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DIRAC COHOMOLOGY, UNITARY REPRESENTATIONS AND A PROOF OF A CONJECTURE OF VOGAN

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1. Introduction

The main result in this paper is a proof of Vogan's conjecture on Dirac cohomology. In the fall of 1997, David Vogan gave a series of talks on the Dirac operator and unitary representations at the MIT Lie groups seminar. In these talks he explained a conjecture which can be stated as follows. Let G be a connected semisimple Lie group with finite center. Let K be the maximal compact subgroup of G corresponding to the Cartan involution θ. Suppose X is an irreducible unitarizable $(\mathfrak{g}, K)$-module. The Dirac operator D acts on $X \otimes S$, where S is a space of spinors for $\mathfrak{p}_0$. Here $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition for the Lie algebra $\mathfrak{g}_0$ of G. The Vogan conjecture says that if D has a non-zero kernel on $X \otimes S$, then the infinitesimal character of X can be described in terms of the highest weight of a $\tilde{K}$-type in $\operatorname{Ker} D$. Here $\tilde{K}$ is a double cover of K corresponding to the group $Spin(\mathfrak{p}_0)$ and hence it acts on $X \otimes S$.

In the most general setting the Dirac operator is defined on a spinor bundle over any smooth manifold with a non-degenerate metric. Its various versions on homogeneous spaces play an important role in representation theory, geometry and topology as well as mathematical physics. For instance, it is well known that the discrete series representations of G can be constructed as kernels of Dirac operators for the corresponding associated twisted spinor bundles ([AS], [H], [P]). The properties of the Dirac operators have been extensively studied and their applications are observed in many branches of mathematics and in theoretical physics. The literature related to the Dirac operator is immense and it is impossible for us to list appropriate references to all important contributions. We merely mention two pieces of recent work known to us: one is Bertram Kostant's work on cubic Dirac operators on equal rank homogeneous spaces, which has applications in physics and topological K-theory [K]; the other is David Vogan's determination of the smallest non-zero eigenvalue of the Laplacian on a locally symmetric space

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by using Parthasarathy's Dirac inequality [V4], which is a fundamental problem in differential geometry and theory of automorphic forms. This far-reaching result was first obtained by Jian-Shu Li by using the dual pair correspondence [L].

We now describe more precisely the conjecture of Vogan which also allows X to be non-unitary. In this case, the kernel of D is replaced by the Dirac cohomology, $\operatorname{Ker} D/(\operatorname{Im} D \cap \operatorname{Ker} D)$; if the Dirac cohomology is non-zero, then the same as before can be concluded about the infinitesimal character of X. In case X is unitary, D is self-adjoint and the Dirac cohomology is just $\operatorname{Ker} D$.

Conjecture (Vogan). *Let X be an irreducible $(\mathfrak{g}, K)$-module, such that the Dirac cohomology is non-zero. Let γ be a $\tilde{K}$-type contained in the Dirac cohomology, where $\tilde{K}$ is the two-fold spin cover of K. Then the infinitesimal character of X is given by $\gamma + \rho_c$. Here ρ_c is the half sum of the compact positive roots.*

For an explanation of how $\gamma + \rho_c$ defines an infinitesimal character, see the discussion before Theorem 2.3. This conjecture was evidenced by W. Schmid's work in [S] in the case when X is the Harish-Chandra module of a discrete series representation. Before that, it was observed for most discrete series representations by Hotta and Parthasarathy in [HP].

Vogan also presented a purely algebraic conjecture that implies the one above. Namely, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ the Clifford algebra of $\mathfrak{p}$. Then one can consider the following version of the Dirac operator:

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \dots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form. Furthermore, one can consider a diagonal embedding of $\mathfrak{k}$ into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, given by the embedding $\mathfrak{k} \to \mathfrak{g} \to U(\mathfrak{g})$ and the map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \to C(\mathfrak{p})$. We denote the image of this diagonal embedding by $\mathfrak{k}_\Delta$. The conjecture now says that every element $z \otimes 1$ of $Z(\mathfrak{g}) \otimes 1 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ can be written as

$$\zeta(z) + Da + bD \tag{1.1}$$

where $\zeta(z)$ is in $Z(\mathfrak{k}_\Delta)$, and $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Our contribution is to introduce a differential d on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ related to D. It turns out that the conjecture follows once we determine the cohomology of this differential. As a matter of fact, we obtain an even stronger result than (1.1), i.e., we show that every element $z \otimes 1$ of $Z(\mathfrak{g}) \otimes 1 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ can be written as

$$\zeta(z) + Da + aD \tag{1.2}$$

where ζ is a homomorphism from $Z(\mathfrak{g})$ to $Z(\mathfrak{k}_\Delta)$, and $a \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Let us also note that besides implying the first conjecture, this conjecture is an analogue for Dirac operators of the Casselman-Osborne theorem about the action of $Z(\mathfrak{g})$ on $\mathfrak{n}$-cohomology; see [CO] or [V1], Theorem 3.1.5. It is also worth mentioning that Vogan's conjecture implies a refinement of the celebrated Parthasarathy's Dirac inequality, which is an extremely useful tool for the classification of irreducible unitary representations of semisimple Lie groups.

Proposition 1.1 (Parthasarathy's Dirac Inequality [P], [VZ])**.** *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Fix a representation of K occurring in X, of highest weight $\mu \in \mathfrak{t}^*$, and a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Write*

$$\rho_c = \rho(\Delta^+(\mathfrak{k})),\ \rho_n = \rho(\Delta^+(\mathfrak{p})).$$

Fix an element $w \in W_K$ such that $w(\mu - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k})$. Then

$$\langle w(\mu - \rho_n) + \rho_c, w(\mu - \rho_n) + \rho_c \rangle \geq \langle \Lambda, \Lambda \rangle.$$

Our main result implies the following theorem.

Theorem 1.2. *The equality in Proposition* 1.1 *holds if and only if some W conjugate of Λ is equal to $w(\mu - \rho_n) + \rho_c$ (and therefore vanishes on $\mathfrak{p}^{\mathfrak{t}}$).*

The paper is organized as follows. In Section 2 we explain in detail the above mentioned conjectures of Vogan, bearing in mind that the content of [V3] is unavailable to most readers. At the end of this section we include some remarks about how to check if the Dirac cohomology is non-zero.

In Section 3 we introduce the differential d, formulate the main result (stating more or less that the cohomology of d is $Z(\mathfrak{k}_\Delta)$), and show how this implies the conjecture.

In Section 4 we prove the main result. The idea is standard: we introduce a filtration and pass to the graded object to calculate the cohomology in the graded setting first. This is easy, and we can also easily come back to the filtered setting.

Section 5 is devoted to calculating the homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$ explicitly. This is done using the fundamental series representations, for which we know both the infinitesimal characters and the Dirac cohomology.

In Section 6 we give a classification of the unitary representations with non-zero Dirac cohomology and strongly regular infinitesimal character. This is similar to the classification of representations with non-zero $(\mathfrak{g}, K)$-cohomology given in [VZ]. The proof can be done by the same technique as in [VZ]. We give a much shorter proof by using Salamanca-Riba's result [SR]. We also give a criterion for a unitary $A_{\mathfrak{q}}(\lambda)$ to have non-zero Dirac cohomology.

Finally in Section 7 we briefly explain the setting of Kostant's cubic Dirac operator from [K], and indicate how our result generalizes to this setting and its consequent topological significance. It was Kostant who pointed out to us the possibility of this generalization. Kostant has announced a paper [K2] which will contain, among other things, these results in more detail.

2. Vogan's conjectures on Dirac cohomology

Let G be a connected real semisimple Lie group with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G. Let θ be a Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Then $\mathfrak{k}_0$ is the Lie algebra of a maximal compact subgroup K of G. Denote by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$; then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

One could also work with a reductive Lie group G with a maximal compact subgroup K, but we prefer the above setting for simplicity.

The Killing form of $\mathfrak{g}_0$ induces a K-invariant inner product on $\mathfrak{p}_0$; let us fix an orthonormal basis $(Z_i; i = 1, \dots, n)$ of $\mathfrak{p}_0$. In particular, we get a map $\gamma : K \to SO(\mathfrak{p}_0)$, whose differential is still denoted by γ.

Consider the algebra $U(\mathfrak{g})\otimes C(\mathfrak{p})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ is the Clifford algebra of $\mathfrak{p}$. The Dirac operator D is an element of this algebra, defined by

$$D=\sum_{i=1}^{n} Z_i\otimes Z_i.$$

It is easy to check that if we change the basis Z_i by an orthogonal transformation, the above expression for D will not change. Thus D does not depend on the choice of the basis Z_i, and moreover it is K-invariant, for the diagonal action of K given by adjoint actions on both factors.

Let X be a $(\mathfrak{g},K)$-module. Let S be a space of spinors for $\mathfrak{p}_0$; it is a complex simple module for $C(\mathfrak{p}_0)$, and hence also for $C(\mathfrak{p})$. In particular, it is a representation of the spin double cover $\tilde{K}$ of K. Namely, $\tilde{K}$ is constructed from the following pullback diagram:

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & Spin(\mathfrak{p}_0) \\ \big\downarrow & & \big\downarrow \\ K & \xrightarrow{\gamma} & SO(\mathfrak{p}_0) \end{array}$$

Since $Spin(\mathfrak{p}_0)$ is constructed as a subgroup of the multiplicative group of invertible elements in $C(\mathfrak{p}_0)$, S is a representation of $Spin(\mathfrak{p}_0)$ and hence also of $\tilde{K}$. It follows that $X\otimes S$ is a $\tilde{K}$-module, where $\tilde{K}$ acts on both factors; on X through K and on S as above.

Furthermore, $X\otimes S$ is a module for the algebra $U(\mathfrak{g})\otimes C(\mathfrak{p})$, with $U(\mathfrak{g})$ acting on X and $C(\mathfrak{p})$ on S. The differential of the $\tilde{K}$-action is the action of a copy of $\mathfrak{k}_0$ diagonally embedded into $U(\mathfrak{g})\otimes C(\mathfrak{p})$ as follows. Recall that there is a Lie algebra map from $\mathfrak{so}(\mathfrak{p}_0)$ into $C(\mathfrak{p}_0)$, given by

$$E_{ij}-E_{ji}\mapsto -\frac{1}{2}Z_iZ_j,$$

where E_{ij} denotes the matrix with all entries equal to 0, except for the ij entry which is equal to 1. Note that this map is K-equivariant. Composing with our $\gamma:\mathfrak{k}_0\to\mathfrak{so}(\mathfrak{p}_0)$, we get a K-equivariant map

$$\alpha:\mathfrak{k}_0\to C(\mathfrak{p}_0).$$

Using this map we embed $\mathfrak{k}_0$ diagonally into $U(\mathfrak{g})\otimes C(\mathfrak{p})$, by

$$\tilde{\alpha}: X\longmapsto X\otimes 1+1\otimes\alpha(X),$$

for $X\in\mathfrak{k}_0$. Clearly, $\tilde{\alpha}$ also defines an embedding of $\mathfrak{k}$ into $U(\mathfrak{g})\otimes C(\mathfrak{p})$. We denote by $\mathfrak{k}_\Delta$ the image of $\mathfrak{k}$ under the map $\tilde{\alpha}$.

Note that $\alpha(\mathfrak{k}_0)$ is contained in the Lie algebra of $Spin(\mathfrak{p}_0)\subset C(\mathfrak{p}_0)$. This immediately implies the following lemma:

Lemma 2.1. *The complexified differential of the $\tilde{K}$-action on $X\otimes S$ corresponds under $\tilde{\alpha}$ to the restriction of the action of $U(\mathfrak{g})\otimes C(\mathfrak{p})$ to the Lie subalgebra $\mathfrak{k}_\Delta$.*

Clearly, the action of $U(\mathfrak{g})\otimes C(\mathfrak{p})$ on $X\otimes S$ is $\tilde{K}$-equivariant. In other words, $X\otimes S$ is a $(U(\mathfrak{g})\otimes C(\mathfrak{p}),\tilde{K})$-module.

The Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ acts on $X \otimes S$; since D is K-invariant, the action of D commutes with the action of $\tilde{K}$ and hence also with the action of $\mathfrak{k}_\Delta$. We define the Dirac cohomology of X to be the $\tilde{K}$-module

$$\operatorname{Ker} D/(\operatorname{Im} D \cap \operatorname{Ker} D).$$

Assume now that X is unitarizable, i.e., there is a positive definite hermitian product $\langle\ ,\ \rangle$ on X, such that every element of $\mathfrak{g}$ acts as a skew hermitian operator on X, and every element of K acts as a unitary operator on X. Let $<,>$ be a hermitian form on S such that every vector in $\mathfrak{p}_0 \subset C(\mathfrak{p})$ acts on S by a skew hermitian operator. The existence of such a hermitian form is shown e.g. in [W], 9.2.3. We define a hermitian form on $X \otimes S$ by

$$\langle v \otimes s, w \otimes s' \rangle = \langle v, w \rangle < s, s' > .$$

It is now clear that every Z_i acts by a skew hermitian operator both on X and on S, and hence we get

Lemma 2.2. *If X is unitarizable, then D is self-adjoint with respect to the hermitian form on $X \otimes S$ defined above.*

In particular, for unitarizable X, $\operatorname{Im} D \cap \operatorname{Ker} D = 0$, and hence the Dirac cohomology is simply $\operatorname{Ker} D$.

Let T be a maximal torus in K, with Lie algebra $\mathfrak{t}_0$. Let $\mathfrak{h}$ be the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$; it is a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Since $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{p}^{\mathfrak{t}}$, we get an embedding of $\mathfrak{t}^*$ into $\mathfrak{h}^*$. Therefore any element of $\mathfrak{t}^*$ determines a character of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Here we are using the standard identification $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ via the Harish-Chandra homomorphism (W is the Weyl group), by which the characters of $Z(\mathfrak{g})$ correspond to the W-orbits in $\mathfrak{h}^*$.

We fix a positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ for $\mathfrak{t}$ in $\mathfrak{k}$; let $\rho_c = \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}))$ be the corresponding half sum of the positive roots. For any finite-dimensional irreducible representation (γ, E_γ) of $\mathfrak{k}$, we denote its highest weight in $\mathfrak{t}^*$ by γ again.

D. Vogan conjectured the following:

Theorem 2.3. *Let X be an irreducible $(\mathfrak{g}, K)$-module, such that the Dirac cohomology is non-zero. Let γ be a $\tilde{K}$-type contained in the Dirac cohomology. Then the infinitesimal character of X is given by $\gamma + \rho_c$.*

In view of the above remarks, in case X is unitarizable, we get the following consequence, which was also conjectured by Vogan:

Corollary 2.4. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module, such that* $\operatorname{Ker} D \neq 0$. *Let γ be a $\tilde{K}$-type contained in* $\operatorname{Ker} D$. *Then the infinitesimal character of X is given by $\gamma + \rho_c$.*

Vogan further reduced the claim of Theorem 2.3 to an entirely algebraic statement in the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Let us first note that the above described map

$$\tilde{\alpha} : \mathfrak{k} \to \mathfrak{k}_\Delta \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

extends to an embedding

$$\tilde{\alpha} : U(\mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

with image equal to $U(\mathfrak{k}_\Delta)$. To see that this is really an embedding, it is enough to note that if $u \in U(\mathfrak{k})$ is a PBW monomial, then $\tilde{\alpha}(u)$ is the sum of $u \otimes 1$ and terms of the form $w \otimes a$, with w having smaller degree than u.

For future use, we note that there is an isomorphism

$$\beta : U(\mathfrak{k}) \otimes 1 \to U(\mathfrak{k}_\Delta);$$

it is the composition of $\tilde{\alpha}$ with the obvious isomorphism $U(\mathfrak{k}) \otimes 1 \to U(\mathfrak{k})$.

We can now state Vogan's algebraic conjecture that implies Theorem 2.3.

Theorem 2.5. *Let $z \in Z(\mathfrak{g})$. Then there is a unique $\zeta(z)$ in the center $Z(\mathfrak{k}_\Delta)$ of $U(\mathfrak{k}_\Delta)$, and there are K-invariant elements $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, such that*

$$z = \zeta(z) + Da + bD.$$

To see that Theorem 2.5 implies Theorem 2.3, let $\tilde{x} \in (X \otimes S)(\gamma)$ be non-zero, such that $D\tilde{x} = 0$ and $\tilde{x} \notin \operatorname{Im} D$. Note that both $z \otimes 1$ and $\zeta(z)$ act as scalars on $\tilde{x}$. The first of these scalars is the infinitesimal character Λ of X applied to z, and the second is the $\mathfrak{k}$-infinitesimal character of γ applied to $\zeta(z)$, that is, $(\gamma + \rho_c)(\zeta(z))$.

On the other hand, since $(z \otimes 1 - \zeta(z))\tilde{x} = Da\tilde{x}$, and $\tilde{x} \notin \operatorname{Im} D$, it follows that $(z \otimes 1 - \zeta(z))\tilde{x} = 0$. Thus the above two scalars are the same, i.e., $\Lambda(z) = (\gamma + \rho_c)(\zeta(z))$.

In Section 5 we will show that under identifications $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W \cong P(\mathfrak{h}^*)^W$ and $Z(\mathfrak{k}_\Delta) \cong Z(\mathfrak{k}) \cong S(\mathfrak{t})^{W_K} \cong P(\mathfrak{t}^*)^{W_K}$ the homomorphism ζ corresponds to the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$. Here the already mentioned inclusion of $\mathfrak{t}^*$ into $\mathfrak{h}^*$ is given by extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$, letting them act by 0 on $\mathfrak{a} = \mathfrak{p}^{\mathfrak{t}}$. This finishes the proof.

We finish this section with the following remark, which indicates how to check if a unitarizable X has non-zero Dirac cohomology.

Proposition 2.6. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume that $X \otimes S$ contains a $\tilde{K}$-type γ, i.e., $(X \otimes S)(\gamma) \neq 0$. Assume further that $||\Lambda|| = ||\gamma + \rho_c||$. Then the Dirac cohomology of X,* Ker D, *contains $(X \otimes S)(\gamma)$. In particular, the Dirac cohomology of X is non-zero.*

Proof. This follows from a formula for the square of D, which is obtained by a direct computation. This formula is stated in Lemma 3.1 in the next section. The formula implies that D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-(||\Lambda||^2 - ||\rho||^2) + (||\gamma + \rho_c||^2 - ||\rho_c||^2) + (||\rho_c||^2 - ||\rho||^2) = 0.$$

By Lemma 2.2, D is self-adjoint. Therefore we have $D = 0$ on $(X \otimes S)(\gamma)$. □

Note that Corollary 2.4 implies the converse of Proposition 2.6: if X is irreducible unitarizable, with Dirac cohomology containing $(X \otimes S)(\gamma)$, then by Corollary 2.4 the infinitesimal character of X is $\Lambda = \gamma + \rho_c$. Hence $||\Lambda|| = ||\gamma + \rho_c||$.

Furthermore, combining Proposition 2.6 with Corollary 2.4, we get the following corollary:

Corollary 2.7. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ, such that $(X \otimes S)(\gamma) \neq 0$. Assume that $||\Lambda|| = ||\gamma + \rho_c||$. Then some W conjugate of Λ is equal to $\gamma + \rho_c$.*

3. The main result

Let us first note that the Clifford algebra $C(\mathfrak{p})$ has a natural $\mathbb{Z}_2$-gradation into even and odd parts:

$$C(\mathfrak{p}) = C^0(\mathfrak{p}) \oplus C^1(\mathfrak{p}).$$

This gradation induces a $\mathbb{Z}_2$-gradation on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in an obvious way.

We define a map d from $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ into itself, as $d = d^0 \oplus d^1$, where

$$d^0 : U(\mathfrak{g}) \otimes C^0(\mathfrak{p}) \to U(\mathfrak{g}) \otimes C^1(\mathfrak{p})$$

is given by

$$d^0(a) = Da - aD, \tag{3.1a}$$

and

$$d^1 : U(\mathfrak{g}) \otimes C^1(\mathfrak{p}) \to U(\mathfrak{g}) \otimes C^0(\mathfrak{p})$$

is given by

$$d^1(a) = Da + aD. \tag{3.1b}$$

In other words, if ϵ_a denotes the sign of a, that is, 1 for even a and -1 for odd a, then $d(a) = Da - \epsilon_a aD$ (for homogeneous a, i.e., those a which have sign).

We will use the following formula for D^2, which can be proved by a straightforward calculation. An analogous formula in a specific representation was first proved by Parthasarathy in [P], Proposition 3.2. See also [W], 9.3.3.

Lemma 3.1. *Let $\Omega_\mathfrak{g}$ be the Casimir operator for $\mathfrak{g}$ (given by $\Omega_\mathfrak{g} = \sum Z_i^2 - \sum W_j^2$, where W_j is an orthonormal basis for $\mathfrak{k}_0$ with respect to the inner product $-B$, where B is the Killing form). Let $\Omega_{\mathfrak{k}_\Delta}$ be the Casimir operator for $\mathfrak{k}_\Delta$ (given by $\Omega_{\mathfrak{k}_\Delta} = \tilde{\alpha}(-\sum W_j^2)$). Then $D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C$, where C is the constant $||\rho_c||^2 - ||\rho||^2$.*

Here ρ_c was defined before Theorem 2.3, and to define ρ, we choose a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ which contains $\Delta^+(\mathfrak{k}, \mathfrak{t})$. In other words, the positive roots for $(\mathfrak{k}, \mathfrak{t})$ are precisely the positive compact roots for $(\mathfrak{g}, \mathfrak{t})$. Finally, the norms on $\mathfrak{t}$ and $\mathfrak{t}^*$ are induced by the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$.

Using Lemma 3.1, we prove that our d induces a differential on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Lemma 3.2. *Let d be the map defined in* (3.1a) *and* (3.1b). *Then*

(i) *d is K-equivariant, hence induces a map from $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ into itself.*

(ii) *$d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.*

Proof. (i) is trivial, since D is K-invariant.

Let $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ be even or odd. Then

$$d^2(a) = d(Da - \epsilon_a aD) = D^2a - \epsilon_{Da} DaD - \epsilon_a(DaD - \epsilon_{aD} aD^2) = D^2a - aD^2,$$

since obviously $\epsilon_{aD} = \epsilon_{Da} = -\epsilon_a$. Using Lemma 3.1, we see that a will commute with D^2 if and only if it commutes with $\Omega_{\mathfrak{k}_\Delta}$. If a is K-invariant, then this clearly holds, as a then commutes with all of $U(\mathfrak{k}_\Delta)$. $\square$

Thus we see that d is a differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, of degree 1 with respect to the above defined $\mathbb{Z}_2$-gradation. Note that we do not have a $\mathbb{Z}$-gradation on

$(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ so that d is of degree 1, i.e., this is not a complex in the usual sense.

We want to calculate the cohomology of d. Before we state the result, let us note the following:

Proposition 3.3. *$Z(\mathfrak{k}_\Delta)$ is in the kernel of d.*

Proof. Since D is K-invariant, it commutes with $\mathfrak{k}_\Delta$, and thus with $U(\mathfrak{k}_\Delta)$ and in particular with $Z(\mathfrak{k}_\Delta)$. Since $Z(\mathfrak{k}_\Delta) \subset (U(\mathfrak{g}) \otimes C^0(\mathfrak{p}))^K$, the claim follows. $\square$

We are ready to state the main result:

Theorem 3.4. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ constructed above. Then $\operatorname{Ker} d = Z(\mathfrak{k}_\Delta) \oplus \operatorname{Im} d$. In particular, the cohomology of d is isomorphic to $Z(\mathfrak{k}_\Delta)$.*

The proof uses the standard method of filtering the algebra (the filtration comes from the usual filtration on $U(\mathfrak{g})$), and then passing to the graded algebra. The analogue of our theorem in the graded setting is easy; the complex we get is closely related to the standard Koszul complex associated to the vector space $\mathfrak{p}$. One can now go back to the original setting by an easy induction on the degree of the filtration.

Before we go into the details of this proof, let us first note a consequence, which immediately proves Vogan's conjecture, Theorem 2.5; just put $b = a$ in Theorem 2.5.

Corollary 3.5. *Let $z \in Z(\mathfrak{g})$. Then there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$, and there is an $a \in (U(\mathfrak{g}) \otimes C^1(\mathfrak{p}))^K$, such that*

$$z \otimes 1 = \zeta(z) + Da + aD.$$

Proof. This follows at once from Theorem 3.4, if we just notice that $z \otimes 1$ commutes with D (indeed, it is in the center of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$), and being even, it is thus in $\operatorname{Ker} d$. Hence, it is of the form $\zeta(z) + d(a) = \zeta(z) + Da + aD$. $\square$

4. The proof of the main result

Let A be an algebra over $\mathbb{C}$ with a filtration

$$0 = F_{-1}A \subseteq F_0 A \subseteq F_1 A \subseteq \cdots$$

such that $F_m A\, F_n A \subseteq F_{m+n} A$. We assume that $\bigcup_n F_n A = A$. We set $\operatorname{Gr}_n A = F_n A / F_{n-1} A$ for $n \geq 0$ and let $\operatorname{Gr} A = \bigoplus_n \operatorname{Gr}_n A$ be the associated graded algebra. We denote by $a \mapsto \bar{a}$ the projection from $F_n A$ to $\operatorname{Gr}_n A$. For $\bar{a} \in \operatorname{Gr}_n A$ and $\bar{b} \in \operatorname{Gr}_m A$, we have $\bar{a}\bar{b} = \overline{ab} \in \operatorname{Gr}_{m+n} A$.

The standard filtration on $U(\mathfrak{g})$ induces a filtration on

$$A = U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

by setting $F_n A = F_n(U(\mathfrak{g}) \otimes C(\mathfrak{p})) = F_n(U(\mathfrak{g})) \otimes C(\mathfrak{p})$.

Note that this filtration is K-invariant. It follows that it induces a filtration on A^K, by $F_n(A^K) = (F_n A)^K$. Clearly, the Dirac operator $D = \sum_i Z_i \otimes Z_i \in F_1 A^K$.

The $\mathbb{Z}_2$-gradation on the Clifford algebra $C(\mathfrak{p})$ induces a $\mathbb{Z}_2$-gradation on A. We set $A^0 = U(\mathfrak{g}) \otimes C^0(\mathfrak{p})$ and $A^1 = U(\mathfrak{g}) \otimes C^1(\mathfrak{p})$. Then $A = A^0 \oplus A^1$. Clearly, this $\mathbb{Z}_2$-gradation is compatible with the above defined filtration.

If $a \in F_nU(\mathfrak{g}) \otimes C^0(\mathfrak{p})$, then $d^0a = Da - aD \in F_{n+1}U(\mathfrak{g}) \otimes C^1(\mathfrak{p})$. If $a \in F_nU(\mathfrak{g}) \otimes C^1(\mathfrak{p})$, then $d^1a = Da + aD \in F_{n+1}U(\mathfrak{g}) \otimes C^0(\mathfrak{p})$. It follows that

$$d^0 \colon F_nA^0 \to F_{n+1}A^1, \quad d^1 \colon F_nA^1 \to F_{n+1}A^0.$$

Thus d^0 and d^1 induce

$$\overline{d^0} \colon \operatorname{Gr}_n A^0 \to \operatorname{Gr}_{n+1} A^1, \quad \overline{d^1} \colon \operatorname{Gr}_n A^1 \to \operatorname{Gr}_{n+1} A^0.$$

If $a \in F_nA^0$, then

$$\overline{d^0}(\bar{a}) = \overline{d^0a} = \overline{Da - aD} = \overline{Da} - \overline{aD} = \bar{D}\bar{a} - \bar{a}\bar{D}.$$

If $a \in F_nA^1$, then $\overline{d^1}(\bar{a}) = \overline{d^1a} = \overline{Da + aD} = \overline{Da} + \overline{aD} = \bar{D}\bar{a} + \bar{a}\bar{D}$. Here $\bar{D} = \sum_i Z_i \otimes Z_i \in (\operatorname{Gr}_1 A^1)^K = (S^1(\mathfrak{g}) \otimes C^1(\mathfrak{p}))^K$.

Therefore $d = d^0 + d^1$ induces

$$\bar{d} = \overline{d^0} + \overline{d^1} \colon \operatorname{Gr} A \to \operatorname{Gr} A.$$

If $\bar{a} = u \otimes Z_{i_1} \cdots Z_{i_k} \in \operatorname{Gr}(U(\mathfrak{g}) \otimes C(\mathfrak{p})) = S(\mathfrak{g}) \otimes C(\mathfrak{p})$, then

$$\begin{aligned}
\bar{d}(\bar{a}) =& \bar{D}\bar{a} - (-1)^k \bar{a}\bar{D} \\
=& \sum_i (Z_i \otimes Z_i)(u \otimes Z_{i_1} \cdots Z_{i_k}) - (-1)^k \sum_i (u \otimes Z_{i_1} \cdots Z_{i_k})(Z_i \otimes Z_i) \\
=& \sum_i [Z_iu \otimes Z_iZ_{i_1} \cdots Z_{i_k} - (-1)^k uZ_i \otimes Z_{i_1} \cdots Z_{i_k}Z_i] \\
=& \sum_{i \neq i_j, \forall j} [Z_iu \otimes Z_iZ_{i_1} \cdots Z_{i_k} - (-1)^k uZ_i \otimes Z_{i_1} \cdots Z_{i_k}Z_i] \\
&+ \sum_{j=1}^{k} [Z_{i_j}u \otimes Z_{i_j}Z_{i_1} \cdots Z_{i_k} - (-1)^k uZ_{i_j} \otimes Z_{i_1} \cdots Z_{i_k}Z_{i_j}] \\
=& \sum_{i \neq i_j} [Z_iu \otimes Z_iZ_{i_1} \cdots Z_{i_k} - Z_iu \otimes Z_iZ_{i_1} \cdots Z_{i_k}] \\
&+ \sum_{j=1}^{k} [(-1)^{j-1} Z_{i_j}u \otimes Z_{i_1} \cdots Z_{i_j}^2 \cdots Z_{i_k} \\
&\qquad - (-1)^k(-1)^{k-j} uZ_{i_j} \otimes Z_{i_1} \cdots Z_{i_j}^2 \cdots Z_{i_k}] \\
=& -2 \sum_{j=1}^{k} (-1)^{j-1} uZ_{i_j} \otimes Z_{i_1} \cdots \widehat{Z_{i_j}} \cdots Z_{i_k}.
\end{aligned}$$

The last equality follows from $Z_{i_j}^2 = -1$.

It follows from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ that

$$S(\mathfrak{g}) = S(\mathfrak{k}) \otimes S(\mathfrak{p}).$$

Then we have

$$S(\mathfrak{g}) \otimes C(\mathfrak{p}) = S(\mathfrak{k}) \otimes S(\mathfrak{p}) \otimes C(\mathfrak{p}).$$

Hence, we have the following vector space isomorphism:

$$S(\mathfrak{g}) \otimes C(\mathfrak{p}) \cong S(\mathfrak{k}) \otimes S(\mathfrak{p}) \otimes \Lambda^{\cdot}(\mathfrak{p}).$$

It follows that $\bar{d} = (-2)\, id \otimes d_{\mathfrak{p}}$, where $d_{\mathfrak{p}}$ is the Koszul differential for the vector space $\mathfrak{p}$. In particular, $\bar{d}$ is a differential on $S(\mathfrak{g}) \otimes C(\mathfrak{p})$, i.e., $(\bar{d})^2 = 0$. Note that

$d_{\mathfrak{p}}$ is exact except at degree 0 and the zeroth cohomology is $\mathbb{C}$, embedded as the constants. It follows that the differential $\bar{d}$ is exact except at degree 0 and the zeroth cohomology is isomorphic to $S(\mathfrak{k})\otimes 1$ embedded in the obvious way. In other words, we see that

$$\operatorname{Ker}\bar{d} = S(\mathfrak{k})\otimes 1 \oplus \operatorname{Im}\bar{d}.$$

Since $\bar{d}$ is K-equivariant, the kernel of $\bar{d}$ on the K-invariants is the same as $(\operatorname{Ker}\bar{d})^K$ which is by the above formula equal to the direct sum of $S(\mathfrak{k})^K\otimes 1$ with the image of $\bar{d}$ on the K-invariants. In other words, we get the following lemma.

Lemma 4.1. *The differential $\bar{d}$ on $(S(\mathfrak{g})\otimes C(\mathfrak{p}))^K$ is exact except at degree 0 and the zeroth cohomology is isomorphic to $S(\mathfrak{k})^K\otimes 1$. More precisely,* $\operatorname{Ker}\bar{d} = S(\mathfrak{k})^K\otimes 1\oplus\operatorname{Im}\bar{d}$.

Now we can prove the main result, Theorem 3.4, i.e.,

$$\operatorname{Ker} d = Z(\mathfrak{k}_\Delta)\oplus\operatorname{Im} d.$$

We only need to show that $a\in\operatorname{Ker} d$ implies that $a\in Z(\mathfrak{k}_\Delta)+\operatorname{Im} d$; the other inclusion follows from Lemma 3.2 and Proposition 3.3, and the fact that the sum is direct is obvious since the sum is direct in Lemma 4.1.

This statement can be proved by induction on the degree of the filtration. Since $F_{-1}(U(\mathfrak{g})\otimes C(\mathfrak{p}))^K = 0$, the statement is true for $a\in F_{-1}A^K$. Assume that the statement is true for all $a\in F_{n-1}A^K$. If $a\in F_nA^K$ and $da=0$, then $\bar{d}(\bar{a}) = \overline{d(a)} = 0$. It follows from Lemma 4.1 that

$$\bar{a} = \bar{z}\otimes 1 + \bar{d}\,\bar{b}$$

for some $\bar{z}\in S(\mathfrak{k})^K$ and $\bar{b}\in\operatorname{Gr}_{n-1}A^K$.

We denote by $\tilde{z}=\sigma(\bar{z})$ the symmetrization of $\bar{z}$ in $U(\mathfrak{k})^K = Z(\mathfrak{k})$. The isomorphism

$$\beta\colon U(\mathfrak{k})\otimes 1\to U(\mathfrak{k}_\Delta)$$

introduced before Theorem 2.5 obviously maps $Z(\mathfrak{k})\otimes 1$ isomorphically onto $Z(\mathfrak{k}_\Delta)$. Let $z=\beta(\tilde{z}\otimes 1)$ and $b=\sigma(\bar{b})$ be the symmetrization of $\bar{b}$ in $F_{n-1}A^K$. Then $a-z-db$ is in $F_{n-1}A^K$, since

$$\overline{a-z-db} = \bar{a}-\bar{z}\otimes 1-\overline{db} = \bar{a}-\bar{z}\otimes 1-\bar{d}\,\bar{b} = 0.$$

We also have

$$d(a-z-db) = da-dz-d^2b = 0-0-0 = 0;$$

namely, $da=0$ by hypothesis, $dz=0$ by Proposition 3.3, and $d^2b=0$ by Lemma 3.2. By the induction hypothesis, $a-z-db = z_1+dc$ for some $z_1\in Z(\mathfrak{k}_\Delta)$ and $c\in F_{n-1}A^K$. It follows that $a = z+z_1+d(b+c)$ is in $Z(\mathfrak{k}_\Delta)+\operatorname{Im} d$. Therefore the statement holds for all $a\in F_nA^K$. This finishes the proof of Theorem 3.4.

5. The fundamental series and the homomorphism ζ

In order to make the action of $z\in Z(\mathfrak{g})$ in Section 2 entirely explicit, we compute the map $\zeta : Z(\mathfrak{g})\to Z(\mathfrak{k}_\Delta)$ explicitly in this section. An outline of this section was suggested to us by Vogan. We first prove that ζ is a homomorphism of algebras.

Lemma 5.1. *The map $\zeta : Z(\mathfrak{g})\to Z(\mathfrak{k}_\Delta)$ defined in Section 2 is a homomorphism of algebras.*

Proof. Let $z_1, z_2 \in Z(\mathfrak{g})$. By Corollary 3.5, there are $a_1, a_2 \in (U(\mathfrak{g}) \otimes C^1(\mathfrak{p}))^K$, such that for $i = 1, 2$, $z_i \otimes 1 = \zeta(z_i) + d(a_i)$. Multiplying these two equations, we get

$$z_1 z_2 \otimes 1 = \zeta(z_1)\zeta(z_2) + d(a_1)\zeta(z_2) + \zeta(z_1)d(a_2) + d(a_1)d(a_2).$$

By Lemma 5.2 below and Proposition 3.3, this is further equal to $\zeta(z_1)\zeta(z_2) + d(a_1\zeta(z_2) + \zeta(z_1)a_2 + a_1 d(a_2))$, and hence the lemma follows from Corollary 3.5. □

Lemma 5.2. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ introduced in Section 3. Then for any homogeneous elements x and y we have*

$$d(xy) = d(x)\, y + \epsilon_x\, x\, d(y).$$

Proof. This follows from a straightforward calculation. □

To compute ζ explicitly, we need to find a collection of "test" representations X with known infinitesimal characters, and Dirac cohomology containing $\tilde{K}$-types γ that range over an algebraically dense set of $\mathfrak{t}^*$. The natural candidates are the fundamental series representations with trivial A-parameter.

Let $H = TA$ be a fundamental Cartan subgroup in G. Let $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be the corresponding θ-stable Cartan subalgebra. Then $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. As usual we drop the subscript 0 for the complexified Lie algebras. Let $X \in i\mathfrak{t}_0$ be such that $\mathrm{ad}(X)$ is semisimple with real eigenvalues. Following [VZ], we define

(i) $\mathfrak{l}$ to be the zero eigenspace of $\mathrm{ad}(X)$,
(ii) $\mathfrak{u}$ to be the sum of positive eigenspaces of $\mathrm{ad}(X)$,
(iii) $\mathfrak{q}$ to be the sum of non-negative eigenspaces of $\mathrm{ad}(X)$.

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a Levi decomposition. Furthermore, $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. We write L for the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. Since $\theta(X) = X$, $\mathfrak{l}$, $\mathfrak{u}$ and $\mathfrak{q}$ are all invariant under θ, so

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}.$$

In particular, $\mathfrak{q} \cap \mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi decomposition

$$\mathfrak{q} \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{u} \cap \mathfrak{k}.$$

We call such a $\mathfrak{q}$ a θ-stable parabolic subalgebra.

Let $\mathfrak{f} \subset \mathfrak{q}$ be any subspace stable under $\mathrm{ad}(\mathfrak{t})$. Then there is a subset $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{t}^*$ and subspaces $\mathfrak{f}_{\alpha_i}$ of $\mathfrak{f}$ such that if $y \in \mathfrak{t}$ and $v \in \mathfrak{f}_{\alpha_i}$, then

$$\mathrm{ad}(y)v = \alpha_i(y)v.$$

We write

$$\Delta(\mathfrak{f}, \mathfrak{t}) = \Delta(\mathfrak{f}) = \{\alpha_1, \ldots, \alpha_r\},$$

the weights or roots of $\mathfrak{t}$ in $\mathfrak{f}$. Here $\Delta(\mathfrak{f})$ is a set with multiplicities, with α_i having multiplicity $\dim \mathfrak{f}_{\alpha_i}$. Then if

$$\rho(\mathfrak{f}) = \rho(\Delta(\mathfrak{f})) = \frac{1}{2} \sum_{\alpha_i \in \Delta(\mathfrak{f}_{\alpha_i})} \alpha_i \in \mathfrak{t}^*,$$

we have

$$\rho(\mathfrak{f})(y) = \frac{1}{2}\, \mathrm{tr}(\mathrm{ad}(y)|_{\mathfrak{t}}) \quad (y \in \mathfrak{t}).$$

Fix a system $\Delta^+(\mathfrak{l}\cap\mathfrak{k})$ of positive roots in the root system $\Delta(\mathfrak{l}\cap\mathfrak{k},\mathfrak{t})$. (Note that we extend the meaning of root system to include the zero weights.) Then

$$\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l}\cap\mathfrak{k}) \cup \Delta(\mathfrak{u}\cap\mathfrak{k})$$

is a positive root system for $\mathfrak{t}$ in $\mathfrak{k}$.

A one-dimensional representation $\lambda\colon \mathfrak{l}\to\mathbb{C}$ is called admissible if it satisfies the following conditions:

(i) λ is the differential of a unitary character of L.

(ii) If $\alpha\in\Delta(\mathfrak{u})$, then $\langle\alpha,\lambda|_{\mathfrak{t}}\rangle\geq 0$.

Given $\mathfrak{q}$ and an admissible λ, define

(5.1) $\mu(\mathfrak{q},\lambda) =$ representation of K of highest weight $\lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p})$.

The following theorem is due to Vogan and Zuckerman.

Theorem 5.3 ([VZ], [V2])**.** *Suppose $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ and $\lambda\colon \mathfrak{l}\to\mathbb{C}$ is admissible as defined above. Then there is a unique unitary $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}(\lambda)$ with the following properties:*

(i) *The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains $\mu(\mathfrak{q},\lambda)$ as defined in* (5.1)*.*

(ii) *$A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda+\rho$.*

(iii) *If the representation of $\mathfrak{k}$ of the highest weight δ occurs in $A_{\mathfrak{q}}(\lambda)$, then*

$$\delta = \mu(\mathfrak{q},\lambda) + \sum_{\beta\in\Delta(\mathfrak{u}\cap\mathfrak{p})} n_\beta\beta$$

with $\mathfrak{n}_\beta$ non-negative integers. In particular, $\mu(\mathfrak{q},\lambda)$ is the lowest K-type of $A_{\mathfrak{q}}(\lambda)$.

We note that the unitarity of $A_{\mathfrak{q}}(\lambda)$ in the above theorem was proved in [V2]. In the context of the definition of θ-stable parabolic subalgebras, if we take X to be a regular element, then we obtain a minimal θ-stable subalgebra $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$. We call such a subalgebra $\mathfrak{b}$ a θ-stable Borel subalgebra. We write Φ^+ for the corresponding system of positive roots. Even though it is not needed in this paper, we note that the corresponding representation $A_{\mathfrak{b}}(\lambda)$ in Theorem 5.3 is the $(\mathfrak{g},K)$-module of a tempered representation of G. If G has a compact Cartan subgroup, then $A_{\mathfrak{b}}(\lambda)$ is the $(\mathfrak{g},K)$-module of a discrete series representation of G. Moreover, all $(\mathfrak{g},K)$-modules of discrete series representations of G are of this form. What we need in this paper is that $A_{\mathfrak{b}}(\lambda)$ have infinitesimal character $\lambda+\rho$ and the lowest K-type $\mu(\mathfrak{b},\lambda)=\lambda+2\rho_n$, where $\rho_n=\rho(\mathfrak{n}\cap\mathfrak{p})$. These facts are contained in Theorem 5.3.

Proposition 5.4. *Let $X=A_{\mathfrak{b}}(\lambda)$ as in Theorem* 5.3 *with $\mathfrak{b}$ a θ-stable Borel subalgebra. Assume that $\lambda|_{\mathfrak{a}}=0$. Then the Dirac cohomology of X contains a $\tilde{K}$-type E_γ of highest weight $\gamma=\lambda+\rho_n$.*

Proof. The lowest K-type $\mu(\mathfrak{b},\lambda)$ has highest weight $\lambda+2\rho_n$. Since $-\rho_n$ is a weight of S, E_γ occurs in $\mu(\mathfrak{b},\lambda)\otimes S$, hence in $X\otimes S$. Since the infinitesimal character of X is $\lambda+\rho=\gamma+\rho_c$, it follows from Proposition 2.6 that E_γ is in the kernel of the Dirac operator D, i.e., in the Dirac cohomology of X. ∎

Now we can describe the homomorphism ζ explicitly.

Theorem 5.5. *The homomorphism ζ satisfies the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{k}) \\ \Big\downarrow & & \Big\downarrow \\ S(\mathfrak{h})^W & \xrightarrow{\mathrm{Res}} & S(\mathfrak{t})^{W_K} \end{array} \tag{5.2}$$

Here the vertical arrows are the Harish-Chandra homomorphisms, and the map Res *corresponds to the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$ under the identifications $S(\mathfrak{h})^W = P(\mathfrak{h}^*)^W$ and $S(\mathfrak{t})^{W_K} = P(\mathfrak{t}^*)^{W_K}$. As before, we can view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$, letting them act by 0 on $\mathfrak{a}$.*

Proof. Let $\bar{\zeta} : P(\mathfrak{h}^*)^W \to P(\mathfrak{t}^*)^{W_K}$ be the homomorphism induced by ζ under the identifications via Harish-Chandra homomorphisms. Furthermore, let $\tilde{\zeta} : \mathfrak{t}^*/W_K \to \mathfrak{h}^*/W$ be the morphism of algebraic varieties inducing the homomorphism $\bar{\zeta}$. We have to show that $\bar{\zeta}$ is the restriction map, or alternatively that $\tilde{\zeta}$ is given by the inclusion map.

We know from Theorem 5.3 that the fundamental series representation $A_{\mathfrak{b}}(\lambda)$ has the lowest K-type

$$\mu(\mathfrak{b}, \lambda) = \lambda + 2\rho_n,$$

and infinitesimal character

$$\Lambda = \lambda + \rho.$$

On the other hand, it follows from Proposition 5.4 that if $\lambda|_{\mathfrak{a}} = 0$, then the Dirac cohomology of $A_{\mathfrak{b}}(\lambda)$ contains the $\tilde{K}$-type of highest weight $\gamma = \lambda + \rho_n$.

When proving that Theorem 2.5 implies Theorem 2.3, we saw that Theorem 2.5 implies $\Lambda(z) = (\gamma + \rho_c)(\zeta(z))$, for all $z \in Z(\mathfrak{g})$. In our present situation we however have

$$\Lambda = \lambda + \rho = (\lambda + \rho_n) + \rho_c = \gamma + \rho_c,$$

so it follows that $\Lambda(\zeta(z)) = \Lambda(z)$ for all $z \in Z(\mathfrak{g})$. This means that $\tilde{\zeta}(\Lambda) = \Lambda$, for all infinitesimal characters Λ of the above fundamental series representations.

It is clear that when λ ranges over all admissible weights in $\mathfrak{h}^*$ such that $\lambda|_{\mathfrak{a}} = 0$, then $\Lambda = \lambda + \rho$ forms an algebraically dense subset of $\mathfrak{t}^*$. To see this, it is enough to note that such λ span a lattice in $\mathfrak{t}^*$. Hence $\tilde{\zeta}$ is indeed the inclusion map. $\square$

Remark 5.6. Both Vogan and Kostant pointed out to us that Theorem 5.5 can also be proved by considering finite-dimensional representations with non-zero Dirac cohomology. Kostant showed us a short proof which we sketch here.

Recall that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ and $\mathfrak{t}$ are orthogonal. Let V_λ be an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$. Then if $\xi \in \mathfrak{t}^*$ is a highest weight for $\mathfrak{k}$, the Dirac operator squared reduces to the scalar

$$||\lambda + \rho||^2 - ||\xi + \rho_{\mathfrak{r}}||^2 \tag{5.3}$$

on the primary component $(V_\lambda \otimes S)(\xi)$. Then for the choice of positive roots arising from a $\mathfrak{g}$-regular hyperbolic element of $\mathfrak{t}$, one has that ρ, ρ_c and ρ_n all lie in $\mathfrak{t}^*$. Now there are an infinite number of choices of λ such that $\lambda \in \mathfrak{t}^*$ (we can simply replace any λ by $\lambda + \theta(\lambda)$, where θ is the Cartan involution). Choose $\xi = \lambda + \rho_n$. Clearly $(V_\lambda \otimes S)(\xi)$ is non-zero. By Proposition 2.6 we see that the vanishing of (5.3) gives rise to Dirac cohomology. It follows that ζ is induced by the Harish-Chandra homomorphism as in (5.2), by the same argument as in the last paragraph of the proof of Theorem 5.5.

6. Unitary representations with non-zero Dirac cohomology

As an application of our main result, in this section we classify irreducible unitary $(\mathfrak{g}, K)$-modules X with non-zero Dirac cohomology, provided that X has strongly regular infinitesimal character. These are $A_{\mathfrak{q}}(\lambda)$-modules, as was proved by S. Salamanca-Riba [SR]. Our classification is analogous to Vogan-Zuckerman's classification of irreducible unitary representations with non-zero $(\mathfrak{g}, K)$-cohomology. We note that a unitary representation with non-zero $(\mathfrak{g}, K)$-cohomology has non-zero Dirac cohomology, but not vice versa. In other words, the Dirac cohomology detects a larger class of irreducible unitary representations. We explain the relations between $(\mathfrak{g}, K)$-cohomology and Dirac cohomology at the end of this section. To classify the irreducible unitary representations which have Dirac cohomology but do not have strongly regular infinitesimal characters is a much more complicated and difficult task, which we hope to achieve later.

Theorem 6.1. *Suppose that $\lambda' \in \mathfrak{t}^*$ is dominant with respect to a positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t}) \subset \Delta^+(\mathfrak{g}, \mathfrak{t})$. Let $\gamma = \lambda' + \rho_n$. Assume that the representation E_γ of $\tilde{K}$ with the highest weight γ is contained in the Dirac cohomology of the irreducible unitary $(\mathfrak{g}, K)$-module X. Then there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an admissible character λ of L such that $\lambda|_{\mathfrak{t}} = \lambda'$ and $X \cong A_{\mathfrak{q}}(\lambda)$.*

This theorem can be proved by the same approach as Vogan-Zuckerman's generalization [VZ] of Kumaresan's result [Ku]. It would take about two to three pages. Here we give a much shorter proof which uses the main result of [SR].

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Given any weight $\Lambda \in \mathfrak{h}^*$, fix a choice of positive roots $\Delta^+(\Lambda, \mathfrak{h})$ for Λ so that

$$\Delta^+(\Lambda, \mathfrak{h}) \subset \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \mathrm{Re}\langle \Lambda, \alpha\rangle \geq 0\}.$$

Set

$$\rho(\Lambda) = \frac{1}{2} \sum_{\alpha \in \Delta^+(\Lambda, \mathfrak{h})} \alpha.$$

Definition 6.2. A weight $\Lambda \in \mathfrak{h}^*$ is said to be real if

$$\Lambda \in \mathfrak{t}_0^* + i\mathfrak{a}_0^*,$$

and to be strongly regular if

$$\langle \Lambda - \rho(\Lambda), \alpha\rangle \geq 0, \quad \forall \alpha \in \Delta^+(\Lambda, \mathfrak{h}).$$

Salamanca-Riba [SR] proved that if an irreducible unitary $(\mathfrak{g}, K)$-module X has strongly regular infinitesimal character, then $X \cong A_{\mathfrak{q}}(\lambda)$ for some θ-stable parabolic subalgebra $\mathfrak{q}$ and admissible character λ of L. Moreover, she proved the following stronger theorem which was conjectured by Vogan.

Theorem 6.3 (Salamanca-Riba)**.** *Suppose that X is an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character $\Lambda \in \mathfrak{h}^*$ satisfying*

$$\mathrm{Re}\langle \Lambda - \rho(\Lambda), \alpha\rangle \geq 0, \quad \forall \alpha \in \Delta^+(\Lambda, \mathfrak{h}).$$

Then there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an admissible character λ of L such that X is isomorphic to $A_{\mathfrak{q}}(\lambda)$.

Proof of Theorem 6.1. Our main result implies that X has infinitesimal character

$$\Lambda = \gamma + \rho_c = \lambda' + \rho_n + \rho_c = \lambda' + \rho.$$

Thus Λ is strongly regular, and therefore, by Theorem 6.3, there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an admissible character λ of L such that X is isomorphic to $A_{\mathfrak{q}}(\lambda)$. The infinitesimal character of $A_{\mathfrak{q}}(\lambda)$ is $\lambda|_{\mathfrak{t}} + \rho$. It follows that $\lambda|_{\mathfrak{t}} = \lambda'$. □

The following theorem gives a criterion for $A_{\mathfrak{q}}(\lambda)$ to have non-zero Dirac cohomology. Combined with Theorem 6.1, this criterion classifies irreducible unitary $(\mathfrak{g}, K)$-modules with strongly regular infinitesimal character and with non-zero Dirac cohomology.

Theorem 6.4. *Let $X = A_{\mathfrak{q}}(\lambda)$ as in Theorem* 5.3. *Then the following two conditions are equivalent:*

(i) *X has non-zero Dirac cohomology.*

(ii) *$\lambda|_{\mathfrak{h}} + \rho_n$ is the highest weight of a $\tilde{K}$-module.*

In particular, X having non-zero Dirac cohomology implies $\lambda|_{\mathfrak{h}}$ vanishes on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$.

Proof. Our main result implies that if X has non-zero Dirac cohomology, then X has infinitesimal character $\Lambda = \gamma + \rho_c$, where γ is the highest weight of a representation of $\tilde{K}$. Therefore, (i) implies (ii). To show that (ii) implies (i) we note that the lowest K-type μ of X can be expressed as

$$\mu = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{p} \cap \mathfrak{u}) = \lambda|_{\mathfrak{t}} + \rho(\Delta^+(\mathfrak{p})) + \rho(\Delta^+(\mathfrak{p})'),$$

for some positive root systems $\Delta^+(\mathfrak{g})$ and $\Delta^+(\mathfrak{g})'$ for $\mathfrak{t}$ in $\mathfrak{g}$. This is possible because for $\Delta^+(\mathfrak{p}) = \Delta^+(\mathfrak{l} \cap \mathfrak{p}) \cup \Delta^+(\mathfrak{u} \cap \mathfrak{p})$ we can set

$$\Delta^+(\mathfrak{p})' = (-\Delta^+(\mathfrak{l} \cap \mathfrak{p})) \cup \Delta^+(\mathfrak{u} \cap \mathfrak{p}).$$

Write ρ_n for $\rho(\Delta^+(\mathfrak{p}))$ and ρ_n' for $\rho(\Delta^+(\mathfrak{p})')$. Let E_γ be the irreducible representation of $\tilde{K}$ with highest weight $\gamma = \lambda|_{\mathfrak{t}} + \rho_n$. Then $E_\gamma \subset \mu \otimes E_{\rho_n'} \subset X \otimes S$ and E_γ is contained in the Dirac cohomology. □

Finally, let us describe the relation between the Dirac cohomology and the $(\mathfrak{g}, K)$-cohomology. We first summarize some of the results in Propositions 5.4, 5.7 & 5.16 and Theorems 5.5 & 5.6 of [VZ] about $(\mathfrak{g}, K)$-cohomology.

Proposition 6.5 ([VZ])**.** *Suppose that X is an irreducible unitary $(\mathfrak{g}, K)$-module and F is a finite-dimensional representation of G. Then $H^*(\mathfrak{g}, K; X \otimes F) = 0$ unless X and F have the same infinitesimal character. If $H^*(\mathfrak{g}, K; X \otimes F) \neq 0$, then there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, and positive root systems $\Delta^+(\mathfrak{g})$ and $\Delta^+(\mathfrak{g})'$ such that*

(i) *$F/\mathfrak{u}F$ is a one-dimensional character of L; write $-\lambda\colon \mathfrak{l} \to \mathbb{C}$ for its differential;*

(ii) *$X \cong A_{\mathfrak{q}}(\lambda)$ with lowest K-type $\mu = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$;*

(iii) *$\lambda|_{\mathfrak{h}}$ is zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$;*

(vi) *$\lambda|_{\mathfrak{h}}$ is a highest weight of F^* with respect to both $\Delta^+(\mathfrak{g})$ and $\Delta^+(\mathfrak{g})'$;*

(v) *the lowest K-type μ of X can be written as $\mu = \lambda|_{\mathfrak{t}} + \rho(\Delta^+(\mathfrak{p})) + \rho(\Delta^+(\mathfrak{p})')$.*

Proposition 6.6. *If X is a unitary $(\mathfrak{g}, K)$-module with non-zero $(\mathfrak{g}, K)$-cohomology, i.e.,*

$$H^i(\mathfrak{g}, K; X \otimes F) \neq 0$$

for some finite-dimensional representation F of G, then X has non-zero Dirac cohomology.

Proof. It follows from Proposition 6.5 that $X \cong A_{\mathfrak{q}}(\lambda)$ with lowest K-type $\mu = \lambda|_{\mathfrak{t}} + \rho(\Delta^+(\mathfrak{g})) + \rho(\Delta^+(\mathfrak{g})')$. It also follows from Proposition 6.5 that $\lambda|_{\mathfrak{t}}$ is the differential of a character of T. Hence, $\gamma = \lambda|_{\mathfrak{t}} + \rho(\Delta^+(\mathfrak{p}))$ is the highest weight of an irreducible representation of $\tilde{K}$. Therefore, E_γ is in the Dirac cohomology of X (this follows as in the proof of Theorem 6.4). $\square$

Note that the converse of Proposition 6.6 is not true. An irreducible unitary representation X of G with non-zero Dirac cohomology may have infinitesimal character different from all finite-dimensional representations of G. Therefore X may have zero $(\mathfrak{g}, K)$-cohomology.

7. A final remark: Kostant's cubic Dirac operator

In this section we show that our proof of Theorem 3.4 can be extended to the more general setting of Kostant's cubic Dirac operator. Our proof of the theorem works perfectly for this setting. This was first pointed out to us by Kostant.

Let G be a compact semisimple Lie group, and let R be a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{r}$ be the complexifications of the corresponding Lie algebras. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$ be the orthogonal decomposition with respect to the Killing form. Choose an orthonormal basis $Z_1, \dots, Z_n$ of $\mathfrak{p}$ with respect to the Killing form $\langle\ ,\ \rangle$. Kostant [K] defines his cubic Dirac operator to be the element

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where $v \in C(\mathfrak{p})$ is the image of the fundamental 3-form $\omega \in \Lambda^3(\mathfrak{p}^*)$,

$$\omega(X, Y, Z) = -\frac{1}{12}\langle X, [Y, Z]\rangle$$

under the Chevalley identification $\Lambda^*(\mathfrak{p}^*) \to C(\mathfrak{p})$. Kostant's cubic Dirac operator reduces to the ordinary Dirac operator when $(\mathfrak{g}, \mathfrak{r})$ is a symmetric pair, since $\omega = 0$ for the symmetric pair. Kostant ([K], Theorem 2.16) shows that

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{r}_\Delta} + C, \tag{7.1}$$

where C is the constant $||\rho||^2 - ||\rho_{\mathfrak{r}}||^2$. We note that the quadratic form on $\mathfrak{p}$ used by Kostant for his cubic Dirac operator is positive definite, while the form we used to define the ordinary Dirac operator for the symmetric pair is negative definite. This makes a sign change in (7.1) comparing with Lemma 3.1.

Now we can define the cohomology of the complex $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$ using Kostant's cubic Dirac operator exactly as in Section 3. Let $a \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$. We define $d(a) = Da - \epsilon_a aD$ in the same way as in (3.1); as before, $d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$. The cubic term involves only $C(\mathfrak{p})$ and not $U(\mathfrak{g})$, so it follows that the cohomology is $Z(\mathfrak{r}_\Delta)$. Namely, by passing to the symbol the cubic term disappears, and the proof in Section 4 works without change in the present case. We summarize this generalization of Theorem 3.4 as the following theorem.

Theorem 7.1. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$ defined by Kostant's cubic Dirac operator as above. Then* $\operatorname{Ker} d = \operatorname{Im} d \oplus Z(\mathfrak{r}_\Delta)$. *In particular, the cohomology of d is isomorphic to $Z(\mathfrak{r}_\Delta)$.*

As a consequence we get an analogous homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ for a reductive subalgebra $\mathfrak{r}$ in a semisimple Lie algebra $\mathfrak{g}$ and a more general version of Vogan's conjecture. If we fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{r}$ and extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then ζ is induced by the Harish-Chandra homomorphism exactly as in (5.2) of Theorem 5.5. Moreover, this homomorphism ζ induces the structure of a $Z(\mathfrak{g})$-module on $Z(\mathfrak{r})$, which has topological significance. Namely, Kostant has shown that from a well-known theorem of H. Cartan [C], which is by far the most comprehensive result on the real (or complex) cohomology of a homogeneous space, one has

Theorem 7.2. *There exists an isomorphism*

$$H^*(G/R, \mathbb{C}) \cong \operatorname{Tor}_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})).$$

These results will appear with all the details in a forthcoming paper by Kostant [K2].

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