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SINGULARITIES OF PAIRS VIA JET SCHEMES

MIRCEA MUSTAŢĂ

Introduction

Let X be a smooth complex variety and Y a closed subscheme of X. The study of the singularities of the pair (X, Y) is a topic which has received a lot of attention, due mainly to applications to the classification of higher dimensional algebraic varieties. Our main goal in this paper is to propose a new point of view in this study, based on the properties of the jet schemes of Y.

For an arbitrary scheme W, the mth jet scheme W_m parametrizes morphisms Spec $\mathbb{C}[t]/(t^{m+1}) \longrightarrow W$. Our main result is the following theorem.

Theorem 0.1. *If X is a smooth variety, $Y \subset X$ a closed subscheme, and $q > 0$ a rational number, then:*

(1) *The pair $(X, q \cdot Y)$ is log canonical if and only if*
$$\dim Y_m \leq (m+1)(\dim X - q),$$
for all m.

(2) *The pair $(X, q \cdot Y)$ is Kawamata log terminal if and only if*
$$\dim Y_m < (m+1)(\dim X - q),$$
for all m.

In fact, it is enough to check the conditions in Theorem 0.1 for one value of m, depending on a log resolution of (X, Y) (see Theorem 3.1 for the precise statement). As a consequence of the above result, we obtain a formula for the log canonical threshold.

Corollary 0.2. *If X is a smooth variety and $Y \subset X$ is a closed subscheme, then the log canonical threshold of the pair (X, Y) is given by*
$$c(X, Y) = \dim X - \sup_{m \geq 0} \frac{\dim Y_m}{m+1}.$$

Similarly to the previous remark, the above supremum can be obtained for specific values of m. A similar formula holds for the log canonical threshold around a closed subset $Z \subset X$. Note that as a consequence of the above formula, the log canonical threshold depends only on Y and $\dim X$, but not on the particular ambient variety or the embedding.

We apply Corollary 0.2 to give simpler proofs of some results on the log canonical threshold proved by Demailly and Kollár in [DK] using analytic techniques. For

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example, we use a semicontinuity statement about the dimension of the jet schemes to deduce the semicontinuity of the log canonical threshold.

The main technique we use in the proof of Theorem 0.1 is motivic integration, a technique due to Kontsevich, Batyrev, and Denef and Loeser. The idea is similar to the one used in our previous paper [Mu]. Here is a brief description of the idea of the proof.

Let $X_\infty = \operatorname{proj\,lim}_m X_m$ be the space of arcs of X. A $\mathbb{C}$-valued point of X_∞ corresponds to a morphism $\operatorname{Spec} \mathbb{C}[[t]] \longrightarrow X$. Let $F_Y : X_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$ be the function such that for an arc γ over $x \in X$ which corresponds to a morphism $\widetilde{\gamma} : \mathcal{O}_{X,x} \longrightarrow \mathbb{C}[[t]]$, we have $F_Y(\gamma) = \operatorname{ord}(\widetilde{\gamma}(\mathcal{I}_{Y,x}))$. For a suitable function $f : \mathbb{N} \longrightarrow \mathbb{N}$, the motivic integral of $f \circ F_Y$ on X_∞ encodes the information about the dimension of Y_m, for all m.

On the other hand, if we take a log resolution $\pi : X' \longrightarrow X$, then by the change of variable formula of Batyrev ([Ba]) and Denef and Loeser ([DL]), this integral can be expressed as the integral on X'_∞ of $f \circ F_{\pi^{-1}(Y)} + F_W$. Here $F_{\pi^{-1}(Y)}$ and F_W are the functions associated as above with the effective divisors $\pi^{-1}(Y)$ and W, where W is the relative canonical divisor of π. Since both W and $\pi^{-1}(Y)$ are divisors with simple normal crossings, this last integral can be explicitly computed by a formula involving the coefficients of W and $\pi^{-1}(Y)$. By an inspection of the monomials which enter in the expression of the two integrals we deduce the equivalence in Theorem 0.1.

The paper is organized as follows. In the first section we review the definitions of log canonical and Kawamata log terminal pairs. In the algebraic context the usual setting in the literature is that of a pair (X, Y), where Y is a divisor on X. Therefore we give the extension to the case of pairs of arbitrary codimension, making reference for the case of divisors to [Kol]. Note, however, that we will always stick to the case of a smooth ambient variety, as our main results are valid only in this context.

The second section contains an overview of the basic definitions and notions related to jet schemes and motivic integration. We also give some further properties of jet schemes. In particular, we prove a semicontinuity result about the dimensions of the jet schemes.

The third section is the heart of the paper. It gives the proof of Theorem 0.1 along the lines described above. In the last section we show how Corollary 0.2 can be used to prove properties of the log canonical threshold. For example, the canonical isomorphism $(Y \times Y')_m \simeq Y_m \times Y'_m$ implies that $c(X \times X', Y \times Y') = c(X, Y) + c(X', Y')$. This can be used to prove the inequality in [DK]:

$$c_z(X, Y_1 \cap Y_2) \leq c_z(X, Y_1) + c_z(X, Y_2),$$

for every $z \in Y_1 \cap Y_2$. Another application is to the semicontinuity theorem in [DK] for log canonical thresholds. We conclude with a concrete example: we use the formula in Corollary 0.2 to compute the log canonical threshold of monomial ideals, recovering in this way a result from [AGV] (see also [Ho] for a proof via multiplier ideals).

1. Log terminal and log canonical pairs

All schemes are of finite type over $\mathbb{C}$. A variety is a reduced, irreducible scheme. We use the theory of singularities of pairs for which our main reference is [Kol]. However, as we work with pairs of arbitrary codimension, we review in this section some extensions to this setting of the definitions we need. For an analytic approach to singularities of pairs of arbitrary codimension, see [DK].

We consider pairs (X, Y), where X is a smooth variety and $Y \hookrightarrow X$ is a closed subscheme with $Y \neq X$. A log resolution of (X, Y) is a proper, birational morphism $\pi : X' \longrightarrow X$ such that X' is smooth, $\pi^{-1}(Y) = D$ is an effective divisor and the union $D \cup Ex(\pi)$ has simple normal crossings. Here $Ex(\pi)$ denotes the exceptional locus of π. The results in [Hi] show that log resolutions exist (in fact, one can assume in addition that π is an isomorphism over $X \setminus Y$).

For an arbitrary proper, birational morphism $\pi : X' \longrightarrow X$, with X' and X smooth, the relative canonical divisor $K_{X'/X}$ is the unique effective divisor supported on $Ex(\pi)$ such that $\mathcal{O}_{X'}(K_{X'/X}) \simeq \omega_{X'} \otimes \pi^*(\omega_X^{-1})$.

Definition 1.1. Let (X, Y) be a pair as above and $q > 0$ a rational number. For a log resolution $\pi : X' \longrightarrow X$ of (X, Y), with $\pi^{-1}(Y) = D$, we write $K_{X'/X} - q \cdot D = \sum_i \alpha_i E_i$. The total discrepancy of $(X, q \cdot Y)$ is defined by

$$\text{totdiscrep}\,(X, q \cdot Y) = \begin{cases} -\infty, & \text{if } \alpha_i < -1, \text{ for some } i; \\ \min_i\{0, \alpha_i\}, & \text{otherwise.} \end{cases} \tag{1.1}$$

The pair $(X, q \cdot Y)$ is called Kawamata log terminal or log canonical if

$$\text{totdiscrep}\,(X, q \cdot Y) > -1 \quad \text{or} \quad \text{totdiscrep}\,(X, q \cdot Y) \geq -1,$$

respectively.

Remark 1.2. Since $q \cdot D - K_{X'/X}$ is a divisor with simple normal crossings, it follows from Lemma 3.11 in [Kol], that we could have defined the total discrepancy of $(X, q \cdot Y)$ as the total discrepancy of $q \cdot D - K_{X'/X}$. In particular, $(X, q \cdot Y)$ is Kawamata log terminal (log canonical) if and only if $(X', q \cdot D - K_{X'/X})$ is Kawamata log terminal (log canonical).

Proposition 1.3. *The total discrepancy does not depend on the particular log resolution, and therefore neither do the notions of log canonical and Kawamata log terminal pairs.*

Proof. By the above remark, it is enough to show that if $\pi' : X'' \longrightarrow X'$ is a log resolution of $(X', D \cup Ex(\pi))$, with ${\pi'}^{-1}(D) = F$, then

$$\text{totdiscrep}\,(q \cdot D - K_{X'/X}) = \text{totdiscrep}\,(q \cdot F - K_{X''/X}).$$

This follows from the fact that $K_{X''/X} = K_{X''/X'} + {\pi'}^*(K_{X'/X})$ and the invariance of the total discrepancy under proper birational morphisms in the case of divisors (see [Kol], Lemma 3.10). □

Definition 1.4. Let (X, Y) be a pair as above, $Z \subset X$ a nonempty closed subset, and $q > 0$ a rational number. The log canonical threshold of $(X, q \cdot Y)$ around Z is defined by

$$c_Z(X, q \cdot Y) = \sup\{q' \in \mathbb{Q}_+ \mid (X, qq' \cdot Y) \text{ is log canonical in an open neighbourhood of } Z\}.$$

When $Z = X$, we omit it from the notation.

As it is obvious that $c_Z(X, q \cdot Y) = (1/q)\, c(X, Y)$, it follows that when computing log canonical thresholds, there is no loss of generality in assuming that $q = 1$. The proposition below is the analogue of Proposition 8.5 in [Kol]. It shows how to compute the log canonical threshold from a log resolution.

Let (X,Y) be a pair and $Z\subseteq X$ a nonempty closed subset. Let $\pi : X'\longrightarrow X$ be a log resolution of (X,Y) which is an isomorphism over $X\setminus Y$. We write $\pi^{-1}(Y)=D=\sum_i a_iD_i$, with $a_i\geq 1$ and $K_{X'/X}=\sum_i b_iD_i$.

Proposition 1.5 ([DK], 1.7, [Kol], 8.5). *With the above notation, the log canonical threshold of (X,Y) around Z is given by*

$$c_Z(X,Y)=\min_i\{(b_i+1)/a_i \mid \pi(D_i)\cap Z\neq\emptyset\}. \tag{1.2}$$

In particular, either $c_Z(X,Y)$ is a positive rational number, or $c_Z(X,Y)=\infty$.

Proof. We have $(X,q\cdot Y)$ log canonical if and only if $b_i-qa_i\geq -1$ or, equivalently, $q\leq (b_i+1)/a_i$, for all i. By replacing X with the complement of the union of those images of D_i which do not intersect Z, we get our result. □

Remark 1.6. The log canonical threshold $c_Z(X,Y)$ is infinite if and only if $Z\cap Y=\emptyset$.

Remark 1.7. We assumed above that for every pair (X,Y), we have $Y\neq X$. We will sometimes find it convenient to also include the case of the pair (X,X), so that we make the convention $c_Z(X,X)=0$, for every nonempty closed subset $Z\subseteq X$.

Though unnecessary for our needs, for the sake of completeness we mention the translation of the above notions into the language of multiplier ideals (see [Ein] or [La]). Recall that if π is a log resolution of (X,Y) such that $\pi^{-1}(Y)=D$, then the multiplier ideal of the pair $(X,q\cdot Y)$ is defined by

$$I(X,q\cdot\mathcal{I}_{Y/X})=\pi_*(\mathcal{O}_{X'}(K_{X'/X}-[q\cdot D])), \tag{1.3}$$

where $[q\cdot D]$ denotes the integral part of the $\mathbb{Q}$-divisor $q\cdot D$.

Proposition 1.8. *The pair $(X,q\cdot Y)$ is Kawamata log terminal if and only if $I(X,q\cdot\mathcal{I}_{Y/X})=\mathcal{O}_X$ and it is log canonical if and only if $I(X,q'\cdot\mathcal{I}_{Y/X})=\mathcal{O}_X$ for every $q'<q$. In particular, if $Z\subseteq X$ is a nonempty closed subset, then*

$$c\,(X,Y)=\sup\{q>0\mid I(X,q\cdot\mathcal{I}_{Y/X})=\mathcal{O}_X\ \mathit{around}\, Z\}. \tag{1.4}$$

Proof. It is enough to prove the first assertion. This follows from the fact that $I(X,q\cdot\mathcal{I}_{Y/X})=\mathcal{O}_X$ if and only if $K_{X'/X}-[q\cdot D]$ is effective or, equivalently, all the coefficients in $K_{X'/X}-q\cdot D$ are >-1. □

2. An overview of jet schemes and motivic integration

In this section we collect the definitions and basic properties of jet schemes. We review also motivic integration on smooth varieties, a technique which plays a major role in the proof of our main result in the next section.

For every scheme W (of finite type over $\mathbb{C}$) and every $m\in\mathbb{N}$, the jet scheme W_m is a scheme of finite type over $\mathbb{C}$ characterized by

$$\mathrm{Hom}(\mathrm{Spec}\ A,W_m)\simeq\mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}),W), \tag{2.1}$$

for every $\mathbb{C}$-algebra A. In particular, the closed points of W_m correspond to $\mathbb{C}[t]/(t^{m+1})$-valued points of W. The correspondence $W\longrightarrow W_m$ is a functor. In fact, it follows from the definition that it is the right adjoint of the functor $Z\longrightarrow Z\times\mathrm{Spec}\ \mathbb{C}[t]/(t^{m+1})$.

It is clear that we have $W_0 \simeq W$ and $W_1 \simeq TW$, the total tangent space of W. In general, we have projections $\phi_m : W_m \longrightarrow W_{m-1}$ which are induced by the projections $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^m)$, for a $\mathbb{C}$-algebra A. By composing these morphisms we get natural projections $\rho_m : W_m \longrightarrow W$. Whenever the variety is not clear from the context, we will add a superscript: for example, ρ_m^W. The projective limit of the schemes W_m is a scheme W_∞, in general not of finite type over $\mathbb{C}$, called the space of arcs of W. Its $\mathbb{C}$-valued points correspond to $\mathbb{C}[[t]]$-valued points of W. There are canonical morphisms $\psi_m : W_\infty \longrightarrow W_m$.

This construction is local in the sense that if $U \subseteq W$ is an open subset, then $U_m \simeq \rho_m^{-1}(U)$, for all m. More generally, if $f : W' \longrightarrow W$ is an étale morphism, then $W'_m \simeq W_m \times_W W'$.

We can therefore reduce the description of W_m to the case when W is affine. In this case one can write down explicit equations for W_m as follows. Suppose that $W \subseteq \operatorname{Spec} \mathbb{C}[X_i; i \in I]$ is defined by a system of polynomials $(f_\alpha)_\alpha$. Consider the ring $R_m = \mathbb{C}[X_i, X'_i, \ldots, X_i^{(m)}; i \in I]$ and D the unique $\mathbb{C}$-derivation of R_m such that $D(X_i^{(j)}) = X_i^{(j+1)}$ for all i and j, where $X_i^{(0)} = X_i$ and $X_i^{(m+1)} = 0$. For a polynomial $f \in \mathbb{C}[X_i; i \in I]$, we put $f^{(j)} = D^j(f)$. With this notation, $W_m \subseteq \operatorname{Spec} R_m$ is defined by $(f_\alpha, f'_\alpha, \ldots, f_\alpha^{(m)})_\alpha$. It follows from this description that if $W' \hookrightarrow W$ is a closed subscheme of W, then the induced morphism $W'_m \longrightarrow W_m$ is also a closed immersion.

If W is a smooth variety of dimension n, then the morphisms ϕ_m are locally trivial with fiber $\mathbb{A}^n$. In particular, W_m is a smooth variety of dimension $(m+1)n$.

We discuss some additional properties of jet schemes which we will apply in the last section to deduce corresponding properties of the log canonical threshold.

Proposition 2.1. *For every two schemes W and Z and every m, there is a natural isomorphism $(W \times Z)_m \simeq W_m \times Z_m$.*

Proof. The assertion follows from the fact that the functor $W \longrightarrow W_m$ has a left adjoint, and therefore it commutes with direct products. □

For every scheme W and every $m \geq 1$ there is a natural "action":

$$\Phi_m : \mathbf{A}^1 \times W_m \longrightarrow W_m$$

over W defined at the level of A-valued points as follows. For an algebra A, an A-valued point in $\mathbf{A}^1 \times W_m$ consists of a pair (a, f), for some $a \in A$ and a morphism $f : \operatorname{Spec} A[t]/(t^{m+1}) \longrightarrow W$. $\Phi_m(a, f)$ is the composition $f \circ g_a$, where g_a is induced by the A-algebra homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{m+1})$ which maps t to at.

It is clear that these "actions" are compatible with the projection ϕ_m. The restriction of Φ_m to $\mathbb{C}^* \times W_m$ induces an action of the torus on W_m. On the other hand, note that $\Phi_m(\{0\} \times W_m)$ is the image of the "zero section" σ_m of ρ_m. This is defined by composition with the scheme morphism induced by the inclusion $A \hookrightarrow A[t]/(t^{m+1})$. It is clear that $W_m \setminus \sigma_m(W)$ is $\mathbb{C}^*$-invariant. With this notation, we have

Lemma 2.2. *For every $w \in W$, the fiber $\rho_m^{-1}(w)$ is the cone over the (possibly empty) projective scheme $(\rho_m^{-1}(w) \setminus \{\sigma_m(w)\})/\mathbb{C}^*$.*

Proof. By restricting to an open neighbourhood of w, we may assume that $W \subseteq \mathbf{A}^N$ and that $w = 0$ is the origin. We have an embedding $W_m \subseteq \mathbf{A}^{(m+1)N}$ which induces an embedding $\rho_m^{-1}(0) \subseteq \mathbf{A}^{mN}$ such that $\sigma_m(0)$ corresponds to the origin. The action of $\mathbb{C}^*$ on $\rho_m^{-1}(0) \setminus \{0\}$ extends to an action on $\mathbf{A}^{mN} = \mathrm{Spec}[X_i', \ldots, X_i^{(m)}; i]$ induced by $\lambda \cdot X_i^{(j)} = \lambda^j X_i^{(j)}$. Therefore $\rho_m^{-1}(0) \setminus \{0\}/\mathbb{C}^*$ is a subscheme of a weighted projective space, hence it is projective, and $\rho_m^{-1}(w)$ is the cone over it. □

Our next goal is the proof of the following semicontinuity result. For a scheme $\pi : \mathcal{W} \longrightarrow S$ over S and a closed point $s \in S$, we denote the fiber of π over s by $\mathcal{W}_s$.

Proposition 2.3. *Let $\pi : \mathcal{W} \longrightarrow S$ be a family of schemes and $\tau : S \longrightarrow \mathcal{W}$ a section of π. For every $m \geq 1$, the function*

$$f(s) = \dim (\rho_m^{\mathcal{W}_s})^{-1}(\tau(s))$$

is upper semicontinuous on the set of closed points of S.

The key to the proof of Proposition 2.3 is to use a relative version of the jet schemes. Suppose we work over a fixed scheme S. If $\mathcal{W} \longrightarrow S$ is a scheme over S, then the mth relative jet scheme $(\mathcal{W}/S)_m$ is characterized by

$$\mathrm{Hom}_S(\mathcal{Z} \times \mathrm{Spec}\ \mathbb{C}[t]/(t^{m+1}), \mathcal{W}) \simeq \mathrm{Hom}_S(\mathcal{Z}, (\mathcal{W}/S)_m),$$

for every scheme $\mathcal{Z}$ over S. Therefore the functor $\mathcal{W} \longrightarrow (\mathcal{W}/S)_m$ is the right adjoint of the functor $\mathcal{Z} \longrightarrow \mathcal{Z} \times \mathrm{Spec}\ \mathbb{C}[t]/(t^{m+1})$ between schemes over S.

The existence of $(\mathcal{W}/S)_m$ can be settled as in the absolute case by giving local equations. More precisely, since the construction is local on $\mathcal{W}$, we may assume that both S and $\mathcal{W}$ are affine: $S = \mathrm{Spec}\ A$ and $\mathcal{W} \hookrightarrow \mathrm{Spec}\ A[X_i; i]$ is defined by $(f_\alpha)_\alpha$. Then $(\mathcal{W}/S)_m \hookrightarrow \mathrm{Spec}\ A[X_i, X_i', \ldots, X_i^{(m)}; i]$ is defined by $(f_\alpha, f_\alpha', \ldots, f_\alpha^{(m)})_\alpha$. Here $f^{(j)} = D^j(f)$, where D is the unique derivation over A of $A[X_i, X_i', \ldots, X_i^{(m)}; i]$ such that $D(X_i^{(p)}) = X_i^{(p+1)}$ for all p (we put $X_i^{(0)} = X_i$ and $X_i^{(m+1)} = 0$). As in the absolute case, we have canonical projections $\rho_m : (\mathcal{W}/S)_m \longrightarrow \mathcal{W}$ and a "zero section" $\sigma_m : \mathcal{W} \longrightarrow (\mathcal{W}/S)_m$.

The following lemma follows immediately from the functorial definition of relative jet schemes.

Lemma 2.4. *For every scheme morphism $S' \longrightarrow S$, if $\mathcal{W}' \simeq \mathcal{W} \times_S S'$, then we have a canonical isomorphism $(\mathcal{W}'/S')_m \simeq (\mathcal{W}/S) \times_S S'$, for every m. In particular, for every closed point $s \in S$, the fiber of $(\mathcal{W}/S)_m$ over s is isomorphic with $(\mathcal{W}_s)_m$.*

Over every closed point $s \in S$, the projection ρ_m and the "zero section" σ_m are the ones we defined in the absolute case. On the other hand, the actions of $\mathbb{C}^*$ on each fiber globalize to an action on $(\mathcal{W}/S)_m$ such that $(\mathcal{W}/S)_m \setminus \sigma_m(\mathcal{W})$ is invariant. After these preparations we can give the proof of Proposition 2.3.

Proof of Proposition 2.3. Consider the projection $(\mathcal{W}/S)_m \setminus \sigma_m(\mathcal{W}) \longrightarrow \mathcal{W}$ and let $\mathcal{Z}$ be the inverse image by this morphism of $\tau(S)$. We have a natural action of $\mathbb{C}^*$ on $\mathcal{Z}$ and the quotient by this action is a scheme $\overline{\mathcal{Z}}$ which is proper over S. The properness follows from the fact that it is locally projective, an assertion which is just the globalization of Lemma 2.2 and can be proved in the same way.

But for every closed point $s \in S$ we have $f(s) = \dim \overline{\mathcal{Z}}_s + 1$ (or $f(s) = 0$ if $\overline{\mathcal{Z}}_s = \emptyset$) and our assertion follows from the semicontinuity of the dimension of the fibers of a proper morphism. □

We review now the basic facts about motivic integration on smooth varieties. This technique was developed by Kontsevich [Kon], Batyrev [Ba] and Denef and Loeser [DL]. There are several possible extensions (see [DL] for the general treatment), but we use only Hodge realizations of motivic integrals on the space of arcs of a smooth variety. For a nice introduction to these ideas we refer to Craw [Cr].

Suppose from now on that X is a smooth variety of dimension n. On X_∞ the theory provides an algebra of sets $\mathcal{M}$ and a finitely additive measure on $\mathcal{M}$ with values in the Laurent power series ring in u^{-1} and v^{-1}:

$$\mu : \mathcal{M} \longrightarrow S = \mathbb{Z}[[u^{-1}, v^{-1}]][u, v].$$

On S we consider the topology defined by the descending sequence of subgroups $\{\bigoplus_{i+j\geq l} \mathbb{Z}u^{-i}v^{-j}\}_l$.

We will be interested in the subalgebra $\mathcal{C}yl$ of $\mathcal{M}$ consisting of cylinders of the form: $\psi_m^{-1}(C)$, for some $m \geq 1$ and some constructible subset $C \subset X_m$. The measure of such a cylinder is given by $\mu(\psi_m^{-1}(C)) = E(C; u, v)(uv)^{-(m+1)n}$, where $E(C; u, v)$ is the Hodge-Deligne polynomial of C. What is important for us is that if $C \subseteq X_m$ is a locally closed subset, then $E(C; u, v)$ is a polynomial of degree $2\,(\dim C)$ and the term of degree $2\,(\dim C)$ is $l(C)(uv)^{\dim C}$, where $l(C)$ is the number of irreducible components of C of maximal dimension.

Besides the cylinders, some sets of measure zero appear in the definition of measurable functions. If $T \subset X_\infty$ is such that there is a sequence of cylinders $W_r \in \mathcal{C}yl$ with $T \subset W_r$ for all r and $\mu(W_r) \longrightarrow 0$, then $T \in \mathcal{M}$ and $\mu(T) = 0$.

A function $F : X_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$ is called measurable if $F^{-1}(s) \in \mathcal{M}$ for every $s \in \mathbb{N} \cup \{\infty\}$ and if $\mu(F^{-1}(\infty)) = 0$. If the sum $\sum_{s\in\mathbb{N}} \mu(F^{-1}(s))(uv)^{-s}$ is convergent in S, then F is called integrable, and the sum is called the motivic integral of F and denoted by $\int_{X_\infty} e^{-F}$.

Every subscheme $Y \hookrightarrow X$ defines a function F_Y on X_∞, as follows. If $\gamma \in X_\infty$ is an arc over $x \in X$, then it can be identified with a ring homomorphism $\widetilde{\gamma} : \mathcal{O}_{X,x} \longrightarrow \mathbb{C}[[t]]$. We define $F_Y(\gamma) := \mathrm{ord}(\widetilde{\gamma}(\mathcal{I}_{Y,x}))$. It follows from this definition that

$$F_Y^{-1}(s) = \psi_{s-1}^{-1}(Y_{s-1}) \setminus \psi_s^{-1}(Y_s),$$

for every integer $s \geq 0$ and $F_Y^{-1}(\infty) = Y_\infty$. Here we make the convention $Y_{-1} = X$ and $\psi_{-1} = \psi_0$. It follows that $F_Y^{-1}(s) \in \mathcal{C}yl$ for every $s \geq 0$. Moreover, the following lemma shows that if $Y \neq X$, then $F_Y^{-1}(\infty) \in \mathcal{M}$ and $\mu(F_Y^{-1}(\infty)) = 0$. Therefore F_Y is measurable.

Lemma 2.5 ([Mu], 3.7)**.** *If $D \subset X$ is a divisor and $x \in X$ is a point such that $\mathrm{mult}_x D = a$, then*

$$\dim\,(\rho_m^D)^{-1}(x) \leq nm - [m/a], \tag{2.2}$$

where $[y]$ denotes the integral part of y. In particular, for every closed subscheme Y of X, with $Y \neq X$, we have $\lim_{m\to\infty}(\dim Y_m - n(m+1)) = -\infty$, so that $F_Y^{-1}(\infty) \in \mathcal{M}$ and $\mu(F_Y^{-1}(\infty)) = 0$.

One of the main results of the theory is the change of variable formula for motivic integrals.

Proposition 2.6 ([Ba], 6.27, [DL], 3.3)**.** *Let $\pi : X' \longrightarrow X$ be a proper, birational morphism of smooth varieties. Let $W = K_{X'/X}$ be the relative canonical divisor* *and* $\pi_\infty : X'_\infty \longrightarrow X_\infty$ *the morphism induced by* π. *For every measurable function* $F : X_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$, *we have*

$$\int_{X_\infty} e^{-F} = \int_{X'_\infty} e^{-(F\circ\pi_\infty + F_W)},$$

meaning that one integral exists if and only if the other one does, and in this case they are equal.

Remark 2.7. In connection with the change of variable formula, note that if $F = F_Y$, for some proper subscheme $Y \hookrightarrow X$, then $F \circ \pi_\infty = F_{\pi^{-1}(Y)}$.

3. Jet schemes of log terminal and log canonical pairs

The main goal of this section is to prove the characterization of log canonical and Kawamata log terminal pairs in terms of jet schemes. First let us fix the notation.

Consider a pair (X, Y) with $\dim X = n$ and fix a log resolution of this pair $\pi : X' \longrightarrow X$ such that π is an isomorphism over $X \setminus Y$. Let $D = \pi^{-1}(Y)$ and $W = K_{X'/X}$ and write $D = \sum_{i=1}^r a_i D_i$, with $a_i \geq 1$ for all i and $W = \sum_{i=1}^r b_i D_i$.

Theorem 3.1. *With the above notation, if* $q \in \mathbb{Q}^*_+$, *then we have the following equivalences.*

1. $(X, q\cdot Y)$ *is log canonical if and only if* $\dim Y_m \leq (m+1)(n-q)$, *for every* $m \geq 0$. *Moreover, it is enough to have this inequality for some* $m \geq 0$ *such that* $a_i \mid (m+1)$ *for all* i.
2. $(X, q\cdot Y)$ *is Kawamata log terminal if and only if* $\dim Y_m < (m+1)(n-q)$, *for every* $m \geq 0$. *As above, it is enough to have this inequality for some* m *such that* $a_i \mid (m+1)$ *for all* i.

Proof. The idea of the proof is similar to that of Theorem 3.1 in [Mu], so that we refer to that paper for some of the details.

With the notation in the theorem, recall that we have $(X, q\cdot Y)$ log canonical (Kawamata log terminal) if and only if $b_i + 1 \geq (>) q a_i$, for all i. Since the case $Y = \emptyset$ is trivial (we follow the convention that $\dim(\emptyset) = -\infty$), we assume from now on that Y is nonempty.

We fix a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, such that for every $s \geq 0$,

$$(\star) \qquad f(s+1) > f(s) + \dim Y_s + C(s+1),$$

where $C \in \mathbb{N}$ is a constant such that $C > |n - (b_i+1)/a_i|$, for all i. We extend this function by defining $f(\infty) = \infty$. For the proof of the "if" part, we will add later an extra condition of the same type on f.

We integrate the function $F = f \circ F_Y$ over X_∞. Computing the integral from the definition, one can see that $\int_{X_\infty} e^{-F} = S_1 - S_2$, where

$$S_1 = \sum_{s\geq 0} E(Y_{s-1}; u, v)(uv)^{-sn-f(s)},$$

$$S_2 = \sum_{s\geq 0} E(Y_s; u, v)(uv)^{-(s+1)n-f(s)}.$$

It is clear that every monomial which appears in the sth term of S_1 has degree bounded above by $2P_1(s)$ and below by $2P_2(s)$, where $P_1(s) = \dim Y_{s-1} - sn - f(s)$ and $P_2(s) = -sn - f(s)$, for every $s \geq 0$ (recall our convention that $Y_{-1} = X$). We have precisely one monomial of degree $2P_1(s)$, namely $(uv)^{P_1(s)}$ whose coefficient is $l(Y_{s-1})$, the number of irreducible components of Y_{s-1} of maximal dimension.

Similarly, every monomial which appears in the sth term of S_2 has degree bounded above by $2Q_1(s)$ and below by $2Q_2(s)$, where

$$Q_1(s) = \dim Y_s - (s+1)n - f(s),$$

$$Q_2(s) = -(s+1)n - f(s).$$

There is exactly one monomial of degree $2Q_1(s)$, namely $(uv)^{Q_1(s)}$ with coefficient $l(Y_s)$.

From the above evaluation of the terms in S_1 and S_2 and Lemma 2.5 we see that F is integrable. Moreover, it follows from condition $(\star)$ that $P_1(s+1) < \min\{P_2(s), Q_2(s)\}$ for every s. We also have $Q_1(s) \leq P_1(s)$ for every s, with equality if and only if $s \geq 1$ and $\dim Y_s = \dim Y_{s-1} + n$. Lemma 2.5 implies therefore that we have strict inequality for infinitely many s. This shows that in $\int_{X_\infty} e^{-F}$ we have monomials of degree $2P_1(s)$ for infinitely many values of s.

We now apply the change of variable formula in Proposition 2.6 for the morphism π to get

$$\int_{X_\infty} e^{-F} = \int_{X'_\infty} e^{-(F\circ\pi_\infty + F_W)}. \tag{3.1}$$

Using the fact that $F\circ\pi_\infty = f\circ F_{\pi^{-1}(Y)}$ and that $W\cup\pi^{-1}(Y)$ has simple normal crossings, one can compute explicitly the integral. For a subset $J\subseteq\{1,\ldots,r\}$ let $D_J^\circ = \bigcap_{i\in J} D_i \setminus \bigcup_{i\not\in J} D_i$. With this notation we have

$$\int_{X_\infty} e^{-F} = \sum_{J\subseteq\{1,\ldots,r\}} S_J, \text{ where}$$

$$S_J = \sum_{\alpha_i\geq 1, i\in J} E(D_J^\circ; u, v)(uv-1)^{|J|}\cdot (uv)^{-n-\sum_{i\in J}\alpha_i(b_i+1)-f(\sum_{i\in J}\alpha_i a_i)}.$$

Every monomial in the term of S_J corresponding to $(\alpha_i)_{i\in J}$ has degree bounded above by $2R_1(\alpha_i; i\in J)$ and below by $2R_2(\alpha_i; i\in J)$, where

$$R_1(\alpha_i; i\in J) = -\sum_{i\in J}\alpha_i(b_i+1) - f(\sum_{i\in J}\alpha_i a_i)$$

and $R_2(\alpha_i; i\in J) = R_1(\alpha_i; i\in J) - n$. Note that $R_1(\emptyset) = -f(0)$.

Let us introduce the notation $\tau(s) = \dim Y_s - (s+1)(n-q)$. We see that if $J\neq\emptyset$, then

$$R_1(\alpha_i; i\in J) = P_1(\sum_{i\in J}\alpha_i a_i) - \tau(\sum_{i\in J}\alpha_i a_i - 1) + \sum_{i\in J}\alpha_i(qa_i - b_i - 1).$$

Moreover, property $(\star)$ implies that if $J\neq\emptyset$, then

$$P_1(\sum_{i\in J}\alpha_i a_i + 1) < R_2(\alpha_i; i\in J) < R_1(\alpha_i; i\in J) \tag{3.2}$$

$$< \min\{P_2(\sum_{i\in J}\alpha_i a_i - 1), Q_2(\sum_{i\in J}\alpha_i a_i - 1)\},$$

and $P_1(1) < R_2(\emptyset)$. This shows that the only monomial of the form $(uv)^{P_1(s)}$ which could appear in the part of S_J corresponding to $(\alpha_i)_i$ is for $s = \sum_{i\in J}\alpha_i a_i$.

We now prove (1). Suppose first that $(X, q \cdot Y)$ is log canonical, so that $b_i + 1 \geq qa_i$, for all i. Assume that for some $m \geq 0$, we have $\tau(m) > 0$. It follows from the above discussion that $(uv)^{P_1(m+1)}$ does not appear in $\int_{X_\infty} e^{-F}$. As we have seen, this implies $\dim Y_{m+1} = \dim Y_m + n$. In particular, we have $\tau(m+1) > 0$. Continuing in this way, we deduce $\dim Y_{s+1} = \dim Y_s + n$ for all $s \geq m$, which is impossible.

Conversely, suppose that for some fixed m such that $a_i \mid (m+1)$ for all i we have $\tau(m) \leq 0$, and that $b_j + 1 < ca_j$, for some j.

We choose the function f such that in addition to $(\star)$ it satisfies the following condition. For every p, let $\mathcal{J}_p$ be the finite set consisting of all the pairs $(J, (\alpha_i)_{i\in J})$ such that $\sum_{i\in J} a_i\alpha_i = p$. The extra condition we require for f is that for every $(J, (\alpha_i)_{i\in J}) \in \mathcal{J}_{m+1}$ and every $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, with $p \leq m$, we have

$$f(m+1) > f(p) - \sum_{i\in J} \alpha_i(b_i + 1) + \sum_{i\in J'} \alpha'_i(b_i + 1) + n. \tag{3.3}$$

This means that $R_1(\alpha_i; i \in J) < R_2(\alpha'_i; i \in J')$.

Note that from (3.2) we can deduce that if $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, with $p \geq m+2$, then

$$P_1(m+1) > R_1(\alpha'_i; i \in J'). \tag{3.4}$$

On the other hand, the top degree monomials which appear in different terms of the sums S_J (for possibly different J) do not cancel each other, as they have positive coefficients. Let d be the highest degree of a monomial which appears in a term corresponding to some $(J, (\alpha_i)_{i\in J}) \in \mathcal{J}_{m+1}$. The previous remark shows that the corresponding monomial does not cancel with the other monomials which appear in terms corresponding to $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_{m+1}$.

We obviously have $(\{j\}, (m+1)/a_j) \in \mathcal{J}_{m+1}$ and therefore our assumption that $\tau(m) \leq 0$ and $b_j + 1 < qa_j$ gives $d > 2P_1(m+1)$. We also deduce from (3.2) that $d < \min\{2P_2(m), 2Q_2(m)\}$.

We see from formulas (3.3) and (3.4) that this monomial of degree d does not cancel with other monomials in terms corresponding to $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, if $p \neq m+1$. Therefore in the integral $\int_{X_\infty} e^{-F}$ we have a monomial of degree d, with $2P_1(m+1) < d < \min\{2P_2(m), 2Q_2(m)\}$, a contradiction.

The proof of (2) is entirely similar. □

In order to state the formula for the log canonical threshold which follows from Theorem 3.1, we make the following definition. Consider a pair (X, Y) as before and $Z \subseteq X$ a nonempty closed subset. We also allow the case $Y = X$.

Definition 3.2. With the above notation, if $m \in \mathbb{N}$, we define $\dim_Z Y_m$ to be the dimension of Y_m along $Y_m \cap \rho_m^{-1}(Z)$, i.e. the maximum dimension of an irreducible component T of Y_m such that $\rho_m(T) \cap Z \neq \emptyset$.

Remark 3.3. It follows from our discussion of jet schemes in the previous section that for every irreducible component T of Y_m, the image $\rho_m(T)$ is closed in Y (and therefore in X). Indeed, since T must be invariant with respect to the "action" of $\mathbf{A}^1$ on Y_m, it follows that $\rho_m(T) = (\sigma_m^Y)^{-1}(T)$.

Therefore there is an open neighbourhood U of Z such that for every irreducible component T of Y_m with $\rho_m(T) \cap Z = \emptyset$, U does not intersect $\rho_m(T)$. For every such U, we have $\dim_Z Y_m = \dim (Y \cap U)_m$.

In the corollary below, we keep the previous notation for a log resolution of the pair (X,Y).

Corollary 3.4. *For every pair (X,Y) and $Z\subseteq X$ as above,*

$$c_Z(X,Y)=n-\sup_{m\geq 0}\frac{\dim_Z Y_m}{m+1}.$$

Moreover, if $m\geq 0$ is such that $a_i\mid (m+1)$, for all i such that $\pi(E_i)\cap Z\neq\emptyset$, then

$$c_Z(X,Y)=n-\frac{\dim_Z Y_m}{m+1}.$$

Proof. Note first that the formula is trivial when $Y=X$, since in this case for every $m\geq 1$, we know that $Y_m=X_m$ is a smooth variety of dimension $(m+1)\dim X$. Suppose from now on that $Y\neq X$.

For a fixed m, we can find an open neighbourhood U of Z such that $\dim_Z Y_m=\dim\,(Y\cap U)_m$ and $c_Z(X,Y)=c(U,Y\cap U)$. Therefore, in order to prove the second assertion of the corollary we may assume $Z=X$, in which case the formula follows from Theorem 3.1.

In order to prove the first assertion, it remains to be shown that $c_Z(X,Y)\geq n-(\dim_Z Y_m)/(m+1)$, for every m. We reduce again to the case when $Z=X$, and we conclude by applying Theorem 3.1. $\square$

Corollary 3.5. *If (X,Y) is a pair and Z a nonempty closed subset of X, then $c_Z(X,Y)$ depends only on Y, $\dim X$ and the closed subset $Y\cap Z\subseteq Y$, but not on the particular ambient variety X or the embedding $Y\hookrightarrow X$.*

Proof. The statement is a consequence of the formula in Corollary 3.4. $\square$

We now give a version of the formula in Corollary 3.4 involving only the fibers of ρ_m^Y over $Y\cap Z$. The drawback is that in this case, the supremum which appears in the formula cannot be obtained, in general, for a specific integer.

Corollary 3.6. *If (X,Y) is a pair and if $Z\subseteq X$ is a closed subset as before, then*

$$c_Z(X,Y)=\dim X-\sup_{m\geq 0}\frac{\dim\,(\rho_m^Y)^{-1}(Y\cap Z)}{m+1}.$$

Proof. Since the cases $Y=X$ and $Y\cap Z=\emptyset$ are trivial, we may assume that we are in neither of them. We obviously have $\dim\,(\rho_m^Y)^{-1}(Z\cap Y)\leq\dim_Z Y_m$, for every m, so that the formula in Corollary 3.4 gives

$$c_Z(X,Y)\leq\dim X-\sup_{m\geq 0}\frac{\dim\,(\rho_m^Y)^{-1}(Z\cap Y)}{m+1}.$$

On the other hand, the formula giving the lower bound for the dimension of the fibers of a morphism implies that

$$\dim_Z Y_m\leq\dim Y+\dim\,(\rho_m^Y)^{-1}(Y\cap Z).$$

Note that by Corollary 3.4, there is a positive integer N such that $c_Z(X,Y)=\dim X-\frac{\dim_Z Y_m}{m+1}$ if $N\,|(m+1)$. Therefore we deduce

$$c_Z(X,Y)\geq\dim X-\frac{\dim Y}{m+1}-\frac{\dim\,(\rho_m^Y)^{-1}(Y\cap Z)}{m+1},$$

for every m, such that $N\,|(m+1)$. This implies that for every $p \geq 1$,

$$c_Z(X,Y) \geq \dim\, X - \frac{\dim\, Y}{pN} - \sup_{m\geq 0} \frac{\dim\,(\rho_m^Y)^{-1}(Y\cap Z)}{m+1},$$

so that in fact

$$c_Z(X,Y) \geq \dim\, X - \sup_{m\geq 0} \frac{\dim\,(\rho_m^Y)^{-1}(Y\cap Z)}{m+1},$$

which completes the proof of the corollary. □

We end this section with the following

Example 3.7. Consider the case of a cusp:

$$Y = Z(u^2 - v^3) \hookrightarrow X = \mathbb{A}^2.$$

Y is a curve and $Y_{\rm sing} = \{0\}$, so that $\dim\, Y_m = \dim(\rho_m^Y)^{-1}(0)$.

In order to describe $(\rho_m^Y)^{-1}(0)$, note that it consists of ring homomorphisms

$$\phi\,:\,\mathbb{C}[u,v]/(u^2-v^3) \longrightarrow \mathbb{C}[t]/(t^{m+1})$$

such that $\mathrm{ord}(\phi(u))$, $\mathrm{ord}(\phi(v)) \geq 1$. If $m > 5$, this implies that $\phi(u) = t^3 f$ and $\phi(v) = t^2 g$, such that the classes $\overline{f}$, $\overline{g}$ of f and g in $\mathbb{C}[t]/(t^{m-5})$ satisfy $\overline{f}^2 - \overline{g}^3 = 0$. In this way we get an isomorphism $(\rho_m^Y)^{-1}(0) \simeq Y_{m-6} \times \mathbb{A}^7$, so that $\dim\, Y_m = \dim\, Y_{m-6} + 7$. This easily gives $\sup_m\{(\dim\, Y_m)/(m+1)\} = 7/6$, so that $c(\mathbb{A}^2, Y) = 5/6$.

4. The log canonical threshold via jets

In this section we apply Corollary 3.4 to deduce properties of the log canonical threshold which appear in the analytic context in [DK]. In the codimension one case, some of these properties are proved also in [Kol] using log resolutions. We believe that our treatment, using properties of the jet schemes, simplifies many of the proofs.

In this section, when we consider pairs (X,Y) we also allow the case $Y = X$. Recall that we follow the convention that $\dim(\emptyset) = -\infty$.

Proposition 4.1 ([DK], 1.4)**.** *If X is a smooth variety, Y', Y'' are two closed subschemes such that $Y' \hookrightarrow Y''$ and $Z \subseteq X$ is a nonempty closed subset, then $c_Z(X,Y') \geq c_Z(X,Y'')$.*

Proof. Since Y' is a subscheme of Y'', it follows that Y'_m is a subscheme of Y''_m, so that $\dim_Z Y'_m \leq \dim_Z Y''_m$ for every m. The assertion now follows from Corollary 3.4. □

Proposition 4.2 ([DK], 1.4)**.** *For every pair (X,Y) and every nonempty closed subset $Z \subseteq X$, we have $c_Z(X,Y) \leq \mathrm{codim}_Z(Y,X)$, where $\mathrm{codim}_Z(X,Y)$ denotes the smallest codimension of an irreducible component of Y meeting Z.*

Proof. Using Corollary 3.4, we have

$$\begin{aligned} c_Z(X,Y) &= \dim\, X - \sup_{m\geq 0} \frac{\dim_Z Y_m}{m+1} \\ &\leq \dim\, X - \dim_Z Y_0 = \mathrm{codim}_Z(Y,X). \end{aligned}$$

□

For a pair (X, Y) and a point $x \in X$, we define $\mathrm{mult}_x Y$ as follows. If $\mathcal{O}_{X,x}$ is the local ring of X at x, with maximal ideal $\underline{m}_{X,x}$, and if $\mathcal{I}_{Y,x} \subseteq \mathcal{O}_{X,x}$ is the ideal of Y at x, then $\mathrm{mult}_x Y$ is the largest $q \in \mathbb{N}$ such that $I_{Y,x} \subseteq \underline{m}_{X,x}^q$. Note that if Y is a divisor, then this is the same with the Hilbert-Samuel multiplicity, but in general the two notions are different.

Proposition 4.3 ([DK], 1.4, [Kol], 8.10)**.** *Let (X, Y) be a pair and $Z \subseteq X$ a nonempty closed subset. If q is the smallest $p \in \mathbb{N}$ such that $\mathrm{mult}_x Y \leq p$ for every $x \in Z$, then*

$$1/q \leq c_Z(X, Y) \leq (\dim X)/q. \tag{4.1}$$

Proof. The cases $Y = X$ (when $q = \infty$) and $Y \cap Z = \emptyset$ (when $q = 0$) are trivial, so that we may suppose that we are in neither of them.

There is $y \in Z$ such that $\mathrm{mult}_y Y = q$. If $\rho_{q-1} : X_{q-1} \longrightarrow X$ is the canonical projection, then $\rho_{q-1}^{-1}(y) \subseteq Y_{q-1}$. Indeed, every local morphism $\mathcal{O}_{X,y} \longrightarrow \mathbb{C}[t]/(t^q)$ factors through $I_{Y,y}$, as $I_{Y,y} \subseteq \underline{m}_{X,y}^q$.

This gives $\dim_Z Y_{q-1} \geq \dim \rho_{q-1}^{-1}(y) = (q-1)\dim X$. Corollary 3.4 implies that $c_Z(X, Y) \leq \dim X - (1/q)(q-1)\dim X = (\dim X)/q$.

For the other inequality we use Lemma 2.5. This shows that for every $x \in X$ and every $m \in \mathbb{N}$, we have $\dim(\rho_m^{-1}(x) \cap Y_m) \leq m \dim X - [m/\mathrm{mult}_x Y]$. Since $\mathrm{mult}_x Y \leq q$ for every $x \in Z$, the same is true on an open neighbourhood U of Z. We deduce

$$\dim (U \cap Y)_m \leq \dim Y + m \cdot \dim X - [m/q],$$

for every m, and an easy computation gives $\dim (U \cap Y)_m/(m+1) \leq \dim X - 1/q$. Corollary 3.4 implies $c_Z(X, Y) \geq c(U, U \cap Y) \geq 1/q$. $\square$

Proposition 4.4 ([DK], 2.7))**.** *If (X', Y') and (X'', Y'') are two pairs and $Z' \subseteq X'$, $Z'' \subseteq X''$ are two nonempty closed subsets, then*

$$c_{Z' \times Z''}(X' \times X'', Y' \times Y'') = c_{Z'}(X', Y') + c_{Z''}(X'', Y''). \tag{4.2}$$

Proof. Proposition 2.1 gives a canonical isomorphism $(Y' \times Y'')_m \simeq Y'_m \times Y''_m$, for every m. Therefore we have $\dim_{Z' \times Z''}(Y' \times Y'')_m = \dim_{Z'} Y'_m + \dim_{Z''} Y''_m$. We pick m such that a'_i, a''_j, $a_k \mid (m+1)$ for all i, j and k, where a'_i, a''_j and a_k are the coefficients appearing in log resolutions of (X', Y'), (X'', Y'') and $(X' \times X''$, $Y' \times Y'')$. We conclude by applying the second assertion in Corollary 3.4. $\square$

Proposition 4.5 ([DK], 2.2)**.** *Let (X, Y) be a pair. If $H \subset X$ is a smooth irreducible divisor and if $Z \subseteq H$ is a nonempty closed subset, then*

$$c_Z(X, Y) \geq c_Z(H, Y \cap H). \tag{4.3}$$

Proof. Let U be an open neighbourhood of Z in X such that $c_Z(H, H \cap Y) = c\,(U \cap H, U \cap H \cap Y)$. Since $c_Z(X, Y) \geq c_{U \cap H}(U, U \cap Y)$, by restricting everything to U, we may assume that $Z = H$.

From the local description of the jet schemes, it follows that since $H \cap Y$ is defined locally in Y by one equation, we have $(H \cap Y)_m$ defined locally in Y_m by $m+1$ equations, for every $m \in \mathbb{N}$. Moreover, if $\rho_m : X_m \longrightarrow X$ is the canonical projection, then for every irreducible component T of Y_m such that $\rho_m(T) \cap H \neq \emptyset$, we have $T \cap (Y \cap H)_m \neq \emptyset$.

Indeed, it follows from our discussion in §2 that if $\sigma_m^Y : Y \longrightarrow Y_m$ is the "zero section", then $\sigma_m^Y(\rho_m(T)) \subseteq T$. Moreover, since the "zero section" is functorial, we have $\sigma_m^Y(Y\cap H)\subset (Y\cap H)_m$, and we deduce that $(Y\cap H)_m\cap T$ contains $\sigma_m^Y(\rho_m(T)\cap H)$.

It follows that $\dim_H Y_m \leq \dim(Y\cap H)+m+1$, so that

$$\sup_{m\geq 0}\frac{\dim_H Y_m}{m+1}\leq \sup_{m\geq 0}\frac{\dim(Y\cap H)_m}{m+1}+1.$$

Applying Corollary 3.4, we deduce the inequality in the proposition. □

Corollary 4.6. *The statement of Proposition 4.5 remains true if we replace H with a smooth variety of arbitrary codimension.*

Proof. If we cover X with open subsets U_i, then we obviously have $c_Z(X,Y)=\inf_i\{c_{Z\cap U_i}(U_i, Y\cap U_i)\}$ and a similar relation for the restrictions to H. Since H is smooth, it is locally a complete intersection, so that we can find an open cover of X such that on each subset U_i, $H\cap U_i$ is an intersection of smooth divisors. It is therefore enough to apply Proposition 4.5 inductively on each of these open subsets. □

This corollary can be used to deduce a formula for the log canonical threshold of an intersection $Y'\cap Y''$ in terms of the dimensions of the jet schemes of Y' and Y''. We give below the case when the closed subset $Z\subseteq X$ is a point, when the formula has a particularly nice form.

Proposition 4.7 ([DK], 2.9)**.** *If X is a smooth variety, Y', Y'' two proper closed subschemes and $z\in Y'\cap Y''$ a point, then*

$$c_z(X,Y')+c_z(X,Y'')\geq c_z(X,Y'\cap Y''). \tag{4.4}$$

Proof. Consider the following cartesian diagram:

$$\begin{array}{ccc} Y'\cap Y'' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle\Delta} \\ Y'\times Y'' & \longrightarrow & X\times X \end{array}$$

where the horizontal maps are the natural embeddings and Δ is the diagonal embedding. Applying Corollary 4.6 to the embedding Δ, the pair $(X\times X, Y'\times Y'')$, and the closed subset $\{(z,z)\}\subset\Delta(X)$, we deduce

$$c_{(z,z)}(X\times X, Y'\times Y'')\geq c_z(X,Y'\cap Y'').$$

On the other hand, Proposition 4.4 implies

$$c_{(z,z)}(X\times X, Y'\times Y'')=c_z(X,Y')+c_z(X,Y''),$$

and these two relations prove the proposition. □

We now use Corollary 3.4 and Proposition 2.3 to prove a result of Demailly and Kollár ([DK]) about the semicontinuity of the log canonical threshold. We first need a preliminary result. For the notation concerning the coefficients of the divisors in a log resolution, see the beginning of the previous section.

Lemma 4.8. *Let $\pi : \mathcal{X}\longrightarrow S$ be a smooth morphism and $\mathcal{Y}\hookrightarrow\mathcal{X}$ a closed subscheme. It is possible to find log resolutions for each fiber $(\mathcal{X}_s,\mathcal{Y}_s)$ such that the coefficients a_i, b_i which appear in all these resolutions form a finite set.*

Proof. We will prove this by induction on $\dim S$. It is enough to find a locally closed cover of S such that the restriction of π over each member of the cover has the desired property. Therefore we may assume that S is a smooth variety and it is enough to find a nonempty open subset U of S over which the restriction of π has this property. By a theorem of Nagata, we can embed $\mathcal{X}$ as an open subscheme in a scheme $\mathcal{X}'$ proper over S. By replacing $\mathcal{X}$ with $\mathcal{X}'$ and $\mathcal{Y}$ by its closure in $\mathcal{X}'$, we may assume that π is proper. After restricting to an open subset of S, we may assume that π is also smooth.

Let $\pi' : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be a log resolution for the pair $(\mathcal{X}, \mathcal{Y})$. It is easy to see that after further restricting over an open subset of S we can assume that for every $s \in S$, the restriction to the fiber $\pi_s : \widetilde{\mathcal{X}}_s \longrightarrow \mathcal{X}_s$ is a log resolution of $(\mathcal{X}_s, \mathcal{Y}_s)$, in which case the assertion is obvious. ☐

We can now give the proof of the semicontinuity result.

Theorem 4.9 ([DK], 3.1)**.** *Let $\pi : \mathcal{X} \longrightarrow S$ be a smooth morphism, $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, and $\tau : S \longrightarrow \mathcal{Y}$ a section of $\pi|_{\mathcal{Y}}$. The function defined by*

$$f(s) = c_{\tau(s)}(\mathcal{X}_s, \mathcal{Y}_s),$$

for every closed point $s \in S$, is lower semicontinuous.

Proof. Lemma 4.8 and Proposition 1.5 show that the set $\{c_{\tau(s)}(\mathcal{X}_s, \mathcal{Y}_s) \mid s \in S\}$ is finite. Moreover, it follows from Lemma 4.8 and Corollary 3.4 that we can find $N \geq 1$ such that for every $s \in S$ and every m with $N | (m+1)$, we have

$$c_{\tau(s)}(\mathcal{X}_s, \mathcal{Y}_s) = \dim \mathcal{X}_s - \frac{\dim_{\tau(s)} (\mathcal{Y}_s)_m}{m+1}.$$

Fix $s_0 \in S$. Since there are only finitely many log canonical thresholds to consider, it is enough to show that for every $\epsilon > 0$, there is an open neighbourhood U of s_0 such that $c_{\tau(s)}(\mathcal{X}_s, \mathcal{Y}_s) \geq c_{\tau(s_0)}(\mathcal{X}_{s_0}, \mathcal{Y}_{s_0}) - \epsilon$, for every $s \in U$. Since π is flat, by restricting to an open neighbourhood of s_0 we may assume that $\dim \mathcal{X}_s$ is constant on S.

Therefore it is enough to find m with $N | (m+1)$ and U such that

$$\dim_{\tau(s)} (\mathcal{Y}_s)_m / (m+1) \leq \dim_{\tau(s_0)} (\mathcal{Y}_{s_0})_m / (m+1) + \epsilon,$$

for all $s \in U$. We fix m such that $N | (m+1)$ and $\dim \mathcal{X}_s / (m+1) \leq \epsilon$ for all s. Using Proposition 2.3, we choose an open neighbourhood U of s_0 such that

$$\dim (\rho_m^{\mathcal{Y}_s})^{-1}(\tau(s)) \leq \dim (\rho_m^{\mathcal{Y}_{s_0}})^{-1}(\tau(s_0)),$$

for all $s \in U$. The following inequalities show that U satisfies our requirement:

$$\dim_{\tau(s)} (\mathcal{Y}_s)_m / (m+1) \leq \dim (\rho_m^{\mathcal{Y}_s})^{-1}(\tau(s))/(m+1) + \dim \mathcal{X}_s/(m+1)$$

$$\leq \dim (\rho_m^{\mathcal{Y}_{s_0}})^{-1}(\tau(s_0))/(m+1) + \epsilon \leq \dim_{\tau(s_0)} (\mathcal{Y}_{s_0})_m/(m+1) + \epsilon.$$

☐

We conclude by showing how the formula in Corollary 3.4 can be used to explicitly compute the log canonical threshold. We consider the case of monomial ideals and derive a formula from [AGV] (see also [Ho] for a formula in this context for all the multiplier ideals).

Let us fix the notation. $X = \mathbf{A}^n$ is the affine space and $Y = V(I)$, where $I \subset R = k[X_1, \ldots, X_n]$ is a monomial ideal such that $(0) \neq I \neq R$. For $\mathbf{a} = (a_i)_i \in \mathbb{Z}^n$, we use the notation $X^{\mathbf{a}} = \prod_i X_i^{a_i}$. The vector $(1, \ldots, 1) \in \mathbb{Z}^n$ is denoted by $\mathbf{e}$. The Newton polyhedron of I, denoted by P_I, is the convex hull (in $\mathbb{R}^n$) of the set $\{\mathbf{a} \in \mathbb{Z}^n \mid X^{\mathbf{a}} \in I\}$.

Proposition 4.10 ([AGV], [Ho]). *With the above notation, we have the following formula for the log canonical threshold of (X, Y):*

$$c(X, Y) = \sup\{r > 0 \mid \mathbf{e} \in rP_I\}.$$

Proof. We will use the formula in Corollary 3.4, so that we first estimate dim Y_m, for every $m \geq 1$. Note that Y_m can be covered by the locally closed subsets $Z_{a_1,\ldots,a_n}$, with $0 \leq a_i \leq m+1$ for all i, where $Z_{a_1,\ldots,a_n}$ is the set of ring homomorphisms $\phi : R/I \longrightarrow k[t]/(t^{m+1})$, with $\mathrm{ord}\,(\phi(X_i)) = a_i$, for every i. We put $\mathrm{ord}(\phi(X_i)) = m+1$ if $\phi(X_i) = 0$.

It is clear that if $Z_{a_1,\ldots,a_n} \neq \emptyset$, then $\dim Z_{a_1,\ldots,a_n} = (m+1)n - \sum_i a_i$. On the other hand, $Z_{a_1,\ldots,a_n} \neq \emptyset$ if and only if for every $\mathbf{b} \in \mathbb{N}^n$ such that $X^{\mathbf{b}} \in I$, we have $\sum_i a_i b_i \geq (m+1)$.

Let P_I° be the polar polyhedron of P_I, defined by

$$P_I^{\circ} = \{\mathbf{u} \in \mathbb{R}^n \mid \sum_i u_i v_i \geq 1, \text{ for all } \mathbf{v} \in P_I\}.$$

We see that $Z_{a_1,\ldots,a_n} \neq \emptyset$ if and only if $(a_1/(m+1), \ldots, a_n/(m+1)) \in P_I^{\circ}$.

From the above discussion, we deduce that

$$\frac{\dim X_m}{m+1} = n - \frac{1}{(m+1)} \inf_{\mathbf{a}} \sum_i a_i,$$

where the infimum is taken over all $\mathbf{a} \in \mathbb{N}^n \cap (m+1)P_I^{\circ}$, such that $a_i \leq m+1$ for all i. The formula in Corollary 3.4 gives

$$c\,(X, Y) = \inf_{\mathbf{a}} \sum_i a_i,$$

the infimum being taken over all $\mathbf{a} \in P_I^{\circ} \cap \mathbb{Q}^n \cap [0, 1]^n$.

We clearly have $P_I^{\circ} \subset \mathbb{R}_+^n$. Note that if $\mathbf{a} \in P_I^{\circ}$ and $\mathbf{a}'$ is defined by $a_i' = \min\{a_i, 1\}$ for all i, then $\mathbf{a}' \in P_I^{\circ}$. We deduce that $c\,(X, Y) = \inf_{\mathbf{a} \in P_I^{\circ}} \sum_i a_i$. In order to complete the proof it is enough to note that $\mathbf{e} \in rP_I$ if and only if for every $\mathbf{a} \in P_I^{\circ}$ we have $\sum_i a_i \geq r$ (this comes from the fact that P_I is the polar polyhedron of P_I°). $\square$

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Department of Mathematics, University of California, Berkeley, California 94720 – and – Institute of Mathematics of the Romanian Academy

Current address: Clay Mathematics Institute, 1770 Massachusetts Avenue, No. 331, Cambridge, Massachusetts 02140

E-mail address: mirceamustata@yahoo.com