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THE GRAFTING MAP OF TEICHMÜLLER SPACE

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§1. Introduction

One of the underlying principles in the study of Kleinian groups is that aspects of the complex projective geometry of quotients of $\hat{\mathbb{C}}$ by these groups reflect properties of the three-dimensional hyperbolic geometry of the quotients of $\mathbb{H}^3$ by these groups. Yet, even though it has been over thirty-five years since Lipman Bers exhibited a holomorphic embedding of the Teichmüller space of Riemann surfaces in terms of the projective geometry of a Teichmüller space of quasi-Fuchsian manifolds, no corresponding embedding in terms of the three-dimensional hyperbolic geometry has been presented. One of the goals of this paper is to give such an embedding. This embedding is straightforward and has been expected for some time ([Ta97], [Mc98]): to each member of a Bers slice of the space QF of quasi-Fuchsian 3-manifolds, we associate the bending measured lamination of the convex hull facing the fixed "conformal" end.

The geometric relationship between a boundary component of a convex hull and the projective surface at infinity for its end is given by a process known as *grafting*, an operation on projective structures on surfaces that traces its roots back at least to Klein [Kl33, p. 230, §50], with a modern history developed by many authors ([Ma69], [He75], [Fa83], [ST83], [Go87], [GKM00], [Ta97], [Mc98]). The main technical tool in our proof that bending measures give coordinates for Bers slices, and the second major goal of this paper, is the completion of the proof of the "Grafting Conjecture". This conjecture states that for a fixed measured lamination λ, the self-map of Teichmüller space induced by grafting a surface along λ is a homeomorphism of Teichmüller space; our contribution to this argument is a proof of the injectivity of the grafting map. While the principal application of this result that we give is to geometric coordinates on the Bers slice of QF, one expects that the grafting homeomorphism might lead to other systems of geometric coordinates for other families of Kleinian groups (see §5.2); thus we feel that this result is of interest in its own right.

We now state our results and methods more precisely. Throughout, S will denote a fixed differentiable surface which is closed, orientable, and of genus $g \geq 2$. Let $\mathcal{T}_g$ be the corresponding Teichmüller space of marked conformal structures on S, and let $\mathcal{P}_g$ denote the deformation space of (complex) projective structures on S (see §2 for definitions).

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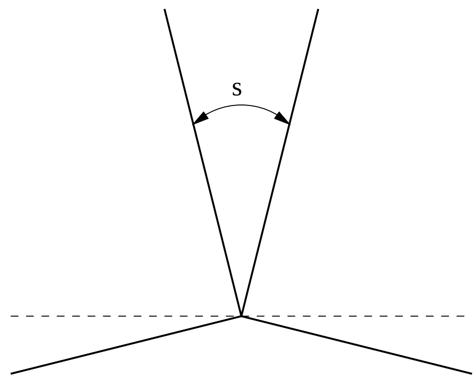

FIGURE 1. The region $\tilde{A}_s$.

There are two well-known parametrizations of $\mathcal{P}_g$, each reflecting a different aspect of the general theory. The first uses the Schwarzian derivative of the developing map to obtain a quadratic differential on S, holomorphic with respect to the complex structure underlying the given projective structure. This identifies $\mathcal{P}_g$ with the total space of the bundle $\mathcal{Q}_g \to \mathcal{T}_g$ of holomorphic quadratic differentials over Teichmüller space. This identification is representative of the complex analytic side of the theory; see for instance [Ea81], [Gu81], [He75], [Kr69], [Kr71], [KM81], [MV94], [Sh87], [ST99].

The second parametrization is due to Thurston and is more geometric in nature. To describe it, fix a hyperbolic metric $\sigma \in \mathcal{T}_g$, a simple closed geodesic $\gamma \subset S$ of length ℓ, and a positive real number s. Let

$$\tilde{A}_s = \left\{ (r, \theta) \in \mathbb{C}^* : |\theta - \pi/2| \le \frac{s}{2} \right\}$$

(see Figure 1) and

$$A_s = \tilde{A}_s / \langle z \mapsto e^{\ell} z \rangle.$$

Of course, if $s \ge 2\pi$, we must interpret the projective structure on $\tilde{A}_s$ as being defined by a developing map which is no longer an embedding; in any case, we call A_s a (projective) *s-annulus*. A new projective structure on S is defined by cutting the original hyperbolic surface (S, σ) open along the simple closed curve γ and gluing in A_s (see Figure 2).

This is the *grafting* operation; it provided the first examples [Ma69] of projective structures for which the developing map is not a covering of its image. Grafting extends by continuity from pairs (γ, s) to general measured laminations, defining a map $\Theta : \mathcal{ML} \times \mathcal{T}_g \to \mathcal{P}_g$. Thurston has shown (in unpublished work) that Θ is a homeomorphism (see [KT92], [La92]).

A natural problem is to understand how these geometric and complex analytic aspects interact. For instance, a measured lamination $\lambda \in \mathcal{ML}$ defines a slice $\Theta(\{\lambda\} \times \mathcal{T}_g) \subset \mathcal{P}_g$; following this inclusion with the projection $\mathcal{P}_g \to \mathcal{T}_g$ defines a self-map of Teichmüller space $\mathrm{Gr}_\lambda : \mathcal{T}_g \to \mathcal{T}_g$. Our main result can be stated concisely as follows:

Theorem A. Gr_λ *is a homeomorphism.*

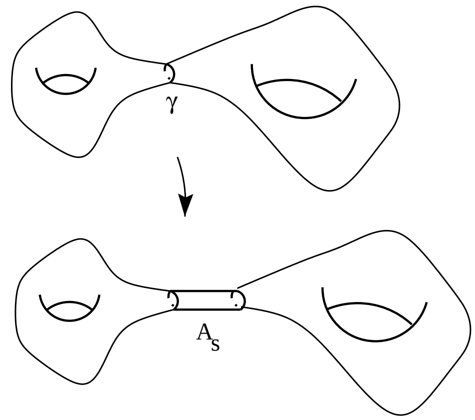

FIGURE 2. The cylinder A_s is glued into (S, σ) along the geodesic γ.

This result was obtained in special cases by McMullen [Mc98] (one-dimensional Teichmüller spaces) and Tanigawa [Ta97] (for integral points of $\mathcal{ML}$, using a result of Faltings [Fa83]). Our result holds for all elements of $\mathcal{ML}$ and all Teichmüller spaces of finitely punctured closed Riemann surfaces of finite genus. For the sake of expositional ease, we write the proof for Teichmüller spaces of closed Riemann surfaces, but the extension to Teichmüller spaces of finitely punctured surfaces is mostly a matter of additional notation; see the remark at the end of §4.

Theorem A allows one to understand various complex analytic constructions in the theory of Teichmüller spaces and Kleinian groups in terms of measured geodesic laminations and the grafting construction. As an example, we obtain the following corollary in §5.1:

Corollary. *Let B_Y be a Bers slice with fixed conformal structure Y, and define a map $\beta : B_Y \to \mathcal{ML}$ which assigns the bending lamination on the component of the convex hull boundary facing Y. Then β is a homeomorphism onto its image.*

The space of projective structures is intimately related with the space of locally convex pleated maps of $\tilde{S}$ into $\mathbb{H}^3$ (as detailed for instance in [EM87]). The dual notions are explored in [Sc99], where it is shown that $\mathcal{P}_g$ classifies causally trivial de Sitter structures on $S \times \mathbb{R}$; here the grafting operation corresponds to a "stretching" of the causal horizon. We give an application of Theorem A to this situation in §5.3.

Finally, in [Mc98] McMullen observes that Theorem A follows from the conjectural rigidity of hyperbolic cone 3-manifolds (see [HK98]); hence our result can be viewed as further positive evidence for the validity of this conjecture.

We prove the theorem by employing standard methods from geometric analysis. There are two cases to consider: a basic case where λ is supported on a simple closed curve, and the general case where λ is an arbitrary measured lamination. Most of the ideas of the proof are contained in the basic case, while the general case is obtained by approximating a general lamination by simple closed curves, and then slightly generalizing the argument for the basic case before taking a limit.

The argument for the basic case of λ supported on a simple closed curve begins by noting that, in view of the work of Tanigawa [Ta97], we need to prove that $d\mathrm{Gr}_\lambda : T_\sigma \mathcal{T}_g \to T_{\mathrm{Gr}_\lambda \sigma} \mathcal{T}_g$ is an isomorphism. To show this, we consider a map from $\mathrm{Gr}_\lambda\sigma_\epsilon$ to $\mathrm{Gr}_\lambda\sigma = \mathrm{Gr}_\lambda\sigma_0$, which is conformal up to a small error of $o(\epsilon)$. (In the general case, we take this map to be possibly slightly non-conformal, of order $\mathcal{O}(\epsilon)$.) We aim to show that this map is an isometry (to order $o(\epsilon)$) of the so-called *grafted metrics*; this metric is Euclidean on the inserted cylinder and hyperbolic elsewhere (see §2.2). A consequence of the isometry is that the grafted cylinder may be excised, proving that σ_ϵ is isometric to σ_0 to order $o(\epsilon)$; thus $d\mathrm{Gr}_\lambda$ has no kernel, proving the result.

The major issues are that a priori the map from $\mathrm{Gr}_\lambda\sigma_\epsilon$ to $\mathrm{Gr}_\lambda\sigma = \mathrm{Gr}_\lambda\sigma_0$ may not preserve the regions where the metric is either hyperbolic or Euclidean, and it is difficult to specify conformally the Euclidean length of the Euclidean cylinder. Still, the Liouville equation and its infinitesimal form (equation (3.1.5)) for prescribed curvature have particularly simple forms, even in this case of the $C^{1,1}$ grafted metric, and we find that the Jacobi equation (2.4.5) governing the infinitesimal variations of the geodesics separating the hyperbolic and Euclidean regions involves precisely the same quantity, i.e. the infinitesimal ratio of the $t = \epsilon$ grafted metric to the $t = 0$ grafted metric. By playing these two sets of equations off against one another, and finding a certain constant of integration from the hypothesis that the Euclidean cylinders are of constant length, we deduce that this infinitesimal ratio of metrics is identically unity, to order $o(\epsilon)$, which is sufficient for our result.

Regularity issues play an important, but technical, role in the argument. They are discussed in §2. The heart of the argument, relating (in the case of the lamination being a simple closed curve) the infinitesimal Liouville equation (3.1.5) and the infinitesimal Jacobi equation (2.4.5), is confined to the reasonably short sections 3.2 and 3.3; those sections are nearly self-contained, and could be profitably read independently of the rest of the paper. The fourth section extends the analysis to general laminations, and the fifth section contains some applications of Theorem A.

§2. Notation and background

2.1. Teichmüller space, Bers embedding. Let S denote a smooth surface of genus $g \geq 2$, and let $\mathcal{M}_{-1} = \mathcal{M}_{-1}(S)$ denote the space of metrics $\rho|dw|^2$ on S with Gaussian curvature identically -1. The group Diff_o of diffeomorphisms of S homotopic to the identity acts on $\mathcal{M}_{-1}$ by pullback: if $\phi \in \mathrm{Diff}_o$, then $\phi \cdot \rho = \phi^*\rho$. We define the Teichmüller space of genus g, $\mathcal{T}_g$, to be the quotient space $\mathcal{T}_g = \mathcal{M}_{-1}/\mathrm{Diff}_o$, i.e. equivalence classes of metrics in $\mathcal{M}_{-1}$ under the action of Diff_o. A metric (S, ρ) represents a conformal class of metrics on S, hence a Teichmüller equivalence class of Riemann surfaces. Let $QD(\sigma)$ denote the $(3g-3)$-dimensional complex vector space of holomorphic quadratic differentials on (S, ρ).

There are a number of continuous and real analytic parametrizations of the Teichmüller space $\mathcal{T}_g$ and a complex analytic parametrization given by Lipman Bers [Be66]. The *Bers embedding*, as it is usually known (see [Na88] for a comprehensive account), is defined as follows. Fix a point Y in $\mathcal{T}_g$. Then, for any (variable) point $X \in \mathcal{T}_g$, there is a quasi-Fuchsian manifold $Q(X, Y)$ with conformal boundaries X and Y and fundamental group $\Gamma(X, Y)$. In addition, there is a simultaneous uniformization homeomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the sphere $\hat{\mathbb{C}}$ which does the following: 1) it conjugates $\Gamma(\bar{Y}, Y)$ to $\Gamma(X, Y)$, 2) it equivariantly and conformally maps the unit disk Δ to the universal cover of Y, and 3) it equivariantly and quasi-conformally maps the complement Δ^* of the unit disk to the universal cover of X. As $F\big|_\Delta$ is conformal, we may take its Schwarzian derivative, say $\mathcal{S}(F\big|_\Delta) = \Psi_X$.

The holomorphic function Ψ_X descends to a holomorphic quadratic differential on the Riemann surface Y: the correspondence $X \in \mathcal{T}_g \mapsto \Psi_X \in QD(Y)$ is the Bers embedding $B_Y : \mathcal{T}_g \to QD(Y)$. As the name suggests, it is an embedding [Be66] of the $(3g-3)$-dimensional Teichmüller space $\mathcal{T}_g$ into the $(3g-3)$-dimensional complex vector space $QD(Y)$; the point Y maps to the origin, and it follows from results of Nehari [Ne49] that the image, with respect to the appropriate norm, is contained in a ball of radius 6 and contains a ball of radius 2.

Within the space QF of quasi-Fuchsian manifolds, the family $\{Q(X, Y)|X \in \mathcal{T}_g\}$ is known as the *Bers slice of QF based at Y.*

2.2. Grafting, Thurston metric. Recall that a *(complex) projective structure* on S is a maximal atlas of charts from S into $\mathbb{C}P^1$ such that all transition maps are restrictions of elements of $PSL(2, \mathbb{C})$. Such a structure yields in the usual way a holonomy representation $hol : \pi_1(S) \to PSL(2, \mathbb{C})$ and an equivariant developing map $dev : \tilde{S} \to \mathbb{C}P^1$. We will write $\mathcal{P}_g$ for the moduli space of projective structures on S (as defined and topologized, for instance, in [Go88]).

Let $\mathcal{S}$ denote the set of isotopy classes of essential simple closed curves on S. There is a well-defined intersection pairing $i : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}$ given by the minimum number of intersection points among pairs of representative curves in the isotopy classes. This in turn defines an embedding of $\mathbb{R}_+ \times \mathcal{S}$ into $\mathbb{R}^{\mathcal{S}}$ by sending a weighted simple closed curve (s, γ) to the $\mathcal{S}$-tuple $(s \cdot i(\gamma, \alpha))_{\alpha \in \mathcal{S}}$. The space of measured laminations $\mathcal{ML}$ is defined to be the closure of $\mathbb{R}_+ \times \mathcal{S}$ in $\mathbb{R}^{\mathcal{S}}$. For simplicity, a measured lamination coming from a pair (s, γ) will be denoted $s\gamma$.

In the presence of a hyperbolic structure on S, it is typical to define measured laminations in terms of geodesic laminations equipped with a measure on transverse arcs (see [Th82] or [Bo88] for more details). We can also use a hyperbolic structure on S to define a notion of the length $L(\lambda)$ of a measured lamination λ: one defines $L(s\gamma)$ to be the product of s and the hyperbolic length of γ on S, and then extends $L : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ to all of $\mathcal{ML}$ by continuity (see e.g. [Ke85]).

In §1, grafting was defined in terms of a map $\Theta : \mathcal{ML} \times \mathcal{T}_g \to \mathcal{P}_g$; for laminations in the subset $\mathbb{R}_+ \times \mathcal{S}$ of weighted simple closed curves the projective structure $\Theta(s\gamma, \sigma)$ was defined by gluing together the Fuchsian projective structure associated to σ and a projective s-annulus along γ. The proof that Θ extends continuously to all of $\mathcal{ML} \times \mathcal{T}_g$ can be found in [KT92].

In order to understand the surjectivity of Θ, let us briefly recall the *canonical stratification* associated to a projective structure (originally due to Thurston – see also [KP94], [Ap88], [Sc99], [KT92]). First note that, via the developing map dev, $\tilde{S}$ inherits a notion of *open round ball* from $\mathbb{C}P^1$. Furthermore, also using dev, we can pull back the spherical metric on $\mathbb{C}P^1$ to an (incomplete) metric on $\tilde{S}$ – the metric completion depends only on the projective structure and is called the *Möbius completion* of $\tilde{S}$ [KP94]. The closure of an open round ball in the Möbius completion is conformally equivalent to the compactified hyperbolic space $\mathbb{H}^2 \cup S^1_\infty$, so the usual notion of "hyperbolic convex hull" transfers. Thus, given an open round ball U, we write $C(U)$ for the intersection of U and the convex hull of $\bar{U} \setminus U$ in $\bar{U}$. The key observation is the following:

Lemma 2.2.1 ([KP94]). *For every $p \in \tilde{S}$, there is a unique open round ball U_p such that $p \in C(U_p)$.*

The sets U_p given by the lemma are called *maximal balls*, and define a stratification of $\tilde{S}$ into the sets $C(U_p)$ (this descends in turn to a stratification of S). It is easy to verify that in the case of a projective structure obtained by grafting along $\lambda \in \mathcal{ML}$, this stratification is basically the same as the one given by the leaves and complementary regions of λ.

We also obtain a canonical Riemannian metric defined to be the restriction to $C(U_p)$ of the hyperbolic metric on the open round ball U_p [KP94]. We call this metric the *grafted metric* or the *Thurston metric*; if the projective structure is obtained by grafting the hyperbolic surface σ along the measured lamination λ (i.e. $\Theta(\lambda, \sigma)$) we write this metric as $\mathrm{gr}_\lambda(\sigma)$. Chasing through the definitions in the case of grafting along a weighted simple closed curve $s\gamma$, one sees that $\mathrm{gr}_\lambda(\sigma)$ coincides with σ on $S \setminus \gamma$ and is flat on the inserted annulus.

2.3. Harmonic maps from surfaces. In this section we develop the facts we need about families of harmonic maps between the mildly singular grafted surfaces. Our goal is a regularity result on the infinitesimal variation of the holomorphic energy density, which, while technical, is quite important in simplifying our approach.

2.3.1. *Theory for fixed domain and range.* Let $(M, \sigma|dz|^2)$ and $(N, \rho|dw|^2)$ denote M and N equipped with Riemannian structures; z refers to a local conformal coordinate on the surface M, and w refers to a local conformal coordinate on the surface N (all Riemannian metrics will be of class $C^{1,1}$ in this paper, although the definitions of this section and basic existence and uniqueness results hold in much greater generality; see e.g. [GS92]). For a Lipschitz map $w : (M, \sigma|dz|^2) \to (N, \rho|dw|^2)$, the differential dw is well defined almost everywhere on M, and thus we may define (compare [EF01, Chapter 9]) the *energy density* $e(w; \sigma, \rho)$ of w at almost every point z by

$$e(w; \sigma, \rho) = e(w) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2 + \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$$

and the *energy* $E(w; \sigma, \rho)$ by

$$\begin{aligned} E(w; \sigma, \rho) &= \int_M e(w; \sigma, \rho)\sigma dz d\bar{z} \\ &= \int_M \rho(w(z))|w_z|^2 + \rho(w(z))|w_{\bar{z}}|^2 dz d\bar{z}. \end{aligned}$$

Evidently, while the total energy depends upon the metric structure of the target surface (N, ρ), it only depends upon the conformal structure of the source (M, σ).

A critical point of this functional is called a *harmonic map.* We will be interested in the situation where $M = N = S$, a fixed surface of finite analytic type, with a fixed homotopy class $w_0 : S \to S$ of maps, and where the target (S, ρ) (though possibly only $C^{1,1}$) is non-positively curved in the sense of Alexandrov. In that case, (see [GS92, Theorem 2.3, Lemma 1.1]) there is a unique (if $w_*(\pi_1 M)$ is non-abelian) harmonic map $w(\sigma) : (S, \sigma) \to (S, \rho)$ in the homotopy class of w_0.

In the next section, we will specialize to a case where we will find additional smoothness for w. To that end, we record the (Euler-Lagrange) equation satisfied by a harmonic map $w(z) : (S, \sigma) \to (S, \rho)$:

$$w_{z\bar{z}} + \frac{\rho(w)_w}{\rho} w_z w_{\bar{z}} = 0. \tag{2.3.1}$$

A priori, this equation only holds in the weak sense, i.e. for every Lipschitz test function $\psi : S \to S$, we have

$$\int_S \left(w_z \bar{\psi}_{\bar{z}} - \frac{\rho(w)_w}{\rho} w_z w_{\bar{z}} \bar{\psi} \right) dz d\bar{z} = 0;$$

standard regularity and bootstrapping arguments (Lemma 2.3.2 below) applied to these equations shows that w is actually of class $C^{2,\alpha}$. As is immediate from equation (2.3.1), any conformal map from (S, σ) to (S, ρ) is automatically harmonic. Indeed, this is the case we will require for our proof in §3 of the basic case where the measured lamination λ is supported on a simple closed curve γ. The general situation of a non-conformal harmonic map will arise only in §4 where we will consider harmonic quasi-conformal approximates to conformal maps.

For harmonic maps $w : (\mathcal{R}, \sigma) \to (N, \rho)$ from a Riemann surface $\mathcal{R}$ to a smooth target, one can characterize the harmonicity of w in terms of conformal objects on $\mathcal{R}$. The pullback metric $w^*\rho$ decomposes by type as

$$\begin{aligned} w^*\rho &= \langle w_*\partial_z, w_*\partial_z \rangle_\rho dz^2 + (\|w_*\partial_z\|_\rho^2 + \|w_*\partial_{\bar{z}}\|_\rho^2)\sigma dz d\bar{z} + \langle w_*\partial_{\bar{z}}, w_*\partial_{\bar{z}} \rangle_\rho d\bar{z}^2 \\ &= \varphi dz^2 + \sigma e(w) dz d\bar{z} + \bar{\varphi} d\bar{z}^2. \end{aligned}$$

It is easy to show (see [Sa78]) that if w is harmonic, then $\Phi = \varphi dz^2$ is a holomorphic quadratic differential on $\mathcal{R}$, called the *Hopf differential* of w. In particular, Schoen [Sc84] has emphasized that even for harmonic maps to singular metric spaces (S, ρ), it is a consequence of Weyl's lemma that the Hopf differential

$$\Phi = \varphi dz^2 = \langle w_*\partial_z, w_*\partial_z \rangle_\rho dz^2 = \rho(w) w_z \overline{w_{\bar{z}}} dz^2$$

is holomorphic.

The expression $\mathcal{H} = \|w_*\partial_z\|_\rho^2$ plays a special role in harmonic maps between surfaces (see, for instance, [Wo91a]). First, we can rewrite the pullback metric $w^*\rho$ entirely in terms of $\Phi = \varphi dz^2$ and $\mathcal{H}$ as follows:

$$w^*\rho = \varphi dz^2 + \left(\mathcal{H} + \frac{|\Phi|^2}{\sigma \mathcal{H}} \right) dz d\bar{z} + \bar{\varphi} d\bar{z}^2.$$

Moreover, the function $\mathcal{H} = \mathcal{H}(z)$ satisfies the Bochner equation (this is basically a Liouville equation for prescribed curvature, using the harmonic map gauge)

$$\Delta_\sigma \log \mathcal{H}(z) = -2K_\rho(w(z)) \left\{ \mathcal{H}(z) - \frac{|\Phi(z)|^2}{\sigma(z)\mathcal{H}(z)} \right\} + 2K_\sigma(z). \tag{2.3.2}$$

Here K_ρ and K_σ refer to the Gauss curvatures of (S, σ) and (S, ρ), respectively, and we are stating the equation only in the context of smooth maps; we will later extend the meaning of this equation to the singular context which is our principal interest in this paper.

2.3.2. *Smoothness of harmonic maps families.* We will be interested in harmonic maps between surfaces equipped with the grafted (Thurston) metrics; in particular, we will carefully study one-parameter families of such maps. This study relies on the background result that these maps are reasonably smooth, and that the family of maps is reasonably smooth in the family parameter, for a smooth family of grafted metrics. In this section, we establish these basic smoothness results: the proofs are straightforward generalizations of those found in the literature (see [Jo97], [EL81], [Sa78]), but as the precise versions we need do not seem to be present already in print, we include them here for the sake of completeness. Also in this section, we

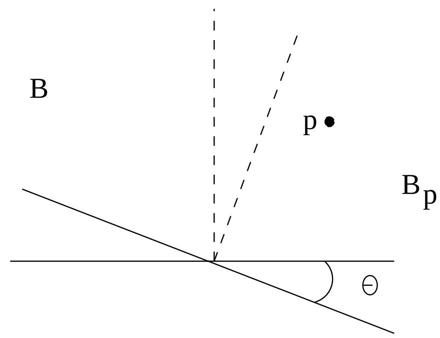

FIGURE 3.

state and prove the main technical regularity device (Lemma 2.3.5) we require for this paper. This lemma does not seem to be a consequence of general regularity theory, but seems instead to require more of the structure of this particular situation of a harmonic map between grafted surfaces.

We begin with the regularity of the Thurston metrics:

Lemma 2.3.1. *For any $\lambda \in \mathcal{ML}$, the grafted metric $\mathrm{gr}_\lambda(\sigma)$ is of class $C^{1,1}$. If λ_n is a sequence of laminations supported on simple closed curves and $\lambda_n \to \lambda$, then the $C^{1,1}$ norms $\|\mathrm{gr}_{\lambda_n}(\sigma)\|$ are uniformly bounded.*

Proof. The first statement is in [KP94, (5.4)]; we will sketch the proof following their notation. Note that it suffices to work locally on the surface; we are assuming in the second statement a uniform bound on the size of the grafting cylinders which implies a uniform upper bound on the diameter of the grafted surface. Thus a bound on the Lipschitz constants of the derivative of the metric over a neighborhood of uniform size on the surface will imply the desired global bound on the $C^{1,1}$ norm.

The idea is to fix a stratum of the canonical stratification and compute the variation of the conformal factor as one moves through nearby strata; here, by the comments in the previous paragraph, we can take "nearby" to mean strata whose maximal balls meet the initial one in an acute angle. Let B be a maximal open round ball in $\tilde{S}$, let p be a point of B, and let B_p be the maximal ball which defines the stratum containing p. Since $p \in B$, the angle θ between B and B_p must be acute [KP94, (4.7)]; we identify these balls with their images under a developing map as in Figure 3; here the ball B maps to the upper half plane.

Note that we are making no assumption on the lamination in this part of the argument; there may be a (unpictured) Cantor set's worth of bending lines between the strata defined by B and B_p; only the total bending measure θ between these strata will matter in estimating the desired conformal factor.

Suppose p has polar coordinates (r, β); then the hyperbolic metric g_B on B at p is given by $\frac{ds^2}{r^2 \sin^2 \beta}$ while the hyperbolic metric (and by definition grafted metric) on B_p is $\frac{ds^2}{r^2 \sin^2(\theta+\beta)}$. Thus $\mathrm{gr}(\sigma)(p) = \rho g_B$ where $\rho = \frac{\sin^2 \beta}{\sin^2(\theta+\beta)}$. This computation is valid only for $\beta < \frac{\pi}{2} - \theta$. However, the obvious inequalities

$$\sin \beta \leq \sin(\theta + \beta) \leq 1$$

(for $\beta < \frac{\pi}{2} - \theta$) give bounds on ρ which are independent of θ:

$$\sin^2 \beta \leq \rho \leq 1.$$

Now let $x(p)$ be the distance from p to the stratum defined by B (the second quadrant in our figure). Then hyperbolic trigonometry gives $\cosh(x(p)) = \frac{1}{\sin \beta}$, and we can rewrite our conformal factor, solely in terms of the coordinate $x(p)$, in the form

$$\rho(x) = \frac{1}{(\cos\theta + \sin\theta \sinh x)^2}.$$

Since $\beta < \frac{\pi}{2} - \theta$, this is valid for $\cosh(x(p)) > \frac{1}{\sin(\frac{\pi}{2}-\theta)} = \sec(\theta)$. In terms of the coordinate x we have, from the bounds on ρ given above, for all $p \in B$,

$$\frac{g_B}{\cosh^2 x(p)} \leq \mathrm{gr}(\sigma)(p) \leq g_B.$$

The differentiability of ρ (and therefore of the metric) follows because $\frac{1}{\cosh^2 x}$ has first order contact with the constant 1 at $x = 0$.

The Lipschitz norm of the derivative can be estimated by examination of the derivative of the factor $\rho(x)$. We compute:

$$\rho'(x) = \frac{-2\sin\theta\cosh x}{(\cos\theta + \sin\theta\sinh x)^3}.$$

Now we consider the difference quotients $\frac{|\rho'(x_1)-\rho'(x_0)|}{|x_1-x_0|}$; recall that the arbitrary choice of p determined the relevant angle θ (as well as the defining maximal balls B and B_p). Thus we regard the angle θ as fixed in this computation. In particular, it is sufficient for the purposes of estimating the $C^{1,1}$ norm of $\mathrm{gr}_\lambda(\sigma)$ that we consider $x_1 \in B_p$ and $x_0 \in B$ in the difference quotient $\frac{|\rho'(x_1)-\rho'(x_0)|}{|x_1-x_0|}$. However, in that case, an elementary calculus argument shows that this difference quotient is bounded by the maximum of 2 and the value of $\frac{|\rho'(x_1)-\rho'(x_0)|}{|x_1-x_0|}$ at the edges of the strata (the dotted lines in Figure 3); i.e. at $x_0 = 0$ and $\cosh x_1 \approx \sec\theta$. But here we have

$$\frac{|\rho'(x_1) - \rho'(x_0)|}{|x_1 - x_0|} = \frac{|\rho'(x_1)|}{|x_1|} \approx \frac{\sin 2\theta}{\cosh^{-1}(\sec\theta)},$$

which is uniformly bounded in θ, for $\theta \in (0, \pi/2]$ (the supremum is 2 and occurs when θ approaches zero, as l'Hôpital's rule confirms). The uniform bound on the $C^{1,1}$ norms over a family of simple closed curve grafting loci follows from the fact that these estimates are independent of the size of the grafting cylinder. □

Next, we consider the regularity of an individual harmonic map $w\colon (S, \mathrm{gr}_\lambda(\sigma_0)) \to (S, \mathrm{gr}_\lambda(\sigma_1))$.

Lemma 2.3.2. *There exists a harmonic map $w : (S, \mathrm{gr}_\lambda(\sigma_0)) \to (S, \mathrm{gr}_\lambda(\sigma_1))$ homotopic to the identity; this map is of class $C^{2,\alpha}$.*

Proof. As S is compact, and $\mathrm{gr}_\lambda(\sigma_1)$ is an NPC space (see [GS92]), it is straightforward that there is an energy minimizer w in the given homotopy class. Then we are able to make considerable use of the literature: Theorem 2.3 of [GS92] then ensures that $w \in H^1(S, S)$ is locally Lipschitz. The rest of the proof is straightforward bootstrapping applied to the harmonic map equation (see, e.g., [Jo97, Theorem 3.5.2b]). In particular, since the map w is locally Lipschitz, equation (2.3.1) shows that $w_{z\bar{z}} = -\frac{\rho_w}{\rho} w_z w_{\bar{z}} \in L^\infty$ where $\rho = \mathrm{gr}_\lambda(\sigma_1)$. Standard elliptic regularity results [GT83, Problem 4.8] then show that $w \in C^{1,\alpha}$. But then, we can redo the estimate, and using that $\rho \in C^{1,1}$, we find that $w_{z\bar{z}} = -\frac{\rho_w}{\rho} w_z w_{\bar{z}} \in C^\alpha$, with the standard Schauder interior estimates [GT83, Thm. 4.6] then giving that $w \in C^{2,\alpha}$, as required. □

Finally, we come to the smoothness of the families of the maps. For the remainder of this section we need only consider grafting along a weighted simple closed curve $s\gamma \in \mathcal{ML}$ and will therefore abbreviate the notation $\mathrm{gr}_{s\gamma}(\sigma)$ by $\mathrm{gr}(\sigma)$. We begin by recording the fact that $\mathrm{gr}(\sigma_t)$ varies analytically in t, for an analytic family of hyperbolic metrics σ_t.

Lemma 2.3.3. *Let $\{\sigma_t\}$ be a real analytic family (in t) of hyperbolic metrics. Then the family $\{\mathrm{gr}(\sigma_t)\}$ of grafted metrics is also real analytic in t.*

Remark. We recall that McMullen shows in [Mc98] that the metrics $\{\mathrm{gr}(\sigma_t)\}$ define a real analytic curve of points in Teichmüller space. We actually need that associated to a neighborhood in Teichmüller space, there is a family of metric tensors (and not just underlying conformal structures) so that the correspondence $\sigma \mapsto \mathrm{gr}(\sigma)$ is real analytic with respect to the real analytic structure on Teichmüller space.

Proof. To see that the metrics themselves vary analytically, we argue directly. Let $\Lambda_t : S^1 \to (S, \sigma_t)$ denote a constant speed parametrization of the simple closed σ_t-geodesic γ_t in the free homotopy class $[\gamma]$ on S; these maps are uniquely determined up to a rotation. To fix the maps uniquely, at least for small t, consider a real analytic transversal α to the image $\Lambda_0(S^1)$ through a point, say $p_0 \in \Lambda_0(S^1)$. Now the images $\Lambda_t(S^1)$ vary analytically in t and meet α in a single point for all small t, and so $\Lambda_t(S^1) \cap \alpha = p_t$ also varies analytically in t (by the analytic implicit function theorem). Thus, normalizing the maps Λ_t to require $\Lambda_t(1) = p_t$ then defines a real-analytic family of parametrized geodesics in the free homotopy class $[\gamma]$ on S.

Define next $(S, \mathrm{gr}(\sigma_t))$ by equipping the topological space

$$\{\overline{S - \gamma_t}\} \sqcup_{\gamma_t} \{[-s/2, s/2] \times S^1\}$$

with a metric as follows. The space $[-s/2, s/2] \times S^1$ is given the product metric, once S^1 is scaled to have length $\ell_{\sigma_t}(\gamma_t)$, and the space $\overline{S - \gamma_t}$ is given the metric σ_t. Furthermore the gluing along the image of γ_t is done isometrically, with the requirement that p_t^ℓ be glued to $(-s/2, 1)$ and p_t^r be glued to $(s/2, 1)$ – here we denote the left and right images of p_t by p_t^ℓ and p_t^r respectively. These metric spaces represent the grafted metrics $(S, \mathrm{gr}(\sigma_t))$ under the obvious identification of homotopy groups, and so we may then label the metric spaces as $(S, \mathrm{gr}(\sigma_t))$.

With these constructions in mind, consider the maps $v_t : (S, \mathrm{gr}(\sigma_0)) \to (S, \mathrm{gr}(\sigma_t))$ defined to be harmonic on the complement of the Euclidean cylinder while satisfying the boundary conditions that the maps of the boundary should be at constant speed, taking p_0^ℓ to p_t^ℓ and p_0^r to p_t^r. By the results of [EL81], these maps v_t are real analytic in t (though probably not along S), and thus the pullback metrics $v_t^* \mathrm{gr}(\sigma_t)$ on S vary real analytically in t. □

Consider such an analytic family $\{\mathrm{gr}(\sigma_t)\}$ and the family $\{w_t\}$ of harmonic maps $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ which we know to exist and be of class $C^{2,\alpha}$.

Lemma 2.3.4. *The family $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ of harmonic maps is analytic in t, for small values of t. Any individual map $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ is a homeomorphism, and $\frac{d}{dt}\big|_{t=0} w_t \in C^{2,\alpha}$.*

Proof. We mimic an allied proof in [EL81]; see also [Sa78]. To avoid cumbersome equations, we will abbreviate the description of metrics by setting $\rho_t = \mathrm{gr}(\sigma_t)$ in the proof which follows. Given such a family of maps $w_t : (S, \rho_t) \to (S, \rho_0)$, we let ζ_t denote a local conformal coordinate on (S, ρ_t) and $z = \zeta_0$ denote a coordinate on (S, ρ_0). The tension $\tau_t = \tau(w_t)$ is a vector field along the map w_t which we write as

$$\tau_t = \tau(w_t) = \frac{1}{\rho_t}\left[\left(\frac{\partial^2}{\partial\zeta_t\partial\bar\zeta_t}w_t\right) + \left(\frac{\partial}{\partial w_t}\log\rho_t(w_t)\right)\left(\frac{\partial}{\partial\zeta_t}w_t\right)\left(\frac{\partial}{\partial\bar\zeta_t}w_t\right)\right].$$

(The reader unaccustomed to such objects might recall the situation of a (possibly non-simple) geodesic on a surface being regarded as the image of a harmonic map of a circle into the surface. There the tension of the map is represented as the geodesic curvature vector field along the geodesic and is defined even at image points where the map is not an embedding.) We then compute the first variation $\delta\tau[\dot w]$ in w at $t = 0$ of the tension (see [Jo97, (3.6.7)]) to be

$$\begin{aligned}\delta\tau[\dot w] &= \frac{1}{\rho_0}\left(\dot w_{z\bar z} + \frac{(\rho_0)_z}{\rho_0}\dot w_{\bar z}\right)\\ &= \frac{1}{\rho_0^2}\left((\rho_0\dot w_{\bar z})_z\right).\end{aligned} \tag{2.3.3}$$

Here we have simplified the formula considerably by applying it at $t = 0$, where $w_0 : (S, \rho_0) \to (S, \rho_0)$ is the identity map with $(w_0)_{\bar z} = 0$.

We aim to apply the analytic implicit function theorem (see [Be77]): the formal setting is that we regard the tension τ as a functional

$$\tau : C^{2,\alpha}(S, S) \times (-\epsilon, \epsilon) \longrightarrow C^{0,\alpha}(T(S))$$

where $C^{k,\alpha}(T(S))$ denotes $C^{k,\alpha}$ sections of the tangent bundle to S, and the map τ associates, to a map $w \in C^{2,\alpha}(S, S)$ and a time t, the tension field $\tau(w, \rho_t)$ of the map $w : (S, \rho_t) \to (S, \rho_0)$. This functional is evidently analytic in t, so our attention turns to formula (2.3.3): we assert that

$$\|\delta\tau\| > 0$$

where the norm is that taken on functionals between $C^{2,\alpha}(T(S))$ and $C^{0,\alpha}(T(S))$. To apply the implicit function theorem in this setting, it is enough to prove that the operator $\delta\tau = \frac{1}{\rho_0}\left(\partial^2_{z\bar z} + \frac{(\rho_0)_z}{\rho_0}\partial_{\bar z}\right)$ is invertible on $C^{2,\alpha}(T(S))$; i.e. that given a vector field $f \in C^{0,\alpha}(T(S))$, there is a vector field $u \in C^{2,\alpha}(T(S))$ so that

$$\frac{1}{\rho_0}u_{z\bar z} + \frac{(\rho_0)_z}{\rho_0^2}u_{\bar z} = \delta\tau(u) = f \tag{2.3.4}$$

and that $\|u\|_{2,\alpha}$ is bounded in terms of $\|f\|_{0,\alpha}$.

With these formal preparations concluded, we begin our principal task of solving (2.3.4).

In preparation for the existence result, we begin by proving a uniqueness result, i.e. we claim that when $f = 0$ in equation (2.3.4), we have that $u = 0$. This follows immediately by an integration by parts; in particular observe that integrating both sides of (2.3.4) against the vector field $\bar{u}$ (after raising indices appropriately) with respect to the (S, ρ_0) volume measure yields

$$\begin{aligned}
0 &= \iint \left(u_{z\bar{z}} + \frac{(\rho_0)_z}{\rho_0} u_{\bar{z}} \right) \bar{u}\rho_0 dz d\bar{z} \\
&= -\iint u_{\bar{z}}(\bar{u}_z)\rho_0 dz d\bar{z} - \iint u_{\bar{z}}\bar{u}(\rho_0)_z dz d\bar{z} + \iint (\rho_0)_z u_{\bar{z}} \bar{u} dz d\bar{z} \\
&= -\iint |u_{\bar{z}}|^2 dA_{\rho_0}
\end{aligned}$$

where we have obtained the second line from the first line by integrating (only) the first term by parts. We conclude that the Beltrami differential $u_{\bar{z}}$ vanishes identically on S, so that u represents a holomorphic vector field in the compact Riemann surface S. It is an easy consequence of Riemann-Roch, for instance, that this forces u to vanish identically on S.

To solve (2.3.4) then, we apply the Fredholm alternative for the elliptic equation (2.3.4). Note that the coefficients of the operator on the left hand side of (2.3.4) are in $C^{0,1} \subset C^\alpha$. It then follows by an application of the Fredholm alternative (for instance, one could adapt the proof given for the Dirichlet problem of such an equation on $\mathbb{R}^n$ in [GT83, Theorem 6.15] to our closed surface setting) that (2.3.4) is solvable in $C^{2,\alpha}$ for the vector field u. The required estimate $\|u\|_{2,\alpha} \le C\|f\|_{0,\alpha}$ then follows from standard elliptic techniques: the maximum principle (see [GT83, Theorem 3.7]) provides a C^0 bound on u in terms of a C^0 bound on f, and then the desired estimate follows from the basic Schauder estimates (see for instance [GT83, Theorem 6.2]).

Thus the analytic implicit function theorem allows us to conclude that for small t, there is an analytic family of harmonic (i.e. $\tau = 0$) maps $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$.

That any individual map $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ is a homeomorphism follows from the map w_t being a perturbation of the identity, and, of course, our use of the analytic implicit function theorem for maps $w(t)$ in $C^{2,\alpha}$ shows that the vector field $\frac{d}{dt}\big|_{t=0} w_t \in C^{2,\alpha}$. □

Remark. Let us review this proof from the point of view of the harmonic maps theory, in hopes of better connecting it to the rest of the argument. The equation for the variation of the harmonic map away from the identity (here, for convenience of computation in this remark, we fix the domain metric σ and vary the target) is

$$\begin{aligned}
0 &= \dot{w}_{z\bar{z}} + \frac{\sigma_z}{\sigma}\dot{w}_{\bar{z}} \\
&= \frac{1}{\sigma}(\sigma \dot{w}_{\bar{z}})_z.
\end{aligned}$$

(As in the previous proof, $\dot{w}$ is a vector field along the map w.) Thus

$$\sigma \dot{w}_{\bar{z}} = \overline{\phi} \tag{2.3.5}$$

where ϕ is an analytic quadratic differential (compare (4.2.1)–(4.2.2)). In particular, by dividing through by σ, we may write an equality of Beltrami differentials:

$$\dot{w}_{\bar{z}} = \overline{\phi}/\sigma. \tag{2.3.6}$$

Not only is the right hand side obviously in $C^{1,1}$, but we can use (2.3.6) and the generalized Cauchy integral formula to write a fairly explicit formula for $\dot{w}$ in a domain Ω:

$$\dot{w} = \psi + \frac{1}{2\pi i}\iint\limits_{\Omega} \frac{(\overline{\phi}/\sigma)(\delta)}{\delta - z} d\delta d\overline{\delta}.$$

Here ψ is an analytic function on Ω. From this expression, we can see that $\dot{w} \in C^{2,\alpha}$ rather directly.

The point to observe though, is that to a one-parameter family of harmonic maps w_t, we have a one-parameter family of Hopf differentials Φ_t. The important fact (see also §2.3.1) to realize about Hopf differentials is that under fairly mild conditions (only enough to ensure that they are weakly holomorphic in the sense of Weyl's Lemma, which is assured, for instance, by a uniform bound on the total energy of the maps) they are very smooth (complex analytic, in fact), independent of the smoothness of the range metric. Now, these differentials have the local expression $\phi_t = \sigma_t(w_t)(w_t)_z\overline{(w_z)_{\bar{z}}}$. If we can differentiate in t at $t = 0$, we would find that $\dot{\phi} = \sigma(z)\dot{w}_{\bar{z}}$; this is expressed by the formula (2.3.5) with ϕ taking the place of $\dot{\phi}$. Thus once we somehow establish that our family of harmonic maps is differentiable, then formula (2.3.6) puts $\dot{w} \in C^{1,1}$, from which our generalized Cauchy integral formula gives $\dot{w} \in C^{2,\alpha}$. Thus it is the existence portion of the proof above which is most crucial, and not the specific Banach spaces we have used.

Our main technical result concerns the time derivative of the holomorphic energy density; recall from §2.3.1 the definition $\mathcal{H} = \|w_*\partial_z\|^2_\rho$ of the holomorphic energy density $\mathcal{H}$ of a harmonic map $w : (S,\sigma) \to (S,\rho)$. We will be interested in the situation where $\sigma = \rho_t$ represents a family of metrics on the domain varying away from $\rho = \rho_0$ in t; this then determines a family $\mathcal{H}_t$ of holomorphic energy densities of the harmonic maps $w_t : (S,\rho_t) \to (S,\rho_0)$, and in particular the infinitesimal change $\dot{\mathcal{H}} = \frac{d}{dt}\big|_{t=0} \mathcal{H}_t$ in t of holomorphic energy densities. The regularity of $\dot{\mathcal{H}}$ is crucial to our argument.

Lemma 2.3.5. *In the notation above, if the metrics ρ_t are an analytic family $\rho_t = \mathrm{gr}(\sigma_t)$ of grafted metrics, then the derivative $\dot{\mathcal{H}}$, as a function on the surface S, is Lipschitz.*

Proof. Of course, our derivative $\frac{d}{dt}\big|_{t=0} \mathcal{H}_t$ is analytic away from the boundary of the grafted cylinder, so our claim is really that $\dot{\mathcal{H}}$ is Lipschitz in a neighborhood of those curves. We see this by a direct computation in coordinates, coupled with our regularity estimates (Lemmas 2.3.1 and 2.3.4) for the harmonic maps and the metrics. To begin, we write in coordinates

$$\begin{aligned}\dot{\mathcal{H}} &= \frac{d}{dt}\Big|_{t=0} \mathcal{H}_t \\ &= \frac{d}{dt}\Big|_{t=0} \frac{\mathrm{gr}(\sigma_0)(w_t)}{\mathrm{gr}(\sigma_t)(\zeta_t)}|\partial_{\zeta_t} w_t|^2\end{aligned}$$

where $w_t : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ is the harmonic map varying away (for $t \neq 0$) from the identity map $\mathrm{id} : (S, \mathrm{gr}(\sigma_0)) \to (S, \mathrm{gr}(\sigma_0))$. Here we are using a conformal coordinate ζ_t on S in which we compute the conformal factor $\mathrm{gr}(\sigma_t)$; it is easy to compute that in terms of a Beltrami differential $\mu_t = (\partial\zeta_t/\partial\bar{z})/(\partial\zeta_t/\partial z)$, we have the coordinate change expression

$$\frac{\partial}{\partial\zeta_t} = \frac{1}{1 - |\mu_t|^2}\frac{1}{(\zeta_t)_z}(\partial_z - \bar{\mu}_t\partial_{\bar{z}}).$$

Naturally at $t = 0$, we have $\zeta_t = z$, and by Lemma 2.3.1, we can construct our family of surfaces so that the family of coordinates ζ_t is smooth in (z, t) and the conformal factors $\mathrm{gr}(\sigma_t)(\zeta_t)$ are an analytic family in $C^{1,1}$. Thus we may assume that $\mu_t = \mathcal{O}(t)$, that $\frac{d}{dt}(\zeta_t)_z$ is smooth in z, and that $\frac{d}{dt}\mathrm{gr}(\sigma_t)(p) \in C^{1,1}$ (here we think of the factor as being determined at a point $p \in S$ by the choice of conformal coordinate ζ_t and a particular choice of representative metric tensor in the Diff_o-orbit of metrics). Also, the identity map $\mathrm{id} : (S, \mathrm{gr}(\sigma_0)) \to (S, \mathrm{gr}(\sigma_0))$ is conformal and hence (locally) complex analytic (in the local coordinates), and by Lemma 2.3.4, the expression $\frac{d}{dt}\partial_z w_t$ is real analytic in t for an analytic path of metrics $\{\sigma_t\}$.

With these considerations in mind, we compute

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{H}_t &= \frac{d}{dt}\Big|_{t=0} \frac{\mathrm{gr}(\sigma_0)(w_t)}{\mathrm{gr}(\sigma_t)(\zeta_t)} |\partial_{z_t} w_t|^2 \\ &= \frac{2}{\mathrm{gr}(\sigma_0)} \mathrm{Re}\left\{ \frac{\partial}{\partial z}\mathrm{gr}(\sigma_0) \cdot \frac{d}{dt} w_t \right\} \\ &\quad - \frac{1}{\mathrm{gr}(\sigma_0)} \frac{d}{dt}\mathrm{gr}(\sigma_t) + 2\,\mathrm{Re}\{\dot{w}_z - \dot{\zeta}_z\}. \end{aligned} \tag{2.3.7}$$

The lemma follows from examining the terms in this expression (2.3.7): we already noted that ζ_t is smooth in (z,t) and that $\frac{d}{dt}\mathrm{gr}(\sigma_t) \in C^{1,1}$; most importantly, the term $\mathrm{gr}(\sigma_0)$ is in $C^{1,1}$ by Lemma 2.3.1, while $\frac{d}{dt} w_t$ is in $C^{2,\alpha}$ from Lemma 2.3.4.

Thus $\frac{\partial}{\partial z}\mathrm{gr}(\sigma_0)$ is in $C^{0,1}$ and the least regular term in (2.3.7) is the first one, but it, and likewise $\dot{\mathcal{H}}$, are hence in $C^{0,1}$. □

Remark. An alternative approach to this result might be to take a time derivative of equation (2.3.2) (see equation (3.1.5)) and then attempt to extract the required regularity of $\dot{\mathcal{H}}$ from standard regularity theory. Yet the regularity we require seems slightly stronger than that which an elementary use of the literature would guarantee; hence we adopt the approach above. [We could have equivalently used the computation of (2.3.7) to identify the least regular term and then used equation (3.1.5) to find an equation for the difference of $\dot{\mathcal{H}}$ and that term; then standard regularity theory could be applied to show that this difference was smooth – but we would have lost the intuition of how that least regular term arose.]

2.4. Variation of geodesics. This section contains a brief discussion of the equations governing the variation fields of a geodesic in a family of conformally related Riemannian metrics. We begin by setting some notation. Consider a smooth family of Riemannian metrics g_t on S and a family of g_t-geodesics $\gamma_t : [0,1] \to S$. We adopt Fermi coordinates along the curve γ_0 so that

$$g_0 = F(x_1, x_2)^2 dx_1^2 + dx_2^2,$$

setting $F(x_1, 0) = 1$. Here γ_0 is parametrized by $\gamma_0 = \{(x_1, x_2) | x_2 = 0\}$ and as γ_0 is a geodesic, we have $\partial_2 \big|_{\gamma_0} F(x_1, x_2) = 0$. The geodesic equation for γ_t in these coordinates is given by

$$\gamma_{t,11}^k + \Gamma_{t,ij}^k(\gamma_t(x_1))\gamma_{t,1}^i \gamma_{t,1}^j = 0 \tag{2.4.1}$$

where $\Gamma^k_{t,ij}$ are the g_t-Christoffel symbols. We differentiate (2.4.1) in time t to obtain the following equation for the vector field $\dot\gamma^k\partial_k = \frac{d}{dt}\big|_{t=0}\gamma^k_t\partial_k$:

$$\dot\gamma^k_{11} + \left(\frac{d}{dt}\Big|_{t=0}\Gamma^k_{t,ij}(\gamma_0(x_1))\right)\gamma^i_{0,1}\gamma^j_{0,1} + \partial_m\Gamma^k_{0,ij}(\gamma_0(x_1))\dot\gamma^m\gamma^i_{0,1}\gamma^j_{0,1}$$
$$+ \Gamma^k_{0,ij}(\gamma_0(x_1))\dot\gamma^i_1\gamma^j_{0,1} + \Gamma^k_{0,ij}(\gamma_0(x_1))\gamma^i_{0,1}\dot\gamma^j_1 = 0.$$

Since we are principally interested in the normal component of the variation field $\frac{d}{dt}\gamma_t$, we set $k = 2$, greatly simplifying the remainder of the computation. In the Fermi coordinates chosen, we have that $\Gamma^2_{0,ij} = 0$ along γ_0 (since $\partial_2\big|_{\gamma_0} F(x_1,x_2) = 0$), and also, for a constant speed geodesic, we have $\gamma^i_{0,1} = \ell\delta^i_1$ where ℓ is the length of the geodesic and δ^i_1 is the Kronecker delta. Thus the previous equation along γ_0 simplifies to

$$\dot\gamma^2_{11} + \left(\frac{d}{dt}\Big|_{t=0}\Gamma^2_{t,11}\right)\ell^2 + \left(\partial_m\Gamma^2_{0,11}\right)\dot\gamma^m\ell^2 = 0 \tag{2.4.2}$$

where of course

$$\Gamma^2_{t,11} = \frac{1}{2}g^{2\alpha}_t\left(2\partial_1 g_{t,1\alpha} - \partial_\alpha g_{t,11}\right).$$

Moreover, we will be interested primarily in the situation where $g_t = \frac{1}{\mathcal{H}_t}g_0 + o(t)$ is a family of metrics conformal to first order (see (3.1.2)) and where g_0, being written in Fermi coordinates, is diagonal; this also forces g_t to be diagonal (to first order) which simplifies the above description to

$$\Gamma^2_{t,11} = -\frac{1}{2\mathcal{H}_t}\,\partial_2\left(\frac{F(x_1,x_2)^2}{\mathcal{H}_t}\right) + o(t).$$

It is then straightforward to compute from this equation and from $\mathcal{H}_0 \equiv 1$ that

$$\partial_m\Gamma^2_{0,11} = K\delta^2_m \tag{2.4.3}$$

and

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Gamma^2_{t,11} &= \frac{\dot{\mathcal{H}}}{2\mathcal{H}_0^2}\,\partial_2\left(\frac{F(x_1,x_2)^2}{\mathcal{H}_0}\right) + \frac{1}{2\mathcal{H}_0}\,\partial_2\left(\frac{F(x_1,x_2)^2\dot{\mathcal{H}}}{\mathcal{H}_0^2}\right)\\ &= \frac{\dot{\mathcal{H}}}{2}\,\partial_2(F(x_1,x_2)^2) + \frac{1}{2}\partial_2(F(x_1,x_2)^2)\dot{\mathcal{H}} + \frac{1}{2}F(x_1,0)^2\partial_2\dot{\mathcal{H}}\\ &= \frac{1}{2}\partial_2\dot{\mathcal{H}}\end{aligned} \tag{2.4.4}$$

where the first and second terms vanish because $\partial_2 F(x_1,x_2) = 0$, and the third term simplifies using $F(x_1,0) = 1$.

We conclude from (2.4.2), (2.4.3) and (2.4.4) that the variational field $V = \frac{d}{dt}\big|_{t=0}\gamma^2_t$ satisfies

$$V_{11} + K_0 V\ell^2 = -\frac{1}{2}\ell^2\partial_2\dot{\mathcal{H}}. \tag{2.4.5}$$

All of this was in the situation of a family of smoothly varying Riemannian metrics. However, our setting involves (Lemma 2.3.3) a real-analytic family of Thurston (grafted) metrics, which are but in the class $C^{1,1}$. Under these hypotheses, there is also a well-defined variational field satisfying equation (2.4.5). We see this on the side where $K_0 \equiv -1$ by considering, for each choice of parameter t, a curve $\gamma^{(\epsilon)}_t$ of constant geodesic curvature ϵ at distance $\sinh\epsilon$ from the hyperbolic geodesic γ_t. These curves are uniquely defined, and the proof of Eells-Lemaire [EL81] that geodesics vary smoothly with parameters extends to prove that these curves also vary smoothly. Examining this proof more carefully, we see that it yields an estimate $\|\frac{d}{dt}\gamma_t^{(\epsilon)}\|_{C^2} < C(\epsilon)$ for the C^2 norm of the variational field $V^{(\epsilon)} = \frac{d}{dt}\gamma_t^{(\epsilon)}$ and this estimate admits a universal bound $C(\epsilon) < C$ for ϵ small. Thus we find that the geodesic γ_t also varies smoothly.

The reader should now recognize that in the case of the Thurston metric defined above, equation (2.4.5) is really a pair of equations for a single variational field V. That is, on the flat cylinder, we have $K_0 \equiv 0$, while on the hyperbolic portion of the surface, we have $K_0 \equiv -1$. In this case it will be convenient to impose (x, y)-coordinates on the cylinder so that the geodesic meridians are parametrized by the (arc-length) coordinate y, and the orthogonal arcs are parametrized by the coordinate x, also by arclength; this choice of coordinates translates ∂_1 derivatives of V into ∂_y derivatives of V via the recipe $V_{yy} = \ell^{-2}V_{11}$. We obtain the following two versions of (2.4.5) on the cylinder and hyperbolic portions, respectively:

$$V_{yy} = -\frac{1}{2}(\partial_x\dot{\mathcal{H}})_0, \tag{2.4.5$_0$}$$

$$V_{yy} - V = -\frac{1}{2}(\partial_x\dot{\mathcal{H}})_{-1}. \tag{2.4.5$_{-1}$}$$

Here we have written the pair of derivatives of $\dot{\mathcal{H}}$ as $(\partial_x\dot{\mathcal{H}})_0$ and $(\partial_x\dot{\mathcal{H}})_{-1}$ depending on which side of the boundary curves we are considering. Subtracting these two versions of (2.4.5) yields the following useful expression for the variation field V:

$$V = -\frac{1}{2}\left((\partial_x\dot{\mathcal{H}})_0 - (\partial_x\dot{\mathcal{H}})_{-1}\right). \tag{2.4.6}$$

The only sign convention worth noting in these equations is that V is measured positively in the direction of increasing x.

§3. The case of simple closed curves

In this section, we prove the main theorem in the model case when the measured lamination is a weighted simple closed curve. We begin by deriving our basic equation of study (3.1.5), after which the proof effectively becomes a computation, undertaken in §3.2. Our setup applies quite generally to families of grafted metrics in which the length of the inserted annulus is allowed to vary. We only use the information that this length is constant in t at the very end of the proof – this is the content of section §3.3.

3.1. Basic equation of study. We begin with a precise statement of our objective.

Theorem 3.1 (Model Case)**.** *Let S be a closed differentiable surface of genus $g > 1$, let γ be an essential simple closed curve on S, and let $s \in \mathbb{R}_+$ be a positive real number. Then the grafting map $\mathrm{Gr}_{s\gamma} : \mathcal{T}_g \to \mathcal{T}_g$ is a homeomorphism.*

Proof of Theorem 3.1. It has been shown that Gr_λ is real analytic [Mc98] and proper [Ta97], therefore it suffices, as $\mathcal{T}_g$ is a cell, to prove

Lemma 3.1.1. *The tangent map $d\mathrm{Gr}_{s\gamma} : T\mathcal{T}_g \to T\mathcal{T}_g$ is injective.*

Proof of Lemma 3.1.1. We suppose, in order to obtain a contradiction, that there is a hyperbolic surface (S, σ_0) and a tangent vector $[\dot{\mu}] \in T_{[\sigma_0]}\mathcal{T}_g$ so that $d\mathrm{Gr}_{s\gamma}[\dot{\mu}] = [0] \in T_{[\mathrm{Gr}_{s\gamma}(\sigma)]}\mathcal{T}_g$. Represent the tangent vector $[\dot{\mu}]$ as the germ of a particular family of hyperbolic metrics σ_t on S.

We find it convenient to use harmonic maps to "fix the gauge" in comparing the grafted surfaces $\mathrm{Gr}_{s\gamma}(\sigma_t)$. In particular, we imagine $\mathrm{Gr}_{s\gamma}(\sigma_t)$ as being realized by the metric $\mathrm{gr}_{s\gamma}(\sigma_t)$ on the underlying differentiable surface S. As in the previous section we will omit the subscript $s\gamma$ in the discussion which follows. Of course, we have a choice for these representative metrics, as the group Diff_o of diffeomorphisms isotopic to the identity acts on metrics on S, with the orbit of $\mathrm{gr}(\sigma_t)$ consisting of isometric metrics. However, by the results in §2, and because $\mathrm{gr}(\sigma_0)$ is non-positively curved, there is a unique harmonic map $w : (S, \mathrm{gr}(\sigma_t)) \to (S, \mathrm{gr}(\sigma_0))$ homotopic to the identity for any of our choices of $\mathrm{gr}(\sigma_t)$. In particular, we can choose this representative metric $\mathrm{gr}(\sigma_t)$ on S so that the identity map

$$\mathrm{id} : (S, \mathrm{gr}(\sigma_t)) \longrightarrow (S, \mathrm{gr}(\sigma_0))$$

is harmonic for all $t \geq 0$.

We next introduce notation parallel to that preceding Lemma 2.3.5. Let

$$\mathcal{H}_t = \| \,\mathrm{id}_{z_t} \|^2 = \frac{\mathrm{gr}(\sigma_0)}{\mathrm{gr}(\sigma_t)} |\,\mathrm{id}_{z_t}|^2 \tag{3.1.1}$$

denote the holomorphic energy density of the harmonic conformal map, where here we have expanded the metric $\mathrm{gr}(\sigma_t)$ in local conformal coordinates z_t (themselves smooth in t) as $\mathrm{gr}(\sigma_t) = \mathrm{gr}(\sigma_t)|dz_t|^2$. Now since the metrics $\mathrm{gr}(\sigma_t)$ are varying as $o(t)$ by hypothesis, the smooth dependence of harmonic maps on metrics (Lemma 2.3.4) shows that the difference (say in the $C^{2,\alpha}$ norm) between the pushforward of $\mathrm{gr}(\sigma_t)$ and $\mathrm{gr}(\sigma_0)$ by the identity map is $o(t)$; since at $t = 0$ the map is an isometry, we have that the identity is conformal to $o(t)$ in the following sense:

$$\mathrm{gr}(\sigma_t)|dz_t|^2 = \frac{1}{\mathcal{H}_t}\mathrm{gr}(\sigma_0)|dz_0|^2 + o(t). \tag{3.1.2}$$

Furthermore, because the identity map is harmonic, we apply the Bochner equations (2.3.2) to conclude that

$$\Delta_{\mathrm{gr}(\sigma_t)} \log \mathcal{H}_t = -2K_0(\mathrm{id}(z_t))\mathcal{H}_t + 2K_t(z_t) + o(t). \tag{3.1.3}$$

The $o(t)$ term comes from the fact that the harmonic map is conformal to $o(t)$, so the Hopf differential term in (2.3.2) is also $o(t)$. We then use (3.1.2) to rewrite (3.1.3) as

$$\mathcal{H}_t \Delta_{\mathrm{gr}(\sigma_0)} \log \mathcal{H}_t = -2K_0(\mathrm{id}(z_t))\mathcal{H}_t + 2K_t(z_t) + o(t).$$

Divide by $\mathcal{H}_t$ to obtain the equation

$$\Delta_{\mathrm{gr}(\sigma_0)} \log \mathcal{H}_t = -2K_0 + \frac{2K_t}{\mathcal{H}_t} + o(t) \tag{3.1.4}$$

which is the precursor to our basic equation of study (the error term remains $o(t)$ here because $\mathcal{H}_t = 1 + \mathcal{O}(t)$). To obtain the basic equation of study, we differentiate equation (3.1.4) in time to obtain an equation for $\dot{\mathcal{H}} = \frac{d}{dt}\big|_{t=0} \mathcal{H}_t$:

$$\Delta_{\mathrm{gr}(\sigma_0)}\dot{\mathcal{H}}/\mathcal{H}_0 - 2\dot{K}/\mathcal{H}_0 + 2K_0\dot{\mathcal{H}}/\mathcal{H}_0^2 = 0$$

where $\dot{K}$ denotes $\frac{d}{dt}\big|_{t=0} K_t$. Since $\mathcal{H}_0 \equiv 1$ by construction and $\dot{K} \equiv 0$ away from the boundary of the inserted cylinder, we may summarize our equation (off the boundary of the inserted cylinder) as

$$\left(\Delta_{\mathrm{gr}(\sigma_0)} + 2K_0\right)\dot{\mathcal{H}} = 0. \tag{3.1.5}$$

We observed in Lemma 2.3.5 that $\dot{\mathcal{H}} \in C^{0,1}$; thus, much depends on understanding the "jump" in derivatives of a solution $\dot{\mathcal{H}}$ across the curves comprising the boundary of the flat cylinder.

We now conclude the proof of Lemma 3.1.1, up to the analysis of $\dot{\mathcal{H}}$. In the next section, we will prove (Lemma 3.2.1) that $\dot{\mathcal{H}} = 0$ and that hence $\mathrm{id} : (S, \mathrm{gr}_{s\gamma}(\sigma_t)) \to (S, \mathrm{gr}_{s\gamma}(\sigma_0))$ is an isometry, to order $o(t)$. From equation (2.4.6), this implies that the variational field V also vanishes identically, so that the geodesic γ is fixed (to order $o(t)$).

With these estimates in mind, we argue that σ_t and σ_0 differ, as points in Teichmüller space, by $o(t)$. To see this, recall [Jo97] that for any neighborhood of Teichmüller space, there are a finite number of free homotopy classes of simple closed curves whose geodesic lengths provide real analytic coordinates for that neighborhood. Thus it is enough to show that the lengths of these geodesics differ by $o(t)$.

Consider such a simple closed geodesic α on (S, σ_t). This determines a geodesic $\hat{\alpha}$ on $(S, \mathrm{gr}(\sigma_t))$ and its image, say α^*, under the $(1+o(t))$-quasi-isometry between $\mathrm{gr}(\sigma_t)$ and $\mathrm{gr}(\sigma_0)$. If α does not meet the geodesic γ, then $\hat{\alpha}$ also does not meet the geodesic γ, and so, for t sufficiently small, the analogous situation holds for α^*. Thus, as α has bounded length, and $\mathrm{gr}(\sigma_t)$ and $\mathrm{gr}(\sigma_0)$ are $(1+o(t))$-quasi-isometric, we have that α and α^* have the same length, to $o(t)$.

If α does meet γ, then the process of grafting determines a broken geodesic, say β, on $\mathrm{gr}(\sigma_t)$: the curve β consists of the arc $\hat{\alpha}$ on the hyperbolic portion of $\mathrm{gr}(\sigma_t)$, together with a straight longitudinal arc on the grafted cylinder connecting the endpoints of the segment $\hat{\alpha}$. Now under the $(1 + o(t))$-quasi-isometry between $\mathrm{gr}(\sigma_t)$ and $\mathrm{gr}(\sigma_0)$, the curve β gets taken to a $(1 + o(t))$-quasi-isometric image of itself: there is a $(1 + o(t))$-quasigeodesic which is almost entirely in the hyperbolic portion of the surface, with endpoints connected by a $(1 + o(t))$-quasi-isometric image of the longitude, and the two segments meeting at angles which are only $o(t)$ different from the angles at the breaks of the original curve β. Thus, there is a $(1 + o(t))$-quasi-isometry of a neighborhood of those breaks which deforms this image of β to a curve β^* which is a $(1 + o(t))$-quasigeodesic in the hyperbolic portion of $\mathrm{gr}(\sigma_0)$, is straight in the flat portion of $\mathrm{gr}(\sigma_0)$, and which has angles at the interfaces which agree to $o(t)$. Thus this curve β^* projects to a curve on (S, σ_0) which is a $(1 + o(t))$-quasigeodesic away from γ and whose angle at γ is $\pi - o(t)$. Hence the σ_0-length of the geodesic in that free homotopy class $[\beta]$ agrees with the σ_t-length of the geodesic in the class $[\beta]$, up to an error of $o(t)$. This concludes the proof of the lemma. □

3.2. Computation of $\dot{\mathcal{H}}$. The goal of this section is a proof of

Lemma 3.2.1. *Any function $\dot{\mathcal{H}}$ which solves (3.1.5) and (3.1.2) must vanish identically on S. Thus* $\mathrm{id} : (S, \mathrm{gr}_{s\gamma}(\sigma_t)) \to (S, \mathrm{gr}_{s\gamma}(\sigma_0))$ *is an isometry, to order $o(t)$.*

Remark. In fact, if $\dot{\mathcal{H}}$ were known to be C^1 on S, then the proof would be standard: we multiply (3.1.5) by $\dot{\mathcal{H}}$, and integrate by parts, obtaining

$$0 = \iint_S -|\nabla\dot{\mathcal{H}}|^2 + 2K|\dot{\mathcal{H}}|^2 dA \leq 0.$$

We would then conclude, effectively proving the lemma and the theorem, that $\dot{\mathcal{H}}$ vanishes identically on S. This conclusion though, for the general case of $\dot{\mathcal{H}}$ not necessarily smooth, is basically the content of the remainder of this section, and the centerpiece of this paper. Indeed, our goal for the remainder of §3 is a generalization of the above argument, without the assumption that $\dot{\mathcal{H}}$ is smooth across those boundary curves.

Proof. Write S as the union of two subsurfaces S_0 (the inserted flat cylinder, where $K \equiv 0$) and S_{-1} (the complement of the cylinder, where $K \equiv -1$). We choose coordinates on S_0 as in §2.4:

$$S_0 = \left\{(x,y) \mid -\frac{s}{2} \leq x \leq \frac{s}{2}, 0 \leq y \leq \ell\right\},$$

where ℓ is the length of the grafting curve γ with respect to the hyperbolic metric σ_0, and the two boundary components of S_0 are denoted γ_- (where $x = -\frac{s}{2}$) and γ_+ (where $x = \frac{s}{2}$).

As remarked above, we first and foremost need to understand the normal derivatives of $\dot{\mathcal{H}}$ across the two boundary curves comprising $\partial S_0 = \partial S_{-1}$. We will write $(\partial_n\dot{\mathcal{H}})_0$ and $(\partial_n\dot{\mathcal{H}})_{-1}$ for these normal derivatives as computed from the flat and hyperbolic sides, respectively (with the normal vector field pointing away from the cylinder).

We multiply (3.1.5) by $\dot{\mathcal{H}}$ and then integrate by parts on S_0 to find

$$0 \leq \iint_{S_0} |\nabla\dot{\mathcal{H}}|^2 = \int_{\partial S_0} \dot{\mathcal{H}}(\partial_n\dot{\mathcal{H}})_0$$

and, similarly, on S_{-1} to find

$$0 \leq \iint_{S_{-1}} |\nabla\dot{\mathcal{H}}|^2 + 2|\dot{\mathcal{H}}|^2 = \int_{\partial S_{-1}} -\dot{\mathcal{H}}(\partial_n\dot{\mathcal{H}})_{-1}.$$

(Here, and for the rest of the paper, surface integrals are taken with respect to area measure and line integrals with respect to arclength measure.) Adding the pair of equations, we find

$$0 \leq \int_{\partial S_0} \dot{\mathcal{H}}\left((\partial_n\dot{\mathcal{H}})_0 - (\partial_n\dot{\mathcal{H}})_{-1}\right).$$

Breaking this boundary integral into a pair of integrals over γ_- and γ_+ and rewriting the normal derivatives in terms of x-derivatives, we obtain a coordinate version of the previous inequality,

$$0 \leq \int_{\gamma_-} \dot{\mathcal{H}}\left((\partial_x\dot{\mathcal{H}})_{-1} - (\partial_x\dot{\mathcal{H}})_0\right) + \int_{\gamma_+} \dot{\mathcal{H}}\left((\partial_x\dot{\mathcal{H}})_0 - (\partial_x\dot{\mathcal{H}})_{-1}\right).$$

We now use the crucial observation that the normal derivatives appear in the inhomogeneous term of our geodesic variation equation. Substituting the expression

(2.4.6) for the variational field V into the integrand yields, as a summary,

$$0 \le 2 \iint_S |\nabla\dot{\mathcal{H}}|^2 - 2K|\dot{\mathcal{H}}|^2 = \int_{\gamma_-} \dot{\mathcal{H}}V - \int_{\gamma_+} \dot{\mathcal{H}}V \tag{3.2.1}$$

which we record for use both later in this section as well as in §4.

Next, extend the function V, defined on ∂S_0, to a harmonic function (also denoted V) defined on the entire cylinder S_0. (It is classical that such a Dirichlet problem is uniquely solvable.)

Remark. Our original 1998 proof involved solving explicitly for V and $\dot{\mathcal{H}}$ in terms of Fourier series on the cylinder. The referee suggested that by extending a function related to V we would be able to simplify our argument. We are grateful for his/her suggestions and encouragement for finding a proof that is less dependent on a choice of coordinates. Our version of the rest of the argument is designed to facilitate the generalization of the case of γ a simple closed curve to the case (in §4) of measured laminations.

Since $\dot{\mathcal{H}}$ is harmonic on S_0 by (3.1.5), we have that both V_{yy} and $-\frac{1}{2}\dot{\mathcal{H}}_x$ are harmonic on S_0 and agree on ∂S_0 by $(2.4.5)_0$. Thus we see that $(2.4.5)_0$ extends to hold on the entire cylinder S_0 for the extended function V. We may then differentiate $(2.4.5)_0$ with respect to x to obtain, in S_0,

$$V_{yyx} = -\frac{1}{2}\dot{\mathcal{H}}_{xx} = \frac{1}{2}\dot{\mathcal{H}}_{yy},$$

using again the harmonicity of $\dot{\mathcal{H}}$. Thus

$$\left(V_x - \frac{1}{2}\dot{\mathcal{H}}\right)_{yy} = -\left(V_x - \frac{1}{2}\dot{\mathcal{H}}\right)_{xx} = 0$$

and so for a fixed path $\{y = const\}$ the expression $V_x - \frac{1}{2}\dot{\mathcal{H}}$ is linear in x:

$$V_x - \frac{1}{2}\dot{\mathcal{H}} = C + C_1 x.$$

Differentiating in x, we get

$$C_1 = V_{xx} - \frac{1}{2}\dot{\mathcal{H}}_x = -\left(V_{yy} + \frac{1}{2}\dot{\mathcal{H}}_x\right),$$

using the harmonicity of V. The quantity on the right vanishes on the boundary of the cylinder by $(2.4.5)_0$, so we conclude that

$$V_x = \frac{1}{2}\dot{\mathcal{H}} + C,$$

for some constant C.

At this point, we still have not used the fact that the length s of the inserted cylinder remains constant in t; it is this fact which will allow us to prove that the constant C is zero:

Lemma 3.2.2. *$V_x = \frac{1}{2}\dot{\mathcal{H}}$.*

We postpone the proof of Lemma 3.2.2 until §3.3, preferring to assume it for now to finish the proof of Lemma 3.2.1.

It is standard practice when working with harmonic functions like V to integrate $V\Delta V$ by parts to obtain

$$0 = \iint_{S_0} V\Delta V = -\iint_{S_0} |\nabla V|^2 + \int_{\partial S_0} V\partial_n V.$$

Then, rearranging, converting to (x, y) coordinates, and applying Lemma 3.2.2 yields

$$0 \le \iint_{S_0} |\nabla V|^2 = -\int_{\gamma_-} VV_x + \int_{\gamma_+} VV_x = -\frac{1}{2}\int_{\gamma_-} \dot{\mathcal{H}}V + \frac{1}{2}\int_{\gamma_+} \dot{\mathcal{H}}V. \tag{3.2.2}$$

Combining (3.2.1) and (3.2.2), we see that

$$0 \le 2\iint_S |\nabla\dot{\mathcal{H}}|^2 - 2K|\dot{\mathcal{H}}|^2 = \int_{\gamma_-} \dot{\mathcal{H}}V - \int_{\gamma_+} \dot{\mathcal{H}}V \le 0. \tag{3.2.3}$$

Thus the inequalities become equalities; it follows that $\dot{\mathcal{H}}$ vanishes on S_{-1} and is constant on S_0. Yet, as $\dot{\mathcal{H}}$ is continuous, we see that $\dot{\mathcal{H}}$ vanishes everywhere on S. □

3.3. Slice condition. We have yet to use the hypothesis that the grafted cylinder has constant length s in the family $\mathrm{gr}(\sigma_t)$. Certainly it is necessary to use this hypothesis to prove Lemma 3.1.1, as Teichmüller space is $6g-6$ (real) dimensional and the space $\mathcal{T}_g \times \mathbb{R}_+$ of grafted hyperbolic metrics (up to Diff_o equivalence) is $6g-5$ (real) dimensional. Thus we might expect that the map $\mathcal{T}_g \times \mathbb{R}_+ \to \mathcal{T}_g$ which records the conformal equivalence class of an equivalence class of grafted metrics would pull back points to one-dimensional families of grafted metrics. The content of Lemma 3.1.1 is that such families would meet level sets $\mathcal{T}_g \times \{s_0\} \subset \mathcal{T}_g \times \mathbb{R}_+$ in points; thus we must somehow use the fact that we are restricted to such a level set in the proof of Lemma 3.1.1.

Let us extend the notation of §3.1 somewhat and allow the Euclidean portion of the grafted metric $\mathrm{gr}(\sigma_t)$ to be a Euclidean cylinder of length $s = s(t)$, where we permit the length $s(t)$ to vary in t. Recall that the length of the grafting curve was ℓ, and we will write $L = s(0)\ell$ for the area of the inserted cylinder at $t = 0$ in all that follows.

We begin this section by observing that, on the cylinder S_0 where $0 = \Delta\dot{\mathcal{H}} = \dot{\mathcal{H}}_{xx} + \dot{\mathcal{H}}_{yy}$, we have

$$\frac{d}{dx}\int_{x=x_0} \dot{\mathcal{H}}_x = \int_{x=x_0} \dot{\mathcal{H}}_{xx} = \int_{x=x_0} -\dot{\mathcal{H}}_{yy} = 0,$$

and so the average of $\dot{\mathcal{H}}_x$ around a meridian is independent of the particular curve $\{x = x_0\}$ around which we average. Specifying to one of the two boundary curves and applying $(2.4.5)_0$, we see that this average is zero:

$$\int_{\gamma_+} \dot{\mathcal{H}}_x = \int_{\gamma_+} -2V_{yy} = 0.$$

It follows that $\frac{d}{dx}\int_{x=x_0} \dot{\mathcal{H}} = 0$, so this integral is also independent of $\{x = x_0\}$. Let

$$d_0 = \frac{1}{\ell}\int_{\gamma_+} \dot{\mathcal{H}} \tag{3.3.1}$$

be the common average value.

In the notation of §3.2, we claim that the derivative $\frac{ds}{dt}$ arises in the following way:

Lemma 3.3.1. $\int_{\gamma_+} V - \int_{\gamma_-} V = \frac{1}{2} d_0 L + L\left(\frac{1}{s}\frac{ds}{dt}\right)$.

From this lemma, we may deduce Lemma 3.2.2 easily:

Proof of Lemma 3.2.2. We need to show that the constant $C = V_x - \frac{1}{2}\dot{\mathcal{H}}$ is zero. Integrate this equation for C over the cylinder S_0:

$$\begin{aligned} CL = \iint_{S_0} C = \iint_{S_0} V_x - \iint_{S_0} \frac{1}{2}\dot{\mathcal{H}} \\ &= \left(\int_{\gamma_+} V - \int_{\gamma_-} V\right) - \frac{1}{2} d_0 L \\ &= \frac{1}{2} d_0 L + L\left(\frac{1}{s}\frac{ds}{dt}\right) - \frac{1}{2} d_0 L \\ &= 0. \end{aligned}$$

Here the second line follows from the fundamental theorem of calculus applied to the lines $\{y = const\}$ and the definitions (3.3.1) of d_0 and $L = s\ell$, the third line follows from Lemma 3.3.1, and the last line follows from applying the hypothesis $\frac{ds(t)}{dt} = 0$. □

Proof of Lemma 3.3.1. The plan is to compute the t-derivative of the area $A(\mathrm{gr}(\sigma_t))$ of the family $\mathrm{gr}(\sigma_t)$ of grafted metrics two ways. In the first method, we use that $\mathrm{gr}(\sigma_t)$ is a piecewise homogeneous metric which is composed of a portion which is hyperbolic with geodesic boundary and a portion which is composed of a Euclidean cylinder, and so the area is computable via Gauss-Bonnet and elementary geometry. In the second method, we use the analytical formulae (3.1.2) and (3.1.5).

First method. If we remove the cylindrical portion of the grafted metric and glue the resulting hyperbolic surface-with-geodesic-boundary together across its pair of geodesic boundary components, we obtain a closed hyperbolic surface of area $-2\pi\chi$, where χ is the Euler characteristic. Thus, using that the cylinder has length $s(t)$, we find that the area $A(\mathrm{gr}(\sigma_t))$ of the grafted metric satisfies

$$A(\mathrm{gr}(\sigma_t)) = -2\pi\chi + \ell_\gamma(\mathrm{gr}(\sigma_t)) \cdot s(t)$$

so that

$$\frac{d}{dt}\Big|_{t=0} A(\mathrm{gr}(\sigma_t)) = \left(\frac{d}{dt}\Big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t))\right) s(0) + \ell_\gamma(\mathrm{gr}(\sigma_0)) \cdot \frac{d}{dt}\Big|_{t=0} s(t). \tag{3.3.2}$$

We wish to find the derivative $\frac{d}{dt}\big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t))$; here both the metric $\mathrm{gr}(\sigma_t)$ and the curve γ_t (the $\mathrm{gr}(\sigma_t)$-geodesic in the homotopy class $[\gamma]$) vary with t (see the discussion following (2.4.5)). Thus

$$\frac{d}{dt}\Big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t)) = \frac{d}{dt}\Big|_{t=0} \ell_{\gamma_t}(\mathrm{gr}(\sigma_0)) + \frac{d}{dt}\Big|_{t=0} \ell_{\gamma_0}(\mathrm{gr}(\sigma_t)).$$

We then observe that since the family γ_t of curves contains the $\mathrm{gr}(\sigma_0)$-*geodesic* γ_0, the term $\frac{d}{dt}\big|_{t=0} \ell_{\gamma_t}(\mathrm{gr}(\sigma_0))$ vanishes, and we must have that

$$\frac{d}{dt}\Big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t)) = \frac{d}{dt}\Big|_{t=0} \int_{\gamma_0} ds_{\mathrm{gr}(\sigma_t)}. \tag{3.3.3}$$

Yet the term $ds_{\mathrm{gr}(\sigma_t)}$ is computable from (3.1.2) as

$$ds_{\mathrm{gr}(\sigma_t)} = \sqrt{\frac{1}{\mathcal{H}_t}} ds_{\mathrm{gr}(\sigma_0)} + o(t),$$

recalling that the metrics $\mathrm{gr}(\sigma_t)$ and $\mathrm{gr}(\sigma_0)$ are only conformal to first order in (3.1.2). We combine this equation with (3.3.3) and differentiate to find

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t)) &= -\frac{1}{2}\int_\gamma \dot{\mathcal{H}}\mathcal{H}_0^{-3/2} ds_{\mathrm{gr}(\sigma_0)} \\ &= -\frac{1}{2}\int \dot{\mathcal{H}} ds_{\mathrm{gr}(\sigma_0)} \\ &= -\frac{d_0 \ell}{2} \end{aligned} \tag{3.3.4}$$

using $\mathcal{H}_0 \equiv 1$ and (3.3.1).

Combining (3.3.2) and (3.3.4) yields

$$\frac{d}{dt} A(\mathrm{gr}(\sigma_t)) = -\frac{1}{2} d_0 L + \ell \frac{d}{dt}\Big|_{t=0} s(t). \tag{3.3.5}$$

Second method. Formula (3.1.2) suggests another method, as the area $A(\mathrm{gr}(\sigma_t))$ may be expressed as

$$\begin{aligned} A(\mathrm{gr}(\sigma_t)) &= \iint_S dA(\mathrm{gr}(\sigma_t)) \\ &= \iint_S \frac{dA(\mathrm{gr}(\sigma_0))}{\mathcal{H}_t} + o(t). \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} A(\mathrm{gr}(\sigma_t)) &= -\iint_S \dot{\mathcal{H}}\mathcal{H}_0^{-2} dA(\mathrm{gr}(\sigma_0)) \\ &= -\iint_{S_{-1}} \dot{\mathcal{H}} dA(\mathrm{gr}(\sigma_0)) - \iint_{S_0} \dot{\mathcal{H}} dA(\mathrm{gr}(\sigma_0)) \\ &= -\iint_{S_{-1}} \dot{\mathcal{H}} dA(\mathrm{gr}(\sigma_0)) - d_0 L \end{aligned} \tag{3.3.6}$$

using again that $\mathcal{H}_0 \equiv 1$ and (3.3.1). To evaluate the first term, begin with equation (3.1.5) and integrate to find

$$0 = \iint_{S_{-1}} (\Delta_{\sigma_0} - 2)\dot{\mathcal{H}} dA_{\mathrm{gr}(\sigma_0)} = \int_{\partial S_{-1}} \partial_n \dot{\mathcal{H}} ds_{\mathrm{gr}(\sigma_0)} - 2\iint_{S_{-1}} \dot{\mathcal{H}} dA_{\mathrm{gr}(\sigma_0)},$$

where here ∂_n refers to the outward pointing normal on ∂S_{-1}. We rearrange to find

$$
\begin{aligned}
-\iint\limits_{S_{-1}} \dot{\mathcal{H}} &= -\frac{1}{2}\int_{\partial S_{-1}} \partial_n \dot{\mathcal{H}} ds_{\mathrm{gr}(\sigma_0)} \\
&= \int_{\gamma_+} \frac{1}{2}(\partial_x \dot{\mathcal{H}})_{-1} - \int_{\gamma_-} \frac{1}{2}(\partial_x \dot{\mathcal{H}})_{-1} \\
&= \int_{\gamma_+} V - \int_{\gamma_-} V
\end{aligned}
\tag{3.3.7}
$$

using $(2.4.5)_{-1}$ and the fact that the V_{yy} terms integrate around the boundary curves to zero. We combine (3.3.6) and (3.3.7) to find

$$
\frac{d}{dt}A(\mathrm{gr}(\sigma_t)) = \left(\int_{\gamma_+} V - \int_{\gamma_-} V\right) - d_0 L. \tag{3.3.8}
$$

Summary. Formulae (3.3.5) and (3.3.8) combine to yield

$$
-\frac{1}{2}d_0 L + \ell \frac{d}{dt}\Big|_{t=0} s(t) = \left(\int_{\gamma_+} V - \int_{\gamma_-} V\right) - d_0 L. \tag{3.3.9}
$$

The statement of the lemma follows immediately from (3.3.9) and the definition $L = s\ell$. ∎

§4. The general case

In this section we will prove the main theorem in the case of grafting on a measured lamination which is not necessarily a weighted simple closed curve.

Theorem A. *For any $\lambda \in \mathcal{ML}$, $\mathrm{Gr}_\lambda : \mathcal{T}_g \to \mathcal{T}_g$ is a homeomorphism.*

As in the model case (Theorem 3.1), we need only prove the local injectivity; to that end, we suppose the theorem is false and get a variation σ_t of σ_0 such that $\mathrm{Gr}_\lambda\sigma_t = \mathrm{Gr}_\lambda\sigma_0$ to first order. The harmonic map setup is the same as the model case; we isotope the grafted metrics and assume that the identity map id : $(S, \mathrm{gr}_\lambda(\sigma_t)) \to (S, \mathrm{gr}_\lambda(\sigma_0))$ is harmonic and conformal to order $o(t)$. The functions $\mathcal{H}_t$ and $\dot{\mathcal{H}}$ are defined in the usual way.

The first step in the proof, of course, is to approximate $\lambda \in \mathcal{ML}$ by a sequence of weighted simple closed curves $s_m\gamma_m \to \lambda$ and attempt to use our computations from the model case. The main difficulty is that the family of grafted metrics $\mathrm{Gr}_{s_m\gamma_m}(\sigma_t)$ can no longer be assumed conformal to $o(t)$ and we must generalize some results from §2.4 and §3 to allow for this possibility. Our method is to carry out the derivation of §3 for a single non-conformal deformation; thus we will suppress the subscripts m in the notation until section §4.5.

Our plan is as follows: in §4.1, we verify that the main equation of study (3.1.5) continues to hold even for a non-conformal deformation. Section 4.2 contains the required generalizations of the geodesic variation equations, and §4.3 generalizes the discussion of the "slice condition" from §3.3. In §4.4, we put these pieces together to obtain a bound on $\dot{\mathcal{H}}$, which is used in §4.5 to complete the proof.

Basically, the proof at this stage amounts to only a modification of the arguments in §2 and §3. We write the present section in the style of a modification, as a collection of "corrections" to the proof in the model case, where the lamination was a simple closed curve. We eventually need to argue (at the end of §4.4) that the resulting formulas are modified only to the order of non-conformality of $\mathrm{Gr}_{s_m\gamma_m}(\sigma_t)$ from $\mathrm{Gr}_{s_m\gamma_m}(\sigma_0)$, with bounds that are uniform in m; beyond simple care with rewriting the model argument to exhibit the uniformity of the bounds, this final argument relies on reinterpreting the slice condition for this general case.

Crucial to all of the following computations is the following non-conformal generalization of (3.1.2) (compare the expression preceding equation (2.3.2)):

$$\mathrm{gr}(\sigma_t) = -\Phi(t)dz_t^2 + \frac{\mathrm{gr}(\sigma_0)}{\mathcal{H}_t} - \overline{\Phi(t)}d\bar{z}_t^2 + \mathcal{O}(t^2). \tag{4.0.1}$$

Here $\Phi(t)$ is a holomorphic quadratic differential on $\mathrm{gr}(\sigma_t)$ which, as a tensor, depends differentiably on t and vanishes when $t = 0$.

4.1. The infinitesimal Bochner equation. In this section, we show that the basic global equations are unchanged.

Lemma 4.1.1. *$\dot{\mathcal{H}}$ satisfies equation (3.1.5).*

Thus our main equation of study is unchanged despite allowing the family of harmonic maps to stray from conformality.

Proof. To see this, we begin with the Bochner equation (2.3.2)

$$\Delta_t \log \mathcal{H}_t = -2K_0\mathcal{H}_t + 2K_0\frac{|\Phi(t)|^2}{\rho_t\mathcal{H}_t} + 2K_t.$$

Since the Hopf differentials vary smoothly in t, we may write the quadratic differential $\Phi(t)$ in (4.0.1) as $\Phi(t) = t\Phi_0 + \mathcal{O}(t^2)$ and then, differentiating once with respect to t at $t = 0$, we see immediately that

$$\frac{d}{dt}\Big|_{t=0} 2K_0\frac{|\Phi(t)|^2}{\rho_t\mathcal{H}_t} = 0$$

so the derivative of the right hand side becomes $-2K_0\dot{\mathcal{H}} + 2\dot{K}$. Because $\log \mathcal{H}_0 \equiv 0$, all terms of $\frac{d}{dt}\Delta_t$ vanish except those involving $\frac{d}{dt}\log \mathcal{H}_t$. Thus

$$\frac{d}{dt}\Delta_t \log \mathcal{H}_t = \Delta_0\dot{\mathcal{H}},$$

yielding (3.1.5). □

4.2. The geodesic variational vector field. From formula (4.0.1), we can compute the expressions we need in order to apply formula (2.4.1) to the present case. In particular, when we differentiate (2.4.1) in time, we observe that formula (2.4.2) (and those formulae following it) continues to be valid; we are left to evaluate $\frac{d}{dt}\big|_{t=0} \Gamma^2_{t,11}$, where we adopt Fermi coordinates for $\mathrm{gr}(\sigma_0)$. In terms of those Fermi coordinates $\{x^1, x^2\}$ for $\mathrm{gr}(\sigma_0)$ along the curve γ (here representing one of $\{\gamma_-, \gamma_+\}$), the formula (4.0.1) becomes

$$\mathrm{gr}(\sigma_t) = \frac{\mathrm{gr}(\sigma_0)}{\mathcal{H}_t} + t\phi_{ij}dx_idx_j + \mathcal{O}(t^2) \tag{4.2.1}$$

where the tensor $\phi_{ij}dx_idx_j$ may be represented as

$$\phi_{ij} = \begin{pmatrix} -2\,\mathrm{Re}\,\phi & 2\,\mathrm{Im}\,\phi \\ 2\,\mathrm{Im}\,\phi & 2\,\mathrm{Re}\,\phi \end{pmatrix} + \mathcal{O}(x_2^2)\mathcal{O}(t) + \mathcal{O}(t^2). \tag{4.2.2}$$

Here we have written $\Phi(t) = \{t\phi + \mathcal{O}(t^2)\}\left(dx_1 + i\frac{dx_2}{F(x_1,x_2)}\right)^2$, noting that $dx_1 + i\frac{dx_2}{F(x_1,x_2)}$ is a conformal coordinate up to order $\mathcal{O}(x_2^2)$. We continue to consider both $\Phi(t)$ and ϕ as small quantities, since we regard $\mathrm{gr}(\sigma_t)$ as nearly conformal to $\mathrm{gr}(\sigma_0)$ (i.e conformal to first order in t). Our expression for ϕ_{ij} clearly depends on the coordinates chosen; in our work with local expressions for $\dot{\Phi}$, we will routinely use the notation ϕ_{ij}, and we will clarify any ambiguities arising from this consistent use as they arise. In particular, we will discuss any issues of changes of values of ϕ_{ij} under analytic continuation along curves as those issues present themselves.

Thus, since $F(x_1, x_2) = 1 + \mathcal{O}(x_2^2)$, we may compute along the curve γ that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Gamma^2_{t,11} &= \frac{1}{2}\partial_2\dot{\mathcal{H}} + \frac{1}{2}\left[2\frac{\partial}{\partial x_1}(2\,\mathrm{Im}\,\phi) + \frac{\partial}{\partial x_2}(2\,\mathrm{Re}\,\phi)\right] + \mathcal{O}(x_2^2)\\ &= \frac{1}{2}\partial_2\dot{\mathcal{H}} - \frac{\partial}{\partial x_2}\mathrm{Re}\,\phi\end{aligned}$$

where we evaluate along the curve $\gamma = \{x_2 = 0\}$ and we use the Cauchy-Riemann equations for ϕ to simplify the second term. Thus, in analogy to equation (2.4.5), V satisfies the amended variational equations

$$(2.4.5)_0^\Phi \qquad V_{yy} = -\frac{1}{2}(\partial_x\dot{\mathcal{H}})_0 + (\mathrm{Re}\,\phi)_x,$$

$$(2.4.5)_{-1}^\Phi \qquad V_{yy} - V = -\frac{1}{2}(\partial_x\dot{\mathcal{H}})_{-1} + (\mathrm{Re}\,\phi)_x.$$

Note that when we subtract these equations, the $(\mathrm{Re}\,\phi)_x$ terms cancel and so equation (2.4.6) holds in the non-conformal case as well.

As a consequence of (3.1.5) and (2.4.6) remaining unchanged in this non-conformal case, the proof of (3.2.1)=$(3.2.1)^\Phi$,

$$(3.2.1)^\Phi \qquad 0 \le 2\iint_S |\nabla\dot{\mathcal{H}}|^2 - 2K|\dot{\mathcal{H}}|^2 = \int_{\gamma_-}\dot{\mathcal{H}}V - \int_{\gamma_+}\dot{\mathcal{H}}V,$$

goes through unchanged, so we may use this inequality in the present section as well. Our goal in the remainder of this section is to argue (by modifying our argument of §3) that the final term vanishes as $\|\dot{\Phi}_m(t)\|$ tends to zero as $m \to \infty$.

4.3. The slice condition. In this section, we generalize §3.3 to the (general) case of a non-conformal deformation. Here the metric $\mathrm{gr}(\sigma_t)$ is involved in the computation of the first variation of arclength (3.3.4) and in the computation of the first variation of area (3.3.6). With respect to the latter, we note from (4.0.1) that

$$dA(\mathrm{gr}(\sigma_t)) = \frac{dA(\mathrm{gr}(\sigma_0))}{\mathcal{H}_t(1 - |\nu(t)|^2)}$$

where $\nu(t) = \mathcal{O}(t)$ denotes the Beltrami differential. Thus

$$(4.3.1) \qquad dA(\mathrm{gr}(\sigma_t)) = \frac{dA(\mathrm{gr}(\sigma_0))}{\mathcal{H}_t} + \mathcal{O}(t^2).$$

In the computation of the first variation of arclength, we have from (4.2.1) and (4.2.2) that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \ell_\gamma(\mathrm{gr}(\sigma_t)) &= \frac{d}{dt}\Big|_{t=0} \int_\gamma ds_{\mathrm{gr}(\sigma_t)} \\
&= \frac{d}{dt}\int_\gamma \sqrt{\mathcal{H}_t^{-1} - 2\operatorname{Re} t\phi + \mathcal{O}(t^2)}\; ds_{\mathrm{gr}(\sigma_0)} \\
&= -\frac{1}{2}\int_\gamma (\dot{\mathcal{H}} + 2\operatorname{Re}\phi) ds_{\mathrm{gr}(\sigma_0)} \qquad (4.3.2)\\
&= -\frac{1}{2}\int_\gamma \dot{\mathcal{H}} ds_{\mathrm{gr}(\sigma_0)} + \ell\mathcal{O}(\|\dot\Phi(t)\|) \\
&= -\frac{1}{2} d_0\ell + \ell\mathcal{O}(\|\dot\Phi(t)\|).
\end{aligned}$$

The penultimate expression may require some explanation. For any given infinitesimal holomorphic quadratic differential $\dot\Phi(t)$, on the fixed compact surface $\mathrm{gr}(\sigma_0)$, the Harnack inequality bounds the supremum of $|\operatorname{Re}\phi|/\sigma_0$ over the curve γ in terms of the integral norm of $\dot\Phi(t)$. As there is but a compact set of such unit norm quadratic differentials $\dot\Phi(t)$, the result holds in general, even in a precompact family $\{\gamma_m\}$ of grafting loci.

Finally we collect terms, as in the summary (3.3.9), but with the addition of the considerations from (4.3.1) and (4.3.2), and using the definition $L = s\ell$, to find

Lemma 4.3.1. $\int_{\gamma_+} V - \int_{\gamma_-} V = \frac{1}{2} d_0 L + L\left(\frac{1}{s}\frac{ds}{dt}\right) + L\mathcal{O}(\|\dot\Phi\|)$.

4.4. The extended identity. This section will closely parallel the computation of $\dot{\mathcal{H}}$ in §3.2. We begin by noting that because (2.4.6) remains true in the non-conformal case, the proof of equation (3.2.1) continues to hold. Similarly, one can check that since $\int_{x=const}(\operatorname{Re}\phi)_x dy = \int_{x=const}(\operatorname{Im}\phi)_y dy = 0$, the additional $(\operatorname{Re}\phi)_x$ term in $(2.4.5)_0^\Phi$ integrates to zero around γ_+, and thus the proof that $\int_{\{x=x_0\}} \dot{\mathcal{H}}$ is independent of x_0 also continues to hold. (The use of the local expressions for $\dot\Phi$ may require some explanation. In particular, the validity of these arguments about integrals involving derivatives of ϕ all rely on the background assumption that the domain is a Euclidean cylinder, so the natural transformations between (x, y) patches are translations. Thus $\dot\Phi(z + i\ell)d(z + i\ell)^2 = \dot\Phi(z)dz^2$ and so $\int_{x=const}(\operatorname{Im}\phi)_y dy = 0$ in the above.)

The computation gets a bit more interesting when we harmonically extend the function V from ∂S_0 to all of S_0; while $(2.4.5)_0^\Phi$ continues to extend to S_0, upon differentiating $(2.4.5)_0^\Phi$ with respect to x, we obtain

$$\begin{aligned}
V_{yyx} &= -\frac{1}{2}\dot{\mathcal{H}}_{xx} + (\operatorname{Re}\phi)_{xx} \\
&= \frac{1}{2}\dot{\mathcal{H}}_{yy} - (\operatorname{Re}\phi)_{yy}
\end{aligned}$$

(by the harmonicity of $\dot{\mathcal{H}}$ and ϕ) from which we conclude that

$$V_x = \frac{1}{2}\dot{\mathcal{H}} - \operatorname{Re}\phi + C \qquad (4.4.1)$$

for some constant C. Once again we can identify the constant C (now possibly non-zero) from the slice condition – here we will apply Lemma 4.3.1 in place of

Lemma 3.3.1:

$$
\begin{aligned}
C = \frac{1}{L}\iint_{S_0} C &= \frac{1}{L}\iint_{S_0}\left(V_x - \frac{1}{2}\dot{\mathcal{H}} + \operatorname{Re}\phi\right)\\
&= \frac{1}{L}\left(\int_{\gamma_+} V - \int_{\gamma_-} V\right) - \frac{1}{2}d_0 + \mathcal{O}(|\operatorname{Re}\phi|)\\
&= \frac{1}{2}d_0 + \left(\frac{1}{s}\frac{ds}{dt}\right) - \frac{1}{2}d_0 + \mathcal{O}(|\operatorname{Re}\phi|)\\
&= \left(\frac{1}{s}\frac{ds}{dt}\right) + \mathcal{O}(|\operatorname{Re}\phi|).
\end{aligned}
$$

Here again, the second line follows from the Fundamental Theorem of Calculus and (3.3.1), and the third line from Lemma 4.3.1. Finally, we integrate by parts and compute as in §3.2:

$$0 = \iint_{S_0} V\Delta V = -\iint_{S_0} |\nabla V|^2 + \int_{\partial S_0} V\partial_n V.$$

Rearranging and using our computation for V_x:

$$
\begin{aligned}
0 \le \iint_{S_0} |\nabla V|^2 &= -\int_{\gamma_-} VV_x + \int_{\gamma_+} VV_x\\
&= -\int_{\gamma_-} V\left(\frac{1}{2}\dot{\mathcal{H}} - \operatorname{Re}\phi + \frac{1}{s}\frac{ds}{dt} + \mathcal{O}(|\operatorname{Re}\phi|)\right)\\
&\quad + \int_{\gamma_+} V\left(\frac{1}{2}\dot{\mathcal{H}} - \operatorname{Re}\phi + \frac{1}{s}\frac{ds}{dt} + \mathcal{O}(|\operatorname{Re}\phi|)\right)\\
&= \left(-\frac{1}{2}\int_{\gamma_-}\dot{\mathcal{H}}V + \frac{1}{2}\int_{\gamma_+}\dot{\mathcal{H}}V\right) + \left(\int_{\gamma_-} V\operatorname{Re}\phi - \int_{\gamma_+} V\operatorname{Re}\phi\right)\\
&\quad - \left(\frac{1}{s}\frac{ds}{dt} + \mathcal{O}(|\operatorname{Re}\phi|)\right)\left(\int_{\gamma_-} V - \int_{\gamma_+} V\right).
\end{aligned}
$$

We shall argue in the next section (Lemma 4.5.1) that in approximating a graft along a general lamination by grafts along simple closed curves, the slice condition translates into a generalized slice condition, i.e. a statement that

$$\frac{d}{dt}\log s = \frac{1}{s}\frac{ds}{dt} = 0.$$

Using this fact and Lemma 4.3.1 on the final term, we have, after applying (3.2.1)= $(3.2.1)^{\Phi}$,

$$
\begin{aligned}
0 \le \iint_{S} |\nabla\dot{\mathcal{H}}|^2 - 2K|\dot{\mathcal{H}}|^2 &= \frac{1}{2}\int_{\gamma_-}\dot{\mathcal{H}}V - \frac{1}{2}\int_{\gamma_+}\dot{\mathcal{H}}V\\
&\le \left(\int_{\gamma_-} V\operatorname{Re}\phi - \int_{\gamma_+} V\operatorname{Re}\phi\right) + \mathcal{O}(|\operatorname{Re}\phi|)L\left(\frac{d_0}{2} + \mathcal{O}(|\operatorname{Re}\phi|)\right). \qquad (4.4.2)
\end{aligned}
$$

We focus first on the final term, and in particular, we seek a bound on d_0. To that end, we plug equation (4.3.2) into (3.3.2) and use $L = s\ell$ to obtain

$$\frac{d}{dt}A(\operatorname{gr}(\sigma_t)) = -\frac{1}{2}d_0 L + L\left(\frac{1}{s}\frac{d}{dt}\Big|_{t=0} s(t)\right) + L\mathcal{O}(|\operatorname{Re}\phi|). \qquad (4.4.3)$$

Now, $\frac{dL}{dt} = \frac{dA}{dt}$, since $A = L - 2\pi\chi$, and we see that after we apply the generalized slice condition to eliminate the middle term in (4.4.3), we may bound d_0 by bounding $\frac{dL}{dt}$. However, the map which assigns to a measured lamination λ the length function $dL_\lambda : T\mathcal{T}_g \to \mathbb{R}$ is continuous in $\lambda \in \mathcal{ML}$ with respect to the topology of C^∞ convergence on compact subsets [Ke85]; here we identify the space of infinitesimal earthquake paths at a point in $\mathcal{T}_g$ with the tangent space to $\mathcal{T}_g$ at that point. Thus $|\frac{dL}{dt}|$ is bounded uniformly in m, say by C_1, and equation (4.4.3) implies that d_0 is bounded as well:

$$|d_0| \leq \frac{2}{L}(C_1 + L\mathcal{O}(|\operatorname{Re}\phi|)). \tag{4.4.4}$$

Thus we are left to bound the term

$$\begin{aligned}
\left|\int_{\gamma_+} V\operatorname{Re}\phi - \int_{\gamma_-} V\operatorname{Re}\phi\right| &= \left|\iint_{S_0} (V\operatorname{Re}\phi)_x\right| \\
&\leq \left|\iint_{S_0} V_x\operatorname{Re}\phi\right| + \left|\iint_{S_0} V(\operatorname{Re}\phi)_x\right| \\
&= \left|\iint_{S_0} \left(\frac{1}{2}\dot{\mathcal{H}} - \operatorname{Re}\phi + \mathcal{O}(|\operatorname{Re}\phi|)\right)\operatorname{Re}\phi\right| \\
&\quad + \left|\iint_{S_0} V(\operatorname{Re}\phi)_x\right| \quad \text{by (4.4.1)} \\
&\leq \frac{1}{2}L||\dot{\mathcal{H}}||_\infty||\operatorname{Re}\phi||_\infty + L\mathcal{O}(||\operatorname{Re}\phi||_\infty^2) + L||V||_\infty||\operatorname{Re}\phi||_{C^1}.
\end{aligned} \tag{4.4.5}$$

Now Lemma 2.3.5 shows that the Lipschitz norm of $\dot{\mathcal{H}}$ is bounded; indeed, given that the proofs of Lemma 2.3.5 and its prerequisite Lemma 2.3.4 depend only on the $C^{1,1}$ norms of $\operatorname{gr}(\sigma_t)$, and by Lemma 2.3.1 these norms are uniformly bounded over our family $\operatorname{Gr}_{s_m\gamma_m}(\sigma_t)$, we see that $||\dot{\mathcal{H}}||_{0,1}$ is bounded uniformly in m, as well. An L^∞ bound follows because the Lipschitz (hence continuous) function is defined on a compact set.

To get a bound on V we use (2.4.6) (which we recall remains true even in the general quasiconformal case):

$$V = -\frac{1}{2}\left((\partial_x\dot{\mathcal{H}})_0 - (\partial_x\dot{\mathcal{H}})_{-1}\right)$$

from which the regularity of $\dot{\mathcal{H}}$ (Lemma 2.3.5) gives

$$||V||_\infty \leq C(S_0), \tag{4.4.6}$$

uniformly in m.

Combining (4.4.2), (4.4.4), (4.4.5) and (4.4.6), we conclude

$$\iint_S |\nabla\dot{\mathcal{H}}|^2 - 2K|\dot{\mathcal{H}}|^2 \leq \left|\int_{\gamma_-} \dot{\mathcal{H}}V - \int_{\gamma_+} \dot{\mathcal{H}}V\right| \leq L\mathcal{O}(|\operatorname{Re}\phi|) \tag{4.4.7}$$

where the bound is uniform in m because, as remarked above, the length function L is a continuous function on $\mathcal{ML}$.

4.5. Conclusion of the proof.

Proof of Theorem A. We have already established (in §3) the result in the case where λ is a lamination supported on a finite set $\{\gamma_1, \ldots, \gamma_k\}$ of disjoint simple closed curves. For the general case consider a sequence $\langle s_m\gamma_m\rangle$ of measured laminations supported on simple closed curves of lengths $\ell_m = \ell(\gamma_m)$ which approximates λ. In our notation, s_m denotes the multiple of the transverse (intersection) measure for the geodesic γ_m, so we may express the transverse measure for λ as $\lim_{m\to\infty} s_m i(\cdot, \gamma_m)$ (see the discussion in §2.2).

In analogy with the opening of the proof of Lemma 3.1.1, we suppose (in order to obtain a contradiction) that there is a family of surfaces σ_t so that

$$\mathrm{Gr}_\lambda(\sigma_t) = \mathrm{Gr}_\lambda(\sigma_0) + o(t) \tag{4.5.1}$$

in $\mathcal{T}_g$. We then consider the family $\mathrm{Gr}_{s_m\gamma_m}(\sigma_t)$: it is of course no longer necessary that $\mathrm{Gr}_{s_m\gamma_m}(\sigma_t)$ should equal $\mathrm{Gr}_{s_m\gamma_m}(\sigma_0)$ to $o(t)$, but the condition (4.5.1) should be asymptotically true in m, by construction. This implies that the Hopf differential (which we will denote as $\Phi_m(t)$ and which measures the quasiconformality between $\mathrm{Gr}_{s_m\gamma_m}(\sigma_t)$ and $\mathrm{Gr}_{s_m\gamma_m}(\sigma_0)$) should have first variation $\frac{d}{dt}\big|_{t=0} \Phi_m(t)$ in t which tends to zero as $m \to \infty$. We write

$$\lim_{m\to\infty} \frac{d}{dt}\Big|_{t=0} \Phi_m(t) = 0$$

which implies that the terms $L\mathcal{O}(\|\phi\|) \to 0$, since L is bounded.

In the derivation of (4.4.7) we used the following:

Lemma 4.5.1. $\frac{d}{dt}\big|_{t=0} \log(s(t)) = 0.$

Proof. We simply need to interpret the "slice condition" for λ correctly. In particular, suppose we allow the transverse measure of λ to vary as our grafted metrics vary, i.e. let $\lambda_t = \rho_t\lambda$ with ρ_t differentiable and $\rho_0 = 1$. The slice condition is given by

$$\dot\rho = \frac{d}{dt}\Big|_{t=0} \rho_t = 0.$$

Now we can approximate λ_t by scaling the weighted simple closed curves $\{s_m\gamma_m\}$ in exactly the same way:

$$\lambda_t = \lim_{m\to\infty} \rho_t s_m \gamma_m.$$

Having done so, we have $s(t) = \rho_t s_m$ in the calculations above, and $0 = \dot\rho = \frac{\dot s}{s_m}$. □

Conclusion of the Proof of Theorem A. Having proved this lemma, we may now apply (4.4.7) to see $\dot{\mathcal{H}}_m = \mathcal{O}(\|\dot\Phi_m(t)\|) \to 0$ as $m \to \infty$ (where we have finally added subscripts to $\dot{\mathcal{H}}$ and $\Phi(t)$ to emphasize the dependence of these quantities upon the approximating sequence). A discussion analogous to that at the conclusion of §3.1 then shows that $\sigma_{t,m}$ agrees with $\sigma_{0,m}$ to $o_m(1) + o(t)$ (where $o_m(1)$ goes to zero as m tends to infinity). As the metrics $\sigma_{t,m}$, $\sigma_{0,m}$, σ_t, and σ_0 are all contained in a single precompact neighborhood of Teichmüller space, we may conclude that σ_t agrees with σ_0 to $o(t)$, as required.

Remark. We have unnecessarily restricted ourselves to unpunctured surfaces, primarily for notational simplicity and expositional cleanliness. The proofs all extend to the punctured case once we make three observations: (i) all of the measured laminations under consideration avoid a uniform neighborhood of the punctures,

(ii) there is a unique harmonic map of finite energy between surfaces of bounded non-positive curvature and some negative curvature [Al64], [Wo91b], and (iii) the holomorphic energy function $\mathcal{H}$ for such a map is bounded in C^1 across the punctures, so no new non-vanishing boundary terms would arise in the integrations by parts preceding formulae (3.2.1) and (3.2.2).

§5. Applications

5.1. Geometric coordinates on the Bers slice. Consider the Bers slice

$$B_Y = \mathcal{T}_g \times \{Y\} \subset \mathcal{T}_g \times \mathcal{T}_g \cong QF.$$

There is a natural map $\beta : B_Y \to \mathcal{ML}$ which assigns to a quasi-Fuchsian group the bending lamination λ on the component of the convex hull boundary facing the fixed structure Y (continuity of β is proved in [KS95]). The hyperbolic structure on this component of the convex hull boundary defines a point μ in Teichmüller space, and the relevant observation is that $Y = \mathrm{Gr}_\lambda(\mu)$. Theorem A shows that the metric μ is determined by Y and λ; therefore since the Thurston homeomorphism $\Theta : \mathcal{ML} \times \mathcal{T}_g \to \mathcal{P}_g$ (described in the introduction, §1) is one-one, the map β is also one-one. Invariance of domain allows one then to conclude that β is a homeomorphism onto its image. This is a simple way of assigning "bending coordinates" to B_Y.

Corollary 5.1.1. *Let B_Y be a Bers slice with fixed conformal structure Y. Then the map assigning the bending lamination on the component of the convex hull boundary facing Y is a homeomorphism onto its image.*

5.2. Generalized Bers slices. Deformation spaces of books of I-bundles. Any geometrically finite, freely indecomposable Kleinian group G has a space $\mathcal{T}(G)$ of quasiconformal deformations parametrized by the product $\prod T(S_i)$ of the Teichmüller spaces $T(S_i)$ of its boundary components $\{S_1, \dots, S_K\}$ at infinity [Ma74]. As in the example of the quasi-Fuchsian groups above, one can define slices $\mathcal{S}(t_1, \dots, t_{K-1})$ of these deformation spaces by simply fixing, say, the conformal structures at infinity of the first $K-1$ boundary components, and letting the last conformal structure vary over its Teichmüller space $T(S_K)$.

Let us focus our attention on a class of geometrically finite three-manifolds homeomorphic to the interior of a book of I-bundles. The deformation spaces of these three-manifolds are studied in detail in [AC96] and are important because of the discovery by Anderson and Canary that the closures of those deformation spaces exhibit previously unexpected phenomena. The simplest of these manifolds has the following description. Begin with a solid torus with three disjoint parallel annuli on the boundary; here we choose the annuli so that their central curve is homotopic within the solid torus to the core curve of the solid torus. Attach, along those annuli, thickenings $\{H_1, H_2, H_3\}$ of one-holed surfaces $\{M_1, M_2, M_3\}$ of genera h_1, h_2, and h_3 (respectively). The new three-manifold N has boundary surfaces $\{S_1, S_2, S_3\}$ of genera $h_1 + h_2, h_2 + h_3$, and $h_3 + h_1$; indeed, these bounding surfaces S_1, S_2, and S_3 are obtained by gluing M_i to M_{i+1} (with cyclic indexing) along the single boundary $\gamma = \partial M_i = \partial M_{i+1}$. Because all the thickenings of the surfaces are glued along neighborhoods of curves which retract to the core curve, we see that all of the curves γ_i are homotopic to each other and to the core curve γ of the central solid torus.

Now consider the space $\mathcal{T}(G)$ of quasi-conformal deformations of the Kleinian group G obtained as the holonomy of the hyperbolization of this three-manifold N.

We consider the slice $\mathcal{S}(t_1,t_2) \subset \mathcal{T}(G) = T(S_1) \times T(S_2) \times T(S_3)$ defined by the coordinate description

$$\mathcal{S}(t_1,t_2) = \{t_1\} \times \{t_2\} \times T(S_3).$$

We then consider a map $\beta_{AC} : \mathcal{S}(t_1,t_2) \to \mathcal{ML} \times \mathcal{ML}$, analogous to the map $\beta : B_Y \to \mathcal{ML}$ above, which assigns to an element $(t_1,t_2,t_3) \in \mathcal{S}(t_1,t_2)$ the pair of bending measures of the boundary components (C_1 and C_2) of the convex hulls facing the conformal structures at infinity represented by t_1 and t_2. Our application of Theorem A is the following

Corollary 5.2.1. *The map $\beta_{AC} : \mathcal{S}(t_1,t_2) \to \mathcal{ML}(S_1) \times \mathcal{ML}(S_2)$ is injective.*

Proof. Suppose $\beta_{AC}(t_1,t_2,t_3) = \beta_{AC}(t_1,t_2,t_3')$. Then by Theorem A, not only do the bending measures on the convex hull boundary components C_i and C_i' coincide ($i = 1, 2$, facing the ends t_i and t_i', respectively), but so do the hyperbolic structures. Lift to the quasi-Fuchsian covers $\mathcal{Q}_i$ and $\mathcal{Q}_i'$ of (t_1,t_2,t_3) and (t_1,t_2,t_3') corresponding to the surfaces C_i and C_i' and observe that these are identical by Corollary 5.1.1. Thus the holonomy representations of $\pi_1(C_i)$ are conjugate to those of $\pi_1(C_i')$. But as there is a common element γ in $\pi_1(C_1) \subset \pi_1(N)$ and $\pi_1(C_2) \subset \pi_1(N)$, we see that the pair of representations of $\pi_1(S_1) *_{[\gamma]} \pi_1(M_3) = \pi_1(N(t_1,t_2,t_3))$ (in the obvious notation) are conjugate. Thus $[(t_1,t_2,t_3)] = [(t_1,t_2,t_3')] \in \mathcal{T}(G)$, as desired. $\square$

5.3. $2+1$ **de Sitter spacetimes.** We finish with an application to the structure of (2+1)-dimensional de Sitter spacetimes, following [Sc99]. Recall that 3-dimensional de Sitter space is defined to be the set of unit spacelike vectors in Minkowski space:

$$\mathbb{S}^3_1 = \{v \in \mathbb{R}^4_1 | \langle v, v\rangle = +1\}.$$

This is the model space for Lorentzian 3-manifolds of constant positive curvature. Projectivizing $\mathbb{R}^4_1$ to $\mathbb{R}P^3$, we get the Klein model of hyperbolic space from the unit timelike vectors, the sphere at infinity S^2_∞ from the light cone, and a projective model of $\mathbb{S}^3_1$ as the remainder of $\mathbb{R}P^n$. Taking polar duals with respect to the sphere at infinity gives a correspondence between points in the projectivized de Sitter space and planes in hyperbolic space (and thus with round circles on S^2_∞).

Now imagine a projective structure on a closed hyperbolic surface S close to a Fuchsian structure (the construction works for any projective structure but is easiest to describe for the quasi-Fuchsian case). Using the polarity mentioned above, the set of all closed round balls contained within $dev(\tilde{S})$ defines a certain open subset $\mathcal{U}$ of $\mathbb{S}^3_1$. The holonomy $hol(\pi_1(S))$ acts discontinuously on $\mathcal{U}$ and the quotient is a de Sitter spacetime homeomorphic to $S \times \mathbb{R}$. Any example arising from a projective structure on S in this way is called a *standard de Sitter spacetime.* Standard de Sitter spacetimes are well behaved from the point of view of causality – in particular, we can choose the product structure so that each slice $S \times \{t\}$ is spacelike and every timelike or lightlike curve crosses $S \times \{t\}$ exactly once (we say $S \times \mathbb{R}$ is a *domain of dependence*).

The main result of [Sc99] is that every de Sitter spacetime $S \times \mathbb{R}$ which is a domain of dependence embeds in a standard de Sitter spacetime. Now suppose we have an example coming from a projective structure with Thurston coordinates $(\lambda, \sigma) \in \mathcal{ML} \times \mathcal{T}_g$. A domain of dependence has a well-defined *causal horizon*; it follows easily that the causal horizon corresponds to the space of *maximal* open round balls, which is in turn isometric to the $\mathbb{R}$-tree dual to λ [Sc99].

We are now able to refine our classification of de Sitter spacetimes, by providing coordinates in terms of naturally-arising data in the future (the future causal horizon) and the past (the conformal structure on S at past infinity). More precisely, we have the following reworking of Theorem A:

Corollary 5.3.1. *Let $\lambda \in \mathcal{ML}$ be a measured lamination with dual $\mathbb{R}$-tree $\tilde{\lambda}$. Let $\mathbb{S}^3_1(S;\tilde{\lambda})$ be the family of standard de Sitter spacetimes with future causal horizon $\tilde{\lambda}$, and define a map $c_\infty : \mathbb{S}^3_1(S;\tilde{\lambda}) \to \mathcal{T}_g$ which assigns the conformal structure on S at past infinity. Then c_∞ is one-one.*

Proof. By definition, any two standard de Sitter spacetimes $\mathcal{M}_i$ $(i = 1, 2)$ in $\mathbb{S}^3_1(S;\tilde{\lambda})$ come from projective structures on S (say with Thurston coordinates (λ, σ_i)). By examining the construction above, we have

$$c_\infty(\mathcal{M}_i) = \mathrm{Gr}_\lambda(\sigma_i).$$

Because Gr_λ is one-one (Theorem A), c_∞ is also one-one. $\square$

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References

[Al64] S.I. Al'ber, *On n-dimensional Problems in the Calculus of Variations*, Sov. Math. Dokl. **5** (1964), 700–704.

[AC96] J. W. Anderson and R. D. Canary, *Algebraic limits of Kleinian groups which rearrange the pages of a book*, Invent. Math. **126** (2) (1996), 205–214. MR **97h:**57025

[Ap88] B. N. Apanasov, *The geometry of Nielsen's hull for a Kleinian group in space and quasiconformal mappings*, Ann. Global Anal. Geom. **6** (3) (1988), 207–230. MR **90k:**57046

[Be77] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977. MR **58:**7671

[Be66] L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134. MR **33:**273

[Bo88] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988), 139–162. MR **90a:**32025

[Ea81] C. J. Earle, *On variation of projective structures*, in Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, 1981, pp. 87–99. MR **83a:**32017

[EF01] J. Eells and B. Fuglede, *Harmonic Maps Between Riemannian Polyhedra*, Cambridge University Press, Cambridge, 2001.

[EL81] J. Eells and L. Lemaire, *Deformations of metrics and associated harmonic maps*, Proc. Indian Acad. Sci. **90** (1) (1981), 33–45. MR **83g:**58013

[EM87] D. B. A. Epstein and A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces*, in Analytical and Geometric Aspects of Hyperbolic Space, London Math. Soc. Lecture Note Ser., vol. 111, Cambridge Univ. Press, Cambridge, 1987, pp. 113–254. MR **89c:**52014

[Fa83] G. Faltings, *Real projective structures on Riemann surfaces*, Compositio Math. **48** (2) (1983), 223–269. MR **85d:**32046

[GKM00] D. M. Gallo, M. E. Kapovich, and A. Marden, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Ann. of Math. **151** (2000), 625–704.

[GT83] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grund. Math. Wiss., vol. 224, Springer-Verlag, New York-Berlin-Heidelberg, 1983. MR **86c:**35035

[Go87] W. M. Goldman, *Projective structures with Fuchsian holonomy*, J. Differential Geom. **25** (1987), 297–326. MR **88i:**57006

[Go88] ———, *Geometric structures on manifolds and varieties of representations*, in Geometry of Group Representations, Contemp. Math., vol. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 169-197. MR **90i:**57024

[GS92] M. L. Gromov and R. M. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 165–246. MR **94e:**58032

[Gu81] R. C. Gunning, *Affine and projective structures on Riemann surfaces*, in Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, 1981, pp. 225–244. MR **83g:**30054

[He75] D. A. Hejhal, *Monodromy groups and linearly polymorphic functions*, Acta Math. **135** (1975), 1–55. MR **57:**3380

[HK98] C. D. Hodgson and S. P. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom. **48** (1) (1998), 1–59. MR **99b:**57030

[Jo97] J. Jost, *Compact Riemann Surfaces*, Springer-Verlag, New York-Berlin-Heidelberg, 1997. MR **2000k:**32011

[KT92] Y. Kamishima and S. P. Tan, *Deformation spaces on geometric structures*, in Aspects of low-dimensional manifolds, Adv. Stud. Pure Math., vol. 20, Kinokuniya, Tokyo, 1992, pp. 263–299. MR **94k:**57023

[KS95] L. Keen and C. M. Series, *Continuity of convex hull boundaries*, Pacific J. Math. **168** (1) (1995), 183–206. MR **96d:**30055

[Ke85] S. P. Kerckhoff, *Earthquakes are analytic*, Comm. Math. Helv. **60** (1) (1985), 17–30. MR **86m:**57014

[Kl33] F. Klein, *Vorlesungen Über die Hypergeometrische Funktion*, Springer-Verlag, Berlin, 1933.

[Kr69] I. Kra, *Deformations of Fuchsian groups*, Duke Math. J. **36** (1969), 537–546. MR **42:**491

[Kr71] ———, *Deformations of Fuchsian groups, II*, Duke Math. J. **38** (1971), 499–508. MR **44:**2951

[KM81] I. Kra and B. Maskit, *Remarks on projective structures*, in Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, 1981, pp. 343–359. MR **83f:**30042

[KP94] R. S. Kulkarni and U. Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z. **216** (1994), 89–129. MR **95b:**53017

[La92] F. Labourie, *Surfaces convexes dans l'espace hyperbolique et* $\mathbb{C}P^1$*-structures*, J. London Math. Soc. **45** (1992), 549–565. MR **93i:**53062

[Ma74] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. **99** (1974), 383–462. MR **50:**2485

[Ma69] B. Maskit, *On a class of Kleinian groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **442** (1969), 1–8. MR **40:**5857

[MV94] K. Matsuzaki and J. A. Velling, *Notes on projective structures and Kleinian groups*, Osaka J. Math. **31** (1) (1994), 165–175. MR **95c:**30054

[Mc98] C. T. McMullen, *Complex earthquakes and Teichmüller theory*, J. Amer. Math. Soc. **11** (2) (1998), 283–320. MR **98i:**32030

[Na88] S. Nag, *The complex analytic theory of Teichmüller spaces*, Wiley, New York, 1988. MR **89f:**32040

[Ne49] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551. MR **10:**696e

[Sa78] J. H. Sampson, *Some properties and applications of harmonic mappings*, Ann. Sci. École Norm. Sup. **11** (2) (1978), 211–228. MR **80b:**58031

[Sc99] K. P. Scannell, *Flat conformal structures and the classification of de Sitter manifolds*, Comm. Anal. Geom. **7** (2) (1999), 325–345. MR **2000a:**53122

[Sc84] R. M. Schoen, *Analytic aspects of the harmonic map problem*, in Seminar on nonlinear partial differential equations, Math. Sci. Res. Inst. Publ., vol. 2, Springer-Verlag, New York-Berlin-Heidelberg, 1984, pp. 321–358. MR **86b:**58032

[Sh87] H. Shiga, *Projective structures on Riemann surfaces and Kleinian groups*, J. Math. Kyoto Univ. **27** (3) (1987), 433–438. MR **88k:**30056

[ST99] H. Shiga and H. Tanigawa, *Projective structures with discrete holonomy representations*, Trans. Amer. Math. Soc. **351** (2) (1995), 813–823. MR **99d:**32025

[ST83] D. P. Sullivan and W. P. Thurston, *Manifolds with canonical coordinates: some examples*, Enseign. Math. **29** (1983), 15–25. MR **84i:**53035

[Ta97] H. Tanigawa, *Grafting, harmonic maps, and projective structures on surfaces*, J. Differential Geom. **47** (3) (1997), 399–419. MR **99c:**32029

[Th82] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton Univ., Princeton, 1982.

[Wo91a] M. Wolf, *High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space*, Topology **30** (4) (1991), 517–540. MR **92j:**32075

[Wo91b] ———, *Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli space*, J. Differential Geom. **33** (1991), 487-539. MR **92b:**58055

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