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THE PLANAR CANTOR SETS OF ZERO ANALYTIC CAPACITY AND THE LOCAL $T(b)$-THEOREM

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1. Introduction

In this paper we characterize the planar Cantor sets of zero analytic capacity. Our main result answers a question of P. Mattila [Ma] and completes the solution of a long-standing open problem with a curious history, which goes back to 1972. We refer the reader to [I2, p. 153] and [Ma] for more details. Moreover, we confirm a conjecture of Eiderman [E] concerning the analytic capacity of the N-th approximation of a Cantor set.

Before formulating our main results, we recall the definition of the basic objects involved.

The analytic capacity of a compact subset E of the complex plane $\mathbb{C}$ is

$$\gamma(E) = \sup |f'(\infty)|, \tag{1}$$

where the supremum is taken over all analytic functions f on $\mathbb{C}\setminus E$ such that $|f| \leqslant 1$ on $\mathbb{C}\setminus E$. Although there has recently been important progress on our understanding of analytic capacity (see the survey papers [D], [V3] and the references given there), many basic questions about γ remain unanswered. One of the oldest is the semi-additivity problem, that is, the problem of showing the existence of an absolute constant C such that

$$\gamma(E \cup F) \leqslant C\,\{\gamma(E) + \gamma(F)\}\,, \tag{2}$$

for all compact sets E and F. If (2) were true, then one would have powerful geometric criteria for rational approximation, which are otherwise missing (see [V2] and [Vi2]).

On the other hand, it has recently been established that a close variant of γ, called positive analytic capacity, is indeed semi-additive. The positive analytic capacity of a compact set E is

$$\gamma^+(E) = \sup \mu(E),$$

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where the supremum is taken over the positive Borel measures μ supported on E such that the Cauchy potential $f = \frac{1}{z} * \mu$ is a function in $L^\infty(\mathbb{C})$ with $\|f\|_\infty \leqslant 1$. Since $(\frac{1}{z} * \mu)'(\infty) = \mu(E)$, we clearly have $\gamma^+(E) \leqslant \gamma(E)$.

Since γ^+ is semi-additive, as shown in [T1], it is clear that (2) follows from the inequality

$$\gamma(E) \leqslant C\gamma^+(E), \quad E \text{ compact } \subset \mathbb{C}, \tag{3}$$

where the positive constant C does not depend on E. We will see below that our main result provides a proof of (3) for a particular (but significant) class of sets E. To the best of our knowledge, the first mention of (3) that can be found in the literature is in [DO]. An equivalent form of (3), which involves Menger curvature, has recently been conjectured by Melnikov (see [D]).

Now we turn our attention to Cantor sets. Given a sequence $\lambda = (\lambda_n)_{n=1}^\infty$, $0 \leqslant \lambda_n \leqslant 1/3$, we construct a Cantor set by the following algorithm. Consider the unit square $Q^0 = [0,1] \times [0,1]$. At the first step we take four closed squares inside Q^0, of side-length λ_1, with sides parallel to the coordinate axes, such that each square contains a vertex of Q^0. At step 2 we apply the preceding procedure to each of the four squares produced at step 1, but now using the proportion factor λ_2. Then we obtain 16 squares of side-length $\sigma_2 = \lambda_1\lambda_2$. Proceeding inductively, we have at the n-th step 4^n squares Q_j^n, $1 \leqslant j \leqslant 4^n$, of side-length $\sigma_n = \prod_{j=1}^n \lambda_j$. Write

$$E_n = E(\lambda_1, \dots, \lambda_n) = \bigcup_{j=1}^{4^n} Q_j^n,$$

and define the Cantor set associated with the sequence $\lambda = (\lambda_n)_{n=1}^\infty$ by the identity

$$E = E(\lambda) = \bigcap_{n=1}^\infty E_n.$$

Our main result reads as follows.

Theorem 1. *The Cantor set $E(\lambda)$ has zero analytic capacity if and only if*

$$\sum_{n=1}^\infty \frac{1}{(4^n\sigma_n)^2} = \infty.$$

The assumption $\lambda_n \leqslant 1/3$ for the Cantor sets $E(\lambda)$ is purely technical. Actually Theorem 1 (as well as Theorem 2 below) holds for any sequence $(\lambda_n)_n$ with $0 < \lambda_n < 1/2$. See Remark 2 at the end of the paper for more details.

Mattila showed in [Ma] that the above condition is necessary and our contribution in this paper is to prove the sufficiency. The special case $\lambda_n = 1/4$, $n \geqslant 1$, was obtained independently by Garnett [G1] and Ivanov [I1] in the 1970's and, since then, the "corner quarters" Cantor set has become the favorite example of a set of zero analytic capacity and positive length. P. Jones gave in [J] an alternative proof of Garnett's result, based on harmonic measure. Recently Jones' approach has been used to establish the vanishing of the analytic capacity of $E(\lambda)$ for some special classes of sequences $\lambda = (\lambda_n)_{n=1}^\infty$ with $4^n\sigma_n$ tending to infinity [GY].

Theorem 1 follows from a more precise result on the analytic capacity of the set $E_N = E(\lambda_1, \dots, \lambda_N)$. The asymptotic behaviour of $\gamma^+(E_N)$ is completely understood: for some constant $C > 1$ and all $N = 1, 2, \dots$ one has

$$C^{-1}\Big(\sum_{n=1}^{N}\frac{1}{(4^n\sigma_n)^2}\Big)^{-1/2} \leqslant \gamma^+(E_N) \leqslant C\Big(\sum_{n=1}^{N}\frac{1}{(4^n\sigma_n)^2}\Big)^{-1/2}. \tag{4}$$

The upper estimate is due to Eiderman [E] and a different proof has been given in [T2]. The lower estimate was proved by Mattila in [Ma]. However, the result was not explicitly stated in [Ma], presumably because at that time the main object of interest was γ rather than γ^+. An indication of how one proves the lower estimate in (4) will be provided at the end of Section 2. See also [E, p. 821].

Theorem 1 follows from the upper estimate in (4) and the next result.

Theorem 2. *There exists a positive constant C_0 such that*

$$\gamma(E_N) \leqslant C_0\gamma^+(E_N),\ N = 1, 2, \dots. \tag{5}$$

If $\lambda_n = 1/4$, $n \geqslant 1$, then combining Theorem 2 with (4), we get

$$\gamma(E_N) \leqslant \frac{C}{\sqrt{N}},\ N = 1, 2, \dots,$$

which improves considerably Murai's inequality [Mu]

$$\gamma(E_N) \leqslant \frac{C}{\log N},\ N = 2, 3, \dots,$$

the best estimate known up to now.

The main tool used in our proof of Theorem 2 is the local $T(b)$-Theorem of M. Christ [CH2], a particular version of which will be discussed and stated in Section 2. Section 2 also contains some basic facts on the Cauchy transform and the Plemelj formulae. The proof of Theorem 2 is presented in Section 3.

Our notation and terminology are standard. For example $D(z, r)$ is the open disk centered at z and of radius r, ds is the arclength measure on a rectifiable arc and $P \simeq Q$ means that $C^{-1}Q \leqslant P \leqslant CQ$ for some absolute constant $C > 1$.

The symbols $C, C', C'', C_0, C_1, \dots$ stand for absolute constants with a definite value. We will also use the symbol A to denote an absolute constant that may vary at different occurrences.

Remark 1. The second author [T3] has recently proved that Theorem 2 also holds for a general compact set. In particular, this implies the semi-additivity of analytic capacity. The proof in [T3] also involves an induction argument and an appropriate $T(b)$-Theorem as in the present paper.

2. Background results

2.1. **Cauchy integrals.** Fix an integer $M > 0$ and let $E_M = E(\lambda_1, \dots, \lambda_M)$ be the M-th approximation of the Cantor set associated with the sequence $(\lambda_n)_{n=1}^{\infty}$. Then E_M is the union of 4^M closed squares Q_j^M, $1 \leqslant j \leqslant 4^M$, and ∂E_M is the union of the 4^M closed piecewise linear curves ∂Q_j^M. For a Borel measure μ supported on ∂E_M set

$$C(\mu)(z) = \int \frac{d\mu(\zeta)}{\zeta - z}, \quad z \notin \partial E_M,$$

and

$$C(\mu)(z) = \lim_{\varepsilon\to 0}\int_{|\zeta - z|>\varepsilon}\frac{d\mu(\zeta)}{\zeta - z}, \quad z \in \partial E_M,$$

whenever the principal value integral exists. Let $C^+(\mu)(z)$ (respectively, $C^-(\mu)(z)$) stand for the non-tangential limit of $C(\mu)(w)$ as w tends to z from the interior of E_M (respectively, from the complement of E_M). It follows from standard classical results that $C\mu(z)$, $C^+\mu(z)$ and $C^-\mu(z)$ exist for almost all z with respect to arclength measure ds on ∂E_M. Moreover one has the Plemelj formulae (see [V3])

$$\begin{cases} C^+\mu(z) = C\mu(z) + \pi i f(z), \\ C^-\mu(z) = C\mu(z) - \pi i f(z), \end{cases} \tag{6}$$

where the identities hold for ds-almost all $z \in \partial E_M$ and $\mu = f(z)dz + \mu_s$, f being integrable and μ_s being singular with respect to ds.

Assume that one has $\mu_s = 0$ and

$$|f(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M,$$

and that one wants to show

$$|C(\mu)(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M. \tag{7}$$

Then one only has to check that

$$|C(\mu)(z)| \leqslant A, \text{ for } z \notin E_M,$$

because then

$$|C^-(\mu)(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M,$$

and thus the second identity in (6) gives (7).

2.2. **The local $T(b)$-Theorem.** The local $T(b)$-Theorem is a criterion for the L^2 boundedness of a singular integral that was proved originally by M. Christ in the setting of homogeneous spaces [CH2]. We state below a very particular version of Christ's result, which is adapted to the principal value Cauchy integral and to a measure μ supported on ∂E_M. The reader may think that μ is of the form $\mu = c ds_{|\partial E_M}$ for some (small) positive constant c. However, one should keep in mind that, for $4^n\sigma_n \nearrow \infty$, $ds_{|\partial E_M}$ does not satisfy condition (i) in the statement below with a constant independent of M. As will become clear later, an appropriate choice of c is required to get (i) and (iii) with absolute constants.

Theorem (Christ)**.** *Let μ be a positive Borel measure supported on ∂E_M satisfying, for some absolute constant C, the following conditions:*

(i) $\mu(D(z,r)) \leqslant Cr$, $z \in \partial E_M$, $r > 0$.
(ii) $\mu(D(z,2r)) \leqslant C\mu(D(z,r))$, $z \in \partial E_M$, $r > 0$.
(iii) *For each disc D centered at a point in ∂E_M there exists a function b_D in $L^\infty(\mu)$, b_D supported on D, satisfying $|b_D| \leqslant 1$ and $|C(b_D\mu)| \leqslant 1$ μ-almost everywhere on ∂E_M, and $\mu(D) \leqslant C\left|\int b_D d\mu\right|$.*

Then

$$\int |C(f\mu)|^2 \, d\mu \leqslant C' \int |f|^2 d\mu, \quad f \in L^2(\mu), \tag{8}$$

for some absolute constant C' (depending only on C).

The relevance of inequality (8) for our problem lies in the fact that it implies

$$\mu(K) \leqslant C''\gamma^+(K), \quad K \text{ compact } \subset \partial E_M, \tag{9}$$

for some absolute constant C'' (depending only on C').

The derivation of (9) from (8) goes through a well-known path: first, by classical Calderón-Zygmund theory one gets a weak $(1,1)$ inequality from (8); then, a surprisingly simple method to dualize a weak $(1,1)$ inequality leads immediately to (9). The original argument is in [DO]. Some years before [DO] Uy found a slightly different way of dualizing a weak $(1,1)$ inequality, which, however, does not yield (9) (see [Uy]). The interested reader will find additional information in [CH1], [T1] and [V1].

Inequality (9) also explains why the lower estimate in (4) follows from Mattila's arguments in [Ma]; see Theorem 3.7 on p. 202 and the first paragraph after it.

3. Proof of Theorem 2

We first give a sketch of the argument. Assume that one can find a positive Borel measure μ supported on ∂E_N, $E_N = E(\lambda_1, \dots, \lambda_N)$, which satisfies (i) and (ii) with M replaced by N, such that $\|\mu\| = \gamma(E_N)$ and the Cauchy integral is bounded on $L^2(\mu)$. Then we get (9) with M replaced by N, as we explained in Section 2. For $K = \partial E_N$ (9) yields

$$\gamma(E_N) = \|\mu\| \leqslant A\gamma^+(E_N),$$

as desired. In the actual argument we do not construct μ on E_N. For reasons that will become clear later, we are forced to work in E_M with M smaller than N. On the other hand, M cannot be much smaller than N, because in the course of the subsequent reasoning one needs to have $\gamma^+(E_M) \leqslant A\gamma^+(E_N)$. Hence M has to be chosen carefully, in such a way that the local $T(b)$-Theorem can be applied to get the boundedness of the Cauchy integral on $L^2(\mu)$, with an absolute constant.

Now we start the proof of Theorem 2.

Set $a_n = 4^n \sigma_n$ and

$$S_n = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}.$$

We can assume, without loss of generality, that for each $N > 1$, there exists M, $1 \leqslant M < N$, such that

$$S_M \leqslant \frac{S_N}{2} < S_{M+1}. \tag{10}$$

Otherwise $\frac{S_N}{2} < S_1$ and thus, by (4), $\gamma^+(E_N) \geqslant A^{-1}\lambda_1$. On the other hand, taking into account the obvious estimate of analytic capacity by length, we clearly have

$$\gamma(E_N) \leqslant \gamma(E_1) \leqslant \frac{1}{2\pi}\,\mathrm{length}(\partial E_1) = \frac{8}{\pi}\lambda_1.$$

Therefore (5) is trivial in the present case, provided C_0 is chosen to satisfy $C_0 \geqslant \frac{8}{\pi}A$.

Assume, then, that (10) holds and let us proceed to prove (5) by induction on N. The case $N = 1$ is obviously true. The induction hypothesis is

$$\gamma(E_n) \leqslant C_0\gamma^+(E_n), \quad 0 < n < N,$$

where the precise value of the absolute constant C_0 will be determined later.

We distinguish two cases.

Case 1: For some absolute constant C_1, to be determined later,

$$a_M\gamma(E(\lambda_{M+1}, \dots, \lambda_N)) \leqslant C_1\gamma(E_N). \tag{11}$$

Case 2: (11) does not hold.

We deal first with Case 2. By the induction hypothesis applied to the sequence $\lambda_{M+1}, \dots, \lambda_N$ and by (4) we have

$$\begin{aligned}\gamma(E_N) &\leqslant C_1^{-1} a_M \gamma(E(\lambda_{M+1}, \dots, \lambda_N)) \\ &\leqslant C_1^{-1} C_0 a_M \gamma^+(E(\lambda_{M+1}, \dots, \lambda_N)) \\ &\leqslant C_1^{-1} C_0 A \frac{a_M}{\left(\displaystyle\sum_{n=M+1}^{N} \frac{1}{(4\lambda_{M+1} \cdots 4\lambda_n)^2}\right)^{1/2}} \\ &= C_1^{-1} C_0 A \frac{1}{\left(\displaystyle\sum_{n=M+1}^{N} \frac{1}{a_n^2}\right)^{1/2}}.\end{aligned}$$

Clearly, the inequality $S_M \leqslant \dfrac{S_N}{2}$ is equivalent to

$$\frac{1}{\left(\displaystyle\sum_{n=M+1}^{N} \frac{1}{a_n^2}\right)^{1/2}} \leqslant \frac{\sqrt{2}}{\left(\displaystyle\sum_{n=1}^{N} \frac{1}{a_n^2}\right)^{1/2}},$$

and so, again by (4),

$$\gamma(E_N) \leqslant C_1^{-1} C_0 A \gamma^+(E_N).$$

If $C_1 = A$, where A is the constant in the preceding inequality, we get (5), as desired.

Now let us now consider Case 1. Set

$$\mu = \frac{\gamma(E_N)}{\text{length}(\partial E_M)} ds_{|\partial E_M},$$

so that $\|\mu\| = \gamma(E_N)$. To check condition (i) in the local $T(b)$-Theorem of Section 2, we consider two cases. If $r \leqslant \sigma_M$, then

$$\mu(D) \leqslant \frac{\gamma(E_N)}{\text{length}(\partial E_M)} 2r \leqslant Cr,$$

because $\gamma(E_N) \leqslant \gamma(E_M) \leqslant \text{length}(\partial E_M)$.

For $r > \sigma_M$ we can replace arbitrary discs centered at points in ∂E_M by the squares Q_j^n, $0 \leqslant n \leqslant M$, $0 \leqslant j \leqslant 4^n$. In other words, it suffices to prove

$$\mu(Q_j^n) \leqslant C\ell(Q_j^n),\ 0 \leqslant n \leqslant M,\ 1 \leqslant j \leqslant 4^n. \tag{12}$$

To show this, given a disc D of radius r centered at $z \in \partial E_M$, one considers a square $Q_j^n \supset D$, where n is chosen so that $\ell(Q_j^n)$ is comparable to r. Then,

$$\mu(Q_j^n) = \gamma(E_N) \frac{1}{4^n} = \frac{4\gamma(E_N)}{\text{length}(\partial E_n)} \ell(Q_j^n) \leqslant \frac{4\gamma(E_n)}{\text{length}(\partial E_n)} \ell(Q_j^n) \leqslant \frac{2}{\pi} \ell(Q_j^n),$$

and so (12) is proved.

It is also a simple matter to ascertain that (ii) holds.

Now our goal is to prove that hypothesis (iii) of the local $T(b)$-Theorem is satisfied. Once this has been verified, (9) applied to $K = \partial E_M$ yields

$$\gamma(E_N) = \|\mu\| \leqslant A\gamma^+(E_M) \leqslant \frac{A}{S_M^{1/2}}. \tag{13}$$

Assuming that (13) holds, to complete the proof we again distinguish two cases, according to whether $1/a_{M+1}^2$ is greater than S_M or not.

If $1/a_{M+1}^2 > S_M$, then

$$S_{M+1} = S_M + \frac{1}{a_{M+1}^2} \simeq \frac{1}{a_{M+1}^2},$$

and so

$$\gamma^+(E_{M+1}) \simeq \frac{1}{S_{M+1}^{1/2}} \simeq a_{M+1} = \frac{1}{4}\,\text{length}(\partial E_{M+1}).$$

Hence

$$\gamma(E_N) \leqslant \gamma(E_{M+1}) \leqslant \frac{1}{2\pi}\,\text{length}(\partial E_{M+1}) \simeq \gamma^+(E_{M+1}) \simeq \gamma^+(E_N),$$

and thus (5) holds for a sufficiently big constant C_0.

If $1/a_{M+1}^2 \leqslant S_M$, then $S_{M+1} \simeq S_M$ and so

$$\gamma^+(E_N) \simeq \frac{1}{S_N^{1/2}} \simeq \frac{1}{S_M^{1/2}},$$

which gives (5), with a big enough C_0, by (13).

Summing up, we have reduced the proof of Theorem 2 to checking that hypothesis (iii) of the local $T(b)$-Theorem is satisfied. As we already remarked when dealing with hypothesis (i), in proving (iii) we can replace discs centered at points in ∂E_M by squares Q_j^n, $1 \leqslant j \leqslant 4^n$, $0 \leqslant n \leqslant M$. In other words, it is enough to show that, given a square Q_j^n, $0 \leqslant n \leqslant M$, $1 \leqslant j \leqslant 4^n$, there exists a function b_j^n in $L^\infty(\mu)$, supported on Q_j^n, satisfying $|b_j^n| \leqslant 1$ and $|C(b_j^n\mu)| \leqslant 1$ $d\mu$-almost everywhere on ∂E_M and such that

$$\mu(Q_j^n) \leqslant C\left|\int b_j^n d\mu\right|.$$

Let f be the Ahlfors function of E_N. Then f is analytic on $\mathbb{C}\setminus E_N$, $|f(z)| \leqslant 1$, $z \notin E_N$, $f(\infty) = 0$ and $f'(\infty) = \gamma(E_N)$. The non-tangential boundary value of f at $\zeta \in \partial E_N$, which exists for ds-almost all $\zeta \in \partial E_N$, is denoted by $f(\zeta)$. Set $\nu = \dfrac{1}{2\pi i} f(\zeta) d\zeta_{|\partial E_N}$, so that

$$f(z) = \int \frac{1}{z-\zeta}\, d\nu(\zeta), \quad z \notin E_N.$$

Fix a generation n, $0 \leqslant n \leqslant M$. Then, for some index k, $1 \leqslant k \leqslant 4^n$,

$$\gamma(E_N) = \sum_{j=1}^{4^n} \nu(Q_j^n) \leqslant 4^n |\nu(Q_k^n)|,$$

or, equivalently, $\mu(Q_k^n) \leqslant |\nu(Q_k^n)|$.

To define the function b_k^n associated with Q_k^n, we need to describe a simple preliminary construction.

Take a compactly supported C^∞ function φ on $\mathbb{C}$, $0 \leqslant \varphi \leqslant 1$, $\int_{\partial Q^0} \varphi ds \geqslant 1$, such that φ vanishes on $\bigcup_{j=1}^4 D(z_j, 1/4)$, where the z_j are the vertices of Q^0. Then $|C(\varphi ds_{|\partial Q_0})| \leqslant A$, as one checks easily. Set

$$\varphi_j^M(z) = \varphi\Big(\frac{z - v_j^M}{\sigma_M}\Big)\chi_{Q_j^M},$$

where v_j^M is the left lower vertex of Q_j^M. Hence $|C(\varphi_j^M d\mu)| \leqslant A$ and

$$\int \varphi_j^M d\mu = \frac{1}{\operatorname{length}(\partial Q_j^M)} \int \varphi_j^M ds, \quad \mu(Q_j^M) \geqslant \frac{1}{4}\mu(Q_j^M).$$

Define

$$b = b_k^n = \sum_{Q_j^M \subset Q_k^n} \nu(Q_j^M) \frac{\varphi_j^M}{\int \varphi_j^M d\mu}.$$

For $j \neq k$ we construct b_j^n by simply translating b_k^n. We have $Q_j^n = w_j^n + Q_k^n$, for some complex number w_j^n. Set

$$b_j^n(z) = b_k^n(z - w_j^n), \quad z \in \mathbb{C}.$$

Now we will prove that b_k^n satisfies condition (iii). Clearly,

$$\left|\int b d\mu\right| = |\nu(Q_k^n)| \geqslant \mu(Q_k^n).$$

To show that b is bounded, it suffices to prove

$$|\nu(Q_j^M)| \leqslant A\mu(Q_j^M), \quad 1 \leqslant j \leqslant 4^M, \tag{14}$$

and for this we first remark that $|C(\chi_{Q_j^M}\nu)(z)| \leqslant A$, $z \notin E_N$. This is proved in [G2, Lemma 2.3 (a), p. 90]. Since $C(\chi_{Q_j^M}\nu)$ is analytic outside $Q_j^M \cap E_N$, we conclude that

$$|\nu(Q_j^M)| \leqslant A\gamma(Q_j^M \cap E_N).$$

Now notice that the set $Q_j^M \cap E_N$ can be obtained from $E(\lambda_{M+1}, \ldots, \lambda_N)$ by a dilation of factor σ_M and a translation. Hence, recalling (11),

$$\gamma(Q_j^M \cap E_N) = \sigma_M \gamma(E(\lambda_{M+1}, \ldots, \lambda_N)) \leqslant C_1 \frac{1}{4^M}\gamma(E_N) = C_1\mu(Q_j^M),$$

which gives (14). It is worth noting at this point that the above inequality explains why M cannot be taken to be N.

Thanks to the discussion in Section 2 on Cauchy integrals and the Plemelj formulae, it becomes clear that we are only left with the task of proving

$$|C(bd\mu)(z)| \leqslant A, \quad z \notin E_M \cap Q_k^n.$$

Since $|C(\chi_{Q_k^n}\nu)(z)| \leqslant A$, $z \notin E_N \cap Q_k^n$, we only need to estimate, for $z \notin E_M \cap Q_k^n$, the difference

$$C(bd\mu)(z) - C(\chi_{Q_k^n}\nu)(z) = \sum_{Q_j^M \subset Q_k^n} C(\alpha_j^M)(z), \tag{15}$$

where

$$\alpha_j^M = \nu(Q_j^M)\frac{\varphi_j^M d\mu}{\int \varphi_j^M d\mu} - \chi_{Q_j^M}\nu.$$

We have $\int d\alpha_j^M = 0$ and $|C(\alpha_j^M)(z)| \leqslant A$, $z \notin Q_j^M$, $1 \leqslant j \leqslant 4^M$, again using [G2, Lemma 2.3 (a), p. 90].

Thus, if z_j^M is the center of Q_j^M,

$$|C(\alpha_j^M)(z)| \leqslant A \frac{\sigma_M^2}{\operatorname{dist}(z, Q_j^M)^2}, \quad |z - z_j^M| > \sigma_M. \tag{16}$$

By the maximum principle, in estimating (15), we can assume that $|z - z_j^M| \leqslant \sigma_M$ for some j with $Q_j^M \subset Q_k^n$. Hence (15) is not greater than

$$A + A \sum_{l \neq j} \frac{\sigma_M^2}{\operatorname{dist}(z, Q_l^M)^2}. \tag{17}$$

For $0 \leqslant n \leqslant M$ let Q^n be the square in the n-th generation that contains Q_j^M. We can estimate (17) by

$$A + A \sum_{n=0}^{M-1} \sum_{Q_l^M \subset Q^n \setminus Q^{n+1}} \frac{\sigma_M^2}{\operatorname{dist}(z, Q_l^M)^2}$$
$$\leqslant A + A \sum_{n=0}^{M-1} \frac{\sigma_M^2}{\sigma_n^2} 4^{M-n} \leqslant A + A \sum_{n=0}^{M-1} \left(\frac{4}{9}\right)^{M-n} \leqslant A,$$

because $\sigma_M = \sigma_n \lambda_{n+1} \cdots \lambda_M \leqslant \dfrac{\sigma_n}{3^{M-n}}$, $0 \leqslant n \leqslant M$.

This completes the construction of the function b_k^n associated with the square Q_k^n as required by hypothesis (iii) in the local $T(b)$-Theorem.

Now, by translation invariance it is clear that b_j^n for $j \leqslant k$ also satisfies (iii).

This shows that the local $T(b)$-Theorem can be applied to μ and thus completes the proof of Theorem 2. □

Remark 2. For simplicity, we assumed above that $\lambda_n \leqslant 1/3$ for all n. However, both Theorem 1 and Theorem 2 hold for $0 < \lambda_n < 1/2$. Let us sketch the changes needed in the proof.

First it should be noticed that the estimate (4) for $\gamma^+(E_N)$ holds for $0 < \lambda_n < 1/2$. Indeed, the arguments for the left inequality in (4) in [Ma] are valid in this case. On the other hand, the right inequality in (4) is also true for $0 < \lambda_n < 1/2$. For example, arguing as in [T2], one can easily check that

$$\gamma^+(E_N) \leqslant A \left(1 + \sum_{\substack{1 \leqslant n \leqslant N, \\ \lambda_n \leqslant 1/3}} \frac{1}{(4^n \sigma_n)^2} \right)^{-1/2} \approx \left(\sum_{1 \leqslant n \leqslant N} \frac{1}{(4^n \sigma_n)^2} \right)^{-1/2}.$$

The other places where the assumption $\lambda_n \leqslant 1/3$ has been used are (14) and (15). The inequality (14) also holds for $0 < \lambda_n < 1/2$. It follows from Vitushkin's estimates for the integral $\int_\Gamma f(z)\, dz$ for piecewise Lyapunov curves Γ [Vi1] (in our case $\Gamma = \partial Q_j^M$). To prove (15), one can use the sharper estimate

$$|C(\alpha_j^M)(z)| \leqslant A \frac{\sigma_M \, \gamma(Q_j^M \cap E_N)}{\operatorname{dist}(z, Q_j^M)^2} \leqslant A C_1 \frac{\sigma_M \, \mu(Q_j^M)}{\operatorname{dist}(z, Q_j^M)^2}, \quad |z - z_j^M| > \sigma_M,$$

instead of (16) (see [G2, pp. 12–13], for example). We leave the details for the reader.

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