. The vector field that he constructs is, up to a constant complex multiple, the same as the one whose real part is Z_2. The above argument shows that compactness actually plays no role; the holomorphicity of $Z_2 - iJZ_2$ is a purely local fact. Apparently, Matsumoto did not realize that some complex multiple of his vector field had a real part whose flow of Z_2 was not only holomorphic but isometric as well.

3.5.2. *The central algebra.* We are now going to show that Z_2 is the first of a sequence of real parts of holomorphic vector fields on M whose representative functions can be written down explicitly in terms of H and T. For example, the next term in the sequence will be seen to be the vector field Z_3 whose representative function is $z_3 = -2i\,(H - \operatorname{tr}(H)\ \mathrm{I}_n)\,T$.

Theorem 3.11. *For every k in the range $0 \le k \le n-1$, the function*

$$z_{k+2} = 2i(-1)^k\big(H^k - h_1\,H^{k-1} + h_2\,H^{k-2} + \cdots + (-1)^k\,h_k\ \mathrm{I}_n\big)\,T$$

is the representative function of a vector field $Z_{k+2} \in \mathfrak{g}$. Moreover, the span $\mathfrak{z}$ of $Z_2, \dots, Z_{n+1}$ lies in the center of $\mathfrak{g}$.

For $x \in M$, the subspace $\mathfrak{z}_x = \operatorname{span}\{Z_2(x), \dots, Z_{n+1}(x)\}$ is the Ω-complement to $\ker df_x$, where $f : M \to \mathbb{R}^{2n+1}$ is the moduli mapping of §3.2.3.

If M is connected, then $\dim\mathfrak{z} \le n$ is the maximum over $x \in M$ of $\dim\mathfrak{z}_x$. If $\dim\mathfrak{z} = n$, then $\mathfrak{g} = \mathfrak{z}$.

Remark 3.12. While the formula for z_{k+2} makes sense for all $k \ge 0$, this expression vanishes identically when $k \ge n$, due to the Cayley-Hamilton Theorem.

Proof. A computation like that done in the proof of (3.12) shows that, for $k \ge 0$,

$$d(H^kT) + \phi\, H^kT \equiv H^kT\,\alpha_0 + H^{k-1}T\,\alpha_1 + \cdots + H^0T\,\alpha_k \mod \omega. \tag{3.18}$$

Using this identity, a calculation analogous to (3.14) yields

$$d(z_{k+2}) + \phi\, z_{k+2} \equiv 0 \mod \omega.$$

Thus, according to §2.1.2, each z_{k+2} is the representative function of a vector field Z_{k+2} on M whose local flow is holomorphic.

Letting $\tilde{Z}_{k+2}$ be any vector field on P so that $\omega(\tilde{Z}_{k+2}) = z_{k+2}$,

$$\begin{aligned} \pi^*\bigl(Z_{k+2} \lrcorner \Omega\bigr) &= \tilde{Z}_{k+2} \lrcorner \bigl(-\tfrac{i}{2}\,\omega^* \wedge \omega\bigr) = -\tfrac{i}{2}\,\bigl(z^*_{k+2} \wedge \omega - \omega^*\, z_{k+2}\bigr) \\ &= -(-1)^k\bigl(T^*(H^k - h_1\,H^{k-1} + \cdots + (-1)^k\,h_k\;\mathrm{I}_n)\,\omega \\ &\qquad\qquad + \omega^*(H^k - h_1\,H^{k-1} + \cdots + (-1)^k\,h_k\;\mathrm{I}_n)\,T\bigr) \\ &= (-1)^{k+1}\bigl(\alpha_{k+1} - h_1\,\alpha_k + h_2\,\alpha_{k-1} + \cdots + (-1)^k h_k\,\alpha_1\bigr) \\ &= -dh_{k+1} = \pi^*\bigl(-dh_{k+1}\bigr)\,, \end{aligned} \tag{3.19}$$

by virtue of (3.13). Thus, $Z_{k+2} \lrcorner \Omega = -dh_{k+1}$, so that Z_{k+2} is the Ω-Hamiltonian vector field associated to h_{k+1}.

Since the flow of Z_{k+2} is both holomorphic and symplectic, Z_{k+2} belongs to $\mathfrak{g}$, as claimed.

Since the representative function z_{k+2} is constructed as a polynomial in H and T, which are invariant under the Y'-flow on P for any vector field $Y \in \mathfrak{g}$, it follows that Z_{k+2} is invariant under the flow of any $Y \in \mathfrak{g}$, i.e., $[Y, Z_{k+2}] = 0$ for any $Y \in \mathfrak{g}$. Thus, the Z_{k+2} for $0 \le k \le n{-}1$ span a central subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$.

For any $x \in M$, the nondegeneracy of Ω implies that $\mathfrak{z}_x \subset T_xM$ is the Ω-dual of

$$\operatorname{span}\{dh_1(x), \ldots, dh_n(x)\} = \operatorname{span}\{dA_1(x), \ldots, dA_n(x)\}$$

(see §3.4.2). The map $f : M \to \mathbb{R}^{2n+1}$ has components given by A_i and B_i. By Theorem 3.8 and §3.4.2, each B_i can be written on each connected component of M as a weighted polynomial in $A_1, \ldots, A_i$ with constant coefficients. Thus, the kernel of df_x is the same as the kernel of $(dA)_x$ where $A = (A_1, \ldots, A_n)$, which establishes the stated Ω-complementarity.

Now suppose that M is connected. For each $k \ge 0$, each $x \in M$, and each $u \in P_x$,

$$u\bigl(\mathfrak{z}_x\bigr) = \operatorname{span}\bigl\{\ iH(u)^k\,T(u) \mid\ 0 \le k \le n{-}1\ \bigr\}.$$

Since $H(u)$ is Hermitian symmetric, it follows that the dimension of $\mathfrak{z}_x$ over $\mathbb{R}$ is the largest integer $m_x \le n$ so that the vectors $T(u), H(u)T(u), \ldots, H(u)^{m_x-1}T(u)$ are linearly independent (over either $\mathbb{C}$ or $\mathbb{R}$) and, moreover, that $\mathfrak{z}_x \cap J\,\mathfrak{z}_x = (0)_x$.

Let $M^\circ \subset M$ be the (nonempty) open set consisting of those $x \in M$ for which m_x achieves the maximum value $m \le n$. If $m = n$, then $\dim\mathfrak{z}_x = n$ for all $x \in M^\circ$ and, since $\dim\mathfrak{z}_x \le \dim\mathfrak{z} \le n$ for all $x \in M$, it follows that $\dim\mathfrak{z} = n$. If $m = 0$, then T vanishes identically, implying that $\mathfrak{z} = (0)$. If $0 < m < n$, let k satisfy $m \le k < n$. Then, on M°, the vector field Z_{k+2} is a linear combination of the independent vector fields $Z_2, \dots, Z_{m+1}$. Thus, there exist smooth real-valued functions $w_2, \dots, w_{m+1}$ on M° so that

$$Z_{k+2} = w_2\, Z_2 + \cdots + w_{m+1}\, Z_{m+1}.$$

Consequently,

$$Z_{k+2} - iJZ_{k+2} = w_2\,(Z_2 - iJZ_2) + \cdots + w_{m+1}\,(Z_{m+1} - iJZ_{m+1}).$$

However, the left hand side of this equation is a holomorphic vector field while the holomorphic vector fields $(Z_2 - iJZ_2), \dots, (Z_{m+1} - iJZ_{m+1})$ are linearly independent (over $\mathbb{C}$) at each point of M°. It follows that the functions $w_2, \dots, w_{m+1}$ are real-valued holomorphic functions on M° and hence must be constants. Since M is connected, the identity

$$Z_{k+2} = w_2\, Z_2 + \cdots + w_{m+1}\, Z_{m+1}$$

must hold on all of M. In other words, the vector fields $Z_2, \dots, Z_{m+1}$ are a basis of $\mathfrak{z}$, as desired.

If $\dim \mathfrak{z} = n$, then at any point $x \in M^\circ$, the vectors $Z_2(x), \dots, Z_{n+1}(x)$ are linearly independent. If $\pi(u) = x$, then $\{\, H^k(u)T(u) \mid 0 \le k \le n{-}1 \,\}$ are linearly independent, implying (since $H(u)$ is diagonalizable) that $T(u)$ does not lie in any sum of fewer than n distinct eigenspaces of $H(u)$. By §3.3.4, this implies that the differential of the mapping $(H, T, V) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ has kernel of dimension equal to n, so $\dim \mathfrak{g} = n$, as desired. □

3.5.3. *The momentum mapping.* The proof of Theorem 3.11 shows that the map $h : M \to \mathbb{R}^n$ defined by

$$h = \bigl(h_1, h_2, h_3, \dots, h_n\bigr) \tag{3.20}$$

is a momentum mapping for the infinitesimal torus action generated by $\mathfrak{z}$.[12]

As already remarked, there is an invertible weighted-homogeneous polynomial mapping $\Delta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ that satisfies

$$\Delta\bigl(A_1, \dots, A_n, B_2, \dots, B_{n+2}\bigr) = \bigl(h_1, \dots, h_n, C_2, \dots, C_{n+2}\bigr).$$

Thus, the fibers of $\Delta \circ f : M \to \mathbb{R}^{2n+1}$ are the orbits of the symmetry pseudo-groupoid of the underlying Bochner-Kähler structure. By Theorem 3.8, if M is connected, then the functions C_k are constant. Thus, for connected M, the fibers of h are the orbits of this symmetry pseudo-groupoid.

3.6. **Cohomogeneity and the momentum polynomial.** Assume that M is connected and that $\dim \mathfrak{z} = m$.

3.6.1. *Cohomogeneity.* The proof of Theorem 3.11 shows that dh_k for $k > m$ is a constant linear combination of the differentials $dh_1, \dots, dh_m$ and that these latter 1-forms are linearly independent on an open subset of M. Thus, $h(M)$ lies in an m-dimensional affine subspace $\mathfrak{a} \subset \mathbb{R}^n$ and, moreover, contains an open subset of $\mathfrak{a}$. Since the fibers of h are the orbits of the symmetry pseudo-groupoid, it is reasonable to call the number m the *cohomogeneity* of the Bochner-Kähler structure.

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[12] More precisely, the infinitesimal n-torus action on M is given by the Lie algebra homomorphism $\mathbb{R}^n \to \mathfrak{z} \subset \mathfrak{X}(M)$ defined by the explicit generators $Z_2, \dots, Z_{n+1}$.

3.6.2. *Cohomogeneity.* Let t be a parameter and define the *momentum polynomial* $p_h(t)$ of M by the formula

$$p_h(t) = t^n - h_1\, t^{n-1} + \cdots + (-1)^n h_n\,.$$

Of course, $p_h(t) = \det(t\,\mathrm{I}_n - H)$ is the characteristic polynomial of the Hermitian symmetric matrix H, so all of its roots are real.

Theorem 3.13. *If (M,Ω) is a connected Bochner-Kähler manifold of cohomogeneity m, then $n{-}m$ of the roots of $p_h(t)$ are constant and, outside a closed, proper, complex analytic subvariety $N \subset M$, the remaining m roots are distinct, real-analytic, and functionally independent.*

Proof. If $m = 0$, then, in particular, h_1 is constant, so (M,Ω) has constant scalar curvature. By Proposition 2.5, (M,Ω) is locally homogeneous, so all of the eigenvalues of H are constant. (By Proposition 2.5, there are at most two distinct eigenvalues.) In this case, N can be taken to be empty.

Suppose from now on that $m > 0$. Technically, we should treat the cases $m = n$ and $m < n$ separately, but the argument for $m = n$ differs from that for $m < n$ by trivial notational changes, so we will not explicitly assume $m < n$ but, rather, let the reader make the necessary modifications for the case $m = n$.

By Theorem 3.11, the differentials $dh_1, \ldots, dh_m$ are linearly independent exactly where the vector fields $Z_2, \ldots, Z_{m+1}$ are linearly independent. In particular, the locus $N \subset M$ where $dh_1\wedge\cdots\wedge dh_m$ vanishes is also where the holomorphic m-vector

$$(Z_2 - \imath J Z_2)\wedge(Z_3 - \imath J Z_3)\wedge\cdots\wedge(Z_{m+1} - \imath J Z_{m+1})$$

vanishes. Thus, N is a closed, proper, complex analytic subvariety of M and so has real codimension at least 2. Consequently, its complement $M^\circ \subset M$ is a connected, open, dense subset of M.

Since $\dim \mathfrak{z}_x = m$ for all $x \in M^\circ$, a subbundle $P_0 \subset P$ can be defined over M° by saying that $u \in \pi^{-1}(M^\circ)$ lies in P_0 if and only if $u(\mathfrak{z}_x) = i\mathbb{R}^m \subset \mathbb{C}^m \subset \mathbb{C}^n$. Then $\pi : P_0 \to M^\circ$ is a smooth principal $\mathrm{O}(m)\times \mathrm{U}(n{-}m)$-bundle over M°.

Pull the forms ω and ϕ and the functions H, T, and V back to P_0. By definition, for every $u \in P_0$, the vectors $T(u), H(u)T(u), \ldots, H(u)^{m-1}T(u)$ span $\mathbb{R}^m \subset \mathbb{C}^n$ and, moreover, $H(u)\cdot\mathbb{R}^m \subset \mathbb{R}^m$. Thus, there exists a function $T' : P_0 \to \mathbb{R}^m$, a function H' on P_0 with values in the open set of symmetric m-by-m (real) matrices with m distinct eigenvalues, and a function H'' on P_0 with values in Hermitian symmetric $(n{-}m)$-by-$(n{-}m)$ matrices so that

$$T = \begin{pmatrix} T' \\ 0 \end{pmatrix}, \qquad H = \begin{pmatrix} H' & 0 \\ 0 & H'' \end{pmatrix}. \tag{3.21}$$

Write $\phi = -\phi^*$ in $(m, n{-}m)$-block form as

$$\phi = \begin{pmatrix} \phi' & \tau^* \\ -\tau & \phi'' \end{pmatrix} \tag{3.22}$$

where, of course, ϕ' and ϕ'' take values in skew-Hermitian matrices of dimensions m and $n{-}m$, respectively. The lower right-hand $(n{-}m)$-by-$(n{-}m)$ block of the dH equation in (2.14) then becomes

$$dH'' = -\phi''\, H'' + H''\, \phi''. \tag{3.23}$$

Consequently, the eigenvalues of H'' are constant on P_0. Let

$$\det(t\,\mathrm{I}_{n-m} - H'') = p_{h''}(t) = t^{n-m} - h''_1\, t^{n-m} + \cdots + (-1)^{n-m}\, h''_{n-m}$$

be the characteristic polynomial of H'', where the h''_i are constants. Then, on M° at least, $p_{h''}(t)$ divides $p_h(t)$. Using the Euclidean algorithm, write

$$p_h(t) = p_{h''}(t)\, q(t) + r(t)$$

where q and t are polynomials in t and where the degree of r is at most $n{-}m{-}1$. The coefficients of q and r are constant linear combinations of the coefficients in p_h and so are continuous. Since the coefficients of r vanish on M°, which is dense in M, it follows that r vanishes identically on M. Thus, $p_{h''}(t)$ divides $p_h(t)$ on all of M.

Defining real-analytic functions $h'_1, \ldots, h'_m$ on M by

$$q(t) = t^m - h'_1\, t^{m-1} + \cdots + (-1)^m\, h'_m\,,$$

one sees that $q(t) = p_{h'}(t) = \det(t\,\mathrm{I}_m - H')$ on M°, i.e., that $p_h(t) = p_{h'}(t)\, p_{h''}(t)$.

Of course the roots of $p_{h'}(t)$ on M° equal the eigenvalues of H' on P_0 and so are distinct and therefore real-analytic on M°. Since the h_i are constant coefficient linear combinations of the h'_j, it follows that there is a constant a so that

$$dh_1 \wedge \cdots \wedge dh_m = a\, dh'_1 \wedge \cdots \wedge dh'_m\,.$$

Obviously, a is nonzero and $dh'_1 \wedge \cdots \wedge dh'_m$ is nonvanishing on M°. Since the roots of $p_{h'}(t)$ are distinct on M°, and the h'_i are the elementary symmetric functions of these roots, it follows that these roots must be functionally independent on M°, as claimed. □

Remark 3.14. Theorem 3.13 accounts for the $n{-}m$ constant coefficient linear relations among the momenta $h_1, \ldots, h_n$ implicit in the initial discussion. They are just the $n{-}m$ coefficients of the remainder polynomial $r(t)$.

3.6.3. *Reduced momentum.* The mapping $h' = (h'_1, \ldots, h'_m) : M \to \mathbb{R}^m$ will be known as the *reduced momentum mapping* of M. The proof of Theorem 3.13 shows that $h' : M^\circ \to \mathbb{R}^m$ is a submersion onto its image.

The polynomial $p_{h'}(t)$ will be referred to as the *reduced momentum polynomial* of M. The roots of $p_{h'}(t)$ are real on M°, which is dense in M, so the roots of $p_{h'}(t)$ are real at every point of M. For each $x \in M$, let

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$$

be the roots of $p_{h'}(t)$, counted with multiplicity. By a standard argument based on the Stone-Weierstraß theorem, the functions $\lambda_i : M \to \mathbb{R}$ are continuous.[13] Thus, the reduced momentum polynomial factors continuously as

$$p_{h'}(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_m).$$

Example 3.15 (Low cohomogeneity)**.** Suppose M is locally isometric to $M^p_c \times M^{n-p}_{-c}$. Then, looking back at the proof of Proposition 2.5 and the definition of H, one computes that

$$p_h(t) = \left(t + \frac{c(n-p+1)}{2(n+2)}\right)^p \left(t - \frac{c(p+1)}{2(n+2)}\right)^{n-p}.$$

Since this example is locally homogeneous, i.e., $m = 0$, it follows that $p_h(t) = p_{h''}(t)$.

[13] Note that λ_i will be real-analytic even at $x \in N$ as long as it is a simple root of $p_{h'}(t)$ at x.

On the other hand, for Example 2.3 (i.e., rotationally symmetric),

$$p_h(t) = \left(t - \frac{k}{(n+2)}\right)^{n-1} \left(t - \frac{k}{(n+2)} - a\,|z|^2 f'\big(|z|^2\big)\right).$$

As long as $a \neq 0$, these examples have cohomogeneity $m = 1$, with

$$p_{h''}(t) = \left(t - \frac{k}{(n+2)}\right)^{n-1} \quad \text{and} \quad p_{h'}(t) = \left(t - \frac{k}{(n+2)} - a\,|z|^2 f'\big(|z|^2\big)\right).$$

4. Global geometry and symmetries

Throughout this section it will be assumed that M is a connected complex n-manifold endowed with a Bochner-Kähler structure Ω. All the notation introduced earlier will be retained.

4.1. **The characteristic polynomials.**

In this section, two constant coefficient polynomials will be introduced that are invariants of the analytically connected equivalence class of the Bochner-Kähler structure. Also a formula (Theorem 4.4) will be developed to compute them from the value of the structure function at a single point.

4.1.1. *The characteristic polynomial.* Let C_k for $k = 2, \ldots, n{+}2$ be the constants introduced in Theorem 3.8. For the sake of convenience, set $C_0 = 1$ and $C_k = 0$ for $k = 1$ and $k > n{+}2$. Let t be a real parameter. Then by Theorem 3.8 (plus the remark following it),

$$\begin{aligned}
\sum_{k=0}^{\infty} C_k\, t^k &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^l h_l\, B_{k-l}\, t^k \\
&= \left(\sum_{k=0}^{\infty} (-1)^l h_l\, t^l\right)\left(\sum_{j=0}^{\infty} B_j\, t^j\right) \\
&= \det(\mathrm{I}_n - t\,H)\left(1 + h_1\, t + V\, t^2 + t^3 \sum_{k=3}^{\infty} T^*(tH)^{k-3}T\right) \\
&= \det(\mathrm{I}_n - t\,H)\left(1 + h_1\, t + V\, t^2 + t^3\, T^*(\mathrm{I}_n - tH)^{-1}T\right) \\
&= \det(\mathrm{I}_n - t\,H)\,(1 + h_1\, t + V\, t^2) + t^3\, T^*\, \mathrm{Cof}(\mathrm{I}_n - tH)\, T
\end{aligned}$$

where $\mathrm{Cof}(\mathrm{I}_n - tH)$ is the signed cofactor matrix[14] of $\mathrm{I}_n - tH$.

The cautious reader may object that the second factors on the second and third lines need not converge for all t. However, every $u \in P$ has an open neighborhood on which T^*T and $\mathrm{tr}(H^*H)$ are bounded, so that the series is bounded by a geometric series and hence converges for $|t|$ sufficiently small. The upshot of this is that the two series converge absolutely and uniformly on compact subsets of a certain open neighborhood of $P \times 0$ in $P \times \mathbb{R}$, so equality of the first and last terms holds on that open subset. The left hand side is evidently a polynomial in t of degree at most $n{+}2$ and the final form of the right hand side is also a polynomial in t, so it follows that these first and last expressions are equal for all t.

[14] The signed cofactor matrix of any n-by-n matrix R is the (unique) homogeneous polynomial matrix of degree $(n{-}1)$ that satisfies the identity $R\,\mathrm{Cof}(R) = \det(R)\,\mathrm{I}_n$.

Replacing t by t^{-1} and multiplying through by t^{n+2} gives the form of the identity that will be most useful, namely:

$$\sum_{k=0}^{n+2} C_k\, t^{n+2-k} = \det(t\,\mathrm{I}_n - H)\,(t^2 + h_1\,t + V) + T^*\ \mathrm{Cof}(t\,\mathrm{I}_n - H)\,T. \tag{4.1}$$

The polynomial $p_C(t) = t^{n+2} + C_2\,t^n + C_3\,t^{n-2} + \cdots + C_{n+2}$ will be said to be the *characteristic polynomial* of the Bochner-Kähler structure.

Example 4.1 (Low cohomogeneity)**.** Suppose M is locally isometric to $M_c^p \times M_{-c}^{n-p}$. Then, looking back at the proof of Proposition 2.5 and the definition of H, one computes that

$$p_C(t) = (t + (n{-}p{+}1)\,r)^{p+1}\,(t - (p{+}1)\,r)^{n-p+1}, \quad \text{where} \qquad r = \frac{c}{2(n+2)}\,.$$

For Example 2.3 (i.e., rotationally symmetric), the formula is

$$p_C(t) = (t - 2r)^n\left[(t + nr)^2 - \tfrac{1}{4}k^2 + a\right], \quad \text{where} \qquad r = \frac{k}{2(n+2)}\,.$$

4.1.2. *The reduced characteristic polynomial.* Let P_1 be the set of those $u \in P_0$ that satisfy the condition that $H'(u)$ be diagonal, with eigenvalues arranged in descending order, and that each of the entries $T_i(u)$ of $T'(u) \in \mathbb{R}^m$ be positive. (See §3.6 for definitions.) Then P_1 is an $\{\mathrm{I}_m\}\times \mathrm{U}(n{-}m)$-bundle over M°. Using the identities derived in §3.6, equation (4.1) can be written as

$$\frac{p_C(t)}{p_{h''}(t)} = \left(t^2 + h_1\,t + V\right)\prod_{j=1}^{m}(t - \lambda_j) \ + \ \sum_{i=1}^{m} T_i^2 \prod_{j\neq i}(t - \lambda_j). \tag{4.2}$$

In particular, $p_{h''}(t)$ divides $p_C(t)$. Denote the quotient by $p_D(t)$. It is a monic polynomial with constant coefficients of degree $m{+}2$ and will be called the *reduced characteristic polynomial*.

Example 4.2 (Low cohomogeneity)**.** Suppose M is locally isometric to $M_c^p \times M_{-c}^{n-p}$. Then, $m = 0$ and

$$p_D(t) = (t + (n{-}p{+}1)\,r)\,(t - (p{+}1)\,r)\,, \quad \text{where} \qquad r = \frac{c}{2(n+2)}\,.$$

For Example 2.3, where $m = 1$, the formula is

$$p_D(t) = (t - 2r)\left[(t + nr)^2 - \tfrac{1}{4}k^2 + a\right], \quad \text{where} \qquad r = \frac{k}{2(n+2)}\,.$$

Proposition 4.3. *Every root of $p_{h''}(t)$ is also a root of $p_D(t)$.*

Proof. Let $p_{h''}(t)$ have roots $\lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_n$, counting multiplicity, and set

$$\Lambda = \begin{pmatrix} \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & \lambda_{m+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{4.3}$$

Let $P_2 \subset P_1$ consist of the coframes $u \in P_1$ for which $H''(u) = \Lambda$. This P_2 is a bundle over M° with structure group $\{\mathrm{I}_m\}\times G_\Lambda$, where $G_\Lambda \subset \mathrm{U}(n{-}m)$ is the group of unitary matrices commuting with Λ. All calculations will now take place on P_2.

Adopt the index range convention $1 \le i, j, k \le m < a, b, c < n$. Thus, for example $T_i > 0$ but $T_a = 0$. Also, $H_{a\bar{\imath}} = 0$. The (a, i)-entry of the structure equation (2.14) for dH becomes (no sum over i)

$$(\lambda_a - \lambda_i)\phi_{a\bar{\imath}} + T_i\,\omega_a = 0. \tag{4.4}$$

Meanwhile, the structure equation for dT_a becomes

$$({\lambda_a}^2 + h_1\,\lambda_a + V)\,\omega_a - \sum_{i=1}^{m} T_i\,\phi_{a\bar{\imath}} = 0. \tag{4.5}$$

Combining these equations yields

$$\begin{aligned}\left(({\lambda_a}^2 + h_1\,\lambda_a + V)\prod_{i=1}^{m}(\lambda_a - \lambda_i)\right)\omega_a &= \left(\prod_{i=1}^{m}(\lambda_a - \lambda_i)\right)\sum_{j=1}^{m} T_j\,\phi_{a\bar{\jmath}}\\ &= \sum_{j=1}^{m}\left(\prod_{i\neq j}(\lambda_a - \lambda_i)\right) T_j\,(\lambda_a - \lambda_j)\phi_{a\bar{\jmath}}\\ &= -\left(\sum_{j=1}^{m} {T_j}^2 \prod_{i\neq j}(\lambda_a - \lambda_i)\right)\omega_a\,.\end{aligned}$$

Since ω_a is nonzero on P_2, it follows that

$$p_D(\lambda_a) = \left({\lambda_a}^2 + h_1\,\lambda_a + V\right)\prod_{j=1}^{m}(\lambda_a - \lambda_j) \;+\; \sum_{i=1}^{m} T_i^2 \prod_{j\neq i}(\lambda_a - \lambda_j) = 0,$$

as desired. □

4.1.3. *Point data.* In this subsubsection, a formula will be developed for the characteristic polynomials of a connected Bochner-Kähler structure in terms of a single value of its structure function. The formula for p_C is, of course, already given by (4.1). However, the formula for p_D is somewhat more subtle.

First, recall the concepts introduced in §3.3, suitably modified for the present section. For any $(H_0, T_0, V_0) \in i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$, let $H_1, H_2, \ldots, H_\delta$ be the distinct eigenvalues of H_0. Let $L_\alpha \subset \mathbb{C}^n$ be the eigenspace of H_0 belonging to the eigenvalue H_α, and let $n_\alpha \ge 1$ be the (complex) dimension of L_α. Write

$$T_0 = T_1 + \cdots + T_\delta \tag{4.6}$$

where T_α lies in L_α for $1 \le \alpha \le \delta$. Define the quantities

$$V_\alpha = {H_\alpha}^2 + (\operatorname{tr} H_0)\,H_\alpha + V_0 + \sum_{\beta\neq\alpha} \frac{|T_\beta|^2}{(H_\alpha - H_\beta)} \tag{4.7}$$

and

$$m_\alpha = \begin{cases} 2 & \text{if } T_\alpha \neq 0 \text{ and } n_\alpha > 1;\\ 1 & \text{if } T_\alpha \neq 0 \text{ and } n_\alpha = 1;\\ 1 & \text{if } T_\alpha = 0 \text{ and } V_\alpha \neq 0;\\ 0 & \text{if } T_\alpha = 0 \text{ and } V_\alpha = 0.\end{cases} \tag{4.8}$$

Theorem 4.4. *If* (M^n, g, Ω) *is a connected Bochner-Kähler manifold whose structure function* (H, T, V) *assumes the value* (H_0, T_0, V_0)*, then*

$$p_C(t) = \prod_{\alpha=1}^{\delta} (t - H_\alpha)^{n_\alpha} \left[t^2 + (\operatorname{tr} H_0)\, t + V_0 + \sum_{\alpha=1}^{\delta} \frac{|T_\alpha|^2}{(t - H_\alpha)} \right] \tag{4.9}$$

and

$$p_D(t) = \prod_{\alpha=1}^{\delta} (t - H_\alpha)^{m_\alpha} \left[t^2 + (\operatorname{tr} H_0)\, t + V_0 + \sum_{\alpha=1}^{\delta} \frac{|T_\alpha|^2}{(t - H_\alpha)} \right]. \tag{4.10}$$

Proof. The formula for p_C follows directly from (4.1), so the formula for p_D will follow from the equivalent statement

$$p_{h''}(t) = \prod_{\alpha=1}^{\delta} (t - H_\alpha)^{n_\alpha - m_\alpha}, \tag{4.11}$$

and this is what will be proved.

By §3.3.3, the generic orbit of the symmetry pseudo-groupoid has codimension equal to $m_1 + \cdots + m_\delta$. By §3.5.3 and Theorem 3.13, the orbits of the symmetry pseudo-groupoid in M° (which is open and dense in M) have codimension m. Consequently, $m = m_1 + \cdots + m_\delta$, so that $n - m = (n_1 - m_1) + \cdots + (n_\delta - m_\delta)$. Moreover, the inequality $n_\alpha \geq m_\alpha$ follows immediately from the definitions.

Thus, by the very definition of $p_{h''}(t)$, it will suffice to show that, for each α, the polynomial $p_h(t)$ has a constant root H_α of multiplicity at least $n_\alpha - m_\alpha$. By Theorem 3.13, it suffices to show this constancy in an open neighborhood of the point $x \in M$ for which there exists a $u \in P_x$ satisfying $\big(H(u), T(u), V(u)\big) = (H_0, T_0, V_0)$.

Thus, let $w \in \mathbb{C}^n$ be a nonzero vector and let $c : (-\varepsilon, \varepsilon) \to M$ be the constant speed geodesic satisfying $u\big(\dot c(0)\big) = w$. Then c can be lifted uniquely to a curve $\gamma : (-\varepsilon, \varepsilon) \to P$ that satisfies $\gamma(0) = u$ and $\gamma^*\phi = 0$ (i.e., the coframe field γ is parallel along c). Because c is a constant speed geodesic, γ also satisfies $\gamma^*(\omega) = w\, ds$, where s is the parameter on $(-\varepsilon, \varepsilon)$.

Because the polynomial $p_h(t)$ is invariant under the action of the symmetry pseudo-groupoid, it suffices to consider only geodesics with initial velocities orthogonal to the subspace $O_x \subset T_xM$ that is the tangent to the orbit through x. Thus, we will assume that if $T_\alpha \neq 0$, then $T_\alpha^* w$ is real, and that if $T_\alpha = V_\alpha = 0$, then w is orthogonal to L_α.

For simplicity, set $H(s) = H\big(\gamma(s)\big)$, $T(s) = T\big(\gamma(s)\big)$, and $V(s) = V\big(\gamma(s)\big)$. Then these functions on $(-\varepsilon, \varepsilon)$ satisfy the initial conditions $\big(H(0), T(0), V(0)\big) = (H_0, T_0, V_0)$ and the system of ordinary differential equations

$$\begin{aligned} \dot H &= T\, w^* + w\, T^*, \\ \dot T &= \big(H^2 + (\operatorname{tr} H)\, H + V\ \mathrm{I}_n\big)\, w, \\ \dot V &= (\operatorname{tr} H)\big(T^* w + w^* T\big) + \big(T^* H w + w^* H T\big). \end{aligned} \tag{4.12}$$

Let $L \in i\,\mathfrak{u}(n)$ be any fixed element that satisfies $Lw = LT_0 = 0$ and $[L, H_0] = 0$. Because of the latter equation, L preserves each of the eigenspaces of H_0, i.e., the subspaces L_α. Consequently, $LT_\alpha = 0$ and $Lw_\alpha = 0$ for all α. Conversely, if $L \in i\,\mathfrak{u}(n)$ preserves the eigenspaces of H_0 and annihilates T_α and w_α for all α, then it satisfies $Lw = LT_0 = 0$ and $[L, H_0] = 0$.

Now, the above differential equations imply the differential equations

$$
\begin{aligned}
[L, \dot{H}] &= LT\, w^* - w\,(LT)^*\,, \\
L\dot{T} &= \big([L,H]H + H[L,H] + (\operatorname{tr} H)\,[L,H]\big)\, w,
\end{aligned}
\tag{4.13}
$$

so that the quantities $\big([L,H(s)], LT(s)\big)$ satisfy a linear system of ordinary differential equations with vanishing initial condition at $s=0$. Consequently $[L,H(s)]$ and $LT(s)$ vanish identically for all s, as does Lw (for trivial reasons). By the above characterization of those $L \in i\,\mathfrak{u}(n)$ that satisfy $[L,H_0] = LT_0 = Lw = 0$, this implies that the subspace $K_\alpha \subset L_\alpha$ that is perpendicular to T_α and w_α is necessarily an eigenspace of $H(s)$ that is perpendicular to $T_\alpha(s)$ and w for all s.

If $K_\alpha \neq 0$, then there is a well-defined eigenvalue $H_\alpha(s)$ of $H(s)$ associated to K_α. In particular,

$$
\dot{H}_\alpha(s)y = \dot{H}(s)y = \big(T(s)\,w^* + w\,T(s)^*\big)y = 0 \tag{4.14}
$$

for all $y \in K_\alpha$. Of course, this implies that $\dot{H}_\alpha(s) = 0$, i.e., that $H_\alpha(s) = H_\alpha(0) = H_\alpha$ for all s.

There are now four cases to consider:

If α is such that $T_\alpha \neq 0$ and $n_\alpha > 1$, then $\dim K_\alpha$ is either $n_\alpha{-}1$ or $n_\alpha{-}2$, depending on whether w_α is zero or not. In either case, $\dim K_\alpha \geq n_\alpha{-}2 = n_\alpha{-}m_\alpha$, so H_α is a root of $p_h(t)$ of multiplicity at least $n_\alpha{-}m_\alpha$, as desired.

If α is such that $T_\alpha \neq 0$ and $n_\alpha = 1$, then $m_\alpha = 1$ and $K_\alpha = 0$. In this case, of course, $n_\alpha{-}m_\alpha = 0$, so H_α is trivially a root of $p_h(t)$ of multiplicity at least $n_\alpha{-}m_\alpha$, as desired.

If α is such that $T_\alpha = 0$ but $V_\alpha \neq 0$, then $\dim K_\alpha$ is either n_α or $n_\alpha{-}1$, depending on whether w_α is zero or not. In either case, $\dim K_\alpha \geq n_\alpha - 1 = n_\alpha - m_\alpha$, so H_α is a root of $p_h(t)$ of multiplicity at least $n_\alpha{-}m_\alpha$, as desired.

Finally, if $T_\alpha = 0$ and $V_\alpha = 0$, then $w_\alpha = 0$ by the above condition on c. Thus $K_\alpha = L_\alpha$, so that $\dim K_\alpha = n_\alpha{-}0 = n_\alpha - m_\alpha$ and, again, H_α is a root of $p_h(t)$ of multiplicity at least $n_\alpha{-}m_\alpha$, as desired. □

The following result will be needed in the next subsection. Its proof follows by inspection of the formula for $p_D(t)$ and the definition of the m_α and so will be omitted.

Corollary 4.5. *No root of $p_{h'}(t)$ is a multiple root of $p_D(t)$.*

4.2. **Momentum cells.** In general, the two characteristic polynomials do not completely determine the analytically connected equivalence class of a Bochner-Kähler structure. However, as will be seen in this subsection, they do determine it up to a finite number (at most $m{+}1$) of possibilities (Theorem 4.10).

4.2.1. *The roots of p_D.* It turns out that the reality and multiplicity properties of the roots of p_D are severely constrained.

Proposition 4.6. *One of the following cases holds:*

1. *p_D has m real, distinct roots, all of order 1;*
2. *p_D has m real, distinct roots, one of order 3 and the rest of order 1;*
3. *p_D has $m{+}1$ real, distinct roots, one of order 2 and the rest of order 1;*
4. *p_D has $m{+}2$ real, distinct roots, all of order 1.*

Proof. Substituting $t = \lambda_i$ into (4.2) yields

$$p_D(\lambda_i) = T_i^2 \prod_{j\neq i}(\lambda_i - \lambda_j). \tag{4.15}$$

Since $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ and $T_i > 0$ on M°, it follows that $(-1)^{i-1}p_D(\lambda_i) > 0$ for $1 \leq i \leq m$ holds on M°.

Equivalently, for every $x \in M^\circ$, the polynomial $p_D(t)$ has an even number of real roots (counted with multiplicity) greater than $\lambda_1(x)$ and an odd number of real roots (counted with multiplicity) in each open interval $\big(\lambda_i(x), \lambda_{i+1}(x)\big)$ for $1 \leq i < m$. Moreover, since $(-1)^m p_D(t)$ is positive for all t sufficiently negative, $p_D(t)$ has an odd number of real roots (counted with multiplicity) less than $\lambda_m(x)$. These considerations imply that $p_D(t)$ has at least m distinct real roots.

If p_D has exactly m real roots, then the above parity conditions show that they must all have odd order. Since $p_D(t)$ has degree $m{+}2$, either Case 1 or Case 2 must hold.

If p_D has exactly $m{+}1$ real, distinct roots, then Case 3 must hold.

If p_D has exactly $m{+}2$ real, distinct roots, then Case 4 must hold. □

Remark 4.7 (Global inequalities). By continuity, the inequality $(-1)^{i-1}p_D(\lambda_i) \geq 0$ holds on M.

Remark 4.8 (Root labeling). We will use the following convention to label the real roots of p_D: When p_D has m distinct real roots, denote them by $r_1 > r_2 > \cdots > r_m$; when p_D has $m{+}1$ distinct real roots, denote them by $r_1 > \cdots > r_{m+1}$; and when p_D has $m{+}2$ distinct real roots, denote them by $r_0 > r_1 > \cdots > r_{m+1}$. In all cases, the list of real roots of p_D, in descending order, will be denoted by r.

If r_i is any real root of $p_D(t)$, the number of roots $\{\lambda_1(x), \ldots, \lambda_m(x)\}$ that are strictly greater than r_i is independent of $x \in M^\circ$, so we will denote this common value by μ_i. The function μ is constrained as follows:

Cases 1 and 2: Necessarily, $\mu_i = i$.

Case 3: Let r_i be the double root. Then μ_i is either i (SubCase 3-i,a; impossible when $i = m{+}1$) or $i{-}1$ (SubCase 3-i,b). Moreover, $\mu_j = j$ for $j < i$, while $\mu_j = j{-}1$ for $j > i$.

Case 4: There is an integer $i \leq m$ so that $\mu_i = i$ (SubCase 4-i). Then $\mu_j = j{+}1$ for $j < i$ while $\mu_j = j{-}1$ for $j > i$.

4.2.2. *Momentum cells.* Since

$$p_{h'}(r_i) = \prod_{j=1}^{m}(r_i - \lambda_j),$$

it follows that $(-1)^{\mu_i}p_{h'}(r_i) > 0$ on M°. By continuity, the inequality

$$(-1)^{\mu_i}\big({r_i}^m - h'_1\,{r_i}^{m-1} + h'_2\,{r_i}^{m-2} - \cdots + (-1)^m\,h'_m\big) \geq 0$$

holds on M, with strict inequality on M°. Moreover, by Corollary 4.5, if r_i is a multiple root of p_D, then it is not a root of $p_{h'}(t)$ at any point of M, so that the above inequality is strict on all of M.

The image $h'(M) \subset \mathbb{R}^m$ therefore lies in the intersection of the closed half-spaces $\overline{H(r_i, \mu_i)}$ defined by the inequalities

$$(-1)^{\mu_i}\big({r_i}^m - {r_i}^{m-1}\,x_1 + {r_i}^{m-2}\,x_2 - \cdots + (-1)^m\,x_m\big) \geq 0 \tag{4.16}$$

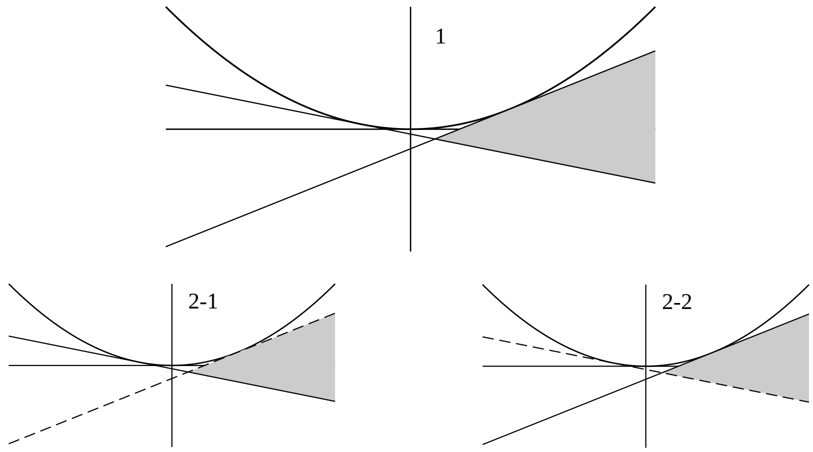

FIGURE 1. Possible momentum cells with $m = 2$ and two distinct real roots: Case 1 (both roots simple), Case 2-1 (r_1 triple), and Case 2-2 (r_2 triple).

as r_i ranges over the simple real roots of p_D and the open half-space $H(r_i, \mu_i)$ defined by

$$(-1)^{\mu_i}\left({r_i}^m - {r_i}^{m-1}\,x_1 + {r_i}^{m-2}\,x_2 - \cdots + (-1)^m\,x_m\right) > 0 \tag{4.17}$$

if r_i is a multiple root of p_D. This intersection will be referred to as the *momentum cell* $C(p_D, \mu) \subset \mathbb{R}^m$. Note that $h'(M^\circ)$ lies in $C(p_D, \mu)^\circ$, the interior of $C(p_D, \mu)$, and that $h' : M^\circ \to C(p_D, \mu)^\circ$ is a submersion onto its image.

4.2.3. *Possible momentum cells.* More generally, if $p_D(t)$ is any monic polynomial of degree $m+2$ with real coefficients that falls into one of the Cases 1 through 4 of Proposition 4.6, define the *possible momentum cells of* p_D as follows:

If p_D falls into Case 1 or Case 2, define $\mu_i = i$ for $1 \le i \le m$ and let $C(p_D, \mu)$ be defined by the inequalities (16) (strict or not depending on the multiplicity of the roots). This cell $C(p_D, \mu)$ is a closed, unbounded, convex polytope in Case 1, but is not closed in Case 2, since one of the faces is missing.

If p_D falls into Case 3, with r_i being the double root, define $\mu_j = j$ for $j < i$ and $\mu_j = j{-}1$ for $j > i$, while μ_i is allowed to be one of i (type a) or $i{-}1$ (type b). Let $C(p_D, \mu)$ be defined by the inequalities (16) (strict or not depending on the multiplicity of the roots). Thus, there are two possible momentum cells except in the case that r_{m+1} is the double root, in which case there is only one possible cell. When there are two cells, neither is closed and their closures share the missing face. The only subcase with a bounded cell is Subcase (3-1,b).

When p_D falls into Case 4, choose an integer i in the range $0 \le i \le m$ and define μ so that $\mu_j = j{+}1$ for $j < i$, while $\mu_i = i$ and $\mu_j = j{-}1$ for $j > i$. Let $C(p_D, \mu)$ be defined by the inequalities (4.16). Thus, there are $m{+}1$ possible momentum cells, one for each possible choice of i. Each of these cells is a closed polytope and they are mutually disjoint. When $\mu_m = m$ (i.e., $\lambda_m > r_{m-1}$ on M°), this 'highest' cell has m faces. Each of the other cells has $m{+}1$ faces. The only bounded cell falls in Subcase 4-0, i.e., $\mu_0 = 0$ (implying that $\mu_i = i{-}1$ for all $1 \le i \le m{+}1$).

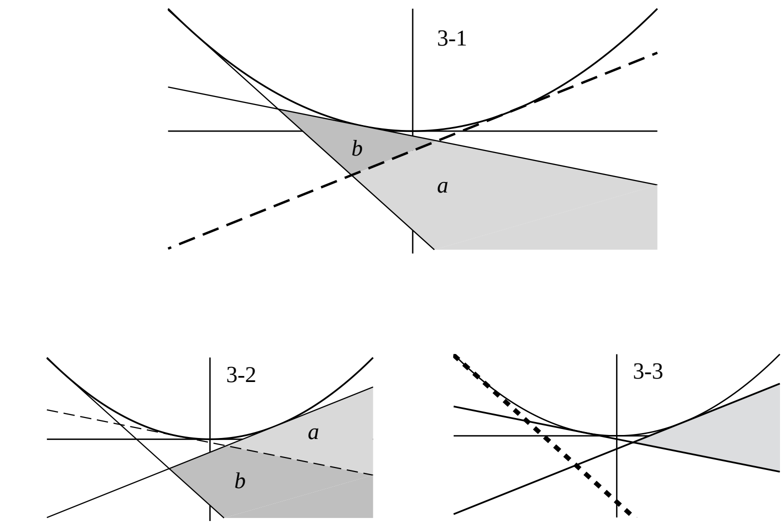

FIGURE 2. Possible momentum cells with $m = 2$ and three distinct real roots: Case 3-1 (r_1 double; two cells, a unbounded, b bounded), Case 3-2 (r_2 double; two cells, both unbounded), and Case 3-3 (r_3 double; one cell, unbounded).

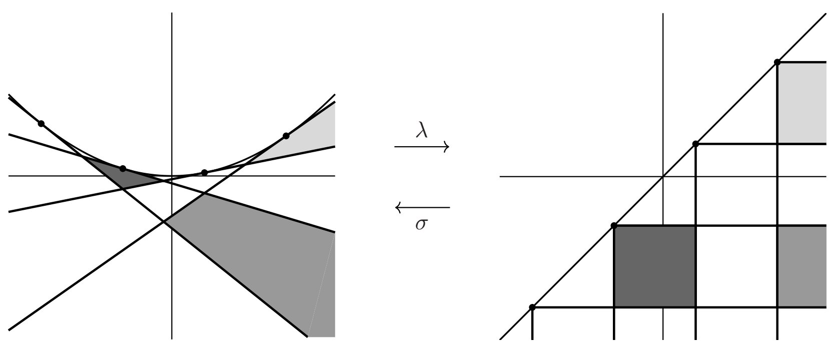

FIGURE 3. Possible momentum cells in Case 4 with $m = 2$ and four distinct real roots and their corresponding spectral products. The 'highest' cell (SubCase 4-2) has only two faces. The 'lowest' cell (SubCase 4-0) is the only bounded one.

Figures 1, 2, and 3 show the possible momentum cells when $m = 2$ in the four cases. The drawn axes are u_1 $(= h_1')$ and u_2 $(= h_2')$. Of course, all of these cells lie below the discriminant parabola $u_2 = \frac{1}{4}\,{u_1}^2$ and are bounded by its tangent lines.

4.2.4. *The product representation.* It will be useful to have another description of the possible momentum cells.

Let $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ be the standard symmetrizing map, so that $\sigma = (\sigma_1, \ldots, \sigma_m)$ where $\sigma_k : \mathbb{R}^m \to \mathbb{R}$ is the k-th elementary symmetric function of its arguments.

The map σ is one-to-one on the closed set

$$\mathbb{R}^m_{\geq} = \{\ (x_1, \dots, x_m) \in \mathbb{R}^m \ \mid\ x_1 \geq x_2 \geq \cdots \geq x_m\ \}.$$

Moreover, $\sigma : \mathbb{R}^m_{\geq} \to \sigma(\mathbb{R}^m_{\geq}) \subset \mathbb{R}^m$ is a homeomorphism onto its image and a real-analytic diffeomorphism on its interior, which will be denoted $\mathbb{R}^m_{>} \subset \mathbb{R}^m_{\geq}$. Denote the inverse of σ by $\lambda : \sigma(\mathbb{R}^m_{\geq}) \to \mathbb{R}^m_{\geq}$.

For any momentum cell $C(p_D, \mu)$, the subset $\lambda\bigl(C(p_D, \mu)\bigr)$ is a product of the form

$$\lambda\bigl(C(p_D, \mu)\bigr) = I_1 \times I_2 \times \cdots \times I_m \tag{4.18}$$

where $I_1, I_2, \dots, I_m$ are (nonempty) intervals in $\mathbb{R}$ with nonoverlapping interiors. The endpoints of the closure $\overline{I_i}$ are roots of p_D and I_i contains such an endpoint if and only if that endpoint is a simple root of p_D.

In Case 1, where the distinct roots $r_1 > \cdots > r_m$ are all simple, $I_1 = [r_1, \infty)$ and $I_k = [r_k, r_{k-1}]$ for $1 < k \leq m$.

In Case 2 in which, say, r_1 is the triple root, $I_1 = (r_1, \infty)$, $I_2 = [r_2, r_1)$, and (assuming $m > 2$) $I_k = [r_k, r_{k-1}]$ for $3 \leq k \leq m$.

The intervals $(I_1, \dots, I_m)$ will be referred to as the *spectral bands* associated to $C(p_D, \mu)$ and $I_1 \times I_2 \times \cdots \times I_m$ will be known as the *spectral product.* Since

$$\sigma(I_1 \times I_2 \times \cdots \times I_m) = C(p_D, \mu), \tag{4.19}$$

specifying the spectral bands is equivalent to specifying $C(p_D, \mu)$. Also, note that σ maps $I_1^\circ \times I_2^\circ \times \cdots \times I_m^\circ$ diffeomorphically onto $C(p_D, \mu)^\circ$, the interior of $C(p_D, \mu)$.

Proposition 4.9. *Suppose that $p_D(t)$ is a polynomial of degree $m{+}2$ that falls into one of the Cases of Proposition* 4.6*. Suppose that there exists a monic polynomial $p_C(t)$ of degree $n{+}2$ with the properties that $p_C(t)/p_D(t)$ is a polynomial all of whose roots are real roots of $p_D(t)$ and that its t^{n+1}-coefficient vanishes.*

Then for every k' in a possible momentum cell $C(p_D, \mu) \subset \mathbb{R}^m$, there exists a Bochner-Kähler n-manifold (M, g, Ω) whose characteristic polynomials are $p_C(t)$ and $p_D(t)$ and whose reduced momentum mapping $h' : M \to \mathbb{R}^m$ assumes the value k'.

Proof. This will be a matter of checking cases.

First, some generalities. Given polynomials $p_D(t)$ and $p_C(t)$ satisfying the hypotheses of the proposition and a $k' \in \mathbb{R}^m$ lying in a possible momentum cell $C(p_D, \mu)$ for p_D, define $p_{h''}(t) = p_C(t)/p_D(t)$. By hypothesis, all the roots of $p_{h''}(t)$ are real and are roots of $p_D(t)$ as well.

Define

$$p_{k'}(t) = t^m - k'_1\, t^{m-1} + \cdots + (-1)^m k'_m.$$

Let $\lambda\bigl(C(p_D, \mu)\bigr) = I_1 \times I_2 \times \cdots \times I_m$, and let $\lambda(k') = (s_1, s_2, \dots, s_m)$, so that there exists a real factorization of the form

$$p_{k'}(t) = (t - s_1)(t - s_2) \cdots (t - s_m)$$

with $s_k \in I_k$ for $1 \leq k \leq m$.

First, assume that k' lies in $C(p_D, \mu)^\circ$, the interior of $C(p_D, \mu)$. Then s_k lies in I_k° for $1 \leq k \leq m$ and, in particular, the s_k are all distinct and are not roots of $p_D(t)$. It follows that the rational function $p_D(t)/p_{k'}(t)$ has a simple pole at $t = s_k$

for $1 \le k \le m$. Since $p_D(t)$ has degree $m{+}2$ and is monic, there is a partial fractions expansion of the form

$$\frac{p_D(t)}{p_{k'}(t)} = t^2 + b_1\,t + b_2 + \sum_{k=1}^{m} \frac{q_k}{(t-s_k)}.$$

Because of the way that the possible momentum cells were defined, the inequality $(-1)^{k-1}p_D(s_k) > 0$ holds for $s_k \in I_k^\circ$, so it follows that $q_k > 0$ for $1 \le k \le m$.

If $n > m$, define $s_{m+1} \ge s_{m+2} \ge \cdots \ge s_n$ so that

$$p_{h''}(t) = (t-s_{m+1})(t-s_{m+2})\cdots(t-s_m).$$

By hypothesis, each root of $p_{h''}(t)$ is a real root of p_D and so is not equal to any of the roots of $p_{k'}(t)$.

Consider the element $(s,t,v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ defined by letting s be the diagonal matrix with entries $s_{i\bar{\imath}} = s_i$ for $1 \le i \le n$; letting $t_i = \sqrt{q_i}$ for $1 \le i \le m$ and $t_i = 0$ for $m < i \le n$ (if $n > m$); and letting $v = b_2$. The hypothesis that p_C have no t^{n+1}-term is then seen to be equivalent to the condition that $b_1 = \operatorname{tr} s$, while the condition that each s_a for $a > m$ be a root of p_D is then equivalent to the condition that

$$v_a = {s_a}^2 + b_1\,s_a + v + \sum_{i=1}^{m} \frac{{t_i}^2}{(s_a - s_i)} = 0.$$

By Theorem 4.4, the Bochner-Kähler structure on a neighborhood M of $0 \in \mathbb{C}^n$ that has a unitary coframe $u_0 : T_0M \to \mathbb{C}^n$ with $\big(H(u_0), T(u_0), V(u_0)\big) = (s,t,v)$ has $p_C(t)$ and $p_D(t)$ as its characteristic polynomials and satisfies $h'(0) = k'$. This establishes existence for the interior points of $C(p_D, \mu)$.

It remains to treat the boundary cases, i.e., cases in which one or more of the s_i are actually roots of $p_D(t)$.

Now, if $s_j = s_{j+1}$ for any j, then $\{s_j\} = I_j \cap I_{j+1}$, so that s_j is a simple root of p_D. In such a case, necessarily, $s_{j-1} > s_j$ (if $j > 1$) since $I_{j-1} \cap I_{j+1} = \emptyset$ and $s_{j+1} > s_{j+2}$ (if $j < m{-}1$) since $I_j \cap I_{j+2} = \emptyset$. Consequently, each s_j is at most a double root of $p_{k'}(t)$ and, if so, it must also be a root of $p_D(t)$. It follows that the rational function $p_D(t)/p_{k'}(t)$ has a simple pole at $t = s_j$ for $1 \le j \le m$. Since $p_D(t)$ has degree $m{+}2$ and is monic, there is a partial fractions expansion

$$\frac{p_D(t)}{p_{k'}(t)} = t^2 + b_1\,t + b_2 + \sum_{j=1}^{m} \frac{q_j}{(t-s_j)},$$

where, in order to make the q_j unique, it is now necessary to add the condition that $q_{j+1} = 0$ if $s_{j+1} = s_j$. If j is such that s_j is not a root of $p_D(t)$, then the inequality $(-1)^{j-1}p_D(s_j) > 0$ holds so that $q_j > 0$. If j is such that s_j is a simple root of both $p_{k'}(t)$ and $p_D(t)$, then $q_j = 0$. If j is such that $s_j = s_{j+1}$, then $(-1)^{j-1}p_D(t)$ and $(-1)^{j-1}p_{k'}(t)$ are both positive on I_j°. Since s_j is a double root of $p_{k'}(t)$ and a simple root of $p_D(t)$, it follows that

$$\lim_{t\to s_j^+} \frac{p_D(t)}{p_{k'}(t)} = +\infty,$$

which can only hold if $q_j > 0$. In particular, $q_j \ge 0$ has been defined for $1 \le j \le m$ so that the above partial fractions expansion is valid.

If $n > m$, again define $s_{m+1} \geq s_{m+2} \geq \cdots \geq s_n$ so that

$$p_{h''}(t) = (t - s_{m+1})(t - s_{m+2}) \cdots (t - s_m).$$

Again, each root of $p_{h''}(t)$ is a real root of p_D but now it may also be a root of $p_{k'}(t)$.

Define an element $(s, t, v) \in i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ by letting s be the diagonal matrix with entries $s_{i\bar{\imath}} = s_i$ for $1 \leq i \leq n$; letting $t_i = \sqrt{q_i}$ for $1 \leq i \leq m$ and $t_i = 0$ for $m < i \leq n$ (if $n > m$); and letting $v = b_2$. It must now be verified that the element (s, t, v) does indeed have $p_C(t)$ and $p_D(t)$ as its characteristic polynomials.

Now, the hypothesis that p_C have no t^{n+1}-term is again seen to be equivalent to the condition that $b_1 = \operatorname{tr} s$, and the condition that $t_i = 0$ for $i > m$ or when $s_i = s_{i-1}$ implies that

$$\begin{aligned} t^2 + (\operatorname{tr} s)\, t + v + \sum_{j=1}^{n} \frac{{t_j}^2}{(t - s_j)} &= t^2 + b_1\, t + b_2 + \sum_{j=1}^{m} \frac{q_j}{(t - s_j)} \\ &= \frac{p_D(t)}{p_{k'}(t)} = \frac{p_C(t)}{p_{h''}(t) p_{k'}(t)}, \end{aligned}$$

so that

$$p_C(t) = \prod_{i=1}^{n} (t - s_i) \left[t^2 + (\operatorname{tr} s)\, t + v + \sum_{j=1}^{n} \frac{{t_j}^2}{(t - s_j)} \right],$$

as desired. It remains to verify that $p_D(t)$ is the reduced characteristic polynomial associated to (s, t, v), i.e., to compute the numbers n_i and m_i for each eigenvalue s_i of s according to the recipe of §3.3 and show that s_i is a root of $p_{h''}(t)$ of multiplicity exactly equal to $n_i - m_i$. This will be done by breaking it down into a number of cases.

If s_a is not s_i for any $1 \leq i \leq m$, then $p_D(s_a) = 0$ is equivalent to

$$v_a = {s_a}^2 + (\operatorname{tr} s)\, s_a + v + \sum_{i=1}^{m} \frac{{t_i}^2}{(s_a - s_i)} = 0,$$

and this implies that s_a is an eigenvalue of s of some multiplicity $n_a \geq 1$ that satisfies $t_a = v_a = 0$, so that $m_a = 0$. Thus, s_a is a root of $p_{h''}(t)$ and has multiplicity $n_a - m_a = n_a$, as desired.

If s_i is not a root of $p_D(t)$, then, by construction, it is a simple eigenvalue of s and also satisfies $t_i = \sqrt{q_i} > 0$, so $m_i = 1$ and $n_i - m_i = 0$, so that $(t - s_i)$ is not a factor of $p_{h''}(t)$, again, as desired.

If s_i is a simple root of $p_D(t)$ and a simple root of $p_{k'}(t)$, then, by construction, $t_i = 0$. The quantity v_i is calculated to be

$$\begin{aligned} v_i &= {s_i}^2 + (\operatorname{tr} s)\, s_i + v + \sum_{j \neq i} \frac{{t_j}^2}{(s_i - s_j)} \\ &= \lim_{t \to s_i} \left(t^2 + (\operatorname{tr} s)\, t + v + \sum_{j=1}^{m} \frac{{t_j}^2}{(t - s_j)} \right) \\ &= \lim_{t \to s_i} \frac{p_D(t)}{p_{k'}(t)} \neq 0. \end{aligned}$$

Thus, the recipe gives $m_i = 1$, again as desired.

If s_i is a simple root of $p_D(t)$ and a double root of $p_{k'}(t)$, then it can be assumed that $s_i = s_{i+1}$, so that $t_i > 0$ (and $t_{i+1} = 0$). Since $n_i \geq 2$, the recipe gives $m_i = 2$, again, as desired.

Finally, if s_i is a multiple root of $p_D(t)$, then it cannot be a root of $p_{k'}(t)$ at all, by the definition of the momentum cell $C(p_D, \mu)$. Consequently, $t_i = 0$ by definition and calculation shows that $v_i = 0$ as well. Thus $m_i = 0$, as desired.

By Theorem 4.4, the Bochner-Kähler structure on a neighborhood M of $0 \in \mathbb{C}^n$ that has a unitary coframe $u_0 : T_0M \to \mathbb{C}^n$ with $\big(H(u_0), T(u_0), V(u_0)\big) = (s, t, v)$ has $p_C(t)$ and $p_D(t)$ as its characteristic polynomials and satisfies $h'(0) = k'$. This establishes existence for the boundary points of $C(p_D, \mu)$. □

The way is now paved for the following result, which, together with the previous proposition, classifies the analytically connected equivalence classes of Bochner-Kähler structures.

Theorem 4.10. *The analytically connected class of a Bochner-Kähler structure is determined by p_C, p_D, and the momentum cell $C(p_D, \mu)$ that contains the reduced momentum image. Moreover, for a Bochner-Kähler structure with data $\big(p_C, p_D, \mu\big)$, the union of the reduced momentum images of the Bochner-Kähler structures that are analytically connected to it is the entire momentum cell $C(p_D, \mu)$.*

Proof. It has been established that p_C and p_D and the momentum cell $C(p_D, \mu)$ are invariants of the analytically connected equivalence class. Moreover, by Proposition 4.9, every point of $C(p_D, \mu)$ lies in the image of the reduced momentum mapping of some Bochner-Kähler structure.

To prove Theorem 4.10, it will thus suffice to show that any two Bochner-Kähler structures with the same data (p_C, p_D, μ) are analytically connected.

Now, if (M, g, Ω) and $(\tilde{M}, \tilde{g}, \tilde{\Omega})$ are connected Bochner-Kähler manifolds with the same data (p_C, p_D, μ) and their reduced momentum images $h'(M)$ and $h'(\tilde{M})$ have nontrivial intersection, then they contain points $x \in M$ and $\tilde{x} \in \tilde{M}$ so that $f(x) = \tilde{f}(\tilde{x})$ where $f : M \to \mathbb{R}^{2n+1}$ and $\tilde{f} : \tilde{M} \to \mathbb{R}^{2n+1}$ are the corresponding moduli maps. By Theorem 3.1 and Corollary 3.4, the germs of Bochner-Kähler structures around $x \in M$ and $\tilde{x} \in \tilde{M}$ are isomorphic. Since M and $\tilde{M}$ are connected, the germ of the Bochner-Kähler structure around any $y \in M$ is analytically connected to the germ of the Bochner-Kähler structure around any $\tilde{y} \in M$.

Now, from Theorem 3.13, it follows that $h'(M^\circ)$ lies in the interior of $C(p_D, \mu)$ and that $h' : M^\circ \to C(p_D, \mu)^\circ$ is a submersion onto its image, which is therefore open.

The union of the open sets $h'(\tilde{M}^\circ)$ as $(\tilde{M}, \tilde{g}, \tilde{\Omega})$ ranges over the Bochner-Kähler structures that are analytically connected to any given (M, g, Ω) is a connected component of $C(p_D, \mu)^\circ$. Since $C(p_D, \mu)^\circ$ is convex and hence connected, this union must be all of $C(p_D, \mu)^\circ$.

By Proposition 4.9, the union of all the sets $h'(M)$ as (M, g, Ω) ranges over the Bochner-Kähler structures with data (p_C, p_D, μ) is equal to the entire cell $C(p_D, \mu)$. Since $h'(M^\circ)$ is a nonempty subset of $C(p_D, \mu)^\circ$ for any such (M, g, Ω), it follows that all of these are analytically connected, as desired. □

Remark 4.11 (Coarse moduli and polytope embeddings). By Theorem 4.10, the analytically connected equivalence classes in F_n correspond to the data (p_C, p_D, μ) that satisfy the conditions of Proposition 4.9. Note that, for any given $p_C(t)$, there are at most a finite number of choices of (p_D, μ) that will satisfy these constraints.

Thus, each value of $C = (C_2, \dots, C_{n+2})$ corresponds to only a finite number of equivalence classes. It is in this sense that the functions $C_i : F_n \to \mathbb{R}$ furnish the complete set of 'coarse moduli' for Bochner-Kähler structures in dimension n.

Moreover, the mapping $\Delta : F_n \to \mathbb{R}^{2n+1}$ of §3.5.3 embeds each analytically connected equivalence class as a (not necessarily closed) convex polytope of some dimension $m \le n$. In particular, each of these equivalence classes is contractible.

Corollary 4.12. *If (M, g, Ω) is a complete, connected Bochner-Kähler structure, then the reduced momentum mapping $h' : M \to C(p_D, \mu)$ is surjective, the submersion $h' : M^\circ \to C(p_D, \mu)^\circ$ is a fibration, and the fibers of h' in M are connected.*

Proof. Fix $x \in M$. By Theorem 4.10, to prove that h' is surjective it suffices to show that if $(\tilde{M}, \tilde{g}, \tilde{\Omega})$ is any Bochner-Kähler structure containing an $\tilde{x}$ with $f(x) = \tilde{f}(\tilde{x})$, then $\tilde{h}'(\tilde{M})$ is a subset of $h'(M)$.

Consider any $\tilde{y} \in \tilde{M}$ and choose a smooth path $\tilde{c} : [0,1] \to \tilde{M}$ with $\tilde{c}(0) = \tilde{x}$ and $\tilde{c}(1) = \tilde{y}$. Choose a $\tilde{u}_0 \in \tilde{P}_{\tilde{x}}$ and let $\tilde{u} : [0,1] \to \tilde{P}$ be the parallel transport of $\tilde{u}_0$ along $\tilde{c}$. Thus $(\tilde{u})^*(\tilde{\phi}) = 0$ while $(\tilde{u})^*(\tilde{\omega}) = v(s)\,ds$ for some $v : [0,1] \to \mathbb{C}^n$.

Since $f(x) = \tilde{f}(\tilde{x})$ by hypothesis, there exists a $u_0 \in P_x$ so that $H(u_0) = \tilde{H}(\tilde{u}_0)$, $T(u_0) = \tilde{T}(\tilde{u}_0)$, and $V(u_0) = \tilde{V}(\tilde{u}_0)$. Since the metric on M is complete, there will exist a unique curve $u : [0,1] \to P$ satisfying the initial condition $u(0) = u_0$ and the ordinary differential equations

$$u^*(\omega) = v(s)\,ds, \qquad u^*(\phi) = 0,$$

i.e., u is the parallel transport of u_0 along the curve $c = \pi \circ u$ in M. The structure equations (2.14) and the Chain Rule now imply that the two curves defined on $[0,1]$

$$\big(H \circ u, T \circ u, V \circ u\big) \qquad \text{and} \qquad \big(\tilde{H} \circ \tilde{u}, \tilde{T} \circ \tilde{u}, \tilde{V} \circ \tilde{u}\big)$$

in $i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ satisfy the same initial conditions and system of ordinary differential equations, so they are equal on $[0,1]$. Now, setting $s = 1$ yields $\tilde{f}(\tilde{y}) = f\big(c(1)\big) \in h'(M)$. Since $\tilde{y} \in \tilde{M}$ was arbitrary, $\tilde{h}'(\tilde{M})$ is a subset of $h'(M)$.

To prove the second part, suppose first that M is simply-connected. Then any local isometry $\psi : U \to M$ defined on an open subset $U \subset M$ extends uniquely to a global isometry of M.[15] Thus $I(M)$, the global holomorphic isometry group of M, acts transitively on the fibers of $h'(M)$.

Now $(h')^{-1}\big(C(p_D, \mu)^\circ\big) = M^\circ$ and, by the first part of the proof, $h' : M^\circ \to C(p_D, \mu)^\circ$ is surjective. Since $h' : M^\circ \to C(p_D, \mu)^\circ$ is also a submersion whose fibers are $I(M)$-orbits, it follows that $h' : M^\circ \to C(p_D, \mu)^\circ$ is a fibration.

To treat the case where M is not simply-connected, pass to the universal cover $\tilde{M}$ and note that $\tilde{h}'$ is invariant under the deck transformations Δ of the cover $\tilde{M} \to M$ (which form a discrete subgroup of $I(\tilde{M})$). Since $\tilde{h}' : \tilde{M}^\circ \to C(p_D, \mu)^\circ$ is a fibration, dividing by the (free) action of Δ yields that $h' : M^\circ \to C(p_D, \mu)^\circ$ is also a fibration.

To prove the connectedness of the fibers of h' it suffices to treat the case where M is simply-connected, so assume this. Again, $I(M)$ acts transitively on the fibers of h'. Since M° is the complement of a codimension 1 complex subvariety in M, it is connected. The exact sequence of the fibration $h' : M^\circ \to C(p_D, \mu)^\circ$ and the contractibility of $C(p_D, \mu)^\circ$ then imply that the fibers of $h' : M^\circ \to C(p_D, \mu)^\circ$ are

[15] Simply choose $x \in U$ and extend ψ so that it commutes with the exponential map at x, i.e., so that $\psi\big(\exp_x(v)\big) = \exp_{\psi(x)}\big(\psi'(x)(v)\big)$. The completeness and analyticity of the metric and the simple connectivity of M imply that such an extension of ψ to all of M exists and is an isometry, as desired.

connected. By Corollary 3.5, the $I(M)$-stabilizer of any point in M is a product of unitary groups and hence is connected. Since the $I(M)$-orbit of any $x \in M^\circ$ has been shown to be connected, it follows that $I(M)$ must be connected as well. Consequently, all of its orbits in M are connected and these are the fibers of h'. $\square$

4.3. **A Riemannian submersion.** The mapping $h' : M \to C(p_D, \mu) \subset \mathbb{R}^m$ can be used to give more detailed information about the Bochner-Kähler metric.

4.3.1. *The cell metric.* The proof of Corollary 4.12 shows that, at least when M is complete, there is a metric on $C(p_D, \mu)^\circ$ for which $h' : M^\circ \to C(p_D, \mu)^\circ$ is a Riemannian submersion. It turns out that this metric exists even when M is not complete and can be identified explicitly.

Theorem 4.13. *Given a monic polynomial $p_D(t)$ of degree $m+2$ that falls into one of the cases of Proposition 4.6, there exist rational functions $R_D^{ij} = R_D^{ji}$ on $\mathbb{R}^m$ with the property that the quadratic form*

$$R_D = R_D^{ij}(u)\, du_i\, du_j \tag{4.20}$$

restricts to be positive definite on the interior of each possible momentum cell for $p_D(t)$ and moreover, so that for any Bochner-Kähler manifold with reduced momentum polynomial $p_D(t)$, the reduced momentum mapping $h' : M \to \mathbb{R}^m$ is a Riemannian submersion when restricted to M°.

Proof. Recall the bundle $P_2 \subset P_1$ that was introduced in the proof of Proposition 4.3 and the notation introduced there. On P_2, the matrix H is diagonal and $H_{i\bar{\imath}} = \lambda_i$ for $1 \le i \le m$. The structure equation for $dH_{i\bar{\imath}}$ then becomes

$$d\lambda_i = T_i(\omega_i + \overline{\omega_i}). \tag{4.21}$$

By equation (4.2),

$$T_i{}^2 = \frac{p_D(\lambda_i)}{\prod_{j\neq i}(\lambda_i - \lambda_j)},$$

so (4.21) can be written in the form

$$\mathrm{Re}(\omega_i) = \frac{1}{2}\sqrt{\frac{\prod_{j\neq i}(\lambda_i - \lambda_j)}{p_D(\lambda_i)}}\, d\lambda_i\,.$$

In other words,

$$\sum_{i=1}^m \mathrm{Re}(\omega_i)^2 = \frac{1}{4}\sum_{i=1}^m \frac{\prod_{j\neq i}(\lambda_i - \lambda_j)}{p_D(\lambda_i)}\, d\lambda_i{}^2\,.$$

Now suppose that $h'(M)$ lies in $C(p_D, \mu)$ and that

$$\lambda\big(C(p_D, \mu)\big) = I_1 \times I_2 \times \cdots \times I_m \subset \mathbb{R}^m_{\geq}$$

as in §4.2.4. Since $(-1)^{i-1} p_D(y_i) > 0$ for $y_i \in I_i^\circ$, the quadratic form

$$S = \frac{1}{4}\sum_{i=1}^m \frac{\prod_{j\neq i}(y_i - y_j)}{p_D(y_i)}\, dy_i{}^2 \tag{4.22}$$

is positive definite on $I_1^\circ \times I_2^\circ \times \cdots \times I_m^\circ \subset \mathbb{R}^m_{>}$. Since S has rational coefficients, is well defined on $\mathbb{R}^m$ minus the union of the hyperplanes $p_D(y_i) = 0$, and is invariant under permutations of the coordinates y_i, it follows that there are unique rational functions $R_D^{ij} = R_D^{ji}$ on $\mathbb{R}^m$ so that

$$S = R_D^{ij}\big(\sigma(y)\big)\, d\big(\sigma_i(y)\big)\, d\big(\sigma_j(y)\big),$$

i.e., setting $R_D = R_D^{ij}(u)\, du_i\, du_j$, the positive definite quadratic form S defined on $I_1^\circ \times I_2^\circ \times \cdots \times I_m^\circ$ is of the form $\sigma^*(R_D)$. Since $\sigma : I_1^\circ \times I_2^\circ \times \cdots \times I_m^\circ \to C(p_D, \mu)^\circ$ is a diffeomorphism, R_D is positive definite on $C(p_D, \mu)^\circ$. Moreover, the above formula on M° can now be written as

$$\sum_{i=1}^{m} \mathrm{Re}(\omega_i)^2 = (h')^*\big(R_D\big),$$

showing that $h' : M^\circ \to C(p_D, \mu)^\circ$ is a Riemannian submersion when the target is given the Riemannian metric R_D. □

4.3.2. *Explicit formulae.* When $m = 2$, with $u_1 = y_1 + y_2$ and $u_2 = y_1 y_2$, the relation between S and R is expressible in the intermediate form

$$\begin{aligned} 4S = \frac{y_1 - y_2}{p_D(y_1)}\, dy_1{}^2 + \frac{y_2 - y_1}{p_D(y_2)}\, dy_2{}^2 = {}& \frac{\big(y_1{}^2 p_D(y_2) - y_2{}^2 p_D(y_1)\big)}{(y_1 - y_2) p_D(y_1) p_D(y_2)}\, du_1{}^2 \\ &- 2\frac{\big(y_1 p_D(y_2) - y_2 p_D(y_1)\big)}{(y_1 - y_2) p_D(y_1) p_D(y_2)}\, du_1\, du_2 \\ &+ \frac{p_D(y_2) - p_D(y_1)}{(y_1 - y_2) p_D(y_1) p_D(y_2)}\, du_2{}^2. \end{aligned}$$

Each of the coefficients on the right is visibly a symmetric rational function of y_1 and y_2 and so can be written as a rational function of u_1 and u_2.

Notice that the expression $p_D(y_1)p_D(y_2)$ is a common denominator of all these rational expressions. In the case where $p_D(t)$ has all four of its roots real, this can be written in the form

$$p_D(y_1)p_D(y_2) = \prod_{\alpha=0}^{3} (y_1 - r_\alpha)(y_2 - r_\alpha) = \prod_{\alpha=0}^{3} (u_2 - r_\alpha\, u_1 + r_\alpha{}^2),$$

so that the denominator is actually a product of linear functions on $\mathbb{R}^2$, indeed, the very linear functions whose vanishing defines the faces of possible momentum cells.

For any m, in Case 4, in which all of the roots of p_D are real and distinct, this generalizes, leading to an expression for the metric R_D that will turn out to be very useful. As usual, let

$$r_0 > r_1 > \cdots > r_m > r_{m+1}$$

be the real roots of p_D. Since the roots are real and distinct, $(-1)^\alpha p_D'(r_\alpha) > 0$ for $0 \le \alpha \le m{+}1$. Note the identity

$$p_D'(r_\alpha) = \prod_{\beta \ne \alpha} (r_\alpha - r_\beta),$$

as well as the classical identities

$$\sum_{\alpha=0}^{m+1} \frac{{r_\alpha}^k}{p_D'(r_\alpha)} = \begin{cases} \dfrac{(-1)^{m+1}}{r_0 r_1 \cdots r_{m+1}} & \text{when } k = -1; \\ 0 & \text{when } 0 \le k \le m; \\ 1 & \text{when } k = m{+}1; \\ r_0 + \cdots + r_{m+1} & \text{when } k = m{+}2. \end{cases}$$

(The cases $k = -1$ and $k = m{+}2$ will be used in a later section.) Using coordinates $(u_1, \ldots, u_m)$ on $\mathbb{R}^m$ as above, define linear functions[16]

$$l_\alpha = -\frac{(r_\alpha^m - r_\alpha^{m-1}\, u_1 + \cdots + (-1)^m u_m)}{p_D'(r_\alpha)} \tag{4.23}$$

for $0 \le \alpha \le m{+}1$, so that the equations $l_\alpha = 0$ define the hyperplanes that are the faces of the various possible momentum cells for p_D. Note that the above classical identities in the range $0 \le k \le m{+}1$ are equivalent to the equations

$$\sum_{\alpha=0}^{m+1} l_\alpha = 0 \qquad \text{and} \qquad \sum_{\alpha=0}^{m+1} r_\alpha\, l_\alpha = -1, \tag{4.24}$$

which are the only linear relations among the l_α. Note also that

$$\sigma^*(l_\alpha) = \frac{\prod_{i=1}^m (y_i - r_\alpha)}{\prod_{\beta\neq\alpha}(r_\beta - r_\alpha)}.$$

The metric R_D then has the simple expression

$$R_D = \sum_{\alpha=0}^{m+1} \frac{{dl_\alpha}^2}{4\, l_\alpha}. \tag{4.25}$$

Indeed,

$$\begin{aligned} \sigma^*\left(\sum_{\alpha=0}^{m+1} l_\alpha \left(\frac{dl_\alpha}{l_\alpha}\right)^2\right) &= \sum_{\alpha=0}^{m+1} \frac{\prod_{i=1}^m (y_i - r_\alpha)}{\prod_{\beta\neq\alpha}(r_\beta - r_\alpha)} \left(\sum_{j=1}^m \frac{dy_j}{(y_j - r_\alpha)}\right)^2 \\ &= \sum_{j,k=1}^m \left(\sum_{\alpha=0}^{m+1} \frac{\prod_{i=1}^m (y_i - r_\alpha)}{(y_j - r_\alpha)(y_k - r_\alpha)\prod_{\beta\neq\alpha}(r_\beta - r_\alpha)}\right) dy_j\, dy_k \\ &= \sum_{i=1}^m \frac{\prod_{j\neq i}(y_i - y_j)}{p_D(y_i)}\, {dy_i}^2, \end{aligned}$$

where the last equality follows from the classical identities above and the Lagrange interpolation identity.

The expression (4.25) can also be written in Hessian form as

$$R_D = R_D^{ij}\, du_i\, du_j = \frac{\partial^2 G}{\partial u_i\, \partial u_j}\, du_i\, du_j$$

[16]The reader will note that the formula for l_α makes sense in general as long as r_α is a simple root of $p_D(t)$. Accordingly, l_α will be taken to be defined by (4.23) in this more general case.

where the potential function G has the form

$$G = \frac{1}{4}\sum_{\alpha=0}^{m+1} l_\alpha\,\big(\log|l_\alpha| - 1\big),$$

a fact that will be useful below.

The formula for R_D is evidently singular along the hyperplanes $l_\alpha = 0$, but this singularity is mild and can be 'resolved' with little difficulty. For simplicity, and since this case will be useful in the analysis below, we will illustrate this for the 'lowest' cell, i.e., Subcase 4-0. The spectral intervals in this subcase are $I_i = [r_i, r_{i+1}]$ and the functions $l_1, \dots, l_{m+1}$ are all nonnegative on this cell $C(p_D, \mu)$, which is an m-simplex. In fact, (4.24) shows that $l_0 = -(l_1 + \cdots + l_{m+1})$ and that

$$1 = \sum_{\alpha=1}^{m+1} (r_0 - r_\alpha)\, l_\alpha\,, \tag{4.26}$$

so that the functions $(r_0 - r_\alpha)\, l_\alpha$ for $1 \le \alpha \le m{+}1$ can be regarded as homogeneous affine coordinates on this simplex. Now let $E \subset \mathbb{R}^{m+1}$ be the m-dimensional ellipsoid defined by

$$1 = \sum_{\alpha=1}^{m+1} (r_0 - r_\alpha)\, {p_\alpha}^2\,. \tag{4.27}$$

There is then a unique smooth map $s : E \to C(p_D, \mu)$ defined by $s^*(l_\alpha) = p_\alpha^2$ for $1 \le \alpha \le m{+}1$. Since $s^*(l_0) = -({p_1}^2 + \cdots + {p_{m+1}}^2)$, the s-pullback metric is

$$s^*(R_D) = s^*\left(\sum_{\alpha=0}^{m+1} \frac{d{l_\alpha}^2}{4\, l_\alpha}\right) = \sum_{\alpha=1}^{m+1} d{p_\alpha}^2 \;-\; \frac{(p_1\, dp_1 + \cdots + p_{m+1}\, dp_{m+1})^2}{({p_1}^2 + \cdots + {p_{m+1}}^2)}. \tag{4.28}$$

The quadratic form on the right hand side is well defined on $\mathbb{R}^{m+1}$ minus the origin and is positive semidefinite there, with the null space of the quadratic form being spanned by the radial vector at each point.[17] Thus, this quadratic form is positive definite and smooth on E, thereby providing the desired 'resolution' of the singularities of R_D on $C(p_D, \mu)$. Note that the rank of the mapping s at $p = (p_\alpha) \in E$ is equal to one less than the number of nonzero entries p_α. This will be useful below.

The analysis of R_D in Cases 1, 2, and 3 can be derived from the Case 4 analysis by either regarding two of the roots as complex conjugates and combining the corresponding terms in the above sums to obtain real expressions (Case 1) or collecting two or three of the terms and taking the limit as the corresponding roots come together (Cases 2 and 3). This will only be needed in Case 3-1b below, so this case will be done here and the others will be left to the interested reader.

Case 3-1 can be regarded as the limit of Case 4 as the root r_0 approaches r_1 while the roots r_1 through r_{m+1} remain fixed. Thus, the relations (4.24) can be solved for l_0 and l_1 in the form

$$(r_1 - r_0)\, l_0 = 1 - \sum_{\alpha=2}^{m+1} (r_1 - r_\alpha)\, l_\alpha, \qquad (r_0 - r_1)\, l_1 = 1 - \sum_{\alpha=2}^{m+1} (r_0 - r_\alpha)\, l_\alpha.$$

[17] In fact, this metric is just the tangential part $r^2\, d\sigma_m^2$ of the expression for the standard metric in polar coordinates $dr^2 + r^2\, d\sigma_m^2$, where $d\sigma_m^2$ is the standard metric on the m-sphere. A curious consequence of this fact is that the metric R_D is conformally flat.

Using these formulae, one computes the limit

$$\lim_{r_0\to r_1}\left(\frac{{dl_0}^2}{l_0}+\frac{{dl_1}^2}{l_1}\right)=\frac{t\,da^2}{a^2}-\frac{2\,da\,dt}{a}$$

where

$$a=1-\sum_{\alpha=2}^{m+1}(r_1-r_\alpha)\,l_\alpha \qquad\text{and}\qquad t=l_2+l_3+\cdots+l_{m+1}, \tag{4.29}$$

and where the formulae (4.23) for l_α for $2\le\alpha\le m{+}1$ remain valid. Thus, the formula for R_D in Case 3-1 is

$$R_D=\frac{t\,da^2}{4a^2}-\frac{da\,dt}{2a}+\sum_{\alpha=2}^{m+1}\frac{{dl_\alpha}^2}{4l_\alpha}\,. \tag{4.30}$$

In Case 3-1*b*, all of the quantities $a, l_2, \dots, l_{m+1}$ are nonnegative. In fact, the quantity $a>0$ together with the quantities $(r_1-r_\alpha)\,l_\alpha$ for $2\le\alpha\le m{+}1$ can be regarded as affine homogeneous coordinates on the momentum cell $C(p_D,\mu)$.

Set $\rho_\alpha=r_1-r_\alpha>0$ and $\rho=(\rho_2,\dots,\rho_{m+1})$. Let $E_\rho\subset\mathbb{R}^m$ be the ellipsoidal domain defined by

$$\sum_{\alpha=2}^{m+1}\rho_\alpha\,{p_\alpha}^2<1.$$

Define a surjective map $s: E_\rho\to C(p_D,\mu)$ by $s^*l_\alpha=p_\alpha^2$ for $2\le\alpha\le m{+}1$. This map s satisfies

$$\bar a=s^*(a)=1-\sum_{\alpha=2}^{m+1}\rho_\alpha\,{p_\alpha}^2 \qquad\text{and}\qquad \bar t=s^*(t)={p_2}^2+\ldots+{p_{m+1}}^2.$$

Thus $R_\rho=s^*(R_D)$ has the form

$$R_\rho=\frac{\bar t\,d\bar a^2}{4\bar a^2}-\frac{d\bar a\,d\bar t}{2\bar a}+\sum_{\alpha=2}^{m+1}{dp_\alpha}^2\,.$$

Since $\bar t$ vanishes to second order at $p=0$ (the center of E_ρ), the quadratic form R_ρ is visibly positive definite and smooth on a neighborhood of $p=0$. Let $\delta>0$ be less than any $1/\sqrt{\rho_\alpha}$ and consider the annular region $A_\delta\subset E_\rho$ defined by $\bar t\ge\delta^2$. On this region, $\bar a$ and $\bar t$ are both positive and R_ρ can be written in the form

$$R_\rho=\frac{\bar t}{4}\left(\frac{d\bar a}{\bar a}-\frac{d\bar t}{\bar t}\right)^2-\frac{{d\bar t}^2}{4{\bar t}^2}+\sum_{\alpha=2}^{m+1}{dp_\alpha}^2=\frac{\bar t}{4}\left(\frac{d\bar a}{\bar a}-\frac{d\bar t}{\bar t}\right)^2+R^*_\rho$$

where R^*_ρ is defined by this last equality. Since $\bar t=|p|^2$, it follows without difficulty that R^*_ρ is positive semidefinite on $\mathbb{R}^m$ minus the origin (where it is singular) and that its null space at each point is one dimensional and is spanned by the radial vector. Since the 1-form $\rho=d\bar a/\bar a-d\bar t/\bar t$ is evidently nonvanishing on the radial vector field, it follows that R_ρ is positive definite (and smooth) everywhere on E_ρ. Thus, R_ρ on E_ρ provides the desired resolution of the boundary singularities of R_D on the momentum cell $C(p_D,\mu)$.

Moreover, on A_δ, whose outer boundary is defined by $\bar{a} = 0$, the inequality

$$R_\rho \geq \frac{\delta^2}{4}\left(\frac{d\bar{a}}{\bar{a}} - \frac{d\bar{t}}{\bar{t}}\right)^2 = \left(\frac{\delta}{2}\, d\left(\log\left(\frac{\bar{t}}{\bar{a}}\right)\right)\right)^2$$

holds. Since $\log(\bar{t}/\bar{a})$ is proper on A_δ, it follows that R_ρ is complete on E_ρ.

This result will be needed in §4.4.3, when completeness is being discussed. For use in that section, we will point out that the above formulae define a convex domain $E_\rho \subset \mathbb{R}^m$ and a complete metric R_ρ on E_ρ for any $\rho = (\rho_2, \ldots, \rho_{m+1})$ satisfying $\rho_\alpha \geq 0$ for all α. (Recall that the metrics that arise as resolutions of singular metrics on $C(p_D, \mu)$ satisfy $0 < \rho_2 < \cdots < \rho_{m+1}$.)

The metric R_ρ is flat only when $\rho_2 = \cdots = \rho_{m+1} = 0$, in which case $E_0 = \mathbb{R}^m$ and R_0 is the standard flat metric. Moreover, the above formulae show that R_ρ is always conformally flat, with (E_ρ, R_ρ) being globally conformal to (E_0, R_0).

4.3.3. *Necessary conditions for completeness.* It turns out that most of the possible momentum cells cannot be the reduced momentum image of a complete Bochner-Kähler manifold.

Proposition 4.14. *If there is a complete Bochner-Kähler (M, g, Ω) whose reduced momentum mapping has image in $C(p_D, \mu)$, then $C(p_D, \mu)$ is bounded.*

Proof. Suppose that (M, g, Ω) is connected and complete, with characteristic polynomials p_C and p_D but that its reduced momentum mapping takes values in an unbounded momentum cell $C(p_D, \mu)$. Let $I_1 \times \cdots \times I_m$ be the corresponding spectral product. The unboundedness of $C(p_D, \mu)$ implies that I_1 is either $[r, \infty)$ or (r, ∞) where r is the largest real root of p_D.

Again, let $P_2 \subset P_1$ be the bundle over M° constructed in the course of the proof of Proposition 4.3. Because the structure group of P_2 is $\mathrm{I}_m \times G_\Lambda$, it follows that the 1-forms ω_i for $1 \leq i \leq m$ are actually well defined on M°. Let E_1 be the vector field on M° that is g-dual to $\mathrm{Re}(\omega_1)$. Then, by the relation $d\lambda_i = T_i(\omega_i + \overline{\omega_i})$, it follows that $d\lambda_i(E_1) = 0$ for $1 < i \leq m$ and that

$$d\lambda_1(E_1) = 2T_1 = 2\sqrt{\frac{p_D(\lambda_1)}{\prod_{j\neq 1}(\lambda_1 - \lambda_j)}} > 0.$$

In particular, along an integral curve of E_1 the functions λ_j for $1 < j \leq m$ are constant while λ_1 is strictly increasing.

Fix $x \in M^\circ$ and let $a : [0, T) \to M$ be the maximal forward integral curve of E_1 with $a(0) = x$. We claim that T cannot be finite. If it were, the fact that E_1 is a unit speed vector field and that M is complete would imply that $a(t)$ approaches a limit $y \in M$ as t approaches T (after all, $d\big(a(t), a(s)\big) \leq |t - s|$). The limit point y could not lie in M° since then $[0, T)$ would not be maximal. By continuity, $\lambda(y)$ must not lie in $I_1^\circ \times \cdots \times I_m^\circ$. However, $\lambda_i(y) = \lambda_i(x)$ for $1 < i \leq m$ while $\lambda_1(y) > \lambda_1(x)$. Since $I_1^\circ = (r, \infty)$ this forces $\lambda(y)$ to lie in $I_1^\circ \times \cdots \times I_m^\circ$, a contradiction since $h'(M^\circ)$ lies in $C(p_D, \mu)^\circ$. Thus $T = \infty$, as claimed. In particular, the forward flow of E_1 exists for all time on M°.

However, this leads to a contradiction: Along a, the element of arc is given by

$$ds = \frac{1}{2}\sqrt{\frac{\prod_{j=2}^{m}\big(\lambda_1 - \lambda_j(x)\big)}{p_D(\lambda_1)}}\, d\lambda_1\,.$$

Let $\lim_{t\to\infty} \lambda_1\big(a(t)\big) = \lambda_\infty \le \infty$. Since $a : [0,\infty) \to M$ has unit speed, the integral

$$\int_{\lambda_1(x)}^{\lambda_\infty} \sqrt{\frac{\prod_{j=2}^{m}\big(\xi - \lambda_j(x)\big)}{p_D(\xi)}}\, d\xi$$

must be infinite. However, this integral is bounded by

$$\int_{\lambda_1(x)}^{\infty} \sqrt{\frac{\prod_{j=2}^{m}\big(\xi - \lambda_j(x)\big)}{p_D(\xi)}}\, d\xi$$

which converges, since $p_D(t)$ has degree $m{+}2$.

This contradiction implies that (M, g) could not have been complete. □

Remark 4.15 (Bounded momentum cells). The discussion in §4.2.3 shows that there are only two cases in which the momentum cell is bounded:

The first case is SubCase 3-1b, i.e., p_D has r_1 as a double root and $\mu_1 = 0$. The spectral bands are $I_1 = [r_2, r_1)$ and $I_j = [r_{j+1}, r_j]$ for $1 < j \le m$. This cell is bounded but not compact.

The second case is SubCase 4-0, i.e., $p_D(t)$ has $m{+}2$ simple roots $r_0 > \cdots > r_{m+1}$ and the spectral bands are $I_j = [r_{j+1}, r_j]$ for $1 \le j \le m$. This cell is compact. However, as the next proposition shows, Subcase 4-0 never contains a complete example when $m > 0$.

Proposition 4.16. *When $m > 0$, there is no complete Bochner-Kähler manifold whose reduced characteristic polynomial p_D has $m{+}2$ distinct roots.*

Proof. In view of Proposition 4.14 and the remark above, what has to be shown is that SubCase 4-0 cannot occur for a complete Bochner-Kähler manifold when $m > 0$. This will involve an interesting examination of the fixed points of the flow of the canonical torus action.

Thus, suppose, to the contrary, that (M, g, Ω) is a complete Bochner-Kähler structure with $m > 0$ and that

$$p_D(t) = (t - r_0)(t - r_1)\cdots(t - r_{m+1})$$

where $r_0 > \cdots > r_{m+1}$. By Proposition 4.14, the momentum cell $C(p_D, \mu)$ must be bounded, which implies that SubCase 4-0 obtains, namely $(-1)^{i-1} p_{h'}(r_i) \ge 0$ for $1 \le i \le m{+}1$.

For $1 \le \alpha \le m{+}1$, let $F_\alpha \subset C(p_D, \mu)$ be the α-th face of this m-simplex, i.e., the intersection of $C(p_D, \mu)$ with the hyperplane $l_\alpha = 0$ (where the functions l_α are as defined in (4.23)). Let $N_\alpha = (h')^{-1}(F_\alpha)$ be the preimage of F_α. Evidently, each N_α is a closed, analytic subset of M and the union of the N_α is the complex subvariety $N \subset M$. Thus, N_α is a (nonempty) complex subvariety of M.

For $0 \le \alpha \le m{+}1$, define functions $w_\alpha = (h')^*(l_\alpha) \ge 0$ on M and then define vector fields $W_\alpha \in \mathfrak{z}$ by $W_\alpha \lrcorner \Omega = -dw_\alpha$. By (4.24), the W_α satisfy

$$\sum_{\alpha=0}^{m+1} W_\alpha = \sum_{\alpha=0}^{m+1} r_\alpha\, W_\alpha = 0. \tag{4.31}$$

Moreover, any m of these vector fields are linearly independent on M°. Note that since w_α reaches its minimum of 0 along N_α, the vector field W_α vanishes along N_α. Since M is complete, the flows of the vector fields W_α are complete.

We will show that the flow of each vector field W_α is periodic of period π by examining the rotation ∇W_α along the fixed hypersurface N_α.

Now, equation (3.15) can be written as

$$t^n\, d\big(p_h(t^{-1})\big) = -t^n\, p_h(t^{-1})\big(t\,\alpha_1 + t^2\,\alpha_2 + \cdots\big), \tag{4.32}$$

where t is replaced by $-t$ and is regarded as a parameter, taken to be sufficiently small so that the series converges in a neighborhood of any given compact domain in M. Using (3.11), this can be written in the form

$$d\big(p_h(t^{-1})\big) = -t\, p_h(t^{-1}) \sum_{k=0}^{\infty} \big(T^*\,(tH)^k\,\omega + \omega^*\,(tH)^k\,T\big) \tag{4.33}$$

and the series can then be summed, yielding the equation

$$d\big(p_h(t^{-1})\big) = -t\, p_h(t^{-1})\,\big(T^*\,(\mathrm{I}_n - tH)^{-1}\,\omega + \omega^*\,(\mathrm{I}_n - tH)^{-1}\,T\big)\,.$$

Replacing t by t^{-1}, this becomes

$$\begin{aligned} d\big(p_h(t)\big) &= -p_h(t)\,\big(T^*\,(t\,\mathrm{I}_n - H)^{-1}\,\omega + \omega^*\,(t\,\mathrm{I}_n - H)^{-1}\,T\big) \\ &= -\,(T^*\,\mathrm{Cof}(t\,\mathrm{I}_n - H)\,\omega + \omega^*\,\mathrm{Cof}(t\,\mathrm{I}_n - H)\,T)\,. \end{aligned} \tag{4.34}$$

The final expression is valid for all t, while the middle expression is valid away from the locus $p_h(t) = 0$ in $P \times \mathbb{R}$.

Since $p_h(t) = p_{h''}(t)p_{h'}(t)$, and since $p_{h''}(t)$ has constant coefficients, (4.34) implies

$$d\big(p_{h'}(t)\big) = -p_{h'}(t)\,\big(T^*\,(t\,\mathrm{I}_n - H)^{-1}\,\omega + \omega^*\,(t\,\mathrm{I}_n - H)^{-1}\,T\big) \tag{4.35}$$

away from the locus $p_h(t) \neq 0$.

Define a vector field $W(t)$ on M by $W(t) \lrcorner\, \Omega = -d\big(p_{h'}(t)\big)$. This vector field depends polynomially on t and lies in $\mathfrak{z}$ for all t. In fact, comparison with (4.23), the definition of l_α, shows that $p_{h'}(r_\alpha) = -p'_D(r_\alpha)\,w_\alpha$, so it follows that

$$W(r_\alpha) = -p'_D(r_\alpha)\,W_\alpha\,, \qquad 0 \le \alpha \le m{+}1. \tag{4.36}$$

By (4.35), the vector field $W(t)$ has representative function $w(t) : P \to \mathbb{C}^n$ given by

$$w(t) = -2i\, p_{h'}(t)\,(t\,\mathrm{I}_n - H)^{-1}T. \tag{4.37}$$

The expression on the left is polynomial in t, so the expression on the right must be also. Since the flow of $W(t)$ is a holomorphic isometry, it follows that

$$d\big(w(t)\big) + \phi\, w(t) = w'(t)\,\omega$$

where $w'(t)$ takes values in $\mathfrak{u}(n)$. In fact, by (2.14),

$$\begin{aligned} w'(t) = 2i\, p_{h'}(t)\,(t\,\mathrm{I}_n - H)^{-1}\Big[&TT^*(t\,\mathrm{I}_n - H)^{-1} - T^*(t\,\mathrm{I}_n - H)^{-1}T\,\mathrm{I}_n \\ &- H^2 - h_1\,H - V\,I_n\Big] \end{aligned} \tag{4.38}$$

and the matrix on the right is visibly skew-Hermitian. When $T(u) = 0$, formula (4.38) simplifies to the form in which it will be the most useful:

$$w'(t)(u) = -2i\, p_{h'(u)}(t)\,\big(t\,\mathrm{I}_n - H(u)\big)^{-1}\Big[H(u)^2 + h_1(u)\,H(u) + V(u)\,I_n\Big]. \tag{4.39}$$

Now fix β in the range $1 \le \beta \le m{+}1$ and let $k_\beta \in C(p_D,\mu)$ be the vertex that lies on the intersection of the faces F_α for $\alpha \neq 0, \beta$, i.e., k_β is the vertex that lies *opposite* the face F_β. Applying Corollary 4.12, choose $x_\beta \in M$ to satisfy $h'(x_\beta) = k_\beta$ and then let $u_\beta \in P$ satisfy $\pi(u_\beta) = x_\beta$. Then $T(u_\beta) = 0$ since the differential of h' vanishes at x_β. In particular, x_β is a zero of W_α for all α.

Now, r_α is a root of $p_{h'(u_\beta)}(t)$ for all $\alpha \neq 0, \beta$ since $h'(u_\beta)$ lies on each F_α with $\alpha \neq \beta$. Thus, the set $\{\lambda_1(x_\beta), \ldots, \lambda_m(x_\beta)\}$ consists of the r_α where $\alpha \neq 0, \beta$. Consequently, since (4.2) now simplifies to

$$\prod_{\alpha=0}^{m+1}(t-r_\alpha) = p_D(t) = p_{h'(u_\beta)}(t)\,\big(t^2 + h_1(u_\beta)\,t + V(u_\beta)\big)$$
$$= \left(\prod_{\alpha\neq 0,\beta}^{m+1}(t-r_\alpha)\right)\,\big(t^2 + h_1(u_\beta)\,t + V(u_\beta)\big),$$

it follows that $\big(t^2 + h_1(u_\beta)\,t + V(u_\beta)\big) = (t - r_0)(t - r_\beta)$. In particular, (4.39) becomes

$$w'(t)(u_\beta) = -2i\,\left(\prod_{\alpha\neq 0,\beta}(t-r_\alpha)\right)\,\big[H(u_\beta)-r_0\,\mathrm{I}_n\big]\big[H(u_\beta)-r_\beta\,I_n\big]\,\big[t\,\mathrm{I}_n - H(u_\beta)\big]^{-1}.$$

Now, any eigenvalue of $H(u_\beta)$ is a root of $p_{h(u_\beta)}(t) = p_{h''}(t)p_{h'(u_\beta)}(t)$ and so, by Proposition 4.3, must be of the form r_γ for some $\gamma = 0, \ldots, m{+}1$. Let $V_{\beta,\gamma} \subset \mathbb{C}^n$ denote the eigenspace of $H(u_\beta)$ belonging to the eigenvalue r_γ. Then the above formula implies that $w'(t)(u_\beta)$ annihilates $V_{\beta,\beta}$ and $V_{\beta,0}$ and that, for $v \in V_{\beta,\gamma}$ with $\gamma \neq 0, \beta$,

$$w'(t)(u_\beta)\,v = -2i\,(r_\gamma-r_0)(r_\gamma - r_\beta)\left(\prod_{\alpha\neq 0,\beta,\gamma}(t-r_\alpha)\right)\,v. \tag{4.40}$$

Since the right hand side of (4.40) is a polynomial in t, it now makes sense to substitute $t = r_\alpha$ for any α. When $\alpha \neq 0, \beta, \gamma$, this gives $w'(r_\alpha)(u_\beta)\,v = 0$ for $v \in V_{\beta,\gamma}$, while, if $\gamma \neq 0, \beta$, this gives

$$w'(r_\gamma)(u_\beta)\,v = -2i\,(r_\gamma-r_0)(r_\gamma - r_\beta)\left(\prod_{\alpha\neq 0,\beta,\gamma}(r_\gamma-r_\alpha)\right)\,v = -2i\,p_D'(r_\gamma)v.$$

In other words, for $\beta \neq \gamma$ in the range $1 \leq \beta, \gamma \leq m{+}1$,

$$w'(r_\gamma)(u_\beta) = -2i\,p_D'(r_\gamma)\,E_{\beta,\gamma} \tag{4.41}$$

where $E_{\beta,\gamma} : \mathbb{C}^n \to V_{\beta,\gamma}$ is the orthogonal projection onto this eigenspace. Thus, the flow of $W(r_\gamma)$ is periodic of period $\pi/p_D'(r_\gamma)$ and so, by (4.36), the flow of W_γ is periodic of period π, as claimed.

Now, further information can be obtained by evaluating $w'(r_\beta)$ at u_β itself. Indeed, in the above formula, if v lies in $V_{\beta,\gamma}$ with $\gamma \neq 0, \beta$, then putting $t = r_\beta$ gives

$$w'(r_\beta)(u_\beta)(v) = -2i\,(r_\gamma-r_0)(r_\gamma - r_\beta)\left(\prod_{\alpha\neq 0,\beta,\gamma}(r_\beta-r_\alpha)\right)\,v$$
$$= 2i\,\frac{(r_\gamma - r_0)}{(r_\beta - r_0)}\,\left(\prod_{\alpha\neq\beta}(r_\beta-r_\alpha)\right)\,v = 2i\,p_D'(r_\beta)\,\frac{(r_\gamma - r_0)}{(r_\beta - r_0)}\,v.$$

In other words, using the projection notation already introduced,

$$w'(r_\beta)(u_\beta) = 2i\, p'_D(r_\beta) \sum_{\gamma\neq 0,\beta} \frac{(r_\gamma - r_0)}{(r_\beta - r_0)}\, E_{\beta,\gamma}.$$

Since the flow of W_β has period π, each of the ratios $(r_\gamma - r_0)/(r_\beta - r_0)$ must be an integer for $1 \leq \beta \neq \gamma \leq m+1$. Since $r_\beta \neq r_\gamma$ when $\beta \neq \gamma$, these ratios cannot be $+1$. Thus, as the inverse of each such ratio is another such ratio, these integers must all be -1. However, this is equivalent to $\frac{1}{2}(r_\beta + r_\gamma) = r_0$, which is impossible, since r_0 is greater than either r_β or r_γ. This contradiction establishes the proposition. □

Proposition 4.16 has an important corollary, which was first proved by Kamishima [14].

Corollary 4.17. *The only connected compact Bochner-Kähler manifolds are the compact quotients of the symmetric Bochner-Kähler manifolds* $M^p_c \times M^{n-p}_{-c}$.

Proof. A compact Bochner-Kähler manifold is necessarily complete and its reduced momentum image is necessarily compact. Proposition 4.16, Corollary 4.12, and the fact that only SubCase 4-0 has a compact momentum cell imply that $m > 0$ is impossible. When $m = 0$, the momentum mapping is constant and the metric is therefore locally homogeneous, so that, by Proposition 2.5, its simply-connected cover (which is complete) must be isometric to $M^p_c \times M^{n-p}_{-c}$, as claimed. □

Remark 4.18 (Orbifolds). While Proposition 4.16 rules out complete Bochner-Kähler manifolds in Subcase 4-0, it does not rule out the existence of complete Bochner-Kähler orbifolds. In fact, the argument in Proposition 4.16 implies that, if there is a complete orbifold with reduced characteristic polynomial $p_D(t)$ as in the proof, then the ratios $(r_\gamma - r_0)/(r_\beta - r_0)$ must all be rational for $1 \leq \beta, \gamma \leq m+1$. A little algebra then leads to the formulae

$$\begin{aligned} p_D(t) &= (t - r_0)\,(t - r_1)\,\cdots\,(t - r_{m+1}),\\ p_C(t) &= (t - r_0)^{\nu_0+1}\,(t - r_1)^{\nu_1+1}\,\cdots\,(t - r_{m+1})^{\nu_{m+1}+1} \end{aligned}$$

with

$$r_\beta = r \sum_{\alpha=0}^{m+1} (\nu_\alpha + 1)(p_\alpha - p_\beta), \qquad 0 \leq \beta \leq m+1,$$

where $r > 0$ is real, $0 = p_0 < p_1 < p_2 < \cdots < p_{m+1}$ is a strictly increasing sequence of integers with no common divisor, and $\nu_0, \ldots, \nu_{m+1}$ are nonnegative integers satisfying

$$n = m + \nu_0 + \cdots + \nu_{m+1}.$$

While we have not done all of the necessary calculations, it appears that, for each choice of r, $p = (p_1, \ldots, p_{m+1})$, and $\nu = (\nu_0, \ldots, \nu_{m+1})$ satisfying the above conditions, there exists a complete orbifold with characteristic polynomials p_C and p_D as above that fits into SubCase 4-0. The case $n = 1$ has already been verified in §3.2.5, and the cases with $n = m$ will be verified in §4.4.6.

The parameter r can be normalized to 1 by scaling the metric. Thus, up to scaling, there exists a countable family of complete Bochner-Kähler orbifolds in each dimension whose momentum cells are compact. It appears that these orbifolds are weighted projective space in most cases. In fact, it will be seen that every weighted projective space carries a Bochner-Kähler metric.

By the same methods as employed in the proof of Proposition 4.16, one can prove the following more general periodicity result. Details will be left to the reader.

Proposition 4.19. *Let r_α be a simple root of p_D, let $w_\alpha = (h')^*(l_\alpha)$, and let W_α be the vector field in $\mathfrak{z}$ defined by $W_\alpha \lrcorner \Omega = -dw_\alpha$. Then on a neighborhood of any zero of W_α, the flow of W_α is periodic, with period π.*

Remark 4.20 (Locality). One must restrict to a neighborhood of a fixed point for the conclusion of Proposition 4.19. In the first place, without some completeness assumptions, there is no reason to believe that the flow of W_α is even defined for all time except near a fixed point. In the second place, even if the flow is defined for all time, by removing the zero locus of W_α and passing to a covering space, one could conceivably arrange that W_α have no closed orbits at all.

4.4. **A geodesic foliation.** By Theorem 3.11, the vector fields $Z_2, \dots, Z_{m+1}$ are linearly independent (over $\mathbb{C}$) on M° and satisfy $[Z_i, Z_j] = 0$ for $2 \le i, j \le m{+}1$. Moreover, since these vector fields are the real parts of holomorphic vector fields, they satisfy

$$[Z_i, Z_j] = [Z_i, JZ_j] = [JZ_i, JZ_j] = 0.$$

Since the $2m$ vector fields $Z_2, \dots, Z_{m+1}, JZ_2, \dots, JZ_{m+1}$ Lie-commute and are linearly independent on M°, they are tangent to a foliation $\mathcal{F}$ of M° whose leaves are complex submanifolds of M (of complex dimension m). Moreover, the vector fields $JZ_2, \dots, JZ_{m+1}$ are tangent to a foliation $\mathcal{E}$ of M° whose tangent spaces are the orthogonal complements to fibers of $h' : M^\circ \to C(p_D, \mu)^\circ$.

4.4.1. *Geometry of the leaves.* It turns out that the $\mathcal{F}$-leaves are themselves rather interesting objects.

Proposition 4.21. *The leaves of the foliation $\mathcal{F}$ are totally geodesic in M° and the induced Kähler structure on each $\mathcal{F}$-leaf is Bochner-Kähler of cohomogeneity m. The characteristic and momentum polynomials of any $\mathcal{F}$-leaf are*

$$p^L_C(t) = p^L_D(t) = p_D(t-\lambda) \qquad \text{and} \qquad p^L_h(t) = p^L_{h'}(t) = p_{h'}(t-\lambda)$$

where the constant λ is defined so that $p_{h''}(t) = t^{n-m} - (m{+}2)\lambda\, t^{n-m-1} + \cdots$.

Proof. Return to the structure equations on P_2 that were introduced in the proof of Proposition 4.3. Since $(-1)^{i-1}p_D(\lambda_i) > 0$ for $1 \le i \le m$ while $p_D(\lambda_a) = 0$ for $a > m$, it follows that $\lambda_i - \lambda_a$ is nonvanishing on M° and hence on P_2. Equation (4.4) can thus be written as

$$\phi_{a\bar{\imath}} = \frac{T_i}{\lambda_i - \lambda_a}\,\omega_a\,.$$

Let $L^\circ \subset M^\circ$ be an $\mathcal{F}$-leaf, and let $P^L_2 \subset P_2$ be the bundle $\pi^{-1}(L^\circ) \cap P_2$, which is a G_Λ-bundle over L°. By the definition of the bundle P_2 and the foliation $\mathcal{F}$, the forms ω_a vanish when pulled back to P^L_2, so, by the above equations, so do the forms $\phi_{a\bar{\imath}}$. Consequently, the complex m-manifold L° is totally geodesic in M°, as claimed.

Denoting pullback to P^L_2 by a superscript L, the formulae

$$\omega^L = \begin{pmatrix}\tilde\omega\\ 0\end{pmatrix}, \qquad \phi^L = \begin{pmatrix}\tilde\phi & 0\\ 0 & \phi''\end{pmatrix}, \qquad H^L = \begin{pmatrix}H' & 0\\ 0 & \Lambda\end{pmatrix}, \qquad T^L = \begin{pmatrix}T'\\ 0\end{pmatrix}$$

hold, where $\tilde\omega$ takes values in $\mathbb{C}^m$ and $\tilde\phi$ takes values in $\mathfrak{u}(m)$. The notations H', T', and Λ are as previously established. The Kähler form Υ induced on L° by pullback from Ω then satisfies $\pi^*(\Upsilon) = -\frac{i}{2}\,\tilde\omega^*{\wedge}\tilde\omega$.

The pullbacks of the structure equations to P_2^L then imply $d\tilde\omega = -\tilde\phi{\wedge}\tilde\omega$, so that $\tilde\phi$ is the connection matrix of the torsion-free Kähler connection of the induced Kähler structure. The pullbacks further imply

$$\begin{aligned} d\tilde\phi + \tilde\phi\wedge\tilde\phi &= H'\,\tilde\omega^*\wedge\tilde\omega - H'\,\tilde\omega\wedge\tilde\omega^* - \tilde\omega\wedge\tilde\omega^*\,H' + \tilde\omega^*\,H'\,\tilde\omega\ \mathrm{I}_m \\ &\qquad + (\operatorname{tr} H' + \operatorname{tr}\Lambda)\left(\tilde\omega^*\wedge\tilde\omega\ \mathrm{I}_m - \tilde\omega\wedge\tilde\omega^*\right) \\ &= \tilde H\,\tilde\omega^*\wedge\tilde\omega - \tilde H\,\tilde\omega\wedge\tilde\omega^* - \tilde\omega\wedge\tilde\omega^*\,\tilde H + \tilde\omega^*\,\tilde H\,\tilde\omega\ \mathrm{I}_m \\ &\qquad + (\operatorname{tr}\tilde H)\left(\tilde\omega^*\wedge\tilde\omega\ \mathrm{I}_m - \tilde\omega\wedge\tilde\omega^*\right) \end{aligned}$$

where $\tilde H = H' + \lambda\,\mathrm{I}_m$ and $(m{+}2)\,\lambda = \operatorname{tr}\Lambda$. Since $p_{h''}(t) = \det\big(tI_{n-m}{-}\Lambda\big)$, this defines λ as in the statement of the proposition.

Thus, by definition, the induced metric on L° is Bochner-Kähler and has momentum polynomial

$$p_h^L(t) = \det\big(t\,I_m - \tilde H\big) = \det\big((t-\lambda)\,I_m - H'\big) = p_{h'}(t-\lambda),$$

as claimed. Moreover, the pullback of the dH equation implies

$$d\tilde H = -\tilde\phi\,\tilde H + \tilde H\,\tilde\phi + T'\,\tilde\omega^* + \tilde\omega\,(T')^*$$

so that $\tilde T = T'$ is the $\mathbb{C}^m$-valued function defined by the structure equations for Υ.

Since the entries of $\tilde T = T'$ are all nonzero and the eigenvalues of $\tilde H$ are distinct, the rank of the momentum mapping for L° is m, implying that $p_{h'}^L(t) = p_h^L(t)$.

Finally, the pullback of the identity for dT becomes

$$d\tilde T = -\tilde\phi\,\tilde T + \big((\tilde H)^2 + \operatorname{tr}(\tilde H)\,\tilde H + (V{-}\lambda\ \operatorname{tr}(H'){-}(m{+}1)\lambda^2)\big)\tilde\omega,$$

so that, setting $\tilde V = V{-}\lambda\ \operatorname{tr}(H'){-}(m{+}1)\lambda^2$, the structure function for the metric on L° takes the form $(\tilde H, \tilde T, \tilde V)$.

The formula for $p_C^L(t)$ then becomes

$$\begin{aligned} p_C^L(t) &= \det(t\,I_m - \tilde H)\big(t^2 + \operatorname{tr}(\tilde H)\,t + \tilde V\big) + (\tilde T)^*\operatorname{Cof}(tI_m - \tilde H)\,\tilde T \\ &= \det((t{-}\lambda)\,I_m - H')\big((t{-}\lambda)^2 + \operatorname{tr}(H)\,(t{-}\lambda) + V\big) \\ &\qquad + (\tilde T)^*\operatorname{Cof}\big((t{-}\lambda)I_m - H'\big)\,\tilde T \\ &= \frac{p_C(t{-}\lambda)}{p_{h''}(t{-}\lambda)} = p_D(t{-}\lambda) \end{aligned}$$

where the last line uses the definition of $p_C(t)$, the identity $p_h(t) = p_{h'}(t)p_{h''}(t)$, and the identity

$$\operatorname{Cof}(tI_n - H) = \begin{pmatrix} p_{h''}(t)\operatorname{Cof}(tI_m{-}H') & 0 \\ 0 & p_{h'}(t)\operatorname{Cof}(tI_{n-m}{-}\Lambda) \end{pmatrix}.$$

These formulae establish the proposition. □

Corollary 4.22. *The momentum mapping $h^L : L^\circ \to \mathbb{R}^m$ is equal to the restriction of $h' : M^\circ \to \mathbb{R}^m$ to L° followed by an invertible linear map $\Phi_\lambda : \mathbb{R}^m \to \mathbb{R}^m$. The corresponding momentum cells satisfy $C(p_D^L, \mu^L) = \Phi_\lambda\big(C(p_D,\mu)\big)$.*

4.4.2. *Completion and real slices.* It will now be shown that the $\mathcal{F}$- and $\mathcal{E}$-leaves can be extended through the locus N where the Z_k become dependent.

Proposition 4.23. *If the metric on M is complete, then the closure of any $\mathcal{F}$-leaf L° is a complete, totally geodesic complex m-manifold $L \subset M$. Moreover, the geodesic completion of any $\mathcal{E}$-leaf $R^\circ \subset L^\circ$ is a totally geodesic real m-manifold R and the mapping $h' : R \to C(p_D, \mu)$ is surjective.*

Proof. Before beginning the proof, it will be useful to establish the following fact. If g is any real-analytic, complete metric on a manifold M and $S \subset M$ is a connected, totally geodesic submanifold of some dimension s, then S can be 'completed': There exists an s-manifold $\bar{S}$ and a totally geodesic immersion $\iota : \bar{S} \to M$ whose image contains S and for which the induced metric $\bar{g} = \iota^* g$ on $\bar{S}$ is complete. This completion $(\bar{S}, \iota)$ is unique up to diffeomorphism.

Here is a sketch of the proof: Fix $x \in S$ and consider, for every $v \in T_xS$, the constant speed geodesic $\gamma_v : \mathbb{R} \to M$ defined by $\gamma_v(t) = \exp_x(tv)$. Let $E(v) \subset T_{\gamma_v(1)}M$ be the parallel translation of $E(0) = T_xS$ along γ_v from $t = 0$ to $t = 1$, and let $\bar{S} \subset \mathrm{Gr}_s(TM)$ be the set of all such $E(v)$. Since S is totally geodesic, when $|v|$ is sufficiently small $E(v)$ is equal to $T_{\gamma_v(1)}S$. It follows, by the real-analyticity of g, that $\exp_{\gamma_v(1)} : E(v) \to M$ embeds $B_\delta(0) \subset E(v)$ into M as a totally geodesic submanifold of M as long as $\delta > 0$ is less than the injectivity radius at $\gamma_v(1)$. From this, it is not hard to prove that $\bar{S}$ is an embedded submanifold of $\mathrm{Gr}_s(TM)$. Moreover, the basepoint projection $\iota : \bar{S} \to M$ defined by $\iota\big(E(v)\big) = \gamma_v(1)$ is a totally geodesic immersion. Since each of the geodesics γ_v lifts to a complete geodesic $t \mapsto E(tv)$ in $\bar{S}$ for the induced metric $\bar{g} = \iota^* g$, all of the $\bar{g}$-geodesics in $\bar{S}$ passing through $E(0)$ are complete. Thus, $\bar{g}$ is complete. The completeness of the metric then ensures that $\iota(\bar{S})$ contains S, as desired.

Now apply this result to the leaf $L^\circ \subset M^\circ$ and consider $\overline{L^\circ} \subset \mathrm{Gr}_{2m}(TM)$. Since the induced metric on $\overline{L^\circ}$ is real-analytic and is Bochner-Kähler on an open set, it is Bochner-Kähler everywhere. Moreover, by Proposition 4.21, it has cohomogeneity m, equal to its complex dimension. Let $h^L : \overline{L^\circ} \to \mathbb{R}^m$ denote its momentum mapping. By Corollary 4.22 and real-analyticity, $h^L = \Phi_\lambda \circ h' \circ \iota$, since this holds on L°. Since the rank of $\mathfrak{z}$ is m, the proof of Theorem 3.11 coupled with the remarks of §3.3.4 show that $\mathfrak{z}$ accounts for all of the infinitesimal symmetries of $\overline{L^\circ}$, i.e., that the full symmetry group of $\overline{L^\circ}$ is generated by the canonical torus action, even if one were to pass to its simply-connected cover. In particular, by Corollary 4.12, the fibers of h^L are connected and are the orbits of the canonical torus action.

Now, h^L is a submersion outside some closed complex submanifold $K \subset \overline{L^\circ}$. Since h' is a submersion only when it is restricted to M°, it follows that $\iota\big(\overline{L^\circ} \setminus K\big)$ must lie in M°. Since $\overline{L^\circ} \setminus K$ is connected, and since it contains the $\mathcal{F}$-leaf L°, it follows from analyticity that $\iota(\overline{L^\circ} \setminus K)$ must be equal to L°. Thus, $\overline{L^\circ} \setminus K$ consists of the tangent planes to L°. It follows that ι is one-to-one on $\overline{L^\circ} \setminus K$.

If ι were not one-to-one on K, this would violate the connectedness of the fibers of h^L, so ι is one-to-one everywhere. In other words, $L = \iota(\overline{L^\circ})$ is a submanifold of M, as claimed. Obviously, L is the closure of L° in M.

Now, turning to the geometry of the leaves of the foliation $\mathcal{E}$, note that these leaves are defined by the equations $\omega_a = \mathrm{Im}(\omega_i) = 0$ (since H and T are real on P_2). To prove that these leaves are totally geodesic, it would suffice to show that the imaginary part of ϕ' vanishes when one restricts to such a leaf. Thus, write $\tilde{\omega} = \xi + i\,\eta$ and $\phi' = \theta + i\,\psi$, where ξ, η, θ, and ψ take values in $\mathbb{R}^m$, $\mathbb{R}^m$,

$\mathfrak{so}(m)$, and the space of real symmetric matrices, respectively. Since H and T are real-valued, the imaginary part of the equation for dH' becomes

$$0 = -H'\,\psi - \psi\,H' - T'\,{}^t\eta + \eta\,{}^tT'.$$

It then follows by linear algebra that on the open set $U \subset M^\circ$ where H' has no two eigenvalues that sum to zero, the components of ψ are linear combinations of the components of η. By the structure equation for dH', the eigenvalues of H' are independent on each leaf of $\mathcal{E}$ since $d\lambda_i = 2T_i, \xi$. Thus, the open set U intersects each $\mathcal{E}$-leaf in a dense open set. Consequently, the components of ψ vanish on each $\mathcal{E}$-leaf, implying that each $\mathcal{E}$-leaf is totally geodesic, as desired.

Now, let R° be an $\mathcal{E}$-leaf and let $\bar{R} \subset \mathrm{Gr}_m(TM)$ be its geodesic completion. By construction, R° meets each isometry orbit in M° orthogonally. Thus, by Theorem 4.13, the map $h' : R^\circ \to C(p_D, \mu)^\circ$ is an isometry when R° is given the induced metric. It remains to show that $h' \circ \iota : \bar{R} \to C(p_D, \mu)$ is surjective. The completeness and real-analyticity of the induced metric $\bar{g}$ on $\bar{R}$ coupled with the analysis of the resolution of the singular cell metric in SubCase 3-1b done at the end of §4.3.2 shows that $(\bar{R}, \bar{g})$ must be an isometric quotient of (E_ρ, R_ρ) for some ρ. The completeness of this mapping and the surjectivity of this resolution imply the desired surjectivity. □

4.4.3. *The leaf metric.* The Bochner-Kähler metric induced on a leaf L° can now be described rather explicitly in terms of the geometry of the momentum cell associated to M.

Theorem 4.24. *Let $\big(R^D_{ij}(u)\big)$ be the inverse of the coefficient matrix $\big(R^{ij}_D(u)\big)$ of the cell metric R_D on $C(p_D, \mu)^\circ$. Then, on the universal cover of L°, there exist functions $\theta^1, \dots, \theta^m$ for which the induced Kähler form and metric are*

$$\Upsilon = dh'_k \wedge d\theta^k \qquad \text{and} \qquad ds^2 = R^{jk}_D(h')\, dh'_j \circ dh'_k + R^D_{jk}(h')\, d\theta^j \circ d\theta^k.$$

Proof. First, it will be useful to take a different basis for $\mathfrak{z}$. Recall that the functions $(h'_1, \dots, h'_m)$ are constant linear combinations of the functions $(h_1, \dots, h_m)$ and vice versa. By equation (3.19), $Z_{k+1} \lrcorner\, \Omega = -dh_k$ for $1 \le k \le m$, so the vector fields $Z'_2, \dots, Z'_{m+1}$ defined by $Z'_{k+1} \lrcorner\, \Omega = -dh'_k$ for $1 \le k \le m$ are also a basis of $\mathfrak{z}$.

Let L° be an $\mathcal{F}$-leaf. The vector fields $\{Z'_k - iJZ'_k \mid 2 \le k \le m{+}1\}$ are a basis for the holomorphic vector fields on L°, so there are unique holomorphic 1-forms $\zeta^1, \dots, \zeta^m$ on L° that satisfy

$$\zeta^j\big(Z'_{k+1} - iJZ'_{k+1}\big) = \imath\,\delta^j_k$$

for $1 \le j, k \le m$. (The introduction of the factor of $\imath$ simplifies formulae to appear below.) Because the vector fields $Z'_k - iJZ'_k$ are Lie-commuting, the 1-forms ζ^j are closed.

Write $\zeta^j = \xi^j + \imath\,\eta^j$ where ξ^j and η^j are real 1-forms. Then the defining equation above is equivalent to

$$\eta^j(Z'_{k+1}) = -\xi^j(JZ'_{k+1}) = \tfrac{1}{2}\,\delta^j_k \quad \text{and} \quad \xi^j(Z'_{k+1}) = \eta^j(JZ'_{k+1}) = 0.$$

Since the ζ^j are a basis for the holomorphic 1-forms on L°, the metric on L° can be written in the form

$$ds^2 = g_{jk}\,\zeta^j \circ \overline{\zeta^k}$$

where $g_{jk} = \overline{g_{kj}}$ and where the pullback of Ω to L° is

$$\Upsilon = \tfrac{i}{2}\, g_{jk}\, \zeta^j \wedge \overline{\zeta^k}.$$

Now,

$$dh'_k = -Z'_{k+1} \lrcorner \Upsilon = \tfrac{1}{2}\big(g_{kj}\, \overline{\zeta^j} + g_{jk}\, \zeta^j\big),$$

or, equivalently,

$$dh'_k = \tfrac{1}{2}(g_{kj} + g_{jk})\, \xi^j + \tfrac{i}{2}(g_{kj} - g_{jk})\, \eta^j.$$

Since Z'_{j+1} for $1 \le j \le m$ is tangent to the fibers of h', the coefficient of η^j in the above equation must vanish, i.e., $g_{kj} = g_{jk} = \overline{g_{kj}}$. Thus,

$$dh'_k = g_{kj}\, \xi^j.$$

Define $g^{ij} = g^{ji}$ so that $g^{ij} g_{jk} = \delta^i_k$. Note, in particular, that $\xi^j = g^{jk}\, dh'_k$. The metric on L° can now be written in the form

$$\begin{aligned} ds^2 &= g_{jk}\ \zeta^j {\circ} \overline{\zeta^k} = g_{jk}\,(\xi^j + i\,\eta^j){\circ}(\xi^k - i\,\eta^k) \\ &= g_{jk}\big(\xi^j{\circ}\xi^k + \eta^j{\circ}\eta^k\big) \\ &= g^{jk}\, dh'_j{\circ}dh'_k + g_{jk}\, \eta^j{\circ}\eta^k. \end{aligned}$$

Since L° is totally geodesic in M°, it follows from Theorem 4.13 that

$$g^{ij}\, dh'_i\, dh'_j = (h')^*(R_D) = R_D^{ij}(h')\, dh'_i\, dh'_j\,.$$

Thus, $g^{ij} = R_D^{ij}(h')$ and so

$$\xi^j = g^{jk}\, dh'_k = R_D^{jk}(h')\, dh'_k = (h')^*\big(R_D^{jk}(u)\, du_k\big).$$

Now lift everything to the universal cover of L°. Since the η^k are closed, there exist functions $\theta^1, \ldots, \theta^m$ on this universal cover so that $\eta^k = d\theta^k$. Then

$$ds^2 = R_D^{jk}(h')\, dh'_j{\circ}dh'_k + R^D_{jk}(h')\, d\theta^j{\circ}d\theta^k$$

where (R^D_{jk}) is the inverse matrix to (R_D^{jk}) and

$$\Upsilon = R^D_{jk}(h')\, \xi^j \wedge d\theta^k = dh'_k \wedge d\theta^k.$$

These are the desired formulae. □

Remark 4.25 (Kähler metrics of Hessian type). The reader may find the metric in the above form to be very familiar. In fact, Kähler metrics of this form are well known in the literature as being of *Hessian type*. Their general form is as follows: Let $D \subset \mathbb{R}^m$ be an open domain in $\mathbb{R}^m$, assumed to be simply-connected for simplicity. Let $x_1, \ldots, x_m$ be any linear coordinates on $\mathbb{R}^m$ and suppose that g is a Riemannian metric on D, written in the form

$$g = g^{jk}(x)\, dx_j \circ dx_k\,.$$

Using the flat affine structure on $\mathbb{R}^m$ restricted to D, one gets a canonical metric on T^*D as follows: Let $y^1, \ldots, y^m$ be the coordinates that are linear on the fibers of $T^*D \to D$ and dual to the coordinates $x_1, \ldots, x_m$ in the sense that the tautological 1-form on T^*D is $y^j\, dx_j$. Let $g_{jk} = g_{kj}$ be the functions on D so that $g^{jk} g_{kl} = \delta^j_l$ and define the metric

$$\hat{g} = g^{jk}(x)\, dx_j \circ dx_k + g_{jk}(x)\, dy^j \circ dy^k.$$

This metric on T^*D does not depend on the choice of coordinates x_i, but only on g and the flat affine structure that D inherits from $\mathbb{R}^m$. Moreover, this metric is compatible with the symplectic form on T^*D given by $\Upsilon = -d\big(y^j\,dx_j\big) = dx_j{\wedge}dy^j$.

Thus, the metric $\hat{g}$ and 2-form Υ define an almost complex structure on T^*D for which the 1-forms

$$\zeta^j = g^{jk}(x)\,dx_k + \imath\,dy^j$$

give a basis for the $(1,0)$-forms. This almost complex structure will be integrable if and only if the forms ζ^j are closed. In other words, the pair $(\hat{g},\Upsilon)$ defines a Kähler metric on T^*D if and only if $d\big(g^{jk}(x)\,dx_k\big) = 0$ for all $1 \le j \le m$. Since D is simply-connected, this closure condition holds if and only if there exists a convex 'potential' function $G : D \to \mathbb{R}$ for which

$$g^{jk} = \frac{\partial^2 G}{\partial x_j\,\partial x_k},$$

i.e., if and only if the metric g is of Hessian type. For this reason, metrics of the form $\hat{g}$ as above are often called Kähler metrics of Hessian type. Note that, for such a metric, translation in the y-variables defines a Hamiltonian torus action that is holomorphic and whose momentum mapping is the basepoint projection $T^*D \to D$.

For further investigation of these metrics, the reader might consult [12] and [1].

In the case of Bochner-Kähler metrics, the formula for the potential function $G : C(p_D,\mu)^\circ \to \mathbb{R}$ has been indicated in §4.3.2. The main problem with this representation is that it only describes the leaf metric away from the singular locus N. More work must now be done to analyze the metric near this locus.

4.4.4. *A partial completion.* By Theorem 4.24, the 2-form and Riemannian metric on $C(p_D,\mu)^\circ \times \mathbb{R}^m$ defined by

$$\Upsilon = du_k \wedge d\theta^k \qquad \text{and} \qquad ds^2 = R_D^{jk}(u)\,du_j{\circ}du_k + R^D_{jk}(u)\,d\theta^j{\circ}d\theta^k$$

define a Bochner-Kähler structure on $C(p_D,\mu)^\circ \times \mathbb{R}^m$. The simply-connected cover of any $\mathcal{F}$-leaf L° has an immersion into $C(p_D,\mu)^\circ \times \mathbb{R}^m$, canonical up to a translation in the θ-coordinates, that pulls this Bochner-Kähler structure back to the induced one on L°. In this sense, this Bochner-Kähler structure on $C(p_D,\mu)^\circ \times \mathbb{R}^m$ is universal for Bochner-Kähler metrics associated to this reduced momentum cell. Under this immersion, which is a local diffeomorphism, the vector field Z'_{k+1} is carried into $\partial/\partial\theta^k$.

Suppose now that r_α is a simple root of $p_D(t)$ such that $l_\alpha = 0$ defines a face of $C(p_D,\mu)$. Then the vector field W_α is defined (Proposition 4.19) and has the expansion

$$W_\alpha = \frac{1}{p_D'(r_\alpha)}\big({r_\alpha}^{m-1} Z'_2 - {r_\alpha}^{m-2} Z'_3 + \cdots + (-1)^{m-1} Z'_{m+1}\big).$$

It follows that, under the canonical immersion, W_α is carried over to the vector field

$$\Theta_\alpha = \frac{1}{p_D'(r_\alpha)}\left({r_\alpha}^{m-1}\frac{\partial}{\partial\theta^1} - {r_\alpha}^{m-2}\frac{\partial}{\partial\theta^2} + \cdots + (-1)^{m-1}\frac{\partial}{\partial\theta^m}\right).$$

Suppose now that M contains points that satisfy $w_\alpha = 0$, i.e., the image of the reduced momentum mapping contains points that lie on the face $l_\alpha = 0$. Then by

Proposition 4.19, near such points the flow of the vector field W_α is periodic with period π. This suggests considering the vector

$$\tau_\alpha = \frac{\pi}{p_D'(r_\alpha)}\bigl((-r_\alpha)^{m-1}, (-r_\alpha)^{m-2}, \dots, 1\bigr) \in \mathbb{R}^m.$$

The above Bochner-Kähler structure is well defined on $C(p_D,\mu)^\circ \times \bigl(\mathbb{R}^m/(\mathbb{Z}\,\tau_\alpha)\bigr)$. It is not hard to see that there exists a simply-connected complex m-manifold X_α endowed with a Bochner-Kähler structure and a totally geodesic hypersurface $Y_\alpha \subset X_\alpha$ so that, first, the Bochner-Kähler structure on $X_\alpha \setminus Y_\alpha$ is isomorphic to the above Bochner-Kähler structure on $C(p_D,\mu)^\circ \times \bigl(\mathbb{R}^m/(\mathbb{Z}\,\tau_\alpha)\bigr)$ and, second, the image of the momentum mapping on X_α is equal to $C(p_D,\mu)^\circ$ union the interior of the face $l_\alpha = 0$.

The argument for this 'face-wise' extension is based on Theorem 4.10, which shows that, for any point $v \in C(p_D,\mu)$, there must exist *some* Bochner-Kähler metric in the given analytically connected equivalence class whose reduced momentum mapping assumes the value v. Taking v to lie in the interior of the face $l_\alpha = 0$ and applying uniqueness, one sees that the extension must exist locally. A simple patching argument then allows one to produce the extension X_α. Details are left to the reader, but see the next section, where the extension is computed explicitly in a couple of cases of interest.

Remark 4.26 (Guillemin's completion). The reader should also compare Guillemin's description [12] of a Kähler metric constructed from the data of a polytope, since the issue of completion across the facets is much the same. However, one big difference in the present case is that the polytopes involved here are not necessarily closed. Another is that they do not generally satisfy the rationality requirements for the global existence theorems that Guillemin is able to cite. Instead, in the present case the singular loci corresponding to the faces are 'filled in' with 'patches' whose existence stems from Theorem 4.10.

This construction generalizes: If $A = \{\alpha_1, \dots, \alpha_k\}$ is a set of $k \le m$ distinct simple roots of $p_D(t)$ for which each hyperplane $l_{\alpha_j} = 0$ defines a face of $C(p_D,\mu)$, then the vectors v_{α_j} as defined above generate a discrete subgroup $\Lambda_A \subset \mathbb{R}^m$ and the Bochner-Kähler structure descends to $C(p_D,\mu)^\circ \times \bigl(\mathbb{R}^m/\Lambda_A\bigr)$. Moreover, there is a simply-connected complex m-manifold X_A endowed with a Bochner-Kähler structure and a (reducible) hypersurface $Y_A \subset X_A$ (whose irreducible components are totally geodesic) so that, first, $X_A \setminus Y_A$ is isomorphic as a Bochner-Kähler manifold to $C(p_D,\mu)^\circ \times \bigl(\mathbb{R}^m/\Lambda_A\bigr)$, and, second, the image of the momentum mapping on X_A is equal to $C(p_D,\mu)^\circ$ union the faces $l_{\alpha_j} = 0$ $(1 \le j \le k)$ and minus any faces omitted from this list.

In cases where $C(p_D,\mu)$ has at most m simple faces (which includes all the cases except SubCase 4-i for $i < m$), one can take A to be the set of all the α for which $l_\alpha = 0$ is a simple face of $C(p_D,\mu)$, and the result will be X_A, whose momentum image is the entire momentum cell. This is, in some sense, 'the maximally complete' Bochner-Kähler structure of dimension m with the given momentum cell as momentum image. By Propositions 4.14 and 4.21, however, X_A cannot be metrically complete unless the cell is bounded and has exactly m simple faces. As has been seen, when $m > 0$ this can only happen in SubCase 3-1b. As will be seen in the next section, the metric on X_A does turn out to be complete in this SubCase.

In SubCase 4-i for $i < m$, the hyperplane $l_\alpha = 0$ is a simple face of $C(p_D, \mu)$ for all $\alpha \neq i$. The relations

$$\sum_\alpha W_\alpha = \sum_\alpha r_\alpha W_\alpha = 0$$

then imply the relation

$$\sum_{\alpha \neq i} (r_i - r_\alpha) W_\alpha = 0$$

among the vector fields that would have to be periodic if one were going to be able to complete the metric across all $m+1$ of the faces simultaneously. This in turn implies that

$$\sum_{\alpha \neq i} (r_i - r_\alpha) \tau_\alpha = 0$$

and this is the unique linear relation among the $\{\tau_\alpha \mid \alpha \neq i\}$. These $m+1$ vectors generate a discrete lattice Λ_i in $\mathbb{R}^m$ if and only if the ratios $(r_i - r_\alpha)/(r_i - r_\beta)$ are rational for all $\alpha, \beta \neq i$.

However, this rationality condition is not sufficient for the Bochner-Kähler structure on $C(p_D, \mu)^\circ \times (\mathbb{R}^m/\Lambda_i)$ to complete to a smooth manifold. In fact, the necessary condition for this is that these ratios all be integers, which an elementary argument shows to be not possible. Instead, the rationality is sufficient to ensure that the metric extends to a smooth orbifold whose momentum mapping is onto $C(p_D, \mu)$. By Propositions 4.14 and 4.21, this metric is complete only in SubCase 4-0, and this returns to the orbifold discussion at the end of §4.3.3.

4.4.5. *Complete examples.* We will now describe a formula that defines an n-parameter family of complete Bochner-Kähler metrics on $\mathbb{C}^n$. We will then state a theorem about these metrics and follow this with a discussion that motivates the derivation of this (rather unlikely looking) formula.

First, fix $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ where each ρ_i is a nonnegative real number. We claim that there is a real-analytic function $s : \mathbb{C}^n \to [0, \infty)$ so that

$$s(z) - \sum_{i=1}^n e^{-\rho_i s(z)} |z_i|^2 = 0$$

for all $z \in \mathbb{C}^n$. (This claim will be justified below.) Of course, the function s depends on ρ, but we will not notate this. When $\rho = 0$, one has $s(z) = |z|^2$, but otherwise this is not an elementary function. By construction, the function s is invariant under the standard n-torus action on $\mathbb{C}^n$ defined by

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) = \left(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n\right).$$

Now set

$$S(z) = 1 + \sum_{i=1}^n \rho_i \, e^{-\rho_i s(z)} |z_i|^2 \geq 1.$$

Define an Hermitian symmetric positive definite matrix $G(z) = \left(G^{ij}(z)\right)$ by

$$G^{ij}(z) = S(z) \left(\delta^{ij} e^{\rho_i s(z)} + \left(\rho_i + \rho_j + \rho_i \rho_j \, s(z)\right) \overline{z_i} \, z_j \right).$$

Write $G(z)^{-1} = \big(G_{ij}(z)\big) > 0$ and define the Hermitian metric

$$g_\rho = G_{ij}(z)\, dz_i \circ d\overline{z_j}.$$

This metric is evidently invariant under the standard n-torus action defined above. Of course, g_0 is the standard flat metric on $\mathbb{C}^n$.

Theorem 4.27. *For every choice of $\rho_i \geq 0$, the metric g_ρ is Bochner-Kähler and complete on $\mathbb{C}^n$. Conversely, every simply-connected, complete Bochner-Kähler manifold in dimension n is either homogeneous or is isometric to $(\mathbb{C}^n, g_\rho)$ for some ρ with $\rho_i \geq 0$. When the ρ_i are distinct and positive, the only symmetries of this metric belong to the standard n-torus action on $\mathbb{C}^n$.*

Proof. The structure of the proof will be as follows: We will first assume that we have a complete Bochner-Kähler metric that is not locally homogeneous and consider the induced metric on a completed $\mathcal{F}$-leaf. Knowing by earlier discussions that the only possibility for this is in SubCase 3-1b, we will use knowledge of the form of p_D and the momentum cell to choose a particularly good basis for $\mathfrak{z}$, one for which each of the vector fields of the basis has a periodic flow of period π. We will then attempt to find global holomorphic coordinates on the leaf that will carry these vector fields into the vector fields that generate the standard m-torus action defined above. Using these calculations as a guide and then comparing with the discussion at the end of §4.3.2 of the 'resolution' of boundary singularities of the cell metric in SubCase 3-1b, we will finally arrive at a candidate for the metric in these good coordinates and finish by showing how completeness and real-analyticity give the conclusions of the theorem.

Thus, suppose that M^n is simply connected and has a complete Bochner-Kähler metric of cohomogeneity $m > 0$. As has already been remarked, the momentum cell must fall into SubCase 3-1b, so that

$$p_D(t) = (t - r_1)^2 (t - r_2) \cdots (t - r_{m+1}) \qquad (r_1 > \cdots > r_{m+1}).$$

For notational simplicity, we will use the index range $2 \leq \alpha, \beta, \gamma \leq m{+}1$ and the abbreviation ρ_α (> 0) for $r_1 - r_\alpha$ in what follows. Use (4.23) as the definition of the linear function $l_\alpha : \mathbb{R}^m \to \mathbb{R}$ for $\alpha \geq 2$, as before, and set

$$a = 1 - \sum_\alpha \rho_\alpha\, l_\alpha \qquad \text{and} \qquad t = l_2 + \cdots + l_{m+1}\,,$$

as was done in §4.3.2 during the analysis of the metric R_D on this momentum cell $C(p_D, \mu) \subset \mathbb{R}^m$, which is defined by the inequalities $l_\alpha \geq 0$ and $a > 0$. Note that this momentum cell contains only one vertex, namely the point k_1 where all of the l_α vanish. Also as before, let $F_\alpha \subset C(p_D, \mu)$ be the face defined by $l_\alpha = 0$ for $2 \leq \alpha \leq m{+}1$.

As was done in the proof of Proposition 4.16, set $w_\alpha = (h')^*(l_\alpha)$ and consider the m vector fields $W_\alpha \in \mathfrak{z}$ defined by $W_\alpha \lrcorner \Omega = -dw_\alpha$. These vector fields are a basis of $\mathfrak{z}$ and are linearly independent on M°.

Fix $q \in M^\circ$, let $L \subset M$ be the leaf of the foliation $\mathcal{F}$ passing through q, and let $E \subset L$ be the leaf of the foliation $\mathcal{E}$ passing through q. On E°, the map $h' : E \to C(p_D, \mu)$ is a local isometry when $C(p_D, \mu)$ is given the metric R_D.

Recall the discussion and notation at the end of §4.3.2 about the metric R_ρ on the ellipsoidal domain $E_\rho \subset \mathbb{R}^m$. The map $s : E_\rho \to C(p_D, \mu)$ is surjective and is isometric and submersive away from the hyperplanes $p_\alpha = 0$. Letting $p \in E_\rho$

be the point with coordinates $p_\alpha = \sqrt{w_\alpha(q)} > 0$, it follows that there is a real-analytic map ψ from a neighborhood of $p \in E_\rho$ to a neighborhood of $q \in E$ satisfying $\psi(p) = q$ and $h'\circ\psi = s$. This map is an isometry when E_ρ is endowed with the metric R_ρ. Since E_ρ is simply-connected and the metric R_ρ is both real-analytic and complete, it follows that ψ can be extended uniquely as a global isometry $\psi : E_\rho \to E$ and that it satisfies $h'\circ\psi = s$. Since the rank of the differential of $s : E_\rho \to C(p_D, \mu)$ at any $p = (p_\alpha)$ is equal to the number of nonzero entries p_α, it follows that the rank of the differential of h' at any $x \in E$ is equal to m minus the number of faces F_α on which $h'(x)$ lies. Since the rank of the differential of h' at x is equal to the dimension of the span of $\{W_\alpha(x) | 2 \le \alpha \le m{+}1\}$, it follows from this discussion that for any x, the nonzero elements in the list $\big(W_2(x), \dots, W_{m+1}(x)\big)$ are linearly independent. This observation will be useful below.

Each W_α vanishes on $N_\alpha = (h')^{-1}(F_\alpha)$, which, since the flow of W_α is isometric, is a totally geodesic submanifold of M and, moreover, is a complex submanifold of M as well (since W_α is the real part of a holomorphic vector field on M). It also follows from the discussion in the previous paragraph that W_α is nonzero off of N_α. Let $L_\alpha = N_\alpha \cap L$. Then L_α is a totally geodesic complex hypersurface in L.

One of the goals of this argument is to show that there are holomorphic coordinates $z_2, \dots, z_{m+1}$ on L for which

$$W_\alpha - iJW_\alpha = 4\, i\, z_\alpha \frac{\partial}{\partial z_\alpha}, \qquad 2 \le \alpha \le m{+}1,$$

and to find the expression for the induced Kähler metric on L in these coordinates. (The choice of the coefficient 4 is dictated by the fact that the flow of the vector field W_α has period π. The proof of this periodicity follows the same lines as the corresponding proof in the SubCase 4-0 situation analyzed in Proposition 4.16. Since it only differs in details from that proof, the argument will be left to the reader.)

Accordingly, let $\zeta^2, \dots, \zeta^{m+1}$ be the holomorphic 1-forms on L° that satisfy

$$\zeta^\alpha(W_\beta - iJW_\beta) = 4i\, \delta^\alpha_\beta.$$

These forms extend meromorphically to L, with simple poles along the hypersurfaces L_α. Since the vector fields W_α Lie-commute, it follows that each ζ^α is closed. Note that, if the coordinates z_α are to exist as claimed, it will have to be true that $\zeta^\alpha = dz_\alpha/z_\alpha$.

Writing $\zeta^\alpha = \xi^\alpha + i\,\eta^\alpha$, the above equations are equivalent to

$$\xi^\alpha(W_\beta) = \eta^\alpha(JW_\beta) = 0, \qquad -\xi^\alpha(JW_\beta) = \eta^\alpha(W_\beta) = 2\,\delta^\alpha_\beta.$$

Again, if the coordinates z_α exist as claimed, it will follow that $2\xi^\alpha = d\big(\log|z_\alpha|^2\big)$.

Since the ζ^α are a basis for the holomorphic 1-forms on L°, the metric on L° can be written in the form

$$ds^2 = g_{\alpha\beta}\, \zeta^\alpha \circ \overline{\zeta^\beta}$$

where $g_{\alpha\beta} = \overline{g_{\beta\alpha}}$ and where the pullback of Ω to L° is

$$\Upsilon = \tfrac{i}{2}\, g_{\alpha\beta}\, \zeta^\alpha \wedge \overline{\zeta^\beta}.$$

Now, the identity $\zeta^\alpha(W_\beta) = 2i\,\delta^\alpha_\beta$ implies

$$dw_\alpha = -W_\alpha \lrcorner \Upsilon = g_{\alpha\beta}\, \overline{\zeta^\beta} + g_{\beta\alpha}\, \zeta^\beta,$$

or, equivalently,

$$dw_\alpha = (g_{\alpha\beta} + g_{\beta\alpha})\,\xi^\beta - i\,(g_{\alpha\beta} - g_{\beta\alpha})\,\eta^\beta.$$

Since W_α is tangent to the fibers of h', and since $w_\alpha = (h')^*(l_\alpha)$ is constant on those fibers, it follows that the coefficient of η^β in the above equation must vanish, i.e., $g_{\alpha\beta} = g_{\beta\alpha} = \overline{g_{\alpha\beta}}$. Thus,

$$dw_\alpha = 2g_{\alpha\beta}\,\xi^\beta.$$

Define $g^{\alpha\beta} = g^{\beta\alpha}$ so that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. Note, in particular, that $\xi^\alpha = \frac{1}{2}g^{\alpha\beta}\,dw_\beta$. The metric on L° can now be written in the form

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\ \zeta^\alpha{\circ}\overline{\zeta^\beta} = g_{\alpha\beta}\,(\xi^\alpha + i\,\eta^\alpha){\circ}(\xi^\beta - i\,\eta^\beta) \\ &= g_{\alpha\beta}\big(\xi^\alpha{\circ}\xi^\beta + \eta^\alpha{\circ}\eta^\beta\big) \\ &= \tfrac{1}{4}g^{\alpha\beta}\,dw_\alpha{\circ}dw_\beta + g_{\alpha\beta}\,\eta^\alpha{\circ}\eta^\beta. \end{aligned}$$

Since L° is totally geodesic in M°, it follows from Theorem 4.13 that

$$\tfrac{1}{4}g^{\alpha\beta}\,dw_\alpha\,dw_\beta = (h')^*(R_D) = (h')^*\big(T^{\alpha\beta}\big)\,dw_\alpha\,dw_\beta$$

where $T^{\alpha\beta} = T^{\beta\alpha}$ for $2 \le \alpha, \beta \le m{+}1$ is defined on $C(p_D,\mu)^\circ$ so that the formula

$$\sum_{\alpha,\beta=2}^{m+1} T^{\alpha\beta}\,dl_\alpha\,dl_\beta = R_D = \frac{t\,da^2}{4a^2} - \frac{da\,dt}{2a} + \sum_{\alpha=2}^{m+1}\frac{{dl_\alpha}^2}{4l_\alpha}$$

holds. Thus, $g^{\alpha\beta} = 4(h')^*\big(T^{\alpha\beta}\big)$.

Using the definitions of a and t, it follows from the formula for R_D that

$$4T^{\alpha\beta} = \frac{\delta_{\alpha\beta}}{l_\alpha} + \frac{(\rho_\alpha{+}\rho_\beta)}{a} + \frac{\rho_\alpha\rho_\beta\,t}{a^2}\,,$$

so that, in particular,

$$\begin{aligned} 4T^{\alpha\beta}\,dl_\beta &= \frac{dl_\alpha}{l_\alpha} - \frac{da}{a} + \rho_\alpha\left(\frac{dt}{a} - \frac{t\,da}{a^2}\right) \\ &= d\left(\log\frac{l_\alpha}{a} + \rho_\alpha\,\frac{t}{a}\right). \end{aligned}$$

Meanwhile, if the coordinates z_α exist as claimed, this will imply that

$$\frac{d|z_\alpha|^2}{|z_\alpha|^2} = 2\xi^\alpha = g^{\alpha\beta}\,dw_\beta = (h')^*\big(4T^{\alpha\beta}\,dl_\beta\big) = d\left((h')^*\left(\log\frac{l_\alpha}{a} + \rho_\alpha\frac{t}{a}\right)\right),$$

i.e., there will exist constants $c_\alpha > 0$ so that

$$|z_\alpha|^2 = c_\alpha(h')^*\left(\frac{l_\alpha}{a}\,e^{\rho_\alpha\,t/a}\right).$$

Since z_α would only be determined up to a multiplicative constant anyway by the above normalizations, one might as well take $c_\alpha = 1$, which will normalize the z_α up to a phase.

These calculations suggest the following construction of a candidate for the leaf metric: Consider the system of equations

$$p_\alpha = \frac{y_\alpha}{b}\,e^{\rho_\alpha\,s/b}, \qquad \text{where} \qquad b = 1 - \sum_{\beta=2}^{m+1}\rho_\beta\,y_\beta \quad \text{and} \quad s = \sum_{\beta=2}^{m+1} y_\beta,$$

relating the m variables $y_2, \ldots, y_{m+1}$ to the variables $p_2, \ldots, p_{m+1}$. These formulae define a real-analytic mapping $\mathbf{p}$ from the open halfspace $H_y \subset \mathbb{R}^m$ defined by $b > 0$ in y-space into p-space.

We claim that the mapping $\mathbf{p}$ is a diffeomorphism from H_y onto its image $D_p \subset \mathbb{R}^m$ and that this open image contains the primary orthant $O_p \subset \mathbb{R}^m$, i.e., the closed domain defined by $p_\alpha \geq 0$. Consequently, $\mathbf{p}$ has an inverse $\mathbf{y} : D_p \to H_y$, i.e., the above equations can be solved real-analytically in the form

$$y_\alpha = \mathbf{y}_\alpha(p_2, \ldots, p_{m+1}).$$

Moreover, this inverse $\mathbf{y}$ takes O_p diffeomorphically onto the partially open simplex $\Sigma \subset H_y$ defined by the inequalities $y_\alpha \geq 0$ and $\sum_\beta \rho_\beta\, y_\beta < 1$.

To prove this claim, it is helpful to introduce the intermediate quantities

$$u_\alpha = \frac{y_\alpha}{(1 - \sum_\beta \rho_\beta\, y_\beta)}\,.$$

These equations can be inverted in the form

$$y_\alpha = \frac{u_\alpha}{(1 + \sum_\beta \rho_\beta\, u_\beta)}\,,$$

thus showing that they define a diffeomorphism from H_y to the half-space $H_u \subset \mathbb{R}^m$ defined by $1 + \sum_\beta \rho_\beta\, u_\beta > 0$. Then the claim above amounts to showing that the mapping defined by

$$p_\alpha = u_\alpha e^{\rho_\alpha(u_2 + \cdots + u_{m+1})}$$

is invertible on the domain H_u and that its image has the desired properties.

Consider the function f on $\mathbb{R} \times \mathbb{R}^m$ defined by

$$f(r, p) = r - \sum_\alpha e^{-\rho_\alpha r} p_\alpha\,.$$

Now, $\partial f / \partial r = 1 + \sum_\alpha \rho_\alpha e^{-\rho_\alpha r} p_\alpha$ is positive on $\mathbb{R} \times O_p$, so that $r \mapsto f(r, \bar{p})$ is a strictly increasing function on $\mathbb{R}$ for every $\bar{p} \in O_p$. Note that $f(0, \bar{p}) < 0$ for $\bar{p} \in O_p$ and that $\lim_{r\to\infty} f(r, \bar{p}) = \infty$ for $\bar{p} \in O_p$ (since each of the ρ_α is positive). It then follows by the intermediate value theorem and the implicit function theorem that the equation $f(r, p) = 0$ can be solved uniquely and real-analytically for $r \geq 0$ on an open set $O_p^* \subset \mathbb{R}^m$ containing the domain O_p.

Thus, let $\mathbf{r} : O_p^* \to \mathbb{R}$ satisfy $f\bigl(\mathbf{r}(p), p\bigr) = 0$ and set

$$u_\alpha = p_\alpha e^{-\rho_\alpha \mathbf{r}(p)}.$$

Then, by construction,

$$\sum_\alpha u_\alpha = \sum_\alpha p_\alpha e^{-\rho_\alpha \mathbf{r}(p)} = \mathbf{r}(p),$$

so that $p_\alpha = u_\alpha e^{\rho_\alpha(u_2 + \cdots + u_{m+1})}$. Moreover, when p lies in O_p,

$$1 + \sum_\beta \rho_\alpha\, u_\alpha = 1 + \sum_\alpha \rho_\alpha e^{-\rho_\alpha \mathbf{r}(p)} p_\alpha = \frac{\partial f}{\partial r}\bigl(\mathbf{r}(p), p\bigr) > 0,$$

so the image point lies in H_u. The inversion of the original system is therefore

$$y_\alpha = \frac{p_\alpha\, e^{-\rho_\alpha \mathbf{r}(p)}}{1 + \sum_\beta \rho_\beta\, p_\beta\, e^{-\rho_\beta \mathbf{r}(p)}} = \mathbf{y}_\alpha\bigl(p_2, \ldots, p_{m+1}\bigr),$$

as was desired.

Now define a metric on $\mathbb{C}^m$ as follows: First, define functions on $\mathbb{C}^m$ by

$$G^{\alpha\beta}(\mathbf{z}) = \overline{z_\alpha}\, z_\beta \left(\frac{\delta_{\alpha\beta}}{\mathbf{y}_\alpha\left(|z_2|^2, \dots, |z_{m+1}|^2\right)} + \frac{(\rho_\alpha + \rho_\beta)}{\mathbf{a}} + \frac{\rho_\alpha \rho_\beta\, \mathbf{t}}{\mathbf{a}^2} \right)$$

where

$$\mathbf{a} = 1 - \sum_\beta \rho_\beta\, \mathbf{y}_\beta\big(|z_2|^2, \dots, |z_{m+1}|^2\big) \qquad \text{and} \qquad \mathbf{t} = \sum_\beta \mathbf{y}_\beta\big(|z_2|^2, \dots, |z_{m+1}|^2\big).$$

Note that $\mathbf{a}$ is strictly positive on $\mathbb{C}^m$. Moreover, since $\mathbf{y}_\alpha = p_\alpha \mathbf{y}^*_\alpha$ where $\mathbf{y}^*_\alpha$ is strictly positive on $\mathbb{C}^m$, the formula for $G^{\alpha\beta} = \overline{G^{\beta\alpha}}$ defines a smooth function on $\mathbb{C}^m$ for all α and β. Moreover, the inequalities satisfied by the $\mathbf{y}_\alpha$ show that the Hermitian matrix $G(\mathbf{z}) = \big(G^{\alpha\beta}(\mathbf{z})\big)$ is positive definite for all $\mathbf{z} \in \mathbb{C}^m$. Let $G_{\alpha\beta}(\mathbf{z})$ denote the components of the inverse matrix and define

$$d\mathbf{s}^2 = G_{\alpha\beta}(\mathbf{z})\, dz_\alpha\, d\overline{z_\beta}.$$

This is an Hermitian metric on $\mathbb{C}^m$. It is visibly invariant under the torus action generated by the real parts of the holomorphic vector fields

$$Z_\alpha = 4i\, z_\alpha \frac{\partial}{\partial z_\alpha}.$$

Setting $\zeta^\alpha = dz_\alpha / z_\alpha$ yields $\zeta^\alpha(Z_\beta) = 4i\, \delta^\alpha_\beta$. Tracing through the construction above, one sees that, away from the complex hyperplanes $z_\alpha = 0$, the metric can be written in the form

$$d\mathbf{s}^2 = f_{\alpha\beta}(\mathbf{z})\, \zeta^\alpha \circ \overline{\zeta^\beta}$$

where the inverse matrix $f^{\alpha\beta}$ has the form

$$f^{\alpha\beta}(\mathbf{z}) = \frac{\delta_{\alpha\beta}}{\mathbf{y}_\alpha\big(|z_2|^2, \dots, |z_{m+1}|^2\big)} + \frac{(\rho_\alpha + \rho_\beta)}{\mathbf{a}} + \frac{\rho_\alpha \rho_\beta\, \mathbf{t}}{\mathbf{a}^2}\,,$$

with

$$\mathbf{a} = 1 - \sum_\beta \rho_\beta\, \mathbf{y}_\beta\big(|z_2|^2, \dots, |z_{m+1}|^2\big) \qquad \text{and} \qquad \mathbf{t} = \sum_\beta \mathbf{y}_\beta\big(|z_2|^2, \dots, |z_{m+1}|^2\big).$$

Thus, the map from $\mathbb{C}^m$ to $C(p_D, \mu)$ defined by $l_\alpha = \mathbf{y}_\alpha\big(|z_2|^2, \dots, |z_{m+1}|^2\big)$ is a Riemannian submersion from the complement of the hyperplanes $z_\alpha = 0$ onto $C(p_D, \mu)$.

It is not difficult now to trace through the construction and see that the restriction of the metric $d\mathbf{s}^2$ to $\mathbb{R}^m \subset \mathbb{C}^m$ is isometric to the metric R_ρ on E_ρ and is hence complete. It now follows without difficulty that $d\mathbf{s}^2$ is complete on $\mathbb{C}^m$. Note that this completeness is a consequence of the completeness of the metric R_ρ on E_ρ and so, by the discussion in §4.3.2, is valid for any ρ all of whose entries are nonnegative, not just for those ρ whose entries are positive and strictly increasing.

Moreover, looking back at the formula for the metric on L and comparing terms, one sees that $\big(\mathbb{C}^m, d\mathbf{s}^2\big)$ is locally and (therefore, by completeness) globally holomorphically isometric to L with its induced metric and that, under this isomorphism, the Kähler form corresponding to $d\mathbf{s}^2$ is simply

$$\Upsilon = \tfrac{i}{2}\, f_{\alpha\beta}(\mathbf{z})\, \zeta^\alpha \wedge \overline{\zeta^\beta} = \tfrac{i}{2}\, G_{\alpha\beta}(\mathbf{z})\, dz_\alpha \wedge \overline{dz_\beta}\,.$$

This provides the desired 'explicit' formula for the metric on the leaf L. (Bear in mind, though, that the function $\mathbf{r}$, which is the crucial ingredient in the recipe for the metric, was found by abstractly solving an implicit equation.)

As the reader can verify, the formula given above simplifies (after some trivial changes in notation) to the formula for g_ρ given before the statement of Theorem 4.27.

The argument to this point shows that the metric g_ρ defined before the statement of Theorem 4.27 is Bochner-Kähler for any ρ all of whose entries are positive and distinct. However, the property of being Bochner-Kähler is preserved in the limit as any of the entries of ρ vanish or become equal. (The curvature tensor is evidently analytic in ρ.) Consequently, the metric g_ρ is Bochner-Kähler and complete for any ρ with nonnegative entries.

Finally, returning to the notation used before the statement of Theorem 4.27, suppose $\rho = (\rho_1, \dots, \rho_n)$ with

$$0 \le \rho_1 \le \rho_2 \le \cdots \le \rho_n\,.$$

Suppose first that these inequalities are strict and set

$$r_1 = \frac{1}{(n+2)}\big(\rho_1 + \cdots + \rho_n\big)$$

and then $r_\alpha = r_1 - \rho_{\alpha-1}$ for $2 \le \alpha \le n{+}1$, so that

$$2r_1 + r_2 + \cdots + r_{n+1} = 0$$

and $r_1 > r_2 > \cdots > r_{n+1}$. From the construction in the first part of the proof, it follows that the metric g_ρ satisfies

$$p_C(t) = p_D(t) = (t - r_1)^2 (t - r_2) \cdots (t - r_{n+1})$$

and has cohomogeneity n. Since the metric is complete, by Propositions 4.14 and 4.16, the momentum cell must fall into the SubCase 3-1b. Moreover, since any strictly decreasing sequence $(r_1, \dots, r_{n+1})$ satisfying $2r_1 + r_2 + \cdots + r_{n+1} = 0$ can be written in the above form for a unique ρ with $0 < \rho_1 < \cdots < \rho_n$, it follows that such parameters account for all of the n-dimensional reduced momentum cells in SubCase 3-1b. Thus, by Theorem 4.10, this formula gives all of the complete, simply-connected cohomogeneity n Bochner-Kähler metrics of dimension n. Note that the origin is the unique fixed point of all of the vector fields in $\mathfrak{z}$, and it follows from (4.1) that

$$p_{h(0)}(t) = (t - r_2) \cdots (t - r_{n+1}) \qquad \text{and} \qquad \big(t^2 + h_1(0)\,t + V(0)\big) = (t - r_1)^2.$$

From this, it follows from Proposition 3.7 that the Lie algebra of the symmetry group is $\mathfrak{z}$. Since the group of symmetries is necessarily connected, it follows that the flows in $\mathfrak{z}$ generate the entire symmetry group.

Now consider what happens as the ρ_i vary. The metric g_ρ depends analytically on ρ, so the formulae for $p_{h(0)}(t)$ and $V(0)$ must remain true for all values of ρ. The vector fields in $\mathfrak{z}$ all vanish at 0, so it follows that $B_3 = |T|^2$ must vanish at 0. Now, applying Theorem 4.4, one sees that, as ρ varies through $\mathbb{R}^n$ satisfying $0 \le \rho_1 \le \cdots \le \rho_n$, the values of the moduli pass through all of the values that can give rise to momentum cells in SubCase 3-1b, with the one exception of $\rho = 0$, since, in this case, there is no such cell. Consequently, as ρ varies in the primary orthant, the g_ρ account for all of the possible analytically connected equivalence classes that can contain a complete metric. Since these metrics are all complete, it follows from Theorem 4.10, and Propositions 4.14 and 4.16, that these contain all of the inhomogeneous complete Bochner-Kähler metrics on simply connected manifolds. □

Remark 4.28 (Existence). Interestingly, the argument above justifies the original assumption that there exists a complete Bochner-Kähler metric that is not locally homogeneous by producing such examples at the end.

4.4.6. *Weighted projective spaces.* A construction similar to that in SubCase 3-1b can be used to express the leaf metric in complex coordinates in SubCase 4-0. Since the details are similar to those in the proof of Theorem 4.27, we will be brief.

Consider a momentum cell $C(p_D,\mu)$ in SubCase 4-0, with

$$p_D(t) = (t-r_0)(t-r_1)\cdots(t-r_{m+1}) \qquad (r_0 > \cdots > r_{m+1}).$$

The cell $C(p_D,\mu)$ is defined by the inequalities $l_\alpha \geq 0$ for $1 \leq \alpha \leq m{+}1$.

The first task is to produce holomorphic coordinates on the completion X_A when $A = \{2,\ldots,m{+}1\}$. In fact, the argument to follow will show that X_A is biholomorphic to $\mathbb{C}^m$. By Proposition 4.21, it suffices to consider the case $n = m$, for one can always reduce to this case by simultaneously translating all of the r_α until $r_0 + \cdots + r_{m+1} = 0$. So assume that this has been done.

For simplicity, use the abbreviation ρ_α (>0) for $r_0{-}r_\alpha$ when $\alpha \geq 1$. Use (4.23) as the definition of the linear function $l_\alpha : \mathbb{R}^m \to \mathbb{R}$ as before, and note that the relations (4.24) can be written as

$$\rho_1\, l_1 = 1 - \sum_{\alpha>1} \rho_\alpha\, l_\alpha \qquad \text{and} \qquad -\rho_1\, l_0 = 1 - \sum_{\alpha>1} (\rho_\alpha{-}\rho_1)\, l_\alpha\,.$$

The functions $l_2,\ldots,l_{m+1}$ are nonnegative coordinates on the cell, the function l_1 is nonnegative, and the function l_0 is strictly negative. Of course, the function l_1 is strictly positive on the cell minus the face $l_1 = 0$, and this will be important below. In what follows, whenever repeated indices invoke the summation convention, the range will be assumed to be $2 \leq \alpha,\beta \leq m{+}1$ unless stated otherwise.

As was done in the proof of Proposition 4.16, set $w_\alpha = (h')^*(l_\alpha)$ and consider the vector fields $W_\alpha \in \mathfrak{z}$ defined by $W_\alpha \lrcorner\, \Omega = -dw_\alpha$. The vector fields $W_2,\ldots,W_{m+1}$ are a basis of $\mathfrak{z}$ and are linearly independent on X_A°. The map $h' : X_A \to C(p_D,\mu)$ is a Riemannian submersion on X_A° when $C(p_D,\mu)$ is given the metric R_D. The image $h'(X_A)$ is equal to $C(p_D,\mu)$ minus the face $l_1 = 0$.

Each W_α vanishes on $N_\alpha = (h')^{-1}(F_\alpha)$, which, since the flow of W_α is isometric, is a totally geodesic complex hypersurface in X_A. Moreover, W_α is nonzero off of N_α for $\alpha \geq 2$.

As before, we will show that there are holomorphic coordinates $z_2,\ldots,z_{m+1}$ on X_A for which

$$W_\alpha - iJW_\alpha = 4\,i\,z_\alpha \frac{\partial}{\partial z_\alpha}\,, \qquad 2 \leq \alpha \leq m{+}1,$$

and find the expression for the Bochner-Kähler metric on X_A in these coordinates.

Accordingly, let $\zeta^2,\ldots,\zeta^{m+1}$ be the holomorphic 1-forms on X_A° that satisfy

$$\zeta^\alpha(W_\beta - iJW_\beta) = 4i\,\delta^\alpha_\beta.$$

These forms extend meromorphically to X_A, with simple poles along the hypersurfaces N_α. Since the vector fields W_α Lie-commute, it follows that each ζ^α is closed. As before, if the coordinates z_α are to exist as claimed, it will have to be true that $\zeta^\alpha = dz_\alpha/z_\alpha$.

Writing $\zeta^\alpha = \xi^\alpha + i\,\eta^\alpha$, the above equations are equivalent to

$$\xi^\alpha(W_\beta) = \eta^\alpha(JW_\beta) = 0, \qquad -\xi^\alpha(JW_\beta) = \eta^\alpha(W_\beta) = 2\,\delta^\alpha_\beta.$$

Again, if the coordinates z_α exist as claimed, it will follow that $2\xi^\alpha = d\big(\log|z_\alpha|^2\big)$.

Since the ζ^α are a basis for the holomorphic 1-forms on X_A°, the metric on X_A° can be written in the form

$$ds^2 = g_{\alpha\beta}\,\zeta^\alpha{\circ}\overline{\zeta^\beta}$$

where $g_{\alpha\beta} = \overline{g_{\beta\alpha}}$ and where the pullback of Ω to X_A° is

$$\Upsilon = \tfrac{i}{2}\,g_{\alpha\beta}\,\zeta^\alpha{\wedge}\overline{\zeta^\beta}.$$

Now, the identity $\zeta^\alpha(W_\beta) = 2i\,\delta^\alpha_\beta$ implies

$$dw_\alpha = -W_\alpha \lrcorner \Upsilon = g_{\alpha\beta}\,\overline{\zeta^\beta} + g_{\beta\alpha}\,\zeta^\beta,$$

or, equivalently,

$$dw_\alpha = (g_{\alpha\beta} + g_{\beta\alpha})\,\xi^\beta - i\,(g_{\alpha\beta} - g_{\beta\alpha})\,\eta^\beta.$$

Since W_α is tangent to the fibers of h', and since $w_\alpha = (h')^*(l_\alpha)$ is constant on those fibers, it follows that the coefficient of η^β in the above equation must vanish, i.e., $g_{\alpha\beta} = g_{\beta\alpha} = \overline{g_{\alpha\beta}}$. Thus,

$$dw_\alpha = 2g_{\alpha\beta}\,\xi^\beta.$$

Define $g^{\alpha\beta} = g^{\beta\alpha}$ so that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. Note, in particular, that $\xi^\alpha = \frac12 g^{\alpha\beta}\,dw_\beta$. The metric on X_A° can now be written in the form

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\;\zeta^\alpha{\circ}\overline{\zeta^\beta} = g_{\alpha\beta}\,(\xi^\alpha + i\,\eta^\alpha){\circ}(\xi^\beta - i\,\eta^\beta)\\ &= g_{\alpha\beta}\big(\xi^\alpha{\circ}\xi^\beta + \eta^\alpha{\circ}\eta^\beta\big)\\ &= \tfrac14 g^{\alpha\beta}\,dw_\alpha{\circ}dw_\beta + g_{\alpha\beta}\,\eta^\alpha{\circ}\eta^\beta. \end{aligned}$$

Since h' is a Riemannian submersion on X_A°, it follows that

$$\tfrac14 g^{\alpha\beta}\,dw_\alpha\,dw_\beta = (h')^*(R_D) = (h')^*\big(T^{\alpha\beta}\big)\,dw_\alpha\,dw_\beta\,,$$

where $T^{\alpha\beta} = T^{\beta\alpha}$ for $2 \le \alpha,\beta \le m{+}1$ is defined on $C(p_D,\mu)^\circ$ so that the formula

$$\sum_{\alpha,\beta=2}^{m+1} T^{\alpha\beta}\,dl_\alpha\,dl_\beta = R_D = \sum_{\alpha=0}^{m+1} \frac{{dl_\alpha}^2}{4l_\alpha}$$

holds. Thus, $g^{\alpha\beta} = 4(h')^*\big(T^{\alpha\beta}\big)$.

Using the relations above that express l_0 and l_1 in terms of $l_2,\dots,l_{m+1}$, it follows from the formula for R_D that

$$4T^{\alpha\beta} = \frac{\delta_{\alpha\beta}}{l_\alpha} + \frac{(\rho_\alpha{-}\rho_1)(\rho_\beta{-}\rho_1)}{{\rho_1}^2\,l_0} + \frac{\rho_\alpha\rho_\beta}{{\rho_1}^2\,l_1}\,,$$

so that, in particular,

$$\begin{aligned} 4T^{\alpha\beta}\,dl_\beta &= \frac{dl_\alpha}{l_\alpha} + \frac{\rho_\alpha{-}\rho_1}{\rho_1}\frac{dl_0}{l_0} - \frac{\rho_\alpha}{\rho_1}\frac{dl_1}{l_1}\\ &= d\left(\log\left(l_\alpha\,(-l_0)^{(\rho_\alpha-\rho_1)/\rho_1}(l_1)^{-\rho_\alpha/\rho_1}\right)\right). \end{aligned}$$

Meanwhile, if the coordinates z_α exist as claimed, this will imply that

$$\frac{d|z_\alpha|^2}{|z_\alpha|^2} = 2\xi^\alpha = g^{\alpha\beta}\,dw_\beta = (h')^*\big(4T^{\alpha\beta}\,dl_\beta\big) = d\left((h')^*\left(\log\frac{l_\alpha\,(-l_0)^{(\rho_\alpha-\rho_1)/\rho_1}}{(l_1)^{\rho_\alpha/\rho_1}}\right)\right),$$

i.e., there will exist constants $c_\alpha > 0$ so that

$$|z_\alpha|^2 = c_\alpha (h')^* \left(\frac{l_\alpha \, (-l_0)^{(\rho_\alpha - \rho_1)/\rho_1}}{(l_1)^{\rho_\alpha/\rho_1}} \right).$$

Since z_α would only be determined up to a multiplicative constant anyway by the above normalizations, one might as well take $c_\alpha = 1$, which will normalize the z_α up to a phase. Writing $x_\alpha = -l_\alpha/l_0 \geq 0$ for $\alpha > 0$, this equation can be written more simply as

$$|z_\alpha|^2 = (h')^* \left(\frac{x_\alpha}{(x_1)^{\rho_\alpha/\rho_1}} \right), \qquad \alpha = 2, 3, \ldots, m{+}1,$$

where the $x_\alpha \geq 0$ satisfy the relation $x_1 + \cdots + x_{m+1} = 1$.

Consider the equation

$$s + \sum_{\alpha=2}^{m+1} |z_\alpha|^2 s^{\rho_\alpha/\rho_1} = 1$$

on $\mathbb{R} \times \mathbb{C}^m$. An analysis similar to the one performed in the proof of Theorem 4.27 shows that when $\rho_\alpha/\rho_1 \geq 0$ for $\alpha \geq 2$ there is a unique real-analytic function $\mathbf{s} : \mathbb{C}^m \to (0, \infty)$ that satisfies

$$\mathbf{s}(z) + \sum_{\alpha=2}^{m+1} |z_\alpha|^2 \big(\mathbf{s}(z)\big)^{\rho_\alpha/\rho_1} = 1$$

for all $z \in \mathbb{C}^m$. Note that the function $\mathbf{s}$ is invariant under the standard m-torus action on $\mathbb{C}^m$ and is algebraic if and only if all of the ratios ρ_α/ρ_1 are rational. Using the function $\mathbf{s}$, the equations above can be solved in the form $(h')^*(x_1) = \mathbf{s}(z)$ and

$$(h')^*(x_\alpha) = |z_\alpha|^2 \big(\mathbf{s}(z)\big)^{\rho_\alpha/\rho_1} \qquad (\alpha > 1),$$

whence, by algebra, follows the formula

$$w_\alpha = (h')^*(l_\alpha) = \frac{|z_\alpha|^2 \big(\mathbf{s}(z)\big)^{\rho_\alpha/\rho_1}}{\rho_1 + \sum_{\beta=2}^{m+1} (\rho_\beta - \rho_1)\, |z_\beta|^2 \big(\mathbf{s}(z)\big)^{\rho_\beta/\rho_1}} \qquad (2 \leq \alpha \leq m{+}1).$$

This motivates defining a metric on $\mathbb{C}^m$ as follows: Set

$$\mathbf{w}_\alpha(z) = \frac{|z_\alpha|^2 \big(\mathbf{s}(z)\big)^{\rho_\alpha/\rho_1}}{\rho_1 + \sum_{\beta=2}^{m+1} (\rho_\beta - \rho_1)\, |z_\beta|^2 \big(\mathbf{s}(z)\big)^{\rho_\beta/\rho_1}}$$

for $2 \leq \alpha \leq m{+}1$ and define functions $\mathbf{w}_1$ and $\mathbf{w}_0$ on $\mathbb{C}^m$ by

$$\rho_1\, \mathbf{w}_1(z) = 1 - \sum_{\alpha>1} \rho_\alpha\, \mathbf{w}_\alpha(z) \qquad \text{and} \qquad -\rho_1\, \mathbf{w}_0(z) = 1 - \sum_{\alpha>1} (\rho_\alpha - \rho_1)\, \mathbf{w}_\alpha(z)\,.$$

Then $\mathbf{w}_1$ and $-\mathbf{w}_0$ are strictly positive on $\mathbb{C}^m$. For $2 \leq \alpha, \beta \leq m{+}1$, define functions on $\mathbb{C}^m$ by

$$G^{\alpha\beta}(z) = \overline{z_\alpha}\, z_\beta \left(\frac{\delta_{\alpha\beta}}{\mathbf{w}_\alpha(z)} + \frac{(\rho_\alpha - \rho_1)(\rho_\beta - \rho_1)}{{\rho_1}^2\, \mathbf{w}_0(z)} + \frac{\rho_\alpha \rho_\beta}{{\rho_1}^2\, \mathbf{w}_1(z)} \right).$$

It is not difficult to show that the Hermitian matrix $G(z) = \big(G^{\alpha\beta}(\mathbf{z})\big)$ is positive definite for all $z \in \mathbb{C}^m$. Let $G_{\alpha\beta}(z)$ denote the components of the inverse matrix and define

$$d\mathbf{s}^2 = G_{\alpha\beta}(z)\, dz_\alpha\, d\overline{z_\beta}.$$

This is an Hermitian metric on $\mathbb{C}^m$. It is visibly invariant under the torus action generated by the real parts of the holomorphic vector fields

$$Z_\alpha = 4i\, z_\alpha \frac{\partial}{\partial z_\alpha}.$$

Setting $\zeta^\alpha = dz_\alpha/z_\alpha$ yields $\zeta^\alpha(Z_\beta) = 4i\,\delta^\alpha_\beta$. Tracing through the construction above, one sees that, away from the complex hyperplanes $z_\alpha = 0$, the metric can be written in the form

$$d\mathbf{s}^2 = f_{\alpha\beta}(z)\,\zeta^\alpha \circ \overline{\zeta^\beta}$$

where the inverse matrix $f^{\alpha\beta}$ has the form

$$f^{\alpha\beta}(z) = \frac{\delta_{\alpha\beta}}{\mathbf{w}_\alpha(z)} + \frac{(\rho_\alpha{-}\rho_1)(\rho_\beta{-}\rho_1)}{{\rho_1}^2\,\mathbf{w}_0(z)} + \frac{\rho_\alpha\rho_\beta}{{\rho_1}^2\,\mathbf{w}_1(z)}\,.$$

In particular, the map from $\mathbb{C}^m$ to $C(p_D,\mu)$ defined by $l_\alpha = \mathbf{w}_\alpha\big(|z_2|^2,\dots,|z_{m+1}|^2\big)$ is a Riemannian submersion from the complement of the hyperplanes $z_\alpha = 0$ onto $C(p_D,\mu)^\circ$ endowed with the metric R_D.

It is not difficult now to trace through the construction and see that the restriction of the metric $d\mathbf{s}^2$ to $\mathbb{R}^m \subset \mathbb{C}^m$ is isometric to the metric R_D on E as defined in §4.3.2.

Moreover, looking back at the formulae for the metric on X_A° and comparing terms, one sees that $\big(\mathbb{C}^m, d\mathbf{s}^2\big)$ must be globally holomorphically isometric to X_A with its Bochner-Kähler metric and that, under this isomorphism, the Kähler form corresponding to $d\mathbf{s}^2$ is simply

$$\Upsilon = \tfrac{i}{2}\, f_{\alpha\beta}(\mathbf{z})\,\zeta^\alpha \wedge \overline{\zeta^\beta} = \tfrac{i}{2}\, G_{\alpha\beta}(\mathbf{z})\, dz_\alpha \wedge \overline{dz_\beta}\,.$$

This provides the desired explicit formula for the metric on the leaf X_A.

Although the derivation provided the inequalities $0 < \rho_1 < \cdots < \rho_{m+1}$, the recipe given for the metric only needs the assumption $\rho_\alpha > 0$ for $1 \le \alpha \le m{+}1$. This means, for example, that the metric makes sense when all of the ρ_α are equal. In this case, the reader can verify that the metric $d\mathbf{s}^2$ on $\mathbb{C}^m$ is simply the Fubini-Study metric on $\mathbb{CP}^m$ restricted to the complement of a hyperplane.

Suppose now that all of the ratios ρ_α/ρ_1 are rational and let $r > 0$ be such that $\rho_\alpha = (m{+}2)r\,p_\alpha$ where the numbers $0 < p_1 < \ldots < p_{m+1}$ are integers with no common divisor. This uniquely defines r and the integers p_α. Moreover, the equations $r_0 - r_\alpha = \rho_\alpha = rp_\alpha$ and $r_0 + r_1 + \cdots + r_{m+1} = 0$ imply

$$r_\alpha = r\left(\sum_{\beta=0}^{m+1} p_\beta - (m{+}2)p_\alpha\right)$$

where, for notational symmetry, we have set $p_0 = 0$.

Recalling that $\rho_1\, W_1 + \rho_2\, W_2 + \cdots + \rho_{m+1} W_{m+1} = 0$, it follows that

$$p_1\big(W_1 - iJW_1\big) = -4i\left(p_2\, z_2 \frac{\partial}{\partial z_2} + \cdots + p_{m+1}\, z_{m+1} \frac{\partial}{\partial z_{m+1}}\right).$$

Now, by setting $[p] = [p_1, \dots, p_{m+1}]$ and considering the weighted projective space $\mathbb{CP}^{[p]}$ one gets by taking the quotient of $\mathbb{C}^{m+1}$ minus the origin by the $\mathbb{C}^*$-action

$$\lambda \cdot (Z_1, Z_2, \dots, Z_{m+1}) = \big(\lambda^{p_1} Z_1, \lambda^{p_2} Z_2, \dots, \lambda^{p_{m+1}} Z_{m+1}\big).$$

This is an orbifold and not a manifold except when $p_1 = \cdots = p_{m+1}$ (in which case, this is $\mathbb{CP}^m$). Let $[Z_1, \dots, Z_{m+1}] \in \mathbb{CP}^{[p]}$ denote the orbit of $(Z_1, \dots, Z_{m+1}) \in \mathbb{C}^m$. Consider the holomorphic mapping $\Phi_1 : \mathbb{C}^m \to \mathbb{CP}^{[p]}$ defined by

$$\Phi_1(Z_2, \dots, Z_{m+1}) = [1, Z_2, \dots, Z_{m+1}].$$

This mapping is a p_1-fold branched covering of its image and the above considerations show that the metric $d\mathbf{s}^2$ extends to be a smooth orbifold metric on $\mathbb{CP}^{[p]}$. The end result is the following:

Theorem 4.29. *Every weighted projective space $\mathbb{CP}^{[p]}$ supports a Bochner-Kähler metric.*

Remark 4.30 (Uniqueness). Of course, in the classical case of projective space, the Bochner-Kähler metric so constructed is a constant multiple of the Fubini-Study metric. By Corollary 4.17, this is the unique Bochner-Kähler metric on $\mathbb{CP}^m$, up to a constant multiple. We suspect, though all of the details have not been checked, that this uniqueness holds for all of the weighted projective spaces.

Remark 4.31 (Reduction). The reader will recall that one way of constructing the Fubini-Study metric is to apply reduction to the flat Kähler metric under the diagonal S^1-action. Given this, one might suspect that the Bochner-Kähler metric on $\mathbb{CP}^{[p]}$ is obtained from the flat Kähler metric by applying reduction to the weighted S^1-action described above. However, calculation shows that, except in the classical case, the reduction metric is *not* Bochner-Kähler.

4.5. **Reduction and the full metric.** Theorem 4.24 provides a formula for the induced metric on the $\mathcal{F}$-leaves of a Bochner-Kähler metric. In the case of maximal cohomogeneity, i.e., $m = n$, the regular set M° constitutes a single $\mathcal{F}$-leaf, so this formula determines the metric completely. In this section, we will indicate how one can reconstruct the full metric from the knowledge of the metric on the $\mathcal{F}$-leaves. Thus, for the rest of this section, we will assume that M^n is endowed with a Bochner-Kähler metric of cohomogeneity m satisfying $0 < m < n$, since otherwise, there is nothing to do.

Let $p_C(t)$ and $p_D(t)$ be the characteristic polynomials of the Bochner-Kähler structure. Write

$$p_{h''}(t) = (t - \lambda_{m+1}) \cdots (t - \lambda_n)$$

where, by Proposition 4.3, the roots $\lambda_{m+1} \geq \cdots \geq \lambda_n$ are also roots of $p_D(t)$. Let $\pi : P_2 \to M^\circ$ be the G_Λ-bundle as described in the proof of Proposition 4.3 and return to that notation, particularly the index ranges. Recall that the λ_i are distinct and not equal to any of the λ_a, and that the T_i are positive and real and satisfy

$${T_i}^2 = \frac{p_D(\lambda_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)}.$$

Also, recall the relations

$$\phi_{a\bar{\imath}} = \frac{T_i\,\omega_a}{\lambda_i{-}\lambda_a}\,,$$

which followed from the structure equations applied to $h_{a\bar{\imath}} = 0$. The structure equations applied to $h_{a\bar{b}} = \delta_{ab}\lambda_a$ yield

$$(\lambda_a - \lambda_b)\phi_{a\bar{b}} = 0,$$

so that $\phi_{a\bar{b}} = 0$ when $\lambda_a \neq \lambda_b$. The structure equations applied to $h_{i\bar{\jmath}} = 0$ for $i \neq j$ yield the relations

$$\phi_{i\bar{\jmath}} = -\frac{T_i\,\overline{\omega_j} + T_j\,\omega_i}{\lambda_i - \lambda_j}, \qquad i \neq j,$$

while the structure equations applied to $h_{i\bar{\imath}} = \lambda_i$ yield

$$d\lambda_i = T_i(\omega_i + \overline{\omega_i}).$$

Meanwhile, the structure equations applied to T_i yield

$$\begin{aligned} dT_i &= -\phi_{i\bar{\jmath}}\,T_j + ({\lambda_i}^2{+}h_1\,\lambda_i{+}V)\,\omega_i \\ &= -\phi_{i\bar{\imath}}\,T_i + \sum_{j\neq i}\left(\frac{T_i\,\overline{\omega_j} + T_j\,\omega_i}{\lambda_i - \lambda_j}\right)T_j + ({\lambda_i}^2{+}h_1\,\lambda_i{+}V)\,\omega_i \end{aligned}$$

which can be rearranged to give

$$\frac{dT_i}{T_i} = -\phi_{i\bar{\imath}} + \sum_{j\neq i}\frac{T_j\,\overline{\omega_j}}{\lambda_i - \lambda_j} + \left(({\lambda_i}^2{+}h_1\,\lambda_i{+}V) + \sum_{j\neq i}\frac{{T_j}^2}{\lambda_i - \lambda_j}\right)\frac{\omega_i}{T_i}\,.$$

Now, the structure equations for ω_a are (summation over i and b)

$$\begin{aligned} d\omega_a &= -\phi_{a\bar{\imath}}\wedge\omega_i - \phi_{a\bar{b}}\wedge\omega_b = \frac{T_i\,\omega_a}{\lambda_a{-}\lambda_i}\,\omega_i - \phi_{a\bar{b}}\wedge\omega_b \\ &= -\left(\phi_{a\bar{b}} + \delta_{a\bar{b}}\,\frac{T_i\,\omega_i}{\lambda_a{-}\lambda_i}\right)\wedge\omega_b = -\left(\varphi_{a\bar{b}} + \frac{1}{2}\delta_{a\bar{b}}\,\frac{d\lambda_i}{\lambda_a{-}\lambda_i}\right)\wedge\omega_b\,, \end{aligned}$$

where

$$\varphi_{a\bar{b}} = -\overline{\varphi_{b\bar{a}}} = \phi_{a\bar{b}} + \frac{1}{2}\delta_{a\bar{b}}\,\sum_i\frac{T_i\,(\omega_i - \overline{\omega_i})}{\lambda_a{-}\lambda_i}\,.$$

Since $p_{h'}(\lambda_a) = \prod_i(\lambda_a{-}\lambda_i)$ and since $\phi_{a\bar{b}} = \varphi_{a\bar{b}} = 0$ when $\lambda_a \neq \lambda_b$, setting

$$\eta_a = |p_{h'}(\lambda_a)|^{-1/2}\omega_a$$

yields $d\eta_a = -\varphi_{a\bar{b}}{\wedge}\eta_b$. This implies that, for each root r of $p_{h''}(t)$, the quadratic form and 2-form

$$g_r = \sum_{\{a:\lambda_a=r\}}\eta_a\circ\overline{\eta_a} \qquad \text{and} \qquad \Omega_r = \frac{\imath}{2}\sum_{\{a:\lambda_a=r\}}\eta_a\wedge\overline{\eta_a}$$

define a Kähler structure on the space of leaves of the system $\{\eta_a = 0 \mid \lambda_a = r\,\}$ in any open set in M° where this leaf space is Hausdorff. If r has multiplicity $\nu > 0$, this leaf space has complex dimension ν.

To compute the curvature of this leaf space, one needs to compute the 2-forms

$$\Phi_{a\bar{b}} = d\varphi_{a\bar{b}} + \varphi_{a\bar{c}}\wedge\varphi_{c\bar{b}},$$

so we now turn to this task. Since $\varphi_{a\bar b} = \phi_{a\bar b}$ when $a \neq b$, the structure equations for $a \neq b$ yield (summation on i and c)

$$\begin{aligned}\Phi_{a\bar b} &= d\phi_{a\bar b} + \phi_{a\bar c}\wedge\phi_{c\bar b} = -\phi_{a\bar i}\wedge\phi_{i\bar b} - (\lambda_a+\lambda_b+h_1)\,\omega_a\wedge\overline{\omega_b}\\ &= \left[\frac{{T_i}^2}{(\lambda_a-\lambda_i)(\lambda_b-\lambda_i)} - (\lambda_a+\lambda_b+h_1)\right]\omega_a\wedge\overline{\omega_b}\\ &= \left[\frac{p_D(\lambda_i)}{(\lambda_a-\lambda_i)(\lambda_b-\lambda_i)\prod_{j\neq i}(\lambda_i-\lambda_j)} - (\lambda_a+\lambda_b+h_1)\right]\omega_a\wedge\overline{\omega_b}.\end{aligned}$$

Rather miraculously, when $\lambda_a \neq \lambda_b$, the classical identities of §4.3.2 show that this expression is zero, as should have been expected. On the other hand, if $\lambda_a = \lambda_b = r_i$ (but still $a \neq b$), the same classical identities show that this expression simplifies to

$$\Phi_{a\bar b} = \frac{p_D'(r_i)}{p_{h'}(r_i)}\,\omega_a\wedge\overline{\omega_b} = \frac{p_D'(r_i)|p_{h'}(r_i)|}{p_{h'}(r_i)}\,\eta_a\wedge\overline{\eta_b} = (-1)^{\mu_i}p_D'(r_i)\,\eta_a\wedge\overline{\eta_b}\,.$$

(Recall that $(-1)^{\mu_i}p_{h'}(r_i) > 0$ on M°.) It remains to compute the quantities $\Phi_{a\bar a}$. This computation is greatly simplified by first observing that $\Phi_{a\bar a}$ must be a sum of terms of the form $\omega_b\wedge\overline{\omega_c}$ where $\lambda_a = \lambda_b = \lambda_c$. Thus, in carrying out the expansion from the definitions, all other terms can be ignored. For simplicity, we will use $\equiv$ to denote equality modulo the ideal generated by the 1-forms ω_i and $\overline{\omega_i}$ for $1 \leq i \leq m$ and the 1-forms $\phi_{a\bar b}$ for $m < a, b \leq n$. Then, first of all (summation over j and b),

$$d\omega_i = -\phi_{i\bar j}\wedge\omega_j - \phi_{i\bar b}\wedge\omega_b \equiv \frac{T_i\,\omega_b\wedge\overline{\omega_b}}{\lambda_b-\lambda_i}\,.$$

Using this and the identities quoted above, the calculation of $\Phi_{a\bar a}$ follows from the structure equations and goes as (summation over j and b)

$$\begin{aligned}\Phi_{a\bar a} &= d\varphi_{a\bar a} + \varphi_{a\bar b}\wedge\varphi_{b\bar a} \equiv d\left(\phi_{a\bar a} + \frac{1}{2}\,\frac{T_j\,(\omega_j-\overline{\omega_j})}{\lambda_a-\lambda_j}\right)\\ &\equiv -\phi_{a\bar j}\wedge\phi_{j\bar a} - (2\lambda_a+h_1)\,\omega_a\wedge\overline{\omega_a} - (\lambda_a+\lambda_b+h_1)\,\omega_b\wedge\overline{\omega_b} + \frac{{T_j}^2\,\omega_b\wedge\overline{\omega_b}}{(\lambda_a-\lambda_j)(\lambda_b-\lambda_j)}\\ &= \frac{p_D'(r_i)}{p_{h'}(r_i)}\left(\omega_a\wedge\overline{\omega_a} \quad + \sum_{\{b:\lambda_a=r_i\}}\omega_b\wedge\overline{\omega_b}\right)\\ &= (-1)^{\mu_i}\,p_D'(r_i)\left(\eta_a\wedge\overline{\eta_a} \quad + \sum_{\{b:\lambda_a=r_i\}}\eta_b\wedge\overline{\eta_b}\right).\end{aligned}$$

These formulae imply that the Kähler structure defined by g_{r_i} and Ω_{r_i} actually has constant holomorphic sectional curvature equal to $(-1)^{\mu_i}\,4\,p_D'(r_i)$.

4.5.1. *Reduction.* Since $h' : M \to C(p_D, \mu)$ is the momentum mapping of the infinitesimal torus action defined by the basis $Z_1', \ldots, Z_m'$ of $\mathfrak{z}$, it is natural to consider the effect of applying symplectic reduction. Since the torus action is not assumed to be globally defined (because no completeness assumptions have been made about the metric), this can only be done locally.

For simplicity, we will only consider reduction at a regular value of the reduced momentum mapping $h' : M \to C(p_D, \mu)$. Recall that $h' : M^\circ \to C(p_D, \mu)^\circ$ is a submersion, let $x \in M^\circ$ be fixed, and let $\kappa = h'(x) \in C(p_D, \mu)^\circ$. The method of symplectic reduction then consists of the following: Consider the codimension m

submanifold $(h')^{-1}(\kappa) \subset M^\circ$. This submanifold is foliated by m-dimensional leaves whose tangent spaces are spanned by the vector fields $Z_2, \dots, Z_{m+1}$. Suppose that this foliation is simple, i.e., its leaf space M_κ is Hausdorff. (This can always be arranged by restricting to a suitable open neighborhood of x.) Then the pullback of Ω to $(h')^{-1}(\kappa) \subset M^\circ$ is a closed 2-form that is the pullback to $(h')^{-1}(\kappa)$ of a symplectic form Ω_κ on M_κ. The symplectic manifold $(M_\kappa, \Omega_\kappa)$ is then called the *symplectic reduction* of (M, Ω) at κ.

Proposition 4.32. *Fix $x \in M^\circ$ and let $\kappa = h'(x) \in C(p_D, \mu)^\circ$. There is a unique metric g_κ on M_κ for which the leaf projection $(h')^{-1}(\kappa) \to M_\kappa$ is a Riemannian submersion.*

The data $(M_\kappa, g_\kappa, \Omega_\kappa)$ defines a Kähler structure on M_κ that is locally isomorphic to a product of complex space forms. Specifically, for each root r of $p_{h''}(t)$ of multiplicity ν, the local product contains a ν-dimensional complex space form of constant holomorphic sectional curvature

$$c(r, \kappa) = \frac{4\, p_D'(r)}{p_{h(x)}(r)}$$

and these are all of the factors.

Proof. Let $P_2(\kappa) \subset P_2$ be the part of P_2 that lies over $(h')^{-1}(\kappa) \subset M^\circ$. The structure equations on $P_2(\kappa)$ are the same as those on P_2 with the difference that, after restriction to $P_2(\kappa)$, the functions λ_i and T_i become constant and the 1-forms ω_i become purely imaginary. Note that the equations $\omega_a = 0$ define the foliation by the torus leaves.

Now going back to the structure equations, just derived above, one sees that, on $P_2(\kappa)$, the equations

$$d\omega_a = -\varphi_{a\bar{b}} \wedge \omega_b$$

hold, where $\varphi = (\varphi_{a\bar{b}}) = -\varphi^*$ is blocked according to the multiplicities in the descending string of eigenvalues

$$\lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_n \,.$$

It follows, of course, that the quadratic form $g_\kappa = \omega_a \circ \overline{\omega_a}$ is well defined on the leaf space M_κ and that this metric and the symplectically reduced 2-form $\Omega_\kappa = \frac{i}{2}\, \omega_a \wedge \overline{\omega_a}$ define a Kähler structure on M_κ.

Finally, the computation of the curvature forms above shows that this Kähler structure is a product of the type described in the proposition. □

4.5.2. *The general metric.* As the preceding formulae and Proposition 4.32 now make clear, a recipe for any Bochner-Kähler metric on its regular locus M° can be constructed as a generalized warped product over a momentum cell, where the fibers are products of so-called Sasakian space forms, i.e., the canonical circle bundles over complex space forms of constant holomorphic sectional curvature. In other words, once the leaf metric has been found, as in Theorem 4.24, the full metric can be constructed by group theoretic means. This is to be expected, since, after all, the pseudo-group of local symmetries of a connected Bochner-Kähler metric acts transitively on the fibers of the momentum mapping.

The explicit formula does not appear to be of great interest. For brevity, we will not go into details.

5. Final remarks

In this last section, we will make some remarks about related geometries.

5.1. **Pseudo-Kähler geometry.** When a complex n-manifold M is endowed with a pseudo-Kähler structure, i.e., a pseudo-Riemannian metric g that is invariant under the complex structure and whose associated 2-form Ω is closed, the structure group of the geometry is $\mathrm{U}(p,q)$ for some $p,q>0$ with $p{+}q=n$. Since this group is simply a different real form of the group $\mathrm{U}(n)$, one would expect a similar decomposition of the curvature tensor. Indeed, this is what happens, the curvature tensor again breaking into the sum of three irreducible tensors. For simplicity of terminology, we will still refer to these as the scalar curvature, the traceless Ricci curvature, and the Bochner curvature and will refer to pseudo-Kähler structures for which the Bochner curvature vanishes as Bochner-Kähler.

The differential analysis of §2.3 extends essentially without change to the pseudo-Kähler case; it is just a matter of changing a few signs. Theorems 3.1 through 3.11 generalize with essentially no change as well. However, past this point, the analysis becomes somewhat more complicated because the orbit structure of the action of $\mathrm{U}(p,q)$ on $\mathfrak{u}(p,q)\oplus\mathbb{C}^n\oplus\mathbb{R}$ is considerably more complicated than before. One must deal with nondiagonalizable elements, nilpotent orbits, and a host of other problems. It seems unlikely that the simple description of the analytically connected equivalence classes found in the Kähler case can be carried through in the pseudo-Kähler case.

5.2. **A split-form analog.** There is another 'real form' of Kähler geometry that has an analog of the Bochner-Kähler condition.

A Kähler structure can be thought of as a symplectic manifold (M^{2n},Ω) endowed with an Ω-skew endomorphism $J: TM\to TM$ that satisfies $J^2=-\mathrm{I}$ and a torsion-free connection ∇ with respect to which both Ω and J are parallel.

A different geometry results if one starts with a symplectic manifold as above and considers an Ω-skew endomorphism $K: TM\to TM$ that satisfies $K^2=+\mathrm{I}$ together with a torsion-free connection ∇ with respect to which both Ω and K are parallel. Some authors [24, 25] call the data (M,Ω,K,∇) a *hyperbolic Kähler structure*, though this terminology seems likely to invite confusion.

Since the null plane fields of $K\pm\mathrm{I}$ are necessarily Ω-Lagrangian plane fields and since the hypothesis that there be a torsion-free connection with respect to which they are parallel implies that these two plane fields are integrable, such a structure endows the symplectic manifold with a pair of transverse, Ω-Lagrangian foliations $\mathcal{F}_\pm$.

Conversely, any symplectic manifold (M^{2n},Ω) endowed with a pair of transverse, Ω-Lagrangian foliations $\mathcal{F}_\pm$ has an Ω-skew endomorphism $K: TM\to TM$ so that the tangent plane fields to the two foliations are the kernels of $K\pm\mathrm{I}$ and a unique torsion-free connection ∇ with respect to which both Ω and K are parallel. Thus, it makes sense to call such a structure a *bi-Lagrangian* structure, which we will do for the rest of this subsection.

Let $\mathbb{R}_n$ denote the space of *row* vectors of length n whose entries are real numbers, so that the natural matrix multiplication $\mathbb{R}_n\times\mathbb{R}^n\to\mathbb{R}$ is a nondegenerate pairing and $\mathbb{R}_n$ is thus identified as the dual space of $\mathbb{R}^n$. Endow $\mathbb{R}_n\oplus\mathbb{R}^n$ with its natural induced symplectic structure. Let $\mathrm{GL}(n,\mathbb{R})$ act on $\mathbb{R}_n\oplus\mathbb{R}^n$ on the left by

$$A\cdot(\xi,x)=\bigl(\xi\,A^{-1},\ A\,x\bigr).$$

This action preserves the symplectic structure on $\mathbb{R}_n \oplus \mathbb{R}^n$ and its bi-Lagrangian splitting into $L_- = \mathbb{R}_n \oplus 0$ and $L_+ = 0 \oplus \mathbb{R}^n$. In fact, $\mathrm{GL}(n,\mathbb{R})$ is the largest subgroup of $\mathrm{Aut}(\mathbb{R}_n \oplus \mathbb{R}^n)$ that preserves these structures.

Now let $(M^{2n}, \Omega, \mathcal{F}_\pm)$ be a bi-Lagrangian manifold. Say that a coframe $u : T_xM \to \mathbb{R}_n \oplus \mathbb{R}^n$ is *adapted* if u is a symplectic isomorphism, identifies $T_x\mathcal{F}_-$ with $\mathbb{R}_n \oplus 0$, and identifies $T_x\mathcal{F}_+$ with $0 \oplus \mathbb{R}^n$. The bundle $\pi : P \to M$ of adapted coframes is then naturally a right $\mathrm{GL}(n,\mathbb{R})$-bundle with the action defined by

$$(u \cdot A)(v) = A^{-1} \cdot u(v).$$

The tautological 1-form of this $\mathrm{GL}(n,\mathbb{R})$-structure can be written in the form (η, ω) where η takes values in $\mathbb{R}_n$ and ω takes values in $\mathbb{R}^n$. One then has the formula $\pi^*(\Omega) = \eta{\wedge}\omega$.

The existence of a torsion-free connection with respect to which Ω and K are parallel is equivalent to the existence of a $\mathfrak{gl}(n,\mathbb{R})$-valued 1-form ϕ on P satisfying the equations

$$d\eta = -\eta \wedge \phi, \qquad d\omega = -\phi \wedge \omega. \tag{5.1}$$

This is the *first structure equation* of Cartan. The 2-form $\Phi = d\phi + \phi{\wedge}\phi$ then satisfies the *first Bianchi identities*

$$\eta \wedge \Phi = \Phi \wedge \omega = 0. \tag{5.2}$$

These identities imply that there is a function $R : P \to \mathrm{Hom}\big(\mathbb{R}_n \otimes \mathbb{R}^n, \mathfrak{gl}(n,\mathbb{R})\big)$ so that the *second structure equation* holds:

$$\Phi = d\phi + \phi \wedge \phi = R(\eta \otimes \omega)$$

and, moreover, that R can be interpreted as taking values in a 'curvature space' $\mathcal{K}$ that is isomorphic as a $\mathrm{GL}(n,\mathbb{R})$-module to $S^2(\mathbb{R}_n) \otimes S^2(\mathbb{R}^n)$. Applying the trace (or 'contraction') maps

$$S^2(\mathbb{R}_n) \otimes S^2(\mathbb{R}^n) \longrightarrow \mathbb{R}_n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$$

then yields, as in the Kähler case, a decomposition of $\mathcal{K}$ into three irreducible, inequivalent $\mathrm{GL}(n,\mathbb{R})$-modules and a corresponding decomposition of the curvature tensor of any bi-Lagrangian structure into three parts. For simplicity, we will refer to these three parts as the scalar curvature, the traceless Ricci curvature, and the Bochner curvature. (In [24, 25], the latter curvature is called the "HB-tensor".)

When the Bochner curvature vanishes, the bi-Lagrangian structure will be said to be *Bochner-bi-Lagrangian.* This vanishing condition is equivalent to the existence of a function $S : P \to \mathfrak{gl}(n,\mathbb{R}) \simeq \mathbb{R}^n \otimes \mathbb{R}_n$ that satisfies

$$d\phi = -\phi \wedge \phi + S\,\eta \wedge \omega - S\,\omega \wedge \eta - \omega \wedge \eta\, S + \eta\, S \wedge \omega\ \mathrm{I}_n\ . \tag{5.3}$$

The reader will note the analogy with the second structure equation for Bochner-Kähler structures.

The same sort of analysis as in §2.3 shows that there exist functions $F : P \to \mathbb{R}^n$ and $G : P \to \mathbb{R}_n$ so that

$$dS = -\phi\, S + S\,\phi + F\,\eta + \omega\ G + \tfrac{1}{2}(G\,\omega + \eta\, F)\, I_n\,; \tag{5.4}$$

that there exists a function $Q : P \to \mathbb{R}$ so that

$$dF = -\phi\, F + \big(Q\ \mathrm{I}_n + S^2\big)\,\omega\,, \qquad dG = G\,\phi + \eta\,\big(Q\ \mathrm{I}_n + S^2\big)\,; \tag{5.5}$$

and that

$$dQ = GS\,\omega + \eta\, SF. \tag{5.6}$$

Moreover, the exterior derivatives of equations (5.1)–(5.6) are identities.

Thus, the system of structure equations (5.1)–(5.6) satisfies the conditions for Cartan's Theorem A.1 to apply (see the appendix). In particular, the analog of Theorem 3.1 holds for Bochner-bi-Lagrangian structures and there is a finite-dimensional moduli space of germs of such structures. The analog of Theorem 3.8 will hold as well, in that there will be $n+1$ polynomials in the functions S, F, G, and Q that are constant on each connected Bochner-bi-Lagrangian structure bundle and the rank of the mapping $(S, F, G, Q) : P \to \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n \oplus \mathbb{R}_n \oplus \mathbb{R}$ is never more than n, implying that the 'group' of local isometries of the structure always acts with local cohomogeneity at most n.

In principle, one could describe the analytically connected equivalence classes for this type of structure and examine completeness questions, and so on. This project is made much more complicated than its Kähler analog by the fact that the $\mathrm{GL}(n, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n \oplus \mathbb{R}_n \oplus \mathbb{R}$ do not separate the $\mathrm{GL}(n, \mathbb{R})$-orbits. This is potentially interesting, since it means that one could possibly have continuous families of Bochner-bi-Lagrangian structures all with the same coarse moduli. Whether this really does happen is an interesting question.

5.3. **Self-dual Kähler metrics.** *This section was added after P. Gauduchon supplied the preprint* [2]. *The author thanks P. Gauduchon for bringing it to his attention.*

The reader will recall that, when $n = 2$, the Bochner tensor is the same as the anti-self-dual part of the Weyl tensor, i.e., when $n = 2$, Bochner-Kähler metrics are the same as self-dual Kähler metrics. The self-dual part of the Weyl curvature in this case is essentially the scalar curvature s. In particular, the squared norm of the Weyl curvature is the same as s^2, up to a universal constant factor.

From this point of view, some of the results in this article in the case of dimension 2 had already been obtained. For example, in [9, Theorem 1] (which also follows from earlier work by B.-Y. Chen [7]) asserts that there are no compact self-dual Kähler manifolds other than the locally symmetric ones. Of course, this is the $n = 2$ case of Corollary 4.17 of the present article. Their proofs use nontrivial global results about complex surfaces, while the proof in the present article is essentially self-contained. It is also interesting to note that, in view of Theorem 4.29, their proofs must make essential use of the hypothesis that the domain of definition of the metric is a compact manifold, rather than just a compact orbifold.

After the initial version of this article was posted to the arXiv, the author was contacted by P. Gauduchon, who explained that he and V. Apostolov had recently obtained a local classification of self-dual Hermitian-Einstein metrics and that this implied a local classification of self-dual Kähler metrics. In particular, they had also proved that such metrics always have local cohomogeneity at most 2. For more information about their version of the local classification, the reader should consult their preprint [2]. In particular, their work provides an independent alternative to the classification derived in this article when $n = 2$.

In fact, a remarkable relation between self-dual Kähler metrics and Einstein metrics follows from the work of Derdzinski [9] and Apostolov and Gauduchon [2]. The interested reader should consult [2] for details, but some of their results will be summarized here as preparation for the remarks to be made at the end of this subsection.

If g is a self-dual Kähler metric on a complex 2-manifold M with scalar curvature s not identically zero, then g is not conformally flat. Apostolov and Gauduchon show that on the open set M^* where s is nonzero, the Hermitian metric $g^* = s^{-2}g$ is Einstein (as well as being self-dual). Of course, unless s is constant (which only happens when g is locally symmetric), g^* will not be Kähler.

Conversely, Apostolov and Gauduchon show that any self-dual Hermitian Einstein metric that is not conformally flat is of the form g^* for a unique self-dual Kähler metric g with nonzero scalar curvature.

However, from the point of view in [2], completeness issues for either self-dual Hermitian Einstein metrics or self-dual Kähler metrics appear not to be easily resolvable. For example, they did not know[18] whether or not there were any complete examples of cohomogeneity 2. Using the description in this article, however, it is easy to see that there are many complete examples of self-dual Hermitian Einstein metrics with cohomogeneity 2.

Before discussing these examples, here are three general observations that will be useful: Let M be a connected complex surface endowed with a Bochner-Kähler metric g and characteristic polynomial $p_C(t) = t^4{+}C_2\,t^2{+}C_3\,t{+}C_4$ and momentum mapping $h = (h_1, h_2) : M \to \mathbb{R}^2$. First, the scalar curvature of g is $s = -24h_1$. Second, the Einstein constant of g^* is $-6912C_3$. Third, the squared norm of the self-dual part of the Weyl curvature of g^* is $c\,s^6 > 0$ for some universal constant $c > 0$.

Now, consider the complete cohomogeneity 2 metrics on $\mathbb{C}^2$ provided by Theorem 4.27, where the parameters ρ_1 and ρ_2 satisfy $0 < \rho_1 < \rho_2$. The characteristic polynomials are

$$p_C(t) = p_D(t) = (t - r_1)^2(t - r_2)(t - r_3)$$

where

$$r_1 = \tfrac{1}{4}(\rho_1{+}\rho_2), \qquad r_2 = \tfrac{1}{4}(\rho_2{-}3\rho_1), \qquad r_3 = \tfrac{1}{4}(\rho_1{-}3\rho_2).$$

The momentum cell $C(p_D, \mu)$ is the bounded cell of SubCase 3-1b (see Figure 2). Since the momentum mapping $h = h' : \mathbb{C}^2 \to C(p_D, \mu)$ is surjective and since the eigenvalues of H satisfy $r_2 \le \lambda_1 < r_1$ and $r_3 \le \lambda_2 \le r_2$, it follows that $h_1 = \operatorname{tr} H = \lambda_1{+}\lambda_2$ varies between an infimum of $r_2{+}r_3$ (achieved only at $0 \in \mathbb{C}^2$) and a supremum of $r_1{+}r_2$ (not achieved). Thus, since $s = -24h_1$, the scalar curvature satisfies the bounds

$$-12(\rho_2{-}\rho_1) < s \le 12(\rho_2{+}\rho_1) = s(0).$$

Moreover, since $C(p_D, \mu)$ has only one vertex and neither of its two faces is vertical, it follows that dh_1 vanishes only at 0. Consequently, the equation $s = 0$ defines a smooth hypersurface $S \subset \mathbb{C}^2$. This hypersurface is unbounded because the u_2-axis (i.e., $u_1 = 0$) cuts through the omitted face of $C(p_D, \mu)$.

Since s is bounded, it follows that $g^*_\rho = s^{-2}g_\rho$ is complete on each of the two domains D_+ (where $s > 0$) and D_- (where $s < 0$). The domain D_+ is contractible, while D_- has the homotopy type of a circle. Thus, this one example of a self-dual Kähler metric gives rise to two *distinct* complete, self-dual Hermitian Einstein manifolds.

[18] P. Gauduchon, private communication.

As another interesting example, consider the Bochner-Kähler metric of Sub-Case 4-1, where $r_0 > r_1 > r_2 > r_3$ are chosen so that $r_0{+}r_3 < 0$. Choosing the 'completion' $X_2 \subset \mathbb{C}^2$ obtained by omitting the face $l_2 = 0$, one sees that the domain $D_+ \subset \mathbb{C}^2$ defined by $s > 0$ is bounded, with boundary a smooth compact hypersurface $S \subset \mathbb{C}^2$. Again, the corresponding self-dual Hermitian-Einstein metric g^* is complete on D_+. This case is interesting because (D_+, g^*) exists even when the ratios of the roots r_i are not rational, so that any attempt to 'complete' the corresponding self-dual Kähler metric (D_+, g) to a maximal domain leads inevitably to worse than orbifold singularities, i.e., to a non-Hausdorff complex space.

By considering Case 1, one can construct an example of a self-dual Hermitian-Einstein manifold (M, g^*) that is maximally extended and the corresponding self-dual Kähler metric (M, g) is maximally extended, but such that neither g nor g^* is complete. Neither can be extended because the scalar curvature of g is proper on M and tends to $-\infty$ while the squared norm of the Weyl curvature of g^* is proper on M and tends to $+\infty$.

What is perhaps more interesting are the Case 4-0 examples, which include the weighted projective planes $\mathbb{CP}^{[p_1,p_2,p_3]}$ where $0 < p_1 < p_2 < p_3$ are integers with greatest common divisor equal to 1. For the Bochner-Kähler metric g on this orbifold, the scalar curvature is everywhere positive as long as $p_3 < p_1{+}p_2$ and the corresponding Hermitian Einstein metric has positive Einstein constant. When $p_3 = p_1{+}p_2$, the scalar curvature is positive except at one point (a singular orbifold point) and the corresponding Hermitian Einstein metric has vanishing Einstein constant and is complete on the (orbifold) complement of this point. Finally, when $p_3 > p_1{+}p_2$, the scalar curvature vanishes along a hypersurface $S \subset \mathbb{CP}^{[p_1,p_2,p_3]}$. The complement consists of two open sets $\mathbb{CP}^{[p_1,p_2,p_3]}_{\pm}$ (labeled according to the sign of s), each endowed with a complete Hermitian Einstein metric with negative Einstein constant. One of these two pieces, $\mathbb{CP}^{[p_1,p_2,p_3]}_{-}$, can be 'unfolded' to become a smooth, complete, Hermitian Einstein manifold that is biholomorphic to a bounded domain in $\mathbb{C}^2$, while the other, $\mathbb{CP}^{[p_1,p_2,p_3]}_{+}$, has unremovable orbifold singularities.

Appendix A. Cartan's generalization of Lie's Third Theorem

This appendix is an exposition of the passage [6, Chapter II, §§17–29] from Cartan's work on a generalization of Lie's Third Fundamental Theorem to the 'intransitive case' together with a few comments of an elementary nature designed to extend the applicability of Cartan's results to the smooth category and to a 'semi-global' setting. (In [6], Cartan worked almost entirely in what would now probably be called the category of real-analytic germs.) These results have, in modern times, been incorporated into the theory of local Lie algebras, Lie algebroids, and Lie groupoids. For references and surveys of this modern work the reader might consult [27] and [18]. The point of view taken in this appendix is decidedly not modern; instead we follow Cartan's exposition and development since Cartan's version of the result is more suited for the application in this article.

A.1. **Cartan's problem.** One is given the following data:

(1) a nonempty open set $X \subset \mathbb{R}^s$ (with coordinates $x = (x^a)$ on $\mathbb{R}^s$),
(2) an integer $n \geq 1$, and
(3) functions F^a_i and $C^i_{jk} = -C^i_{kj}$ on X, for $1 \leq i, j, k \leq n$ and $1 \leq a \leq s$.

The goal is to describe the solutions to the following 'realization problem': Find

(1) a manifold N^n,
(2) a coframing $\eta = (\eta^i)$ of N, and
(3) a mapping $h = (h^a) : N \to X \subset \mathbb{R}^s$

satisfying

$$dη^i = \tfrac{1}{2}C^i_{jk}(h)\,\eta^j \wedge \eta^k, \qquad dh^a = F^a_i(h)\,\eta^i\,. \tag{A.1}$$

Example A.1 (Lie's Third Fundamental Theorem)**.** Consider the simple case where the F^a_i are all zero. Then the mapping $h : N \to X$ of any realization must be constant, say $h = \bar{h}$. A necessary condition on the constants $\bar{C}^i_{jk} = C^i_{jk}(\bar{h})$ can then be found by computing the exterior derivatives of the equations

$$d\eta^i = \tfrac{1}{2}\bar{C}^i_{jk}\,\eta^j \wedge \eta^k.$$

These give $0 = \bar{C}^i_{jl}\,d\eta^j \wedge \eta^l$, which, in view of the above relations, can be rewritten (after an index substitution and skewsymmetrization) in the form

$$0 = \tfrac{1}{2}\,\bar{C}^i_{pl}\bar{C}^p_{jk}\,\eta^j \wedge \eta^k \wedge \eta^l = \tfrac{1}{6}\big(\bar{C}^i_{pj}\bar{C}^p_{kl} + \bar{C}^i_{pk}\bar{C}^p_{lj} + \bar{C}^i_{pl}\bar{C}^p_{jk}\big)\,\eta^j \wedge \eta^k \wedge \eta^l.$$

Using the linear independence of the η^i, one derives the *Jacobi conditions*

$$\bar{C}^i_{pj}\bar{C}^p_{kl} + \bar{C}^i_{pk}\bar{C}^p_{lj} + \bar{C}^i_{pl}\bar{C}^p_{jk} = 0$$

as necessary conditions for the existence of a solution to the problem. In other words, any realization (N, η, h) must have h be constant and take values in the locus $X' \subset X$ defined by the equations

$$C^i_{pj}C^p_{kl} + C^i_{pk}C^p_{lj} + C^i_{pl}C^p_{jk} = 0.$$

Conversely, Lie's Third Fundamental Theorem asserts that the Jacobi conditions suffice to ensure the existence of a solution to the realization problem, i.e., if $\bar{h}$ lies in X', then there exists a realization (N, η, h) with $h \equiv \bar{h}$. Moreover, any two realizations assuming the same value $\bar{h}$ are locally equivalent in the obvious sense.

A.2. **Differential conditions in the general case.** Even when the F^a_i are not assumed to be zero, exterior differentiation of the equations (A.1) of a realization (N, η, h) yields a set of necessary conditions on the map $h : N \to X$. Namely, it must satisfy

$$F^b_i(h)\frac{\partial F^a_j}{\partial x^b}(h) - F^b_j(h)\frac{\partial F^a_i}{\partial x^b}(h) = -C^l_{ij}(h)\,F^a_l(h)$$

(which is equivalent to $d(dh^a) = 0$) and

$$\begin{aligned} F^a_j(h)\frac{\partial C^i_{kl}}{\partial x^a}(h) + F^a_k(h)\frac{\partial C^i_{lj}}{\partial x^a}(h) + F^a_l(h)\frac{\partial C^i_{jk}}{\partial x^a}(h)& \\ = -\big(C^i_{mj}(h)C^m_{kl}(h) + C^i_{mk}(h)C^m_{lj}(h) + C^i_{ml}(h)C^m_{jk}(h)\big)& \end{aligned}$$

(which is equivalent to $d(d\eta^i) = 0$). Unless these equations are identities, they place restrictions on the range of h.

A.3. **Cartan's existence theorem.** On the other hand, if the above equations *are* identities on the functions F_i^a and C_{jk}^i, then one might hope to find realizations of (A.1) without placing any further restrictions on the range of h.

In [6], Cartan proved[19] just such a result in the real-analytic category.

Theorem A.1 (Cartan). *Suppose that $X \subset \mathbb{R}^s$ is an open set and suppose that F_i^a and $C_{jk}^i = -C_{kj}^i$ for $1 \le a \le s$ and $1 \le i, j, k \le n$ are real-analytic functions on X that satisfy*

$$F_i^b \frac{\partial F_j^a}{\partial x^b} - F_j^b \frac{\partial F_i^a}{\partial x^b} = -F_l^a C_{ij}^l \tag{A.2}$$

and

$$F_j^a \frac{\partial C_{kl}^i}{\partial x^a} + F_k^a \frac{\partial C_{lj}^i}{\partial x^a} + F_l^a \frac{\partial C_{jk}^i}{\partial x^a} = -\left(C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m\right). \tag{A.3}$$

Then for every $h_0 \in X$, there exists a real-analytic realization (N, η, h) satisfying the structure equations

$$d\eta^i = \tfrac{1}{2} C_{jk}^i(h)\, \eta^j \wedge \eta^k, \qquad dh^a = F_i^a(h)\, \eta^i,$$

and a $p_0 \in N$ for which $h(p_0) = h_0$.

Moreover, this realization is locally unique in the following sense: Given any other real-analytic realization $(\tilde{N}, \tilde{\eta}, \tilde{h})$ satisfying the corresponding structure equations that contains a point $\tilde{p}_0 \in \tilde{N}$ satisfying $\tilde{h}(\tilde{p}_0) = h_0$, there exists a p_0-neighborhood $U \subset N$, a $\tilde{p}_0$-neighborhood $\tilde{U} \subset \tilde{N}$, and a real-analytic diffeomorphism $\phi : \tilde{U} \to U$ so that

$$\phi(\tilde{p}_0) = p_0, \qquad \phi^*(\eta) = \tilde{\eta}, \qquad \textit{and} \quad \phi^*(h) = \tilde{h}.$$

Remark A.2 (A paraphrase). Informally, one can state Cartan's result in the following way: There is a 'solution' of the structure equations (A.1) provided that the exterior derivatives of these equations are identities, i.e., $d^2 = 0$ is a formal consequence of (A.1). A solution is uniquely specified by choosing the values of the 'invariants' $h = (h^a)$ at one point in the domain of the solution.

A.3.1. *Real-analyticity.* The full theorem that Cartan proves in the cited passage is more general than Theorem A.1 and has to do with existence of so-called 'infinite groups' (nowadays called pseudo-groups) satisfying a given set of structure equations. However, Theorem A.1 is all that is needed in this article. Cartan's proof is via the Cartan-Kähler Theorem, which is only valid in the real-analytic category. While the general theorem that Cartan proves really does need real-analyticity, the special case being discussed here as Theorem A.1 can be proved without recourse to the Cartan-Kähler Theorem. Indeed, it can be proved using only the Frobenius Theorem, the Poincaré Lemma, and Lie's Third Fundamental Theorem (the classical one). See the work of Pradines [21, 22] for this development.

Thus, the above theorem (both existence and uniqueness) is actually valid in the smooth category. However, note that, in the case where F and C actually are real-analytic and satisfy (A.2) and (A.3), it follows from Cartan's uniqueness result

[19] It would be more accurate to say that Cartan only outlined the proof of this result. However, the reader knowledgeable about Cartan-Kähler theory will have no trouble supplying the details. Also, while Cartan does not always explicitly state the assumption of real-analyticity, it is clear from context that he intended this assumption to be in force.

that any sufficiently differentiable realization of (A.1) is real-analytic in suitable coordinates.

Example A.3 (Application)**.** In this article, Theorem A.1 will be applied to the equations (2.14). In that case, the functions F and C are polynomial (in fact, either linear or quadratic) in the linear coordinates on $X = \mathbb{R}^s = i\,\mathfrak{u}(n)\oplus\mathbb{C}^n\oplus\mathbb{R}$ (here $s = n^2{+}2n{+}1$). Thus, the realizations are all real-analytic in this case.

A.4. **A coordinate-free reformulation.** Cartan's conditions can be recast into a somewhat more geometric form as follows: Suppose there are given functions F^a_i and $C^i_{jk} = -C^i_{kj}$ on a domain $X \subset \mathbb{R}^s$. Define n vector fields on X by

$$F_i = F^b_i \,\frac{\partial}{\partial x^b}$$

for $1 \le i \le n$. Then (A.2) can be written in terms of the Lie bracket as

$$[F_i, F_j] = -C^k_{ij}\, F_k\,. \tag{A.4}$$

Also, (A.3) can be written as

$$F_jC^i_{kl} + F_kC^i_{lj} + F_lC^i_{jk} = -\big(C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk}\big). \tag{A.5}$$

When the vector fields F_i are linearly independent, (A.5) follows directly from (A.4); it is simply the Jacobi identity for the Lie bracket. However, when the F_i are everywhere linearly dependent (as is the case in the application in this article) the equations (A.5) are not consequences of (A.4).

In (A.4) and (A.5), no explicit reference is made to the coordinates x^a on X. Thus, it makes sense to speak of systems (X, F, C) satisfying (A.4) and (A.5) where X is any smooth manifold, the F_i are smooth vector fields on X, and the C^k_{ij} are smooth functions on X. Such a system (X, F, C) is an example of what has since become known as a *local Lie algebra* [26] or a *Lie algebroid* [27, 18]. The notion of a realization (N, η, h) generalizes as well, with the formula for $d\eta^i$ remaining the same but the formula $dh = F_i\,\eta^i$ now being interpreted as a formula for $dh : TN \to TX$ in the obvious sense. This 'coordinate free' formulation of Cartan's problem will not be needed in this article, so it will not be discussed any further here.

A.5. **The leaves of F.** For each $x \in X$, let $r(x)$ be the dimension of the span of the vectors $\{F_i(x)\}_{1\le i\le n}$. When r is a constant function on X and (A.4) holds, the Frobenius theorem asserts that the vector fields F_i are tangent to a foliation of X whose leaves have dimension r.

In most applications, however, the function r is not constant on X. (Indeed, it is not constant for the system (2.14).) Nevertheless, there is a simple generalization of the Frobenius theorem that does hold whenever (A.4) holds.

Say that a smooth curve $\xi : [a, b] \to X$ is an F-*curve*[20] if there exist smooth functions v^i on $[a, b]$ for which $\xi'(t) = v^i(t)\, F_i\big(\xi(t)\big)$, and say that x_1 and x_2 in X are F-*equivalent* if they can be joined by a smooth F-curve.

The generalized Frobenius theorem says that, if the vector field system F satisfies (A.4), then the F-equivalence class $[x]_F$ of $x \in X$ is a smooth, connected submanifold of X of dimension $r(x)$. It is called the F-*leaf* through x. This generalized notion of a foliation is sometimes known as a *Stefan foliation* in the literature. For further discussion of this singular leaf structure, which is virtually

[20]When r is not constant, this condition is *a priori* stronger than the mere condition that $\xi'(t)$ lie in the span of $\{F_i\big(\xi(t)\big)\}_{1\le i\le n}$ for all $t \in [a, b]$.

the same as the sort of singular leaf structure that one encounters in the theory of Poisson manifolds, see [27].

A.6. **The rank of a realization.** Suppose now that (X, F, C) satisfies (A.2) and (A.3) (or, equivalently, (A.4) and (A.5)).

Then, for any realization (N, η, h) of the structure equations (A.1) with N connected, the map $h : N \to X$ will have its image lie in a single F-leaf $L \subset X$, whose dimension will be $r(N) = r\big(h(p)\big)$ for some (and hence any) $p \in N$. Moreover, the structure equations (A.1) imply that the map $h : N \to L$ has constant rank $r(N)$ and hence is a submersion onto its (open) image in L.

In [6], Cartan assumes these results without proof or remark. It is not clear whether he knew these facts (which, even in the real-analytic case, require argument it seems to me) or merely assumed that he was in some 'generic' case where they held. In any case, he does not make an issue of it.

The integer $r(N)$ will be referred to as the *rank* of the realization (N, η, h).

Example A.4 (Application)**.** For the system (2.14), the dimension of an F-leaf can be as low as 0 or as high as $n(n{+}1)$.

A.7. **The symmetry algebra of a leaf.** Let (X, F, C) satisfy (A.2) and (A.3). Let $L \subset X$ be an F-leaf of rank r, and let (N, η, h) be a realization of the structure equations (A.1) whose image $h(N)$ is an open subset of L. Then by Theorem A.1, given any $\bar h \in h(N)$ and any two points p_1 and p_2 in the fiber $h^{-1}(\bar h) \subset N$, there is a locally defined 'symmetry' of the realization that carries p_1 to p_2. This locally defined symmetry is unique in a neighborhood of p_1.

Cartan might have expressed this fact by saying something like 'the group of symmetries of the system (η, h) acts simply transitively on the fibers of h'. In the modern literature, this sort of vagueness about the domain of the 'group' of 'local symmetries' of such data is usually avoided by giving a more precise statement using the language of (finite-dimensional) pseudo-groups. Rather than introduce this sort of terminology, we will give the corresponding infinitesimal formulation, which is simpler.

Theorem A.2. *If N is connected and simply-connected and (N, η, h) is a realization of* (A.1) *of rank r, then the subset $\mathfrak{h} \subset \mathfrak{X}(N)$ consisting of the vector fields on N whose* (*local*) *flows on N preserve η and h is a Lie algebra of dimension $n{-}r$. Moreover, for any $x \in N$, the evaluation map $e_x : \mathfrak{h} \to T_xN$ is a vector space isomorphism onto the kernel of $h'(x) : T_xN \to \mathbb{R}^s$.*

Up to isomorphism, the Lie algebra $\mathfrak{h}$ depends only on the leaf L that contains $h(N)$. It will be referred to as the *symmetry algebra* of L.

It is useful to note that the symmetry algebra of a leaf L can be computed without actually having to find a realization (N, η, h) with $h(N) \subset L$. In fact, for any $\bar h \in L$, define a skewsymmetric bilinear pairing $[,]_{\bar h} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$[E_i, E_j]_{\bar h} = C^k_{ij}(\bar h)\, E_k\,,$$

where E_i is the standard basis of $\mathbb{R}^n$. Let $\mathfrak{h}_{\bar h} \subset \mathbb{R}^n$ be the subspace that is the kernel of the (surjective) linear map $\lambda_{\bar h} : \mathbb{R}^n \to T_{\bar h}L$ that satisfies $\lambda_{\bar h}(E_i) = F_i(\bar h)$. Then the restriction of $[,]_{\bar h}$ to $\mathfrak{h}_{\bar h}$ defines a Lie algebra structure on $\mathfrak{h}_{\bar h}$. One can verify that, up to isomorphism, this Lie algebra does not depend on the choice of $\bar h \in L$ and that this is indeed the symmetry algebra of L.

A.8. **A semi-global realization.** With these concepts, a 'semi-global' version of Cartan's existence and uniqueness result can be stated. For lack of space, we will not discuss the (relatively straightforward) proof, which, in any case, can be found in the above cited references.

Theorem A.3. *Let* (X, F, C) *satisfy* (A.2) *and* (A.3)*, and let* $L \subset X$ *be an* F*-leaf with symmetry algebra* $\mathfrak{h}$*. Let* H *be a Lie group whose Lie algebra is* $\mathfrak{h}$*.*

Then over any contractible open subset $U \subset L$ *there exists a principal left* H*-bundle* $(h^a) = h : N \to U$ *together with an* H*-invariant coframing* $\eta = (\eta^i)$ *on* N *so that* (N, η, h) *satisfies* (A.1)*. This realization is unique up to isomorphism.*

Simple examples show that existence and/or uniqueness can fail when U has nontrivial homotopy groups. In fact, this is the source of the orbifold singularities encountered in §4.3.3.

When H is abelian, the obstruction to global existence on a leaf L can be formulated as the vanishing of an element of an appropriate cohomology group on L. When H is nonabelian, there is still a cohomological condition, but it takes values in a certain nonabelian cohomology set. Since this refinement will not be needed in this article, it will not be discussed.

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