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# SIMPLE HOMOGENEOUS MODELS

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This paper attempts to give a self-contained development of dividing theory (also called forking theory) in a strongly homogeneous structure. Dividing is a combinatorial property of the invariant relations on a structure that have yielded deep results for the models of so-called "simple" first-order theories. Below we describe for the nonspecialist how this paper fits in the broader context of geometrical stability theory. Naturally, some background in first-order model theory helps to understand these motivating results; however virtually no knowledge of logic is assumed in this paper. Readers desiring a more thorough description of geometrical stability theory are referred to the surveys [Hru97] and [Hru98].

Traditionally, geometrical stability theory is a collection of results that apply to definable relations on arbitrary models of a complete first-order theory. It is equivalent and convenient to restrict our attention to the definable relations on a fixed representative model of the theory, called a universal domain. Using the terminology of the abstract, a *universal domain* is an uncountable model $M$ equipped with the first-order definable relations $\mathcal{R}$ which is strongly $|M|$-homogeneous and *compact*; i.e., if $\{ X_i : i \in I \}$, where $|I| < |M|$, is a family of definable relations on $M$ so that $\bigcap_{i \in F} X_i \neq \emptyset$ for any finite $F \subset I$, then $\bigcap_{i \in I} X_i \neq \emptyset$. For our purposes the reader can assume there is a one-to-one correspondence between (complete first-order) theories and universal domains.

A massive amount of abstract model theory was developed en route to Shelah's proof of Morley's Conjecture about the number of models, ranging over uncountable cardinals, of a fixed first-order theory [She90]. Most of the work concerned the case of a stable theory, which will not be defined here for the sake of brevity. What is relevant is that most theorems describing the models of a stable theory rely on the forking independence relation. The forking independence relation, $F$, is a ternary relation on the subsets of the universal domain of a theory, where $F(A, B; C)$ is read "$A$ is forking independent from $B$ over $C$" (see Remark 2.2). In a stable theory forking independence is symmetric (over $C$), has finite character (in $A$ and $B$), bounded dividing, the free extension property and is transitive. (See Definition 2.5, Theorem 2.14 and Corollary 2.15 for precise statements of these properties.) These properties facilitate the introduction of several notions of dimension that lead to procedures for determining when two models are isomorphic. The combinatorial-geometric properties of the definable relations reflected in these dimensions profoundly impact the structure of the models beyond the question of fixing an isomorphism type. The results connected to algebra, known as geometrical

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stability theory, have led to new theorems in number theory, in particular the first proof of the Mordell-Lang Conjecture for function fields in positive characteristics.

In hindsight the origins of geometrical stability theory can be traced to the work of Zilber [Zil93], Chelin, Harrington and Lachlan [CHL85] in the late 1970s and early 1980s. The impact of the area escalated in the mid-1980s with Hrushovski's discovery that natural algebraic objects (groups, fields, vector spaces, etc.) are definable in a universal domain satisfying various abstract model-theoretic hypotheses. This interplay between the combinatorial geometry of abstract stability theory and the model theory of algebraic objects significantly deepened our understanding of stable theories and gave new insights into differential fields [Pil98], [Pil97], [MP97] and even algebraically closed fields [HZ96].

Looking towards applications to number theory, Hrushovski realized the need for an analysis of a "generic" algebraically closed difference field (i.e., an algebraically closed field adjoined with an automorphism that is in some sense universal for difference fields). While the relevant universal domain is clearly unstable, Chatzidakis and Hrushovski showed [CH99] that it does admit a notion of independence reminiscent of forking. After this work was well underway it was realized that their notion of dependence agrees with forking independence and the relevant universal domain is "simple". In [She80] Shelah defines a simple theory in terms of a combinatorial property on families of definable sets and proves that forking independence has some of the nice properties found in a stable theory. Every stable theory is simple but not conversely. Our understanding of simple theories increased dramatically when Byunghan Kim showed in his doctoral research [Kim98] that forking in a simple theory is symmetric, transitive, and satisfies type amalgamation (which compensates for the loss of the definability of types, true in a stable theory but not in a simple unstable theory). Since Kim's seminal work more of the machinery of geometrical stability, such as canonical bases and generics in groups and fields, has been generalized to simple theories ([HKP00], [BPW01], [Wag01]). It is hoped that the geometrical stability theory of simple theories will have additional applications to number theory.

All of the model theory discussed above takes place in the context of a universal domain of a first-order theory. Its applicability to number theory depends on the first-order axiomatizability of fundamental concepts in algebraic geometry. In this paper we begin laying the foundation for the application of geometrical stability theory to some mathematical areas that cannot be captured with first-order logic. Whereas Kim's development of forking takes place in a universal domain, the context here is a strongly homogeneous model. That is, we drop the requirement that the model is compact. Using a definition of simplicity for a strongly homogeneous model that specializes to simplicity for a universal domain, it is shown here that forking independence satisfies the same basic properties (symmetry and transitivity, e.g.) as in a simple universal domain. Moreover, the higher-order theorems like type amalgamation and parallelism of types that are critical to geometrical stability theory also hold in the simple strongly homogeneous setting. In Section 6 examples are given of strongly homogeneous models that are simple, although they are not models of simple theories. In particular, it is shown that any Hilbert space is a subspace of a strongly homogeneous Hilbert space and the latter is simple. In [BB02] Berenstein lays the groundwork for deeper applications of model theory to functional analysis by showing that many structures of the form $(H, T)$, where $H$

is a strongly homogeneous Hilbert space and $T$ is a bounded linear operator on $H$, are strongly homogeneous and simple. Forking independence in a Hilbert space is equivalent to a notion of independence based on orthogonal projection, a very natural geometrical relation. At this early "proof of concept" stage it is difficult to gauge the potential impact of this work on the understanding of these examples from analysis. Berenstein also shows in [Ber02] that analogues of theorems from stable group theory extend to the homogeneous setting.

This paper is far from the first investigation of stability-theoretic concepts in models that are not universal domains. Shelah, Grossberg, Hyttinen, Lessmann and others have extensively studied the spectrum function of a class of models that is not the class of models of first-order theory. (The spectrum function of a class of models assigns to a cardinal number $\lambda$ the number of models in the class of cardinality $\lambda$, up to isomorphism.) Analogues of forking independence are found in many of these papers, especially [HS00], although fewer properties can be expected in this setting than for forking independence in a simple first-order theory. From a hypothesis about the spectrum function of a class of models it is normally impossible to show that the class contains a strongly homogeneous model. Thus, from the perspective of that body of research, the context of this paper is very limiting. However, there are very natural mathematical objects that are strongly homogeneous and simple although not the models of simple first-order theories, making this work worthwhile.

This work was strongly influenced by [Pil] and its precursor [Hru]. The proofs of the basic properties of forking independence in those papers showed that Kim's treatment could be reproduced in a non-first-order setting.

## 1. Logical structures and homogeneity

1.1. **Definition of a logical structure.** Let $M$ be a model in a language $L$. That is, $M$ is a set together with finitary relations and functions corresponding to symbols in $L$.

**Definition 1.1.** The pair $(M, \mathcal{R})$ is a *logical structure* if $M$ is a structure for a first-order language $L$ and $\mathcal{R}$ is a collection of finitary relations on $M$ satisfying the following requirements.

(1) The elements of $\mathcal{R}$ are invariant under automorphisms of $M$.
(2) $\mathcal{R}$ is closed under finite unions and intersections.
(3) If $R \in \mathcal{R}$ is $n$-ary and $\pi$ is a permutation of $n$, then $\{ (a_{\pi(1)}, \ldots, a_{\pi(n)}) : R(a_1, \ldots, a_n) \}$ is also in $\mathcal{R}$.

*Remark* 1.1. For the reader unfamiliar with first-order languages an equivalent formulation can be used. Instead of $M$ being a structure in a language we consider a faithful group action $(G, M)$ and view $G$ as the automorphism group of the structure. Then $\mathcal{R}$ is a class of $G$-invariant relations satisfying (2) and (3).

**Definition 1.2.** Let $\bar{a} \in A^\alpha$, $\bar{b} \in A^\beta$, where $\alpha$, $\beta$ are ordinals. For $i < \alpha$, let $v_i$ be a variable that ranges over the elements of $M$. Let $tp_{\mathcal{R}}(\bar{a}/\bar{b})$, called the *$\mathcal{R}$-type of $\bar{a}$ over $\bar{b}$ in $M$*, be the set of expressions $R(v_{i_1}, \ldots, v_{i_m}, b_{j_1}, \ldots, b_{j_l})$, where $R$ is an $m+l$-ary relation in $\mathcal{R}$ and $R(a_{i_1}, \ldots, a_{i_m}, b_{j_1}, \ldots, b_{j_l})$. If $\bar{b} = \emptyset$ it may be omitted.

A set $p$ is an *$\mathcal{R}$-type in $\bar{v}$ over $A$* if it consists of expressions of the form $R(x_1, \ldots, x_n, a_1^R, \ldots, a_m^R)$, where $R \in \mathcal{R}$, $x_1, \ldots, x_n$ are variables from $\bar{v}$ and $a_1^R, \ldots, a_m^R \in A \subset M$. An $\mathcal{R}$-type is *consistent* if it is realized in $M$. If $p(\bar{v})$ is an $\mathcal{R}$-type in $\bar{v}$ over $\bar{b}$, the sequence $\bar{c}$ *realizes* $p$ if $tp_{\mathcal{R}}(\bar{c}/\bar{b}) \supset p$, and $p(M)$ denotes $\{\, \bar{c} : \bar{c}$ realizes $p\,\}$. An $\mathcal{R}$-type $p$ over $A$ is *complete* if for some $\bar{a}$ and $q = tp(\bar{a}/A)$, $p(M) = q(M)$.

If the class of relations $\mathcal{R}$ is clear from context we will drop it from the notation and write $M$ for $(M, \mathcal{R})$ and $tp(a/b)$ for $tp_{\mathcal{R}}(a/b)$.

Looking ahead, dividing will be defined for consistent $\mathcal{R}$-types over subsets of $M$.

**Example 1.1.** (i) Let $K$ be a field and $\mathcal{R}$ the collection of all constructible sets on $K$. Then $(M, \mathcal{R})$ is a logical structure.

(ii) Let $M$ be a structure in the language $L$ and $\mathcal{R}$ the class of first-order definable relations on $M$. Then $(M, \mathcal{R})$ is a logical structure.

(iii) Let $M$ be a structure in the language $L$. Then $(M, \mathcal{R})$ is a logical structure if $\mathcal{R}$ is the collection of all sets of realizations of types in one of the following families:

- formulas of first-order logic;
- quantifier-free formulas of first-order logic;
- finite unions and intersections of complete first-order types, quantifier-free types or existential types;
- types in the logic $L_{\kappa\omega}$, where $\kappa$ is infinite;
- types in $L_k$, the logic with only $k < \omega$ variables.

(iv) If $R$ is a real-closed field and $\mathcal{R}$ is the collection of semi-algebraic sets over $R$, then $(R, \mathcal{R})$ is a logical structure.

*Remark* 1.2. The definition of a logical structure is designed to encompass all of the examples in (iii). This prohibits us from requiring $\mathcal{R}$ to be closed under projection, composition or negation.

1.2. **Homogeneous logical structures.** In this paper our attention will focus on the following kind of objects.

**Definition 1.3.** Let $(M, \mathcal{R})$ be a logical structure, where $M$ is infinite and $\lambda \leq |M|$ is infinite. Then $(M, \mathcal{R})$ is *strongly $\lambda$-homogeneous* if for all $\alpha < \lambda$ and $\bar{a}$, $\bar{b} \in A^{\alpha}$, if $tp(\bar{a}) = tp(\bar{b})$, then there is an automorphism $f$ of $(M, \mathcal{R})$ with $f(\bar{a}) = \bar{b}$.

For brevity we write *$s\lambda$-homogeneous* for strongly $\lambda$-homogeneous.

When $\mathcal{R}$ is the collection of first-order definable relations, the $s\lambda$-homogeneity of $(M, \mathcal{R})$ is equivalent to $M$ being $s\lambda$-homogeneous as a first-order structure. The following is trivial but helps to connect $s\lambda$-homogeneity to a more familiar concept.

**Proposition 1.1.** *An $s\lambda$-homogeneous logical structure $(M, \mathcal{R})$ is $\lambda$-homogeneous in the sense that for $\alpha < \lambda$, $\bar{a}$, $\bar{b} \in A^{\alpha}$ with $tp(\bar{a}) = tp(\bar{b})$ and $c \in M$, there is $d \in M$ such that $tp(\bar{a}c) = tp(\bar{b}d)$.*

Strongly $\lambda$-homogeneous models are ubiquitous in first-order model theory:

**Lemma 1.2.** *For $\lambda$ an infinite cardinal and $T$ a complete first-order theory of cardinality $\leq \lambda$, there is an $s\lambda$-homogeneous model $M$ of $T$ of cardinality $\leq 2^{\lambda}$.*

This is [Bue96, Proposition 2.2.7].

Homogeneity can be used to obtain the consistency of unions of chains of complete types. Such a result is most commonly proved with compactness; however, it also holds in this setting.

**Lemma 1.3.** *Let $(M, \mathcal{R})$ be an $s\lambda$-homogeneous logical structure, $A \subset M$, $|A| < \lambda$. Let $A_0 \subset A_1 \subset \cdots \subset A_\alpha \subset \cdots$, $\alpha < \beta$, be a chain of sets with $A = \bigcup_{\alpha<\beta} A_\alpha$ and $p_\alpha(\bar{v}) \in S(A_\alpha)$, $|\bar{v}| < \lambda$, such that $p_\alpha \subset p_\gamma$, for any $\alpha < \gamma < \beta$. If each $p_\alpha$, $\alpha < \beta$ is consistent in $M$, then $\bigcup_{\alpha<\beta} p_\alpha$ is consistent in $M$.*

*Proof.* This follows quickly from [CK73, Lemma 5.1.18]. □

**Definition 1.4.** Given an $s\lambda$-homogeneous logical structure $(M, \mathcal{R})$, an $\mathcal{R}$-type $p$ is called *large* if the set of realizations of $p$ has cardinality $\geq \lambda$.

**Definition 1.5.** Let $(M, \mathcal{R})$ be a logical structure, $X$ an ordered set, $A \subset M$ and $I = \{ a_i : i \in X \}$ a set of sequences from $M$ indexed by $X$. Then, $I$ is called *$A$-indiscernible* or *indiscernible over $A$* if for all $n < \omega$, and $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ from $X$, $tp(a_{i_1}, \ldots, a_{i_n}/A) = tp(a_{j_1}, \ldots, a_{j_n}/A)$.

If $I$ is $A$-indiscernible the *type diagram* of $I$ over $A$ is the collection of all types $tp(\bar{a}/A)$, as $\bar{a}$ ranges over finite sequences from $I$ whose indices are increasing in $X$.

*Remark* 1.3. Let $(M, \mathcal{R})$ be an $s\lambda$-homogeneous logical structure, $X$ an ordered set, $A \subset M$ and $I = \{ a_i : i \in X \}$, $J = \{ b_i : i \in X \}$ $A$-indiscernible sequences with the same type diagram over $A$ such that $|\bigcup I| < \lambda$. Then, there is an automorphism $f$ of $(M, \mathcal{R})$ fixing $A$ such that $f(a_i) = b_i$, for $i \in X$.

The existence and ubiquity of indiscernible sequences in $s\lambda$-homogeneous models will play a big role in this study. The following result guarantees that when $\lambda$ is sufficiently large, the set of realizations of a large type over a relatively small set will contain an infinite sequence of indiscernibles.

**Lemma 1.4.** *Let $(M, \mathcal{R})$ be $s\lambda$-homogeneous and $\lambda_0$ be the cardinality of the set of complete $\mathcal{R}$-types over $\emptyset$ in finitely many variables realized in $M$. For each $\lambda_1$ and $\lambda$ sufficiently large, there is a $\lambda_2 \leq \lambda$ (depending on $\lambda_1$ and $\lambda_0$) such that if $A \subset M$, $|A| \leq \lambda_1$, $X$ is an ordered set of size at least $\lambda_2$ and $I = \{ \bar{a}_i : i \in X \} \subset M$, where $|\bar{a}_i| \leq \lambda_1$, for $i \in X$,* then *there is $J = \{ \bar{b}_i : i < \omega \} \subset M$, indiscernible over $A$ such that for every $n < \omega$, there exists $i_0 < \cdots < i_n$ in $X$ with*

$$tp(\bar{b}_0, \ldots, \bar{b}_n/A) = tp(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/A).$$

*Proof.* Regarding $M$ as an ordinary first-order structure in a language $L_0$ in which it has elimination of quantifiers, let $T_1$ be an expansion of $Th(M)$ with Skolem functions, $L_1$ the language of $T_1$. Let $\lambda_2 = \beth_{(2^{\lambda_1+\lambda_0})^+}$. Using a standard application of the Erdos-Rado Theorem there exists, in some model $N$ of $T_1$, a sequence $\{ \bar{d}_i : i < \omega \}$ indiscernible over $A$ in $T_1$ such that for every $n < \omega$, there exists $i_0 < \cdots < i_n$ in $X$ with

$$tp_{L_1}(\bar{d}_0, \ldots, \bar{d}_n/A) = tp_{L_1}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/A).$$

Without loss of generality, $N$ is the Skolem hull of $A \cup J$, hence every complete type in $L_0$ in finitely many variables realized in $N$ is also realized in $M$. Assuming that $|N| < \lambda$, by the $s\lambda$-homogeneity of $M$ there is $\{ \bar{b}_i : i < \omega \} \subset M$ such that

$$tp(\bar{b}_0, \ldots, \bar{b}_n/A) = tp(\bar{d}_0, \ldots, \bar{d}_n/A).$$

This completes the proof. □

As a consequence of the preceding lemma, when $M$ is $s\lambda$-homogeneous and $\lambda$ is sufficiently large, sets of realizations of large types over small sets contain infinite indiscernible sequences. The converse follows from the next lemma.

**Lemma 1.5.** *Let $(M, \mathcal{R})$ be $s\lambda$-homogeneous, $A \subset M$ with $|A| < \lambda$, and suppose $I = \{ \bar{a}_i : i \in X \} \subset M$ is an indiscernible sequence over $A$, where $|\bar{a}_0| < \lambda$ and $|X| < \lambda$.*

*(i) For any ordered set $Y$ extending $X$ with $|Y| \leq \lambda$, there are sequences $\bar{a}_j \in M$, for $j \in Y$, such that $J = \{ \bar{a}_i : i \in Y \}$ is $A$-indiscernible. A fortiori, $tp(\bar{a}_0/A)$ is large.*

*(ii) For any linearly ordered set $X'$ of cardinality $\leq \lambda$ there is $J = \{ \bar{b}_i : i \in X' \} \subset M$ indiscernible over $A$ with the same diagram over $A$ as $I$.*

*Proof.* (i) follows from Lemma 1.3 and (ii) follows from (i) by taking $Y = X + X'$. $\square$

**Definition 1.6.** A homogeneous logical structure $(M, \mathcal{R})$ is *compact* if for each $n$, and each set $S$ of relations of the form $R(x_1, \dots, x_n, a_1^R, \dots, a_m^R)$, where $R \in \mathcal{R}$, $x_1, \dots, x_n$ are fixed and $a_1^R, \dots, a_m^R \in M$ can vary with $R$: if $|S| < |M|$ and every finite subset of $S$ is realized in $(M, \mathcal{R})$, then $S$ is realized in $(M, \mathcal{R})$.

*Remark* 1.4. Let $M$ be a saturated model of a first-order theory. Let $\mathcal{R}$ be the collection of definable relations on $M$. Then $(M, \mathcal{R})$ is a compact homogeneous logical structure. Conversely, if $(M, \mathcal{R})$ is a compact homogeneous logical structure, consider $M$ in an expanded language with a predicate symbol for every relation in $\mathcal{R}$. In this language $M$ is a saturated model.

*Remark* 1.5. Let $M$ be a homogeneous model of a first-order theory. Let $\mathcal{R}$ be the collection of first-order definable relations on $M$. Then $(M, \mathcal{R})$ is a homogeneous logical structure. The study of *stability* for such structures was initiated by Shelah in [She70]. Our context is formally more general (as we allow $\mathcal{R}$ to stand for more general relations), but we have phrased our definitions so that the existing stability machinery holds in our context with obvious minor modifications. (See Section 5 on Stability for details.)

## 2. Dividing and simplicity in a homogeneous logical structure

Throughout this section $(M, \mathcal{R})$ is an $s\lambda$-homogeneous logical structure such that

**(P):** for some infinite cardinals $\pi \leq \pi' \leq \lambda$ and every $\mathcal{R}$-type $p(\bar{v})$ over $A \subset M$, $|A| < \pi$ and $|\bar{v}| < \pi$, if $X$ is a sequence of realizations of $p$ of length $\geq \pi'$, then there is an $A$-indiscernible sequence $\{ \bar{b}_i : i < \omega \}$ such that $tp(\bar{b}_0, \dots, \bar{b}_n/A)$ is realized by an increasing sequence in $X$, for each $n < \omega$.

This convention may be restated in important definitions and results for clarity.

Indiscernible sequences play an integral role in this treatment of dividing theory. Indeed, even the definition of dividing involves indiscernibles. For sufficiently large $\lambda$, (**P**) holds for any $s\lambda$-homogeneous logical structure by Lemma 1.4.

As stated in the Introduction, when studying the models of a first-order theory it is common to restrict attention to a fixed universal domain $N$. When working in a universal domain a common fact of model theory such as "a consistent type over a subset of a model can be realized in some other model" is replaced by "a consistent type over a subset of $N$ of cardinality $< |N|$ is realized in $N$. That is, only types over subsets of cardinality $< |N|$ are studied. This restriction is realized in a convention that the terms "set" and "model" only refer to objects of size $< |N|$.

In analog to the first-order conventions, given $(M, \mathcal{R})$ satisfying (**P**), the term "set" will refer to a subset of $M$ of cardinality $< \pi$. By extension the term $\mathcal{R}$-type will only apply to an $\mathcal{R}$-type in $< \pi$ variables over a set of cardinality $< \pi$. We may restate the restriction "of cardinality $< \pi$" for clarity in a context where $(\mathbf{P})$ is being explicitly used.

2.1. **Main definitions.** The principal concepts in this paper are "an $\mathcal{R}$-type $p$ divides over $A \subset M$" and "$M$ is simple".

**Definition 2.1.** Given an $s\lambda$-homogeneous logical structure $(M, \mathcal{R})$ satisfying $(\mathbf{P})$, an $\mathcal{R}$-type $p(\bar{v}, \bar{b})$ over $\bar{b}$ *divides over* $A \subset M$, if there is an infinite $A$-indiscernible sequence $\{ \bar{b}_i : i \in X \}$, with $tp(\bar{b}_0/A) = tp(\bar{b}/A)$, such that $\bigcup_{i \in X} p(\bar{v}, \bar{b}_i)$ is inconsistent.

*Remark* 2.1. (i) Suppose $p(\bar{v}, \bar{b})$ divides over $A$. Since $X$ in the definition is infinite, Lemma 1.5 implies that $tp(\bar{b}/A)$ is large. Consequently, when $tp(\bar{d}/A)$ is small, any $\mathcal{R}$-type $q(\bar{v}, \bar{d})$ does not divide over $A$.

(ii) The inconsistency of $\bigcup_{i \in X} p(\bar{v}, \bar{b}_i)$ in the definition depends on $tp(\{ \bar{b}_i : i \in X \}/A)$. The definition does not exclude the possibility of an infinite $A$-indiscernible sequence $\{ \bar{c}_i : i \in Y \}$ having the same diagram over $A$ as $\{ \bar{b}_i : i \in X \}$, where $\bigcup_{i \in Y} p(\bar{v}, \bar{c}_i)$ consistent.

(iii) $tp(a/b)$ divides over $C$ if and only if $tp(a/bC)$ divides over $C$.

(iv) If $tp(a/b)$ divides over $A$ and $C \subset A$, then $tp(a/b)$ divides over $C$.

The following basic properties of dividing do not require any additional properties of $M$; they are properties of dividing itself.

**Lemma 2.1.** *Let $p(\bar{v}, \bar{b})$ be an $\mathcal{R}$-type over $\bar{b}$ and $A \subset M$. The following are equivalent.*

1. *$p(\bar{v}, \bar{b})$ does not divide over $A$.*
2. *For any infinite $A$-indiscernible sequence $I$ there is $J$, an infinite indiscernible sequence over $A$, and an $\bar{a}'$ realizing $p(\bar{v}, \bar{b}')$, for $\bar{b}' \in J$, such that $J$ has the same type diagram over $A$ as $I$ and $J$ is indiscernible over $A \cup \{\bar{a}'\}$.*
3. *For each infinite indiscernible $I$ over $A$ with $\bar{b} \in I$ there is an $\bar{a}'$ realizing $p(\bar{v}, \bar{b})$ such that $I$ is indiscernible over $A \cup \bar{a}'$.*

*Proof.* (2) $\Longrightarrow$ (1) follows from the definition of dividing and (3) $\Longrightarrow$ (2) is trivial. To prove that (1) implies (2) let $I = \{ \bar{b}_i : i \in X \}$ be an infinite $A$-indiscernible sequence with $tp(\bar{b}_i/A) = tp(\bar{b}/A)$. By Lemma 1.5, there is an ordered set $X'$ extending $X$ with $\pi' \leq |X'| < \lambda$ (see $(\mathbf{P})$ at the beginning of the section for the definition of $\pi'$) such that $I' = \{ \bar{b}_i : i \in X' \}$ is $A$-indiscernible. Since $p(\bar{v}, \bar{b})$ does not divide over $A$, there is $\bar{a}$ in $M$ realizing $\bigcup_{i \in X'} p(\bar{v}, \bar{b}_i)$. Lemma 1.4 gives the existence of $\{ \bar{d}'_i : i < \omega \}$ indiscernible over $A \cup \bar{a}$ such that for $n < \omega$ there exists $i_0 < \cdots < i_n$ in $X'$ satisfying

$$tp(\bar{d}'_0, \ldots, \bar{d}'_n/A) = tp(\bar{b}_{i_0}, \ldots, \bar{b}_{i_n}/A).$$

Since $p(\bar{a}, \bar{d}'_i)$ for each $i$, we have proved that (1) implies (2).

For (2) implies (3), let $I = \{ \bar{b}_i : i \in X \}$ and let $\{ \bar{d}'_i : i < \omega \}$ be indiscernible over $A \cup \bar{a}$ with the same diagram over $A$ as $I$ such that $p(\bar{a}, \bar{d}'_i)$, for $i < \omega$. By Lemma 1.5(ii) there is a sequence $\{ \bar{d}_i : i \in X \}$ indiscernible over $A \cup \bar{a}$ indexed by $X$ with the same diagram over $A \cup \bar{a}$ as $\{ \bar{d}'_i : i < \omega \}$. Now $I$ and $\{ \bar{d}_i : i \in X \}$ are both indiscernibles indexed by $X$ with the same diagram over $A$. Thus, there is an automorphism $f$ of $M$ fixing $A$ and taking $\bar{d}_i$ to $\bar{b}_i$, for $i \in X$. Then $f(\bar{a}) = \bar{a}'$ realizes $\bigcup_{i \in X} p(\bar{v}, \bar{b}_i)$ and $I$ is indiscernible over $A \cup \bar{a}'$, proving the lemma. $\square$

**Proposition 2.2** (Pairs Lemma)**.** *If $tp(\bar{a}/A \cup \bar{b})$ does not divide over $A$ and $tp(\bar{c}/A \cup \bar{b}\bar{a})$ does not divide over $A \cup \bar{a}$, then $tp(\bar{a}\bar{c}/A \cup \bar{b})$ does not divide over $A$.*

*Proof.* Let $I$ be any infinite indiscernible sequence over $A$ containing $\bar{b}$. By the preceding lemma there is $\bar{a}_0$ in $M$ realizing $tp(\bar{a}/A \cup \bar{b})$ such that $I$ is indiscernible over $A \cup \bar{a}_0$. Let $f$ be an automorphism fixing $A \cup \bar{b}$ and sending $\bar{a}$ to $\bar{a}_0$. Then, $tp(f(\bar{c})/A \cup \bar{b}\bar{a}_0)$ does not divide over $A \cup \bar{a}_0$, hence there is $\bar{c}_0$ realizing $tp(f(\bar{c})/A \cup \bar{b}\bar{a}_0)$ such that $I$ is indiscernible over $A \cup \bar{a}_0\bar{c}_0$. Since $\bar{a}_0\bar{c}_0$ realizes $tp(\bar{a}\bar{c}/A \cup \bar{b})$ and $\bar{b} \in I$, the proposition is proved. □

Our goal is to find minimal properties of $M$ on which dividing defines a symmetric and transitive dependence relation, ultimately leading to a dimension theory. Defining this property, simplicity, will take a couple of preliminary notions.

**Definition 2.2.** Given an infinite cardinal $\chi$ and $A$, $B$, $C$ subsets of $M$, $A$ *is $\chi$-free from $B$ over $C$* if for all sequences $\bar{a}$ from $A$ and $\bar{b}$ from $B \cup C$ with $|\bar{a}|,\ |\bar{b}| < \chi$, $tp(\bar{a}/\bar{b})$ does not divide over $C$.

*Remark* 2.2. When $(M, \mathcal{R})$ is a universal domain of a first-order theory, $\aleph_0$-freeness agrees with the forking independence relation mentioned in the paper's introduction. This is slightly inaccurate since "$tp(\bar{a}/\bar{b})$ does not divide over $C$" is equivalent to "$tp(\bar{a}/\bar{b})$ does not fork over $C$" only in a simple theory.

In an arbitrary model $\kappa$-freeness depends on types in $< \kappa$ variables over sequences of length $< \kappa$, which may be infinite if $\kappa$ is uncountable.

**Definition 2.3.** Let $\mathcal{F}$ denote the $\chi$-freeness relation in $(M, \mathcal{R})$. The *character* of $\mathcal{F}$ is the least cardinal $\mu$ such that for all sets $A$, $B$, $C$ in $M$, if $A$ is $\mu$-free from $B$ over $C$, then $A$ is $\chi$-free from $B$ over $C$. $\mathcal{F}$ has *finite character* when the character is $\aleph_0$.

*Remark* 2.3. (i) In most natural instances $\kappa$-freeness has finite character. In models in which $\kappa$-freeness does not have finite character we have to assume outright additional properties of $\kappa$-freeness that lead to a notion of freeness that is symmetric, transitive and has type amalgamation. Moreover, the resulting notion of freeness is so esoteric that a rich theory of dependence is unlikely. For these reasons we include finite character in the definition of simplicity in this paper. With the proper assumptions on $\kappa$-freeness replacing finite character the same proofs used here work more generally. It should be noted that finite character holds when $\kappa = \aleph_0$ and it is in this context in which we can expect the most powerful tools of geometrical stability theory to generalize.

(ii) When $(M, \mathcal{R})$ is compact and $\chi$ is an infinite cardinal, $\chi$-freeness has finite character. To prove this suppose $\bar{a}$ is a sequence of length $< \chi$, $B \subset A$ and $\bar{a}$ is not $\chi$-free from $A$ over $B$. Let $\bar{c}$ be a sequence of length $< \chi$ such that $p(\bar{x}, \bar{c}) = tp(\bar{a}/\bar{c})$ divides over $B$. Let $I = \{\, \bar{c}_i : i \in X \,\}$ be an infinite $B$-indiscernible sequence with $\bar{c}_0 = \bar{c}$ such that $\bigcup_{i \in X} p(\bar{x}, \bar{c}_i)$ is inconsistent. By compactness, for each $i \in X$ there is a finite $\bar{d}_i \subset \bar{c}_i$ such that $\bigcup_{i \in X} p(\bar{x}, \bar{d}_i)$, and without loss of generality, $J = \{\, \bar{d}_i : i \in X \,\}$ is indiscernible. Thus, $tp(\bar{a}/\bar{d}_0)$ divides over $B$; i.e., $\bar{a}$ is not $\aleph_0$-free from $A$ over $B$.

**Definition 2.4.** An infinite cardinal $\kappa$ is given. A set $A \subset M$ is a *$\kappa$-extension base* if for any sequence $\bar{a}$ of length $< \kappa$ with $tp(\bar{a}/A)$ large, and $B \subset A$ such that $\bar{a}$ is

$\kappa$-free from $A$ over $B$, and any set $C$, $A \subset C \subset M$, there is a $\bar{c}$ realizing $tp(\bar{a}/A)$ such that $\bar{c}$ is $\kappa$-free from $C$ over $B$.

*Remark* 2.4. Colloquially speaking, over an extension base any large type has $\kappa$-free extensions over any larger set.

**Definition 2.5.** Let $(M, \mathcal{R})$ be $s\lambda$-homogeneous, $\kappa$ an infinite cardinal.

(i) $(M, \mathcal{R})$ is *almost $\kappa$-simple* if it satisfies

(1) (Finite Character) $\kappa$-freeness has finite character.
(2) (Bounded Dividing Property) For any sequence $\bar{a}$ of length $< \kappa$ and set $A$, there is $B \subset A$, $|B| < \kappa$, such that $\bar{a}$ is $\kappa$-free from $A$ over $B$.
(3) (Free Extension Property) Given a set $A \subset M$, $|A| < \pi$, there is a $\kappa$-extension base $A'$, $A \subset A' \subset M$ and $|A'| < \pi$.

(ii) $(M, \mathcal{R})$ is *$\kappa$-simple* if it is almost $\kappa$-simple and every set $A \subset M$, $|A| < \pi$, is a $\kappa$-extension base.

(iii) $(M, \mathcal{R})$ is *almost simple* (*simple*) if it is almost $\kappa$-simple ($\kappa$-simple) for some infinite $\kappa$, with $\kappa(M)$ denoting the least such cardinal.

(iv) When $(M, \mathcal{R})$ is *almost $\aleph_0$-simple* ($\aleph_0$-simple) it is also called *almost supersimple* (*supersimple*).

*Remark* 2.5. (i) First, let us compare the definition of $\kappa$-simple with the ordinary definition of a simple theory. To distinguish from the term defined above, *classically simple* will be used for the concept normally applied to a first-order theory. Classically simple is defined as follows. Let $(M, \mathcal{R})$ be a universal domain of a first-order theory viewed as a logical structure; i.e., $\mathcal{R}$ is the class of definable relations and $(M, \mathcal{R})$ is compact. A relation $R(\bar{x}, \bar{a})$ *forks over* $A$ if there are $R_0(\bar{x}, \bar{y}_0), \ldots, R_n(\bar{x}, \bar{y}_n) \in \mathcal{R}$ (for some $n$) and $\bar{b}_0, \ldots, \bar{b}_n$ such that (1) $R(\bar{x}, \bar{a}) \longrightarrow \bigvee_{i \leq n} R(\bar{x}, \bar{b}_i)$ and (2) $R(\bar{x}, \bar{b}_i)$ divides over $A$ for all $i$. Then, $(M, \mathcal{R})$ is *classically simple* if there is a cardinal $\kappa$ such that for every finite sequence $\bar{a}$ and set $A$ there is $B \subset A$, $|B| < \kappa$, and every $R(\bar{x}, \bar{b}) \in tp(\bar{a}/A)$ does not fork over $B$. Note: by the compactness of $(M, \mathcal{R})$, $R(\bar{x}, \bar{a})$ does not fork over $A$ if and only if for any set $B \supset A$ there is a $\bar{c}$ satisfying $R(\bar{x}, \bar{a})$ such that no $S(\bar{x}, \bar{b}) \in tp(\bar{c}/B)$ divides over $B$, equivalently, $\bar{c}$ is $\aleph_0$-free from $B$ over $A$. By compactness $\bar{c}$ is $\kappa$-free from $B$ over $A$. So, the existence of free extensions is built into the definition of not forking. It is routine to show that a classically simple $(M, \mathcal{R})$ is $\kappa$-simple as defined in Definition 2.5.

(ii) When $\kappa = \aleph_0$, $\kappa$-freeness has finite character vacuously. Thus, a definition of supersimple could be restated without this condition.

(iii) The authors' experience with dividing outside of the first-order context suggests that there may be important examples of models that are almost $\kappa$-simple but not $\kappa$-simple. While little will be done with the concept here we feel it is worthwhile examining the most fundamental properties of dividing under this hypothesis as well as under $\kappa$-simplicity.

**Notation.** When $(M, \mathcal{R})$ is almost $\kappa$-simple, we drop $\kappa$ from the term $\kappa$-free and simply say *free* or *independent*. Remember that part of the definition of almost $\kappa$-simple is the assumption that $\kappa$-freeness has finite character. So, in this setting, $\aleph_0$-free implies free.

If $p = tp(\bar{a}/A)$ and $\bar{a}$ is free from $B$ over $A$, then we say $q = tp(\bar{a}/A \cup B)$ is a *free extension of* $p$, or $q$ is *free from $B$ over $A$*.

If $(M, \mathcal{R})$ is almost $\kappa$-simple, $A \underset{C}{\downarrow} B$ denotes $A$ is free from $B$ over $C$.

**Notation.** If $(M, \mathcal{R})$ is $\kappa$-simple, letters $a$, $b$, $c, \ldots$ denote sequences of length $< \kappa$ from $M$ and $x$, $y$, $v, \ldots$ corresponding sequences of variables.

Our goal is to prove that $\kappa$-freeness is a well-behaved dependence relation when $(M, \mathcal{R})$ is $\kappa$-simple. Symmetry and transitivity are two critical properties. These are straightforward consequences of Proposition 2.5. To prove that proposition we need to prove that the class of free extensions of a large type is sufficiently rich. The relevant definitions and lemmas follow.

Given an ordered set $X$ and a set $I = \{ a_i : i \in X \}$, $a_{<i}$ denotes $\{ a_j : j < i \}$. If $Y$, $Z \subset X$, then $Z < Y$ means $z < y$, for all $z \in Z$ and for all $y \in Y$. $a_Z$ denotes $\{ a_i : i \in Z \}$.

**Definition 2.6.** Let $p = tp(\bar{a}/A)$, let $\chi$ be an infinite cardinal and $B \subset A$ be such that $\bar{a}$ is $\chi$-free from $A$ over $B$. Let $X$ be an ordered set. A sequence $\{ \bar{a}_i : i \in X \}$ is a *$\chi$-Morley sequence in $p$ over $B$* if

- each $\bar{a}_i$, $i \in X$, realizes $p$,
- $\{ \bar{a}_i : i \in X \}$ is indiscernible over $A$, and
- for each $Y$, $Z \subset X$, $|Y|$, $|Z| < \chi$ and $Z < Y$, $\bar{a}_Y$ is $\chi$-free from $A \cup \bar{a}_Z$ over $B$.

If $(M, \mathcal{R})$ is almost $\kappa$-simple, $I$ is called a *Morley sequence* if $I$ is a $\kappa$-Morley sequence.

**Lemma 2.3.** *Let $(M, \mathcal{R})$ be $s\lambda$-homogeneous, $B \subset A \subset M$, $|B| < \kappa$, and $I = \{ \bar{a}_i : i \in X \}$ an infinite $A$-indiscernible sequence such that for any $i \in X$, $\bar{a}_i$ is $\aleph_0$-free from $A \cup \bar{a}_{<i}$ over $B$. Then $I$ is an $\aleph_0$-Morley sequence over $B$.*

*Proof.* Suppose to the contrary that $I$ is not an $\aleph_0$-Morley sequence over $B$. Let $Y < Z$ be finite subsets of $X$ such that for some finite subsets $\bar{c} \subset \bar{a}_Y$ and $\bar{d} \subset \bar{a}_Z$, $tp(\bar{d}/B \cup \bar{c})$ divides over $B$, and $|Z|$ is minimal with this property. Let $i$ be the largest element of $Z$ and $W = Z \setminus \{i\}$. Let $\bar{d} \cap \bar{a}_i = \bar{e}$ and $\bar{d} \cap \bar{a}_W = \bar{f}$. Then, $tp(\bar{f}/B \cup \bar{c})$ does not divide over $B$ by the minimality assumption on $Z$. By the hypotheses of the lemma, $tp(\bar{e}/B \cup \bar{c}\bar{f})$ does not divide over $B$. So, by Pairs Lemma (Proposition 2.2), $tp(\bar{e}\bar{f}/B \cup \bar{c})$ does not divide over $B$. This contradicts our assumption that $tp(\bar{d}/B \cup \bar{c})$ divides over $B$ to prove the lemma. ☐

**Lemma 2.4.** *Let $(M, \mathcal{R})$ be almost $\kappa$-simple, $A$ an extension base, $\bar{a}$ a sequence of length $< \kappa$ such that $p = tp(\bar{a}/A)$ is large, and $B \subset A$ such that $\bar{a}$ is $\kappa$-free from $A$ over $B$. Let $X$ be any infinite linear order with $|X| \leq \lambda$. Then, $M$ contains an $A$-indiscernible sequence $I = \{ \bar{a}_i : i \in X \}$ which is a $\kappa$-Morley sequence in $p$ over $B$.*

*Proof.* First find an infinite $A$-indiscernible sequence that is $\aleph_0$-free using Lemma 1.4 as follows. Let $\alpha$ be an ordinal $< \lambda$ and suppose sequences $\bar{a}_i$, $i < \alpha$, have been chosen so that $\bar{a}_i$ realizes $p$ and $\bar{a}_i$ is $\kappa$-free from $A \cup \bar{a}_{<i}$ over $B$. By the Extension Property in the definition of $\kappa$-simple there is $\bar{a}_\alpha$ realizing $p$ which is $\kappa$-free from $A \cup \bar{a}_{<\alpha}$ over $B$. Since $(M, \mathcal{R})$ satisfies (**P**) there is $J = \{ \bar{b}_i : i < \omega \}$ such that $J$ is $A$-indiscernible and for any $n < \omega$ there are $i_1, \ldots, i_n < \lambda$ with $tp(\bar{b}_1, \ldots, \bar{b}_n/A) = tp(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/A)$. This latter property controls the type diagram of $J$ and by Lemma 2.3 guarantees that $J$ is an $\aleph_0$-Morley sequence.

Given an infinite linear order $X$, $|X| \leq \lambda$, by Lemma 1.5(ii) there is a sequence $K = \{\bar{c}_i : i \in X\}$ indiscernible over $A$ with the same type diagram as $J$. Thus, $K$ is also an $\aleph_0$-Morley sequence in $p$ over $B$. Since $(M, \mathcal{R})$ is $\kappa$-simple, $\kappa$-freeness has finite character. Thus, $K$ is also a $\kappa$-Morley sequence, proving the lemma. □

*Remark* 2.6. The proof of the preceding lemma illustrates the problems circumvented by assuming finite character. Being a $\kappa$-Morley sequence depends on all subsequences of length $< \kappa$. Without finite character the partition calculus required to get a $\kappa$-Morley sequence could easily be independent of set theory (depending on $\kappa$ and $|M|$). For the same reason it is unclear that a $\kappa$-Morley sequence can be extended to a larger $\kappa$-Morley sequence. While the definition of simplicity could be rewritten to force the existence of enough Morley sequences this would simply hide the complexity in a definition. We believe the class of models satisfying the necessary conditions (without assuming finite character) is very thin, hence our assumption of finite character as part of Definition 2.5.

The following proposition is the key to obtaining symmetry and transitivity of freeness. It says that when $p(\bar{x}, \bar{b})$ divides over $A$, this fact can be witnessed with a Morley sequence in $tp(\bar{b}/A)$.

**Proposition 2.5.** *Let $(M, \mathcal{R})$ be almost $\kappa$-simple, $A$ an extension base, $|b| < \kappa$, and $p(x, b)$ a type over $b$ with $|x| < \kappa$. An indiscernible sequence $I$ indexed by $X$ satisfies $(*)$ if for any suborder $Y$ of $X$ with $|Y| < \kappa$ there is $i \in X$, $i < Y$. Suppose $b$ is free from $A$ over $B \subset A$, $|B| < \kappa$. The following are equivalent.*

(1) *$p(x, b)$ divides over $A$.*
(2) *There is a Morley sequence $I$ in $tp(b/A)$ over $B$ satisfying $(*)$ such that $\bigcup_{d \in I} p(x, d)$ is inconsistent.*
(3) *For any Morley sequence $I$ in $tp(b/A)$ over $B$ satisfying $(*)$, $\bigcup_{d \in I} p(x, d)$ is inconsistent.*

*Proof.* (2) $\Longrightarrow$ (1) is simply by the definition of dividing. (3) $\Longrightarrow$ (2) because such a Morley sequence exists by Lemma 2.4.

We now prove (1) $\Longrightarrow$ (3). Let $I = \{b_i : i \in X\}$ be the given Morley sequence and suppose $p(x, b)$ divides over $A$. Notice that $I$ is also a Morley sequence in $tp(b/B)$ over $B$ and satisfies $(*)$, and $p(x, b)$ divides over $B$. So, replacing $A$ by $B$ if necessary, we may as well assume $|A| < \kappa$.

*Claim.* For each $i \in X$ and $Y \subset X$ with $|Y| < \kappa$ and $i < Y$, $p(x, b_i)$ divides over $A \cup b_Y$.

Simply because $b_i$ realizes $tp(b/A)$, $p(x, b_i)$ divides over $A$. This fact is witnessed by an infinite $A$-indiscernible sequence $J$ containing $b_i$. Let $q(z, y) = tp(b_i, b_Y/A)$. Since $b_Y$ is $\kappa$-free from $\{b_j : j < Y\}$ over $A$ and $|A| < \kappa$, $q(b_i, y)$ does not divide over $A$. Thus, there is a sequence $c$ such that $\bigcup_{d \in J} q(d, c)$. By Lemma 2.1, we can assume that $J$ is indiscernible over $A \cup c$. Since $tp(b_i c/A) = tp(b_i b_Y/A)$ there is an automorphism $f$ fixing $A \cup b_i$ and taking $c$ to $b_Y$. Let $J' = f(J)$. Then, $J'$ is infinite, indiscernible over $A \cup b_Y$, and $\bigcup_{d \in J'} p(x, d)$ is inconsistent. This witnesses that $p(x, b_i)$ divides over $A \cup b_Y$.

To continue with the proof suppose to the contrary that $\bigcup_{d \in I} p(x, d)$ is realized by some $a$. By the Bounded Dividing Property there is $Y \subset X$ of cardinality $< \kappa$ such that $a$ is $\kappa$-free from $A \cup I$ over $A \cup b_Y$. Since $X$ satisfies $(*)$ there is $i \in X$,

$i < Y$. By the claim, $p(x, b_i)$ divides over $A \cup b_Y$. Since $a$ satisfies $p(x, b_i)$ we contradict that $a$ is $\kappa$-free from $A \cup I$ over $A \cup b_Y$. This proves the proposition. $\square$

2.2. **Small types, large types and dividing.** In a saturated model algebraic types and algebraic closure play a very special role. On the one hand, dividing trivializes for algebraic types: If $tp(a/A)$ is algebraic, then $tp(a/A \cup \{b\})$ and $tp(b/A \cup \{a\})$ do not divide over $A$ for all $b$. On the other hand, the close relationship between algebraic closure and dividing in a supersimple model (through minimal types and canonical bases) is the basis for geometrical stability theory.

In a saturated model the algebraic types are exactly the small types (see Definition 1.4). We will show here that small types and small closure act much like algebraic types in a simple model, and to a lesser degree in an almost simple model.

**Notation.** Let $sc\ell(-)$ denote the *small closure operator* defined by $\bar{a} \in sc\ell(A)$ if $tp(\bar{a}/A)$ is small.

*Remark* 2.7. For any $s\lambda$-homogeneous $(M, \mathcal{R})$, $sc\ell(-)$ is a closure operator. That is, $sc\ell(-)$ satisfies:

(1) $A \subset sc\ell(A)$;
(2) $A \subset B \implies sc\ell(A) \subset sc\ell(B)$;
(3) $sc\ell(sc\ell(A)) = sc\ell(A)$.

[The proof is left to the reader.]

In any saturated model, (1) if $a \in acl(A)$ then for all $b$, neither $tp(b/A \cup \{a\})$ nor $tp(a/A \cup \{b\})$ divides over $A$, and (2) $a \in acl(A \cup \{b\}) \setminus acl(A)$ implies that $tp(a/A \cup \{b\})$ and $tp(b/A \cup \{a\})$ both divide over $A$. Property (1) extends to small closure directly, while Proposition 2.7 serves as the generalization of (2).

**Lemma 2.6.** *Let $(M, \mathcal{R})$ be $s\lambda$-homogeneous and $a \in sc\ell(A)$. Then for any $b$, $tp(a/A \cup \{b\})$ and $tp(b/A \cup \{a\})$ do not divide over $A$.*

*Proof.* If $tp(b/A)$ is small then $tp(a/A \cup \{b\})$ does not divide over $A$ (Remark 2.1). Suppose $q(x, b) = tp(a/A \cup \{b\})$ is large and let $I$ be an arbitrary $A$-indiscernible sequence in $tp(b/A)$. For any $d \in I$, any realization of $q(x, d)$ also realizes the small type $tp(a/A)$. Using Lemma 1.4 and cardinality properties we find a $c$ realizing $tp(a/A)$ and an infinite sequence $J$ indiscernible over $A \cup \{c\}$ with the same type diagram over $A$ as $I$, such that $c$ realizes $\bigcup_{d \in J} q(x, d)$. By Lemma 2.1, $q(x, b)$ does not divide over $A$.

Using $a \in sc\ell(A)$ and Lemma 1.5 there is no infinite $A$-indiscernible sequence in $tp(a/A)$. Thus, $tp(b/A \cup \{a\})$ does not divide over $A$. $\square$

**Proposition 2.7.** *Let $(M, \mathcal{R})$ be $\kappa$-simple, and $a$, $b$ and $c$ sequences of length $< \kappa$ such that $tp(a/c)$ is large and $tp(a/b, c)$ is small. Then, $tp(a/b, c)$ divides over $c$ and $tp(b/a, c)$ divides over $c$.*

The complete proof will take several lemmas. Proving that the second of the types divides is straightforward and holds in an almost simple model.

**Lemma 2.8.** *Let $(M, \mathcal{R})$ be almost $\kappa$-simple, and $a$, $b$ and $c$ sequences of cardinality $< \kappa$ with $tp(a/c)$ large and $tp(a/b, c)$ small. Then, $tp(b/a, c)$ divides over $c$.*

*Proof.* This is simply a counting argument. Let $q(x, y) = tp(a, b/c)$. Since $tp(a/c)$ is large, there is an infinite $c$-indiscernible sequence $I$ in $tp(a/c)$. Since $q(x, d)$ is small for any $d$, there cannot be a $d$ realizing $tp(b/A)$ such that $I$ is indiscernible over $c, d$ and $q(a', d)$, for $a' \in I$. By Lemma 2.1, $q(a, y)$ divides over $c$. $\square$

Completing the proof requires a couple of lemmas. While none of the results are especially significant, they are separated into independent lemmas for ease of reference. The following lemma is immediate by the definition of dividing.

**Lemma 2.9.** *If $tp(a/A)$ is large, then $x = a$ divides over $A$.*

**Lemma 2.10.** *Let $(M, \mathcal{R})$ be almost $\kappa$-simple, $a$, $b$ sequences of length $< \kappa$, and $A$ an extension base such that $tp(a/A)$ is large and $tp(a/B \cup \{b\})$ is small for some $B \subset A$, $|B| < \kappa$. Then, $tp(a/B \cup \{b\})$ divides over $A$.*

*Proof.* Let $q(x, b) = tp(a/B \cup \{b\})$. Let $C \subset A$, $|C| < \kappa$, be such that $b$ is free from $A$ over $C$ and $B \subset C$. Notice that $tp(b/A)$ is large since $tp(a/A)$ is large and $tp(a/A \cup \{b\})$ is small. Let $\mu < \lambda$ be a cardinal $> \kappa$ and $I = \{ b_i : i < \mu^* \}$ be a Morley sequence in $tp(b/A)$ over $C$, where $\mu^*$ is the reverse order on $\mu$. For $i < \mu^*$ let $\bar{c}_i$ enumerate the set of realizations of $q(x, b_i)$. Since $q(x, b_i)$ is small we can choose $I$ so that $\{ b_j : j > i \}$ is indiscernible over $A \cup \{ \bar{c}_k b_k : k \leq i \}$ and free from $A \cup \{ \bar{c}_k b_k : k \leq i \}$ over $C$. Given $c' \subset \bar{c}_i$, an arbitrary sequence of length $< \kappa$, $c'$ is free from $A \cup \{ \bar{c}_k b_k : k < i \}$ over $B \cup \{b_i\}$ since $tp(c'/B \cup \{b_i\})$ is small (Lemma 2.6). By Pairs Lemma (Proposition 2.2), $c' b_i$ is free from $A \cup \{ \bar{c}_k b_k : k < i \}$ over $C$, hence, $\bar{c}_i b_i$ is free from $A \cup \{ \bar{c}_k b_k : k < i \}$ over $C$. In particular, $\bar{c}_i$ is free from $\{ \bar{c}_k : k < i \}$ over $C$. By Lemma 2.9 the sequence $\bar{c}_i$ is disjoint from $\bigcup_{k<i} \bar{c}_k$. We conclude that $q(x, b_i) \cup q(x, b_j)$ is inconsistent for $i < j < \lambda^*$. Since $I$ is a Morley sequence, $q(x, b)$ divides over $A$ (Proposition 2.5). $\square$

*Remark* 2.8. Let $(M, \mathcal{R})$ be almost $\kappa$-simple. Properties of the small closure operator significantly affect dividing and the overall structure of the model. The preceding lemma and Remark 2.1(i) show there are relationships between freeness and small closure, but the picture may be very complicated. To organize this discussion, let $\chi_s$ denote the *character of smallness*, i.e., the least cardinal $\chi$ such that if $|\bar{a}| < \kappa$ and $tp(\bar{a}/A)$ is small; then there is $B \subset A$, $|B| < \chi$, such that $tp(\bar{a}/B)$ is small. If $\chi_s \leq \kappa$, then $tp(\bar{a}/A)$ large and $\bar{a}$ free from $B$ over $A$ implies that $tp(\bar{a}/B)$ is large. When $\chi_s > \kappa$, which cannot be ruled out in general, certain results require restricting explicitly to large types or small types. Better results are possible in stable theories.

2.3. **Symmetry and transitivity.** We begin with versions of symmetry and transitivity for sequences of length $< \kappa$ and generalize subsequently.

**Lemma 2.11** (Local Symmetry Lemma)**.** *Let $(M, \mathcal{R})$ be $\kappa$-simple, and $a$, $b$ and $c$ sequences of length $< \kappa$ such that $tp(a/bc)$ does not divide over $c$. Then $tp(b/ac)$ does not divide over $c$.*

*Proof.* If $tp(a/c)$ is small then $tp(b/ac)$ does not divide over $c$ (Lemma 2.6). Suppose $tp(a/c)$ is large. Since $tp(a/bc)$ does not divide over $c$, $tp(a/bc)$ is large (Proposition 2.7). Thus, there is a Morley sequence $I = \{ a_i : i \in \kappa^* \}$ in $tp(a/bc)$ over $c$, where $\kappa^*$ is the reverse order on $\kappa$. Let $q(x, a) = tp(b/ac)$. Since $I$ is a sequence of indiscernibles over $bc$ in $tp(a/bc)$, $b$ realizes $\bigcup_{i \in \kappa^*} q(x, a_i)$. By Proposition 2.5, $q(x, a)$ does not divide over $c$, proving the lemma. $\square$

Following is the version of transitivity that holds for sequences of length $< \kappa$ in a simple theory. As this lemma illustrates, transitivity is simply a combination of symmetry and Pairs Lemma.

**Lemma 2.12** (Local Transitivity). *Let $(M, \mathcal{R})$ be $\kappa$-simple, $c \subset b \subset a$ sequences of length $< \kappa$ and $d$, $|d| < \kappa$, such that $tp(d/a)$ does not divide over $b$ and $tp(d/b)$ does not divide over $c$. Then $tp(d/a)$ does not divide over $c$.*

*Proof.* By Local Symmetry $tp(b/cd)$ does not divide over $c$ and $tp(a/bd)$ does not divide over $b$. Applying Pairs Lemma and $c \subset b \subset a$, $tp(ab/cd)$ does not divide over $c$; hence, $tp(a/cd)$ does not divide over $c$. Thus, $tp(d/a)$ does not divide over $c$, again by Local Symmetry. □

Extending these results about dividing on sequences of length $< \kappa$ to properties of freeness on arbitrary sets requires the following.

**Lemma 2.13** (Weak Transitivity). *Let $(M, \mathcal{R})$ be $\kappa$-simple, $a$, $b$ sequences of length $< \kappa$ and $A$ a set such that $a$ is free from $b$ over $A$. Suppose $ab$ is free from $A$ over $c \subset A$, $|c| < \kappa$. Then $a$ is free from $A \cup \{b\}$ over $c$.*

*Proof.* To begin we claim that $tp(a/cb)$ does not divide over $c$. Suppose, to the contrary, that $tp(a/cb)$ divides over $c$. Let $q(x, y) = tp(ab/c)$. Since $ab$ is free from $A$ over $c$, $b$ is free from $A$ over $c$. Since $tp(b/c)$ must be large (or $a$ would be free from $b$ over $c$), there is an infinite $I$ that is a Morley sequence over $c$ in $tp(b/A)$ and indexed by $\kappa^*$. Using that $a$ is free from $b$ over $A$ and $I$ is indiscernible over $A$, $\bigcup_{d \in I} q(x, d)$ is consistent. Since $I$ is a Morley sequence over $c$, this contradicts that $q(x, b)$ divides over $c$ and Proposition 2.5.

Let $d \subset A$ be a sequence of length $< \kappa$. By symmetry and the choice of $c$, $tp(d/cab)$ does not divide over $cb$. (In fact, the type does not divide over $c$, but that is more than we need.) Combining this fact and the claim with Pairs Lemma, $tp(db/ca)$ does not divide over $c$. By symmetry, $a$ is free from $bd$ over $c$. Thus, $a$ is free from $A \cup \{b\}$ over $c$. □

**Theorem 2.14** (Symmetry Lemma). *Let $(M, \mathcal{R})$ be $\kappa$-simple, and $A$, $B$ and $C$ such that $A$ is free from $B$ over $C$. Then $B$ is free from $A$ over $C$.*

*Proof.* Let $a \subset A$, $b \subset B$ and $c_0 \subset C$ be arbitrary sequences of length $< \kappa$. Let $c_0 \subset c \subset C$, $|c| < \kappa$, be such that $ab$ is free from $C$ over $c$. Since $a$ is free from $b$ over $C$, Lemma 2.13 implies that $tp(a/bc)$ does not divide over $c$. By Local Symmetry (Lemma 2.11) $tp(b/ac)$ does not divide over $c$. Thus $tp(b/a)$ does not divide over $c$ and $tp(b/a)$ does not divide over $C$ (see Remark 2.1). Thus, $b$ is free from $a$ over $C$. This proves the theorem. □

**Corollary 2.15** (Transitivity). *Let $(M, \mathcal{R})$ be $\kappa$-simple, $C \subset B \subset A$ and $D$ such that $D$ is free from $A$ over $B$ and $D$ is free from $B$ over $C$. Then, $D$ is free from $A$ over $C$.*

*Proof.* Let $d \subset D$ and $a \subset A$ be arbitrary sequences of length $< \kappa$. Let $b \subset B$, $|b| < \kappa$, be so that $da$ is free from $B$ over $b$, and let $c \subset C$, $|c| < \kappa$, be such that $db$ is free from $C$ over $c$. By Weak Transitivity (Lemma 2.13) $d$ is free from $a$ over $b$ and $d$ is free from $b$ over $c$. By Local Transitivity, $d$ is free from $a$ over $c$. Thus, $d$ is free from $a$ over $C$, proving the corollary. □

The following applications of symmetry and transitivity will be used later.

**Corollary 2.16.** *Let $(M, \mathcal{R})$ be $\kappa$-simple and $I$ an infinite Morley sequence over $A$ where $a \in I$ has length $< \kappa$.*

*(i) Given $a \in I$, let $A_0 \subset A$ be a set of cardinality $< \kappa$ such that $a$ is free from $A$ over $A_0$. Then, $I$ is a Morley sequence in $tp(a/A)$ over $A_0$.*

*(ii) If $I$ is indiscernible over $B \supset A$, then $I$ is a Morley sequence in $tp(a/B)$ over $A$.*

*Proof.* Let $I = \{ a_i : i \in X \}$. (i) By Transitivity, for any $i \in X$, $a_i$ is free from $A \cup a_{<i}$ over $A_0$. By Lemma 2.3, $I$ is a Morley sequence in $tp(a/A)$ over $A_0$.

(ii) Without loss of generality, $X$ is isomorphic to $\kappa^*$. Let $b \subset B$ and $c \subset A$ be sequences of length $< \kappa$, $Y \subset X$, $|Y| < \kappa$, $d = I_Y$ and $q(x, y, z) = tp(b, d, a_i/c)$, for $i \in X$, $Y < i$. Let $J = \{ a_i : i \in X, Y < i \}$. Since $q(b, d, a)$ holds for all $a \in J$, $q(xy, a)$ does not divide over $c$ by Proposition 2.5. By symmetry, $tp(a_i/bdc)$ does not divide over $c$, for $Y < i$. Thus, $I$ is a Morley sequence in $tp(a/B)$ over $A$ (using Lemma 2.3). □

In an almost simple model, symmetry holds for some special sets.

**Lemma 2.17** (Almost Symmetry Lemma). *Let $(M, \mathcal{R})$ be almost $\kappa$-simple, and $A \subset B$ with $B$ an extension base and $b$ such that $tp(b/B)$ is large and $b$ is free from $B$ over $A$. Then, $B$ is free from $b$ over $A$.*

*Proof.* The hypotheses of the lemma and Lemma 2.4 yield a Morley sequence $I = \{ a_i : i \in \kappa^* \}$ in $tp(b/B)$ over $A$, where $\kappa^*$ is the reverse order on $\kappa$. A fortiori, $I$ is a Morley sequence over $A$. Let $q(x, b)$ be a type over $c \cup b$ for some $c \subset A$, $|c| < \kappa$, satisfied by some $d \in B$, $|d| < \kappa$. Since $I$ is indiscernible in $tp(b/B)$, $q(d, a_i)$ holds for all $i \in \kappa^*$. Since $I$ is a Morley sequence over $A$, $q(x, b)$ does not divide over $A$ by Proposition 2.5. Thus, $B$ is free from $b$ over $A$. □

More general forms of transitivity in arbitrary almost simple theories come down to applying Pairs Lemma and the form of the Symmetry Lemma that holds in that context.

**Definition 2.7.** Let $(M, \mathcal{R})$ be simple. A set of elements $I$ is *independent* over $A$ if for all $a \in I$, $a$ is free from $I \setminus \{a\}$ over $A$.

*Remark* 2.9. By Symmetry and Transitivity, if $I$ is a Morley sequence over $A$ then $I$ is independent over $A$.

## 3. Type amalgamation

The main result in the section is colloquially known as "type amalgamation" and stated as Theorem 3.8. This result lends insight into the question of when distinct free extensions of a type have a common free extension. This is critical to applying the freeness relation to induce geometrical structure properties. In particular, properties of the parallelism relation (see Section 4.1) depend heavily on type amalgamation. The theorem involves the concept of a Lascar strong type, developed in the next subsection.

In the first-order setting the Type Amalgamation Theorem was originally called the Independence Theorem. A rudimentary version was found in [She80]. The theorem here generalizes the first-order version proved by Kim and Pillay [KP].

3.1. **Lascar strong types.** In this subsection $(M, \mathcal{R})$ is an arbitrary $s\lambda$-homogeneous model satisfying $(\mathbf{P})$. As usual, all sets referenced are considered to be subsets of $M$ of cardinality $< \pi$. Dividing theory is not used here. The notion of Lascar strong type was introduced by Lascar in [Las82].

We let $SE^\mu(A)$ be the set of $A$-invariant equivalence relations on $M^\mu$ with a bounded ($< \lambda$) number of equivalence classes. Let $SE(A) = \bigcup_\mu SE^\mu(A)$.

**Definition 3.1.** Tuples $a$, $b$ of the same length have the same *Lascar strong type* over $A \subset M$, written $lstp(a/A) = lstp(b/A)$, if $E(a,b)$ whenever $E \in SE^{\ell(a)}(A)$.

**Lemma 3.1.** *If $tp(a/A)$ is small and $lstp(b/A) = lstp(a/A)$, then $b = a$.*

*Proof.* Let $p = tp(a/A)$. The equivalence relation defined by

$$(p(x) \longleftrightarrow p(y)) \wedge (p(x) \longrightarrow x = y)$$

is $A$-invariant with a bounded number of classes (since $p$ is small). □

**Lemma 3.2.** *If $I$ is an infinite indiscernible sequence over $A$, then $E(a,b)$, for any $E \in SE^{\ell(a)}(A)$ and $a, b \in I$.*

*Proof.* By Lemma 1.5 and the $A$-invariance of $E$, $I$ can be extended to any length $\leq \lambda$. If $\neg E(a,b)$ then $\neg E(c,d)$ for any $c \neq d$ in $I$. Hence, there are unboundedly many equivalence classes. □

The (proof of the) next lemma shows that equality of Lascar strong types over $A$ is the finest equivalence relation in $SE(A)$. Thus, there are fewer than $\lambda$ Lascar strong types over $A$.

**Lemma 3.3.** *For tuples $a$, $b$ of the same length and a set $A$ with $tp(a/A)$ and $tp(b/A)$ large, the following are equivalent:*

(1) *There exists $n < \omega$ and $a = a_0, a_1, a_2, \ldots, a_n = b$ such that for each $i < n$ there exists an infinite $A$-indiscernible sequence containing $a_i$ and $a_{i+1}$;*
(2) *$lstp(a/A) = lstp(b/A)$.*

*Proof.* (1) implies (2) follows from the previous lemma and transitivity of equivalence. For (2) implies (1), call $E$ the equivalence relation defined by (1). Notice that $E$ is $A$-invariant. Suppose $E$ had unboundedly many equivalence classes and let $\{ a_i : i < \mu \}$ be inequivalent elements for some suitably large $\mu \geq \lambda$. By Lemma 1.4, there exists $\{ d_i : i < \omega \}$ indiscernible over $A$, such that $tp(d_0, d_1/A) = tp(a_{j_0}, a_{j_1}/A)$ for some $j_0 < j_1 < \mu$. By the definition of $E$, we have $E(d_0, d_1)$. Hence, $E(a_{j_0}, a_{j_1})$, by $A$-invariance, contradicting the choice of $\{ a_i : i < \mu \}$. Thus, $E \in SE^{\ell(a)}(A)$. Therefore, if $lstp(a/A) = lstp(b/A)$, then $E(a,b)$ holds so that (1) holds. □

**Definition 3.2.** Let $\mathrm{Saut}_A(M)$ be the set of $f \in \mathrm{Aut}_A(M)$ such that for each $a \in M$, $lstp(f(a)/A) = lstp(a/A)$.

$\mathrm{Saut}_A(M)$ is a group, called the *group of strong automorphisms over $A$*. Furthermore, $\mathrm{Saut}_A(M)$ is a normal subgroup of $\mathrm{Aut}_A(M)$: if $f \in \mathrm{Saut}_A(M)$ and $g \in \mathrm{Aut}_A(M)$, then $E(g(a), fg(a))$ by definition of $f$, hence $E(a, g^{-1}fg(a))$, by $A$-invariance.

**Lemma 3.4.** *The following conditions are equivalent:*

(1) $lstp(a/A) = lstp(b/A)$.
(2) *There exists* $f \in \mathrm{Saut}_A(M)$ *such that* $f(a) = b$.

*Proof.* (2) implies (1) follows immediately from the definition. We show (1) implies (2). Define $E(a, b)$ if there exists $f \in \mathrm{Saut}_A(M)$ with $f(a) = b$. This is clearly an equivalence relation since $\mathrm{Saut}_A(M)$ is a group. Notice that it is also $A$-invariant since $\mathrm{Saut}_A(M)$ is normal in $\mathrm{Aut}_A(M)$. Hence, it is enough to show that $E \in SE(A)$. Suppose not and let $\{ a_i : i < \lambda^+ \}$ be a large set of $E$-inequivalent elements. Let $B$ be a bounded set extending $A$, containing a representative of every Lascar strong type over $A$. By the pigeonhole principle, there exists $i < j$ such that $tp(a_i/B) = tp(a_j/B)$. Let $g \in \mathrm{Aut}_B(M)$ such that $g(a_i) = a_j$. But $g \in \mathrm{Saut}_A(M)$, a contradiction. □

**Corollary 3.5.** *If* $lstp(a/A) = lstp(b/A)$ *and* $c \in sc\ell(A)$*, then* $lstp(a/A \cup \{c\}) = lstp(b/A \cup \{c\})$.

*Proof.* This is immediate by Lemma 3.4 and Lemma 3.1. □

**Lemma 3.6.** *If* $lstp(a/A) = lstp(a'/A)$ *and* $b$ *is given, then there is* $b' \in M$ *such that* $lstp(ab/A) = lstp(a'b'/A)$.

*Proof.* By Lemma 3.4, there exists $f \in \mathrm{Saut}_A(M)$ such that $f(a) = a'$. Let $b' = f(b)$. Another application of the that lemma shows the conclusion. □

In a simple model the Extension Property extends to Lascar strong types.

**Lemma 3.7** (Strong Extension)**.** *Let* $(M, \mathcal{R})$ *be* $\kappa$*-simple,* $B \supset A$ *and* $a$ *such that* $tp(a/A)$ *is large. Then, there is* $b$ *such that* $lstp(b/A) = lstp(a/A)$ *and* $b$ *is free from* $B$ *over* $A$.

*Proof.* Since $tp(a/A)$ is large, there is an infinite Morley sequence $I$ over $A$ with $a \in I$. Using Lemma 1.5 there is an infinite sequence of indiscernibles $J$ such that $I\hat{\ }J$ is indiscernible over $A$ and $J$ is indiscernible over $B$. Then $J$ is a Morley sequence over $A$ and $b \in J \implies lstp(b/A) = lstp(a/A)$. By Corollary 2.16, given $b \in J$, $J$ is a Morley sequence in $tp(b/B)$ over $A$. In particular, $b$ is free from $B$ over $A$. □

3.2. **Type amalgamation theorem.** This subsection is devoted to the proof of

**Theorem 3.8** (Type Amalgamation)**.** *Let* $(M, \mathcal{R})$ *be* $\kappa$*-simple, and let* $c$*, and* $b_i$*,* $a_i$*, for* $i = 1, 2$*, be sequences of length* $< \kappa$ *such that*

(1) $tp(b_1/cb_2)$ *does not divide over* $c$*,*
(2) $lstp(a_1/c) = lstp(a_2/c)$*, and*
(3) $tp(a_i/cb_i)$ *does not divide over* $c$.

*Then, there is* $a$ *realizing* $lstp(a_i/cb_i)$*, for* $i = 1, 2$*, such that* $tp(a/cb_1b_2)$ *does not divide over* $c$.

Throughout the subsection $(M, \mathcal{R})$ is $\kappa$-simple. The bulk of the proof will be found in preliminary lemmas that are actually special cases of the theorem. Proposition 2.5 is the main preliminary result here.

**Lemma 3.9.** *Let* $p(x, b)$ *be a type over* $A \cup \{b\}$*, where* $|b| < \kappa$*, which does not divide over* $A$*, and* $I$ *an infinite Morley sequence in* $tp(b/A)$*. Then, for any* $b_0, b_1 \in I$*,* $p(x, b_0) \cup p(x, b_1)$ *does not divide over* $A$.

*Proof.* Without loss of generality, $I = \{ a_i : i \in \kappa^* \}$ where $\kappa^*$ is the reverse order on $\kappa$. Let $X$ be a suborder of $\kappa^*$ such that $X$ is coinitial in $\kappa^* \setminus X$, $\kappa^* \setminus X$ is coinitial in $X$ and $X$ is isomorphic to $\kappa^*$. Let $f$ be an injective function from $X$ into $\kappa^* \setminus X$ such that $i < f(i)$, for $i \in X$. Let $J = \{ a_i a_{f(i)} : i \in X \}$. It is routine to show that $J$ is a Morley sequence in $tp(d/A)$, for $d \in J$. For $d = a_i a_{f(i)} \in J$, let $q(x, d) = p(x, a_i) \cup p(x, a_{f(i)})$.

Since $p(x, b)$ does not divide over $A$, there is a $c$ realizing $\bigcup_{e \in I} p(x, e)$. Thus, $c$ realizes $\bigcup_{d \in J} q(x, d)$. By Proposition 2.5, $q(x, d)$ does not divide over $A$, proving the lemma. □

**Lemma 3.10.** *Let $A$ be a set of cardinality $< \kappa$, and $I = \{ a_i : i \in X \}$ an infinite $A$-indiscernible sequence with $|a| < \kappa$, for $a \in I$. Then there is $Y \subset X$, $|Y| < \kappa$, such that $K = \{a_i : Y < i, i \in X \}$ is a Morley sequence over $A \cup \{ a_i : i \in Y \}$.*

*Proof.* Let $X' = X + \{x\}$ be the order obtained by adding a single element $x$ to the end of $X$. Let $a_x$ be a sequence in $M$ such that $I' = \{ a_i : i \in X' \}$ is $A$-indiscernible. By the $\kappa$-simplicity of $M$ there is $Y \subset X$, $|Y| < \kappa$, such that $a_x$ is $\kappa$-free from $A \cup I$ over $A \cup J$, $J = \{ a_i : i \in Y \}$. Let $Z = \{ j \in X : Y < j \}$ and $K = \{ a_i : i \in Z \}$. If $i \in Z$, then $a_i$ and $a_x$ have the same type over $A \cup J \cup (K \cap a_{<i})$. Thus, $a_i$ is $\kappa$-free from $K \cap a_{<i}$ over $A \cup J$. By Lemma 2.3 and $\kappa$-simplicity, $K$ is a Morley sequence over $A \cup J$. This proves the lemma. □

We now weaken the hypothesis in Lemma 3.9 from a Morley sequence to an arbitrary indiscernible sequence.

**Lemma 3.11.** *Let $|A| < \kappa$ and $p(x, b)$, $|b| < \kappa$, a type over $b$ that does not divide over $A$. Let $I$ be an infinite $A$-indiscernible sequence in $tp(b/A)$. Then, for any $b_0$, $b_1 \in I$, $p(x, b_0) \cup p(x, b_1)$ does not divide over $A$.*

*Proof.* Without loss of generality, $I$ is indexed by $\kappa$. By Lemma 3.10, there is $\alpha < \kappa$ so that $K = \{a_i : \alpha \leq i < \kappa\}$ is a Morley sequence over $A \cup J$, for $J = \{ a_i : i < \alpha \}$. For $a \in K$, $p(x, a)$ does not divide over $A$, hence it does not divide over $A \cup J$. By the Extension Property there is a $c$ realizing $p(x, a)$, such that $q(x, a) = tp(c/A \cup J \cup \{a\})$ does not divide over $A$. Since $K$ is a Morley sequence over $A \cup J$, for $a$, $a' \in K$, $q(x, a) \cup q(x, a')$ does not divide over $A \cup J$. Thus, there is $d$ realizing $q(x, a) \cup q(x, a')$ such that $tp(d/A \cup J \cup \{a, a'\})$ does not divide over $A \cup J$. Since $d$ realizes $q(x, a)$, $tp(d/A \cup J)$ does not divide over $A$. By Transitivity (Corollary 2.15), $tp(c/A \cup J \cup \{a, a'\})$ does not divide over $A$. A fortiori, $p(x, a) \cup p(x, a')$ does not divide over $A$. □

**Lemma 3.12.** *Let $|A| < \kappa$ and $p(x, a)$ and $q(x, b)$ be types over $A \cup \{a\}$ and $A \cup \{b\}$ respectively. Assume that $lstp(b/A) = lstp(b'/A)$ and that $tp(a/Abb')$ does not divide over $A$. If $p(x, a) \cup q(x, b)$ does not divide over $A$, then $p(x, a) \cup q(x, b')$ does not divide over $A$.*

*Proof.* By Lemma 3.3, let $b_0 = b$, $b_1, \ldots, b_n = b'$ be a sequence such that there exist $A$-indiscernible sequences containing $b_i$ and $b_{i+1}$, for $i < n$. By the extension property, we may find $a'$ realizing $tp(a/Abb')$ such that $tp(a'/Ab_0 \ldots b_n)$ does not divide over $A$. Hence, by using an automorphism fixing $b$ and $b'$, we may assume that $tp(a/Ab_0 \ldots b_n)$ does not divide over $A$. Hence, $tp(a/Ab_i b_{i+1})$ does not divide over $A$, so it is enough to show the conclusion when $b$, $b'$ belong to the same indiscernible sequence $I$ (since then $p(x, a) \cup q(x, b_1)$ does not divide over $A$, and so $p(x, a) \cup q(x, b_2)$ does not divide over $A$, etc.).

*Claim.* There is an $A$-indiscernible sequence $\{ a^i b^i : i \in \mathbb{Z} \}$ such that for each $i \in \mathbb{Z}$, $tp(a^i, b^i, b^{i+1}/A) = tp(a, b, b'/A)$.

Let $\alpha$ be a cardinal, $\pi < \alpha \leq \lambda$, and $\alpha^*$ the order with $\alpha$ reversed with the elements of $\alpha^*$ denoted $-i$, for $i \in \alpha$. Since $I$ is $A$-indiscernible, we may in fact assume that it is of the form $\{ b_j : j \in \alpha^* + \alpha \}$, $b = b_{-0}$ and $b' = b_0$. Notice that $I' = \{ b_{-n}\hat{\ } b_n : n < \alpha \}$ is also indiscernible over $A$. Since $tp(a/bb'A)$ does not divide over $A$, Lemma 2.1 says we can choose $I$ so that $I'$ is indiscernible over $A \cup \{a\}$. Notice that $tp(ab_{-n}/A) = tp(ab/A)$ and $tp(ab_n/A) = tp(ab'/A)$, for each $n < \alpha$. By the indiscernibility of $I$ and the homogeneity of $M$, for $j \in \alpha^* + \alpha$ there is an $a_j$ such that $tp(a_j b_k/A) = tp(ab/A)$ for $k \leq j$ and $tp(a_j b_l/A) = tp(ab'/A)$ for $l > j$. By Lemma 1.4, there is an $A$-indiscernible sequence $\{ a^i b^i : i \in \mathbb{Z} \}$ such that $tp(a^0 b^0, \ldots, a^n b^n/A) = tp(a_{i_0} b_{i_0}, \ldots, a_{i_n} b_{i_n}/A)$ for some $i_0 < \cdots < i_n \in \alpha^* + \alpha$. In particular, $tp(a^i b^i b^{i+1}/A) = tp(abb'/A)$, completing the proof of the claim.

Since $p(x, a^0) \cup q(x, b^0)$ does not divide over $A$, $p(x, a^0) \cup q(x, b^0) \cup p(x, a^1) \cup q(x, b^1)$ does not divide over $A$, by Lemma 3.11. In particular, $p(x, a^0) \cup q(x, b^1)$ does not divide over $A$. Since $tp(a^0 b^1/A) = tp(ab'/A)$, the lemma is proved. □

*Proof of Theorem 3.8.* We seek an $a$ realizing $lstp(a_i/cb_i)$, for $i = 1, 2$, such that $tp(a/cb_1 b_2)$ does not divide over $c$. We *claim* that it is enough to find an $a$ realizing $tp(a_i/cb_i)$, for $i = 1, 2$, such that $tp(a/cb_1 b_2)$ does not divide over $c$. As we will see below, the hypotheses imply that $tp(a_i/cb_i)$ is large. For $i = 1, 2$ let $e_i, f_i$ be such that $\{a_i, e_i, f_i\}$ is contained in a Morley sequence in $tp(a_i/cb_i)$ over $c$. We can choose these new elements so that $tp(e_1 f_1/cb_1 b_2 e_2 f_2)$ does not divide over $c$. Let $b'_i = b_i e_i$ and consider $q_i = tp(f_i/cb'_i)$. Noting also that $lstp(f_1/c) = lstp(f_2/c)$, the special case assumed in this claim implies there is an $a$ realizing $q_1 \cup q_2$ such that $tp(a/cb'_1 b'_2)$ does not divide over $c$. Since $e_1$, $f_1$, $a_1$ are in an indiscernible sequence over $cb_1$, $lstp(a/cb_1) = lstp(e_1/cb_1) = lstp(a_1/cb_1)$, and correspondingly over $cb_2$. This proves the claim.

We first have to deal with the cases when some element is in the small closure of another. Let $p_i(x, y) = tp(a_i b_i/c)$, for $i = 1, 2$. If $a_i \in sc\ell(c)$, then $a_1 = a_2$ (by Lemma 3.1) and we are done. So, we can assume that $tp(a_1/c)$ is large, hence $tp(a_1/cb_1)$ and $tp(a_2/cb_2)$ are large (Proposition 2.7). Let $b'_2$ be such that $lstp(a_1 b'_2/c) = lstp(a_2 b_2/c)$ (Lemma 3.6). Suppose $b'_2 \in sc\ell(a_1 c)$. Since $tp(b'_2/a_1 c)$ does not divide over $c$, $b'_2 \in sc\ell(c)$ , implying that $b'_2 = b_2$, from which the lemma follows easily. So, we are left with the case when $tp(b'_2/ca_1)$ is large. By Strong Extension (Lemma 3.7) we can require that $tp(b'_2/ca_1 b_1 b_2)$ does not divide over $ca_1$. By Transitivity, $tp(b'_2/ca_1 b_1 b_2)$ does not divide over $c$, hence $tp(b'_2/cb_1 b_2)$ does not divide over $c$. By Symmetry and Transitivity, $tp(b_1/cb_2 b'_2)$ does not divide over $c$. Also, by several applications of symmetry and transitivity, $tp(a_1/cb_1 b'_2)$ does not divide over $c$. Thus, $p_1(x, b_1) \cup p_2(x, b'_2)$ does not divide over $c$. By Lemma 3.12, $p_1(x, b_1) \cup p_2(x, b_2)$ does not divide over $c$. □

The most important application of Type Amalgamation is the connection of parallelism with freeness found in the next section.

## 4. Parallelism, imaginary elements and canonical bases

One of the central concepts of geometrical stability theory (for first-order stable theories) is the notion of a definable family of uniformly definable sets. In algebraic geometry a uniform family of plane curves is a family $\{ X_{\bar{d}} : \bar{d} \in Y\}$ of one-dimensional subsets $X_{\bar{d}}$ of $K^2$, where $X_{\bar{d}}$ is, for some polynomial $f(x, y, \bar{z})$, the solution set of $f(x, y, \bar{d}) = 0$, and $\bar{d}$ ranges over the elements of the variety $Y \subset K^n$. Of course, for a smooth theory distinct elements of the family should be almost disjoint; i.e., have finite intersection. In this case, the dimension of $Y$ is also called the dimension of the family, a number which affects the intersection theory of the family and other properties. This is one reason for normally restricting to irreducible varieties $X_{\bar{d}}$. Also we require $f$ to be normalized so that $\bar{d}$ is uniquely determined by $X_{\bar{d}}$.

In a simple model we will have notions of dimension that mimic the dimension of varieties without a great deal of work. However, new concepts need to be developed to play the part of the Zariski topology. The first notion is parallelism, an equivalence relation that expresses when types have a common extension. When it exists in the model, a *canonical base* is an equivalence class of the parallelism relation and acts like the field of definition of a variety. However, in this most general setting of a simple $s\lambda$-homogeneous model the introduction of canonical bases as hyperimaginary elements does not proceed as it does in the first-order case (i.e., when the model is saturated). More will be said on this in the next subsection.

Throughout the section $(M, \mathcal{R})$ is an $s\lambda$-homogeneous $\kappa$-simple logical structure.

**Definition 4.1.** A large type $p \in S(A)$ is an *amalgamation base* if type amalgamation holds for $p$; i.e., for any $B$, $C$, with $|B|$, $|C| < \kappa$ and $B$ free from $C$ over $A$, and types $q$ over $B$ and $r$ over $C$ such that $p$ has a free extension over $A \cup B$ containing $q$ and a free extension over $A \cup C$ containing $r$, $p$ has a free extension over $A \cup B \cup C$ containing $q \cup r$.

*Remark* 4.1. When $|A| < \kappa$, a large $p \in S(A)$ is an amalgamation base if and only if for any $a$, $b$ realizing $p$, $lstp(a/A) = lstp(b/A)$. [The direction from right to left is virtually a restatement of the Type Amalgamation Theorem. To prove the other direction, suppose that $a$ and $b$ are any two realizations of $p$ such that $tp(a/bA)$ does not divide over $A$. There are types $q(x, a)$ and $r(x, b)$ over $A \cup \{a\}$ and $A \cup \{b\}$, respectively, which do not divide over $A$, such that if $c$ realizes $q(x, a)$ and $d$ realizes $r(x, b)$, then $lstp(c/A) = lstp(a/A)$ and $lstp(d/A) = lstp(b/A)$. Since $p$ is an amalgamation base there is a $c$ realizing $q(x, a) \cup r(x, b)$. Thus, $lstp(a/A) = lstp(b/A)$.]

4.1. **Parallelism.** In a stable theory stationary types are parallel if they have a common free extension. In a simple homogeneous model (or even a simple theory) the role of parallelism is played by a slightly more complicated concept about amalgamation bases.

**Definition 4.2.** Given $p \in S(A)$ and $q \in S(B)$ amalgamation bases, we write $p \sim_1 q$ if $p \cup q$ does not divide over $A$ and does not divide over $B$. We write $p \sim q$ and say $p$ is *parallel* to $q$ if there are amalgamation bases $q_0, \dots, q_k$ such that $p = q_0 \sim_1 q_1 \sim_1 \cdots \sim_1 q_k = q$.

*Remark* 4.2. Parallelism is clearly an equivalence relation on the class of amalgamation bases. In fact it is the transitive closure of $\sim_1$. Parallelism is invariant under automorphisms.

Suppose $p \in S(a)$ is an amalgamation base and $p' \in S(a')$ is conjugate to $p$, where $a$ is free from $a'$ over $\emptyset$. If $p$ does not divide over $\emptyset$ then Type Amalgamation says that $p \cup p'$ does not divide over $\emptyset$ (assuming there are $b$ realizing $p$ and $b'$ realizing $p'$ with $lstp(b) = lstp(b')$). In this case $p \sim p'$. This suggests that "most" conjugates are parallel if $p$ does not divide over $\emptyset$. This is formalized in the next result, which guides all uses of parallelism.

**Proposition 4.1.** *Let $p \in S(a)$ be an amalgamation base with $|a| < \kappa$, $b$ a sequence and $\mathbf{P}$ the class of types conjugate to $p$ over $b$. Then $p$ does not divide over $b$ if and only if $\mathbf{P}$ contains a bounded number of parallelism classes.*

*Proof* (of the left to right direction). Suppose that $p$ does not divide over $b$. There is a subsequence $b_0$ of $b$ of length $< \kappa$ such that $p$ does not divide over $b_0$ so we may as well assume $|b| < \kappa$. For $q$ a Lascar strong type over $b$ let $\mathbf{P}_q$ be the set of $p' \in \mathbf{P}$, $p' \in S(a')$ such that there is $c$ realizing $p' \cup q$ and $tp(c/a'b)$ does not divide over $b$.

*Claim.* If $p_0$, $p_1 \in \mathbf{P}_q$, for some Lascar strong type $q$ over $b$, then $p_0 \sim p_1$.

Given $p_i \in S(a_i)$ in $\mathbf{P}_q$, for $i = 0, 1$, let $r \in S(c)$ in $\mathbf{P}_q$ such that $tp(c/ba_0a_1)$ does not divide over $b$. By Type Amalgamation there are, for $i = 0, 1$, $c_i$ realizing $p_i \cup r$ such that $tp(c_i/cba_i)$ does not divide over $b$. Thus, $p_i \sim_1 r$, for $i = 0, 1$, hence $p_0 \sim p_1$, proving the claim.

Since the class of Lascar strong types over $b$ is bounded, $\mathbf{P}$ contains a bounded number of parallelism classes. $\square$

The opposite direction of the proposition requires some detailed analysis of the behavior of amalgamation bases. Recall that if $(M, \mathcal{R})$ is almost $\kappa$-simple, $A \underset{C}{\perp} B$ denotes $A$ is free from $B$ over $C$.

**Lemma 4.2.** *Let $r \in S(d)$ be an amalgamation base, where $|d| < \kappa$, $I$ a Morley sequence in $r$ over $d$ indexed by $\kappa^*$ and $c$ a realization of $r$ free from $I$ over $d$. Then, $c$ is free from $d$ over $I$.*

*Proof.* In the special case where $I\hat{}\langle c\rangle$ is $d$-indiscernible, this follows basically from Lemma 3.10. In general we will use this fact and that $r$ is an amalgamation base. Recall Remark 2.9 about independence of Morley sequences.

*Claim.* There is a sequence $J$ of length $< \kappa$ such that $I\hat{}J$ is $d$-indiscernible, $I$ is independent over $J$ and $d$ is free from $I\hat{}J$ over $J$.

To see this, first let $I'$ be a sequence indexed by $\kappa^*$ such that $I\hat{}I'$ is indiscernible over $d$. Consider $I'$ as an indiscernible sequence in the reverse order (indexed by $\kappa$) and apply Lemma 3.10 to obtain a sequence $J \subset I'$ of length $< \kappa$ such that, letting $i_0 = \inf\{\, j : a_j \in J \,\}$, $I'' = \{\, a_i \in I' : i < i_0 \,\}$ is a Morley sequence over $J$ (under the reverse order). Thus, $I''$ is independent over $J$. Since $I\hat{}I''$ is indiscernible over $J$, $I$ is independent over $J$. Now suppose $\bar{a}$ is a finite subset of $\bigcup I \cup \bigcup J$ and suppose, towards a contradiction, that $tp(\bar{a}/J \cup d)$ divides over $J$. There is a $K \subset I$, $|K| < \kappa$, such that $d$ is free from $I \cup J$ over $K \cup J$. By the indiscernibility of $I\hat{}J$ over $d$ we can assume $\bar{a}$ is disjoint from $K$, hence free from $K$ over $J$. Since $\bar{a}$ is free from $d$ over $J \cup K$, transitivity of independence implies that $\bar{a}$ is free from $d$ over $J$. This contradiction proves the claim.

Arguing as in the claim there is also a (nonempty) $K \subset I$, $|K| < \kappa$, such that $I \cup J$ is free from $d$ over $K$. Let $s(x, z) = tp(J, K/d)$. Let $c'$ be any element of $K$ and $q(x, y) = tp(J, c'/d)$.

*Claim.* There is $J''$ free from $K \cup c$ over $d$ such that $q(J'', c)$ and $s(J'', K)$ hold.

We first show there is $J'$ realizing $lstp(J/d)$ and $q(x, c)$. Since $r$ is an amalgamation base, $lstp(c/d) = lstp(c'/d)$. So, there is $f$, a strong automorphism over $d$ taking $c'$ to $c$. Then $lstp(f(J)/d) = lstp(J/d)$ and $q(f(J), c)$ holds. Since $J$ is free from $c'$ over $d$ and $K$ is free from $c$ over $b$, Type Amalgamation yields $J''$ free from $K \cup c$ over $d$ such that $q(J'', c)$ and $s(J'', K)$ hold.

The following chain of arguments shows that $d \underset{K}{\downarrow} c$: $J'' \underset{d}{\downarrow} K \cup \{c\} \implies J'' \underset{K \cup \{d\}}{\downarrow} c$ (transitivity) $\implies c \underset{K \cup \{d\}}{\downarrow} J''$ (symmetry) $\implies c \underset{d}{\downarrow} J'' \cup K$ (transitivity and the independence of $c$ and $K$ over $d$) $\implies c \underset{J'' \cup \{d\}}{\downarrow} K$ (transitivity) $\implies c \underset{J''}{\downarrow} K \cup \{d\}$ (transitivity and the fact that $q(J'', c)$ holds) $\implies c \underset{K \cup J''}{\downarrow} d$ (transitivity) $\implies d \underset{K \cup J''}{\downarrow} c$ (symmetry) $\implies d \underset{K}{\downarrow} \{c\} \cup J''$ (transitivity and the fact that $s(J'', K)$ holds) $\implies d \underset{K}{\downarrow} c$ (a fortiori).

Finally, to show that $d \underset{I}{\downarrow} c$, it suffices to prove that $d \underset{K}{\downarrow} I \cup \{c\}$ by transitivity and the independence of $d$ and $I$ over $K$. It is enough to show that $d \underset{K}{\downarrow} I' \cup \{c\}$, for any $I' \subset I$ with $|I'| < \kappa$. The argument above can be repeated with $K \cup I'$ replacing $K$ to prove that $d \underset{K \cup I'}{\downarrow} c$. An application of transitivity then completes the proof. □

**Lemma 4.3.** *If $q \in S(c)$ is an amalgamation base, $q' \in S(c')$ is conjugate to $q$ and $q \sim q'$, then there is a realizing $q'$ such that $a \underset{c'}{\downarrow} c$ and $a \underset{c}{\downarrow} c'$.*

*Proof.* By the definition of parallelism there are $q_i$, $i \leq k$, such that $q = q_0 \sim_1 \cdots \sim_1 q_{k-1} \sim_1 q_k = q'$. The proof is by induction on $k$, so we assume there is $r \in S(b)$ conjugate to $q$, $r \sim_1 q'$, and there is $d$ realizing $r$ such that $tp(d/cb)$ does not divide over $b$ and does not divide over $c$. Since $r \sim_1 q'$ there is $a$ realizing $r \cup q'$ such that $a \underset{c'}{\downarrow} b$ and $a \underset{b}{\downarrow} c'$. Without loss of generality, $a \underset{b}{\downarrow} cc'$. To prove that $a \underset{c}{\downarrow} c'$ a Morley sequence in $r$ must be introduced and the preceding lemma applied.

Since $r$ does not divide over $c$, there is a sequence $I$ of length $\kappa$ which is a Morley sequence in $r$, is free from $c$ over $b$, and free from $b$ over $c$. Without loss of generality, $I$ is free from $ac'$ over $bc$. Thus, $a$ is free from $Icc'$ over $b$ (by symmetry and transitivity), and $a$ is free from $cc'$ over $Ib$. $a$ is free from $b$ over $I$ by Lemma 4.2. Also by transitivity, $a$ is free from $cc'b$ over $I$, hence $a$ is free from $c'b$ over $Ic$. Since $I$ is free from $a$ over $c$, $a$ is free from $I$ over $c$, hence $a$ is free from $Ibc'$ over $c$. In particular, $a$ is free from $c'$ over $c$, proving the lemma. □

*Proof of Proposition 4.1* (of right to left direction). Since there are boundedly many parallelism classes in $\mathbf{P}$ it contains $p' \in S(a')$ and $p'' \in S(a'')$ such that $p' \sim p''$ and $a' \underset{b}{\downarrow} a''$. By Lemma 4.3 there is $c$ realizing $p''$ such that $c \underset{a''}{\downarrow} a'$ and $c \underset{a'}{\downarrow} a''$. Taking a free extension of $tp(c/a'a'')$ if necessary, we can require that $c$ is free from $b$ over $a'a''$. Since $c$ is free from $a''$ over $a'b$, $a''$ is free from $c$ over $a'b$.

Using $a' \underset{b}{\downarrow} a''$ and transitivity, $a''$ is free from $c$ over $b$, hence $c$ is free from $a''$ over $b$. Since $tp(c/a''b)$ is a free extension of $p''$, $p''$ does not divide over $b$, completing the proof. Since $p''$ is conjugate to $p$ over $b$, $p$ does not divide over $b$. $\square$

4.2. **Canonical bases.** In a simple theory classes of the parallelism relation are captured as hyperimaginary elements. This "internalizes" parallelism and allows the study of the properties of canonical bases with respect to dividing. Bypassing the question of existence in an arbitrary simple $s\lambda$-homogeneous model, a canonical base is defined as follows.

**Definition 4.3.** Let $(M, \mathcal{R})$ be $\kappa$-simple.

(i) Given $p \in S(a)$ an amalgamation base, a set $C$ is a *canonical base for* $p$ if for any automorphism $\alpha$, $\alpha(p)$ is parallel to $p$ if and only if $\alpha$ fixes $C$ pointwise.

(ii) $(M, \mathcal{R})$ is said to have *built-in canonical bases* if for any amalgamation base $p$ there is $C \subset M$ that is a canonical base for $p$.

If $M$ is a simple saturated model, $M^h$, the expansion of $M$ with hyperimaginaries is simple and has built-in canonical bases. While such an expansion does not seem possible for an arbitrary simple $s\lambda$-homogeneous model, there are natural examples with built-in canonical bases. In particular, Berenstein showed in [BB02] that many pairs $(H, T)$ where $H$ is an $s\lambda$-homogeneous Hilbert space and $T$ is a bounded linear operator on $H$, are simple and have built-in canonical bases.

4.3. **Imaginary and hyperimaginary elements.** Imaginary elements were introduced by Shelah to enable us to work with the classes of first-order definable equivalence relations as if they were elements of the model. Hyperimaginary elements were introduced in [HKP00] to capture canonical bases in all simple theories. In that setting a hyperimaginary element is a class of a type-definable equivalence relation in possibly infinitely many variables.

In general there is no reason to think the canonical base exists as a tuple from the model, as it does in an algebraically closed field. In a stable theory this deficiency is removed by expanding from $M$ to $M^{eq}$, which contains the classes of any definable equivalence relation in $M$. When $M$ is saturated so is $M^{eq}$, thus the expanded universe satisfies the conditions under which stability theory is developed. In fact, it is standard to simply work in an expanded universe containing classes for each definable equivalence relation; i.e., a model with built-in imaginary elements. In a simple theory parallelism may only be type-definable. When the classes of a type-definable equivalence relation are added to a saturated model the resulting model may not be saturated, hence the classical theory cannot be applied here. This is handled with so-called hyperimaginary elements [HKP00].

In a simple homogeneous model that is not saturated it is not clear that we can add classes for equivalence relations in $\mathcal{R}$ and obtain a homogeneous model or that the expanded model is simple when it is homogeneous. Even in the settings where this is possible, the most important equivalence relation, parallelism, may not be $\mathcal{R}$-type definable, much less an element of $\mathcal{R}$. Thus, it seems unreasonable to expect that any simple $s\lambda$-homogeneous model can be expanded to one which is simple and has built-in canonical bases.

Since there seem to be few nontrivial results concerning imaginary and hyperimaginary elements in the general homogeneous setting we will skip the definitions.

The reader is referred to on-going research by Ben-Yaacov [BY02] for results about ultraimaginaries; i.e., classes of invariant equivalence relations.

## 5. Stability

An $s\lambda$-homogeneous model $M$ is *$\mu$-stable* if for any $A \subset M$ with $|A| \leq \mu$, $M$ realizes $\leq \mu$ complete types over $A$. An $s\lambda$-homogeneous model is *stable* if it is $\mu$-stable for some $\mu$ ($< |M|$). As we pointed out in the first section, stability of $s\lambda$-homogeneous models was initiated by Shelah and has been studied extensively by Shelah, Hyttinen, Grossberg, Lessmann and others. A first-order theory is stable if and only if it is simple and there is a uniform bound on the number of free extensions of a complete type. For $s\lambda$-homogeneous models, this fails: stability does not imply simplicity. An example due to Shelah of a stable homogeneous model where the free extension property fails can be found in [HL]; a description of this example is given in the next section. We will therefore study stability under the additional assumption of simplicity. We make the following definition.

**Definition 5.1.** An $s\lambda$-homogeneous model $(M, \mathcal{R})$ is *$\kappa$-simply stable in $\mu < |M|$* if $(M, \mathcal{R})$ is $\kappa$-simple and, if $p$ is a complete $\mathcal{R}$-type over $A \subset M$, $|A| \leq \lambda$, then for any $B$, $A \subset B \subset M$ and $|B| \leq \mu$, there are $\leq \mu$ complete types over $B$ which are free extensions of $p$. $(M, \mathcal{R})$ is *$\kappa$-simply stable* if it is $\kappa$-simply stable in some $\mu < |M|$; and *simply stable* is defined similarly.

*Remark* 5.1. If $M$ is the universal domain of a simple first-order theory and $\mathcal{R}$ is the class of definable relations, then $(M, \mathcal{R})$ is simply stable if and only if $Th(M)$ is stable; i.e., $(M, \mathcal{R})$ is stable as an $s\lambda$-homogeneous model.

We recall a few facts about stability of $s\lambda$-homogeneous models. Suppose $(M, \mathcal{R})$ is a stable $s\lambda$-homogeneous model. We will use the notation of logical structures where Shelah, *et al*, consider structures in a first-order language, however, the proofs are the same, and although we quote the general results, we will only use particular cases.

If $(M, \mathcal{R})$ is stable, then there is a first cardinal, written $\lambda(M)$, for which $(M, \mathcal{R})$ is stable in $\lambda(M)$. A complete type $tp(a/A)$ is said to *split strongly over* $B$, if there is an $R \in \mathcal{R}$ and an infinite $B$-indiscernible sequence $\{ b_i : i < \omega \}$ such that if $b_0, b_1 \in A$, then $R(x, b_0) \in p$ if and only if $R(x, b_1) \in p$. There is a least cardinal, written $\kappa(M)$, such that for each finite $a$, and set $C$, there exists $B \subset C$ of size less than $\kappa(M)$ such that $tp(a/C)$ does not split strongly over $B$. In [She70] (although with these definitions this is done in [GL]), Shelah proves that, with $T = Th(M)$, $\kappa(M) \leq \lambda(M) < \beth_{(2^{|T|})^+}$ and that $M$ is stable in $\mu$ if and only if $\mu \geq \lambda(M)$ and $\mu^{<\kappa(M)} = \mu$. This is the *Stability Spectrum Theorem*. Also in [She70], Shelah showed that if $M$ is stable in $\mu$, $I$ is a set of size $\mu^+$ containing finite sequences, and $A$ has size at most $\mu$, then there is an $A$-indiscernible subset of $I$ of size $\mu^+$.

If $I$ is an $A$-indiscernible sequence (hence set), with $\ell(a) < \kappa(M)$ for each $a \in I$, then if $b$ has size less than $\kappa(M)$, there exists $J \subset I$ of size less than $\kappa(M)$ such that $I \setminus J$ is $Ab$-indiscernible (see [GL]). In particular, for an indiscernible sequence $I$, $R \in \mathcal{R}$ and a sequence $b$, if $R(b, c)$ holds for at least $\kappa(M)$ many $c \in I$, then it holds for all but possibly fewer than $\kappa(M)$ many $c \in I$. This is a property of stability that we will use in the proof of Lemma 5.2.

We will show the following proposition, which is easily obtained from [HS00] and [She90], Lemma III 1.11 (3), given the subsequent lemma. By *Lascar strong* *types are stationary in* $M$, we mean that for each finite $a$ and set $A$, if $b_\ell$ realizes $lstp(a/A)$ and $b_\ell$ is free from $C$ over $A$, for $\ell = 1, 2$, then $tp(b_1/C) = tp(b_2/C)$.

**Proposition 5.1.** *Let* $(M, \mathcal{R})$ *be simple. The following conditions are equivalent:*

(1) $M$ *is stable.*
(2) *Lascar strong types are stationary in* $M$.
(3) *There exist cardinals* $\nu \leq \mu < \beth_{(2^{|\mathcal{R}|^+})}$ *such that for each* $A$ *and for each* $B$ *containing* $A$, *the set*

$$\{ tp(a/B) : tp(a/B) \text{ is free over } A \}$$

*has size at most* $(|A| + \mu)^{<\nu}$.

We first show:

**Lemma 5.2.** *Suppose that* $(M, \mathcal{R})$ *is simple and stable.*

(1) *If* $p$ *splits strongly over* $A$ *then* $p$ *divides over* $A$.
(2) *If* $A$ *is free from* $C$ *over* $B$ *then* $tp(a/B \cup \{c\})$ *does not split strongly over* $B$ *for each finite* $a \in A$, *and* $c \in C$.

*Proof.* (1) We show the contrapositive. Suppose that $p$ does not divide over $A$ and let $I = \{ a_i : i < \mu \}$ be indiscernible over $A$ where $a_0, a_1$ are in the parameters of $p$. Let $R \in \mathcal{R}$ be such that $R(x, a_0) \in p$. We must show that $R(x, a_1) \in p$. Consider $p(x, a_0, a_1) = p \upharpoonright Aa_0a_1$. By $s\lambda$-homogeneity, we may assume that $\mu \geq \kappa(M)$. Notice that $\{ a_{2i}\hat{\ } a_{2i+1} : i < \mu \}$ is indiscernible over $A$. Then $\bigcup_{i<\mu} p(x, a_{2i}, a_{2i+1})$ is realized by some $b \in M$, since $p$ does not divide over $A$. Hence $R(b, a_{2i})$ holds for all $i < \mu$, and by stability, $R(b, a_i)$ holds for all but fewer than $\kappa(M)$ elements $a_i \in I$. This implies that $R(b, a_{2i+1})$ holds for some $i < \mu$. Hence $R(x, a_{2i}), R(x, a_{2i+1}) \in p(x, a_{2i}, a_{2i+1})$, so in particular $R(x, a_1) \in p(x, a_0, a_1)$, i.e. $R(x, a_1) \in p$. The proof that $R(x, a_1) \in p$ implies that $R(x, a_0) \in p$ is similar.

(2) Follows from (1) by definition of freeness (freeness has finite character). $\square$

*Proof of the proposition.* Assume that $(M, \mathcal{R})$ is simple. (1) follows from (2) or (3) by counting types just like in the first-order case (we use the fact that each type is free over a subset of size less than $\kappa$ of its parameters). To see that (1) implies (2), use the preceding lemma and Theorem 3.12 of [HS00], which says that if $tp(a/Ac)$ and $tp(b/Ac)$ do not split strongly over $A$ and have nonsplitting extensions over any set, and if in addition $lstp(a/A) = lstp(b/A)$ then $tp(a/Ac) = tp(b/Ac)$. (1) implies (3) is also as in the first-order case: Let $A$ and $B$ containing $A$ be given. Assume $(M, \mathcal{R})$ is stable. We show that (3) holds with $\nu = \kappa(M)$ and $\mu = \lambda(M)$. By the preceding lemma, it is enough to show that the number of extensions in $S(B)$ which do not split strongly over $A$ is at most $\epsilon = (|A| + \lambda(M))^{<\kappa(M)}$. Let $B' \subset B$ contain a realization for each $lstp(b/A)$ for $b$ a finite sequence in $B$. Then, $B'$ has size at most $\epsilon$: By the stability spectrum theorem, $(M, \mathcal{R})$ is stable in $\epsilon$, so any set of size $\epsilon^+$ contains an infinite $A$-indiscernible subset, and thus contains different realizations of the same Lascar strong type over $A$. Notice that if $p$, $q \in S(B)$ do not split strongly over $A$ and $p \upharpoonright B' = q \upharpoonright B'$, then $p = q$. To see this, suppose $R(x, b) \in p$. Let $b' \in B'$ realize $lstp(b/A)$. Then there exist $n < \omega$ and $b_i$ for $i \leq n$ such that $b_0 = b$, $b' = b_n$ and $b_i, b_{i+1}$ belong to an infinite $A$-indiscernible sequence. Since $p$ does not split strongly over $A$, we have $R(x, b_i) \in p$, for $i \leq n$. Hence $R(x, b') \in p$, so $R(x, b') \in q$, and so $R(x, b) \in q$ by the same argument applied to

$q$. Hence, the number of free extensions over $B$ is bounded by $|S(B')|$ which is at most $\epsilon$, since $(M, \mathcal{R})$ is stable in $\epsilon$. $\square$

## 6. Examples

In this section, we give several examples of simple, or simply stable mathematical structures. Some of these examples clarify the connection between stability and simplicity of a homogeneous structure and its first-order theory. In particular, stability does not imply simplicity and the homogeneous models of a stable first-order theory may not be simple. We also give examples which are simply stable, but whose first-order theory is not. Several examples of groups also illustrate limitations of possible generalizations; we may not have generics, or large abelian subgroups, even under strong stability assumptions. Finally, some of these examples have a natural and immediate description at this level of generality, but a less amenable first-order behavior.

6.1. **Models of simple and stable theories.** The first place to look for examples of simple $s\lambda$-homogeneous models is, of course, the homogeneous models of a simple first-order theory. However, the situation is not so clear. Saturated models $M$ of simple or stable theories are clearly simple, or simply stable homogeneous structures $(M, \mathcal{R})$, where $\mathcal{R}$ is the collection of definable relations. Moreover, freeness agrees with nondividing in this case. It is also clear that a homogeneous model of a stable first-order theory is stable in the sense of homogeneous structures (as it realizes fewer types). However, homogeneous models of even a stable theory may fail to be simple, because they may fail to have the extension property (see the example after the lemma). We have:

**Lemma 6.1.** *Let $M$ be a large uncountable homogeneous model of a stable theory $T$, and $\mathcal{R}$ the collection of definable relations on $M$. For $A \cup \{\bar{a}, \bar{b}\} \subset M$, $tp_{\mathcal{R}}(\bar{a}/\bar{b})$ divides over $A$ implies $tp(\bar{a}/\bar{b})$ divides over $A$ in $T$. Thus, if $\kappa = \kappa(T)$, $M$ is almost $\kappa$-simple.*

*Proof.* We are assuming $|M| > |T|^+$, hence $|M| > \kappa$. Suppose $tp(\bar{a}/\bar{b})$ does not divide over $A$ in $T$. If $tp_{\mathcal{R}}(\bar{b}/A)$ is small, then $tp_{\mathcal{R}}(\bar{a}/\bar{b})$ does not divide over $A$. So, suppose $tp_{\mathcal{R}}(\bar{b}/A)$ is large and let $I$ be any indiscernible sequence in $tp_{\mathcal{R}}(\bar{b}/A)$ of cardinality $> \kappa$. For $\bar{b}_0 \in I$ there is $\bar{a}'$ such that $q(\bar{x}, \bar{y}) = tp(\bar{a}'\bar{b}_0/A) = tp(\bar{a}\bar{b}/A)$. There is $J \subset I$, $|J| < \kappa$, such that $\bar{a}'$ is independent from $I' = I \setminus J$ over $A$. By the stationarity of strong types in a stable theory, $\bar{a}'$ realizes $\bigcup_{\bar{c} \in I'} q(\bar{x}, \bar{c})$. Since indiscernible sequences in a stable theory are indiscernible sets, $I$ and $I'$ have the same isomorphism type over $A$. Thus, there is $\bar{a}'' \in M$ realizing $\bigcup_{\bar{c} \in I} q(\bar{x}, \bar{c})$. This shows that $tp_{\mathcal{R}}(\bar{a}/\bar{b})$ does not divide over $A$. $\square$

The following example is due to Shelah. See [HL] for details. This example shows that some homogeneous models of a stable theory may fail to be simple. Since a homogeneous model of a stable first-order theory is stable as a homogeneous structure, this example shows also that stability does not imply simplicity at this level of generality.

The language consists of an infinite number of binary relations $E_n(x, y)$, for $n < \omega$. The first-order theory $T$ asserts that each $E_n(x, y)$ is an equivalence relation on the models of $T$ with an infinite number of equivalence classes, all of which are infinite. Furthermore, each $E_{n+1}$-class is partitioned into infinitely many $E_n$-classes. It is easy to see that $T$ is complete and $\omega$-stable. Let $\bar{M}$ be a large saturated model of $T$ and $a \in \bar{M}$. Let $M = \bigcup_{n<\omega} a/E_n$, where $a/E_n$ is the $E_n$-class of $a$. Then $M$ is a large homogeneous model of $T$ ($\mathcal{R}$ is the set of first-order definable relations of $M$). $M$ is $\omega$-stable, and so almost $\omega$-simple. However, for any $b, c \in M$, the type $tp(b/c)$ divides over the empty set. This shows that no type over the empty set has a free extension, so $M$ fails to be simple.

6.2. **Trees.** Let $\beta$ be an ordinal and $M = A^{\leq\beta}$ viewed as a tree structure, where $|A| > |\beta|$. ($A^{\leq\beta}$ is the set of sequences from $A$ indexed by ordinals $\leq \beta$.) The language under which the tree is formulated is immaterial; let us say it is a partial order $\leq$ interpreted by the subsequence relation. It is not difficult to show that $M$ is homogeneous. (All elements of the same length are in the same orbit.) Dividing in $M$ is understood as follows. Let $c \in M$ be a sequence of length $\gamma < \beta$ and suppose $c < b$ and $c < a$. If there is a $d$, $c < d < a$ and $c < d < b$, then $tp(b/a)$ divides over $c$. To see this consider an indiscernible sequence $I = \{ d_i : i < \omega \}$ in $tp(d/c)$ such that the $d_i$'s are distinct. Since the elements of $I$ must lie on distinct branches in the tree, there is no $x \in M$ satisfying $x > d_i \wedge x > d_j$, for $i \neq j$. Thus, the formula $x > d$ divides over $c$. Continuing for any $a \in M$, $A \subset M$, let $B \subset A$ be a minimal set such that if $c < a$ and $c \leq b$, for some $b \in A$, then $c < b$, for some $b \in B$. Then, $tp(a/A)$ does not divide over $B$. From this observation it is easy to show that $M$ is $\kappa$-simple for any $\kappa \leq |\beta|$. $M$ is also stable.

6.3. **A torsion abelian group.** For $p$ a prime number let $\mathbb{Z}_p$ be the cyclic group with $p$ elements. Let $\lambda$ be an uncountable cardinal. Let

$$M = \bigoplus_{p \text{ prime}} \bigoplus_{\lambda \text{ copies}} \mathbb{Z}_p,$$

and $\mathcal{R}$ the class of existentially definable relations on $M$ in the language with just $+$ and $0$. For $p$ a prime let $M_p$ be the elements of order $p$ in $M$. Then, two tuples from $M_p$ with the same quantifier-free type are in the same orbit of $\mathrm{Aut}(M)$. In other words, the only invariant structure on $M_p$ is as a vector space over the field with $p$ elements. From this observation it is easy to see that $M$ is homogeneous and for any large type $r$ over $A$, where $|A| < |M|$, $M$ contains an infinite $A$-indiscernible sequence of realizations of $r$. Moreover, for any prime $p$, if $\{a\} \cup A \subset M_p$, $B \subset A$, and $tp(a/A)$ divides over $B$, then $a$ is linearly dependent on $A$ over $B$. Given an arbitrary $a \in M$ there are primes $p_1, \ldots, p_n$ and $b_i \in M_{p_i}$ such that $a = b_1 + \cdots + b_n$. In this situation, $tp(a/b_i)$ divides over $\emptyset$. It follows that $M$ is simply superstable. Notice that the first-order theory of $M$ is only stable.

This example illustrates a complication of dealing with simply stable homogeneous groups, namely, generic types may not exist. A type $q \in S(A)$ in a simple homogeneous group $G$ is *generic* if for all $a \in G$, $aq$ is free from $A \cup \{a\}$ over $\emptyset$. For any $q \in S(\emptyset)$ there are primes $p_1, \ldots, p_n$ such that a realization of $q$ is in $M_{p_1} + \cdots + M_{p_n}$. If $b \in M_p$, for $p \neq p_i$, $i = 1, \ldots, n$, then $bq$ divides over $b$. Thus, there are no generic types over $\emptyset$ in $M$.

This context is too general to guarantee the existence of generic types as in simple first-order theories. In his thesis, A. Berenstein [Ber02] considers simply stable homogeneous groups under the assumption that generics exist and proves that they behave as in the first-order (i.e., compact) case.

6.4. **Free groups.** Another natural example of simply stable homogeneous groups are free groups. Let $G$ be an uncountable free group generated by the subset $S$ of $G$, in the language of groups $(*, ^{-1}, 1)$, and a unary predicate $S$ for the set of generators. Then $G$ is a homogeneous structure (which is not saturated, since the type of an element not generated by $S$ is omitted). Any two uncountable such groups of the same size are isomorphic, and it was observed by H. J. Keisler [Kei71] that these groups are an example of his categoricity theorem for $L_{\omega_1,\omega}$. Uncountable categoricity implies that $G$ is $\omega$-stable as a homogeneous structure (see Section 5); this can also be seen directly by counting types. $G$ is also supersimple. The freeness relation can be described easily. For $A \subseteq G$, let $A' \subseteq S$ be those elements generating the elements of $A$. $A'$ is uniquely determined from $A$ and this operation commutes with automorphisms of $G$. For $B \subseteq A, C \subseteq G$, the reader can check that $A \underset{B}{\downarrow} C$ if and only if $A'$ is free from $C'$ over $B'$ in the sense of the trivial pregeometry $S$.

This is again a case where we fail to have generics. Moreover, each abelian subgroup of $G$ is countable, so we cannot expect a generalization of Cherlin's theorem stating that an uncountable saturated group whose first-order theory is $\omega$-stable has a definable abelian subgroup of the same size.

6.5. **Hilbert spaces.** Some of these ideas appear earlier in the work of Henson and Iovino on Banach space structures; see for example [Iov99]. The main technical difference is that the logic of Banach space structures is set up so as to keep compactness. They have several ways of measuring the space of types according to the density character of these spaces with respect to various topologies. However, they show that stability is independent of the topology. Moreover, Iovino showed recently that a compact, homogeneous Banach space structure is stable in their sense if and only if it is stable in the sense developed in the previous section. Some of the material below was also worked out simultaneously, as well as extended, by A. Berenstein in his Ph.D. thesis [Ber02]. He produced several simply stable expansions of Hilbert spaces and proved that the $C^*$ algebra of square integrable functions on a set $X$ is also a simply stable homogeneous structure. He also observed that, as a group, a Hilbert space does not have generics. See also [BB02] for more simply stable expansions of Hilbert spaces.

Let $K$ be the real or complex numbers. An *inner product space* (or *pre-Hilbert space*) over $K$ is a $K$-vector space $V$ equipped with a map $\langle -, - \rangle$ from $V \times V$ into $K$ satisfying, for $x$, $y$, $z \in V$ and $r \in K$:

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
- $\langle rx, y \rangle = r\langle x, y \rangle$,
- $\langle y, x \rangle = \overline{\langle x, y \rangle}$,
- $\langle x, x \rangle$ is a real number $> 0$ if $x \neq 0$.

If $V$ is an inner product space it is a normed linear space under $\| - \| = \sqrt{\langle x, x \rangle}$. $V$ is a *Hilbert space* if it is a Banach space under $\| - \|$; i.e., $V$ is a complete metric space under the norm.

For simplicity suppose $K = \mathbb{R}$. This can be formulated as a model in a first-order language by having one sort, $S_V$, for $V$ and one, $S_F$, for $\mathbb{R}$, a binary map $\langle -, - \rangle$ from $V \times V$ into $\mathbb{R}$, the field operations on $\mathbb{R}$, $<$ on $\mathbb{R}$, and a function giving scalar multiplication. For simplicity we assume there is a constant in the language for each element of $\mathbb{R}$. Let $T_H$ be the theory in this language $L_H$ expressing the itemized properties above for the field $S_F$ and that $S_F$ is an ordered field containing $\mathbb{R}$.

In this section we formulate a Hilbert space $H$ as a logical structure $(H, \mathcal{R})$, where $\mathcal{R}$ consists of the quantifier-free relations in the language described above.

Here are some of the basic facts about Hilbert spaces that go into the following proofs. As a reference we suggest [HS65, Chapter 16].

*Remark* 6.1. (i) If $H$ is a Hilbert space and $A \subset H$, the minimal Hilbert space $H' \subset H$ that contains $A$ is denoted $\hat{A}$. Since $\hat{A}$ is obtained by successively closing under the vector space operations and taking the limit of Cauchy sequences, $\hat{A} \subset dcl(A)$; i.e., it is invariant under the automorphisms of $H$ that fix $A$.

(ii) If $H$ is an inner product space and $A \subset H$ then $A$ contains an orthogonal set $E$ such that every $x \in A$ is nonorthogonal to some element of $E$ (by Zorn's Lemma). Such an $E$ is called a *complete orthogonal set for $A$*.

**Lemma 6.2.** *Let $H$ be a Hilbert space of cardinality $\lambda > 2^{\aleph_0}$. Then $H$ is $s\lambda$-homogeneous.*

*Proof.* Let $\mu < \lambda$ and $\bar{u}, \bar{v} \in H^\mu$ sequences with the same quantifier-free type. Consider the partial map $f$ from $H$ to $H$ defined by $f(u_i) = v_i$, for $i < \mu$. Let $\hat{A}$ and $\hat{B}$ be the Hilbert subspaces generated by $\{ u_i : i < \mu \}$ and $\{ v_i : i < \mu \}$, respectively. Then $f$ can be extended uniquely to a partial isomorphism from $\hat{A}$ onto $\hat{B}$. Since $|A| < |H|$, basic facts about Hilbert spaces (see [HS65]) imply that $f$ can be extended to an automorphism of $H$. $\square$

Recall the condition (**P**) defined at the beginning of Section 2. We assume from here on that $M$ is a homogeneous Hilbert space of cardinality $\lambda$, where $\lambda$ is sufficiently large so that (**P**) holds for $\pi = \aleph_1$ and $\pi'$ some uncountable cardinal $\leq \lambda$. Here, of course, we mean that $M$ is homogeneous with respect to quantifier-free relations.

**Theorem 6.3.** *$M$ is $\aleph_1$-simply stable.*

The proof will be in a series of lemmas.

**Notation.** Given an inner product space $H$, $x \in H$ and $Y \subset H$, $x \perp Y$ if $\langle x, y \rangle = 0$, for all $y \in Y$.

The following proposition summarizes the elementary results that enter the proof.

**Proposition 6.4.** *(i) (Bessel's Inequality) Let $E$ be a nonempty orthonormal set in an inner product space $H$, and let $x \in H$. Then $\sum_{z \in E} |\langle x, z \rangle|^2 \leq \|x\|^2$, hence $\{ z \in E : \langle x, z \rangle \neq 0 \}$ is countable.*

*(ii) Let $H$ be a Hilbert space and $E$ a complete orthogonal subset of $H$. Then for all $x \in H$, $x = \sum_{z \in E} \frac{\langle x, z \rangle}{\langle z, z \rangle} z$.*

**Lemma 6.5.** *Given $A \cup E \subset M$, with $E$ an orthogonal set over $A$, and $a \in M$ such that $a \perp E$, $tp(a/A \cup E)$ does not divide over $A$. If $B \subset dcl(A \cup E)$, then also $tp(a/A \cup E \cup B)$ does not divide over $A$.*

*Proof.* Let $\{ E_i : i \in X \}$ be an $A$-indiscernible sequence in $tp(E/A)$. The construction of $M$ at the beginning of this section, and the fact that $E$ is orthogonal over $A$ guarantees the existence of an $a'$ realizing $tp(a/A)$ such that $a' \perp E_i$, for all $i \in X$. Checking the possible relations on $A \cup E \cup \{a\}$, this implies that

$tp(E_i \cup \{a'\}/A) = tp(E \cup \{a\}/A)$, for all $i \in X$. This witnesses that $tp(a/A \cup E)$ does not divide over $A$.

Continuing, for each $i \in X$, there is $B_i$ such that $tp(E_i \cup B_i/A) = tp(E \cup B/A)$. Since $B_i \subset dcl(E_i \cup A)$, $tp(B_i/E_i \cup A)$ has a unique extension over $A \cup E_i \cup \{a'\}$. Thus, for each $i \in X$, $tp(B_i \cup E_i/A \cup \{a'\}) = tp((B \cup E/A \cup \{a'\})$. We conclude that $tp(a/A \cup E \cup B)$ does not divide over $A$. $\square$

**Lemma 6.6.** *Suppose $a \in M$ and $A \subset M$, where $|A| < |M|$. Then there is a countable $B \subset A$ such that $tp(a/A)$ does not divide over $B$.*

*Proof.* Let $E$ be a complete orthogonal subset of $\hat{A}$, chosen so that $E \cap A$ is a complete orthogonal subset of $A$. Let $F = \{ z \in E : \langle a, z \rangle \neq 0 \}$, which is countable by Bessel's Inequality. Let $F_0 = F \cap A$ and $F_1 = F \setminus F_0$. Let $A_0$ be a minimal subset of $A$ such that $F_1 \subset dcl(A_0)$. Since $F$ is countable, $A_0$ is countable by Remark 6.1(i), and we may assume that $F_0 \subset A_0$. By enlarging $A_0$ and $F$ if necessary we can assume that $e \in E \setminus F$ implies that $e \perp A_0$. By Lemma 6.5, $tp(a/\hat{A})$ does not divide over $A_0$, proving the lemma. $\square$

**Lemma 6.7.** *Given $a \in M$ and $A \subset B \subset M$ there is a unique complete type over $B$ which is an $\aleph_0$-free extension of $tp(a/A)$, and this type does not divide over $A$.*

*Proof.* Consider the Hilbert spaces $\hat{A} \subset \hat{B}$. Let $E$ be a complete orthogonal subset of $\hat{A}$ and $F \supset E$ a complete orthogonal subset of $\hat{B}$. Notice that each $f \in F \setminus E$ is orthogonal to $\hat{A}$. Since $\hat{A} \subset dcl(A)$, $tp(a/A)$ has a unique extension over $\hat{A}$, namely $tp(a/\hat{A})$, which also does not divide over $A$. There is a $b$ realizing $t(a/\hat{A})$ such that each $f \in F \setminus E$ is orthogonal to $b$, and among such $b$ there is a unique type over $\hat{A} \cup F$. There is a unique extension of $tp(b/\hat{A} \cup F)$ over $\hat{B}$, and by Lemma 6.5 $tp(b/\hat{B})$ does not divide over $A$. This proves the lemma. $\square$

Combining Lemma 6.6 and Lemma 6.7 shows that $M$ is $\aleph_1$-simple and stable, proving Theorem 6.3.

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