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BLOW UP IN FINITE TIME AND DYNAMICS OF BLOW UP SOLUTIONS FOR THE L^2–CRITICAL GENERALIZED KDV EQUATION

YVAN MARTEL AND FRANK MERLE

1. Introduction

In this paper, we are interested in the phenomenon of blow up in finite time (or formation of singularity in finite time) of solutions of the critical generalized KdV equation. Few results are known in the context of partial differential equations with a Hamiltonian structure. For the semilinear wave equation, or more generally for hyperbolic systems, the finite speed of propagation allows one to build blowing up solutions by reducing the problem to an ordinary differential equation. For the nonlinear Schrödinger equation,

$$\mathrm{i}u_t = -\Delta u - |u|^{p-1}u, \quad \text{where } u : \mathbf{R} \times \mathbf{R}^N \to \mathbf{C}, \tag{1}$$

the formation of singularity is related to the existence of a conformal invariance of the equation in the critical case, namely,

$$\mathrm{i}u_t = -\Delta u - |u|^{\frac{4}{N}}u \quad (\text{for } N=1\text{: } \mathrm{i}u_t = -u_{xx} - |u|^4 u). \tag{2}$$

Indeed, in this case if $u(t,x)$ is a solution of equation (2), then

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} e^{\frac{\mathrm{i}|x|^2}{4t}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)$$

is also a solution. Observe that this invariance relates regular solutions defined for all time to singular solutions. (Note that in the supercritical case, i.e. $p > 1 + \frac{4}{N}$, the Virial identity, which provides blow up solutions with negative energy, is itself a consequence of the conformal invariance in the critical case.) For more detail see for example [10]. In different contexts, these questions are mostly open.

We consider in this paper the case of the critical generalized Korteweg–de Vries equation

$$\begin{cases} u_t + (u_{xx} + u^5)_x = 0, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{cases} \tag{3}$$

for $u_0 \in H^1(\mathbf{R})$. For this equation, we suggest an approach based on a different idea (due to the lack of invariance for this equation). Let us recall a few facts

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concerning (3). It is a special case of the generalized Korteweg–de Vries equations, for $p \geq 2$ an integer:

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{4}$$

Cases $p = 2$ and $p = 3$, which correspond respectively to the KdV equation and modified KdV equation, have been studied extensively for being completely integrable (see for example Lax [6] and Miura [12]). From the Hamiltonian structure, there are two conservation laws

$$\int u^2(t) = \int u_0^2 \quad \text{(mass conservation)}, \tag{5}$$

$$\begin{aligned} &\frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) \\ &\quad = \frac{1}{2}\int u_{0x}^2 - \frac{1}{p+1}\int u_0^{p+1} \quad \text{(energy conservation)}. \end{aligned} \tag{6}$$

In this paper, we consider only the critical case $p = 5$ and solutions in the energy space $H^1(\mathbf{R})$. We define the energy

$$E(u) = \frac{1}{2}\int u_x^2 - \frac{1}{6}\int u^6.$$

In [5], Kenig, Ponce and Vega prove the following existence and uniqueness result in the energy space $H^1(\mathbf{R})$: for $u_0 \in H^1(\mathbf{R})$, there exist $T > 0$ and a unique maximal solution $u \in C([0,T), H^1(\mathbf{R}))$ of (4) on $[0,T)$. Moreover, either $T = +\infty$, or $T < +\infty$, and then $|u(t)|_{H^1} \to +\infty$, as $t \uparrow T$. In addition, for all $t \in [0,T)$, (5) and (6) are satisfied. Note that for equation (3), the local Cauchy problem is also well posed in $L^2(\mathbf{R})$ (see [5]). We refer to Kato [4] and Ginibre and Tsutsumi [3] for previous results on the well-posedness of the Cauchy problem for (4) and to Bourgain [2] for the periodic case.

For $p < 5$ (the subcritical case), as a consequence of the Gagliardo–Nirenberg inequality, all solutions in H^1 are global and bounded in time. For $p \geq 5$, the situation is different, blow up in finite time is suspected and for this problem $p = 5$ appears as a critical power.

We now fix $p = 5$ and we consider the problem of blow up for (3). Let us introduce the ground state Q, the unique positive solution up to translation of

$$Q_{xx} + Q^5 = Q, \quad Q \in H^1(\mathbf{R}), \quad \text{or equivalently} \quad Q(x) = \frac{3^{1/4}}{\mathrm{ch}^{1/2}(2x)}.$$

Note that $u(t,x) = Q(x-t)$ and $\forall c > 0$, $u_c(t,x) = c^{1/4}Q(c^{1/2}(x-ct))$ are special solutions of (3), satisfying $E(u_c) = E(Q) = 0$.

The variational characterization of Q gives the following Gagliardo–Nirenberg inequality (see Weinstein [13]):

$$\forall v \in H^1(\mathbf{R}), \quad \frac{1}{6}\int v^6 \leq \frac{1}{2}\left(\frac{\int v^2}{\int Q^2}\right)^2 \int v_x^2. \tag{7}$$

In particular, if $|u_0|_{L^2} < |Q|_{L^2}$, then by (5), (6) and (7), the solution $u(t)$ is global and uniformly bounded in H^1. On the contrary, for $|u_0|_{L^2} > |Q|_{L^2}$ there is no obstruction to blow up from energy type arguments. Note that numerical observations suggest existence of blow up in finite time; see Bona et al. [1].

Our approach to study blow up for solutions of (3) is based on a qualitative description of the solutions, either when the solution is global and bounded, or when blow up occurs. We focus on the case where the nonlinear dynamics plays a role and the L^2 norm of the solution is small, i.e.

$$\int Q^2 < \int u_0^2 < \int Q^2 + \alpha_0,$$

where $\alpha_0 > 0$ small.

In the case $E(u_0) < 0$, from the conservation laws, the solution remains close to the function Q in H^1 up to scaling and translation, so that we are able to define a continuous decomposition of the solution of the type

$$u(t,x) = \frac{1}{\widetilde{\lambda}^{1/2}(t)}(Q + \widetilde{\varepsilon}\,)\left(t, \frac{x - \widetilde{x}(t)}{\widetilde{\lambda}(t)}\right), \tag{8}$$

with $|\widetilde{\varepsilon}\,(t)|_{H^1} \leq \delta(\alpha_0)$, where $\delta(\alpha_0) \to 0$ where $\alpha_0 \to 0$. Here, we use the fact that equation (3) is invariant under the scaling transform

if $u(t,x)$ is a solution of (3), then $\forall \lambda > 0$, $\lambda^{1/2}u(\lambda^3 t, \lambda x)$ is also a solution of (3),

which lets the L^2 norm be invariant. This property of closeness to Q up to $\widetilde{\lambda}(t)$, $\widetilde{x}(t)$ gives a nonvanishing property and allows us to define an asymptotic object recurrent in time as $t \to +\infty$ or $t \to T$ (T being the blow up time). The key idea is to show that the recurrence in time yields some rigidity on this object. In fact, we are then able to prove both elliptic type and oscillatory integral type estimates for this limit solution to give the desired result. We now recall the results on blow up obtained so far following this approach in Martel and Merle [8], [9] and Merle [11].

First, existence of solutions of (3) blowing up in finite or infinite time in the energy space H^1 has been proved in [11].

Blow up result ([11])**.** *There exists $\alpha_1 > 0$ such that the following is true. Let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \leq \int Q^2 + \alpha_1$ and $E(u_0) < 0$. Then the corresponding solution $u(t)$ of* (3) *blows up in finite or infinite time, i.e. there exists $0 < T \leq +\infty$ such that*

$$\lim_{t \uparrow T} |u_x(t)|_{L^2} = +\infty \quad \textit{or equivalently} \quad \widetilde{\lambda}(t) \to 0 \textit{ as } t \uparrow T.$$

Note that elliptic estimates (exponential decay in $|x|$) and oscillatory estimates, together with the three conservation laws (mass, energy and $\int u(t,x)dx$), give complete information on the variation of the size of the limit solutions and prove the theorem.

Next, in [9], we have addressed the question of the blow up profile, i.e. the asymptotic form of the solutions after rescaling. We have a characterization of the blow up profile, up to the invariances of the equation. This result is also based on mixed elliptic and oscillatory estimates on the limit object.

Blow up profile ([9])**.** *There exists $0 < \alpha_2 < \alpha_1$ such that the following is true. Let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \leq \int Q^2 + \alpha_2$ and $E(u_0) < 0$. Let $u(t)$ be the corresponding solution of* (3)*, and let $0 < T \leq +\infty$ be its blow up time. Then for all $t \in [0,T)$ there exist $\widetilde{\lambda}(t) > 0$ and $\widetilde{x}(t) \in \mathbf{R}$ such that*

$$\textit{either} \quad \widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)x + \widetilde{x}(t)) \rightharpoonup Q \quad \textit{or} \quad -\widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)x + \widetilde{x}(t)) \rightharpoonup Q$$

in $H^1(\mathbf{R})$ weak, as $t \uparrow T$ with $\widetilde{\lambda}(t) \to 0$, as $t \uparrow T$.

Remark. Note that the alternative in the previous result comes from the fact that if $u(t,x)$ is a solution of (3), then $-u(t,x)$ is also a solution. This is, in some sense, a generalization of the Liouville theorem, and of its corollary in [8], which says that any solution bounded in H^1 from below and from above, starting close to Q in H^1, converges locally in space to Q for large time.

Therefore, for

$$\int u_0^2 \leq \int Q^2 + \alpha_2 \quad \text{and} \quad E(u_0) < 0,$$

we have

$$\widetilde{\lambda}(t) \to 0 \quad \text{and} \quad \widetilde{\varepsilon}(t) \rightharpoonup 0 \quad \text{as } t \to T,$$

where $\widetilde{\lambda}(t)$ and $\widetilde{\varepsilon}(t)$ are related to the decomposition of $u(t)$. Now, with this information on the asymptotic behavior of the solutions, we look at the asymptotic dynamics to obtain a control on the blow up rate (by studying time relations between key quantities of the problem).

More precisely, assume in addition that the initial data has a decay in L^2 at the right, in the following sense:

$$\exists \theta > 0, \quad \text{such that} \quad \forall x_0 > 0, \quad \int_{x \geq x_0} u_0^2(x)dx \leq \frac{\theta}{x_0^6}.$$

Then blow up occurs in finite time. Moreover, we have an upper bound on the blow up rate for a special subsequence of time. The main result is the following.

Theorem 1 (Blow up in finite time and dynamics of blow up solutions)**.** *There exists $0 < \alpha_3 < \alpha_2$ such that the following is true. Let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \leq \int Q^2 + \alpha_3$ and $E(u_0) = E_0 < 0$. Let $u(t)$ be the corresponding solution of* (3)*. Assume in addition that for some $\theta > 0$,*

$$\forall x_0 > 0, \quad \int_{x \geq x_0} u_0^2(x)dx \leq \frac{\theta}{x_0^6}. \tag{9}$$

(i) *Then $u(t)$ blows up in finite time, i.e.*

$$\text{for } T < +\infty \quad \lim_{t \uparrow T} |u_x(t)|_{L^2} = +\infty.$$

(ii) *Moreover, let $t_n \to T$ be the sequence defined as*

$$|u_x(t_n)|_{L^2} = 2^n |Q_x|_{L^2} \quad \text{and} \quad \forall t \in (t_n, T),\ |u_x(t)|_{L^2} > 2^n |Q_x|_{L^2}. \tag{10}$$

Then, there exists $n(u_0)$ such that

$$\forall n \geq n(u_0), \quad |u_x(t_n)|_{L^2} \leq \frac{C_0}{|E_0|(T - t_n)}, \tag{11}$$

where $C_0 = 4(\int Q)^2 |Q_x|_{L^2}$.

Comments and remarks on Theorem 1. We give some comments on the hypothesis and on the conclusion of Theorem 1. In particular, we point out some variants of the assumptions of the theorem.

1. *Existence of such initial data.* This follows from the fact that $E(Q) = 0$, $\nabla E(Q) = -Q$ and Q decays exponentially.

2. *Polynomial decay* (9). Note first that α_3 given in Theorem 1 does not depend on the size of θ. Note also that, for example, condition (9) is implied by

$$\int_{x>0} u_0^2(x)\, x^6 dx < +\infty.$$

This is related to the classical assumption $\int |x|^2 |u_0(x)|^2 dx < +\infty$ in the context of the critical and supercritical nonlinear Schrödinger equations. Recall that in this case, $E(u_0) < 0$ and $\int |x|^2 |u_0|^2 < +\infty$ imply blow up in finite time by the Virial identity. Remark that the power 6 is not optimal from the proof of Theorem 1.

3. Note that the only requirement on the initial data is the fact that the blow up dynamics is close to Q, up to scaling and translation parameters, which is implied by some energy conditions on the initial data. In particular, the proof of Theorem 1 is not based on linearization close to a formal asymptotic profile and we do not require that the initial data is close in a certain sense to a formal blow up solution.

4. *Condition on the L^2 norm of the initial data.* One can expect that the proof also produces blow up solutions which have similar behavior but with large L^2 mass. Indeed, the important quantity in all the proof is the local norm close to the soliton. By the technique of L^2 localization used in [11] and [9], for any $u_0^1 \in H^1(\mathbf{R})$ (not necessarily small) such that $\forall x > 0$, $u_0^1(x) = 0$, and for u_0^2 satisfying the assumptions of Theorem 1, one expects that there exist $B_0 > 0$ large and $0 < t_0 < T(u_0^2)$ ($T(u_0^2)$ is the blow up time for the solution $u^2(t)$ corresponding to u_0^2) such that for the initial data $u_0(x) = u_0^1(x + B_0) + u^2(t_0, x)$, the conclusion of Theorem 1 holds.

We also think that for any initial data u_0 such that $E(u_0) < 0$, $\int u_0^2 x_+^6 < +\infty$, with large L^2 mass, there is a t_1 such that $u(t_1)$ satisfies the previous assumptions and then the solution blows up in finite time.

5. *Comments on the conclusion of Theorem 1.* We shall note that there is rather little control on how long it takes for the solution to reach its asymptotic dynamics, where $|u_x(t_n)|_{L^2}$ is controlled. Indeed, from the proof, it takes a time $t(u_0)$ which depends mainly on θ and $E(u_0)$. Note that the same problem exists for the critical nonlinear Schrödinger equation (at least in the nonradial case). This remark points out why numerical computations do not suggest any blow up rate.

Now, we finish the introduction by some comments on the conclusion of Theorem 1 concerning the blow up rate. First, recall the following corollary of the blow up profile result obtained in [9].

Lower bound on the blow up rate ([9]). *If $\int u_0^2 < \int Q^2 + \alpha_2$ and if the solution $u(t)$ of* (3) *blows up in finite time $T > 0$, then*

$$\lim_{t \uparrow T} (T - t)^{1/3} |u_x(t)|_{L^2} = +\infty.$$

This means that under the assumptions of the blow up result, we exclude the possibility of blow up at the self-similar rate such as $u(t,x) \sim \frac{1}{(T-t)^{1/6}} g\left(\frac{x - x(t)}{(T-t)^{1/3}}\right)$.

Now, we give two corollaries of Theorem 1 related to the blow up speed, under the assumptions of Theorem 1, and coming from the facts that the upper bound obtained in Theorem 1 is for the particular sequence (t_n) defined in (10), and that the proof provides more information. In fact, one difficulty of the proof of Theorem 1 is to control oscillations in time of the size of the solution. Under various assumptions, one can obtain an upper bound for all time.

Corollary 1. (Upper bounds on the blow up rate under the monotonicity assumption.) *Under the assumptions of Theorem 1, suppose in addition that for some* $t_1 \in [0,T)$,

$$\text{if } t_1 < t < t' < T, \text{ then } |u_x(t')|_{L^2} \geq \tfrac{1}{2}|u_x(t)|_{L^2}. \tag{12}$$

Then, there exists $t(u_0) \in (0,T)$ *such that*

$$\forall t \in (t(u_0),T), \quad |u_x(t)|_{L^2} \leq \frac{4C_0}{|E_0|(T-t)}. \tag{13}$$

Corollary 2. (Upper bounds on the blow up rate under the lower bounds assumption.) *Under the assumptions of Theorem 1, suppose in addition that for some* $B>0$, $n_1>0$, *we have*

$$\forall n \geq n_1, \quad |u_x(t_n)|_{L^2} \geq \frac{B}{(T-t_n)}. \tag{14}$$

Then, there exists $t(u_0) \in (0,T)$ *such that*

$$\forall t \in (t(u_0),T), \quad |u_x(t)|_{L^2} \leq \frac{8C_0}{|E_0|(T-t)}. \tag{15}$$

In fact, the proof of Theorem 1 is far more precise. Indeed, under some assumptions on the size of the gradient of $\widetilde{\varepsilon}$ in L^2 (recall that the perturbation term $\widetilde{\varepsilon}$ is defined in (8)), the proof provides the exact blow up rate. Nevertheless these estimates are an open problem. Assuming mainly that for $t(u_0)>0$, $K_0>0$, we have

$$\forall t \in (t(u_0),T), \quad \int \widetilde{\varepsilon}^2 e^{-|y|} \geq K_0 \int \widetilde{\varepsilon}_y^2 \quad \text{or} \quad \frac{1}{10}\left|\int \widetilde{\varepsilon} Q\right| \geq \int \widetilde{\varepsilon}_y^2, \tag{16}$$

where $\widetilde{\varepsilon}$ is defined in (8) with suitable orthogonality conditions, for some $C_0'>0$, we have

$$\forall t \in (t(u_0),T), \quad \frac{C_0'}{|E_0|(T-t)} \leq |u_x(t)|_{L^2} \leq \frac{8C_0}{|E_0|(T-t)}.$$

(See the Remark at the end of Section 4 for a precise statement.)

Let us now sketch the proof of the main result. Consider an initial data with negative energy which is close to Q in the L^2 norm and has an additional decay property for $x>0$, such as (9). Define the sequence (t_n) as in the statement of Theorem 1:

$$|u_x(t_n)|_{L^2} = 2^n |Q_x|_{L^2} \quad \text{and} \quad \forall t \in (t_n,T),\ |u_x(t)|_{L^2} > 2^n|Q_x|_{L^2}.$$

The existence of such a sequence is given by the blow up result [11] (in particular, we do not need to initialize any estimate in the proof).

First, we reduce the proof of Theorem 1 to the proof of the following estimate of $t_{n+1}-t_n$ for n large ($n>n_0$ where n_0 is such that for all $t>t_{n_0}$, the local L^2 norm of the solution is small, the existence of n_0 is a consequence of the blow up profile result):

$$t_{n+1}-t_n \leq \frac{(\int Q)^2}{|E_0|}\widetilde{\lambda}(t_n). \tag{17}$$

By elementary computations, from (17) it is easy to see that the sequence (t_n) is bounded, thus the blow up time T is finite, and then estimate (11) follows again from (17). Note that the proof is not based on a linearization close to a formal solution and that there is no initialization of the estimates (in fact the results of [11] and [9] replace the initialization).

To prove (17), we argue by contradiction, assuming that for some n arbitrarily large, $t_{n+1} - t_n > \frac{(\int Q)^2}{|E_0|}\widetilde{\lambda}(t_n)$. Under the constraint of closeness to Q in L^2, we have the decomposition of the solution

$$\lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) = Q(y) + \varepsilon(t, y),$$

where the function $\varepsilon(t, y)$ is small in H^1 and satisfies for all time t two suitable orthogonality conditions.

The assumption of polynomial decay on the initial data in L^2 at the right, through the use of the stability in time of such a property on the solution $u(t)$ of (3), allows us to obtain decay at the right on ε. This decay for $y > 0$ on ε allows us to control $\int_{y>0} |\varepsilon|$, by a *nonlinear* estimate. Note that the space in which this estimate holds is $L^2(\mathbf{R}) \cap L^1(\mathbf{R}^+)$.

From the proof, four quantities are important: $\int \varepsilon Q$, $\int \varepsilon^2 e^{-\frac{|y|}{100}}$, $\frac{\lambda_s}{\lambda}$, $\int \varepsilon_y^2$. The proof is based on the study of conservation laws (and in fact, local conservation laws close to Q), which will give relations between these four quantities. For each conservation law, a suitable orthogonality condition is adapted.

(i) By the additional decay property for $y > 0$, we are able to choose the orthogonality conditions on ε so that we have a relation of the following type (s is time related to the equation of ε):

$$-\frac{\lambda_s}{\lambda} = \int \varepsilon Q + O(\int \varepsilon^2 e^{-\frac{|y|}{100}}),$$

where $\int \varepsilon Q$ has a slow variation in time. (Indeed, at the linear level, this quantity is invariant, and seems to contain no oscillatory integrals.)

(ii) Next, for other orthogonality conditions, related to another decomposition

$$\widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)y + \widetilde{x}(t)) = Q(y) + \widetilde{\varepsilon}(t, y),$$

quadratic terms $\int_{t_n}^{t_{n+1}} \int \varepsilon^2 e^{-\frac{|y|}{100}}$ are controlled in some sense by $\int_{t_n}^{t_{n+1}} \int \widetilde{\varepsilon} Q$. This result is obtained by a local Virial identity on $\widetilde{\varepsilon}$ (related to orthogonality conditions on $\widetilde{\varepsilon}$). This points out the fact that time oscillations, involving oscillatory integrals, are controlled by quantities that are not oscillatory such as $\int \widetilde{\varepsilon} Q$.

(iii) Under this regime, we are able to compare the functions ε and $\widetilde{\varepsilon}$ coming from the two decompositions, by estimates involving L^2 exponential decay at the right of the soliton, and an additional surprising degeneracy in the relation between ε and $\widetilde{\varepsilon}$ of the conservation law. Thus, in some sense on the time interval (t_n, t_{n+1}), we obtain

$$-\frac{\lambda_s}{\lambda} = \int \widetilde{\varepsilon} Q + O(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}).$$

The small term at the right can again be controlled when $\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}$ is small (implied for a sufficiently large time by the result of the asymptotic profile [11]).

(iv) Now using the energy estimate for $\widetilde{\varepsilon}$ and the fact that $\int \widetilde{\varepsilon}_y^2 \geq 0$, we obtain

$$-\frac{\lambda_s}{\lambda} \geq C|E_0|\lambda^2,$$

in some integral in time sense between t_n, t_{n+1}. This allows us to obtain a contradiction, and thus to prove (17).

The paper is organized as follows. Section 2 is devoted to the introduction of two decompositions of the solution and of their respective properties. Section 3 is devoted to the proof of Theorem 1 assuming the estimate on $t_{n+1} - t_n$. This estimate is then proved in Section 4. Finally, the control of the decay on the right is proved in Section 5.

2. Two decompositions of the solution and related structure

In this section, we recall from [8], [11] and [7] some useful preliminary properties of equation (3).

Let

$$\alpha_0 = \int u_0^2 - \int Q^2 \quad \text{and} \quad E_0 = E(u_0) = \frac{1}{2}\int u_{0x}^2 - \frac{1}{6}\int u_0^6.$$

2.1. **Decomposition of $u(t)$ in L^2 and first properties.** We begin by recalling the following result from [11], [9].

Lemma 1 (Decomposition of the solution related to L^2 dispersion)**.** *There exists $a_1 > 0$ such that if $\alpha_0 < a_1$ and $E(u_0) < 0$, then there exist two C^1 functions $\widetilde{\lambda} : [0,T) \to (0,+\infty)$, $\widetilde{x} : [0,T) \to \mathbf{R}$, such that, for $w \equiv u$ or $w \equiv -u$,*

$$\forall t \in [0,T), \quad \widetilde{\varepsilon}(t,y) = \widetilde{\lambda}^{1/2}(t) w(t, \widetilde{\lambda}(t) y + \widetilde{x}(t)) - Q(y) \tag{18}$$

satisfies the following orthogonality conditions: $\forall t \in [0,T)$,

$$\int \left(\frac{Q}{2} + yQ_y\right) \widetilde{\varepsilon}(t,y)dy = \int y\left(\frac{Q}{2} + yQ_y\right) \widetilde{\varepsilon}(t,y)dy = 0. \tag{19}$$

Moreover, with $\frac{ds}{dt} = \frac{1}{\widetilde{\lambda}^3}$ and $s(0) = 0$, we have the following properties:

(i) *Equation of $\widetilde{\varepsilon}(s)$. The function $\widetilde{\varepsilon}(s)$ satisfies, for $s \in \mathbf{R}^+, y \in \mathbf{R}$,*

$$\begin{aligned} \widetilde{\varepsilon}_s = \quad & (L\widetilde{\varepsilon})_y + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\left(\frac{Q}{2} + yQ_y\right) + \left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right) Q_y \\ & + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\left(\frac{\widetilde{\varepsilon}}{2} + y\widetilde{\varepsilon}_y\right) + \left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right)\widetilde{\varepsilon}_y - (F(\widetilde{\varepsilon}) + \widetilde{\varepsilon}^5)_y, \end{aligned} \tag{20}$$

where $L\widetilde{\varepsilon} = -\widetilde{\varepsilon}_{yy} + \widetilde{\varepsilon} - 5Q^4\widetilde{\varepsilon}$ and $F(\widetilde{\varepsilon}) = 10Q^3\widetilde{\varepsilon}^2 + 10Q^2\widetilde{\varepsilon}^3 + 5Q\widetilde{\varepsilon}^4$.

(ii) *Smallness properties.*

$$\forall t \in [0,T), \quad \left|1 - \widetilde{\lambda}(t)\frac{|u_x(t)|_{L^2}}{|Q_x|_{L^2}}\right| \le \delta(\alpha_0), \quad \text{where } \delta(\alpha) \to 0 \text{ as } \alpha \to 0, \tag{21}$$

$$\forall t \in [0,T), \quad |\widetilde{\varepsilon}(t)|_{L^2} + |\widetilde{\varepsilon}_y(t)|_{L^2} \le C\sqrt{\alpha_0}, \quad \text{where } C > 0. \tag{22}$$

(iii) *Control of the geometrical parameters.*

$$\forall s \ge 0, \quad \left|\frac{\widetilde{\lambda}_s}{\widetilde{\lambda}} - \frac{1}{\mu_0}\int \widetilde{\varepsilon}(s) L\left(\left(\frac{Q}{2} + yQ_y\right)_y\right)\right| \le C\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}}, \tag{23}$$

$$\forall s \ge 0, \quad \left|\left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right) - \frac{1}{\mu_0}\int \widetilde{\varepsilon}(s) L\left[\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y\right]\right| \le C\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}}, \tag{24}$$

where $\mu_0 = \int \left(\frac{Q}{2} + yQ_y\right)^2$. In particular,

$$\forall s \ge 0, \quad \left|\frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\right| + \left|\left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right)\right| \le C\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}}\right)^{1/2} \le C\sqrt{\alpha_0}. \tag{25}$$

In the rest of this paper, we assume that $w = u$ in Lemma 1.

Remark. In the rest of this paper, we choose a_1 small enough so that by (21), (22) and (25), we have

$$\forall t \in [0,T), \quad \frac{|u_x(t)|_{L^2}}{(1.01)|Q_x|_{L^2}} \le \frac{1}{\widetilde{\lambda}(t)} \le \frac{(1.01)|u_x(t)|_{L^2}}{|Q_x|_{L^2}}, \tag{26}$$

$$\forall s > 0, \quad |\widetilde{\varepsilon}(s)|^2_{L^2} \le \frac{1}{4}\int Q^2, \tag{27}$$

$$\forall s > 0, \quad \frac{1}{1.01} \le \frac{\widetilde{x}_s(s)}{\widetilde{\lambda}(s)} \le 1.01. \tag{28}$$

Proof. We give a sketch of the proof of all these results. For more details, we refer the reader to [11] and [9].

(a) Existence and uniqueness of the decomposition.

First, we recall the following claim. Let $\alpha(u) = \int u^2 - \int Q^2$.

Claim 1. There exists $a_2 > 0$ such that the following property is true. For all $0 < \alpha < a_2$, there exists $\delta = \delta(\alpha) > 0$, with $\delta(\alpha) \to 0$ as $\alpha \to 0$, such that $\forall u \in H^1(\mathbf{R})$, $u \not\equiv 0$, if

$$\alpha(u) \le \alpha, \quad E(u) \le 0,$$

then there exist $x_0 \in \mathbf{R}$ and $\epsilon_0 \in \{-1, 1\}$ such that

$$|Q - \epsilon_0 \lambda_0^{1/2} u(\lambda_0 x + x_0)|_{H^1} \le \delta(\alpha),$$

with $\lambda_0 = \frac{|Q_x|_{L^2}}{|u_x|_{L^2}}$.

The proof of this claim is based on variational arguments, and in particular on the following characterization of Q: if a sequence (v_n) of H^1 satisfies

$$\lim_{n\to+\infty} \int v_n^2 = \int Q^2, \quad \lim_{n\to+\infty} \int v_{nx}^2 = \int Q_x^2, \quad \overline{\lim_{n\to+\infty}} E(v_n) \le 0,$$

then there exist a sequence (x_n) of $\mathbf{R}$ and $\epsilon_0 \in \{-1, 1\}$ such that

$$\lim_{n\to+\infty} \epsilon_0 v_n(\cdot + x_n) = Q \quad \text{in } H^1(\mathbf{R}) \text{ as } n \to +\infty.$$

See [11], proof of Lemma 1.

Let $u(t)$ be the solution of (3) on $[0,T)$ corresponding to $u_0 \in H^1(\mathbf{R})$, where $\alpha_0 = \alpha(u_0) < a_2$ and $E_0 = E(u_0) < 0$. By Claim 1, there exist $x_0(t)$ and $\epsilon_0(t)$ such that, with $\lambda_0(t) = |Q_x|_{L^2}/|u_x(t)|_{L^2}$, we have

$$\forall t \in [0,T), \quad |Q - \epsilon_0(t)\lambda_0^{1/2}(t) u(t, \lambda_0(t)x + x_0(t))|_{H^1} \le \delta(\alpha_0).$$

With no restriction, by (27), we can assume $\epsilon_0(t) \equiv 1$ (see [11]), using the fact that if $u(t,x)$ is a solution of (3), then $-u(t,x)$ is also a solution.

Now, we claim that we can sharpen the decomposition in the following sense.

Claim 2. There exists $0 < a_3 < a_2$ such that if $\alpha_0 < a_3$, then there exist unique C^1 functions $\widetilde{\lambda}(t)$ and $\widetilde{x}(t)$ such that

$$\widetilde{\varepsilon}(t,y) \equiv \widetilde{v}(t,y) - Q(y) \equiv \widetilde{\lambda}^{1/2}(t) u(t, \lambda(t)y + \widetilde{x}(t)) - Q(y)$$

satisfies the following properties: $\forall t \in [0,T)$,

$$\int \left(\frac{Q}{2} + yQ_y\right) \widetilde{\varepsilon}(t) = \int y \left(\frac{Q}{2} + yQ_y\right) \widetilde{\varepsilon}(t) = 0, \quad |\widetilde{\varepsilon}(t)|_{H^1} \le \delta(\alpha_0),$$

where $\delta(\alpha) \to 0$ as $\alpha \to 0$.

This claim is proved by using the implicit function theorem; see [9] and [7]. Note that the nondegeneracy conditions are satisfied from

$$\int \left(\frac{Q}{2} + yQ_y\right) y \left(\frac{Q}{2} + yQ_y\right) = 0, \quad \int (Q_y) y \left(\frac{Q}{2} + yQ_y\right) = \int \left(\frac{Q}{2} + yQ_y\right)^2 \ne 0,$$

$$\int \left(\frac{Q}{2} + yQ_y\right) yQ_y = \int \left(\frac{Q}{2} + yQ_y\right)^2 \ne 0, \quad \int (Q_y) yQ_y = 0,$$

where we have used $\int Q\left(\frac{Q}{2} + yQ_y\right) = 0$ and parity properties.

(b) Equation of $\widetilde{\varepsilon}$.

Now, let

$$s = \int_0^t \frac{dt'}{\widetilde{\lambda}^3(t')}, \quad \text{or equivalently} \quad \frac{ds}{dt} = \frac{1}{\widetilde{\lambda}^3} \text{ and } s(0) = 0.$$

Then s is defined in all $[0,+\infty)$. Indeed, recall that by a scaling argument and the resolution of the Cauchy problem in $H^1(\mathbf{R})$, it is proved in [11] (see the Introduction) that for a solution which blows up in finite time $T > 0$, there exists $C = C(u_0) > 0$ such that

$$\forall t \in [0,T), \quad |u_x(t)|_{L^2} \ge \frac{C}{(T-t)^{1/3}}.$$

Therefore, by (26), $\frac{(T-t)}{\widetilde{\lambda}^3(t)} \ge C > 0$, and when $t \in [0,T)$, the variable s is defined in all $[0,+\infty)$. In the case where $T = +\infty$, we have from energetic arguments that $\forall t > 0$, $\widetilde{\lambda}(t) \le \widetilde{\lambda}_0$, for some $\widetilde{\lambda}_0$ and so s is defined in all $[0,+\infty)$.

We have the following equation for $\widetilde{v} = \widetilde{\varepsilon} + Q$, in the s variable:

$$\widetilde{v}_s + (\widetilde{v}_{yy} + \widetilde{v}^5)_y - \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\left(\frac{\widetilde{v}}{2} + y\widetilde{v}_y\right) - \left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right)\widetilde{v}_y = 0.$$

Therefore, we obtain the following equation for $\widetilde{\varepsilon}$:

$$\begin{aligned} \widetilde{\varepsilon}_s \quad &= \quad (L\widetilde{\varepsilon})_y + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\left(\frac{Q}{2} + yQ_y\right) + \left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right) Q_y \\ &\quad + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\left(\frac{\widetilde{\varepsilon}}{2} + y\widetilde{\varepsilon}_y\right) + \left(\frac{\widetilde{x}_s}{\widetilde{\lambda}} - 1\right)\widetilde{\varepsilon}_y - (F(\widetilde{\varepsilon}) + \widetilde{\varepsilon}^5)_y, \end{aligned}$$

where $L\widetilde{\varepsilon} = -\widetilde{\varepsilon}_{xx} + \widetilde{\varepsilon} - 5Q^4\widetilde{\varepsilon}$ and $F(\widetilde{\varepsilon}) = 10Q^3\widetilde{\varepsilon}^2 + 10Q^2\widetilde{\varepsilon}^3 + 5Q\widetilde{\varepsilon}^4$.

Let us recall the following structural properties of L (see [7]):

$$L(Q^3) = -8Q^3, \quad L(Q_y) = 0, \tag{29}$$

$$\forall \varepsilon \in H^1(\mathbf{R}), \text{ if } \int Q^3\varepsilon = \int Q_y\varepsilon = 0, \text{ then } (L\varepsilon,\varepsilon) \ge \int \varepsilon^2. \tag{30}$$

(c) Smallness properties.

The smallness of $\left|1 - \widetilde{\lambda}(t)\frac{|u_x(t)|_{L^2}}{|Q_x|_{L^2}}\right|$ is given by the implicit function theorem in (a), $\widetilde{\lambda}(t)$ being close to $\lambda_0(t)$. Next, we have

$$\forall t \in [0,T), \quad |\widetilde{\varepsilon}(t)|_{L^2} + |\widetilde{\varepsilon}_y(t)|_{L^2} \le C\sqrt{\alpha_0}. \tag{31}$$

To prove (31), we use some ideas from [11]. By the definition of $\widetilde{\varepsilon}$ and the conservation of mass, we have

$$2\int Q\widetilde{\varepsilon}+\int\widetilde{\varepsilon}^2=\int u^2-\int Q^2=\int u_0^2-\int Q^2=\alpha_0.$$

By the conservation of the energy, we have $\widetilde{\lambda}^2E(Q+\widetilde{\varepsilon})=E(u(t))=E_0<0$, and by direct calculations,

$$\begin{aligned}&E(Q+\widetilde{\varepsilon})+\left(\int Q\widetilde{\varepsilon}+\frac{1}{2}\int\widetilde{\varepsilon}^2\right)\\ &\quad=\frac{1}{2}(L\widetilde{\varepsilon},\widetilde{\varepsilon})-\frac{1}{6}\left[20\int Q^3\widetilde{\varepsilon}^3+15\int Q^2\widetilde{\varepsilon}^4+6\int Q\widetilde{\varepsilon}^5+\int\widetilde{\varepsilon}^6\right].\end{aligned}\tag{32}$$

Therefore, we have

$$(L\widetilde{\varepsilon},\widetilde{\varepsilon})\le\alpha_0+C|\widetilde{\varepsilon}|_{H^1}|\widetilde{\varepsilon}|_{L^2}^2.\tag{33}$$

Note that by the choice of orthogonality conditions on $\widetilde{\varepsilon}$, this is not sufficient to conclude directly. Indeed, these are suitable for the Virial identity but not for the energy identity. Nevertheless, consider an auxiliary function ε_1:

$$\varepsilon_1=\widetilde{\varepsilon}-a\left(\frac{Q}{2}+yQ_y\right)-bQ_y.$$

We have

$$\int\varepsilon_1Q^3=\int\varepsilon_1Q_y=0,$$

by taking $a=\frac{\int\widetilde{\varepsilon}Q^3}{\int\left(\frac{Q}{2}+yQ_y\right)Q^3}$, $b=\frac{\int\widetilde{\varepsilon}Q_y}{\int Q_y^2}$ (note that $\int\left(\frac{Q}{2}+yQ_y\right)Q^3=\frac{1}{4}\int Q^4\not\equiv0$).

Note that we also have $\widetilde{\varepsilon}=\varepsilon_1+a\left(\frac{Q}{2}+yQ_y\right)+bQ_y$, so that by orthogonality conditions on $\widetilde{\varepsilon}$ we have $a=\frac{\int\varepsilon_1\left(\frac{Q}{2}+yQ_y\right)}{\int\left(\frac{Q}{2}+yQ_y\right)^2}$, $b=\frac{\int\varepsilon_1y\left(\frac{Q}{2}+yQ_y\right)}{\int\left(\frac{Q}{2}+yQ_y\right)^2}$.

Now, by $\int\left(\frac{Q}{2}+yQ_y\right)Q=0$, $L\left(\frac{Q}{2}+yQ_y\right)=-2Q$, $LQ_y=0$, we find after some elementary calculations:

$$\int\widetilde{\varepsilon}Q=\int\varepsilon_1Q,\quad(L\widetilde{\varepsilon},\widetilde{\varepsilon})=(L\varepsilon_1,\varepsilon_1)-4a(\widetilde{\varepsilon},Q).$$

By the expressions of a and b, we have for some constant K,

$$\frac{1}{K}(\varepsilon_1,\varepsilon_1)\le(\widetilde{\varepsilon},\widetilde{\varepsilon})\le K(\varepsilon_1,\varepsilon_1).\tag{34}$$

Thus, from (30) and (33),

$$\frac{1}{K}(\widetilde{\varepsilon},\widetilde{\varepsilon})\le(\varepsilon_1,\varepsilon_1)\le(L\varepsilon_1,\varepsilon_1)\le\alpha_0+4|a||(\widetilde{\varepsilon},Q)|+C|\widetilde{\varepsilon}|_{L^2}^2|\widetilde{\varepsilon}|_{H^1}.$$

For α_0 small, $|\widetilde{\varepsilon}|_{H^1}$ and $|a|$ are small and from the conservation of mass, we have $2|(\widetilde{\varepsilon},Q)|\le\alpha_0+\int\widetilde{\varepsilon}^2$; thus

$$\frac{1}{K}(\widetilde{\varepsilon},\widetilde{\varepsilon})\le2\alpha_0+C|\widetilde{\varepsilon}|_{L^2}^2(|a|+|\widetilde{\varepsilon}|_{H^1})\le2\alpha_0+\frac{1}{2K}|\widetilde{\varepsilon}|_{L^2}^2.$$

Therefore, $(\widetilde{\varepsilon},\widetilde{\varepsilon})\le4K\alpha_0$ and from (33), $(L\widetilde{\varepsilon},\widetilde{\varepsilon})\le C\alpha_0$. The conclusion then comes from $|\widetilde{\varepsilon}|_{H^1}^2\le(L\widetilde{\varepsilon},\widetilde{\varepsilon})+5\int Q^4\widetilde{\varepsilon}^2\le(L\widetilde{\varepsilon},\widetilde{\varepsilon})+c(\widetilde{\varepsilon},\widetilde{\varepsilon})$.

(d) By multiplying the equation of $\widetilde{\varepsilon}$ by $\left(\frac{Q}{2}+yQ_y\right)$ and then by $y\left(\frac{Q}{2}+yQ_y\right)$, and after integration by parts, we obtain (formally) the following estimates (note that we also use the decay property of Q and its derivatives). The calculations can be justified rigorously by regularization arguments.

$$\forall s\geq 0,\quad \left|\frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}-\frac{1}{\mu_0}\int \widetilde{\varepsilon}(s)L\left(\left(\frac{Q}{2}+yQ_y\right)_y\right)\right|\leq C\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}}, \tag{35}$$

$$\forall s\geq 0,\quad \left|\left(\frac{\widetilde{x}_s}{\widetilde{\lambda}}-1\right)-\frac{1}{\mu_0}\int \widetilde{\varepsilon}(s)L\left(\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right)\right|\leq C\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}}, \tag{36}$$

where $\mu_0=\int\left(\frac{Q}{2}+yQ_y\right)^2$.

This concludes the proof of Lemma 1.

Now, we recall the main properties of this decomposition.

Lemma 2 (Mass and energy relations)**.**

$$\text{(i) } \textit{mass conservation,}\quad 2\int \widetilde{\varepsilon} Q+\int \widetilde{\varepsilon}^2=\alpha_0, \tag{37}$$

$$\text{(ii) } \textit{energy relation,}\quad \left|\widetilde{\lambda}^2E_0+\int \widetilde{\varepsilon} Q-\left(\frac{1}{2}\int \widetilde{\varepsilon}_y^2-\frac{1}{6}\int \widetilde{\varepsilon}^6\right)\right|\leq C\int \widetilde{\varepsilon}^2e^{-|y|}. \tag{38}$$

Proof. (i) is the conservation of the L^2 norm of $u(t)$, written in terms of $\widetilde{\varepsilon}$.

(ii) is obtained by the conservation of energy for $u(t)$, $E(u(t))=E(u_0)$ and by expanding $E(Q+\widetilde{\varepsilon})=\widetilde{\lambda}^2E(u_0)$; see formula (32). For more details, see [9].

For future reference, we give a corollary of these conservation laws and of the fact that we assume $E_0<0$.

Corollary 3 (Estimates on $\widetilde{\varepsilon}$)**.**

$$\int \widetilde{\varepsilon} Q\geq \left|\int \widetilde{\varepsilon} Q\right|-C\int \widetilde{\varepsilon}^2e^{-|y|}, \tag{39}$$

$$\left|\int \widetilde{\varepsilon} Q\right|\geq \widetilde{\lambda}^2|E_0|-C\int \widetilde{\varepsilon}^2e^{-|y|}, \tag{40}$$

$$\left|\int \widetilde{\varepsilon} Q\right|\leq \widetilde{\lambda}^2|E_0|+\frac{1}{2}\int \widetilde{\varepsilon}_y^2+C\int \widetilde{\varepsilon}^2e^{-|y|}, \tag{41}$$

$$\widetilde{\lambda}^2\leq \frac{C}{|E_0|}\left(\int \widetilde{\varepsilon}^2e^{-|y|}\right)^{1/2}. \tag{42}$$

Proof. Proof of (39). Since $\int \widetilde{\varepsilon}^2\leq \frac{1}{4}\int Q^2<\int Q^2$, by the Gagliardo–Nirenberg inequality (7), we have $\frac{1}{2}\int \widetilde{\varepsilon}_y^2-\frac{1}{6}\int \widetilde{\varepsilon}^6>0$. By (38), since $E_0<0$, we have

$$\left|\int \widetilde{\varepsilon} Q\right|\leq \widetilde{\lambda}^2|E_0|+\frac{1}{2}\int \widetilde{\varepsilon}_y^2-\frac{1}{6}\int \widetilde{\varepsilon}^6+C\int \widetilde{\varepsilon}^2e^{-|y|} \tag{43}$$

and

$$\int \widetilde{\varepsilon} Q\geq \widetilde{\lambda}^2|E_0|+\frac{1}{2}\int \widetilde{\varepsilon}_y^2-\frac{1}{6}\int \widetilde{\varepsilon}^6-C\int \widetilde{\varepsilon}^2e^{-|y|}. \tag{44}$$

Therefore,

$$\int \widetilde{\varepsilon} Q \geq \left| \int \widetilde{\varepsilon} Q \right| - 2C \int \widetilde{\varepsilon}^2 e^{-|y|},$$

which proves (39).

Proof of (40). By (44) and $\frac{1}{2}\int \widetilde{\varepsilon}_y^2 - \frac{1}{6}\int \widetilde{\varepsilon}^6 > 0$, we have

$$\begin{aligned} \left| \int \widetilde{\varepsilon} Q \right| &\geq \int \widetilde{\varepsilon} Q \geq \widetilde{\lambda}^2 |E_0| + \tfrac{1}{2} \int \widetilde{\varepsilon}_y^2 - \tfrac{1}{6} \int \widetilde{\varepsilon}^6 - C \int \widetilde{\varepsilon}^2 e^{-|y|} \\ &\geq \widetilde{\lambda}^2 |E_0| - C \int \widetilde{\varepsilon}^2 e^{-|y|}, \end{aligned}$$

which proves (40).

Proof of (41). It is an obvious consequence of (38).

Proof of (42). We have $\left|\int \widetilde{\varepsilon} Q\right| \leq C \left(\int \widetilde{\varepsilon}^2 e^{-|y|}\right)^{1/2}$, by $|Q(y)| \leq Ce^{-|y|}$ and the Cauchy–Schwarz inequality. Thus, by (40), we obtain

$$\widetilde{\lambda}^2 |E_0| \leq C \left(\int \widetilde{\varepsilon}^2 e^{-|y|} \right)^{1/2},$$

and (42) follows.

This completes the proof of Corollary 3.

Now, let us give some dispersive relations in L^2. Let $\Phi \in C^2$, $\Phi(x) = \Phi(-x)$, $\Phi' \leq 0$ on $\mathbf{R}^+$, such that

$$\Phi(x) = 1 \text{ on } [0,1], \quad \Phi(x) = e^{-x} \text{ on } [2, +\infty),$$

and

$$e^{-x} \leq \Phi(x) \leq 3e^{-x} \quad \text{on } \mathbf{R}^+.$$

Let

$$\Psi(x) = \int_0^x \Phi(y) dy.$$

Note that Ψ is an odd function, $\Psi(x) = x$ on $[-1,1]$, and $|\Psi(x)| \leq 3$ on $\mathbf{R}$.

For a parameter $A_0 > 0$, we set

$$\Psi_{A_0}(x) = A_0 \Psi\left(\frac{x}{A_0}\right), \quad \text{so that} \quad \Psi'_{A_0}(x) = \Phi\left(\frac{x}{A_0}\right) = \Phi_{A_0}(x), \quad \text{and}$$

$$\Psi_{A_0}(x) = x \text{ on } [-A_0, A_0], \quad |\Psi_{A_0}(x)| \leq 3A_0 \text{ on } \mathbf{R},$$

$$e^{-\frac{|x|}{A_0}} \leq \Phi_{A_0}(x) \leq 3e^{-\frac{|x|}{A_0}} \text{ on } \mathbf{R}.$$

Lemma 3 (Local Virial identity on $\widetilde{\varepsilon}$)**.** *There exist $A_0 \geq 100$, $0 < a_4 < a_1$, $\delta_0 > 0$ such that if $\alpha_0 < a_4$, then*

$$\left(\int \Psi_{A_0} \widetilde{\varepsilon}^2 \right)_s \leq -\delta_0 \int (\widetilde{\varepsilon}^2 + \widetilde{\varepsilon}_y^2) e^{-\frac{|y|}{A_0}} + \frac{1}{\delta_0} \left(\int \widetilde{\varepsilon} Q \right)^2. \tag{45}$$

Proof. Note that (45) holds because of the orthogonality conditions we have chosen on $\widetilde{\varepsilon}$. The proof of Lemma 3 is completely given in [9]. Note that from the proof it is possible to choose $A_0 \geq 100$.

Recall that (45) is reminiscent of the following (global) Virial identity for $\widetilde{\varepsilon}$, in the case $\Psi \equiv y$, which is formally obtained by multiplying the equation of $\widetilde{\varepsilon}$ by $y\widetilde{\varepsilon}$ and integrating by parts,

$$\begin{aligned}
&\left(\frac{1}{2}\int y\widetilde{\varepsilon}^2\right)_s + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\frac{1}{2}\int y\widetilde{\varepsilon}^2 \\
&\quad = \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\int y\left(\tfrac{Q}{2}+yQ_y\right)\widetilde{\varepsilon} + \left(\tfrac{\widetilde{x}_s}{\widetilde{\lambda}}-1\right)\left(\int yQ_y\widetilde{\varepsilon} - \frac{1}{2}\int \widetilde{\varepsilon}^2\right) \\
&\quad\quad -\frac{3}{2}(L\widetilde{\varepsilon},\widetilde{\varepsilon}) + \int \widetilde{\varepsilon}^2 - 10\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\widetilde{\varepsilon}^2 \\
&\quad\quad +10\int\left(\tfrac{2Q^3}{3} - yQ_yQ^2\right)\widetilde{\varepsilon}^3 + 5\int\left(\tfrac{3Q^2}{2} - yQ_yQ\right)\widetilde{\varepsilon}^4 \\
&\quad\quad +\int (4Q - yQ_y)\widetilde{\varepsilon}^5 + \frac{5}{6}\int \widetilde{\varepsilon}^6.
\end{aligned}$$

Using the orthogonality conditions on $\widetilde{\varepsilon}$, we obtain

$$\left(\frac{1}{2}\int y\widetilde{\varepsilon}^2\right)_s + \frac{\widetilde{\lambda}_s}{\widetilde{\lambda}}\frac{1}{2}\int y\widetilde{\varepsilon}^2 \quad \leq \quad -H_\infty(\widetilde{\varepsilon},\widetilde{\varepsilon}) + C\left|\left(\tfrac{\widetilde{x}_s}{\widetilde{\lambda}}-1\right)\right|\left|\int \widetilde{\varepsilon}Q\right| + O(|\widetilde{\varepsilon}|_{H^1}^3),$$

where

$$H_\infty(\widetilde{\varepsilon},\widetilde{\varepsilon}) = \frac{3}{2}(L\widetilde{\varepsilon},\widetilde{\varepsilon}) - \int \widetilde{\varepsilon}^2 + 10\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\widetilde{\varepsilon}^2.$$

Then the proof of the Virial relation follows from the positivity of H_∞ under two orthogonality conditions on $\widetilde{\varepsilon}$. More precisely, in [8] we have established the following. There exists $\delta_1 > 0$ such that

$$\text{if} \quad \int Q\varepsilon = \int y\left(\tfrac{Q}{2}+yQ_y\right)\varepsilon = 0, \quad \text{then} \quad H_\infty(\varepsilon,\varepsilon) \geq \delta_1\int(\varepsilon_y^2 + \varepsilon^2).$$

2.2. **Control of the solution on the right of $\widetilde{x}(t)$ up to exponential corrections.** Here, we recall results of stability in time of exponential or polynomial decay in L^2 at the right of the soliton.

We claim the following lemma on the solution $u(t)$.

Lemma 4 (Stability of polynomial decay on the right)**.** *There exists $0 < a_5 < a_1$ such that if $\alpha_0 < a_5$, then the following is true. Suppose that for some $\theta > 0$,*

$$\forall x_0 > 0, \quad \int_{x\geq x_0} u_0^2(x)dx \leq \frac{\theta}{x_0^6}. \tag{46}$$

Suppose in addition that $\forall t \in [0,T)$, $\widetilde{\lambda}(t) \leq \lambda_0$, for some $\lambda_0 > 0$. Then, there exists $\theta' > 0$ such that

$$\forall t \in [0,T), \forall x_0 > 0, \quad \int_{x\geq x_0} u^2(t, x+\widetilde{x}(t))dx \leq \frac{\theta'}{x_0^6}. \tag{47}$$

This is a direct consequence of the following more general result.
For $K > 0$, let

$$\forall x \in \mathbf{R}, \quad \phi(x) = \phi_K(x) = cQ\left(\frac{x}{K}\right), \quad \psi(x) = \psi_K(x) = \int_{-\infty}^{x} \phi(y)dy,$$

and $1/c = K\int_{-\infty}^{+\infty} Q$, so that

$$\forall x \in \mathbf{R}, \quad 0 \le \psi(x) \le 1, \quad \lim_{x\to-\infty} \psi(x) = 0, \quad \lim_{x\to+\infty} \psi(x) = 1. \tag{48}$$

Consider $z(t)$ a solution of (3) on $[0, T_z)$ satisfying $\int z^2(0) < \int Q^2 + a_1$ and $E(z(0)) < 0$ and thus admitting a decomposition as in Lemma 1, with parameters ε_z, λ_z and x_z. For $x_0 \in \mathbf{R}$ and $t_0 \in (0, T_z)$, define

$$\forall t \in [0, t_0], \quad \mathbf{I}_{x_0,t_0}(t) = \int z^2(t,x)\psi\left(x - x_z(t_0) - x_0 - \tfrac{1}{4}(x_z(t) - x_z(t_0))\right)dx.$$

Then, we have the following lemma, based on nonlinear arguments.

Lemma 5 (Almost monotonicity of the mass on the left [11])**.** *There exists* $0 < a_5 < a_1$ *such that the following is true. Suppose* $\int z^2(0) < \int Q^2 + a_5$ *and* $E(z(0)) < 0$. *Assume in addition that* $\forall t \in [0, t_0]$, $0 < \lambda_z(t) \le \lambda_0$, *for some* λ_0. *Then, for any* $K > 2(1.01)\lambda_0$, *there exists* $C_0 = C_0(\lambda_0, K) > 0$ *such that*

$$\forall x_0 \in \mathbf{R}, \forall t \in [0, t_0], \quad \mathbf{I}_{x_0,t_0}(t_0) - \mathbf{I}_{x_0,t_0}(t) \le C_0 e^{-\frac{x_0}{K}}.$$

Proof of Lemma 4. Assuming Lemma 5, we prove Lemma 4. Let $K = 4\lambda_0$ and $t_0 \in (0, T)$. By applying Lemma 5 on $u(t)$ between 0 and t_0, we obtain:

$$\forall x_0 \in \mathbf{R}, \quad \mathbf{I}_{x_0,t_0}(t_0) - \mathbf{I}_{x_0,t_0}(0) \le C_0 e^{-\frac{x_0}{K}}.$$

From the fact that $\psi' > 0$, $\psi > 0$, and $\widetilde{x}(t_0) \ge \widetilde{x}(0)$ (note that $\widetilde{x}_t > 0$ by (26)), and then by $\psi(x) \le Ce^{\frac{x}{K}}$, we have

$$\begin{aligned}
\mathbf{I}_{x_0,t_0}(0) &= \int u^2(0,x)\psi(x - \widetilde{x}(t_0) - x_0 - \tfrac{1}{4}(\widetilde{x}(0) - \widetilde{x}(t_0)))dx \\
&\le \int u^2(0,x)\psi(x - \widetilde{x}(0) - x_0)dx \\
&\le \int_{x<x_0/2} u^2(0, x + \widetilde{x}(0))\psi(x - x_0)dx \\
&\quad + \int_{x>x_0/2} u^2(0, x + \widetilde{x}(0))\psi(x - x_0)dx \\
&\le \psi(-x_0/2)\int u^2(0) + \int_{x>x_0/2} u^2(0, x + \widetilde{x}(0)) \\
&\le Ce^{-\frac{x_0}{2K}} + \int_{x>x_0/2} u^2(0, x + \widetilde{x}(0)).
\end{aligned}$$

Since $\psi(x) \ge 1/2$ for $x \ge 0$, we have

$$\int_{x>x_0} u^2(t_0, x + \widetilde{x}(t_0))dx \le 2\mathbf{I}_{x_0,t_0}(t_0),$$

and thus

$$\int_{x>x_0} u^2(t_0, x + \widetilde{x}(t_0))dx \le Ce^{-\frac{x_0}{2K}} + 2\int_{x>x_0/2} u^2(0, x + \widetilde{x}(0)). \tag{49}$$

Thus Lemma 4 is proved. (Note that exponential decay on u_0 on the right would have also been preserved through time.)

Now, we recall the proof of Lemma 5, from [11].

Proof of Lemma 5. Recall that this property is related to two properties of equation (3):

(a) The existence of a Lyapunov function in L^2 for solutions that are small in L^2 (the function $\mathbf{I}_{x_0,t_0}$).

(b) The fact that the solution decomposes itself into a localized part moving at speed $\widetilde{x}_t(t) > \sigma_0 > 0$ and another part small in L^2.

The proof is based on a localization of the interactions between the two parts.

Recall first that if $\varphi : \mathbf{R} \to \mathbf{R}$ is a C^3 function such that, for $C > 0$, $\forall x \in \mathbf{R}$, $|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \le C$, then $t \mapsto \int z^2(t,x)\varphi(x)dx$ is a C^1 function and moreover

$$\frac{d}{dt}\int z^2(t)\varphi = -3\int z_x^2(t)\varphi' + \int z^2(t)\varphi^{(3)} + \frac{5}{3}\int z^6(t)\varphi';$$

see the proof of Lemma 5 in [7].

Let $x_0 \in \mathbf{R}$, $t_0 \in (0, T_z)$, and $K > 2(1.01)\lambda_0$. If we denote $\widetilde{x} = x - x_z(t_0) - x_0 - \frac{1}{4}(x_z(t) - x_z(t_0))$, for any $t \in [0, t_0]$, we obtain

$$\begin{aligned} \frac{d}{dt}\mathbf{I}_{x_0,t_0}(t) &= -3\int z_x^2(t)\psi'(\widetilde{x}) - \frac{(x_z)_t(t)}{4}\int z^2(t)\psi'(\widetilde{x}) \\ &\quad + \int z^2(t)\psi^{(3)}(\widetilde{x}) + \frac{5}{3}\int z^6(t)\psi'(\widetilde{x}). \end{aligned}$$

We have $\psi' = \phi$ and $\psi^{(3)} = \phi''$ so that $\forall x \in \mathbf{R}$,

$$\psi^{(3)}(x) = \frac{c}{K^2}Q_{xx}\left(\frac{x}{K}\right) = \frac{c}{K^2}(Q - Q^5)\left(\frac{x}{K}\right) \le \frac{c}{K^2}Q\left(\frac{x}{K}\right) = \frac{1}{K^2}\phi(x).$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\mathbf{I}_{x_0,t_0}(t) &\le -3\int z_x^2(t)\phi(\widetilde{x}) - \frac{(x_z)_t(t)}{4}\int z^2(t)\phi(\widetilde{x}) \\ &\quad + \frac{1}{K^2}\int z^2(t)\phi(\widetilde{x}) + \frac{5}{3}\int z^6(t)\phi(\widetilde{x}). \end{aligned}$$

By (26) and $(x_z)_t = \frac{(x_z)_s}{\lambda_z^3}$, we have $\forall t \in [0, t_0]$, $(x_z)_t \ge \frac{1}{(1.01)\lambda_z^2} \ge \frac{1}{(1.01)\lambda_0^2}$, and so

$$\frac{d}{dt}\mathbf{I}_{x_0,t_0}(t) \le -3\int z_x^2(t)\phi(\widetilde{x}) - \sigma_0\int z^2(t)\phi(\widetilde{x}) + \frac{5}{3}\int z^6(t)\phi(\widetilde{x}),$$

where $\sigma_0 = \frac{1}{4(1.01)\lambda_0^2} - \frac{1}{K^2} > 0$ ($\sigma_0 > 0$ by the choice of K in the lemma).

From [11], we recall that there exists a constant $\widetilde{c} > 0$ such that for all $v \in H^1$ and $a \in \mathbf{R}$, we have

$$|v^2\phi^{1/2}|^2_{L^\infty(x>a)} \le \widetilde{c}\left(\int v_x^2\phi + \int v^2\phi\right)\int_{x>a} v^2,$$

$$|v^2\phi^{1/2}|^2_{L^\infty(x<a)} \le \widetilde{c}\left(\int v_x^2\phi + \int v^2\phi\right)\int_{x<a} v^2.$$

Choose $X_0 > 0$ and a_5 such that

$$(50)\quad 16\widetilde{c}\left(\int_{\lambda_0|x|>X_0} Q^2\right)^2 < \min(1,\sigma_0/2), \quad \forall t\in[0,t_0],\ \int \varepsilon_z^2(t) < \int_{\lambda_0|x|>X_0} Q^2.$$

Now, we proceed in two steps.

(i) Variational estimates and localization. We claim

$$(51)\quad \forall t\in[0,t_0],\quad \frac{5}{3}\int_{|x-x_z(t)|\geq X_0} z^6(t)\phi(\widetilde{x}) \leq 2\int z_x^2(t)\phi(\widetilde{x}) + \sigma_0\int z^2(t)\phi(\widetilde{x}).$$

Indeed, we have

$$\begin{aligned}\int_{|x-x_z(t)|\geq X_0} z^6(t)\phi(\widetilde{x}) &\leq \left(\int_{|x-x_z(t)|\geq X_0} z^2(t)\right)|z^2(t)\phi^{1/2}|^2_{L^\infty(|x-x_z(t)|\geq X_0)}\\ &\leq \widetilde{c}\left(\int_{|x-x_z(t)|\geq X_0} z^2(t)\right)^2\int(z_x^2+z^2)\phi.\end{aligned}$$

Next, by using

$$z^2(t,x)\leq 2\lambda_z^{-1}(t)\left(Q^2(\lambda_z^{-1}(t)(x-x_z(t)))+\varepsilon_z^2(t,\lambda_z^{-1}(t)(x-x_z(t)))\right),$$

and (50), we obtain

$$\begin{aligned}\int_{|x-x_z(t)|\geq X_0} z^6(t)\phi(\widetilde{x}) &\leq 16\widetilde{c}\left(\int_{\lambda_0|x|>X_0} Q^2\right)^2\int(z_x^2+z^2)\phi\\ &\leq \int z_x^2\phi+\frac{\sigma_0}{2}\int z^2\phi,\end{aligned}$$

thus (51) is proved.

(ii) Conclusion using criticality. From (i), we have

$$\frac{d}{dt}\mathbf{I}_{x_0,t_0}(t)\leq\frac{5}{3}\int_{|x-x_z(t)|<X_0} z^6(t)\phi(\widetilde{x}).$$

Now, by $|z|^2_{L^\infty}\leq|z|_{L^2}|z_x|_{L^2}$, (26), (28), and $(x_z)_t=\frac{(x_z)_s}{\lambda_z^3}$, we have

$$\begin{aligned}\int_{|x-x_z(t)|<X_0} z^6(t)\phi(\widetilde{x}) &\leq |z(t)|^4_{L^2}|z_x(t)|^2_{L^2}|\phi(\widetilde{x})|_{L^\infty(|x-x_z(t)|<X_0)}\\ &\leq \frac{C}{\lambda_z^2(t)}\mathrm{Max}_{|x-x_z(t)|<X_0}\left\{e^{-\frac{|\widetilde{x}|}{K}}\right\}\\ &\leq C(x_z)_t(t)\mathrm{Max}_{|x-x_z(t)|<X_0}\left\{e^{-\frac{|\widetilde{x}|}{K}}\right\}.\end{aligned}$$

Assume that $x_0\geq X_0$. For x such that $|x-x_z(t)|<X_0$, we have

$$\forall t\in[0,t_0],\quad \widetilde{x}=x-x_z(t_0)-x_0-\frac{1}{4}(x_z(t)-x_z(t_0))\leq x-x_z(t)-x_0\leq 0,$$

and so

$$|\widetilde{x}|=-x+x_z(t_0)+x_0+\frac{1}{4}(x_z(t)-x_z(t_0))\geq(x_0-X_0)+\frac{3}{4}(x_z(t_0)-x_z(t)).$$

Thus,

$$\frac{d}{dt}\mathbf{I}_{x_0,t_0}(t)\leq C(x_z)_t(t)e^{-\frac{3}{4K}(x_z(t_0)-x_z(t))}e^{-\frac{x_0}{K}}.$$

By integration between t and t_0, we obtain

$$\forall t \in [0,t_0], \forall x_0 > X_0, \quad \mathbf{I}_{x_0,t_0}(t_0) - \mathbf{I}_{x_0,t_0}(t) \leq Ce^{-\frac{x_0}{K}}.$$

Finally, if $x_0 \leq X_0$, we have $\mathbf{I}_{x_0,t_0}(t_0) \leq \int u^2(0) \leq C_0 e^{-\frac{x_0}{K}}$ by choosing $C_0 \geq e^{\frac{X_0}{K}} \int u^2(0)$. Thus Lemma 5 is proved.

2.3. **Second decomposition of the solution under a priori control of $\int_{y>0} \widetilde{\varepsilon}$.** This decomposition will be adapted to kill the oscillations in time in the variation of λ. Indeed, we define $\varepsilon(t)$, $\lambda(t)$, $x(t)$ so that

$$\tfrac{\lambda_s}{\lambda} \sim -C \int \varepsilon Q,$$

instead of $\frac{\widetilde{\lambda}_s}{\widetilde{\lambda}} \sim \frac{1}{\mu_0} \int \widetilde{\varepsilon} L \left(\left(\left(\frac{Q}{2} + yQ_y \right) \right)_y \right)$ by the previous choice of orthogonality conditions (see Lemma 1). By using the energy relation, we will then relate $\int \varepsilon Q$ to λ^2 to obtain the lower bound of $|\frac{\lambda_s}{\lambda}|$ in some time integral form.

To do this, we impose that

$$\varepsilon(s) \perp \int_{-\infty}^{y} \left(\tfrac{Q}{2} + yQ_y \right).$$

Since $\int_{-\infty}^{+\infty} \left(\frac{Q}{2} + yQ_y \right) = -\frac{1}{2} \int Q \neq 0$, the function $\int_{-\infty}^{y} \left(\frac{Q}{2} + yQ_y \right)$ is not going to 0 as $y \to +\infty$ and thus we need a control on $\int_{y>0} |\varepsilon|$ to make the orthogonality condition rigorous. From the fact that the problem is critical, it will not be continuous in L^2 in the sense that $\int \varepsilon^2 e^{-\frac{|y|}{4}} >> \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}$. However, if one controls $\int_{y>0} |\widetilde{\varepsilon}|$ in some sense, this approach can be successful.

Note that in this new framework, we have

$$J(s) = \int \int_{-\infty}^{y} \left(\tfrac{Q}{2} + yQ_y \right) \varepsilon(s) \equiv 0,$$

which is in fact related to an $L^1(\mathbf{R}^+)$ property.

Let us first give an existence result of such a decomposition under some smallness condition in $L^1(\mathbf{R}^+)$.

For $v \in H^1(\mathbf{R})$, $\lambda_1 > 0$, $x_1 \in \mathbf{R}$, we define

$$\varepsilon_{\lambda_1,x_1}(y) = \lambda_1^{1/2} v(\lambda_1 y + x_1) - Q(y). \tag{52}$$

For $\beta > 0$, we consider

$$\mathbf{V}_\beta = \{ v \in H^1(\mathbf{R}) \cap L^1(\mathbf{R}^+),\ |v - Q|_{H^1} \leq \beta \text{ and } |v-Q|_{L^1(\mathbf{R}^+)} \leq \beta \}.$$

For $v \in \mathbf{V}_\beta$, let $\widetilde{\varepsilon} = v - Q$. Let $\varepsilon = \varepsilon_{\lambda_1,x_1}$ as defined in (52). Then,

$$Q(y) + \varepsilon(y) = \lambda_1^{1/2} Q(\lambda_1 y + x_1) + \lambda_1^{1/2} \widetilde{\varepsilon}(\lambda_1 y + x_1). \tag{53}$$

Lemma 6 (Decomposition related to $L^1(\mathbf{R}^+)$)**.** *There exists $\beta_1 > 0$ and a unique map $(\lambda_1, x_1) : \mathbf{V}_{\beta_1} \to (\frac{1}{2}, 2) \times (-1,1)$ such that if $v \in \mathbf{V}_{\beta_1}$ and $\varepsilon_{\lambda_1,x_1}$ is defined as in* (52)*, then*

$$\int \int_{-\infty}^{y} \left(\tfrac{Q}{2} + yQ_y \right) \varepsilon_{\lambda_1,x_1} = \int y \left(\tfrac{Q}{2} + yQ_y \right) \varepsilon_{\lambda_1,x_1} = 0. \tag{54}$$

Moreover, there exists $C_1 > 0$ such that if $v \in \mathbf{V}_{\beta_1}$, then

$$|\lambda_1 - 1| \leq C_1 \left(\int \widetilde{\varepsilon}^2 e^{-|y|} \right)^{1/2} + C_1 \int_{y>0} |\widetilde{\varepsilon}|, \tag{55}$$

$$|x_1| \leq C_1 \left(\int \widetilde{\varepsilon}^2 e^{-|y|} \right)^{1/2} + C_1 \left(\int_{y>0} |\widetilde{\varepsilon}| \right)^2, \tag{56}$$

$$|\varepsilon_{\lambda_1,x_1}|_{H^1} \leq C_1 |\widetilde{\varepsilon}|_{H^1} + C_1 \int_{y>0} |\widetilde{\varepsilon}|. \tag{57}$$

Note that $|x_1| \leq C_1 \left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{2}} \right)^{1/2} + C_1 \left(\int_{y>0} |\widetilde{\varepsilon}| \right)^2$ will be implied by the criticality of the problem.

Proof. The proof is similar to the one of Proposition 1 in [7]. We apply the implicit function theorem to

$$\rho^1_{\lambda_1,x_1}(v) = \int \int_{-\infty}^{y} \left(\frac{Q}{2} + yQ_y \right) \varepsilon_{\lambda_1,x_1},$$

$$\rho^2_{\lambda_1,x_1}(v) = \int y \left(\frac{Q}{2} + yQ_y \right) \varepsilon_{\lambda_1,x_1}.$$

From $\frac{d}{d\lambda} \lambda^{1/2} w(\lambda x)_{|\lambda=1} = \frac{w}{2} + x w_x$, $\frac{d}{dx_1} w(x+x_1)_{|x_1=0} = w_x(x)$, the nondegeneracy conditions are given by:

$$\begin{aligned} \frac{\partial \rho^1_{\lambda_1,x_1}}{\partial x_1}_{|\lambda_1=1,x_1=0,v=Q} &= \int Q_y \int_{-\infty}^{y} \left(\frac{Q}{2} + yQ_y \right) \\ &= -\int Q \left(\frac{Q}{2} + yQ_y \right) = 0, \end{aligned} \tag{58}$$

$$\begin{aligned} \frac{\partial \rho^1_{\lambda_1,x_1}}{\partial \lambda_1}_{|\lambda_1=1,x_1=0,v=Q} &= \int \left(\frac{Q}{2} + yQ_y \right) \int_{-\infty}^{y} \left(\frac{Q}{2} + yQ_y \right) \\ &= \frac{(\int \left(\frac{Q}{2} + yQ_y \right))^2}{2} = \frac{(\int Q)^2}{8}, \end{aligned} \tag{59}$$

$$\frac{\partial \rho^2_{\lambda_1,x_1}}{\partial x_1}_{|\lambda_1=1,x_1=0,v=Q} = \int y \left(\frac{Q}{2} + yQ_y \right) Q_y = \int \left(\frac{Q}{2} + yQ_y \right)^2, \tag{60}$$

$$\frac{\partial \rho^2_{\lambda_1,x_1}}{\partial \lambda_1}_{|\lambda_1=1,x_1=0,v=Q} = \int y \left(\frac{Q}{2} + yQ_y \right) \left(\frac{Q}{2} + yQ_y \right) = 0. \tag{61}$$

Now, let us obtain (55)–(57). By linearization of (53), we have, with $\varepsilon = \varepsilon_{\lambda_1,x_1}$,

$$\begin{aligned} \varepsilon(y) = (\lambda_1 - 1) \left(\frac{Q}{2} + yQ_y \right) + x_1 Q_y + \lambda_1^{1/2} \widetilde{\varepsilon}(\lambda_1 y + x_1) \\ + (O(|\lambda_1 - 1|^2) + O(|x_1|^2)) e^{-\frac{|y|}{2}}. \end{aligned} \tag{62}$$

By taking the scalar product of the preceding relation with $\int_{-\infty}^{y} \left(\frac{Q}{2} + yQ_y \right)$ and then with $y\left(\frac{Q}{2}+yQ_y\right)$, we obtain, using (58)–(61),

$$(\lambda_1-1)\frac{(\int Q)^2}{8}+\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)\int_{-\infty}^{y}\left(\frac{Q}{2}+yQ_y\right)=O(|\lambda_1-1|^2)+O(|x_1|^2),$$

$$x_1\int\left(\frac{Q}{2}+yQ_y\right)^2+\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)y\left(\frac{Q}{2}+yQ_y\right)=O(|\lambda_1-1|^2)+O(|x_1|^2).$$

Since $\left|\int_{-\infty}^{y}\left(\frac{Q}{2}+yQ_y\right)\right|\leq C$ and $\forall y<0$, $\left|\int_{-\infty}^{y}\left(\frac{Q}{2}+yQ_y\right)\right|\leq Ce^{-\frac{2|y|}{3}}$, we have by the Cauchy–Schwartz inequality

$$\begin{aligned}\left|\int\widetilde{\varepsilon}(\lambda_1 y+x_1)\int_{-\infty}^{y}\left(\frac{Q}{2}+yQ_y\right)\right| &\leq C\left(\left|\int\widetilde{\varepsilon}\,e^{-\frac{2|y|}{3}}\right|+\int_{y>0}|\widetilde{\varepsilon}|\right)\\ &\leq C\left(\left(\int\widetilde{\varepsilon}^2e^{-|y|}\right)^{1/2}+\int_{y>0}|\widetilde{\varepsilon}|\right).\end{aligned}$$

Thus,

$$|\lambda_1-1|\leq C|x_1|^2+C\left(\left(\int\widetilde{\varepsilon}^2e^{-|y|}\right)^{1/2}+\int_{y>0}|\widetilde{\varepsilon}|\right),$$

$$|x_1|\leq C\left(\int\widetilde{\varepsilon}^2e^{-|y|}\right)^{1/2}+C|\lambda_1-1|^2,$$

and (55)–(56) follow. Finally, the estimate on $|\varepsilon|_{H^1}$ is clear from (62). This concludes the proof of Lemma 6.

By performing this new decomposition on some interval of time (s_1,s_2), on which $\int_{y>0}|\widetilde{\varepsilon}(s)|$ is sufficiently small, the functions ε, λ and x have C^1 regularity in time, and satisfy the same equation as $\widetilde{\varepsilon}$, i.e. equation (20). Moreover, multiplying this equation by $\int_{-\infty}^{y}\left(\frac{Q}{2}+yQ_y\right)$ and then by $y\left(\frac{Q}{2}+yQ_y\right)$, we obtain the following properties.

Lemma 7 (Properties of decomposition related to $L^1(\mathbf{R}^+)$)**.**

$$\begin{aligned}&\frac{\lambda_s}{\lambda}\frac{(\int Q)^2}{8}+2\int\varepsilon Q-\left(\frac{x_s}{\lambda}-1\right)\int\varepsilon\left(\frac{Q}{2}+yQ_y\right)\\ &\quad+10\int Q^3\left(\frac{Q}{2}+yQ_y\right)\varepsilon^2+\int G(\varepsilon)\left(\frac{Q}{2}+yQ_y\right)=0\end{aligned}\tag{63}$$

and

$$\begin{aligned}&\left(\frac{x_s}{\lambda}-1\right)\mu_0-\int\varepsilon L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]\\ &\quad-\frac{\lambda_s}{\lambda}\int\varepsilon\left[y\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]-\left(\frac{x_s}{\lambda}-1\right)\int\varepsilon\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\\ &\quad+10\int Q^3\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\varepsilon^2+\int G(\varepsilon)\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y=0,\end{aligned}\tag{64}$$

where $|G(\varepsilon)|\leq C(|\varepsilon|^3+|\varepsilon|^5)$ *and* $\mu_0=\int\left(\frac{Q}{2}+yQ_y\right)^2$.

In particular, we have

Corollary 4.

$$\left|\frac{\lambda_s}{\lambda}\frac{(\int Q)^2}{8} + 2\int \varepsilon Q\right| + \left|\left(\frac{x_s}{\lambda} - 1\right)\mu_0 - \int \varepsilon L\left[\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y\right]\right| \le C\int \varepsilon^2 e^{-\frac{|y|}{2}}. \tag{65}$$

3. Reduction of the proof of Theorem 1

Recall that we have the following results from [11] and [9] concerning the dynamics for initial data with negative energy.

There exists $\alpha_1 > 0$ such that the following two properties are true.

Blow up result ([11])**.** *Let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \le \int Q^2 + \alpha_1$ and $E(u_0) < 0$. Then the corresponding solution $u(t)$ of* (3) *blows up in finite or infinite time $0 < T \le +\infty$.*

Blow up profile ([9])**.** *Let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \le \int Q^2 + \alpha_1$ and $E(u_0) < 0$. Let $u(t)$ be the corresponding solution of* (3)*, and let $0 < T \le +\infty$ be its blow up time. Then for all $t \in [0, T)$ there exist $\widetilde{\lambda}(t) > 0$ and $\widetilde{x}(t) \in \mathbf{R}$ such that*

$$\text{either}\quad \widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)x + \widetilde{x}(t)) \rightharpoonup Q \quad \text{or} \quad -\widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)x + \widetilde{x}(t)) \rightharpoonup Q$$

in $H^1(\mathbf{R})$ weak, as $t \uparrow T$ with $\widetilde{\lambda}(t) \to 0$, as $t \uparrow T$.

From the choice of the decomposition in Section 2 (since if $u(t,x)$ is a solution of (3), then $-u(t,x)$ is also a solution), we have in fact $\widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)x + \widetilde{x}(t)) \rightharpoonup Q$.

For $0 < \alpha_2 < \alpha_1$ to be chosen later, let $u_0 \in H^1(\mathbf{R})$ be such that $\int u_0^2 \le \int Q^2 + \alpha_2$ and $E(u_0) < 0$. Let $u(t)$ be the corresponding solution of (3). From the blow up result, we know that there exists $0 < T \le +\infty$, such that

$$|u_x(t)|_{L^2} \to +\infty \quad \text{as } t \uparrow T.$$

We suppose $\alpha_2 < a_1$ and we consider the decomposition of $u(t)$ from Lemma 1. Thus we define $\widetilde{\varepsilon}$, $\widetilde{\lambda}$ and $\widetilde{x}$ such that

$$Q(y) + \widetilde{\varepsilon}(t,y) = \widetilde{\lambda}^{1/2}(t)u(t, \widetilde{\lambda}(t)y + \widetilde{x}(t)),$$

with $\int \left(\frac{Q}{2} + yQ_y\right)\widetilde{\varepsilon} = \int y\left(\frac{Q}{2} + yQ_y\right)\widetilde{\varepsilon} = 0$. By (26), we have

$$\widetilde{\lambda}(t) \to 0 \quad \text{as } t \uparrow T.$$

From this, there exists $t_0 \ge 0$ such that $\forall t \in [t_0, T)$, $\widetilde{\lambda}(t) \le 1$. Such $t_0 \ge 0$ being fixed, there exists β_0 such that $\forall t \in [0, t_0]$, $\widetilde{\lambda}(t) \le \beta_0$. Assume in addition that u_0 satisfies a property of decay in L^2 on the right: for some $\theta > 0$,

$$\forall x_0 > 0, \quad \int_{x \ge x_0} u_0^2(x)dx \le \frac{\theta}{x_0^6}.$$

Then, by Lemma 4, there exists $\theta' > 0$ depending on θ, β_0, $\widetilde{x}(t_0)$ and $\widetilde{x}(0)$ such that

$$\forall x_0 > 0, \quad \int_{x \ge x_0} u^2(t_0, x)dx \le \frac{\theta'}{x_0^6}.$$

Now, by changing $u(0)$ to $u(t_0)$, we assume that $t_0 = 0$, and we denote $\theta' = \theta$ in the rest of this paper. Therefore, we have $\forall t \in [0, T)$, $\widetilde{\lambda}(t) \le 1$.

Let $t_n \to T$ be the sequence defined by, for all n large,

$$|u_x(t_n)|_{L^2} = 2^n |Q_x|_{L^2} \quad \text{and} \quad \forall t \in (t_n, T),\ |u_x(t)|_{L^2} > 2^n |Q_x|_{L^2}. \tag{66}$$

Note that the existence of such a sequence (t_n) is guaranteed by the result of blow up in finite or infinite time. Recall that from (26), we have $\forall t \in [0,T)$,

$$\frac{|Q_x|_{L^2}}{(1.01)|u_x(t)|_{L^2}} \leq \widetilde{\lambda}(t) \leq \frac{(1.01)|Q_x|_{L^2}}{|u_x(t)|_{L^2}},$$

and then (66) implies that

$$\forall n, \quad \frac{1}{(1.01)2^n} \leq \widetilde{\lambda}(t_n) \leq \frac{1.01}{2^n}, \quad \forall t \geq t_n,\ \widetilde{\lambda}(t) < \frac{1.01}{2^n} \leq (1.01)^2 \widetilde{\lambda}(t_n). \tag{67}$$

Then we claim the following:

There exists $0 < \alpha_2 < \alpha_1$ such that the following is true.

(i) *The sequence (t_n) is bounded, in particular*

$$T < +\infty,$$

and thus $u(t)$ blows up in finite time.

(ii) *Moreover, there exists $n_0 > 0$ such that*

$$\forall n \geq n_0, \quad |u_x(t_n)|_{L^2} \leq \frac{C_0}{|E_0|(T - t_n)}, \tag{68}$$

where $C_0 = 4(\int Q)^2 |Q_x|_{L^2}$.

First, we claim that (i)–(ii) and thus Theorem 1 follow from the following control from above of $t_{n+1} - t_n$ by $\widetilde{\lambda}(t_n)$:

Theorem 2 (Upper bounds on $t_{n+1} - t_n$)**.** *There exists $0 < \alpha_2 < \alpha_1$, such that if $\int u_0^2 \leq \int Q^2 + \alpha_2$ and $E(u_0) < 0$, then the following property is true. Assume that u_0 satisfies for some $\theta > 0$,*

$$\forall x_0 > 0, \quad \int_{x \geq x_0} u_0^2(x) dx \leq \frac{\theta}{x_0^6}.$$

Then, there exists $n_0 > 0$ such that

$$\forall n \geq n_0, \quad t_{n+1} - t_n \leq \frac{(\int Q)^2}{|E_0|} \widetilde{\lambda}(t_n). \tag{69}$$

Remark. Note that the choice of α_2 is independent of θ. However $n_0 = n_0(u_0)$ depends on some constants related to the initial data. Indeed, we consider n_0 such that

$$\forall t \geq t_{n_0}, \quad \int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \leq C(|E_0|, \theta),$$

and therefore $n_0 = n_0(|E_0|, \theta)$. Note that the existence of such n_0 follows from the result of asymptotic profile [9]. Indeed, from the blow up profile result, we have

$$\widetilde{\varepsilon}(t) \rightharpoonup 0 \quad \text{in } H^1(\mathbf{R}) \text{ weak as } t \uparrow T,$$

and so, from the fact that $H^1(\mathbf{R})$ is compactly embedded in $L^2((-R,R))$, for all $R > 0$, and $\int \widetilde{\varepsilon}^2 \leq \frac{1}{4} \int Q^2$, we have

$$\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \to 0 \quad \text{as } t \uparrow T. \tag{70}$$

We prove that Theorem 2 implies Theorem 1 and next we prove Theorem 2.

Theorem 2 implies Theorem 1. Let us first show that the blow up occurs in finite time, i.e.

$$T < +\infty.$$

Assuming that (69) holds, there exists $n_0 > 0$ such that $\forall n \geq n_0$, $\forall m > 0$, we have

$$t_{m+n} - t_n = \sum_{k=n}^{m+n-1} (t_{k+1} - t_k) \leq 1.01 \frac{(\int Q)^2}{|E_0|} \sum_{k=n}^{+\infty} \frac{1}{2^k} = 1.01 \frac{(\int Q)^2}{|E_0|} \frac{1}{2^{n-1}}. \tag{71}$$

This implies $\forall m > 0$,

$$t_{m+n_0} \leq t_{n_0} + 1.01 \frac{(\int Q)^2}{|E_0|} \frac{1}{2^{n_0-1}}. \tag{72}$$

Therefore, the sequence $(t_{m+n_0})_{m>0}$ is increasing and bounded, and since $\lim_{n\to+\infty} t_n = T$ by the definition of the sequence (t_n) and the well posedness of the Cauchy problem in H^1, we obtain $T < +\infty$ and thus $u(t)$ blows up in finite time.

Moreover, from (71), letting $m \to +\infty$, we have

$$\forall n \geq n_0, \quad T - t_n \leq 1.01 \frac{(\int Q)^2}{|E_0|} \frac{1}{2^{n-1}} \leq (1.01)^2 \frac{2(\int Q)^2}{|E_0|} \widetilde{\lambda}(t_n),$$

and so by (26),

$$\forall n \geq n_0, \quad |u_x(t_n)|_{L^2} \leq \frac{(1.01)|Q_x|_{L^2}}{\widetilde{\lambda}(t_n)} \leq \frac{2(1.01)^3(\int Q)^2 |Q_x|_{L^2}}{|E_0|(T - t_n)} \leq \frac{C_0}{|E_0|(T - t_n)},$$

where $C_0 = 4(\int Q)^2 |Q_x|_{L^2}$, which is the desired result.

Therefore, we are reduced to understanding the dynamics on the time interval (t_n, t_{n+1}) and to prove Theorem 2. For this, we will use the following two fundamental propositions, which will be proved in the next sections.

Proposition 1 (Integration of conservation laws)**.** *There exist* $0 < \alpha_I < \alpha_1$ *and* $\delta_I > 0$ *such that if* $\int u_0^2 \leq \int Q^2 + \alpha_I$ *and* $E(u_0) < 0$*, then the following is true. If* $0 < t_1 < t_2 < T$ *are such that*

$$\forall t \geq t_1, \quad \int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \leq \delta_I \tag{73}$$

and

$$\forall t \in (t_1, t_2), \quad \int_{y \geq 0} |\widetilde{\varepsilon}(t)| \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \right)^{3/8}, \tag{74}$$

then

$$\frac{1}{100} + \log \frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)} \geq \frac{8|E_0|}{(\int Q)^2} \int_{t_1}^{t_2} \frac{dt}{\widetilde{\lambda}(t)}.$$

Remark. In (74), the power 3/8 has no particular significance. Indeed, any power strictly less than 1/2 close to 1/2 would work, if one also modifies the power $\frac{1}{100}$ in the local L^2 norm of ε. This would not affect the rest of the proof.

Proof. See Section 4.

Proposition 2 (L^1-control on the right for slow dynamics)**.** *Suppose that for some* $\theta > 0$,

$$\forall x_0 > 0, \quad \int_{x \geq x_0} u_0^2(x)dx \leq \frac{\theta}{x_0^6}$$

and

$$\forall t \in [0, T), \quad \widetilde{\lambda}(t) \leq 1.$$

Let $A > 0$*. There exists* $\delta_{II} = \delta_{II}(A, |E_0|, \theta)$ *such that the following is true. Suppose that* $0 < t_1 < t_2 < T$ *satisfy*

$$\forall t \in (t_1, t_2), \quad \widetilde{\lambda}(t) \leq (4.1)\widetilde{\lambda}(t_2) \tag{75}$$

and

$$t_2 - t_1 \geq A\widetilde{\lambda}(t_2). \tag{76}$$

If in addition $\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{100}} \leq \delta_{II}$*, then*

$$\int_{y>0} |\widetilde{\varepsilon}(t_2, y)|dy \leq \left(\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{100}}\right)^{3/8}. \tag{77}$$

Proof. See Section 5.

Assume these two propositions and let us prove Theorem 2.

Proof of Theorem 2. Suppose $\int u_0^2 \leq \int Q^2 + \alpha_2$. By (70), there exists $n_0 > 0$ such that

$$\forall t \geq t_{n_0}, \quad \int \widetilde{\varepsilon}^2(t)e^{-\frac{|y|}{100}} \leq \min(\delta_I, \delta_{II}),$$

where δ_I, $\delta_{II} = \delta_{II}(A, |E_0|, \theta)$, for $A = (\int Q)^2/(4|E_0|)$, are defined in Propositions 1 and 2, respectively.

Now, n_0 being fixed, we want to prove that (69) is true for all $n \geq n_0$. We argue by contradiction. Assume that we can find $n_1 \geq n_0$ so that

$$t_{n_1+1} - t_{n_1} > \frac{(\int Q)^2}{|E_0|}\widetilde{\lambda}(t_{n_1}). \tag{78}$$

The contradiction is obtained in two steps.

Step 1. Monotonicity property on $(\frac{t_{n_1+1}+t_{n_1}}{2}, t_{n_1})$.

Claim.

$$\forall t \in \left(\frac{t_{n_1+1} + t_{n_1}}{2}, t_{n_1+1}\right), \quad \widetilde{\lambda}(t) > \frac{\widetilde{\lambda}(t_{n_1})}{4}. \tag{79}$$

This follows from the fact that under the regime of Proposition 1, $\widetilde{\lambda}$ is almost monotone in time.

Suppose for the sake of contradiction that (79) is not true. Since $\widetilde{\lambda}$ is a continuous function, and $\widetilde{\lambda}(t_{n+1}) \geq \frac{1}{2(1.01)^2}\widetilde{\lambda}(t_n)$, there exists

$$\bar{t}_{n_1} \in \left(\frac{t_{n_1+1} + t_{n_1}}{2}, t_{n_1+1}\right)$$

such that

$$\widetilde{\lambda}(\bar{t}_{n_1}) = \frac{\widetilde{\lambda}(t_{n_1})}{4} \quad \text{and} \quad \forall t \in (\bar{t}_{n_1}, t_{n_1+1}),\ \widetilde{\lambda}(t) > \widetilde{\lambda}(t_{n_1})/4.$$

For all $t \in (\bar{t}_{n_1}, t_{n_1+1})$, we have

$$t - t_{n_1} \geq \bar{t}_{n_1} - t_{n_1} \geq \frac{t_{n_1+1} - t_{n_1}}{2} \geq \frac{(\int Q)^2}{2|E_0|}\widetilde{\lambda}(t_{n_1})$$

by hypothesis of contradiction (78). By (67), we have $\widetilde{\lambda}(t) \leq (1.01)^2\widetilde{\lambda}(t_{n_1})$, and so

$$t - t_{n_1} \geq \frac{(\int Q)^2}{4|E_0|}\widetilde{\lambda}(t).$$

Moreover, $\forall t' \in (t_{n_1}, t)$, by using (67) and the definition of $\bar{t}_{n_1}$, we have

$$\widetilde{\lambda}(t') \leq (1.01)^2\widetilde{\lambda}(t_{n_1}) \leq (4.1)\widetilde{\lambda}(t).$$

Therefore, we can apply Proposition 2 between t_{n_1} and t, with $A = \frac{(\int Q)^2}{4|E_0|}$. We obtain

$$\forall t \in (\bar{t}_{n_1}, t_{n_1+1}), \quad \int_{y>0} |\widetilde{\varepsilon}(t)| \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}}\right)^{3/8}.$$

Now, we are able to apply Proposition 1 between $\bar{t}_{n_1}$ and t_{n_1+1} and we obtain

$$\frac{1}{100} + \log\frac{\widetilde{\lambda}(\bar{t}_{n_1})}{\widetilde{\lambda}(t_{n_1+1})} \geq \frac{8|E_0|}{(\int Q)^2}\int_{\bar{t}}^{t_{n_1+1}} \frac{dt}{\widetilde{\lambda}(t)} \geq 0.$$

Since

$$\widetilde{\lambda}(\bar{t}_{n_1}) = \frac{\widetilde{\lambda}(t_{n_1})}{4} \leq \frac{(1.01)^2}{2^{n_1-2}} \leq \frac{(1.01)^4}{2}\widetilde{\lambda}(t_{n_1+1}),$$

we obtain

$$-\frac{\log 2}{2} > \frac{1}{100} - \log\frac{2}{(1.01)^4} > \frac{1}{100} + \log\frac{\widetilde{\lambda}(\bar{t}_{n_1})}{\widetilde{\lambda}(t_{n_1+1})} > 0,$$

which is a contradiction. Thus,

$$\forall t \in \left(\frac{t_{n_1+1} + t_{n_1}}{2}, t_{n_1+1}\right), \quad \widetilde{\lambda}(t) > \frac{\widetilde{\lambda}(t_{n_1})}{4}. \tag{80}$$

Step 2. Conclusion by integration of the conservation laws.

Now let

$$\bar{t}_{n_1} = \frac{t_{n_1+1} + t_{n_1}}{2}.$$

Since we have the monotonicity properties on $(\bar{t}_{n_1}, t_{n_1+1})$, we are able from Proposition 1 to obtain a control from above of $t_{n_1+1} - \bar{t}_{n_1}$ which will give a contradiction.

As in Step 1, for any $t \in (\bar{t}_{n_1}, t_{n_1+1})$, we apply Proposition 2 between t_{n_1} and t. We obtain

$$\forall t \in (\bar{t}_{n_1}, t_{n_1+1}), \quad \int_{y>0} |\widetilde{\varepsilon}(t)| \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}}\right)^{3/8}.$$

We apply Proposition 1 between $\bar{t}_{n_1}$ and t_{n_1+1}, and we obtain

$$\frac{1}{100} + \log\frac{\widetilde{\lambda}(\bar{t}_{n_1})}{\widetilde{\lambda}(t_{n_1+1})} \geq \frac{8|E_0|}{(\int Q)^2}\int_{\bar{t}_{n_1}}^{t_{n_1+1}} \frac{dt}{\widetilde{\lambda}(t)}.$$

Since by definition of the sequence (t_n),

$$\widetilde{\lambda}(\bar{t}_{n_1}) \leq \frac{(1.01)^2}{2^{n_1}} \leq 2(1.01)^4\widetilde{\lambda}(t_{n_1+1}),$$

we have

$$1 \geq \frac{1}{100} + \log(2(1.01)^4) \geq \frac{1}{100} + \log \frac{\widetilde{\lambda}(\overline{t}_{n_1})}{\widetilde{\lambda}(t_{n_1+1})} \geq \frac{8|E_0|}{(\int Q)^2} \int_{\overline{t}_{n_1}}^{t_{n_1+1}} \frac{dt}{\widetilde{\lambda}(t)}.$$

Since by the definition of the sequence (t_n), $\forall t \in (\overline{t}_{n_1}, t_{n_1+1})$, $\widetilde{\lambda}(t) \leq (1.01)^2 \widetilde{\lambda}(t_{n_1})$, we obtain

$$1 \geq \frac{8|E_0|}{(1.01)^2(\int Q)^2} \frac{1}{\widetilde{\lambda}(t_{n_1})}(t_{n_1+1} - \overline{t}_{n_1}) = \frac{4|E_0|}{(1.01)^2(\int Q)^2} \frac{1}{\widetilde{\lambda}(t_{n_1})}(t_{n_1+1} - t_{n_1}),$$

and so

$$t_{n_1+1} - t_{n_1} \leq \frac{(\int Q)^2}{2|E_0|} \widetilde{\lambda}(t_{n_1}),$$

which contradicts (78), and concludes the proof of Theorem 2.

Now, we prove Corollary 1.

Proof of Corollary 1. In the context of Theorem 1, we assume in addition that for some $t_1 \in [0, T)$,

$$\text{if } t_1 < t < t' < T, \text{ then } |u_x(t')|_{L^2} \geq \tfrac{1}{2} |u_x(t)|_{L^2}. \tag{81}$$

Then, we claim that there exists $t(u_0) \in (0, T)$ such that

$$\forall t \in (t(u_0), T), \quad |u_x(t)|_{L^2} \leq \frac{4C_0}{|E_0|(T-t)}. \tag{82}$$

Recall that in the proof of Theorem 1, we have proved that there exists $n_0 = n_0(u_0)$ such that

$$\forall n \geq n_0, \quad |u_x(t_n)|_{L^2} \leq \frac{C_0}{|E_0|(T-t_n)}, \tag{83}$$

where $C_0 = 4(\int Q)^2 |Q_x|_{L^2}$.

Set $t(u_0) = \max(t_1, t_{n_0})$. Let $t \in (t(u_0), T)$, and let $n_1 \geq n_0$ such that

$$t_{n_1} \leq t < t_{n_1+1}.$$

By (81), and then by the definition of the sequence (t_n) in Theorem 1, we have

$$|u_x(t)|_{L^2} \leq 2|u_x(t_{n_1+1})|_{L^2} = 4|u_x(t_{n_1})|_{L^2}.$$

Therefore, by (83), and since $T - t \leq T - t_{n_1}$, we have

$$|u_x(t)|_{L^2} \leq \frac{4C_0}{|E_0|(T-t_{n_1})} \leq \frac{4C_0}{|E_0|(T-t)},$$

and Corollary 1 is proved.

Now, we prove Corollary 2.

Proof of Corollary 2. Still in the context of Theorem 1, we assume in addition that for some $B > 0$, $n_1 > 0$, we have

$$\forall n \geq n_1, \quad |u_x(t_n)|_{L^2} \geq \frac{B}{(T-t_n)}. \tag{84}$$

Then, we claim that there exists $t(u_0) \in (0, T)$ such that

$$\forall t \in (t(u_0), T), \quad |u_x(t)|_{L^2} \leq \frac{8C_0}{|E_0|(T-t)}. \tag{85}$$

By the proof of Theorem 1, there exists $n_0 = n_0(u_0)$ such that

$$\forall n \geq n_0, \quad |u_x(t_n)|_{L^2} \leq \frac{C_0}{|E_0|(T-t_n)}, \tag{86}$$

where $C_0 = 4(\int Q)^2 |Q_x|_{L^2}$. Let $n_2 = \max(n_0, n_1)$.

Step 1. First, we claim that there exists $k_0 > 0$ such that $\forall n \geq n_2$, we have

$$t_{n+k_0} - t_n \geq \frac{C_0}{(1.01)|E_0|} \widetilde{\lambda}(t_{n+k_0}). \tag{87}$$

Indeed, let $k_0 > 0$ be the smallest integer such that $2^{k_0} \geq \frac{2C_0}{B|E_0|}$. For any $n \geq n_2$, by (84), we have $T - t_n \geq \frac{B}{|u_x(t_n)|_{L^2}} = \frac{B\, 2^{-n}}{|Q_x|_{L^2}}$, and, by Theorem 1, $T - t_{n+k_0} \leq \frac{C_0}{|E_0||u_x(t_{n+k_0})|_{L^2}} = \frac{C_0 2^{-n-k_0}}{|E_0||Q_x|_{L^2}}$. Therefore,

$$\begin{aligned} t_{n+k_0} - t_n &= (T-t_n) - (T - t_{n+k_0}) \geq \frac{2^{-n-k_0}}{|Q_x|_{L^2}} \left(2^{k_0} B - \frac{C_0}{|E_0|} \right) \\ &\geq \frac{C_0\, 2^{-n-k_0}}{|E_0||Q_x|_{L^2}} \geq \frac{C_0 \widetilde{\lambda}(t_{n+k_0})}{(1.01)|E_0|}, \end{aligned}$$

by (26) and thus (87) is proved.

Step 2. Let $n \geq n_2$. We claim that

$$\forall t \in (t_{n+k_0}, t_{n+k_0+1}), \quad \widetilde{\lambda}(t) \geq \frac{1}{4} \widetilde{\lambda}(t_{n+k_0}). \tag{88}$$

The argument is the same as the one of Step 1 in the proof of Theorem 2. If we suppose that (88) fails, since $\widetilde{\lambda}(t_{n+k_0+1}) \geq \frac{1}{2(1.01)^2} \widetilde{\lambda}(t_{n+k_0})$, there exists $\bar{t}_n \in (t_{n+k_0}, t_{n+k_0+1})$ such that

$$\widetilde{\lambda}(\bar{t}_n) = \frac{1}{4} \widetilde{\lambda}(t_{n+k_0}), \quad \forall t \in (\bar{t}_n, t_{n+k_0+1}),\ \widetilde{\lambda}(t) > \frac{1}{4} \widetilde{\lambda}(t_{n+k_0}).$$

Then, $\forall t \in (\bar{t}_n, t_{n+k_0+1})$, we have by Step 1,

$$t - t_n \geq \bar{t}_n - t_n \geq t_{n+k_0} - t_n \geq \frac{C_0 \widetilde{\lambda}(t_{n+k_0})}{(1.01)|E_0|} \geq \frac{C_0 \widetilde{\lambda}(t)}{(1.01)^3 |E_0|}.$$

Moreover, $\forall t' \in (t_n, t)$, we have

$$\widetilde{\lambda}(t') \leq (1.01)^2 \widetilde{\lambda}(t_n) \leq (1.01)^4 2^{k_0} \widetilde{\lambda}(t_{n+k_0}) \leq (1.01)^4 2^{2+k_0} \widetilde{\lambda}(t),$$

by the definition of the sequence (t_n) and the definition of $\bar{t}_n$. Now, we apply a variant of Proposition 2 between t and t_n, and conclude that

$$\forall t \in (\bar{t}_n, t_{n+k_0+1}), \quad \int_{y>0} |\widetilde{\varepsilon}(t,y)| dy \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{K_0}} \right)^{3/8}, \tag{89}$$

for some constant $K_0 > 0$ (the constant K_0 is related to the fact that instead of 4.1 in (75), we have $(1.01)^4 2^{2+k_0}$).

Therefore, we can now apply Proposition 1 between $\bar{t}_n$ and t_{n+k_0+1}. We find a contradiction exactly in the same way as in Step 1 of the proof of Theorem 2.

Thus, claim (88) is proved.

Step 3. Conclusion.

Set $t(u_0) = t_{n_2+k_0}$. Let $t \in (t(u_0), T)$, and consider $n \geq n_2 + k_0$ such that $t \in [t_n, t_{n+1})$. By Step 2, (26) and Theorem 1, we have

$$|u_x(t)|_{L^2} \leq 4(1.01)^2 |u_x(t_n)|_{L^2} \leq \frac{4(1.01)^2 C_0}{|E_0|(T - t_n)} \leq \frac{8C_0}{|E_0|(T - t)},$$

and Corollary 2 is proved.

4. Proof of a time integral form of the control from below of the focusing speed

In this section, we prove Proposition 1, i.e. we prove that under a control on the right in L^1, and smallness conditions in L^2_{loc}, $\widetilde{\lambda}(t)$ satisfies in an integral form in time the following inequality:

$$-\frac{\widetilde{\lambda}_t}{\widetilde{\lambda}} \geq \frac{8|E_0|}{(\int Q)^2} \frac{1}{\widetilde{\lambda}}.$$

More precisely, we want to prove that:

There exist $\alpha_I > 0$ and $\delta_I > 0$ such that if $\int u_0^2 \leq \int Q^2 + \alpha_I$ and $E(u_0) < 0$, then the following is true. If $0 < t_1 < t_2 < T$ are such that

$$\forall t \geq t_1, \quad \int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \leq \delta_I \tag{90}$$

and

$$\forall t \in (t_1, t_2), \quad \int_{y \geq 0} |\widetilde{\varepsilon}(t)| \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \right)^{3/8}, \tag{91}$$

then

$$\frac{1}{100} + \log \frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)} \geq \frac{8|E_0|}{(\int Q)^2} \int_{t_1}^{t_2} \frac{dt}{\widetilde{\lambda}(t)}.$$

Proof. Let us first give the strategy of the proof. We begin by introducing the decomposition of Section 2.3, i.e.

$$\varepsilon(t, y) = \lambda^{1/2}(t) u(t, \lambda(t) y + x(t)) - Q(y), \tag{92}$$

such that

$$\int \int_{-\infty}^{y} \left(\tfrac{Q}{2} + y Q_y \right) \varepsilon(t, y) dy = \int y \left(\tfrac{Q}{2} + y Q_y \right) \varepsilon(t, y) dy = 0. \tag{93}$$

The fundamental advantage of this decomposition is to be adapted to the equation of the scaling parameter.

First, applying Lemma 6 to $v(t) = Q + \widetilde{\varepsilon}(t)$, by (90) and (91), for α_I and δ_I small enough (related to β_1), for all $t \in (t_1, t_2)$, we consider $\varepsilon(t)$, $\lambda_1(t)$ and $x_1(t)$ with

$$\varepsilon(t, y) = \lambda_1^{1/2}(t) \widetilde{\varepsilon}(t, \lambda_1(t) y + x_1(t)) + \lambda_1^{1/2}(t) Q(\lambda_1(t) y + x_1(t)) - Q(y),$$

such that (93) holds. Then, by (55)–(57), λ_1, x_1 satisfy

$$|\lambda_1 - 1| \leq C\left(\int \widetilde{\varepsilon}^2 e^{-|y|}\right)^{1/2} + C\int_{y>0} |\widetilde{\varepsilon}|,$$

$$|x_1| \leq C\left(\int \widetilde{\varepsilon}^2 e^{-|y|}\right)^{1/2} + C\left(\int_{y>0} |\widetilde{\varepsilon}|\right)^2.$$

From (91), we obtain

$$(94)\qquad |\lambda_1 - 1| \leq C\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}\right)^{3/8}, \quad |x_1| \leq C\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}\right)^{1/2}.$$

Note that we have (92) where $\lambda(t) = \lambda_1(t)\widetilde{\lambda}(t)$, $x(t) = \widetilde{\lambda}(t)x_1(t) + \widetilde{x}(t)$. Therefore, $\lambda(t)$ and $x(t)$ satisfy the relations given in Lemma 7 (see also Step 1 below) and by (94), we have

$$(95)\qquad \left|\frac{\lambda}{\widetilde{\lambda}} - 1\right| = |\lambda_1 - 1| \leq C\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}\right)^{3/8} \leq C\delta_I^{3/8}.$$

Thus we can choose $\delta_I > 0$ so that in the rest of the proof

$$\forall t \in (t_1, t_2), \quad \frac{1}{1.01} \leq \left|\frac{\lambda}{\widetilde{\lambda}}\right| \leq 1.01.$$

Let us call s the new time variable defined as follows:

$$s = \int_{t_1}^{t} \frac{dt'}{\lambda^3(t')}$$

(we use the same notation as for the time variable defined from $\widetilde{\lambda}(t)$; this will not lead to confusion). We call $s_1 = 0$ and s_2 the time associated to t_1 and t_2 respectively.

From the fact that $J(\varepsilon) = \int\int_{-\infty}^{y}\left(\frac{Q}{2} + yQ_y\right)\varepsilon = 0$ and $\int y\left(\frac{Q}{2} + yQ_y\right)\varepsilon = 0$, we have the following relations, for $s \in (s_1, s_2)$:

$$(96)\qquad \begin{aligned} &\frac{\lambda_s}{\lambda}\frac{(\int Q)^2}{8} + 2\int \varepsilon Q - \left(\frac{x_s}{\lambda} - 1\right)\int \varepsilon\left(\frac{Q}{2} + yQ_y\right) \\ &+ 10\int Q^3\left(\frac{Q}{2} + yQ_y\right)\varepsilon^2 + \int G(\varepsilon)\left(\frac{Q}{2} + yQ_y\right) = 0 \end{aligned}$$

and

$$(97)\qquad \begin{aligned} &\left(\frac{x_s}{\lambda} - 1\right)\mu_0 - \int \varepsilon L\left[\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y\right] \\ &- \frac{\lambda_s}{\lambda}\int \varepsilon\left[y\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y\right] - \left(\frac{x_s}{\lambda} - 1\right)\int \varepsilon\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y \\ &+ 10\int Q^3\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y\varepsilon^2 + \int G(\varepsilon)\left(y\left(\frac{Q}{2} + yQ_y\right)\right)_y = 0, \end{aligned}$$

where $|G(\varepsilon)| \leq C(|\varepsilon|^3 + |\varepsilon|^5)$ (see Lemma 7).

The fact that $\int \varepsilon Q$ has a slow variation in time and is related to $\widetilde{\lambda}^2$ by the energy relation together with equations (96)–(97) will allow us to give an equation in $\widetilde{\lambda}$.

To have such a property, we have to compare $\int \varepsilon Q$ and $\int \varepsilon^2 e^{-\frac{|y|}{100}}$. Here, we see that without more information, we cannot reach a conclusion. Thus we have to use the decomposition suitable for the Virial identity, that is, the function $\widetilde{\varepsilon}$, to obtain a comparison between $\int \widetilde{\varepsilon} Q$ and $\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}$. Indeed, the Virial identity will give a certain control of $\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}$ by $\left(\int \widetilde{\varepsilon} Q\right)^2$.

From the critical structure which gives cancellations at the first order and a surprising additional degeneracy, which gives cancellations at the second order, we are able to use this information in order to conclude.

Step 1. Relation between $\frac{\lambda_s}{\lambda}$ and $\int \widetilde{\varepsilon} Q$.

We claim the following relation between $\frac{\lambda_s}{\lambda}$ and $\int \widetilde{\varepsilon} Q$ coming from a surprising degeneracy at the second order of the relation between ε and $\widetilde{\varepsilon}$.

Lemma 8 (Relation between $\frac{\lambda_s}{\lambda}$ and $\int \widetilde{\varepsilon} Q$)**.** *We have* $\forall s \in (s_1, s_2)$,

$$\left|\frac{\lambda_s}{\lambda}\frac{(\int Q)^2}{8} + 2\int \widetilde{\varepsilon} Q + 5(\lambda_1 - 1)\int \widetilde{\varepsilon} Q\right| \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \tag{98}$$

Proof of Lemma 8. First, replacing ε by its expression in terms of $\widetilde{\varepsilon}$ and λ_1, x_1, in all terms of (96), (97), we find the following estimates.

Lemma 9 (Relation between ε and $\widetilde{\varepsilon}$, λ_1, x_1)**.**

$$\begin{aligned}&\left|\int \varepsilon Q - \int \widetilde{\varepsilon} Q + (\lambda_1 - 1)\int \widetilde{\varepsilon}\left(\tfrac{Q}{2} + yQ_y\right) + \tfrac14(\lambda_1 - 1)^2\int y^2 Q^2\right| \\ &\qquad \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).\end{aligned} \tag{99}$$

$$\begin{aligned}&\left|\int \varepsilon\left(\tfrac{Q}{2} + yQ_y\right) - \int \widetilde{\varepsilon}\left(\tfrac{Q}{2} + yQ_y\right) - (\lambda_1 - 1)\int\left(\tfrac{Q}{2} + yQ_y\right)^2\right| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).\end{aligned} \tag{100}$$

$$\begin{aligned}&\Bigg|\int \varepsilon L\left[\left(y\left(\tfrac{Q}{2} + yQ_y\right)\right)_y\right] - \int \widetilde{\varepsilon} L\left[\left(y\left(\tfrac{Q}{2} + yQ_y\right)\right)_y\right] \\ &\qquad\qquad - 2(\lambda_1 - 1)\int\left(\tfrac{Q}{2} + yQ_y\right)^2\Bigg| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).\end{aligned} \tag{101}$$

$$\begin{aligned}&\Bigg|\int Q^3\left(\tfrac{Q}{2} + yQ_y\right)\varepsilon^2 - 2(\lambda_1 - 1)\int Q^3\left(\tfrac{Q}{2} + yQ_y\right)^2\widetilde{\varepsilon} \\ &\qquad\qquad - |\lambda_1 - 1|^2\int Q^3\left(\tfrac{Q}{2} + yQ_y\right)^3\Bigg| \\ &\qquad \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).\end{aligned} \tag{102}$$

$$\begin{aligned}&\left|\int Q^3\left(y\left(\tfrac{Q}{2} + yQ_y\right)\right)_y\varepsilon^2\right| + \left|\frac{\lambda_s}{\lambda}\int \varepsilon\left[y\left(y\left(\tfrac{Q}{2} + yQ_y\right)\right)_y\right]\right| \\ &\qquad + \left|\left(\frac{x_s}{\lambda} - 1\right)\int \varepsilon\left(y\left(\tfrac{Q}{2} + yQ_y\right)\right)_y\right| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).\end{aligned} \tag{103}$$

$$\left|\int G(\varepsilon)\left(\tfrac{Q}{2}+yQ_y\right)\right|+\left|\int G(\varepsilon)\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right|$$
$$\leq C(|x_1|^3+|\lambda_1-1|^3+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}). \tag{104}$$

Proof of Lemma 9. See the Appendix. To find these estimates, we replace $\varepsilon(y)$ by

$$\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)+\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y),$$

and we use the following expansion of $\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y)$:

$$\begin{aligned}\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y)&=(\lambda_1-1)\left(\tfrac{Q}{2}+yQ_y\right)+x_1Q_y\\ &+\frac{(\lambda_1-1)^2}{2}\left(-\frac{Q}{4}+yQ_y+y^2Q_{yy}\right)+(\lambda_1-1)x_1\left(\frac{Q_y}{2}+yQ_{yy}\right)+\frac{x_1^2}{2}Q_{yy}\\ &+O(|\lambda_1-1|^3)e^{-|y|/2}+O(|x_1|^3)e^{-|y|/2}.\end{aligned}$$

Note that to use cancellations of equation (96), we have to expand at the second order (since relation (96) has linear and quadratic terms) and use the orthogonality conditions on ε and $\widetilde{\varepsilon}$ to simplify some expressions at the linear level. We refer to the Appendix for the rest of the proof of Lemma 9.

In particular, by using (101)–(104), we have from (97) (recall $\mu_0=\int\left(\frac{Q}{2}+yQ_y\right)^2$)

$$\left|\left(\tfrac{x_s}{\lambda}-1\right)-\tfrac{1}{\mu_0}\int\widetilde{\varepsilon}L\left[\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right]-2(\lambda_1-1)\right|$$
$$\leq C(|\lambda_1-1|^2+|x_1|^2+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}).$$

Then, by using (100), we have

$$\begin{aligned}&\Bigg|\left(\tfrac{x_s}{\lambda}-1\right)\left(\int\varepsilon\left(\tfrac{Q}{2}+yQ_y\right)\right)\\ &-\left(\tfrac{1}{\mu_0}\int\widetilde{\varepsilon}L\left[\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right]+2(\lambda_1-1)\right)\left(\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)+\mu_0(\lambda_1-1)\right)\Bigg|\\ &\leq C\left(|\lambda_1-1|^3+|x_1|^3+\left(\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}\right)^{3/2}\right),\end{aligned}$$

and so

$$\begin{aligned}&\Bigg|\left(\tfrac{x_s}{\lambda}-1\right)\left(\int\varepsilon\left(\tfrac{Q}{2}+yQ_y\right)\right)-2\mu_0(\lambda_1-1)^2-2(\lambda_1-1)\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)\\ &-(\lambda_1-1)\int\widetilde{\varepsilon}L\left[\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right]\Bigg|\leq C\left(|\lambda_1-1|^3+|x_1|^3+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}\right).\end{aligned}\tag{105}$$

Finally, by using (99), (102), (104), and (105), we have from (96)

$$\left|\tfrac{\lambda_s}{\lambda}\tfrac{(\int Q)^2}{8}+2\int\widetilde{\varepsilon}Q+|\lambda_1-1|^2\mu^*+(\lambda_1-1)\int\widetilde{\varepsilon}W^*\right|$$
$$\leq C(|x_1|^2+|\lambda_1-1|^3+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}), \tag{106}$$

where

$$\mu^* = -\tfrac{1}{2}\int y^2Q^2 - 2\int \left(\tfrac{Q}{2}+yQ_y\right)^2 + 10\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^3,$$
$$W^* = -4\left(\tfrac{Q}{2}+yQ_y\right) - L\left[\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right] + 20Q^3\left(\tfrac{Q}{2}+yQ_y\right)^2.$$

A crucial fact is the following claim which gives for formula (106) a degeneracy at the second order. Note that this degeneracy is not related to the choice of orthogonality conditions on $\widetilde{\varepsilon}$.

Lemma 10 (Cancellations of second order terms in $(\lambda_1 - 1)$)**.**

$$\mu^* = 0 \quad \textit{and} \quad W^* = 5Q. \tag{107}$$

Assuming this lemma, we have from (106),

$$\left|\tfrac{\lambda_s}{\lambda}\tfrac{(\int Q)^2}{8} + 2\int\widetilde{\varepsilon}Q + 5(\lambda_1-1)\int\widetilde{\varepsilon}Q\right| \le C(|x_1|^2 + |\lambda_1-1|^3 + \int\widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}),$$

which proves (98). Thus Lemma 8 is proved. Therefore, we are now reduced to prove Lemma 10.

Proof of Lemma 10. (i) $\mu^* = 0$.

We begin with two elementary calculations. Since $Q^5 = Q - Q_{yy}$, we have by integration

$$Q_y^2 = Q^2 - \frac{1}{3}Q^6. \tag{108}$$

By integration by parts, we have

$$\begin{aligned}\int y^2Q^6 &= \int y^2Q^2 - \int y^2Q_{yy}Q = \int y^2Q^2 + 2\int yQ_yQ + \int y^2Q_y^2\\ &= 2\int y^2Q^2 - \int Q^2 - \frac{1}{3}\int y^2Q^6.\end{aligned}$$

Thus,

$$\int y^2Q^6 = \frac{3}{2}\int y^2Q^2 - \frac{3}{4}\int Q^2.$$

By a similar calculation and using $\int Q^6 = \frac{3}{2}\int Q^2$ (by $E(Q)=0$ and (108)), we have

$$\int y^2Q^{10} = \frac{9}{4}\int y^2Q^6 - \frac{1}{8}\int Q^6 = \frac{9}{4}\int y^2Q^6 - \frac{3}{16}\int Q^2.$$

Thus,

$$\begin{aligned}\int\left(\tfrac{Q}{2}+yQ_y\right)^2 &= \frac{1}{4}\int Q^2 + \int yQ_yQ + \int y^2Q_y^2 = -\frac{1}{4}\int Q^2 + \int y^2Q_y^2\\ &= -\frac{1}{4}\int Q^2 + \int y^2Q^2 - \frac{1}{3}\int y^2Q^6 = \frac{1}{2}\int y^2Q^2.\end{aligned}$$

Next,

$$\begin{aligned}\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^3 &= \frac{1}{8}\int Q^6+\frac{3}{4}\int yQ_yQ^5+\frac{3}{2}\int y^2Q_y^2Q^4+\int y^3Q_y^3Q^3\\ &= \frac{1}{8}\int Q^6-\frac{1}{8}\int Q^6+\frac{3}{2}\int y^2Q^6-\frac{1}{2}\int y^2Q^{10}\\ &\quad+\int y^3Q_yQ^5-\frac{1}{3}\int y^3Q_yQ^9\\ &= \int y^2Q^6-\frac{2}{5}\int y^2Q^{10}\\ &= \frac{1}{10}\left(\int y^2Q^6+\frac{3}{4}\int Q^2\right)=\frac{3}{20}\int y^2Q^2.\end{aligned}$$

Therefore, $\mu^*=0$.

(ii) $W^*=5Q$.

By the expression $Lv=-v_{yy}+v-5Q^4v$, we have for any function v,

$$L(yv_y)=-2v_{yy}+yL(v_y)=-2v_{yy}+y((Lv)_y+5(Q^4)_yv).$$

Thus, from $L\left(\frac{Q}{2}+yQ_y\right)=-2Q$, $Q_{yy}=Q-Q^5$, $Q_y^2=Q^2-\frac{Q^6}{3}$, we have

$$\begin{aligned}&L\left[\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right]\\ &\quad=L\left(\tfrac{Q}{2}+yQ_y\right)+L\left(y\left(\tfrac{Q}{2}+yQ_y\right)_y\right)\\ &\quad=-2Q-2\left(\tfrac{Q}{2}+yQ_y\right)_{yy}+y\left(-2Q_y+20Q_yQ^3\left(\tfrac{Q}{2}+yQ_y\right)\right)\\ &\quad=-2Q-5Q_{yy}-2y(Q-Q^5)_y-2yQ_y+10yQ_yQ^4\\ &\qquad+20y^2Q^3(Q^2-\tfrac{Q^6}{3})\\ &\quad=-7Q+5Q^5-4yQ_y+20y^2Q^5-\frac{20}{3}y^2Q^9+20yQ_yQ^4.\end{aligned}$$

Since

$$\begin{aligned}20Q^3\left(\tfrac{Q}{2}+yQ_y\right)^2 &= 5Q^5+20yQ_yQ^4+20y^2Q^3(Q^2-\tfrac{Q^6}{3})\\ &= 5Q^5+20y^2Q^5-\frac{20}{3}y^2Q^9+20yQ_yQ^4,\end{aligned}$$

we obtain from the definition of W^* that

$$W^*=5Q.$$

Step 2. Inequality satisfied by $\widetilde{\lambda}$.

Now, we claim that from Step 1 and the energy identity in $\widetilde{\varepsilon}$, we have

(109)
$$\frac{1}{200}+\log\frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)}\geq\frac{13|E_0|}{(\int Q)^2}\int_{t_1}^{t_2}\frac{dt}{\widetilde{\lambda}}+\frac{1}{(\int Q)^2}\int_{t_1}^{t_2}\frac{\left|\int\widetilde{\varepsilon}Q\right|}{\widetilde{\lambda}^3}dt-C\int_{t_1}^{t_2}\frac{\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{100}}}{\widetilde{\lambda}^3}dt.$$

By (94), we have

$$|\lambda_1-1|^3\leq C\left(\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{100}}\right)^{9/8}\leq C\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{100}},\quad |x_1|^2\leq C\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{100}},$$

and

$$|\lambda_1 - 1| \left|\int \widetilde{\varepsilon} Q\right| \leq |\lambda_1 - 1|^3 + \left|\int \widetilde{\varepsilon} Q\right|^{3/2}.$$

Thus, from Step 1, $\forall s \in (s_1, s_2)$,

$$-\frac{\lambda_s}{\lambda} \geq \frac{16}{(\int Q)^2} \int \widetilde{\varepsilon} Q - C \left|\int \widetilde{\varepsilon} Q\right|^{3/2} - C \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}. \tag{110}$$

One difficulty here is that $\int \widetilde{\varepsilon} Q$ is not supposed to be nonnegative for all time. However, using the energy identity, we are able to control $\int \varepsilon Q$ when it is negative. Indeed, from (39), we have

$$\int \widetilde{\varepsilon} Q \geq \left|\int \widetilde{\varepsilon} Q\right| - C \int \widetilde{\varepsilon}^2 e^{-|y|}. \tag{111}$$

Therefore,

$$-\frac{\lambda_s}{\lambda} \geq \frac{16}{(\int Q)^2} \left|\int \widetilde{\varepsilon} Q\right| - C \left|\int \widetilde{\varepsilon} Q\right|^{3/2} - C \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}.$$

Since, by the Cauchy–Schwarz inequality,

$$\left|\int \widetilde{\varepsilon} Q\right|^{1/2} \leq C \left(\int \widetilde{\varepsilon}^2 e^{-|y|}\right)^{1/4} \leq C \delta_I^{1/4}, \tag{112}$$

for δ_I small enough, we obtain

$$-\frac{\lambda_s}{\lambda} \geq \frac{15}{(\int Q)^2} \left|\int \widetilde{\varepsilon} Q\right| - C \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}.$$

Now, we go back to the t variable (recall $dt = \lambda^3 ds$), dividing by $\lambda^3(t)$:

$$-\frac{\lambda_t}{\lambda} \geq \frac{15}{(\int Q)^2} \frac{\left|\int \widetilde{\varepsilon} Q\right|}{\lambda^3} - C \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\lambda^3}.$$

Integrating between t_1 and t_2, we obtain

$$\log \frac{\lambda(t_1)}{\lambda(t_2)} \geq \frac{15}{(\int Q)^2} \int_{t_1}^{t_2} \frac{\left|\int \widetilde{\varepsilon} Q\right|}{\lambda^3} dt - C \int_{t_1}^{t_2} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\lambda^3} dt. \tag{113}$$

Therefore, by (95), for δ_I small,

$$\frac{1}{200} + \log \frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)} \geq \log \frac{\lambda(t_1)}{\lambda(t_2)} \geq \frac{14}{(\int Q)^2} \int_{t_1}^{t_2} \frac{\left|\int \widetilde{\varepsilon} Q\right|}{\widetilde{\lambda}^3} dt - C \int_{t_1}^{t_2} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\widetilde{\lambda}^3} dt. \tag{114}$$

Recall that from (40) we have $\left|\int \widetilde{\varepsilon} Q\right| \geq \widetilde{\lambda}^2 |E_0| - C \int \widetilde{\varepsilon}^2 e^{-|y|}$, and claim (109) follows.

Step 3. Conclusion using the local Virial identity on $\widetilde{\varepsilon}$.

We have from (45), written in the t variable, for some $A_0 \geq 100$,

$$\left(\int \psi_{A_0} \widetilde{\varepsilon}^2\right)_t \leq -\delta_0 \frac{\int (\widetilde{\varepsilon}^2 + \widetilde{\varepsilon}_y^2) e^{-\frac{|y|}{A_0}}}{\widetilde{\lambda}^3} + \frac{1}{\delta_0} \frac{\left(\int \widetilde{\varepsilon} Q\right)^2}{\widetilde{\lambda}^3}.$$

Integrating between t_1 and t_2, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\widetilde{\lambda}^3} dt &\leq \int_{t_1}^{t_2} \frac{\int (\widetilde{\varepsilon}^2 + \widetilde{\varepsilon}_y^2) e^{-\frac{|y|}{A_0}}}{\widetilde{\lambda}^3} dt \\ &\leq C \left| \int \psi_{A_0} \widetilde{\varepsilon}^2(t_1) \right| + C \left| \int \psi_{A_0} \widetilde{\varepsilon}^2(t_2) \right| + C \int_{t_1}^{t_2} \frac{\left(\int \widetilde{\varepsilon} Q\right)^2}{\widetilde{\lambda}^3} dt.\end{aligned}$$

From (25), we have $\left(\int \widetilde{\varepsilon}^2\right)^{1/2} \leq C\sqrt{\alpha_I}$, and so

$$\int_{t_1}^{t_2} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\widetilde{\lambda}^3} dt \leq C \int_{t_1}^{t_2} \frac{\left(\int \widetilde{\varepsilon} Q\right)^2}{\widetilde{\lambda}^3} dt + C\alpha_I. \tag{115}$$

From (109) and (115), and taking α_I small enough, we have

$$\frac{1}{100} + \log \frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)} \geq \frac{13|E_0|}{(\int Q)^2} \int_{t_1}^{t_2} \frac{dt}{\widetilde{\lambda}} + \frac{1}{(\int Q)^2} \int_{t_1}^{t_2} \frac{\left|\int \widetilde{\varepsilon} Q\right|}{\widetilde{\lambda}^3} dt - C \int_{t_1}^{t_2} \frac{\left(\int \widetilde{\varepsilon} Q\right)^2}{\widetilde{\lambda}^3} dt.$$

Therefore, by choosing δ_I small enough, by (112), we obtain

$$\frac{1}{100} + \log \frac{\widetilde{\lambda}(t_1)}{\widetilde{\lambda}(t_2)} \geq \frac{13|E_0|}{(\int Q)^2} \int_{t_1}^{t_2} \frac{dt}{\widetilde{\lambda}},$$

and Proposition 1 is proved.

Remark. Blow up rate under control on $\int \widetilde{\varepsilon}_y^2$. From the proof of Theorem 1, we can obtain the blow up rate, if we assume in addition that the gradient term $\int \widetilde{\varepsilon}_y^2$ is controlled by $\left|\int \widetilde{\varepsilon} Q\right|$, and a control of $\int_{y>0} |\widetilde{\varepsilon}|$. More precisely, we claim the following.

Under the assumptions of Theorem 1, suppose in addition that there exists $t_0 \in (0,T)$ such that $\forall t \in (t_0,T)$,

$$\int \widetilde{\varepsilon}_y^2(t) \leq \frac{1}{10} \left| \int \widetilde{\varepsilon}(t) Q \right| \quad \textit{and} \quad \int_{y>0} |\widetilde{\varepsilon}(t)| \leq \left(\int \widetilde{\varepsilon}^2(t) e^{-\frac{|y|}{100}} \right)^{3/8}. \tag{116}$$

Then, for some $t_1 \in (t_0,T)$, we have $\forall t \in (t_1,T)$,

$$\frac{C_0'}{|E_0|(T-t)} \leq |u_x(t)|_{L^2} \leq \frac{8C_0}{|E_0|(T-t)}, \tag{117}$$

where $C_0' = \frac{(\log 2)|Q_x|_{L^2}(\int Q)^2}{2000}$.

Proof of (117). *Reduction to a control of $t_{n+1}-t_n$.* First, we note that (117) is implied by the following control from below of $t_{n+1}-t_n$, for $n_1 > 0$ large enough:

$$\forall n > n_1, \quad t_{n+1} - t_n \geq \frac{8C_0'}{|Q_x|_{L^2}|E_0|} \widetilde{\lambda}(t_n). \tag{118}$$

Indeed, if (118) is satisfied for n large by similar arguments as before, then

$$\frac{C_0 \widetilde{\lambda}(t_n)}{4|Q_x|_{L^2}|E_0|} \geq T - t_n \geq \frac{4C_0'}{(1.01)^2|Q_x|_{L^2}|E_0|} \widetilde{\lambda}(t_n)$$

and

$$\frac{C_0}{|E_0|(T-t_n)} \geq |u_x(t_n)|_{L^2} \geq \frac{2C_0'}{|E_0|(T-t_n)}.$$

Thus, from the definition of the sequence (t_n), we have, for all $t \in (t_n, t_{n+1})$, where $n > n_1$,

$$|u_x(t)|_{L^2} \geq |u_x(t_n)|_{L^2} = \frac{|u_x(t_{n+1})|_{L^2}}{2} \geq \frac{C_0'}{|E_0|(T - t_{n+1})} \geq \frac{C_0'}{|E_0|(T-t)}.$$

Moreover, from Corollary 2, for t close to T, $|u_x(t)|_{L^2} \leq \frac{8C_0}{|E_0|(T-t)}$. Therefore, we are reduced to proving (118) under assumptions (116).

Proof of (118). Let $\delta > 0$ be fixed later, and take $t_0 > 0$ large enough so that $\forall t > t_0$, $\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}} \leq \delta$. Recall that by Lemma 8, in the proof of Proposition 1, we have

$$\left| \frac{\lambda_s}{\lambda} \frac{(\int Q)^2}{8} + 2 \int \widetilde{\varepsilon} Q + 5(\lambda_1 - 1) \int \widetilde{\varepsilon} Q \right| \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).$$

From this, as in Step 2 of the proof of Proposition 1, by using the control of $|\lambda_1 - 1|$ and $|x_1|$ in (94), and the assumption (116) on $\int_{y>0} |\widetilde{\varepsilon}|$, we obtain, in the s variable defined by $\frac{ds}{dt} = 1/\lambda^3$,

$$-\frac{\lambda_s}{\lambda} \leq \frac{17}{(\int Q)^2} \frac{\left|\int \widetilde{\varepsilon} Q\right|}{\lambda^3} + C \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\lambda^3}.$$

By integration between t_n and t_{n+1}, and (95), we obtain for δ small,

$$\begin{aligned} \log \frac{\widetilde{\lambda}(t_n)}{\widetilde{\lambda}(t_{n+1})} &\leq \frac{1}{200} + \log \frac{\lambda(t_n)}{\lambda(t_{n+1})} \\ &\leq \frac{1}{200} + \frac{18}{(\int Q)^2} \int_{t_n}^{t_{n+1}} \frac{|\int \widetilde{\varepsilon} Q|}{\widetilde{\lambda}^3} dt + C \int_{t_n}^{t_{n+1}} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\lambda^3} dt. \end{aligned}$$

Now, by the energy relations (41) and (116), $\forall t \in (t_n, t_{n+1})$,

$$\left| \int \widetilde{\varepsilon} Q \right| \leq \widetilde{\lambda}^2 |E_0| + \frac{1}{2} \int \widetilde{\varepsilon}_y^2 + C \int \widetilde{\varepsilon}^2 e^{-|y|} \leq \widetilde{\lambda}^2 |E_0| + \frac{1}{20} \left| \int \widetilde{\varepsilon} Q \right| + C \int \widetilde{\varepsilon}^2 e^{-|y|},$$

we obtain

$$\begin{aligned} \log \frac{\widetilde{\lambda}(t_n)}{\widetilde{\lambda}(t_{n+1})} \leq \frac{1}{200} + \frac{20|E_0|}{(\int Q)^2} \int_{t_n}^{t_{n+1}} \frac{dt}{\widetilde{\lambda}} - \frac{1}{(\int Q)^2} \int_{t_n}^{t_{n+1}} \frac{|\int \widetilde{\varepsilon} Q|}{\widetilde{\lambda}^3} dt \\ + C \int_{t_n}^{t_{n+1}} \frac{\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}}{\lambda^3} dt. \end{aligned}$$

As in Step 3 of the proof of Proposition 1, we use the Virial identity on $\widetilde{\varepsilon}$ to control the local L^2 norm of $\widetilde{\varepsilon}$ in integral in time, and from the control of $\int_{y>0} |\widetilde{\varepsilon}|$, we have $\forall t \in (t_n, t_{n+1})$, $\widetilde{\lambda}(t) \geq \frac{1}{4} \widetilde{\lambda}(t_n)$. Thus,

$$t_{n+1} - t_n \geq \frac{(\int Q)^2 \log 2}{200|E_0|} \widetilde{\lambda}(t_n),$$

which proves (118).

5. Control of the L^1 norm of $\widetilde{\varepsilon}$ on the right by L^2 polynomial decay

In this section, we prove the following result, which gives a control of $\int_{y>0}|\widetilde{\varepsilon}|$ in terms of $\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}\right)^{3/8}$ in the situation of a slow dynamics. This control is based on the conservation of L^2 polynomial decay on the right. It allows us to control the L^1 norm on the right of the soliton pointwise in time by a sublinear estimate close enough to the linear estimate. For this, as in [11], we need a control from above of $|u_x(t)|_{L^2}$. Let us recall the statement of Proposition 2.

Suppose that for some $\theta > 0$,

$$\forall x_0 > 0, \quad \int_{x \geq x_0} u_0^2(x)dx \leq \frac{\theta}{x_0^6}$$

and

$$\forall t \in [0, T), \quad \widetilde{\lambda}(t) \leq 1.$$

Let $A > 0$. There exists $\delta_{II} = \delta_{II}(A, |E_0|, \theta)$ such that the following is true. Suppose that $0 < t_1 < t_2 < T$ satisfy

$$\forall t \in (t_1, t_2), \quad \widetilde{\lambda}(t) \leq (4.1)\widetilde{\lambda}(t_2) \tag{119}$$

and

$$t_2 - t_1 \geq A\widetilde{\lambda}(t_2). \tag{120}$$

If in addition $\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{100}} \leq \delta_{II}$, then

$$\int_{y>0} |\widetilde{\varepsilon}(t_2, y)|dy \leq \left(\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{100}}\right)^{3/8}. \tag{121}$$

Remark. Assumption (119) on (t_1, t_2) concerns only the size of the soliton to which the solution is close. Note that the conclusion of the proposition is a control of $\int_{y>0}|\widetilde{\varepsilon}|$ by $\left(\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{100}}\right)^{3/8}$ pointwise in time (not in the form of a space-time integral). We do not expect a linear control in (121), since the proof is based on nonlinear properties. Note that the result is also valid for any power of the form $\frac{1}{2}(1-\eta_0)$, for $\eta_0 \in (0,1)$, instead of $3/8$, with the following conclusion:

$$\int_{y>0} |\widetilde{\varepsilon}(t_2, y)|dy \leq \left(\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{B_0}}\right)^{\frac{1}{2}(1-\eta_0)},$$

where $B_0 = B_0(\eta_0)$.

Proof. **Step 1.** Reduction to exponential estimates.

We control the integral $\int_{y>0}|\widetilde{\varepsilon}(t_2, y)|dy$ by considering two regions: a part on the compact set and a part for y large. Let

$$y_{t_2} = -10 \log\left(\int \widetilde{\varepsilon}^2(t_2)e^{-\frac{|y|}{100}}\right).$$

Let $0 < \delta_{II} < 1$ be such that $y_{t_2} \geq -10 \log \delta_{II} > 1$. Then

$$\int_{y>0} |\widetilde{\varepsilon}(t_2, y)| dy = I_1 + I_2, \tag{122}$$

where $I_1 = \int_{0<y<y_{t_2}} |\widetilde{\varepsilon}(t_2, y)| dy$ and $I_2 = \int_{y>y_{t_2}} |\widetilde{\varepsilon}(t_2, y)| dy$.

(i) Estimates on the compact set. First, we have

$$\begin{aligned} I_1 = \int_{0<y<y_{t_2}} |\widetilde{\varepsilon}(t_2, y)| dy &\leq e^{\frac{y_{t_2}}{100}} \int_{y>0} |\widetilde{\varepsilon}(t_2)| e^{-\frac{y}{100}} \\ &\leq e^{\frac{y_{t_2}}{100}} \left(\int_{y>0} e^{-\frac{y}{100}}\right)^{1/2} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}}\right)^{1/2} \\ &\leq 10\, e^{\frac{y_{t_2}}{100}} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}}\right)^{1/2} \\ &\leq 10 \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}}\right)^{2/5} \\ &\leq 10 \delta_{II}^{1/40} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}}\right)^{3/8} \\ &\leq \frac{1}{2} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}}\right)^{3/8}, \end{aligned} \tag{123}$$

for δ_{II} satisfying $\delta_{II}^{1/40} < 1/20$.

(ii) Estimates for y large. Now, to control the second term $I_2 = \int_{y>y_{t_2}} |\widetilde{\varepsilon}(t_2)|$, we proceed in two steps:

(a) We first use L^2 estimates in rescaled variables from t_1 to t_2. Here, we use a similar idea as in [11]. The strength of these estimates is that they are of nonlinear type and allow singular behavior (no control from below of $\widetilde{\lambda}(t)$). In fact, linear estimates break down in this situation.

(b) Then, we prove L^2 estimates in the original variables from 0 to t_1.

Note that to conclude the proof, we need that t_2 and t_1 are not too close to each other so that the estimates obtained in the first step allow us to reduce the problem to estimates of $u(t,x)$ itself in the second step.

First, we use a dyadic decomposition of the half line $[y_{t_2}, +\infty)$ to reduce ourselves to control L^2 expressions.

$$I_2 = \sum_{k=0}^{+\infty} \int_{2^k y_{t_2} < y < 2^{k+1} y_{t_2}} |\widetilde{\varepsilon}(t_2)| \leq \sum_{k=0}^{+\infty} \sqrt{2^k y_{t_2}} \left(\int_{y > 2^k y_{t_2}} \widetilde{\varepsilon}^2(t_2, y) dy\right)^{1/2}.$$

Step 2. Exponential estimates in the rescaled variable from t_2 to t_1.

For all $k \geq 0$, we control $\int_{y>2^k y_{t_2}} \widetilde{\varepsilon}^2(t_2)$ going back to the time t_1 in the rescaled variable. Unfortunately, in this variable, we cannot go directly backwards from t_2 to 0 because of some interactions in the exponential estimates.

Here, we need the assumption of the proposition on $\widetilde{\lambda}(t)$ on this interval. For $t' \in (\frac{t_1 - t_2}{\widetilde{\lambda}^3(t_2)}, 0)$, $x' \in \mathbf{R}$, let

$$z(t', x') = \widetilde{\lambda}^{1/2}(t_2) u(\widetilde{\lambda}^3(t_2) t' + t_2, \widetilde{\lambda}(t_2) x' + \widetilde{x}(t_2)).$$

By the invariances of the critical KdV equation by scaling and translation, z satisfies (3) on $(\frac{t_1-t_2}{\widetilde{\lambda}^3(t_2)}, 0)$. Moreover,

$$z(0,x') = Q + \widetilde{\varepsilon}(t_2, x'),$$

$$z(\frac{t_1-t_2}{\widetilde{\lambda}^3(t_2)}, x') = \widetilde{\lambda}^{1/2}(t_2) u(t_1, \widetilde{\lambda}(t_2)x' + \widetilde{x}(t_2)),$$

and $\lambda_z(t') = \frac{\widetilde{\lambda}(\widetilde{\lambda}^3(t_2)t'+t_2)}{\widetilde{\lambda}(t_2)} \leq 4.1$, $x_z(t') = \frac{\widetilde{x}(\widetilde{\lambda}(t_2)t'+t_2)-\widetilde{x}(t_2)}{\widetilde{\lambda}(t_2)}$.

Now, we use the notation from Section 2.2, considering the function $\psi = \psi_K$, for $K = 9 > 2(1.01)(4.1)$ (see the assumption in Lemma 5), and the quantity $\mathbf{I}_{x_0,t_0}(t)$ defined for the purposes of Lemma 5.

First, let us remark that, since for $x \geq 0$, $\psi(x) \geq 1/2$,

$$\begin{aligned}
\int_{y>2^k y_{t_2}} \widetilde{\varepsilon}^2(t_2, y)dy &\leq 2\int_{y>2^k y_{t_2}} (Q + \widetilde{\varepsilon}(t_2))^2 + 2\int_{y>2^k y_{t_2}} Q^2 \\
&\leq 4\int (Q + \widetilde{\varepsilon}(t_2))^2 \psi(y - 2^k y_{t_2})dy + Ce^{-2^k y_{t_2}} \\
&= 4\int z^2(0,x')\psi(x' - 2^k y_{t_2})dx' + Ce^{-2^k y_{t_2}}. \qquad (124)
\end{aligned}$$

Next, by applying Lemma 5 on z with $t_0 = 0$, $t = \frac{t_1-t_2}{\widetilde{\lambda}(t_2)} < 0$, and with $x_0 = 2^k y_{t_2}$, we have

$$\begin{aligned}
&\int z^2(0,x')\psi(x' - 2^k y_{t_2})dx' \\
&\quad \leq Ce^{-\frac{2^k y_{t_2}}{9}} + \int z^2(\frac{t_1-t_2}{\widetilde{\lambda}(t_2)}, x')\psi(x' - 2^k y_{t_2} - \frac{1}{4}x_z(\frac{t_1-t_2}{\widetilde{\lambda}(t_2)}))dx' \qquad (125)
\end{aligned}$$

(note that $x_z(0) = 0$). Next,

$$\begin{aligned}
&\int z^2(\frac{t_1-t_2}{\widetilde{\lambda}(t_2)}, x')\psi(x' - 2^k y_{t_2} - \frac{1}{4}x_z(\frac{t_2-t_1}{\widetilde{\lambda}(t_2)}))dx' \\
&\quad = \widetilde{\lambda}(t_2)\int u^2(t_1, \widetilde{\lambda}(t_2)x' + \widetilde{x}(t_2))\psi(x' - 2^k y_{t_2} - \frac{1}{4}\frac{\widetilde{x}(t_1)-\widetilde{x}(t_2)}{\widetilde{\lambda}(t_2)})dx' \\
&\quad = \int u^2(t_1, x + \widetilde{x}(t_1))\psi(\frac{x}{\widetilde{\lambda}(t_2)} - 2^k y_{t_2} - \frac{3}{4}\frac{\widetilde{x}(t_2)-\widetilde{x}(t_1)}{\widetilde{\lambda}(t_2)})dx. \qquad (126)
\end{aligned}$$

Now, by (26), (119) and (120),

$$\forall t \in (t_1, t_2), \quad \widetilde{x}_t(t) \geq \frac{1}{(1.01)\widetilde{\lambda}^2(t)} \geq \frac{1}{18\widetilde{\lambda}^2(t_2)} \quad \text{and} \quad t_2 - t_1 \geq A\widetilde{\lambda}(t_2).$$

Therefore,

$$\frac{\widetilde{x}(t_2)-\widetilde{x}(t_1)}{\widetilde{\lambda}(t_2)} \geq \frac{A}{18\widetilde{\lambda}^2(t_2)}.$$

By using the monotonicity of ψ, $\psi<1$, $\forall x<0$, $\psi(x)\le Ce^{\frac{x}{9}}$, and $\widetilde\lambda(t_2)\le 1$, we have

$$\begin{aligned}
&\int u^2(t_1,x+\widetilde x(t_1))\psi(\frac{x}{\widetilde\lambda(t_2)}-2^ky_{t_2}-\frac34\frac{\widetilde x(t_2)-\widetilde x(t_1)}{\widetilde\lambda(t_2)})dx\\
&\le \int u^2(t_1,x+\widetilde x(t_1))\psi(\frac{x}{\widetilde\lambda(t_2)}-2^ky_{t_2}-\frac{A}{24\widetilde\lambda^2(t_2)})dx\\
&\le \int_{x<\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u^2(t_1,x+\widetilde x(t_1))\psi(\frac{x}{\widetilde\lambda(t_2)}-2^ky_{t_2}-\frac{A}{24\widetilde\lambda^2(t_2)})dx\\
&\quad+\int_{x>\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u^2(t_1,x+\widetilde x(t_1))\psi(\frac{x}{\widetilde\lambda(t_2)}-2^ky_{t_2}-\frac{A}{24\widetilde\lambda^2(t_2)})dx\\
&\le \left(\int u^2(t_1)\right)|\psi|_{L^\infty(x<-\frac12(2^ky_{t_2}+\frac{A}{24\widetilde\lambda^2(t_2)}))}\\
&\quad+\int_{x>\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u^2(t_1,x+\widetilde x(t_1))dx\\
&\le C\left(\int u_0^2\right)e^{-\frac1{18}(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})}\\
&\quad+\int_{x>\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u^2(t_1,x+\widetilde x(t_1))dx.
\end{aligned}\tag{127}$$

Here, we use in a crucial way that t_2-t_1 is large enough. Note that the remaining estimates concern values of $x>\frac{A}{48\widetilde\lambda(t_2)}$ far away from 0.

Step 3. L^2 exponential property in the u variable and final estimates.

We use Lemma 5 with $K=3>2(1.01)$ (recall that $\widetilde\lambda(t)\le 1$) directly on $u(t)$ between t_1 and 0. As in the proof of Lemma 4 (see (49)), and then by the property of the initial data u_0, we obtain

$$\begin{aligned}
&\int_{x>\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u^2(t_1,x+\widetilde x(t_1))dx\\
&\quad\le 2\int_{x>\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})} u_0^2(x+\widetilde x(0))dx+Ce^{-\frac16(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})}\\
&\quad\le \frac{2\theta}{\left(\frac12(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})\right)^6}+Ce^{-\frac16(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{24\widetilde\lambda(t_2)})}.
\end{aligned}\tag{128}$$

Since for $x\ge 0$, $e^{-\frac1x}\le Cx^6$, for some $C>0$, we obtain, for a constant $C=C(\theta)>0$ independent of A and α_0, from (124)–(128),

$$\left(\int_{y>2^ky_{t_2}}\widetilde\varepsilon^2(t_2,y)dy\right)^{1/2}\le Ce^{-\frac{2^ky_{t_2}}{18}}+\frac{C}{\left(2^ky_{t_2}\widetilde\lambda(t_2)+\frac{A}{\widetilde\lambda(t_2)}\right)^3}.$$

In conclusion, for $C = C(\theta) > 0$, we have

$$
\begin{aligned}
I_2 &\le \sum_{k=0}^{+\infty} \sqrt{2^k y_{t_2}} \left(\int_{y > 2^k y_{t_2}} \widetilde{\varepsilon}^2(t_2, y) dy \right)^{1/2} \\
&\le C \sum_{k=0}^{+\infty} \sqrt{2^k y_{t_2}} e^{-\frac{2^k y_{t_2}}{18}} + C \sum_{k=0}^{+\infty} \frac{\sqrt{2^k y_{t_2}}}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^3}. \qquad (129)
\end{aligned}
$$

Step 4. Conclusion of the proof.

We conclude the proof by elementary calculations and an estimate of $\widetilde{\lambda}^2$ by $\int \widetilde{\varepsilon}^2 e^{-|y|}$. Indeed, recall from (42) that from the energy relation and $E_0 < 0$, we have

$$
\widetilde{\lambda}^2(t_2) \le \frac{C}{|E_0|} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-|y|} \right)^{1/2}. \qquad (130)
$$

Since there exists $c > 0$ such that $\forall x > 0$, $\sqrt{x} e^{-\frac{x}{18}} \le c e^{-\frac{x}{20}}$, we have

$$
\begin{aligned}
\sum_{k=0}^{+\infty} \sqrt{2^k y_{t_2}} e^{-\frac{2^k y_{t_2}}{18}} &\le C \sum_{k=0}^{+\infty} e^{-\frac{2^k y_{t_2}}{20}} \le C \sum_{n=1}^{+\infty} e^{-\frac{n y_{t_2}}{20}} = \frac{C e^{-\frac{y_{t_2}}{20}}}{1 - e^{-\frac{y_{t_2}}{20}}} \le C e^{-\frac{y_{t_2}}{20}} \\
&\le C \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{1/2}.
\end{aligned}
$$

Finally, by the inequality $b^2 a + \frac{1}{a} \ge 2b$, we have

$$
2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \ge 2\sqrt{A} \sqrt{2^k y_{t_2}}.
$$

Therefore, by applying the preceding inequality twice and by $y_{t_2} > 1$,

$$
\begin{aligned}
\frac{\sqrt{2^k y_{t_2}}}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^3} &\le \frac{1}{2\sqrt{A}} \frac{1}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^2} \\
&\le \left(\frac{1}{2\sqrt{A}} \right)^{5/4} \left(\frac{1}{2^k y_{t_2}} \right)^{1/8} \frac{1}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^{7/4}} \\
&\le \left(\frac{1}{2\sqrt{A}} \right)^{5/4} \left(\frac{1}{2^k} \right)^{1/8} \left(\frac{\widetilde{\lambda}(t_2)}{A} \right)^{7/4},
\end{aligned}
$$

and thus

$$
\sum_{k=0}^{+\infty} \frac{\sqrt{2^k y_{t_2}}}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^3} \le \frac{C}{A^{19/8}} \widetilde{\lambda}^{7/4}(t_2).
$$

Therefore, by (130),

$$
\sum_{k=0}^{+\infty} \frac{\sqrt{2^k y_{t_2}}}{\left(2^k y_{t_2} \widetilde{\lambda}(t_2) + \frac{A}{\widetilde{\lambda}(t_2)} \right)^3} \le \frac{C}{A^{19/8} |E_0|^{7/8}} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{7/16}.
$$

In conclusion

$$\begin{aligned} I_2 &\leq C_1 \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{1/2} + \frac{C_2}{A^{19/8}|E_0|^{7/8}} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{7/16} \\ &\leq \left(C_1 \delta_{II}^{1/8} + \frac{C_2 \delta_{II}^{1/16}}{A^{19/8}|E_0|^{7/8}} \right) \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{3/8}. \end{aligned}$$

By taking δ_{II} satisfying $\delta_{II}^{1/8} \leq \frac{1}{4C_1}$ and $\delta_{II}^{1/16} < \frac{A^{19/8}|E_0|^{7/8}}{4C_2}$, we obtain

$$I_2 \leq \frac{1}{2} \left(\int \widetilde{\varepsilon}^2(t_2) e^{-\frac{|y|}{100}} \right)^{3/8}.$$

Thus Proposition 2 is proved.

Appendix: Proof of Lemma 9

With the notation of Section 4, we want to prove the following estimates, for $|\lambda_1 - 1| + |x_1|$ small enough:

$$\begin{aligned} &\left| \int \varepsilon Q - \int \widetilde{\varepsilon} Q + (\lambda_1 - 1) \int \widetilde{\varepsilon} \left(\tfrac{Q}{2} + yQ_y \right) + \tfrac{1}{4}(\lambda_1 - 1)^2 \int y^2 Q^2 \right| \\ &\qquad \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \end{aligned} \tag{131}$$

$$\begin{aligned} &\left| \int \varepsilon \left(\tfrac{Q}{2} + yQ_y \right) - \int \widetilde{\varepsilon} \left(\tfrac{Q}{2} + yQ_y \right) - (\lambda_1 - 1) \int \left(\tfrac{Q}{2} + yQ_y \right)^2 \right| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \end{aligned} \tag{132}$$

$$\begin{aligned} &\left| \int \varepsilon L \left[\left(y \left(\tfrac{Q}{2} + yQ_y \right) \right)_y \right] - \int \widetilde{\varepsilon} L \left[\left(y \left(\tfrac{Q}{2} + yQ_y \right) \right)_y \right] \right. \\ &\qquad \left. -2(\lambda_1 - 1) \int \left(\tfrac{Q}{2} + yQ_y \right)^2 \right| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \end{aligned} \tag{133}$$

$$\begin{aligned} &\left| \int Q^3 \left(\tfrac{Q}{2} + yQ_y \right) \varepsilon^2 - 2(\lambda_1 - 1) \int Q^3 \left(\tfrac{Q}{2} + yQ_y \right)^2 \widetilde{\varepsilon} \right. \\ &\qquad \left. -|\lambda_1 - 1|^2 \int Q^3 \left(\tfrac{Q}{2} + yQ_y \right)^3 \right| \\ &\qquad \leq C(|x_1|^2 + |\lambda_1 - 1|^3 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \end{aligned} \tag{134}$$

$$\begin{aligned} &\left| \int Q^3 \left(y \left(\tfrac{Q}{2} + yQ_y \right) \right)_y \varepsilon^2 \right| + \left| \tfrac{\lambda_s}{\lambda} \int \varepsilon \left[y \left(y \left(\tfrac{Q}{2} + yQ_y \right) \right)_y \right] \right| \\ &\qquad + \left| \left(\tfrac{x_s}{\lambda} - 1 \right) \int \varepsilon \left(y \left(\tfrac{Q}{2} + yQ_y \right) \right)_y \right| \\ &\qquad \leq C(|\lambda_1 - 1|^2 + |x_1|^2 + \int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \end{aligned} \tag{135}$$

$$\left|\int G(\varepsilon)\left(\tfrac{Q}{2}+yQ_y\right)\right|+\left|\int G(\varepsilon)\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right|$$
$$\leq C(|x_1|^3+|\lambda_1-1|^3+\int \widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}). \tag{136}$$

Recall that

$$\varepsilon(y)=\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)+\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y). \tag{137}$$

Moreover, if for $\lambda_1>0$, $x_1\in\mathbf{R}$, we define

$$v_{\lambda_1,x_1}=\lambda_1^{1/2}v(\lambda_1 x+x_1),$$

then

$$\frac{\partial}{\partial\lambda_1}v_{\lambda_1,x_1}|_{\lambda_1=1,x_1=0}=\frac{v}{2}+xv_x,\quad \frac{\partial}{\partial x_1}v_{\lambda_1,x_1}|_{\lambda_1=1,x_1=0}=v_x, \tag{138}$$

$$\frac{\partial^2}{\partial\lambda_1^2}v_{\lambda_1,x_1}|_{\lambda_1=1,x_1=0}=-\frac{v}{4}+xv_x+x^2v_{xx},\quad \frac{\partial^2}{\partial x_1^2}v_{\lambda_1,x_1}|_{\lambda_1=1,x_1=0}=v_{xx}, \tag{139}$$

$$\frac{\partial^2}{\partial\lambda_1\partial x_1}v_{\lambda_1,x_1}|_{\lambda_1=1,x_1=0}=\frac{v_x}{2}+xv_{xx}. \tag{140}$$

In particular, by the decay properties of Q and its derivatives, we have

$$\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y)=(\lambda_1-1)\left(\frac{Q}{2}+yQ_y\right)+x_1Q_y$$
$$+\frac{(\lambda_1-1)^2}{2}\left(-\frac{Q}{4}+yQ_y+y^2Q_{yy}\right)+(\lambda_1-1)x_1\left(\frac{Q_y}{2}+yQ_{yy}\right)+\frac{x_1^2}{2}Q_{yy}$$
$$+O(|\lambda_1-1|^3)e^{-|y|/2}+O(|x_1|^3)e^{-|y|/2}. \tag{141}$$

Proof of (131). By (137), we have

$$\int\varepsilon Q-\int\widetilde{\varepsilon}Q=\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)Q-\int\widetilde{\varepsilon}Q+\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))Q(y).$$

First, by using (141) at the first order in (λ_1-1), x_1, we have

$$\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)Q-\int\widetilde{\varepsilon}Q=\int\widetilde{\varepsilon}(\lambda_1^{-1/2}Q(\lambda_1^{-1}(y-x_1))-Q(y))$$
$$=(\lambda_1^{-1}-1)\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)-\frac{x_1}{\lambda_1}\int\widetilde{\varepsilon}Q_y+\int\kappa_1(y)e^{-\frac{|y|}{2}}\widetilde{\varepsilon}(y),$$

where κ_1 is a function of y satisfying $\forall y\in\mathbf{R}$, $|\kappa_1(y)|\leq C(|\lambda_1-1|^2+|x_1|^2)$. We obtain

$$\left|\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)Q-\int\widetilde{\varepsilon}Q+(\lambda_1-1)\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)\right|$$
$$\leq C(|\lambda_1-1|^2+|x_1|)\left(\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{2}}\right)^{1/2}\leq C(|\lambda_1-1|^4+|x_1|^2+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}). \tag{142}$$

Second, by using (141) at the second order, we have

$$\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))Q(y)=(\lambda_1-1)\int\left(\tfrac{Q}{2}+yQ_y\right)Q+x_1\int Q_yQ$$
$$+\frac{(\lambda_1-1)^2}{2}\int\left(-\tfrac{Q}{4}+yQ+y^2Q_{yy}\right)Q+(\lambda_1-1)x_1\int\left(\tfrac{Q_y}{2}+yQ_{yy}\right)Q$$
$$+O(|x_1|^2)+O(|\lambda_1-1|^3).$$

We have $\int\left(\frac{Q}{2}+yQ_y\right)Q=\int Q_yQ=\int\left(\frac{Q_y}{2}+yQ_{yy}\right)Q=0$, and

$$\begin{aligned}\int\left(-\tfrac{Q}{4}+yQ+y^2Q_{yy}\right)Q &= -\frac{3}{4}\int Q^2-2\int yQ_yQ-\int y^2Q_y^2\\ &= \frac{1}{4}\int Q^2-\int y^2Q^2+\frac{1}{3}\int Q^6=-\frac{1}{2}\int y^2Q^2,\end{aligned}$$

using $\int y^2Q^6=\frac{3}{2}\int y^2Q^2-\frac{3}{4}\int Q^2$ (see the proof of Lemma 10). Therefore

(143)
$$\left|\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))Q(y)+\frac{(\lambda_1-1)^2}{4}\int y^2Q^2\right|\le C(|x_1|^2+|\lambda_1-1|^3).$$

Finally, (142) and (143) give (131).

Proof of (132). By (137), we have

$$\begin{aligned}\int\varepsilon\left(\tfrac{Q}{2}+yQ_y\right) &= \lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)\left(\tfrac{Q}{2}+yQ_y\right)\\ &\quad+\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\left(\tfrac{Q}{2}+yQ_y\right).\end{aligned}$$

First, we have

$$\left|\lambda_1^{1/2}\int\widetilde{\varepsilon}(\lambda_1 y+x_1)\left(\tfrac{Q}{2}+yQ_y\right)-\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)\right|$$
$$=\left|\int\widetilde{\varepsilon}\left(\lambda_1^{-1/2}\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{-1}(y-x_1))\right)-\int\widetilde{\varepsilon}\left(\tfrac{Q}{2}+yQ_y\right)\right|$$
$$=\left|\int\widetilde{\varepsilon}\left(\lambda_1^{-1/2}\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{-1}(y-x_1))-\left(\tfrac{Q}{2}+yQ_y\right)(y)\right)\right|$$

(144)
$$\le C(|\lambda_1-1|+|x_1|)\int e^{-\frac{|y|}{2}}\widetilde{\varepsilon}(y)\le C(|\lambda_1-1|^2+|x_1|^2+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{2}}).$$

Second, by using (141) at the first order, we have

$$\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\left(\tfrac{Q}{2}+yQ_y\right)$$
$$=(\lambda_1-1)\int\left(\tfrac{Q}{2}+yQ_y\right)^2+x_1\int Q_y\left(\tfrac{Q}{2}+yQ_y\right)$$
$$+O(|x_1|^2)+O(|\lambda_1-1|^2).$$

Since $\int Q_y\left(\frac{Q}{2}+yQ_y\right)=0$, we obtain

$$\left|\int(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\left(\tfrac{Q}{2}+yQ_y\right)-(\lambda_1-1)\int\left(\tfrac{Q}{2}+yQ_y\right)^2\right|$$

(145)
$$\le C(|x_1|^2+|\lambda_1-1|^2).$$

Finally, (144) and (145) give (132).

Proof of (133). By (137), we have

$$\int \varepsilon L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] = \lambda_1^{1/2}\int \widetilde{\varepsilon}(\lambda_1 y + x_1) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]$$
$$+\int (\lambda_1^{1/2} Q(\lambda_1 y + x_1) - Q(y)) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right].$$

First, we have

$$\begin{aligned}
&\lambda_1^{1/2}\int \widetilde{\varepsilon}(\lambda_1 y + x_1) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]\\
&\quad= \int \widetilde{\varepsilon}\left(\lambda_1^{-1/2} L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right](\lambda_1^{-1}(y-x_1))\right)\\
&\quad= \int \widetilde{\varepsilon} L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] + \int \kappa_4(y) e^{-\frac{|y|}{2}}\widetilde{\varepsilon}(y),
\end{aligned}$$

where κ_4 is a function of y satisfying $\forall y \in \mathbf{R}$, $|\kappa_4(y)| \le C(|\lambda_1 - 1| + |x_1|)$. Second, by using (141) at the first order, we have

$$\begin{aligned}
&\int (\lambda_1^{1/2} Q(\lambda_1 y + x_1) - Q(y)) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]\\
&\quad= (\lambda_1 - 1)\int \left(\frac{Q}{2}+yQ_y\right) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right]\\
&\qquad+ x_1 \int Q_y L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] + O(|x_1|^2) + O(|\lambda_1 - 1|^2).
\end{aligned}$$

Since

$$\begin{aligned}
&\int \left(\frac{Q}{2}+yQ_y\right) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] = -2\int Q\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\\
&\quad= 2\int Q_y\left(y\left(\frac{Q}{2}+yQ_y\right)\right) = 2\int\left(\frac{Q}{2}+yQ_y\right)^2
\end{aligned}$$

and $\int Q_y L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] = 0$, we obtain

$$\begin{aligned}
&\left|\int (\lambda_1^{1/2} Q(\lambda_1 y + x_1) - Q(y)) L\left[\left(y\left(\frac{Q}{2}+yQ_y\right)\right)_y\right] - 2(\lambda_1 - 1)\int\left(\frac{Q}{2}+yQ_y\right)^2\right|\\
&\quad\le C(|x_1|^2 + |\lambda_1 - 1|^2).
\end{aligned}$$

Thus (133) is proved.

Proof of (134). By (137), we have

$$\begin{aligned}
&\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\varepsilon^2\\
&=\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)[\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)+\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y)]^2\\
&=\lambda_1\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\widetilde{\varepsilon}^2(\lambda_1 y+x_1)\\
&\quad+2\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)\\
&\quad+\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))^2.
\end{aligned}$$

First,

$$\begin{aligned}
\left|\lambda_1\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\widetilde{\varepsilon}^2(\lambda_1 y+x_1)\right| &= \left|\int\left(Q^3\left(\tfrac{Q}{2}+yQ_y\right)\right)(\lambda_1^{-1}(y-x_1))\widetilde{\varepsilon}^2(y)\right|\\
&\leq C\int\widetilde{\varepsilon}^2 e^{-|y|}.
\end{aligned}$$

Second,

$$\begin{aligned}
&\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)\\
&\;=\int\left(Q^3\left(\tfrac{Q}{2}+yQ_y\right)\right)(\lambda_1^{-1}(y-x_1))(Q(y)-\lambda_1^{-1/2}Q(\lambda_1^{-1}(y-x_1)))\widetilde{\varepsilon}(y)\\
&\;=(\lambda_1-1)\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^2\widetilde{\varepsilon}+x_1\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)Q_y\widetilde{\varepsilon}+\int\kappa_3(y)e^{-\frac{|y|}{2}}\widetilde{\varepsilon}(y),
\end{aligned}$$

where $\forall y\in\mathbf{R}$, $|\kappa_3(y)|\leq C(|x_1|^2+|\lambda_1-1|^2)$. Therefore,

$$\begin{aligned}
&\Bigg|\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))\lambda_1^{1/2}\widetilde{\varepsilon}(\lambda_1 y+x_1)\\
&\qquad-(\lambda_1-1)\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^2\widetilde{\varepsilon}\Bigg|\\
&\leq C\left(|x_1|^2+|\lambda_1-1|^2\left(\int\widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}\right)^{1/2}+\int\widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}\right)\\
&\leq C(|x_1|^2+|\lambda_1-1|^4+\int\widetilde{\varepsilon}^2 e^{-\frac{|y|}{4}}).
\end{aligned}$$

Third,

$$\begin{aligned}
&\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))^2\\
&\;=\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\left((\lambda_1-1)\left(\tfrac{Q}{2}+yQ_y\right)+x_1Q_y\right)^2+O(|\lambda_1-1|^3)+O(|x_1|^3),
\end{aligned}$$

so that, since $\int Q^3\left(\frac{Q}{2}+yQ_y\right)^2Q_y=0$, we have

$$\begin{aligned}
&\left|\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)(\lambda_1^{1/2}Q(\lambda_1 y+x_1)-Q(y))^2-(\lambda_1-1)^2\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^3\right|\\
&\quad\leq C(|x_1|^2+|\lambda_1-1|^3).
\end{aligned}$$

Therefore,

$$\left|\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)\varepsilon^2-2(\lambda_1-1)\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^2\widetilde{\varepsilon}-(\lambda_1-1)^2\int Q^3\left(\tfrac{Q}{2}+yQ_y\right)^3\right|$$
$$\leq C(|x_1|^2+|\lambda_1-1|^3+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{2}}),$$

and so (134) is proved.

Proof of (135). First, recall that from (65), we have

$$\left|\tfrac{\lambda_s}{\lambda}\right|+\left|\left(\tfrac{x_s}{\lambda}-1\right)\right|\leq C\left(\int\varepsilon^2e^{-\frac{|y|}{2}}\right)^{1/2}.$$

Therefore, by the decay property of Q and its derivatives, we have

$$\left|\int Q^3\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\varepsilon^2\right|+\left|\tfrac{\lambda_s}{\lambda}\int\varepsilon\left[y\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right]\right|$$
$$+\left|\left(\tfrac{x_s}{\lambda}-1\right)\int\varepsilon\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right|\leq C\int\varepsilon^2e^{-\frac{|y|}{2}}.$$

By (137) and (141), it is clear that

$$\int\varepsilon^2e^{-\frac{|y|}{2}}\leq C(|\lambda_1-1|^2+|x_1|^2+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}),$$

so that we obtain (135).

Proof of (136). Recall that $|G(\varepsilon)|\leq C(|\varepsilon|^3+|\varepsilon|^5)$. We can assume $|\varepsilon|_{L^\infty}\leq 1$, by the Gagliardo–Nirenberg inequality, so that $|G(\varepsilon)|\leq C|\varepsilon|^3$. Therefore,

$$\left|\int G(\varepsilon)\left(\tfrac{Q}{2}+yQ_y\right)\right|+\left|\int G(\varepsilon)\left(y\left(\tfrac{Q}{2}+yQ_y\right)\right)_y\right|$$
$$\leq C\int|\varepsilon|^3e^{-\frac{|y|}{2}}\leq C(|x_1|^3+|\lambda_1-1|^3+\int\widetilde{\varepsilon}^2e^{-\frac{|y|}{4}}).$$

Thus (136) is proved.

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Département de Mathématiques, Université de Cergy–Pontoise, 2, av. A. Chauvin, 95302 Cergy Pontoise, France

E-mail address: `Yvan.Martel@math.u-cergy.fr`

Département de Mathématiques, Université de Cergy–Pontoise, 2, av. A. Chauvin, 95302 Cergy Pontoise, France – and – Institut Universitaire de France

E-mail address: `Frank.Merle@math.u-cergy.fr`