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EXISTENCE OF BLOW-UP SOLUTIONS IN THE ENERGY SPACE FOR THE CRITICAL GENERALIZED KDV EQUATION

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1. Introduction

In this paper we consider

$$\begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1}$$

for $u_0 \in H^1(\mathbf{R})$. This model is called the *critical* generalized Korteweg–de Vries equation.

Indeed, let us consider the generalized KdV equation, for any integer $p > 1$:

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{2}$$

This kind of problem appears in Physics, for example in the study of waves on shallow water (see Korteweg and de Vries [13]). These equations, with nonlinear Schrödinger equations, are considered as universal models for Hamiltonian systems in infinite dimension. From this Hamiltonian structure, we have formally the two following conservation laws in time:

$$\int u^2(t) = \int u_0^2 \tag{3}$$

and

$$\frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = \frac{1}{2}\int u_{0x}^2 - \frac{1}{p+1}\int u_0^{p+1}. \tag{4}$$

From these conservation laws, H^1 appears as an energy space, so that it is a natural space in which to study the solutions.

Note that $p = 2$ is a special case for equation (2). Indeed, from the integrability theory (see Lax [14]), we have for suitable u_0 (u_0 and its derivatives with fast decay at infinity) an infinite number of conservation laws.

The general question is to understand the dynamics induced by such equations.

Local existence in time of solutions of (2) in the energy space is now well understood; see Kato [10], Ginibre and Tsutsumi [8] for the H^s theory ($s > \frac{3}{2}$), Kenig, Ponce and Vega [11] for the L^2 theory in the case of equation (1) and sharp H^s theory for (2), and Bourgain [3] and [4] for the periodic case.

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In particular, we have the following existence and uniqueness result in $H^1(\mathbf{R})$: for $u_0 \in H^1(\mathbf{R})$, there exist $T > 0$ and a unique maximal solution $u \in C([0,T), H^1(\mathbf{R}))$ of (2) on $[0,T)$, satisfying (3)–(4) for all $0 \le t < T$. Moreover, either $T = +\infty$, or $T < +\infty$, and then $|u(t)|_{H^1} \to +\infty$, as $t \uparrow T$ (see [11], Corollary 2.11 and Corollary 2.12, for the fact that $|u(t)|_{H^1} \to +\infty$, as $t \uparrow T$).

From a variational argument and the Sobolev embedding, it is clear that:

For $p < 5$, we have $T = +\infty$ and $\forall t \in \mathbf{R},\ |u(t)|_{H^1} < C(u_0)$.

For $p > 5$, blow up in finite time, i.e. $T < +\infty$, may occur.

For $p = 5$, from Weinstein [25], we have the following Gagliardo–Nirenberg inequality:

$$\forall v \in H^1(\mathbf{R}), \quad \frac{1}{6}\int v^6 \le \frac{1}{2}\left(\frac{\int v^2}{\int Q^2}\right)^2 \int v_x^2, \tag{5}$$

where $Q(x) = R_1(x) = \frac{3^{1/4}}{\mathrm{ch}^{1/2}(2x)}$ (called the *ground state*) is the solution (up to a sign) of

$$Q_{xx} + Q^5 = Q.$$

Note that the constant is optimal in (5) since the Pohozaev identity yields

$$E(Q) = 0, \quad \text{where} \quad E(v) = \frac{1}{2}\int v_x^2 - \frac{1}{6}\int v^6.$$

If u_0 is such that $|u_0|_{L^2} < |Q|_{L^2}$, then for all $0 \le t < T$,

$$\frac{1}{2}\left(1 - \left(\frac{\int u_0^2}{\int Q^2}\right)^2\right)\int u_x^2(t) \le E_0,$$

where $E_0 = E(u_0)$, and then $u(t)$ is globally defined in time.

Note that existence of singularity in finite time for u (i.e. $T < +\infty$) in the space H^1 is still an open problem for $p \ge 5$. To understand this type of phenomenon, we need a more qualitative approach which includes an understanding of singularity formation.

In a different context, if we consider the nonlinear Schrödinger equation in dimension one (NLSE), for $p > 1$,

$$\begin{cases} i\,u_t = -u_{xx} - |u|^{p-1}u, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{6}$$

and the critical NLSE

$$\begin{cases} i\,u_t = -u_{xx} - |u|^4 u, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{7}$$

then the study of the formation of singularities has been more successful. Indeed, this equation has the same invariants in time (3)–(4). Therefore, variational arguments give the same results for existence of global solutions as for the generalized KdV equation.

Nevertheless, the nonlinear Schrödinger equation has a conformal structure in the following sense: for $p = 5$, if $u(t,x)$ is a solution, then

$$\frac{1}{t^{1/2}} e^{\frac{ix^2}{4t}}\,\overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)$$

is also a solution.

On the one hand, this implies obstruction to global existence for some initial data, for $p \geq 5$. Note that for u_0 of negative energy and of fast decay at infinity, the solution $u(t)$ of (7) blows up in finite time. Indeed, we have the so-called Virial identity which implies that

$$\frac{d^2}{dt^2} \int x^2 |u(t,x)|^2 dx \leq c(p) E_0,$$

where $c(p) > 0$ for $p \geq 5$.

On the other hand, in the case $p = 5$, it gives explicit blow up solutions. Indeed, we have special solutions for equation (7),

$$P(t,x) = e^{ict} R_c(x),$$

where R_c satisfies the equation

$$R_{cxx} + R_c^p = cR_c. \tag{8}$$

Applying the conformal transformation to this special solution, it follows that

$$S(t,x) = \frac{1}{t^{1/2}} e^{-i\frac{c}{t} + i\frac{x^2}{4t}} R_c\left(\frac{x}{t}\right)$$

is also a solution, which blows up at $t = 0$. For more details on this approach, see for example Merle [19], and the references therein.

Note that if we set, for any $t_0 \in \mathbf{R}$, $t_0 \neq 0$,

$$S_{t_0}(x) = c^{1/2}(t_0) S(t_0, xc(t_0)) = e^{-i\frac{c}{t_0} + i\frac{t_0 x^2}{4}} R_c(x),$$

with $c(t) = t$, then

$$S_{t_0}(t,x) = c^{1/2}(t_0) S(t_0 + tc^2(t_0), xc(t_0))$$

is the solution of (7) with S_{t_0} as initial data, by scaling invariance of the equation. In conclusion, for any $t_0 \neq 0$, $S_{t_0}(t,x)$ blows up in finite time, and $S_{t_0}(x) \to R_c(x)$ in H^1 as $t_0 \to 0$ (up to phase). It follows that R_c is unstable and we see from this example that the understanding of the flow close to R_c in a certain sense is linked to the singularity formation.

For the generalized KdV equation (2), special types of solutions which are the only explicit solutions, called *solitons*, play an important role. Indeed, there exist solutions of (2) of the form

$$u(t,x) = R_c(x - ct), \quad \text{where } c > 0,$$

where R_c is defined in (8) or, equivalently, $R_c(x) = \left(\frac{c(p+1)}{2\mathrm{ch}^2\left(\frac{p-1}{2}\sqrt{c}\,x\right)}\right)^{\frac{1}{p-1}}$. Note that $R_c(x) > 0$, $\forall x > 0$, and the set R_c, $c \in \mathbf{R}^+$, is a continuum of traveling waves.

In the subcritical case $p < 5$, it follows for energetic arguments that the solitons are H^1 stable (see Cazenave and Lions [5], Weinstein [27], and Bona, Souganidis and Strauss [2]). In [17] Martel and Merle prove in H^1 the asymptotic completeness of the solitons (see also [21]).

In the supercritical case $p > 5$, in [2] Bona, Souganidis, and Strauss prove, using Grillakis, Shatah, and Strauss [9] type arguments, the H^1 instability of solitons. Numerical simulations (see Bona et al. [1] and references therein) suggest that blow up in finite time occurs for some initial data close to solitons.

In the case $p = 5$ (called the critical case), $Q(x)$, where $Q(x) = R_1(x) = \frac{3^{1/4}}{\mathrm{ch}^{1/2}(2x)}$ (called the *ground state*), is the solution of

$$Q_{xx} + Q^5 = Q.$$

From the Gagliardo–Nirenberg inequality, if u_0 is such that $|u_0|_{L^2} < |Q|_{L^2}$, then $u(t)$ is globally defined and bounded in time.

It was conjectured that there exist blow up solutions of (1) such that $|u_0|_{L^2} \geq |Q|_{L^2}$ (see also the numerical results of [1]). However, unlike NLSE, there is no conformal invariance or variance identity with constant sign which would allow us to have explicit blow up solutions. Note that by energy arguments (similar to the ones for the NLSE), we already know that if blow up in H^1 occurs, then we can at least show a result of concentration of L^2 norm at the blow up time

$$\exists x(t), \quad \text{such that } \forall R > 0,\ \overline{\lim}_{t\to T} \int_{|x-x(t)|\leq R} u^2(t,x)dx \geq \int Q^2.$$

(Note that this result was extended in the H^s case for $s > 0$ in [12].)

One can remark that the flow of equation (1) close to R_c (at least generically) should make precise the flow for all initial data in H^1. Indeed, one can expect that in the case of a global bounded solution $u(t)$,

$$u(t) \sim \sum_i R_{c_i}(x - x_i(t)) + u_R,$$

as $t \to +\infty$, where u_R is a dispersive part, $|u_R(t)|_{L^\infty} \to 0$, as $t \to +\infty$, and $0 < C_1 < c_i < C_2$; in the case of initial data u_0 such that $|u_0|_{L^2}$ is of order $|Q|_{L^2}$, we have

$$u(t) \sim R_c(x - x(t)) + u_R.$$

In the case of a solution blowing up at $t = 0$ (with one blow up point), one can conjecture from the scaling properties that

$$u(t) \sim u^*(x) + R_{c(t)}(x - x(t)) \ \text{ or } \ u(t) \sim u^*(x) + \frac{1}{c^{\frac{1}{2}}(t)} g(\frac{x - x(t)}{c(t)}),$$

where $|g|_{L^2} \geq |Q|_{L^2}$ and $c(t) \to 0$, as $t \to 0$. In this case, $u^* \in H^1$ and corresponds to the regular part of the solution, and $R_c(t)$ or $\frac{1}{c^{\frac{1}{2}}(t)} g(\frac{x-x(t)}{c(t)})$ is the singular part which concentrates in one point a certain part of the L^2 mass. Therefore, from this picture, one can see that the instability of the solitons is linked to blow up in finite time and this points out the importance of understanding the flow close to R_c or $Q^* = \{c_0^{1/4} Q(\sqrt{c_0}(x - x_0))\ ,\ c_0 > 0\}$.

For $p = 5$, a first result in this direction has been established in Martel and Merle [15] by showing that $u(t,x) = Q(x - t)$ is an unstable solution (note that the instability result of Bona et al. [2] does not apply to the critical case). This suggests the existence of blow up solutions close to Q. This result was established in a qualitative way by finding the interior of a parabola as the instability region, and was a consequence of a Virial type identity, energy arguments, and a property of decay of the linearized flow around Q.

In [16], Martel and Merle analyzed in the equation the role of the dispersion in a neighborhood of

$$Q^*_{c_1,c_2} = \{c_0^{1/4} Q(\sqrt{c_0}(x - x_0))\ ,\ c_0 \in (c_1, c_2),\ x_0 \in \mathbf{R}\}.$$

In particular, we proved a rigidity theorem of equation (1) close to $Q^*_{c_1,c_2}$ in the energy space (i.e. a characterization of the soliton) related to the notion of dispersion in L^2.

Liouville property close to $Q^*_{c_1,c_2}$ ($p = 5$). *Let $u_0 \in H^1(\mathbf{R})$ and suppose that the solution $u(t)$ of* (1) *is defined for all time $t \in \mathbf{R}$. Assume that for some $c_1, c_2 > 0$,*

$$\forall t \in \mathbf{R}, \quad c_1 \leq |u(t)|_{H^1} \leq c_2, \tag{9}$$

and there exists $x(t)$ such that $v(t,x) = u(t, x + x(t))$ satisfies

$$\forall \varepsilon_0, \exists R_0 > 0, \forall t \in \mathbf{R}, \quad \int_{|x|>R_0} v^2(t,x)dx \leq \varepsilon_0. \tag{10}$$

There exists $\alpha_0 > 0$, such that if $|u_0 - Q|_{H^1} < \alpha_0$, then there exist λ_0, x_0 such that

$$u(t,x) = \lambda_0^{1/2} Q(\lambda_0(x - x_0) - \lambda_0^3 t).$$

Remark. The classification of entire PDE has been established for some problems. In elliptic problems, the moving plane technique (related to the maximum principle) has been applied successfully to find all positive solutions of problems of the type

$$\Delta u + u^p = 0, \quad u > 0, \quad x \in \mathbf{R}^n.$$

(See Gidas, Ni and Nirenberg [6], and Gigas and Spruck [7].) In the parabolic situation, for the blow up solution of

$$u_t = \Delta u + |u|^{p-1}u,$$

where $u : \mathbf{R}^N \to \mathbf{R}^M$, a Liouville theorem has been established by Merle and Zaag [20] (using blow-up arguments).

In some sense, the preceding property says that if the solution is not a traveling wave, then it has to disperse in L^2. From this result, Martel and Merle in [16] have derived an asymptotic stability property of Q (and of R_c by rescaling).

Asymptotic stability of $Q^*_{c_1,c_2}$. *Let $u_0 \in H^1$, suppose that the solution $u(t)$ of* (1) *is defined for all $t \geq 0$, and assume that for some $c_1, c_2 > 0$,*

$$\forall t \geq 0, \quad c_1 \leq |u(t)|_{H^1} \leq c_2.$$

There exists $\alpha_1 > 0$, such that if $|u_0 - Q|_{H^1} < \alpha_1$, then there exist $\lambda(t)$, $x(t)$ such that

$$\lambda^{1/2}(t) u(t, \lambda(t)(x - x(t))) = Q(x) + u_R(t,x),$$

where $u_R(t) \rightharpoonup 0$ in H^1 as $t \to +\infty$.

In this paper, by carefully using the results of [16], the techniques introduced in this paper, and some additional ideas, we are able to prove the existence of a large class of blow-up solutions in the energy space H^1 of equation (1). The result is in some sense a small initial data result (for the blowing up solution). For an $\alpha_0 > 0$, assume that $\int {u_0}^2 < \int Q^2 + \alpha_0$ (note that if $\int {u_0}^2 < \int Q^2$, then blow-up does not occur). Then if in addition we suppose that the energy of the initial data is nonpositive (which is expected to be the standard blow-up criterion for equation (1)), then the solution of equation (1) blows up in finite time or infinite time. We claim the following theorem.

Theorem (Blow-up results for the critical KdV equation)**.** *There exists* $\alpha_0 > 0$ *such that the following property is true. Let* $u_0 \in H^1(\mathbf{R})$*, and let* $u(t)$ *be the solution of* (1)*. Assume that*

$$E(u_0) < 0 \quad \textit{and} \quad \int {u_0}^2 < \int Q^2 + \alpha_0;$$

then the solution $u(t)$ *blows up in* H^1 *in finite or infinite time.*

Remark. Since we have $E(Q) = 0$ from Pohozaev's identity, and $\nabla E(Q) = -Q$ by direct calculation, we have produced a large class of blow-up solutions (open set) close to soliton. Note that the smallness condition is reasonable from the physical point of view since the generic behavior of a blow-up solution is conjectured to be a local perturbation of the function Q (up to scaling and translation) plus some residual mass far in space from it.

Remark. From the proof of the Theorem, we will recover the fact that the blow-up solution is concentrated in L^2 at the blow-up time. In particular, blow-up occurs in H^s for $s > 0$. However, it is an open problem to show that the blow-up occurs in finite time.

The proof of such a result is not a nonexistence proof given by global obstruction for existence for all time of a solution of equation (1), such as a result of this type in PDE. It is in some sense one of the first results in the direction of understanding a blow-up mechanism in the Hamiltonian context (except in the case where explicit invariance of the equation yields explicit blow-up solutions). It decomposes the mechanism of blow-up of such an equation relating the nonlinear dynamic and the mechanism of dispersion in some sense. The proof shows in fact, that the blow-up is a combination of two effects:

- a process of ejection of mass at infinity in some suitable coordinates system,
- the conservation of the energy.

In section 2, we will explain the strategy of the proof of the main result. In section 3, we prove some basic estimates and establish some relations related to the equation. Section 4 will be devoted to the proof of the main Theorem. The author thanks Stanford University where part of this work was done.

2. Strategy of the proof

2.1. **Parameterization of the problem.** Note that R_c is a family of traveling waves which have the same L^2 norm and energy. Thus the conservation laws are not an obstruction to the following strategy: construct blow-up solutions such that for all time there are $c(t)$ and $x(t)$, such that

$$|R_{c(t)}(x - x(t)) - u(t)|_{L^2}$$

is uniformly small in time. Variational arguments give such a property. Consider now an initial data $u_0 \in H^1(\mathbf{R})$ such that

$$E(u_0) < 0 \quad \text{and} \quad \int {u_0}^2 < \int Q^2 + \alpha_0,$$

for $\alpha_0 > 0$ small enough, and let u be the solution of (1).

We then use the structure of the equation around Q (or $Q_c = c^{1/4}Q(c^{1/2}x)$), which allows us to do explicit calculations on the flow, following the different directions of instability using modulation theory as in [15]. To be more precise, let

$$v(t,y) = \lambda^{1/2}(t)u(t,\lambda(t)y + x(t)) \tag{11}$$

and

$$\varepsilon(t,y) = v(t,y) - Q(y) = \lambda^{1/2}(t)u(t,\lambda(t)y + x(t)) - Q(y), \tag{12}$$

for u a solution of (1), and for $\lambda(t) > 0$ and $x(t)$ two C^1 functions to be chosen later. If we change the time variable as follows:

$$s = \int_0^t \frac{dt'}{\lambda^3(t')} \quad \text{or, equivalently,} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \tag{13}$$

then $\varepsilon(s)$ satisfies, for $s \geq 0$, $y \in \mathbf{R}$,

(14)

$$\begin{aligned} \varepsilon_s \quad &= (L\varepsilon)_y + \frac{\lambda_s}{\lambda}\left(\frac{Q}{2} + yQ_y\right) + \left(\frac{x_s}{\lambda} - 1\right)Q_y \\ &+ \frac{\lambda_s}{\lambda}\left(\frac{\varepsilon}{2} + y\varepsilon_y\right) + \left(\frac{x_s}{\lambda} - 1\right)\varepsilon_y - (10Q^3\varepsilon^2 + 10Q^2\varepsilon^3 + 5Q\varepsilon^4 + \varepsilon^5)_y, \end{aligned}$$

where

$$L\varepsilon = -\varepsilon_{xx} + \varepsilon - 5Q^4\varepsilon = -\varepsilon_{xx} + \varepsilon - \frac{15}{\mathrm{ch}^2(2x)}\varepsilon. \tag{15}$$

(See Lemma 1 in [15].) Recall that $x(t)$ and $\lambda(t)$ are geometrical parameters related to the two invariances of equation (1) (respectively, translation and dilatation invariance), and that the operator L is a classical operator (see for example Titchmarsh [24]).

If, for all $t \geq 0$, $u(t)$ is sufficiently close to Q in H^1, up to translation and scaling, we can define $s \to (\lambda(s), x(s))$ such that

$$\forall s \geq 0, \quad \int Q^3\varepsilon(s) = \int Q_y\varepsilon(s) = 0.$$

A relation between λ, x and their derivatives and ε is given later (see section 3). Recall also that by the invariance of equation (1), we can assume $\lambda(0) = 1$ and $x(0) = 0$, so that $u_0 = Q + \varepsilon(0)$ (see the beginning of section 5 in [15]). The reason we choose such orthogonality conditions on $\varepsilon(s)$ is the fact that by Lemma 2 in [15], we have

$$(LQ^3, Q^3) < 0, \quad (LQ_y, Q_y) = 0,$$

$$\forall \varepsilon \in H^1(\mathbf{R}), \quad \text{if} \int Q^3\varepsilon = \int Q_x\varepsilon = 0, \quad \text{then} \quad (L\varepsilon,\varepsilon) \geq (\varepsilon,\varepsilon),$$

where $(L\varepsilon,\varepsilon)$ is a quantity related to the energy of the solution.

2.2. **Ideas for the proof of the Theorem.** We claim now that under the assumptions of the Theorem, the solution $u(t)$ has to blow-up in finite time or infinite time in H^1. That is, for T finite or infinite, we have

$$|u(t)|_{H^1} \to +\infty \text{ or, equivalently, } \lambda(t) \to 0 \text{ as } t \to T.$$

We argue by contradiction. We assume that $u(t)$ is defined for all $t > 0$ and, for a sequence $t_n \to +\infty$, we have that $|u(t_n)|_{H^1} \le C(u_0)$ for a constant $C(u_0)$. It corresponds in the variable s to a sequence $s_n \to +\infty$ such that for a constant $\lambda(u_0) > 0$,

$$\lambda(s_n) \ge \lambda(u_0).$$

The first idea is to use in various ways that the Airy equation (the linear part of the generalized KdV equation) pushes the mass on the left side, and that the nonlinear soliton travels to the right, which means that, in some sense, linear and nonlinear effects are decoupled. From this fact, we introduce a notion of the L^2 local norm of $\varepsilon(s)$ and see it is a monotonic function of s. Then the assumption on $\lambda(s_n)$ and energy identities imply that the L^2 local norm of $\varepsilon(s)$ goes to the limit as s goes to infinity which is not zero. Consider now a limit object as the time goes to infinity. From the recurrence in time of this object, we are able to show that it satisfies a surprising decay property in space. Indeed, in this problem related to the oscillatory integral, we obtain a decay in space related to the elliptic problem.

Let us consider $\widehat{\varepsilon}_0 \in H^1(\mathbf{R})$ and $\widehat{\lambda}_0 > 0$ such that

$$\varepsilon(s_n) \rightharpoonup \widehat{\varepsilon}_0 \quad \text{in } H^1 \quad \text{and} \quad \lambda(s_n) \to \widehat{\lambda}_0, \tag{16}$$

and also the associated functions $\widehat{\varepsilon}(s)$, $\widehat{u}(t)$ with $(-\widehat{T}_1, \widehat{T}_2)$ the maximal time existence interval. Note that, in particular,

$$E(\widehat{u}(t)) < 0.$$

Delicate estimates involving front type estimates and monotonicity of mass imply

$$\forall x \in \mathbf{R}, \quad |\widehat{\varepsilon}(t,x)| \le c(\delta_0)^{\frac{1}{4}} e^{-c_2 \widehat{\lambda}_0(t)|x|}.$$

This is the key part of the proof which says that, in fact, independent of the oscillation in time of the function $\varepsilon(s)$ (compared to the case [16]) the L^2 compactness in a certain sense $\widehat{u}(t)$ will imply some exponential estimates on $\widehat{u}(t)$ on both sides.

Now, we see that the exponential estimates will give a rigidity in the time oscillation of $\widehat{u}(t)$. Indeed, these imply that

$$\widehat{u}(t) \in L^1.$$

In L^1, equation (1) has a third invariant which is the following:

$$\forall t \in (-\widehat{T}_1, \widehat{T}_2), \quad \int \widehat{u}(t,x)dx = \int \widehat{u}(0,x)dx.$$

Using the exponential estimates again together with this invariant, we obtain that $\widehat{u}(t)$ is defined for all time and

$$\forall t \in \mathbf{R}, \quad \widehat{\lambda}_1 \le \widehat{\lambda}(t) \le \widehat{\lambda}_2,$$

for some $\widehat{\lambda}_2 > \widehat{\lambda}_1 > 0$.

Since the solution $\widehat{u}(t)$ is L^2 compact, it follows from the Liouville theorem obtained in a regular regime by Martel and Merle in [16] that

$$\forall s \in \mathbf{R}, \quad \widehat{\varepsilon}(s) = 0,$$

which is a contradiction with the energy condition on the function $\widehat{u}$ since $E(Q) = 0$. (Note that only parts A, B of [16] are used.) This concludes the proof.

3. Some qualitative properties of the solutions

In this section, we consider $u_0 \in H^1(\mathbf{R})$ such that

$$E(u_0) < 0 \quad \text{and} \quad \alpha(u_0) = \int {u_0}^2 - \int Q^2 < \alpha_1,$$

where α_1 is a constant to be chosen later. For α_1 small enough, let us give some properties of the solution $u(t)$ of equation (1).

3.1. Decomposition of the solution and related variational structures.

Let us start with a classical lemma of proximity of the solution up to scaling and translation factors to the function Q related to the variational structure of Q and the energy condition. We then deduce from this result a decomposition of the solution involving translation and size parameters.

Lemma 1 (Variational estimates on u)**.** *Let $u \in H^1(\mathbf{R})$. There exists a $\delta_1 > 0$ such that the following property is true. Assume that $E(u) < 0$ and $\alpha(u) < \delta_1$; then there exist some parameters $x_0 \in \mathbf{R}$, $\lambda_0 > 0$ and $\epsilon_0 \in \{1, -1\}$ such that*

$$|Q - \epsilon_0 \lambda_0^{1/2} u(\lambda_0(x + x_0))|_{H^1(\mathbf{R})} < \delta(\alpha(u)),$$

where $\delta(\alpha(u)) \to 0$ as $\alpha(u)$ goes to 0.

Proof of Lemma 1. Recall from the variational characterization of the function Q (following from the Gagliardo–Nirenberg inequality) that, for $u \in H^1(\mathbf{R})$, we have

$$E(u) = 0, \qquad \int u^2 = \int Q^2, \qquad \int {u_x}^2 = \int {Q_x}^2$$

is equivalent to

$$u = \epsilon_0 Q(\cdot + x_0) \quad \text{for } \epsilon_0 = 1, -1 \text{ and } x_0 \in \mathbf{R}.$$

Arguing by contradiction, assume that there is a sequence $u_n \in H^1(\mathbf{R})$ such that

$$E(u_n) < 0 \quad \text{and} \quad \int {u_n}^2 \to \int Q^2 \quad \text{as } n \text{ goes to infinity.}$$

Consider now $v_n = \lambda_n^{1/2} u_n(\lambda_n x)$, where $\lambda_n = \frac{|Q_x|_{L^2}}{|u_{nx}|_{L^2}}$. We have the following properties for v_n:

$$\int {v_n}^2 \to \int Q^2 \quad \text{and} \quad \int v_{nx}^2 = \int {Q_x}^2 \quad \text{and} \quad E(v_n) < 0.$$

From the Gagliardo–Nirenberg inequality, we then have $E(v_n) \to 0$. Using a classical concentration compactness procedure, we are able to show that there exist $x_n \in \mathbf{R}$ and $\epsilon_n \in \{1, -1\}$ such that $\epsilon_n v_n(x + x_n) \to Q$ in H^1. See for example [18] and [25]. Note that in dimension one the nonvanishing property of the sequence v_n comes from the facts that $|v_n|_{L^\infty} > c_0 > 0$ for n large and that H^1 bounded functions are uniformly continuous. Indeed, a subsequence $v_n(x + x_n) \rightharpoonup V$ in H^1 which satisfies

$$0 < \int V^2 \leq \int Q^2 \ , \ E(V) \leq 0.$$

Thus, $\int V^2 = \int Q^2, E(V) = 0$, and $v_n(x + x_n) \to V$ in H^1. The conclusion follows from the characterization of Q, and this concludes the proof of the lemma.

We are now able to have the following decomposition of the solution $u(t,x)$ of (1) for $\alpha(u_0)$ small enough. Let us first remark that for $\delta(\delta_1) < \frac{|Q|_{L^2}}{4}$, we have that $\epsilon_0(t)$ defined with the function $u(t)$ by Lemma 1 is independent of time. Indeed, let us remark that $\epsilon_0(t)$ is uniquely defined by

$$|\epsilon_0 Q - \lambda_0^{1/2} u(\lambda_0(x+x_0))|_{H^1(\mathbf{R})} < \delta(\delta_1).$$

If not, after rescaling we will have some λ_0 and x_0, such that

$$|Q - \lambda_0 Q(\lambda_0(+x_0))|_{L^2(\mathbf{R})} < \frac{|Q|_{L^2}}{2};$$

thus $|Q|_{L^2(\mathbf{R})} < \frac{|Q|_{L^2}}{2}$, which is a contradiction. Then using the fact that the function $u(t,x)$ is continuous in H^1, it is easy to conclude from the uniqueness of $\epsilon_0(t)$ that $\epsilon_0(t)$ is continuous in time and constant.

Therefore, under the assumptions of the theorem, the function u is close up to scaling and translation parameters Q for all time (or for all time to $-Q$). Since if $u(t,x)$ is a solution of equation (1), then $-u(t,x)$ is also a solution of equation (1), we can alway assume (taking eventually δ_1 smaller and $-u$) from now on that for $\alpha(u_0) < \delta_1$, for all time $u(t)$ is defined and there exist from Lemma 1 some parameters $x_0(t) \in \mathbf{R}$, $\lambda_0(t) > 0$ such that

$$|Q - \lambda_0(t)^{1/2} u(t, \lambda_0(t)x + x_0(t))|_{H^1(\mathbf{R})} < \delta(\alpha(u_0)), \tag{17}$$

where $\delta(\alpha(u_0)) \to 0$ as $\alpha(u_0)$ goes to 0.

Lemma 2 (Decomposition and modulation of u)**.** *There exists a $\delta_2 > 0$ such that if $\alpha(u_0) < \delta_2$, then for all time and for some continuous functions $\lambda(t)$ and $x(t)$ we have that*

$$\varepsilon(t,y) = \lambda^{1/2}(t) u(t, \lambda(t)y + x(t)) - Q(y) \tag{18}$$

satisfies the properties

$$(Q^3, \varepsilon(t)) = (Q_y, \varepsilon(t)) = 0.$$

Moreover, we have that
$|\frac{\lambda_0(t)}{\lambda(t)}| + |x_0(t) - x(t)| + |\varepsilon(t,y)|_{H^1(\mathbf{R})} < \delta(\alpha(u_0))$, *where* $\delta(\alpha(u_0)) \to 0$ *as* $\alpha(u_0) \to 0$.

Proof of Lemma 2. The existence of the decomposition is a consequence of the implicit function theorem (see [15] for more details). For $\alpha > 0$, let

$$U_\alpha = \left\{ u \in H^1(\mathbf{R});\ \ |u - Q|_{H^1} \le \alpha \right\},$$

and for $u \in H^1(\mathbf{R})$, $\lambda_1 \in \mathbf{R}$, and $x_1 \in \mathbf{R}$, with $\lambda_1 > 0$, we define

$$\varepsilon_{\lambda_1, x_1}(y) = \lambda_1^{1/2} u(\lambda_1 y + x_1) - Q. \tag{19}$$

We claim that there exist $\overline{\alpha} > 0$ and a unique C^1 map: $U_{\overline{\alpha}} \to (1 - \overline{\lambda}, 1 + \overline{\lambda}) \times (-\overline{x}, \overline{x})$, such that if $u \in U_{\overline{\alpha}}$, then there is a unique (λ_1, x_1) such that $\varepsilon_{\lambda_1, x_1}$ defined as in (19) is such that

$$\varepsilon_{\lambda_1, x_1} \perp Q^3 \quad \text{and} \quad \varepsilon_{\lambda_1, x_1} \perp Q_x. \tag{20}$$

Moreover, there exists a constant $C_1 > 0$, such that if $u \in U_{\overline{\alpha}}$, then

$$|\varepsilon_{\lambda_1, x_1}|_{H^1} + |\lambda_1 - 1| + |x_1| \le C_1 \overline{\alpha}. \tag{21}$$

Indeed, we define the following functionals:

$$\rho^1_{\lambda_1,x_1}(u) = \int \varepsilon_{\lambda_1,x_1} Q_x \quad \text{and} \quad \rho^2_{\lambda_1,x_1}(u) = \int \varepsilon_{\lambda_1,x_1} Q^3. \tag{22}$$

Since $\frac{\partial \varepsilon_{\lambda_1,x_1}}{\partial x_1}|_{\lambda_1=1,x_1=0} = u_x$ and $\frac{\partial \varepsilon_{\lambda_1,x_1}}{\partial \lambda_1}|_{\lambda_1=1,x_1=0} = \frac{u}{2} + xu_x$, we obtain at the point $(\lambda_1, x_1, u) = (1, 0, Q)$:

$$\frac{\partial \rho^1_{\lambda_1,x_1}}{\partial x_1} = \int Q_x^2 \quad \text{and} \quad \frac{\partial \rho^1_{\lambda_1,x_1}}{\partial \lambda_1} = \int Q_x \left(\frac{Q}{2} + xQ_x\right) = 0,$$

$$\frac{\partial \rho^2_{\lambda_1,x_1}}{\partial x_1} = \int Q_x Q^3 = 0 \quad \text{and} \quad \frac{\partial \rho^2_{\lambda_1,x_1}}{\partial \lambda_1} = \int \left(\frac{Q}{2} + xQ_x\right) Q^3 = \frac{1}{4}\int Q^2.$$

By the implicit function theorem, there exist $\overline{\alpha} > 0$, a neighborhood $V_{1,0}$ of $(1,0)$ in $\mathbf{R}^2$, and a unique C^1 map $(\lambda_1, x_1) : \{u \in H^1;\ |u - Q|_{H^1} < \overline{\alpha}\} \to V_{1,0}$, such that (20) holds. Now, consider $\delta_2 > 0$ such that $\delta(\alpha(u_0)) < \overline{\alpha}$. For all time, there are parameters $x_0(t) \in \mathbf{R}$, $\lambda_0(t) > 0$ such that

$$|Q - \lambda_0(t)^{1/2} u(t, \lambda_0(t)x + x_0(t))|_{H^1(\mathbf{R})} < \overline{\alpha}.$$

Existence and local uniqueness follows from the previous result applied to the function $\lambda_0(t)^{1/2} u(t, \lambda_0(t)x + x_0(t))$. Smallness estimates follow from direct calculations.

Let us now give some properties of the decomposition. We introduce

$$s = \int_0^t \frac{dt'}{\lambda^3(t')} \quad \text{or, equivalently,} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}. \tag{23}$$

The functions ϵ, λ, and x are now functions of s. Let us check that $\{s(t)\} = [0, +\infty)$. On one hand, the fact that the energy is negative and Gagliardo–Nirenberg implies that λ is bounded from above and if u is defined for $t > 0$ then the conclusion follows. If the u blow-up in finite time T, scaling estimates imply that $\lambda(t) \geq c(T-t)^{\frac{1}{3}}$ (if not $v_{t_0}(\tau, x) = (T - t_0)^{\frac{1}{6}} u(t_0 + (T - t_0)\tau, (T - t_0)^{\frac{1}{3}} x)$ is defined for τ greater than a constant by the fact that the Cauchy problem is well-posed in H^1) and again the conclusion follows. We now have the following properties:

Lemma 3 (Properties of the decomposition)**.** *i) (Equation for ϵ, λ, and x.) The function $\varepsilon(s)$ satisfies equation* (14)*, for $s \in \mathbf{R}$ and $y \in \mathbf{R}$.*

Moreover λ and x are C^1 functions of s and

$$\begin{aligned} \frac{\lambda_s}{\lambda} \quad & \left(\frac{1}{4}\int Q^4 - \int y(Q^3)_y \varepsilon\right) - \left(\frac{x_s}{\lambda} - 1\right) \int (Q^3)_y \varepsilon \\ & = \int L((Q^3)_y)\varepsilon - 10 \int (Q^3)_y Q^3 \varepsilon^2 - \int (Q^3)_y R(\varepsilon), \end{aligned} \tag{24}$$

$$\begin{aligned} -\frac{\lambda_s}{\lambda} \int y Q_{yy} \varepsilon + \left(\frac{x_s}{\lambda} - 1\right) & \left(\frac{1}{2}\int Q^2 - \int Q_{yy}\varepsilon\right) \\ & = 20 \int Q^3 Q_y^2 \varepsilon - 10 \int Q_{yy} Q^3 \varepsilon^2 - \int Q_{yy} R(\varepsilon), \end{aligned} \tag{25}$$

where

$$L\varepsilon = -\varepsilon_{xx} + \varepsilon - 5Q^4\varepsilon \quad \textit{and} \quad R(\varepsilon) = 10Q^3\varepsilon^2 + 10Q^2\varepsilon^3 + 5Q\varepsilon^4 + \varepsilon^5.$$

ii) (Smallness properties.) There exists a $\delta_3 > 0$ such that if $\alpha(u_0) < \delta_3$, then, for a constant $C > 0$,

$$\forall s \in [0, \infty), \quad |\varepsilon(s)|_{L^2} \ + \ |\varepsilon_y(s)|_{L^2} \leq C\sqrt{\alpha(u_0)}.$$

Proof of Lemma 3. i) The equation of $\varepsilon(s,y)$ follows from direct substitution in equation (1). Formulas (24) and (25) are obtained formally by multiplying the equation of ε by the functions Q^3 and Q_y, respectively, and integrating by parts. (Regularization arguments can make it rigorous; see [15] for more details).

ii) The result follows from the conservation of energy and mass of the solution of (1). Indeed from (3) and (4), we have the following relations, for all $s \geq 0$ (Lemma 3 (i)-(iii) in [15]):

$$\int Q\varepsilon(s) + \frac{1}{2}\int \varepsilon^2(s) = \frac{1}{2}\{\int (Q+\varepsilon(s))^2 - \int Q^2\} = \frac{\alpha(u_0)}{2}, \tag{26}$$

$$E(Q+\varepsilon(s)) = \lambda^2(s)E_0. \tag{27}$$

Let us show first that for $\alpha(u_0)$ small, ii) is satisfied. By straightforward calculations, we have from (27),

$$\begin{aligned} E(Q+\varepsilon) + \left(\int Q\varepsilon + \frac{1}{2}\int \varepsilon^2\right) &= \frac{1}{2}(L\varepsilon,\varepsilon) \\ &-\frac{1}{6}\left[20\int Q^3\varepsilon^3 + 15\int Q^2\varepsilon^4 + 6\int Q\varepsilon^5 + \int \varepsilon^6\right]. \end{aligned} \tag{28}$$

Therefore, for all s,

$$\left|E(Q+\varepsilon(s)) + \frac{\alpha(u_0)}{2} - \frac{1}{2}(L\varepsilon(s),\varepsilon(s))\right| \leq C|\varepsilon(s)|_{H^1}|\varepsilon(s)|^2_{L^2}, \tag{29}$$

and for $\alpha(u_0)$ small enough (using the estimates on $|\varepsilon(s)|_{H^1}$ of Lemma 2), we have

$$(L\varepsilon(s),\varepsilon(s)) < \alpha(u_0) + E(Q+\varepsilon(s)) + \frac{(\varepsilon(s),\varepsilon(s))}{2} < \alpha(u_0) + \frac{(\varepsilon(s),\varepsilon(s))}{2}. \tag{30}$$

From the spectral properties of the operator L and the fact $(Q^3,\varepsilon(s)) = (Q^3,\varepsilon(s)) = 0$, we have $(\varepsilon(s),\varepsilon(s)) < (L\varepsilon(s),\varepsilon(s))$. Thus, for all s,

$$(\varepsilon(s),\varepsilon(s)) < 2\alpha(u_0) \quad \text{and} \quad (L\varepsilon(s),\varepsilon(s)) < 3\alpha(u_0).$$

From the fact that

$$|\varepsilon(s)|^2_{H^1} = (L\varepsilon(s),\varepsilon(s)) + 5\int Q^4\varepsilon^2(s) \leq (L\varepsilon(s),\varepsilon(s)) + c_1|\varepsilon(s)|^2_{L^2} \leq C(L\varepsilon,\varepsilon),$$

we have the conclusion. This concludes the proof of Lemma 3.

As a corollary, we have

Corollary 1. *There is a constant δ_3 such that for $\alpha(u_0) < \delta_3$ we have for all $s \in [0,\infty)$,*

$$|\frac{\lambda_s}{\lambda} - 4\frac{\int (Q^3)_y\varepsilon}{\int Q^4}| + |(\frac{x_s}{\lambda} - 1) - \frac{40\int Q^3 Q_y^2\varepsilon}{\int Q^2}| < C|\varepsilon|^2_{L^2} \leq C\alpha(u_0).$$

3.2. **Monotonicity of the mass at the left of the soliton.** We claim in this section that in some sense the mass in L^2 of the solution u of equation (1) close to the soliton is in a certain sense a decreasing function of time. Let us define for $K > 0$,

$$\forall x \in \mathbf{R}, \quad \phi(x) = cQ\left(\frac{x}{K}\right),\ \psi(x) = \int_{-\infty}^{x}\phi(y)dy, \quad \text{where} \quad c = \frac{1}{K\int_{-\infty}^{+\infty} Q},$$

so that

$$\forall x \in \mathbf{R}, \quad 0 \le \psi(x) \le 1, \quad \lim_{x\to-\infty} \psi(x) = 0, \quad \lim_{x\to+\infty} \psi(x) = 1. \tag{31}$$

First we have a monotonicity result concerning small solutions in L^2 of the generalized KdV equation. This monotonicity property says that in some sense the mass in L^2 of a small solution cannot travel fast to the right. (Note that the result is not true for a large solution like $u_c(t,x) = Q_c(x - ct)$ which is such that the L^2 norm is independent of c.) Consider a solution z of equation (1) with initial data z_0, and define for $\sigma > 0$,

$$\forall t, \quad \mathcal{I}_\sigma(t) = \int z^2(t,x)\psi(x - \sigma t)dx.$$

Lemma 4 (Monotonicity of $\mathcal{I}$ for small solutions in L^2 of (1))**.** *For any $\sigma > 0$ and $K \ge \sqrt{\frac{2}{\sigma}}$, there is a constant $c(\sigma) > 0$ such that if $|z_0|_{L^2} \le c(\sigma)$, then the function $\mathcal{I}_\sigma$ is nonincreasing in t.*

We now consider the solution $u(t,x)$ of the equation (1) and functions $x(t)$ and $\lambda(t)$ such as in Lemma 2. We claim that the mass of solution around $x(t)$ (at a distance of order $\sup_{t>t_0} \lambda(t)$) and at the right of the soliton is in some sense a decreasing function of t_0. Note that to the right of the soliton the solution is small in L^2 and the dynamic is more linear. This says that, in an L^2 sense, the linear dynamic which moves the mass slower decoupled from the nonlinear dynamic which moves the mass at a faster speed.

From now on fix

$$K = 2\sqrt{3}.$$

Let us define for $x_0 \in \mathbf{R}, t_0 \in \mathbf{R}$ and $\forall t \ge t_0$,

$$\mathcal{I}_{x_0,t_0}(t) = \int z^2(t,x)\psi(x - x(t_0) - x_0 - \frac{1}{4}(x(t) - x(t_0)))dx.$$

Lemma 5 (Almost monotonicity of the mass)**.** *Let $t_0 \in \mathbf{R}$ such that $\lambda(t_0) = 1$ and $0 < \lambda(t) < 1.1$ for all $t \ge t_0$. There exist $\delta_4 > 0$ and $a_0 > 0$ such that $0 < \alpha(u_0) < \delta_4$. Then for a $C > 0$,*

$$\forall x_0 \le -a_0, \quad \forall t \ge t_0, \quad \mathcal{I}_{x_0,t_0}(t) - \mathcal{I}_{x_0,t_0}(t_0) \le Ce^{\frac{x_0}{3}}.$$

Remark. Note that the case for $t > t_0,\ 0 < \lambda(t) < \lambda_0$, can be treated by rescaling the solution. The result was first proved in a regular regime, that is, for $t > t_0,\ \lambda_1 < \lambda(t) < \lambda_2$, where the constants $0 < \lambda_1 < \lambda_2$, by Martel and Merle in [16]. The result up to a singular regime (λ close to zero) is in fact a consequence of techniques introduced in the context of nonlinear Schrödinger equations in [19] to localize the L^2 conservation law (space where the solution is bounded).

Let us prove this lemma.

Lemma 6. *There is a constant $\tilde{c}$ such that for $u \in H^1$ and $a \in \mathbf{R}$,*

i) $|u^2\phi^{\frac{1}{2}}|^2_{L^\infty(x>a)} \le \tilde{c}\ (\int_{x>a} u^2)(\int u_x^2\phi + \int u^2\phi)$,

ii) $|u^2\phi^{\frac{1}{2}}|^2_{L^\infty(x<a)} \le \tilde{c}\ (\int_{x<a} u^2)(\int u_x^2\phi + \int u^2\phi)$,

iii) $|u^2\phi^{\frac{1}{2}}|^2_{L^\infty} \le \tilde{c}\ (\int u^2)(\int u_x^2\phi + \int u^2\phi)$.

Proof. i) (Same arguments apply for ii) and iii)). For $x > a$, we have

$$|u^2\phi^{\frac{1}{2}}(x)| \;<\; \int_{x>a} |2uu'\phi^{\frac{1}{2}}| + \int_{x>a} |\frac{1}{2}u^2\phi^{-\frac{1}{2}}\phi'|.$$

Since $\frac{1}{2}Q_x^2 = \frac{1}{2}Q^2 - \frac{1}{6}Q^6$, we have from the definition of ϕ, $|\phi'| < \phi$, and the Cauchy Schwartz inequality,

$$|u^2\phi^{\frac{1}{2}}|^2_{L^\infty(x>a)} < c(\int_{x>a} |u||u'|\phi^{\frac{1}{2}} + \int_{x>a} u^2\phi^{\frac{1}{2}})^2 < c(\int_{x>a} u^2)(\int u_x^2\phi + \int u^2\phi).$$

Proof of Lemma 4. Note that if $|z_0|_{L^2} \le c(\sigma)$, then $z \in L^\infty([0,+\infty), H^1(\mathbf{R}))$. We recall (Lemma 5 in [15] for example) that if $x \mapsto \varphi(x)$ is a C^3 function such that for a constant C, $|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \le C$, then $t \mapsto \int z^2(t,x)\varphi(x)dx$ is C^1 and

$$\frac{d}{dt}\int z^2(t)\varphi = -3\int z_x^2(t)\varphi' + \int z^2(t)\varphi^{(3)} + \frac{5}{3}\int z^6(t)\varphi'. \tag{32}$$

Therefore, if we denote $x_\sigma = x - \sigma t$ we have

$$\begin{aligned} \mathcal{I}_\sigma{}'(t) = -3\int z_x^2(t)\psi'(x_\sigma) - \sigma\int z^2(t)\psi'(x_\sigma) \\ + \int z^2(t)\psi^{(3)}(x_\sigma) + \frac{5}{3}\int z^6(t)\psi'(x_\sigma). \end{aligned} \tag{33}$$

Note that $\psi''(x) = \phi'(x) = \frac{c}{K}Q_x\left(\frac{x}{K}\right)$ and $\psi^{(3)}(x) = \frac{c}{K^2}Q_{xx}\left(\frac{x}{K}\right)$. Since $Q_{xx} = Q - Q^5 \le Q$, we have

$$\forall x \in \mathbf{R}, \quad \phi''(x) \le \frac{c}{K^2}Q\left(\frac{x}{K}\right) = \frac{1}{K^2}\phi(x). \tag{34}$$

Thus, from Lemma 6 we have

$$\begin{aligned} \mathcal{I}_\sigma{}'(t) &= -3\int z_x^2(t)\phi(x_\sigma) - \sigma\int z^2(t)\phi(x_\sigma) + \int z^2(t)\phi''(x_\sigma) + \frac{5}{3}\int z^6(t)\phi(x_\sigma) \\ &\le -3\int z_x^2(t)\phi(x_\sigma) - \frac{\sigma}{2}\int z^2(t)\phi(x_\sigma) + \frac{5}{3}\int z^6(t)\phi(x_\sigma) \\ &\le -3\int z_x^2(t)\phi(x_\sigma) - \frac{\sigma}{2}\int z^2(t)\phi(x_\sigma) + \frac{5}{3}|z^2(t,x)\phi^{\frac{1}{2}}(x_\sigma)|^2_{L^\infty}\int z^2(t) \\ &\le -3\int z_x^2(t)\phi(x_\sigma) - \frac{\sigma}{2}\int z^2(t)\phi(x_\sigma) + c|z_0|^4_{L^2}(\int z_x^2\phi(x_\sigma) + \int z^2\phi(x_\sigma)). \end{aligned}$$

The conclusion follows for $|z_0|_{L^2}$ small enough and Lemma 4 is proved.

Proof of Lemma 5. Assume $t_0 = 0$. We now define $\tilde{x} = x - x(0) - \frac{1}{4}(x(t) - x(0)) - x_0$. As before, for $t \ge 0$ we have

$$\begin{aligned} \mathcal{I}'_{x_0,t_0}(t) \le -3\int u_x^2(t,x)\phi(\tilde{x}) - \frac{x_t}{4}\int u^2(t,x)\phi(\tilde{x}) \\ + \frac{1}{18}\int u^2(t,x)\phi(\tilde{x}) + \frac{5}{3}\int u^6(t,x)\phi(\tilde{x}). \end{aligned}$$

Let us remark that, from Lemma 3, we have $|\frac{x_s}{\lambda} - 1| \;<\; C|\varepsilon(s)|_{L^2} \;<\; C\sqrt{\alpha(u_0)}$ for all $s \ge 0$. For $\alpha(u_0)$ small enough, we have $\frac{9}{10} < \frac{x_s}{\lambda} < \frac{11}{10}$. Since $x_t = \frac{x_s}{\lambda^3}$, we have

$$\frac{9}{10\lambda^2} < x_t < \frac{11}{10\lambda^2}.$$

In particular, $\frac{x_t}{8} > \frac{9}{80(1.1)^2} > \frac{1}{18}$, and

$$\mathcal{I}'_{x_0,t_0}(t) \leq -3\int u_x^2(t,x)\phi(\tilde{x}) - \frac{1}{18}\int u^2(t,x)\phi(\tilde{x}) + \frac{5}{3}\int u^6(t,x)\phi(\tilde{x}).$$

Here, since u is not small, the way we treat $\int u^6(t,x)\phi(\tilde{x})$ is different from before. We will treat the regions where $|u(t,x)|$ is large compared to $|u(t)|_{L^\infty}$ and where it is small in a different way. We will see that the contribution that makes $\mathcal{I}_{x_0,t_0}$ increase is controlled by a term which is integrable in time, which will allow us to conclude the proof.

Let us consider a_0 and $\alpha(u_0)$ small enough such that

$$16\tilde{c}(\int_{1.1|x|>a_0} Q^2)^2 < \frac{1}{36} \quad \text{and} \quad \int \varepsilon^2(t) < \int_{1.1|x|>a_0} Q^2,$$

where the constant $\tilde{c}$ is defined in Lemma 6. We then have that

$$\int u^6(t,x)\phi(\tilde{x}) = (I) + (II),$$

where

$$(I) = \int_{|x-x(t)|>a_0} u^6(t,x)\phi(\tilde{x}) \quad \text{and} \quad (II) = \int_{|x-x(t)|<a_0} u^6(t,x)\phi(\tilde{x}).$$

On one hand, from Lemma 6,

$$\begin{aligned}
(I) &= \int_{x-x(t)>a_0} u^6(t,x)\phi(\tilde{x}) + \int_{x-x(t)<-a_0} u^6(t,x)\phi(\tilde{x}) \\
&\leq \tilde{c}(\int_{|x-x(t)|>a_0} u^2)\ |u^2\phi^{\frac{1}{2}}|^2_{L^\infty(|x-x(t)|>a_0)} \\
&< \tilde{c}\ (\int_{|x-x(t)|>a_0} u^2)^2(\int u_x^2\phi + \int u^2\phi).
\end{aligned}$$

Since

$$\begin{aligned}
\int_{|x-x(t)|>a_0} u^2 &\leq 2\int_{|x-x(t)|>a_0} \lambda(t)^{-1}Q(\lambda(t)^{-1}(x-x(t)))^2 \\
&\qquad +\lambda(t)^{-1}\varepsilon(\lambda(t)^{-1}(x-x(t)))^2 \\
&\leq 2\int_{1.1|x|>a_0} Q(x)^2 + 2\int \varepsilon^2(t,x) \quad \leq \quad 4\int_{1.1|x|>a_0} Q(x)^2,
\end{aligned}$$

we have

$$(I) \leq 16\tilde{c}(\int_{1.1|x|>a_0} Q(x)^2)^2(\int u_x^2\phi + \int u^2\phi) \leq \frac{1}{36}(\int u_x^2\phi + \int u^2\phi).$$

On the other hand, we have

$$u(t,x) = \lambda^{-1/2}Q(\lambda^{-1}(x-x(t))) + \lambda^{-1/2}\varepsilon(t,\lambda^{-1}(x-x(t))),$$

and

$$|\varepsilon(t)|^4_{L^\infty} \leq |\varepsilon(t)|^2_{L^2}|\varepsilon_x(t)|^2_{L^2} \quad \text{and} \quad |u(t)|^4_{L^\infty} \leq \frac{c}{\lambda(t)^2} \leq cx_t.$$

Therefore,

$$\begin{aligned}(II) \quad &< \quad \int_{|x-x(t)|<a_0} u^6(t,x)\phi(\tilde{x}) \ < \ |u(t)|^2_{L^2}|u(t)|^4_{L^\infty}|\phi(\tilde{x})|_{L^\infty(|x-x(t)|<a_0)} \\ &< \quad cx_t|\phi(\tilde{x})|_{L^\infty(|x-x(t)|<a_0)} \\ &< \quad cx_t\text{Max}_{|x-x(t)|<a_0}\{e^{-\frac{1}{K}|x-x(0)-\frac{1}{4}(x(t)-x(0))-x_0|}\}.\end{aligned}$$

Note that if $x_0 \le -a_0$, then $|x-x(t)| < a_0$ implies from the fact that the function $x(t)$ is increasing in time that $x - x(0) - \frac{1}{4}(x(t) - x(0)) - x_0 \ge \frac{3}{4}(x(t) - x(0)) \ge 0$ and

$$(II) \le cx_te^{-\frac{3}{4K}(x(t)-x(0))+\frac{x_0}{K}}.$$

To conclude, we have $\forall t \ge 0$, $\mathcal{I}'_{x_0,t_0}(t) \le (II) \le Cx_te^{\frac{x_0}{K}}e^{-\frac{3}{4K}(x(t)-x(0))}$. By integration between 0 and t, and by using the exponential decay in time, it follows that

$$\forall t \ge 0, \quad \mathcal{I}_{x_0,t_0}(t) - \mathcal{I}_{x_0,t_0}(t_0) \le Ce^{\frac{x_0}{K}} \le Ce^{\frac{x_0}{32^{\frac{1}{2}}}},$$

where C is a given constant and Lemma 5 is proved.

4. Blow-up results for critical GKDV

In this section, our purpose is to prove the main Theorem.

Proof of the Theorem. Let us consider $u_0 \in H^1(\mathbf{R})$, and $u(t)$ the solution of (1). Assume that

$$E(u_0) < 0 \quad \text{and} \quad \alpha(u_0) = \int {u_0}^2 - \int Q^2 < \alpha_1,$$

where α_1 is a constant to be chosen later. We assume that $\alpha_1 \le \text{Min}_{i=1,4} \quad \delta_i$, where δ_i is defined in section 3. We claim that for α_1 small enough, there exists $T < +\infty$ or $T = +\infty$ such that

$$|u(t)|_{H^1} \to +\infty, \text{ as } t \uparrow T.$$

The proof is divided into several steps. We argue by contradiction. Let us assume that there is a sequence of solutions of (1), u_n, with initial data $u_n(0)$ such that for each given n,

$$E(u_n(0)) < 0, \quad \alpha_n = \alpha(u_n(0)) \to 0 \ \text{ as } n \ \to \infty \ , \ \ u_n(t) \ \text{ is defined for } t \ge 0 \ \text{ and}$$

$$\text{there is } \ t_{n,m} \to \infty \ \text{ as } m \to \infty \text{ and } \ c_n > 0 \ \text{ such that } \ |u_{nx}(t_{n,m})|_{L^2} \le c_n.$$

Note that the constant c_n depends on n. We want to find a contradiction for n large.

Using the asymptotic stability of solitons in the regular regime (bounded oscillations in time), we obtain a contradiction if the solution satisfies such a property (from energy arguments). The problem is to avoid solutions which have large oscillations in time. For this purpose, we consider some asymptotic regime as the time goes to infinity and, on this object, we will prove some estimates which will remove the problem of large oscillations; then we will find a contradiction.

Step 1. *Renormalisation and reduction of the problem.*
We first consider the following asymptotic regime. Define $l_n = \text{liminf}_{t\to\infty}|u_{nx}(t)|_{L^2} < \infty$. Let us remark from energy arguments that l_n cannot be zero. (Indeed, we

have for all time that $\int u_n^6 \geq -6E(u_n)$ and by the Gagliardo–Nirenberg inequality $|u_{nx}(t)|_{L^2} \geq c$, for a $c>0$.)

From the definition of l_n, there is a $\bar{t}_n$ such that

$$|u_{nx}(\bar{t}_n)|_{L^2} \leq l_n(1+\frac{1}{n}) \quad \text{and} \quad \forall t \geq \bar{t}_n, \quad |u_{nx}(t)|_{L^2} \geq l_n(1-\frac{1}{n}).$$

Using the scaling invariance, we consider

$$\bar{u}_n(t,x) \;=\; (\frac{|Q_x|_{L^2}}{l_n})^{\frac{1}{2}} u_n((\frac{|Q_x|_{L^2}}{l_n})^3 t+\bar{t}_n, \frac{|Q_x|_{L^2}}{l_n}x).$$

We have that $\bar{u}_n$ is also a sequence of solutions of (1) with initial data $\bar{u}_n(0)$ such that $E(\bar{u}_n(0))<0$, $\alpha(\bar{u}_n(0))=\alpha(u_n(0))\to 0$ as n goes to $+\infty$, $\bar{u}_n(t)$ is defined for $t\geq 0$, $\forall t>0$, $|\bar{u}_{nx}(t)|_{L^2} \geq (1-\frac{1}{n})|Q_x|_{L^2}$ and there is $\bar{t}_{n,m}\to+\infty$ such that $|\bar{u}_{nx}(\bar{t}_{n,m})|_{L^2}$ goes to $|Q_x|_{L^2}$ as $m\to+\infty$. With no restriction, we can assume that

$$\bar{t}_{n,m+1}-\bar{t}_{n,m}\to\infty \;\text{ as } m \;\to+\infty.$$

From now on, we omit the bar and we consider a sequence of solutions of (1), u_n, with initial data $u_n(0)$ and $t_{n,m}\to\infty$ as m goes to infinity such that

(H1):

(35)
$$E(u_n(0))<0,\;\; \bar{\alpha}_n=\alpha(u_n(0))\to 0\;,\;\; |u_{nx}(0)|_{L^2}\to|Q_x|_{L^2} \quad \text{as } n \text{ goes to } +\infty,$$

(H2):

(36) $$u_n(t) \;\text{ is defined for }\; t\geq 0, \quad |u_{nx}(t)|_{L^2}\geq(1-\frac{1}{n})|Q_x|_{L^2},$$

(H3):

(37) $$|u_{nx}(t_{n,m})|_{L^2}\to|Q_x|_{L^2} \;\text{ and }\; t_{n,m+1}-t_{n,m}\to\infty \quad \text{as } m \;\to+\infty.$$

From assumption $(H2)$ and Lemma 3, for α_1 small and n large, we have

$$\forall n, \quad \forall t\geq 0, \quad \lambda_n(t)\leq 1.1.$$

Under assumptions $(H1)-(H3)$, we are able to define an object at infinity in time for each n. Indeed, we can assume that for a $\tilde{u}_n(0)\in H^1$,

$$u_n(t_{n,m},x+x(t_{n,m}))\rightharpoonup\tilde{u}_n(0) \quad \text{as } m \text{ goes to infinity.}$$

From Lemma 3, taking α_1 small enough and n large we have that $\tilde{u}_n(0)$ is different from zero. We now consider the solution of the equation (1), $\tilde{u}_n(t)$, with initial data $\tilde{u}_n(0)$.

Various properties on $u_n(t)$ will give properties on $\tilde{u}_n(t)$ at the limit, which will be very restrictive for n large and we will see that such a function $\tilde{u}_n(t)$ cannot exist for n large, which will be a contradiction. In particular, we will show that the function $\tilde{u}_n(t)$ satisfies the decay property in the space of exponential type by using the fact that the object is recurrent in time.

Step 2. *First properties of the limit problem as time goes to infinity.* First, let us give some energy type and convergence properties on $\tilde{u}_n(t)$.

Lemma 7 (Energy constrains on $\tilde{u}_n(t)$)**.** *We have the following properties:*

i) For all n, we have $E(\tilde{u}_n(0))\leq E(u_n(0))<0$ and $0<\alpha(\tilde{u}_n(0))\leq\alpha_n$.

ii) $\tilde{u}_n(0)\to Q$ in H^1 as n goes to infinity.

In particular, the function $\tilde{u}_n(t)$ is defined on $(-t_1(n), t_2(n))$ in H^1, for some $t_1(n) > 0$, $t_2(n) > 0$, from the result of [11]. Applying Lemma 2, since $\alpha(\tilde{u}_n(0)) < \delta_2$, then for all $t \in (-t_1(n), t_2(n))$ and for some $\mathcal{C}^1$ functions $\tilde{\lambda}(t)$ and $\tilde{x}(t)$ we have that

$$\tilde{\varepsilon}_n(t, y) = \tilde{\lambda}_n^{1/2}(t)\tilde{u}_n(t, \tilde{\lambda}_n(t)y + \tilde{x}_n(t)) - Q(y) \tag{38}$$

satisfies $(Q^3, \tilde{\varepsilon}_n(t)) = (Q_x, \tilde{\varepsilon}_n(t)) = 0$. We denote by $s(n)$ the time related to t through the change of variable (13). We will then omit the indices for $\lambda, x, \varepsilon, s$.

Proof of Lemma 7. i) If we denote $v_{n,m} = u_n(t_{n,m}, x + x_n(t_{n,m})) \rightharpoonup \tilde{u}_n(0)$ as m goes to infinity, then by energy and mass conservation we have

$$E(v_{n,m}) = E(u_n(0)) < 0 \quad \text{ and } \quad \alpha(v_{n,m}) = \alpha(u_n(0)).$$

On one hand, $\tilde{u}_n(0) \in H^1$. Let us define the function ρ such that

$$0 \le \rho \le 1\ ,\ \rho(x) = 1 \text{ for } |x| \le 1,\ \rho(x) = 0 \text{ for } |x| \ge 2, \text{ and } \sqrt{\rho},\ \sqrt{1-\rho} \in \mathcal{C}^2,$$

and for $k \in \mathbf{N}$, $\rho_k(x) = \rho(\frac{x}{k})$.

Then, by direct calculations and the fact that weak convergence in H^1 implies strong convergence in L^∞_{loc}, we have

$$E(u_n(0)) = E(v_{n,m}) = E(v_{n,m}\sqrt{\rho_k}) + E(v_{n,m}\sqrt{1-\rho_k}) + R_{m,n,k},$$

where $R_{m,n,k} = -\frac{1}{8}\int v_{m,n}^2 {\rho'_k}^2(\frac{1}{\rho_k} + \frac{1}{1-\rho_k}) - \frac{1}{2}\int v_{m,n}^6 \rho_k(1-\rho_k)$. Note that $R_{m,n,k} \to R_{n,k} = -\frac{1}{8}\int \tilde{u}_n^2 {\rho'_k}^2(\frac{1}{\rho_k} + \frac{1}{1-\rho_k}) - \frac{1}{2}\int \tilde{u}_n^6 \rho_k(1-\rho_k)$ as $m \to +\infty$ and $R_{n,k} \to 0$ as $k \to +\infty$. Taking α_1 small enough and for n large, we have $|v_{n,m} - Q|_{L^2} \le \frac{1}{2}|Q|_{L^2}$ for all m using Lemma 1; there is k_0 such that $|v_{n,m}\sqrt{1-\rho_k}|_{L^2} \le |Q|_{L^2}$ for $n \ge n_0$, $k \ge k_0$, and $m \ge 0$. Using Gagliardo–Nirenberg,

$$E(v_{n,m}\sqrt{1-\rho_k}) \ge 0 \text{ and } E(u_n(0)) \ge E(v_{n,m}\sqrt{\rho_k}) + R_{m,n,k}.$$

We pass to the limit as $m \to +\infty$. From the fact that $v_{n,m}\sqrt{\rho_k}$ converges to $\tilde{u}_n(0)\sqrt{\rho_k}$ in L^6, we have for $n \ge n_0,\ k \ge k_0$,

$$E(u_n(0)) \ge E(\tilde{u}_n(0)\sqrt{\rho_k}) + R_{n,k}.$$

Letting k go to infinity, we then obtain

$$E(u_n(0)) \ge E(\tilde{u}_n(0)).$$

On the other hand, let us remark that the Gagliardo–Nirenberg inequality and $E(\tilde{u}_n(0)) < 0$ imply that $\alpha(\tilde{u}_n(0)) > 0$. In addition, from convexity properties

$$\alpha(\tilde{u}_n(0)) \le \text{liminf}_{m\to\infty}\alpha(v_{n,m}) = \alpha(u_n(0)).$$

ii) From assumption $(H1)$, the fact that by definition $|u_{nx}(0)|_{L^2} \to |Q_x|_{L^2}$ as $n \to \infty$, Lemma 1 and property (17) (which pass to the weak limit), we have $\tilde{u}_n(0) \to Q$ in H^1 as n goes to infinity. This conclude the proof of Lemma 7.

Let us recall some stability properties of the weak convergence for solutions of the equation (at least in the region in H^1 where we consider the flow).

Lemma 8 (Stability of the weak convergence)**.** *For all n and, as m goes to infinity, for all $t \in (-t_1(n), t_2(n))$ and $-t_1(n) < -t_1 < t_2 < t_2(n)$, we have*

$$u_n(t_{n,m} + t, x_n(t_{n,m}) + \cdot) \rightharpoonup \tilde{u}_n(t, \cdot) \quad \textit{in } H^1(\mathbf{R}), \tag{39}$$

$$u_n(t_{n,m} + t, x_n(t_{n,m}) + \cdot) \to \tilde{u}_n(t, \cdot) \quad \textit{in } C([-t_1, t_2], L^2_{loc}(\mathbf{R})). \tag{40}$$

Proof of Lemma 8. See Appendix D of [16]. Key to the proof are the fact that the Cauchy problem for equation (1) is well posed in L^2 which contains H^1 and a local virial identity of the type (33) (Lemma 4).

More precisely, for $[-t_1, t_2] \subset (-t_1(n), t_2(n))$, let us consider M such that $|\tilde{u}_n(t)|_{H^1} \leq M$ $\forall t \in [-t_1, t_2]$. It is sufficient to prove (39) and (40) on an interval $[-t_0, t_0]$, with $t_0 = t_0(M) > 0$; then Lemma 8 is obtained by iteration in time. The norms and the estimates introduced in [11] to solve the local Cauchy problem in $H^1(\mathbf{R})$ for equation (1) are in some sense locally in space control by virial type information.

Corollary 2 (Convergence of the geometric parameters)**.** *We have as m goes to infinity, for all n and $-t_1(n) < -t_1 < t_2 < t_2(n)$,*

$$\lambda_n(t_{n,m}+t) \to \tilde{\lambda}_n(t) \quad \textit{and} \quad x_n(t_{n,m}+t) \to \tilde{x}_n(t) \quad \textit{in} \quad C([-t_1, t_2], \mathbf{R}). \tag{41}$$

Proof of Corollary 2. The proof follows directly from Lemma 8, Lemma 3, parts i)-ii), and straightforward calculations (see Appendix D of [16] for more details).

In particular, for all n, $\forall t \in (-t_1(n), t_2(n))$, $\quad \tilde{\lambda}_n(t) \leq 1.1$.

Step 3. *Exponential decay properties of the limit problem.*
Now from this convergence result and properties related to the almost monotonicity of the local mass of $u_n(t)$, we have the two important exponential decay properties on $\tilde{u}_n(t,x)$ for $x > \tilde{x}_n(t)$ and for $x < \tilde{x}_n(t)$. Here, the strategy of the proof will differ from the one of [16]. We derived these estimates as a consequence of the almost monotonicity property of quantity of the type

$$\mathcal{I}_{x_0,t_0}(t) = \int u_n^2(t,x)\psi(x - x_n(t_0) - x_0 - \frac{1}{4}(x_n(t) - x_n(t_0)))dx.$$

From this property and the fact that $\tilde{u}_n(t,x)$ is a recurrent object in time, we obtain an exponential decay property in L^2 uniformly in time (but without smallness). A pointwise exponential decay with smallness is then obtained on $\tilde{\varepsilon}_n$ by interpolation using the Gagliardo–Nirenberg inequality. Note that the method used in [16] gives the optimal decay rate. But it was based on a dichotomy technique for the solution around the soliton between a purely nonlinear part and an interacting one. Exponential decay properties of the linear semigroup give at the limit the desired estimates directly in L^∞. In the situation where the parameter $\lambda(t)$ is not bounded from below (which is our situation), the strategy of the proof in [16] seems to break down. (The approach was too linear.) However, using this other technique where the space L^2 plays an important role, exponential decay can be proved (even if it is not optimal) and is enough to conclude.

Let a_0, $K = 23^{\frac{1}{2}}$, δ_4, and $c_1 = C$ be the constants defined in Lemma 5. We consider $\alpha(u_0) \leq \delta_4$ so that Lemma 5 applies. In addition, we can assume from Corollary 1 and Lemma 3 that

$$\forall t \in (-t_1(n), t_2(n)), \quad \frac{9}{10\lambda^2} < x_t(t) = \frac{x_s}{\lambda^3} < \frac{11}{10\lambda^2}. \tag{42}$$

We first claim

Lemma 9 (L^2 exponential decay at the right of the soliton)**.** *For $x_0 \geq 10a_0$, $\forall t \in (-t_1(n), t_2(n)), \forall n$, we have*

$$|\tilde{u}_n(t, \tilde{x}_n(t) + x)|^2_{L^2(x \geq x_0)} \leq 10c_1 e^{-\frac{x_0}{6}}. \tag{43}$$

Remark that from the proof, for $x_0 \geq 10a_0$, there is $m(x_0)$ such that $\forall t \in (-t_1(n), t_2(n)), \forall m \geq m(x_0)$,

$$|u_n(t_{n,m}+t_0, x_n(t_{n,m}+t_0)+\cdot)|_{L^2(x\geq x_0)} \leq 10c_1 e^{-\frac{x_0}{6}}.$$

Proof of Lemma 9. Assume by contradiction that there exist $t_0 \in (-t_1(n), t_2(n))$ and $x_0 \geq 10a_0$, such that $|\tilde{u}_n(t_0, \tilde{x}_n(t_0)+x)|^2_{L^2(x\geq x_0)} \geq 10c_1 e^{-\frac{x_0}{6}}$. From Lemma 8 and Corollary 2, we have

$$u_n(t_{n,m}+t_0, x_n(t_{n,m}+t_0)+\cdot) \to \tilde{u}_n(t_0,\cdot) \quad \text{in } L^2_{loc}(\mathbf{R}),$$

$$\lambda_n(t_{n,m}+t_0) \to \tilde{\lambda}_n(t_0), \qquad x_n(t_{n,m}+t_0) \to \tilde{x}_n(t_0).$$

In particular for $m \geq m(x_0)$ we have $|u_n(t_{n,m}+t_0, x(t_{n,m}+t_0)+\cdot)|^2_{L^2(x\geq x_0)} \geq 5c_1 e^{-\frac{x_0}{6}}$. We now claim that by solving the equation backwards from $t_{n,m}+t_0$ where m is large to $t=0$ and using this almost monotonicity property of the mass we see a contradiction for m large. From the fact that equation (1) invariant under the transformation $u(t,x)$ gives $u(-t,-x)$, we have that for $\bar{t} \in (0, t_{n,m}+t_0)$,

$$\bar{u}_{n,m}(\bar{t},x) = u_n(t_{n,m}+t_0-\bar{t}, -x+x(t_{n,m}+t_0))$$

is a solution of equation (1). Let us define

$$\bar{I}_{x_0,m}(\bar{t}) = \int \bar{u}^2_{n,m}(\bar{t},x)\psi(x+x_0+\frac{1}{4}(x_n(t_{n,m}+t_0-\bar{t})-x_n(t_{n,m}+t_0)))dx,$$

where the function ψ is defined in the previous section.

On one hand, for $m \geq m(x_0)$, from the properties of the function ψ, the conservation of the L^2 norm and a change of variable, we have

$$\bar{I}_{x_0,m}(0) \leq \int \bar{u}^2_{n,m}(0) - \frac{1}{2}\int_{x\leq -x_0} \bar{u}^2_{n,m}(0),$$

$$\begin{aligned}\bar{I}_{x_0,m}(0) &\leq \int u_n^2(0) - \frac{1}{2}|u_n(t_{n,m}+t_0, x_n(t_{n,m}+t_0)+\cdot)|^2_{L^2(x\geq x_0)} \\ &\leq \int u_n^2(0) - 2c_1 e^{-\frac{x_0}{6}}.\end{aligned}$$

On the other hand, we are able to apply Lemma 5 from (42), and for $m \geq m(x_0)$, we obtain

$$\begin{aligned}\bar{I}_{x_0,m}(t_{n,m}+t_0) &\leq I_{x_0,m}(0) + c_1 e^{-\frac{x_0}{6}} \\ &\leq \int u_n^2(0) + c_1 e^{-\frac{x_0}{6}} - 2c_1 e^{-\frac{x_0}{6}} \leq \int u_n^2(0) - c_1 e^{-\frac{x_0}{6}}.\end{aligned}$$

Since $\bar{I}_{x_0,m}(t_{n,m}+t_0) = \int u_n^2(0, -\theta+x_n(0))\psi(\theta+x_1+\frac{3}{4}x_n(t_{n,m}+t_0)) \to \int u_n^2(0,x)$ as $m\to\infty$, after a change of variable $\theta = x - x_n(t_{n,m}+t_0)+x_n(0)$, where x_1 is fixed, we have at the limit

$$\int u_n^2(0,x) \leq \int u_n^2(0,x) - c_1 e^{-\frac{x_0}{6}},$$

and a contradiction follows. This concludes the proof of Lemma 9.

Again, strongly using the monotonicity properties of $u_n(t)$ and the previous lemma (the L^2 norm is controlled at the right of the soliton), we obtain

Lemma 10 (L^2 exponential decay at the left of the soliton)**.** *For $x_0 \geq 10a_0$, $\forall t \in (-t_1(n), t_2(n)), \forall n$, we have*

$$|\tilde{u}_n(t, \tilde{x}_n(t) + x)|^2_{L^2(x \leq -x_0)} \leq 40c_1 e^{-\frac{x_0}{6}}. \tag{44}$$

Proof of Lemma 10. Again by contradiction, assume that there exist $x_0 \geq 10a_0$ and $t_0 \in (-t_1(n), t_2(n))$ such that

$$|\tilde{u}_n(t, \tilde{x}_n(t_0) + x)|^2_{L^2(x \leq -x_0)} \geq 40c_1 e^{-\frac{x_0}{6}}.$$

As before, $u_n(t_{n,m} + t_0, x_n(t_{n,m} + t_0) + \cdot) \to \tilde{u}_n(t_0, \cdot)$ in $L^2_{loc}(\mathbf{R})$ as $m \to \infty$, and from the proof of Lemma 9 (see the remark above), we have

$$|u_n(t_{n,m} + t_0, x_n(t_{n,m} + t_0) + \cdot)|_{L^2(x \geq x_0)} \leq 10c_1 e^{-\frac{x_0}{6}}.$$

Therefore, there exists $m \geq m(x_0)$ such that we have

$$|u_n(t_{n,m} + t_0, x_n(t_{n,m} + t_0) + \cdot)|^2_{L^2(x \geq -x_0)} \leq |\tilde{u}_n(t, x)|^2_{L^2} - 5c_1 e^{-\frac{x_0}{6}}.$$

We now claim that by solving equation (1) from $t_{n,m(x_0)} + t_0$ to $t_{n,m}$ where m is very large, we find a contradiction with the fact that $u_n(t_{n,m}, x_n(t_{n,m}) + .) \rightharpoonup \tilde{u}_n(0)$ as $m \to \infty$. Define $\bar{I}_{x_0}(t)$ by

$$\int u_n^2(t_{n,m(x_0)} + t_0 + t, x)\psi(x - x_n(t_{n,m(x_0)} + t_0) + x_0 \\ - \frac{1}{4}(x_n(t_{n,m(x_0)} + t_0 + t) - x_n(t_{n,m(x_0)} + t_0)))dx.$$

As before, we have that $\bar{I}_{x_0,m(x_0)}(0) \leq \int \tilde{u}_n^2(0) - 2c_1 e^{-\frac{x_0}{6}}$, and from Lemma 5 for all $m \geq m(x_0)$, $\bar{I}_{x_0}(t_{n,m} - t_{n,m(x_0)} - t_0) \leq \bar{I}_{x_0}(0) + c_1 e^{-\frac{x_0}{6}} \leq \int \tilde{u}_n^2(0) - c_1 e^{-\frac{x_0}{6}}$.

Since $\bar{I}_{x_0}(t_{n,m} - t_{n,m(x_0)} - t_0)$ is equal to (for x_1 constant), it follows that

$$\int u_n^2(t_{n,m}, x)\psi(x + x_1 - \frac{1}{4}x(t_{n,m})) = \int u_n^2(t_{n,m}, x + x(t_{n,m}))\psi(x + x_1 + \frac{3}{4}x(t_{n,m})).$$

Then, at the limit as $m \to \infty$, we have $x(t_{n,m}) \to \infty$ and

$$\int \tilde{u}_n^2(0, x) \leq \int \tilde{u}_n^2(0, x) - c_1 e^{-\frac{x_0}{6}},$$

and a contradiction follows. This concludes the proof of Lemma 10.

As a corollary of these two exponential estimates and the smallness of $\tilde{\varepsilon}_n$ in H^1, we have

Corollary 3 (L^∞ control on $\tilde{\varepsilon}_n$)**.** *For $x_0 \geq x_0(a_0, c_1)$, $\forall t \in (-t_1(n), t_2(n))$, $\forall x \in \mathbf{R}, \forall n$, we have*

i) $|\tilde{u}_n(t, x)|^2_{L^2(|x| \geq x_0)} \leq 50c_1 e^{-\frac{x_0}{6}}$,

ii) $|\tilde{\varepsilon}_n(t, x)| \leq C\alpha(u_n(0))^{\frac{1}{4}} e^{-\frac{\tilde{\lambda}_n(t)|x|}{24}}$.

Proof of Corollary 3. i) follows from Lemmas 9 and 10. For part ii), we have

$$\frac{1}{\tilde{\lambda}_n(t)^{\frac{1}{2}}}\tilde{\varepsilon}_n(t, \frac{x}{\tilde{\lambda}_n(t)}) = \tilde{u}_n(t, \tilde{x}_n(t) + x) - \frac{1}{\tilde{\lambda}_n(t)^{\frac{1}{2}}}Q(\frac{x}{\tilde{\lambda}_n(t)})$$

and $\frac{1}{\tilde\lambda_n(t)^{\frac12}}Q(\frac{x}{\tilde\lambda_n(t)}) \leq \frac{C}{\tilde\lambda_n(t)^{\frac12}}e^{-\frac{1}{2\tilde\lambda_n(t)}}e^{-\frac{|x|}{2}} \leq Ce^{-\frac{|x|}{2}}$, which imply that for $x_0 \geq x_0(a_0, c_1)$,

$$\forall t \in (-t_1(n), t_2(n)), \forall n, \quad |\frac{1}{\tilde\lambda_n(t)^{\frac12}}\tilde\varepsilon_n(t, \frac{x}{\tilde\lambda_n(t)})|^2_{L^2(|x|\geq x_0)} \leq Ce^{-\frac{x_0}{6}},$$

and by the scaling invariance of the L^2 norm, the fact that $\tilde\lambda_n(t) \leq 1.1$, we have

$$|\tilde\varepsilon_n(t,x)|^2_{L^2(|x|\geq y_0)} \leq Ce^{-\frac{\tilde\lambda_n(t)y_0}{12}} \quad \text{for } \; y_0 \geq 2\frac{x_0(a_0,c_1)}{\tilde\lambda_n(t)}.$$

On one hand, from Gagliardo–Nirenberg, estimates on ε_n and Lemma 3, we have

$$|\tilde\varepsilon_n(t,x)|^2_{L^\infty} \leq |\tilde\varepsilon_{nx}(t,x)|_{L^2}|\tilde\varepsilon_n(t,x)|_{L^2} \leq C\alpha(u_n(0))^{\frac14},$$

and the desired estimates hold for $|x_1| \leq 2\frac{x_0(a_0,c_1)}{\tilde\lambda_n(t)}$.

On the other hand, for $|x_1| \geq 2\frac{x_0(a_0,c_1)}{\tilde\lambda_n(t)}$, we have from the proof of Gagliardo–Nirenberg that

$$|\tilde\varepsilon_n(t,x_1)| \leq |\tilde\varepsilon_n(t,x)|^{\frac12}_{L^2(|x|\geq x_1)}|\tilde\varepsilon_{nx}(t,x)|^{\frac12}_{L^2(|x|\geq x_1)} \leq C\alpha(u_n(0))^{\frac14}e^{-\frac{\lambda_n(t)x_1}{24}}.$$

Step 4. *Conclusion of the proof from rigidity properties.*
We are now able to find a contradiction for n large on $\tilde u_n$ (that is, on the asymptotic regime). For n large, the exponential estimates on $\tilde u_n$ will give rigidity on the variation of the norm of the solution $\tilde u_n(t)$ (and $\tilde\lambda_n(t)$) in time through the last invariant of the equation (1) not yet used: the space average of the solution. We then conclude using rigidity of the regular regime ([16])

i) Rigidity on the norm of $\tilde u_n$.
For n large, we claim that

$$t_1(n) = t_2(n) = +\infty \quad \text{ and } \quad \forall t \in \mathbf{R}, \quad \frac12 \leq \tilde\lambda_n(t) \leq 2.$$

Indeed, from Corollary 3, we have that for all n, $t \in (-t_1(n), t_2(n))$, $\tilde u_n(t) \in L^1$. Now, we consider the other invariant of the equation not defined in the energy space (and therefore in general of no use). We have

$$\forall t \in (-t_1(n), t_2(n)), \quad \int_{\mathbf{R}} \tilde u_n(t,x)dx = \int_{\mathbf{R}} \tilde u_n(0,x)dx. \tag{45}$$

This conservation quantity and the smallness of the exponential estimates on $\tilde\varepsilon_n$ allow us to control $\tilde\lambda_n(t)$. From the fact that $\tilde u_n(0) \to Q$ in H^1 as n goes to infinity, we have that $\tilde\lambda_n(0) \to 1$ by continuity arguments. From the invariance in time of the average and the exponential estimates Corollary 3, (45), (12), and the equality $\int(\tilde\varepsilon_n(t,y) + Q(y))dy = \int \tilde\lambda_n^{1/2}(t)\tilde u_n(t, \tilde\lambda_n(t)y + \tilde x_n(t))dy$ imply that for a θ independent of n,

$$|\int Q - \tilde\lambda_n^{-1/2}(t)\int \tilde u_n(t)| \leq \theta\frac{\alpha(u_n(0))^{\frac14}}{\tilde\lambda_n(t)}.$$

Moreover, since $\tilde\lambda_n(0) \to 1$, $\int \tilde u_n(t) = \int \tilde u_n(0) \to \int Q$ as $n \to +\infty$. Therefore, we have for n large enough,

$$|\tilde\lambda_n^{1/2}(t) - 1| \leq \frac{c\alpha(u_n(0))^{\frac14}}{\tilde\lambda_n^{1/2}(t)} + \frac{1}{10} \leq \frac{1}{20\tilde\lambda_n^{1/2}(t)} + \frac{1}{10}.$$

Using the continuity with respect to time of $\tilde{\lambda}_n(t)$ and that $\frac{9}{10} \leq \tilde{\lambda}_n(0) \leq \frac{11}{10}$ for n large yield by apriori estimates that

$$\forall t \in (-t_1(n), t_2(n)), \quad \frac{1}{2} \leq \tilde{\lambda}_n(t) \leq 2.$$

From the fact that the H^1 norm of $\varepsilon_n(s)$ is uniformly small (see Lemma 3), and for n large and constants c_1, c_2 such that $0 < c_1 \leq c_1'$, we have $\tilde{\lambda}_n^{-1}(t) \leq |\tilde{u}_{nx}(t)|_{L^2} \leq c_2'\tilde{\lambda}_n^{-1}(t) \leq c_2$, which implies in particular from the local wellposedness in H^1 of the Cauchy problem for equation (1) that $t_1(n) = t_2(n) = +\infty$, and the desired estimate follows.

ii) Rigidity on $\tilde{u}_n$.

Now a contradiction follows from the Theorem of Liouville of Martel and Merle in [16] which classifies regular regimes close to the function Q up to invariance. Indeed, on one hand for n large, there is c_1, c_2 (independant of n) such that for all time $t \in \mathbf{R}$,

$$c_1 \leq |\tilde{u}_n(t)|_{H^1} \leq c_2. \tag{46}$$

Corollary 3 implies in particular that

$$\forall \varepsilon_0, \exists R_0 > 0, \forall t \in \mathbf{R}, \quad \int_{|x-\tilde{x}_n(t)|>R_0} \tilde{u}_n^2(t,x)dx \leq \varepsilon_0.$$

Since $\tilde{u}_n(0) \to Q$, for n large, we will have that $|\tilde{u}_n(0) - Q|_{H^1} \leq \alpha_0(c_1, c_2)$, where $\alpha_0(c_1, c_2)$ is defined in the Liouville property close to $Q^*_{c_1,c_2}$ (see the introduction). From this result, for n large the function $\tilde{u}_n$ is a soliton: there are $\hat{\lambda}_n$, $\hat{x}_n$ such that

$$\tilde{u}_n(t,x) = \hat{\lambda}_n^{1/2} Q(\hat{\lambda}_n(x - \hat{x}_n) - \hat{\lambda}_n^3 t)$$

and

$$E(\tilde{u}_n(0)) = \hat{\lambda}_n^2 E(Q) = 0.$$

On the other hand, from Lemma 7, we have $E(\tilde{u}_n(0)) < 0$, which is a contradiction. This concludes the proof of the Theorem.

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