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# FUNCTORIALITY FOR THE EXTERIOR SQUARE OF $GL_4$ AND THE SYMMETRIC FOURTH OF $GL_2$

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## 1. Introduction

Let $\wedge^2 : GL_n(\mathbb{C}) \longrightarrow GL_N(\mathbb{C})$, where $N = \frac{n(n-1)}{2}$, be the map given by the exterior square. Then Langlands' functoriality predicts that there is a map from cuspidal representations of $GL_n$ to automorphic representations of $GL_N$, which satisfies certain canonical properties. To explain, let $F$ be a number field, and let $\mathbb{A}$ be its ring of adeles. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal (automorphic) representation of $GL_n(\mathbb{A})$. In what follows, a cuspidal representation always means a unitary one. Now by the local Langlands correspondence, $\wedge^2\pi_v$ is well defined as an irreducible admissible representation of $GL_N(F_v)$ for all $v$ (the work of Harris-Taylor [H-T] and Henniart [He2] on $p$-adic places and of Langlands [La4] on archimedean places). Let $\wedge^2\pi = \bigotimes_v \wedge^2\pi_v$. It is an irreducible admissible representation of $GL_N(\mathbb{A})$. Then Langlands' functoriality in this case is equivalent to the fact that $\wedge^2\pi$ is automorphic.

Note that $\wedge^2(GL_2(\mathbb{C})) \simeq GL_1(\mathbb{C})$ and in fact for a cuspidal representation $\pi$ of $GL_2(\mathbb{A})$, $\wedge^2\pi = \omega_\pi$, the central character of $\pi$. Furthermore, $\wedge^2(GL_3(\mathbb{C})) \simeq GL_3(\mathbb{C})$. In this case, given a cuspidal representation $\pi$ of $GL_3(\mathbb{A})$, $\wedge^2\pi = \tilde{\pi} \otimes \omega_\pi$, where $\tilde{\pi}$ is the contragredient of $\pi$.

In this paper, we look at the case $n = 4$. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. What we prove is weaker than the automorphy of $\wedge^2\pi$. We prove (Theorem 5.3.1)

**Theorem A.** *Let $T$ be the set of places where $v|2,3$ and $\pi_v$ is a supercuspidal representation. Then there exists an automorphic representation $\Pi$ of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \wedge^2\pi_v$ if $v \notin T$. Moreover, $\Pi$ is of the form $Ind\, \tau_1 \otimes \cdots \otimes \tau_k$, where the $\tau_i$'s are all cuspidal representations of $GL_{n_i}(\mathbb{A})$.*

The reason why we have the exceptional places $T$, especially for $v|2$, is due to the fact that supercuspidal representations of $GL_4(F_v)$ are very complicated when $v|2$. We use the Langlands-Shahidi method and a converse theorem of Cogdell-Piatetski-Shapiro to prove the above theorem (cf. [Co-PS1], [Ki-Sh2]). We expect

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many applications of this result. Among them, we mention two: First, we prove the weak Ramanujan property of cuspidal representations of $GL_4(\mathbb{A})$ (Proposition 6.3; see Definition 3.6 for the notation).

Second, we prove the existence of the symmetric fourth lift of a cuspidal representation of $GL_2(\mathbb{A})$ as an automorphic representation of $GL_5(\mathbb{A})$. More precisely, let $GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the symmetric $m$th power (the $m+1$-dimensional irreducible representation of $GL_2(\mathbb{C})$ on symmetric tensors of rank $m$). Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ with central character $\omega_\pi$. By the local Langlands correspondence, $Sym^m(\pi_v)$ is well defined for all $v$. Hence Langlands' functoriality predicts that $Sym^m(\pi) = \bigotimes_v Sym^m(\pi_v)$ is an automorphic representation of $GL_{m+1}(\mathbb{A})$. Gelbart and Jacquet [Ge-J] proved that $Sym^2(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$. We proved in [Ki-Sh2] that $Sym^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$ as a consequence of the functorial product $GL_2 \times GL_3 \longrightarrow GL_6$, corresponding to the tensor product map $GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$.

We prove (Theorem 7.3.2)

**Theorem B.** *$Sym^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$. If $Sym^3(\pi)$ is cuspidal, $Sym^4(\pi)$ is either cuspidal or induced from cuspidal representations of $GL_2(\mathbb{A})$ and $GL_3(\mathbb{A})$.*

Here we stress that there is no restriction on the places as opposed to the case of the exterior square lift.

Theorem B is obtained by applying Theorem A to $Sym^3(\pi)\otimes\omega_\pi^{-1}$. For simplicity, we write $A^m(\pi) = Sym^m(\pi) \otimes \omega_\pi^{-1}$. We prove that

$$\wedge^2(A^3(\pi)) = A^4(\pi) \boxplus \omega_\pi.$$

This implies that $A^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$, and so is $Sym^4(\pi)$.

An immediate corollary is that we have a new estimate for Ramanujan and Selberg's conjectures using [Lu-R-Sa]. Namely, let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $\pi_v$ be a local (finite or infinite) spherical component, given by $\pi_v = Ind(|\ |_v^{s_{1v}}, |\ |_v^{s_{2v}})$. Then $|Re(s_{iv})| \le \frac{3}{26}$. If $F = \mathbb{Q}$ and $v = \infty$, this condition implies that $\lambda_1 \ge \frac{40}{169} \approx 0.237$, where $\lambda_1$ is the first positive eigenvalue for the Laplace operator on the corresponding hyperbolic space.

In a joint work with Sarnak in Appendix 2 [Ki-Sa], by considering the twisted symmetric square $L$-functions of the symmetric fourth (cf. [BDHI]), we improve the bound further, at least over $\mathbb{Q}$, namely, $Re(s_{ip}) \le \frac{7}{64}$. As for the first positive eigenvalue for the Laplacian, we have $\lambda_1 \ge \frac{975}{4096} \approx 0.238$.

In [Ki-Sh3], we determine exactly when $A^4(\pi)$ is cuspidal. We show that $A^4(\pi)$ is not cuspidal and $A^3(\pi)$ is cuspidal if and only if there exists a non-trivial quadratic character $\eta$ such that $A^3(\pi) \simeq A^3(\pi) \otimes \eta$, or equivalently, there exists a non-trivial grössencharacter $\chi$ of $E$ such that $(Ad(\pi))_E \simeq (Ad(\pi))_E \otimes \chi$, where $E/F$ is the quadratic extension, determined by $\eta$. We refer to that paper for many applications of symmetric cube and symmetric fourth: The analytic continuation and functional equations are proved for the 5th, 6th, 7th, 8th and 9th symmetric power $L$-functions of cuspidal representations of $GL_2$. It has immediate application for Ramanujan and Selberg's bounds and the Sato-Tate conjecture: Let $\pi_v$ be an unramified local component of a cuspidal representation $\pi = \bigotimes_v \pi_v$. Then it is shown that $q_v^{-\frac{1}{9}} < |\alpha_v|, |\beta_v| < q_v^{\frac{1}{9}}$, where the Hecke conjugacy class of $\pi_v$ is given by $diag(\alpha_v, \beta_v)$. Furthermore, if $a_v = \alpha_v + \beta_v$, then for every $\epsilon > 0$, there are sets $T^+$ and $T^-$ of positive lower (Dirichlet) density such that $a_v > 1.68... - \epsilon$ for all $v \in T^+$ and $a_v < -1.68... + \epsilon$ for all $v \in T^-$.

In [Ki5], we give an example of automorphic induction for a non-normal quintic extension whose Galois closure is not solvable. In fact, the Galois group is $A_5$, the alternating group on five letters. The key observation, due to Ramakrishnan is that the symmetric fourth of the 2-dimensional icosahedral representation is equivalent to the 5-dimensional monomial representation of $A_5$ (see [Bu]). It should be noted that the only complete result for non-normal automorphic induction before this is for non-normal cubic extension due to [J-PS-S2] as a consequence of the converse theorem for $GL_3$.

We now explain the content of this paper. In Section 2, we recall a converse theorem of Cogdell and Piatetski-Shapiro and the definition of weak lift and strong lift. In Section 3, we study the analytic properties of the automorphic $L$-functions which we need for the converse theorem, namely, $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$, where $\sigma$ is a cuspidal representation of $GL_m(\mathbb{A})$, $m = 1, 2, 3, 4$, and $\pi$ is a cuspidal representation of $GL_4(\mathbb{A})$. The automorphic $L$-functions appear in the constant term of the Eisenstein series coming from the split spin group $Spin(2n)$ (the $D_n - 3$ case in [Sh3]). Hence we can apply the Langlands-Shahidi method [Ki1], [Ki2], [Ki-Sh2], [Sh1]–[Sh3].

In Section 4, we first obtain a weak exterior square lift by applying the converse theorem to $\wedge^2 \pi = \bigotimes_v \wedge^2 \pi_v$, with $S$ being a finite set of finite places, where $\pi_v$ is unramified for $v < \infty$ and $v \notin S$. In this case, the situation is simpler because if $\sigma \in \mathcal{T}^S(m)$ as in the statement of the converse theorem, one of $\sigma_v$ or $\pi_v$ is in the principal series for $v < \infty$. Here one has to note the following: In the converse theorem, the $L$-function $L(s, \sigma_v \times \Pi_v)$ is the Rankin-Selberg $L$-function defined by either integral representations [J-PS-S] or the Langlands-Shahidi method. They are the same, and they are an Artin $L$-function due to the local Langlands correspondence. However, the $L$-function $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4)$ is defined by the Langlands-Shahidi method [Sh1] as a normalizing factor of intertwining operators which appear in the constant term of the Eisenstein series. The equality of two $L$-functions which are defined by completely different methods is not obvious at all. The same is true for $\epsilon$-factors. Indeed, a priori we do not know the equality when $\pi_v$ is a supercuspidal representation, even if $\sigma_v$ is a character of $F_v^\times$. Hence we need to proceed in two steps as in [Ra1], namely, first, we do the good case when none of $\pi_v$ is supercuspidal, and then we do the general case, following Ramakrishnan's idea of descent [Ra1]. It is based on the observation of Henniart [He1] that a supercuspidal representation of $GL_n(F_v)$ becomes a principal series after a solvable base change. Here one needs an extension of Proposition 3.6.1 of [Ra1] to isobaric automorphic representations (from cuspidal automorphic representations). Appendix 1 provides the extension. We may avoid using the descent method, hence Appendix 1 altogether, by using the stability of $\gamma$-factors as in [CKPSS] (see Remark 4.1 for more detail). We hope to pursue this in the future. Indeed, for the special case of the functoriality of $\wedge^2(A^3(\pi))$, hence the symmetric fourth of $GL_2$, we do not need it. (See Remark 7.2.)

The converse theorem only provides a weak lift $\Pi$ which is equivalent to a subquotient of $Ind\, |det|^{r_1} \tau_1 \otimes \cdots \otimes |det|^{r_k} \tau_k$, where the $\tau_i$'s are (unitary) cuspidal representations of $GL_{n_i}$ and $r_i \in \mathbb{R}$. If $\pi$ satisfies the weak Ramanujan property, it immediately implies $r_1 = \cdots = r_k = 0$. In general, we show that $r_1 = \cdots = r_k = 0$ by comparing the Hecke conjugacy classes of $\wedge^2\pi$ and $\Pi$.

In Section 5.1, we give a new proof of the existence of the functorial product corresponding to the tensor product map $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$. It is originally due to Ramakrishnan [Ra1]. However, we give a proof, based entirely on the Langlands-Shahidi method. As a corollary, we obtain the Gelbart-Jacquet lift $Ad(\pi)$ [Ge-J] as an automorphic representation of $GL_3(\mathbb{A})$ for a cuspidal representation $\pi$ of $GL_2(\mathbb{A})$ by showing that $\pi \boxtimes \tilde{\pi} = Ad(\pi) \boxplus 1$.

In Section 5.2, we construct all local lifts $\Pi_v$ in the sense of Definition 2.2 and show that unless $v|2,3$ and $\pi_v$ is a supercuspidal representation, $\Pi_v$ is in fact $\wedge^2\pi_v$, the one given by the local Langlands correspondence [H-T], [He2]. Here is how it is done: Note that if $v \nmid 2$, any supercuspidal representation of $GL_4(F_v)$ is induced, i.e., corresponds to $Ind(W_{F_v}, W_K, \mu)$, where $K/F_v$ is an extension of degree 4 (not necessarily Galois) and $\mu$ is a character of $K^\times$. (This is the so-called tame case. See, for example, [H, p. 179] for references.) Also thanks to Harris' work [H], we have automorphic induction for non-Galois extensions. Namely, there exists a cuspidal representation $\pi$ which corresponds to $Ind(W_F, W_E, \chi)$, where $E_w = K$, $w|v$, and $\chi$ is a grössencharacter of $E$ such that $\chi_w = \mu$. Likewise, if $v \nmid 2,3$, any supercuspidal representation $\sigma_v$ of $GL_m(F_v)$, $m = 1,2,3,4$, is induced. We embed $\sigma_v$ as a local component of a cuspidal representation using automorphic induction. We can compare the functional equations of $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4)$ and the corresponding Artin $L$-function and obtain our assertion that the local lift we constructed is equivalent to the one given by the local Langlands correspondence. (If $v|3$, we need to twist by supercuspidal representations of $GL_3(F_v)$, where there can be supercuspidal representations which are not induced. The global Langlands correspondence is not available for them.)

In Section 5.3, by applying the converse theorem twice to $\Pi = \bigotimes_v \Pi_v$ with $S_1 = \{v_1\}$, $S_2 = \{v_2\}$, where $v_1, v_2$ are any finite places, we prove that $\Pi$ is an automorphic representation of $GL_6(\mathbb{A})$.

In Section 7, we prove that if $\pi$ is a cuspidal representation of $GL_2(\mathbb{A})$, then $A^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$. Here we need to be careful because of the exceptional places $T$ in the discussion of the exterior square lift. We first prove that there exists an automorphic representation $\Pi$ of $GL_5(\mathbb{A})$ such that $\Pi_v \simeq A^4(\pi_v)$ if $v \notin T$. Next we show that this is true for $v \in T$. If $v|3$, any supercuspidal representation of $GL_2(F_v)$ is monomial, and hence it can be embedded into a monomial cuspidal representation of $GL_2(\mathbb{A})$. If $v|2$, any extraordinary supercuspidal representation of $GL_2(F_v)$ is of tetrahedral type or octahedral type (see [G-L, p. 121]). Hence in this case, the global Langlands correspondence is available [La3], [Tu]. We can compare the functional equations of $L(s, \sigma \times A^4(\pi))$ and the corresponding Artin $L$-function and obtain our assertion.

Finally, we emphasize that for the functoriality of $A^4(\pi)$, we do not need the full functoriality of the exterior square of $GL_4$; first of all, one does not need the comparison of Hecke conjugacy classes in Section 4.1, since $A^3(\pi)$ satisfies the weak Ramanujan property. Secondly, one does not need the method of base change and Ramakrishnan's descent argument (hence Appendix 1), because we can prove the equality of $\gamma$-factors for supercuspidal representations directly (see Remark 7.2 for the details).

## 2. Converse theorem

Throughout this paper, let $F$ be a number field, and let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles. We fix an additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/F$. Let $\rho_m$ be the standard representation of $GL_m(\mathbb{C})$.

First recall a converse theorem from [Co-PS1].

**Theorem 2.1** ([Co-PS1])**.** *Suppose $\Pi = \bigotimes_v \Pi_v$ is an irreducible admissible representation of $GL_n(\mathbb{A})$ such that $\omega_\Pi = \bigotimes_v \omega_{\Pi_v}$ is a grössencharacter of $F$. Let $S$ be a finite set of finite places, and let $\mathcal{T}^S(m)$ be a set of cuspidal representations of $GL_m(\mathbb{A})$ that are unramified at all places $v \in S$. Suppose $L(s, \sigma \times \Pi)$ is nice (i.e., entire, bounded in vertical strips and satisfies a functional equation) for all cuspidal representations $\sigma \in \mathcal{T}^S(m)$, $m < n-1$. Then there exists an automorphic representation $\Pi'$ of $GL_n(\mathbb{A})$ such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.*

Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. In order to apply the converse theorem, we need to do the following:

(1) For all $v$, find an irreducible representation $\Pi_v$ of $GL_6(F_v)$ such that
$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4) = L(s, \sigma_v \times \Pi_v),$$
for all $\sigma_v$, where $\sigma = \bigotimes_v \sigma_v \in \mathcal{T}^S(m)$, $m = 1, 2, 3, 4$.

(2) Prove the analytic continuation and functional equation of the $L$-functions $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$.

(3) Prove that $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is entire for $\sigma \in \mathcal{T}^S(m)$, $m = 1, 2, 3, 4$.

(4) Prove that $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is bounded in vertical strips for $\sigma \in \mathcal{T}^S(m)$, $m = 1, 2, 3, 4$.

Recall the equalities:
$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v)$$
$$= \epsilon(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) \frac{L(1-s, \tilde{\sigma}_v \otimes \tilde{\pi}_v, \rho_m \otimes \wedge^2 \rho_4)}{L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4)},$$
$$\gamma(s, \sigma_v \times \Pi_v, \psi_v) = \epsilon(s, \sigma_v \times \Pi_v, \psi_v) \frac{L(1-s, \tilde{\sigma}_v \times \tilde{\Pi}_v)}{L(s, \sigma_v \times \Pi_v)}.$$
Hence the equalities of $\gamma$ and $L$-factors imply the equality of $\epsilon$-factors.

The $L$-function $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ and the $\gamma$-factor $\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v)$ are available from the Langlands-Shahidi method, by considering the split spin group $Spin(2n)$ with the maximal Levi subgroup $\mathbf{M}$ whose derived group is $SL_{n-3} \times SL_4$. We will study the analytic properties of the $L$-functions in the next section; (2) is well known by Shahidi's work [Sh3]; (4) is the result of [Ge-Sh]. We will especially study (3); in general, the $L$-functions $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ may not be entire. Our key idea is to apply the converse theorem to the twisting set $\mathcal{T}^S(m) \otimes \chi$, where $\chi_v$ is highly ramified for $v \in S$. Then for $\sigma \in \mathcal{T}^S(m) \otimes \chi$, the $L$-function $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is entire. Observe that $L(s, (\sigma \otimes \chi) \times \Pi) = L(s, \sigma \times (\Pi \otimes \chi))$. Hence applying the converse theorem with the twisting set $\mathcal{T}^S(m) \otimes \chi$ is equivalent to applying the converse theorem for $\Pi \otimes \chi$ with the twisting set $\mathcal{T}^S(m)$ (see [Co-PS2]).

We will address problem (1) in Section 4. We have a natural candidate for $\Pi_v$, namely, $\wedge^2 \pi_v$, the one given by the local Langlands correspondence (see Section

4 for the detail). However, proving the equalities in (1) is not so obvious due to the fact that two $L$-functions on the left and on the right are defined in completely different manners. The right-hand side is the Rankin-Selberg $L$-function [J-PS-S] defined by either integral representations or the Langlands-Shahidi method, which in turn is an Artin $L$-function due to the local Langlands correspondence. We note that if $\Pi_v$ is not generic, then we write $\Pi_v$ as a Langlands quotient of an induced representation $\Xi_v$, which is generic, and we define the $\gamma$- and $L$-factors $\gamma(s,\sigma_v\times\Pi_v,\psi_v)=\gamma(s,\sigma_v\times\Xi_v,\psi_v)$ and $L(s,\sigma_v\times\Pi_v)=L(s,\sigma_v\times\Xi_v)$.

The left-hand side is defined in the Langlands-Shahidi method [Sh1] as a normalizing factor of intertwining operators which appear in the constant term of the Eisenstein series. Proving (1) is equivalent to the fact that Shahidi's $L$-functions and $\gamma$-factors on the left are those of Artin factors. It is clearly true if $\sigma_v\otimes\pi_v$ is unramified. Shahidi has shown that (1) is true when $v=\infty$ [Sh7].

*Remark 2.1.* Eventually we are going to prove in Section 5 that $\Pi_v$ on the right side of (1) is generic in our case. However, $\Pi_v$ is not generic in general. For example, if $\pi_v$ is given by the principal series $Ind_B^{GL_4}|\ |^{\frac14}\otimes|\ |^{\frac14}\otimes|\ |^{-\frac14}\otimes|\ |^{-\frac14}$, then $\Pi_v=\wedge^2\pi_v$ is the unique quotient of $Ind_B^{GL_6}|\ |^{\frac12}\otimes|\ |^{-\frac12}\otimes 1\otimes 1\otimes 1\otimes 1$, namely, $Ind_{GL_2\times GL_1\times GL_1\times GL_1\times GL_1}^{GL_6}|det|\otimes 1\otimes 1\otimes 1\otimes 1$. Hence in the course of applying the converse theorem, we need to deal with such non-generic representations on the right side of (1). However, in the definition of Shahidi's $\gamma$- and $L$-factors on the left side of (1), we only deal with generic representations, since any local components of a cuspidal representation of $GL_n(\mathbb{A})$ are generic. By a well-known result, any generic representation of $GL_n(F_v)$ is always a full induced representation.

We were not able to prove (1) for $\Pi_v=\wedge^2\pi_v$ when $v|2,3$ and $\pi_v$ is a supercuspidal representation of $GL_4(F_v)$. Hence we make the following definition.

**Definition 2.2.** Let $\pi=\bigotimes_v\pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. We say that an automorphic representation $\Pi$ of $GL_6(\mathbb{A})$ is a strong exterior square lift of $\pi$ if for every $v$, $\Pi_v$ is a local lift of $\pi_v$ in the sense that

$$\gamma(s,\sigma_v\otimes\pi_v,\rho_m\otimes\wedge^2\rho_4,\psi_v)=\gamma(s,\sigma_v\times\Pi_v,\psi_v),$$
$$L(s,\sigma_v\otimes\pi_v,\rho_m\otimes\wedge^2\rho_4)=L(s,\sigma_v\times\Pi_v),$$

for all generic irreducible representations $\sigma_v$ of $GL_m(F_v)$, $1\le m\le 4$.

If the above equality holds for almost all $v$, then $\Pi$ is called weak lift of $\pi$.

In Section 4, we apply the converse theorem with $S$ being a finite set of finite places such that $\pi_v$ is unramified for $v\notin S$, $v<\infty$. Then if $\pi_v$ is ramified, the local components of the twisting representations at $S$ are unramified and hence the equalities in (1) become simpler. In this way, we first find a weak lift in Section 4 and use it to define all local lifts in Section 5 and to obtain the strong lift.

We record the following proposition which is very useful in proving (1).

**Proposition 2.3** ([Sh4])**.** *Let $\sigma_{1v}$ ($\sigma_{2v}$, resp.) be an irreducible generic admissible representation of $GL_k(F_v)$ ($GL_l(F_v)$, resp.) with parametrization $\phi_i: W_{F_v}\times SL_2(\mathbb{C})\longrightarrow GL_k(\mathbb{C})$ ($GL_l(\mathbb{C})$, resp.) by the local Langlands correspondence* [H-T], [He2]. *Let $L(s,\phi_1\otimes\phi_2)$ be the Artin $L$-function; let $L_1(s,\sigma_{1v}\times\sigma_{2v})$ be the Rankin-Selberg $L$-function defined by integral representation* [J-PS-S]*; and let $L_2(s,\sigma_{1v}\times\sigma_{2v})$ be the Langlands-Shahidi $L$-function defined as a normalizing factor* *for intertwining operators* [Sh1]*. Then we have the equality*

$$L(s,\phi_1\otimes\phi_2)=L_1(s,\sigma_{1v}\times\sigma_{2v})=L_2(s,\sigma_{1v}\times\sigma_{2v}).$$

*We have similar equalities for $\gamma$- and $\epsilon$-factors.*

*Proof.* The equality $L(s,\phi_1\otimes\phi_2)=L_1(s,\sigma_{1v}\times\sigma_{2v})$ is the local Langlands correspondence (the work of Harris-Taylor [H-T] and Henniart [He2] on $p$-adic places and of Langlands [La4] on archimedean places). Similar equalities hold for $\gamma$- and $\epsilon$-factors.

The equality $L_1(s,\sigma_{1v}\times\sigma_{2v})=L_2(s,\sigma_{1v}\times\sigma_{2v})$ is due to Shahidi ([Sh7] for archimedean places and [Sh4, Theorem 5.1] for $p$-adic places; see [Sh6, p 282] for the explanation of why the constant $\omega_2^m(-1)$ disappears). Similar equalities hold for $\gamma$- and $\epsilon$-factors. □

For the sake of completeness, we recall how $L$- and $\epsilon$-factors are defined from the Langlands-Shahidi method [Sh1, Section 7]. Let $\mathbf{G}$ be a quasi-split reductive group defined over a number field $F$. Let $\mathbf{M}$ be a maximal Levi subgroup. Let $\pi$ be a generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. From the theory of local coefficients, which come from intertwining operators, a $\gamma$-factor $\gamma(s,\pi_v,r_i,\psi_v)$ is defined for every generic irreducible admissible representation $\pi_v$ and certain finite-dimensional representation $r_i$'s. If $\pi_v$ is tempered, $L(s,\pi_v,r_i)$ is defined to be

$$L(s,\pi_v,r_i)=P_{\pi_v,i}(q_v^{-s})^{-1},$$

where $P_{\pi_v,i}$ is the unique polynomial satisfying $P_{\pi_v,i}(0)=1$ such that $P_{\pi_v,i}(q_v^{-s})$ is the numerator of $\gamma(s,\pi_v,r_i,\psi_v)$. We define the $\epsilon$-factor using the identity $\gamma(s,\pi_v,r_i,\psi_v)=\epsilon(s,\pi_v,r_i,\psi_v)\frac{L(1-s,\tilde\pi_v,r_i)}{L(s,\pi_v,r_i)}$. Hence if $\pi_v$ is tempered, then the $\gamma$-factor canonically defines both the $L$-factor and the $\epsilon$-factor. If $\pi_v$ is non-tempered, write it as a Langlands quotient of an induced representation and we can write the corresponding intertwining operator as a product of rank-one operators. For these rank-one operators, there correspond $\gamma$- and $L$-factors and we define $\gamma(s,\pi_v,\psi_v)$ and $L(s,\pi_v,r_i)$ to be the product of these rank-one $\gamma$- and $L$-factors. We then define $\epsilon$-factor to satisfy the above relation.

Recall the multiplicativity of $\gamma$-factors (cf. [Sh7]). We suppress the subscript $v$ until the end of Section 2. Let $\pi$ be an irreducible generic admissible representation of $M=\mathbf{M}(F)$. Suppose $\pi\subset Ind_{M_\theta N_\theta}^M\,\sigma\otimes 1$, where $M_\theta N_\theta$, $\theta\subset\Delta$, is a parabolic subgroup of $M$ and $\sigma$ is an irreducible generic admissible representation of $M_\theta$. Let $\theta'=w(\theta)\subset\Delta$ and fix a reduced decomposition $w=w_{n-1}\cdots w_1$ of $w$ as in [Sh2, Lemma 2.1.1]. Then for each $j$, there exists a unique root $\alpha_j\in\Delta$ such that $w_j(\alpha_j)<0$. For each $j$, $2\le j\le n-1$, let $\bar w_j=w_{j-1}\cdots w_1$. Set $\bar w_1=1$. Also let $\Omega_j=\theta_j\cup\{\alpha_j\}$, where $\theta_1=\theta$, $\theta_n=\theta'$, and $\theta_{j+1}=w_j(\theta_j)$, $1\le j\le n-1$. Then the group $M_{\Omega_j}$ contains $M_{\theta_j}N_{\theta_j}$ as a maximal parabolic subgroup and $w_j(\sigma)$ is a representation of $M_{\theta_j}$. The $L$-group ${}^LM_\theta$ acts on $V_i$. Given an irreducible constituent of this action, there exists a unique $j$, $1\le j\le n-1$, which under $w_j$ is equivalent to an irreducible constituent of the action of ${}^LM_{\theta_j}$ on the Lie algebra of ${}^LN_{\theta_j}$. We denote by $i(j)$ the index of this subspace of the Lie algebra of ${}^LN_{\theta_j}$. Finally, let $S_i$ denote the set of all such $j$'s where $S_i$, in general, is a proper subset of $1\le j\le n-1$.

**Proposition 2.4** ([Sh1, (3.13)] (multiplicativity of $\gamma$-factors))**.** *For each $j \in S_i$ let $\gamma(s, w_j(\sigma), r_{i(j)}, \psi)$ denote the corresponding factor. Then*

$$\gamma(s, \pi, r_i, \psi) = \prod_{j \in S_i} \gamma(s, w_j(\sigma), r_{i(j)}, \psi).$$

We follow the exposition in [Sh6, p. 280]. Let $\phi : W_F \times SL_2(\mathbb{C}) \longrightarrow {}^L M$ be the parametrization of $\pi$. Then $\phi$ factors through ${}^L M_\theta$, i.e., there exists $\phi'$ : $W_F \times SL_2(\mathbb{C}) \longrightarrow {}^L M_\theta$ such that $\phi = i \circ \phi'$, where $i : {}^L M_\theta \hookrightarrow {}^L M$. Let $r_i' = r_i|_{{}^L M_\theta}$. Then $r_i' = \bigoplus_j r_{i(j)}$, and

$$\gamma(s, \phi, r_i, \psi) = \prod_j \gamma(s, \phi', r_{i(j)}, \psi).$$

Given an irreducible component of $r_i|_{{}^L M_\theta}$, there exists a unique $j$, which under $w_j$ makes this component equivalent to an irreducible constituent of the action of ${}^L M_{\theta_j}$ on the Lie algebra of ${}^L N_{\theta_j}$. Hence we have

**Proposition 2.5.** *Let $\pi, \sigma$ be as in Proposition 2.4. Suppose $\pi$ is tempered and $\gamma(s, w_j(\sigma), r_{i(j)}, \psi)$ is an Artin factors for each $j \in S_i$, namely, $\gamma(s, w_j(\sigma), r_{i(j)}, \psi)$ $= \gamma(s, \phi', r_{i(j)}, \psi)$ for each $j$. Then $\gamma(s, \pi, r_i, \psi)$ and $L(s, \pi, r_i)$ are also Artin factors.*

*Proof.* Clear from the multiplicativity formulas. Since $\pi$ is tempered, $\gamma$-factors determine the $L$-factors uniquely. □

Because of Proposition 2.5, we are reduced to the supercuspidal case when verifying that Shahidi's $\gamma$- and $L$-factors are Artin factors. Later on, in many situations, all the rank-one factors in Proposition 2.5 are the Rankin-Selberg $\gamma$- and $L$-factors for $GL_n \times GL_m$, and by Proposition 2.3, they are Artin factors.

Next we have [Sh6, Theorem 5.2]

**Proposition 2.6** (multiplicativity of $L$-factors)**.** *Suppose $\pi, \sigma$ to be as in Proposition 2.4. Suppose $\pi$ is tempered and $\sigma$ is a discrete series. Suppose Conjecture 7.1 of* [Sh1] *is valid for every $L(s, w_j(\sigma), r_{i(j)})$, $j \in S_i$. Then*

$$L(s, \pi, r_i) = \prod_{j \in S_i} L(s, w_j(\sigma), r_{i(j)}).$$

Now let $\pi$ be a non-tempered irreducible generic admissible representation of $M = M(F_v)$. Then $\pi$ is the unique quotient of an induced representation $Ind_{M_\theta N_\theta}^M \sigma$ $\otimes$ 1, where $M_\theta N_\theta$, $\theta \subset \Delta$, is a parabolic subgroup of $M$ and $\sigma$ is an irreducible generic quasi-tempered representation of $M_\theta$. (In many cases when the standard module conjecture is known, $\pi = Ind_{M_\theta N_\theta}^M (\sigma \otimes 1)$.) Then by the definition of $L$-factors,

**Proposition 2.7.** *Let $\pi, \sigma$ be as above. Then*

$$L(s, \pi, r_i) = \prod_{j \in S_i} L(s, w_j(\sigma), r_{i(j)}), \quad \gamma(s, \pi, r_i, \psi) = \prod_{j \in S_i} \gamma(s, w_j(\sigma), r_{i(j)}, \psi).$$

*Remark 2.2.* In the multiplicativity of $\gamma$-factors (Proposition 2.4), we realized $\pi$ as a subrepresentation of an induced representation. On the other hand, in the above, $\pi$ is realized as a quotient. However, this does not matter, since local coefficients of two equivalent representations are the same.

*Remark 2.3.* Even though it is not necessary, we remark that we can define $L(s,\pi,r_i)$, even when $\pi$ is non-generic as long as it has generic supercuspidal support. Write $\pi$ as the Langlands quotient of $\Xi = Ind_{M_\theta N_\theta}^{M}\,\sigma\otimes 1$. Just define $\gamma(s,\pi,r_i,\psi)=\gamma(s,\Xi,r_i,\psi)$ using the formula in Proposition 2.4, and define $L(s,\pi,r_i)$ using the formula in Proposition 2.5. These definitions agree with those of the Rankin-Selberg $\gamma$- and $L$-factors in the sense of [J-PS-S] (see the paragraph before Remark 2.1), and hence Proposition 2.3 holds without the genericity condition.

For example, let $\pi_v=\mu\circ det$ be a character of $GL_2(F_v)$, which is the Langlands quotient of $Ind\,\mu|\ |^{\frac12}\otimes\mu|\ |^{-\frac12}$. Then the standard $L$-function $L(s,\pi_v)$ is obtained by considering the induced representation $Ind_{GL_2\times GL_1}^{GL_3}\,\pi_v|det|^{\frac{s}{2}}\otimes|\ |^{-\frac{s}{2}}$, which is a quotient of $Ind_B^{GL_3}\,\mu|\ |^{\frac12+\frac{s}{2}}\otimes\mu|\ |^{-\frac12+\frac{s}{2}}\otimes|\ |^{-\frac{s}{2}}$. Hence $\gamma(s,\pi_v,\psi_v)=\gamma(s+\frac12,\mu,\psi_v)\gamma(s-\frac12,\mu,\psi_v)$, and $L(s,\pi_v)=L(s+\frac12,\mu)L(s-\frac12,\mu)$ if $\mu$ is unramified. On the other hand, if $\sigma_v$ is the Steinberg representation, which is the subrepresentation of $Ind\,\mu|\ |^{\frac12}\otimes\mu|\ |^{-\frac12}$, then $\gamma(s,\sigma_v,\psi_v)=\gamma(s,\pi_v,\psi_v)$. However, by the definition of the $L$-factor, there is a cancellation, and $L(s,\sigma_v)=L(s+\frac12,\mu)$.

## 3. Analytic properties of the $L$-functions

Consider the $D_n-3$ case in [Sh3], $n=4,5,6,7$: Let $\mathbf{G}=Spin(2n)$ be the split spin group. It is, up to isomorphism, the unique simply-connected group of type $D_n$. We can think of it as a two-fold covering group of $SO(2n)$, namely, there is a 2 to 1 map $\phi: Spin(2n)\longrightarrow SO(2n)$. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$.

Let $\theta=\{\alpha_1=e_1-e_2,...,\alpha_{n-4}=e_{n-4}-e_{n-3},\alpha_{n-2}=e_{n_2}-e_{n-1},\alpha_{n-1}=e_{n-1}-e_n,\alpha_n=e_{n-1}+e_n\}=\Delta-\{\alpha_{n-3}\}$. Let $\mathbf{T}\subset\mathbf{M}_\theta=\mathbf{M}$ be the Levi subgroup of $\mathbf{G}$ generated by $\theta$, and let $\mathbf{P}=\mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$:

$$\mathbf{A}=(\bigcap_{\alpha\in\theta}ker\,\alpha)^0$$
$$=\begin{cases}\{H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-3}}(t^{n-3})H_{\alpha_{n-2}}(t^{n-3})H_{\alpha_{n-1}}(t^{\frac{n-3}{2}})H_{\alpha_n}(t^{\frac{n-3}{2}}): t\in\overline{F}^*\}, & \\ \hfill\text{for } n \text{ odd,} & \\ \{H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-3}}(t^{2(n-3)})H_{\alpha_{n-2}}(t^{2(n-3)})H_{\alpha_{n-1}}(t^{n-3})H_{\alpha_n}(t^{n-3}): t\in\overline{F}^*\}, & \\ \hfill\text{for } n \text{ even.} & \end{cases}$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D\simeq SL_{n-3}\times SL_4$. Then

$$\mathbf{A}\cap\mathbf{M}_D=\begin{cases}\{H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-4}}(t^{n-4})H_{\alpha_{n-1}}(t^{\frac{n-3}{2}})H_{\alpha_n}(t^{\frac{n-3}{2}}): t^{n-3}=1\}, & \\ \hfill\text{for } n \text{ odd,} & \\ \{H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-4}}(t^{2(n-4)})H_{\alpha_{n-1}}(t^{n-3})H_{\alpha_n}(t^{n-3}): t^{2(n-3)}=1\}, & \\ \hfill\text{for } n \text{ even.} & \end{cases}$$

We identify $\mathbf{A}$ with $GL_1$. Then

$$\mathbf{M}\simeq(GL_1\times SL_{n-3}\times SL_4)/(\mathbf{A}\cap\mathbf{M}_D).$$

We define a map $\bar f:\mathbf{A}\times\mathbf{M}_D\longrightarrow GL_1\times GL_1\times SL_{n-3}\times SL_4$ by

$$\bar f:(a(t),x,y)\longmapsto\begin{cases}(t,t^{\frac{n-3}{2}},x,y), & \text{for } n \text{ odd,}\\ (t^2,t^{n-3},x,y), & \text{for } n \text{ even,}\end{cases}$$

which induces a map

$$f : \mathbf{M} \longrightarrow GL_{n-3} \times GL_4.$$

Under the identification $\mathbf{M}_D \simeq SL_{n-3} \times SL_4$, $H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-4}}(t^{n-4})$ is an element in $SL_{n-3}$, and $H_{\alpha_{n-1}}(t)H_{\alpha_{n-2}}(t^2)H_{\alpha_n}(t)$ is an element in $SL_4$. Using this, it is easy to see that

$$f(H_{\alpha_{n-3}}(t)) = (diag(1, ..., 1, t),\, diag(1, 1, t, t)).$$

We note that it is independent of the choices of the roots of unity which show up.

Let $\sigma, \pi$ be cuspidal representations of $GL_{n-3}(\mathbb{A}), GL_4(\mathbb{A})$ with central characters $\omega_1, \omega_2$, resp. Let $\Sigma$ be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by $f$ and $\sigma, \pi$. (More precisely,[1] we need to proceed in the following way: $\mathbf{M}(\mathbb{A})\mathbb{A}^*$ is co-compact in $GL_{n-3}(\mathbb{A}) \times GL_4(\mathbb{A})$, where $\mathbb{A}^*$ is embedded as the center of, say, the first factor. Consequently $\sigma \otimes \pi|_{f(M)},\ M = \mathbf{M}(\mathbb{A})$, decomposes to a direct sum of irreducible cuspidal representations of $M$. Let $\Sigma$ be any irreducible constituent of this direct sum. As we shall see, its choice is irrelevant.)

The central character of $\Sigma$ is

$$\omega_\Sigma = \begin{cases} \omega_1 \omega_2^{\frac{n-3}{2}}, & \text{for } n \text{ odd,} \\ \omega_1^2 \omega_2^{n-3}, & \text{for } n \text{ even.} \end{cases}$$

Now suppose $\sigma_v, \pi_v$ are unramified representations, given by

$$\sigma_v = \pi(\mu_1, ..., \mu_{n-3}), \quad \pi_v = \pi(\nu_1, \nu_2, \nu_3, \nu_4).$$

Let $\Sigma_v$ be the unramified representation of $\mathbf{M}(F_v)$, given by $\sigma_v, \pi_v$'s. Then $\Sigma_v$ is induced from the character $\chi$ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \ldots, \chi \circ H_{\alpha_{n-4}}(t) = \mu_{n-4}\mu_{n-3}^{-1}(t),$$
$$\chi \circ H_{\alpha_{n-1}}(t) = \nu_1\nu_2^{-1}(t), \quad \chi \circ H_{\alpha_{n-2}}(t) = \nu_2\nu_3^{-1}(t), \quad \chi \circ H_{\alpha_n}(t) = \nu_3\nu_4^{-1}(t),$$
$$\chi(a(t)) = \omega_{\Sigma_v}(t).$$

Since $f(H_{\alpha_{n-3}}(t)) = (diag(1, ..., 1, t),\, diag(1, 1, t, t))$, we have

$$\chi \circ H_{\alpha_{n-3}}(t) = \mu_{n-3}\nu_3\nu_4.$$

Hence, we see that, for almost all $v$,

$$L(s, \Sigma_v, r_1) = L(s, \sigma_v \otimes \pi_v, \rho_{n-3} \otimes \wedge^2 \rho_4),$$
$$L(s, \Sigma_v, r_2) = L(s, \sigma_v, \wedge^2 \otimes \omega_{2v}).$$

For ramified places, let $L(s, \Sigma_v, r_1)$ and $L(s, \Sigma_v, r_2)$ be the ones defined in [Sh1, Section 7]. Observe that in particular, if $v = \infty$, then $L(s, \pi_v, r_i)$ is the corresponding Artin $L$-function (cf. [Sh7]) in each case.

Let $I(s, \Sigma_v)$ be the induced representation, and let $N(s, \Sigma_v, w_0)$ be the normalized local intertwining operator [Ki1, (2.1)]:

$$A(s, \Sigma_v, w_0) = \frac{L(s, \Sigma_v, r_1)L(2s, \Sigma_v, r_2)}{L(1+s, \Sigma_v, r_1)L(1+2s, \Sigma_v, r_2)} \frac{N(s, \Sigma_v, w_0)}{\epsilon(s, \Sigma_v, r_1, \psi_v)\epsilon(2s, \Sigma_v, r_2, \psi_v)},$$

where $A(s, \Sigma_v, w_0)$ is the unnormalized intertwining operator. In [Ki4], we showed that $N(s, \Sigma_v, w_0)$ is holomorphic and non-zero for $Re(s) \geq \frac{1}{2}$ for all $v$. For the sake of completeness, we give a proof.

[1] Thanks are due to Prof. Shahidi who pointed this out.

**Proposition 3.1.** *The normalized local intertwining operators* $N(s,\Sigma_v,w_0)$ *are holomorphic and non-zero for* $Re(s)\geq\frac{1}{2}$ *for all* $v$.

*Proof.* We proceed as in [Ki2, Proposition 3.4]. If $\Sigma_v$ is tempered, then the unnormalized operators are holomorphic and non-zero for $Re(s)>0$. We only need to verify Conjecture 7.1 of [Sh1], namely, $L(s,\Sigma_v,r_i)$ is holomorphic for $Re(s)>0$: for archimedean places, $L(s,\Sigma_v,r_i)$ is an Artin $L$-function, and hence our assertion follows. For $p$-adic places, by the multiplicativity of $L$-factors (Proposition 2.6), $L(s,\Sigma_v,r_i)$ is a product of rank-one $L$-functions for discrete series. The rank-one factors are Rankin-Selberg $L$-functions for $GL_k\times GL_l$, and the cases $D_n-2$ and $D_n-3$. The first two cases are well known ([Sh1, Proposition 7.2]). The $D_n-3$ case is a result of [As].

If $\Sigma_v$ is non-tempered, we write $I(s,\Sigma_v)$ as in [Ki1, p. 841],

$$I(s,\Sigma_v)=I(s\tilde{\alpha}+\Lambda_0,\pi_0)=Ind_{\mathbf{M}_0(F_v)\mathbf{N}_0(F_v)}^{\mathbf{G}(F_v)}\,\pi_0\otimes q^{\langle s\tilde{\alpha}+\Lambda_0,\,H_{P_0}(\ )\rangle},$$

where $\pi_0$ is a tempered representation of $\mathbf{M}_0(F_v)$ and $\mathbf{P}_0=\mathbf{M}_0\mathbf{N}_0$ is another parabolic subgroup of $\mathbf{G}$. We can identify the normalized operator $N(s,\Sigma_v,w_0)$ with the normalized operator $N(s\tilde{\alpha}+\Lambda_0,\pi_0,w_0)$, which is a product of rank-one operators attached to tempered representations (cf. [Zh, Proposition 1]).

Here $\tilde{\alpha}=e_1+\cdots+e_{n-3}$; $\Lambda_0=r_1e_1+r_2e_2+\cdots+(-r_2)e_{n-4}+(-r_1)e_{n-3}+(r_1'+r_2')e_{n-2}+(r_1'-r_2')e_{n-1}$, where $\frac{1}{2}>r_1\geq\cdots\geq r_{[\frac{n-3}{2}]}\geq 0$, $\frac{1}{2}>r_1'\geq r_2'\geq 0$. Here $r_i=0$ if $\pi_{1v}$ is tempered. The same is true for $\pi_{2v}$. Hence

$$s\tilde{\alpha}+\Lambda_0=(s+r_1)e_1+\cdots+(s-r_1)e_{n-3}+(r_1'+r_2')e_{n-2}+(r_1'-r_2')e_{n-1}.$$

All the rank-one operators are operators attached to tempered representations of a parabolic subgroup whose Levi subgroup has a derived group isomorphic to $SL_k\times SL_l$ inside a group whose derived group is $SL_{k+l}$, unless $r_1'=r_2'\neq 0$, in which case the rank-one operator is for $D_k-2$. It is the case when $\pi_2'=Ind\,|det|^{r'}\rho\otimes|det|^{-r'}\rho$, where $\rho$ is a tempered representation of $GL_2$.

In the first case, the operators are restrictions to $SL_{k+l}$ of corresponding standard operators for $GL_{k+l}$. By [M-W2, Proposition I.10] one knows that these rank-one operators are holomorphic for $Re(s)>-1$. Hence by identifying roots of $G$ with respect to a parabolic subgroup with those of $G$ with respect to the maximal torus, it is enough to check $Re(\langle s\tilde{\alpha}+\Lambda_0,\beta^\vee\rangle)>-1$ for all positive roots $\beta$ if $Re(s)\geq\frac{1}{2}$. We observed that the least value of $Re(\langle s\tilde{\alpha}+\Lambda_0,\beta^\vee\rangle)$ is $Re(s)-r_1-(r_1'+r_2')$ which is larger than $-1$, if $Re(s)\geq\frac{1}{2}$.

Now suppose we are in the exceptional case, namely, $\pi_2'=Ind\,|det|^{r'}\rho\otimes|det|^{-r'}\rho$, where $\rho$ is a tempered representation of $GL_2$. Then by direct computation, we see that $N(s\tilde{\alpha}+\Lambda_0,\pi_0,\tilde{w})$ is a product of the following three operators:
$N(s\tilde{\alpha}'+\Lambda_0',\pi_{1v}\otimes\rho\otimes\rho,w_0')$,
$N((s-2r')\tilde{\alpha}'+\Lambda_0',\pi_{1v}\otimes\omega_\rho,w_0')$, and
$N((s+2r')\tilde{\alpha}'+\Lambda_0',\pi_{1v}\otimes\omega_\rho,w_0')$,
where $s\tilde{\alpha}'+\Lambda_0'=(s+r_1)e_1+\cdots+(s-r_1)e_{n-3}$ and $\omega_\rho$ is the central character of $\rho$. The first operator is the operator for $D_k-2$ and it is in the corresponding positive Weyl chamber and is holomorphic for $Re(s)\geq\frac{1}{2}$ ([Ki1, Lemma 2.4]). The last two operators are the operators for $GL_k\times GL_1$. Since $Re(s-2r'-r_1)>-1$ if $Re(s)\geq\frac{1}{2}$, they are holomorphic. Consequently, $N(s\tilde{\alpha}+\Lambda_0,\pi_0,\tilde{w}_0)$ is holomorphic for $Re(s)\geq\frac{1}{2}$. By Zhang's lemma (cf. [Ki2, Lemma 1.7], [Zh]), it is non-zero as well. □

We recall some general results in the next two propositions. Let $\mathbf{G}$ be a quasi-split group defined over a number field $F$, and let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup over $F$. Let $\Sigma$ be a cuspidal representation of $\mathbf{M}(\mathbb{A})$.

**Proposition 3.2** (Langlands [La2, Lemma 7.5] or [Ki1, Proposition 2.1])**.** *Unless $w_0\Sigma \simeq \Sigma$, the global intertwining operator $M(s, \Sigma, w_0)$ is holomorphic for $Re(s) \geq 0$.*

**Proposition 3.3** ([Ki1, Lemma 2.3])**.** *If $w_0\Sigma \not\simeq \Sigma$, $\prod_{i=1}^m L_S(1 + is, \Sigma, r_i)$ has no zeros for $Re(s) > 0$.*

*Remark 3.1.* Since the Eisenstein series $E(s, f, g, P)$ is holomorphic for $Re(s) = 0$, we see that $\prod_{i=1}^m L_S(1+is, \Sigma, r_i)$ has no zeros for $Re(s) = 0$ either. Since the local $L$-functions $L(s, \Sigma_v, r_i)$ have no zeros, the completed $L$-function $\prod_{i=1}^m L(1+is, \Sigma, r_i)$ has no zeros for $Re(s) \geq 0$.

Let $S$ be a finite set of finite places where $\pi_v$ is unramified if $v < \infty$ and $v \notin S$. Fix $\chi$ be a grössencharacter of $F$ such that $\chi_v$ is highly ramified for at least one $v \in S$. Let $\Sigma_\chi$ be the cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map $f : \mathbf{M} \longrightarrow GL_{n-3} \times GL_4$ and $\sigma \otimes \chi, \pi$. Then the central character of $\Sigma_\chi$ is

$$\omega_{\Sigma_\chi} = \begin{cases} \omega_1 \chi^m \omega_2^{\frac{n-3}{2}}, & \text{for } n \text{ odd}, \\ \omega_1^2 \chi^{2m} \omega_2^{n-3}, & \text{for } n \text{ even}. \end{cases}$$

Note that $w_0(\omega_{\Sigma_\chi}) = \omega_{\Sigma_\chi}^{-1}$. Hence if $\chi_v$ is highly ramified (say, $\chi_v^{24}$ is ramified), then

$$w_0(\omega_{\Sigma_\chi}) \neq \omega_{\Sigma_\chi},$$

for $m = 1, 2, 3, 4$. Therefore,

$$w_0(\Sigma_\chi) \not\simeq \Sigma_\chi,$$

for $m = 1, 2, 3, 4$. Hence by Propositions 3.1 and 3.2,

**Proposition 3.4.** *Let $\chi$ be as above. Then for all cuspidal representations $\sigma \in \mathcal{T}^S(m)$, $m = 1, 2, 3, 4$, $L(s, (\sigma \otimes \chi) \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is entire.*

*Proof.* For simplicity, we denote $\sigma \otimes \chi$ by $\sigma$. Then $w_0(\Sigma) \not\simeq \Sigma$, where $\Sigma$ is the cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map $f : \mathbf{M} \longrightarrow GL_{n-3} \times GL_4$ and $\sigma, \pi$.

We proceed as in [Ki-Sh, Proposition 3.8]. From [Ki2, (1.2)], we have

$$M(s, \Sigma, w_0) = \frac{L(s, \Sigma, r_1) L(2s, \Sigma, r_2)}{L(1 + s, \Sigma, r_1) L(1 + 2s, \Sigma, r_2) \epsilon(s, \Sigma, r_1) \epsilon(2s, \Sigma, r_2)} N(s, \Sigma, w_0).$$

By Proposition 3.3, $M(s, \Sigma, w_0)$ is holomorphic for $Re(s) > 0$. By Proposition 3.1, $N(s, \Sigma, w_0)$ is non-zero for $Re(s) \geq \frac{1}{2}$. Hence $\frac{L(s,\Sigma,r_1)L(2s,\Sigma,r_2)}{L(1+s,\Sigma,r_1)L(1+2s,\Sigma,r_2)}$ is holomorphic for $Re(s) \geq \frac{1}{2}$. Starting with $Re(s)$ large where both $L$-functions converge absolutely, one can argue inductively that $L(s, \Sigma, r_1) L(2s, \Sigma, r_2)$ is holomorphic for $Re(s) \geq \frac{1}{2}$. We only need to prove that $L(s, \Sigma, r_2)$ has no zeros for $Re(s) \geq 1$. Then by the functional equation, we conclude that $L(s, \Sigma, r_1)$ is entire.

Note that $L(s, \Sigma, r_2) = L(s, \sigma, \wedge^2 \otimes \omega_2)$. So if $m = 1, 2$, it is well known. If $m = 3$, note that $L(s, \sigma, \wedge^2 \otimes \omega_2) = L(s, \tilde{\sigma} \otimes \omega_1 \omega_2)$. Hence it has no zeros for $Re(s) \geq 1$. If $m = 4$, apply Proposition 3.3 to the $D_4 - 3$ case, in which case only one $L$-function, namely, $L(s, \sigma, \wedge^2 \otimes \omega_2)$, shows up in the constant term of the Eisenstein series. Hence it has no zeros for $Re(s) \geq 1$. $\square$

The following theorem was proved in [Ge-Sh] by assuming Proposition 3.1.

**Theorem 3.5** ([Ge-Sh])**.** *Let $\chi$ be as above. Then for all cuspidal representations $\sigma \in \mathcal{T}^S(m)$, $m = 1, 2, 3, 4$, $L(s, (\sigma \otimes \chi) \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is bounded in vertical strips.*

Recall the weak Ramanujan property of automorphic representations of $GL_n(\mathbb{A})$: Let $\pi = \bigotimes \pi_v$ be an automorphic representation of $GL_n(\mathbb{A})$. Let $\pi_v$ be unramified for $v \notin S$, where $S$ is a finite set of places, including all archimedean places. Suppose, for each $v \notin S$, the Hecke conjugacy class attached to $\pi_v$ is given by $diag(\alpha_{1v}, ..., \alpha_{nv})$.

**Definition 3.6.** We say that $\pi$ satisfies the weak Ramanujan property if given $\epsilon > 0$,

$$\max_i \{|\alpha_{iv}|, |\alpha_{iv}^{-1}|\} \leq q_v^{\epsilon},$$

for $v \notin T$, where $T$ is a set of density zero.

If $\pi = \bigotimes_v \pi_v$ is a cuspidal representation of $GL_n(\mathbb{A})$, we can formulate this in the following way. In this case, since $\pi_v$ is generic and unitary, if $v \notin S$, $\pi_v$ is given by ([Ta]) $\pi_v = Ind\, \mu_1|\ |^{r_1} \otimes \cdots \otimes \mu_k|\ |^{r_k} \otimes \nu_1 \otimes \cdots \otimes \nu_l \otimes \mu_k|\ |^{-r_k} \otimes \cdots \otimes \mu_1|\ |^{-r_1}$, where $0 < r_k \leq \cdots \leq r_1 < \frac{1}{2}$, and the $\mu_i$'s, $\nu_j$'s are unramified unitary characters of $F_v^{\times}$. Then $\pi$ satisfies the weak Ramanujan property if given $\epsilon > 0$, the set of places where $r_1 > \epsilon$ has density zero.

**Proposition 3.7.** *(Unitary) cuspidal representations of $GL_2(\mathbb{A}), GL_3(\mathbb{A})$ satisfy the weak Ramanujan property.*

*Proof.* Let $a_v = u_1 q^{r_1} + \cdots + u_k q^{r_k} + b_1 + \cdots + b_l + u_1 q^{-r_1} + \cdots + u_k q^{-r_k}$, where $u_i = \mu_i(\varpi)$ and $b_j = \nu_j(\varpi)$. Let $\epsilon > 0$. Then by Lemma 3.1 of [Ra2], the set of places where $|a_v| \geq q^{\epsilon}$ has density zero.

We first look at $GL_2$. Then $a_v = u_1 q^{r_1} + u_1 q^{-r_1}$. Note that $|u_1 q^{-r_1}| \leq 1$. Hence $|a_v| \geq q^{r_1} - 1$. Hence our result follows.

For $GL_3$, we have $a_v = u_1 q^{r_1} + b + u_1 q^{-r_1}$. Then $|a_v| \geq q^{r_1} - 2$. Hence our result follows again. □

The following proposition is not relevant to our purpose. However, we state it here in order to show the importance of the weak Ramanujan property.

**Proposition 3.8.** *Let $\sigma$ be a cuspidal representation of $GL_m(\mathbb{A})$, $m = 1, 2, 3$, and let $\pi$ be a cuspidal representation of $GL_4(\mathbb{A})$ which satisfies the weak Ramanujan property. Then the L-function $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is holomorphic for $Re(s) > 1$.*

*Proof.* By the weak Ramanujan property, we can find a place $v$ where $\sigma_v, \pi_v$ are unramified and $I(s, \Sigma_v)$ is irreducible for $Re(s) > 1$ (see [Ki2, Theorem 3.1]). Hence it cannot be unitary. By applying [Ki2, Observation 1.3], we see that $M(s, \Sigma, w_0)$ is holomorphic for $Re(s) > 1$. By arguing inductively as in Proposition 3.4, and noting that $L(s, \Sigma, r_2)$ has no zeros for $Re(s) \geq 1$ (see the proof of Proposition 3.4), we conclude that $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2 \rho_4)$ is holomorphic for $Re(s) > 1$. □

**Proposition 3.9** ([J-S, Theorem 1, Section 8])**.** *Let $\chi$ be any grössencharacter, and let $\pi$ be a (unitary) cuspidal representation of $GL_4(\mathbb{A})$. Then a partial L-function $L_S(s, \chi \otimes \pi, \rho_1 \otimes \wedge^2 \rho_4)$ is holomorphic for $Re(s) > 1$. It has a pole at $s = 1$ if and only if $\chi^2 \omega_\pi = 1$ and a certain period integral is not zero.*

*Proof.* In [J-S, Theorem 1, Section 8], that $L_S(s, \chi \otimes \pi, \rho_1 \otimes \wedge^2\rho_4)$ is holomorphic for $Re(s) > 1$ is not stated explicitly. However, the global integral $I(s, \chi, \phi, \Phi)$ is holomorphic for $Re(s) > 1$ since the singularities of the integral are those of the Eisenstein series, which are absolutely convergent for $Re(s) > 1$ (see p. 179 of [J-S]). □

## 4. Exterior square lift; weak lift

Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $GL_4(\mathbb{A})$. Let $\phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$ be the parametrization of $\pi_v$ for each $v$, given by the local Langlands correspondence [H-T], [He2], [La4]. Then we obtain a map $\wedge^2 \circ \phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$. Let $\wedge^2\pi_v$ be the irreducible admissible representation attached to $\wedge^2 \circ \phi_v$ by the local Langlands correspondence. It is obvious that if $\pi_v$ is an unramified representation, given by $\pi_v = Ind_B^{GL_4}\, \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4$, where the $\eta_i$'s are unramified quasi-characters of $F_v^\times$, then $\wedge^2\pi_v$ is the unique unramified subquotient of the principal series $Ind_B^{GL_6}\, \eta_1\eta_2 \otimes \eta_1\eta_3 \otimes \eta_1\eta_4 \otimes \eta_2\eta_3 \otimes \eta_2\eta_4 \otimes \eta_3\eta_4$.

Then $\wedge^2\pi = \bigotimes_v \wedge^2\pi_v$ is an irreducible admissible representation of $GL_6(\mathbb{A})$. In this section we apply the converse theorem (Theorem 2.1) to $\wedge^2\pi$ with $S$ being a finite set of finite places, where $\pi_v$ is unramified for $v < \infty$ and $v \notin S$. We obtain a weak lift of $\pi$, namely, we prove that there exists an automorphic representation $\Pi' = \bigotimes_v \Pi'_v$ such that $\Pi'_v \simeq \wedge^2\pi_v$ for $v \notin S$.

In Section 5, we construct all local lifts $\Pi_v$ in the sense of Definition 2.2, using weak lifts, with the property that $\Pi_v \simeq \wedge^2\pi_v$, if $v \notin T$, where $T$ is the set of places such that $v|2,3$ and $\pi_v$ is a supercuspidal representation of $GL_4(F_v)$. We apply the converse theorem again, to conclude that $\Pi = \bigotimes_v \Pi_v$ is an automorphic representation of $GL_6(\mathbb{A})$.

First we show

**Proposition 4.1.** *Let $\sigma \in \mathcal{T}^S(m) \otimes \chi$ for a grössencharacter $\chi$. Then for $v \notin S$, $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ and $\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ are Artin factors, i.e.,*

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4, \psi_v) = \gamma(s, \sigma_v \times \wedge^2\pi_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4) = L(s, \sigma_v \times \wedge^2\pi_v).$$

*Proof.* When $v = \infty$, this follows from the result of [Sh7]. Suppose $v < \infty$. Then by the assumption, $\pi_v$ is unramified for $v \notin S$. Since $\pi_v$ is also generic, we can write it as $\pi_v = Ind_B^{GL_4}\, \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4$, where the $\eta_i$'s are unramified quasi-characters of $F_v^\times$. Then by the multiplicativity of $\gamma$-factors (cf. Proposition 2.4) and by the definition of $L$-factors (cf. Proposition 2.5), $\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4, \psi_v)$ and $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ are products of $\gamma(s, \sigma_v \otimes \eta_i\eta_j, \psi_v)$ and $L(s, \sigma_v \otimes \eta_i\eta_j)$ for $1 \le i < j \le 4$, resp. By Theorem 3.1 and Theorem 9.5 of [J-PS-S], the same multiplicativity formulas hold for the right-hand side. Shahidi (Proposition 2.3) has shown that in the case of $GL_k \times GL_l$, his $L$- and $\gamma$-factors are those of Artin. Our assertion follows. □

It would be useful to have the above identity for all $v \in S$. However, it is not even known that Shahidi's exterior square $L$-function $L(s, \pi_v, \wedge^2\rho_4)$ is an Artin $L$-function when $\pi_v$ is a supercuspidal representation. But we have

**Proposition 4.2.** *Let $\sigma \in \mathcal{T}^S(m) \otimes \chi$ for a grössencharacter $\chi$, and suppose that for $v \in S$, $\pi_v$ is not supercuspidal. Then $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ and*

$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ *are Artin factors, i.e.,*

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4, \psi_v) = \gamma(s, \sigma_v \times \wedge^2\pi_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4) = L(s, \sigma_v \times \wedge^2\pi_v).$$

*Proof.* Since $v \in S$, $\sigma_v$ is in the principal series. Since $\sigma_v$ is unramified and generic, we can write it as $\sigma_v = Ind_B^{GL_m}\, \eta_1 \otimes \cdots \otimes \eta_m$, where the $\eta_i$'s are unramified quasi-characters of $F_v^\times$. Then by the multiplicativity of $\gamma$-factors (Proposition 2.4) and by the definition of $L$-factors (Proposition 2.5), $\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4, \psi_v)$ and $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ are products of $\gamma(s, \pi_v, \wedge^2 \otimes \eta_i, \psi_v)$ and $L(s, \pi_v, \wedge^2 \otimes \eta_i)$ for $1 \le i \le m$, resp. By Theorem 3.1 and Theorem 9.5 of [J-PS-S], the same multiplicativity formulas hold for the right-hand side.

Hence it is enough to prove the equalities when $\sigma_v = \eta$ is a character of $F_v^\times$. Note that $\pi_v$ is generic. By a well-known result, it is a full induced representation. Since $\pi$ is not supercuspidal, by the multiplicativity of $\gamma$-factors and $L$-factors (cf. Propositions 2.4 and 2.5), $\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4, \psi_v)$ and $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2\rho_4)$ are products of $\gamma$- and $L$-factors for $GL_k \times GL_l$. We have the same multiplicativity formula for the right-hand side. By Proposition 2.3, we have the equality of Langlands-Shahidi $L$-functions and those of Artin for $GL_k \times GL_l$. Hence our result follows. $\square$

Later in Lemma 5.2.1, we will extend the above result to any generic irreducible representation $\sigma_v$. In light of the above proposition, we need to proceed in two steps as in [Ra1], namely, first, we do the good case when none of $\pi_v$ is supercuspidal, and then we do the general case, following Ramakrishnan's idea of descent [Ra1]. It is based on the observation of Henniart [He1] that a supercuspidal representation of $GL_n(F_v)$ becomes a principal series after a solvable base change.

*Remark 4.1.* In actuality, in establishing a weak lift, we do not need the local Langlands correspondence. At bad places $S$, we take the candidate $\Pi_v$ to be arbitrary, except that the central character of $\Pi_v$ is the same as $\wedge^2\pi_v$, namely, $\omega_{\pi_v}^3$. Then we would apply the stability of $\gamma$-factors by using highly ramified characters as in [CKPSS]. Namely, given two irreducible admissible representations $\pi_{1v}, \pi_{2v}$ of $GL_4(F_v)$, $\gamma(s, \pi_{1v}, \wedge^2 \otimes \chi_v) = \gamma(s, \pi_{2v}, \wedge^2 \otimes \chi_v)$ for every highly ramified character $\chi_v$. We hope to be able to prove this in the future. Once it is done, we may avoid using the descent argument, and hence Appendix 1 altogether.

Once we obtain a weak lift, we will construct $\Pi_v$ for $v \in S$ in Section 5.2 such that the equalities of $\gamma$- and $L$-factors in Definition 2.2 hold.

**4.1 Lift in the good case.** Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. Following [Ra1], we say $\pi$ is good if none of $\pi_v$ is supercuspidal.

**Theorem 4.1.1.** *Suppose $\pi$ is good. Then there exists a weak exterior square lift $\Pi = \bigotimes_v \Pi_v$ of $\pi$, i.e., $\Pi_v \simeq \wedge^2\pi_v$ for almost all $v$. It is an automorphic representation of $GL_6(\mathbb{A})$ of the form $Ind\, \tau_1 \otimes \cdots \otimes \tau_k$, where $\tau_i$ is a cuspidal representation of $GL_{n_i}(\mathbb{A})$.*

In the notation of [J-S3], $Ind\, \tau_1 \otimes \cdots \otimes \tau_k = \tau_1 \boxplus \cdots \boxplus \tau_k$. The proof of this theorem will occupy this subsection.

Choose $\chi$ so that Proposition 3.4 and Theorem 3.5 hold. Then by Propositions 4.1 and 4.2, we can apply the converse theorem (Theorem 2.1) to $\wedge^2\pi$ and $S$, where

$S$ is a finite set of finite places such that $\pi_v$ is unramified for $v \notin S$, $v < \infty$. We obtain that $\wedge^2\pi \otimes \chi$ is quasi-automorphic, and hence $\wedge^2\pi$ is as well, i.e., there exists an automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \wedge^2\pi_v$ for all $v \notin S$.

By the classification of automorphic representations of $GL_n$ [J-S3], $\Pi$ is equivalent to a subquotient of

$$Ind\, |det|^{r_1}\tau_1 \otimes \cdots \otimes |det|^{r_k}\tau_k, \tag{4.1}$$

where $\tau_i$ is a (unitary) cuspidal representation of $GL_{n_i}(\mathbb{A})$ and $r_i \in \mathbb{R}$. Note that for almost all places, $\Pi_v$ is the unique unramified subquotient of $\Xi_v = Ind\, |det|^{r_1}\tau_{1v} \otimes \cdots \otimes |det|^{r_k}\tau_{kv}$. Hence the Hecke conjugacy class of $\Pi_v$ is the same as that of $\Xi_v$. Note also that the central character of $\Pi$ is $\omega_\Pi = \omega_\pi^3$. In particular, it is unitary. Hence $n_1r_1 + \cdots + n_kr_k = 0$. We want to show that all the $r_i$'s are zero.

The following proposition illustrates the importance of the weak Ramanujan property. We may use it instead of Proposition 4.1.6 in Section 7 since the symmetric cube of a cuspidal representation of $GL_2$ satisfies the weak Ramanujan property.

**Proposition 4.1.2.** *Suppose $\pi$ satisfies the weak Ramanujan property. Then $r_1 = \cdots = r_k = 0$.*

*Proof.* By the assumption, $\Pi$ also satisfies the weak Ramanujan property. Suppose the $r_i$'s are not all zero. From the relation $n_1r_1 + \cdots + n_kr_k = 0$, it follows that there is an $i$ such that $r_i > 0$. But then this contradicts the weak Ramanujan property with $\epsilon = r_i$. □

We will show $r_1 = \cdots = r_k = 0$, without assuming the weak Ramanujan property of $\pi$. First we have

**Lemma 4.1.3** ([Ra2, Lemma 3.1])**.** *Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. Let $\pi_v$ be an unramified component with the trace $a_v$, i.e., $a_v = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, where the Hecke conjugacy class of $\pi_v$ is given by $diag(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Then given $\epsilon > 0$, the set of places where $|a_v| \geq q_v^\epsilon$ has density zero.*

Note that at a place where $\pi_v$ is non-tempered, the trace $a_v$ has one of the three forms below. Here $u_1, u_2, u_3$ are complex numbers with absolute value one. We suppress the dependence of all the factors on $v$ for simplicity of notation, except $a_v$.

$S_1$; $a_v = u_1q^a + u_2q^a + u_1q^{-a} + u_2q^{-a}$, where $0 < a < \frac{1}{2}$;
$S_2$; $a_v = u_1q^a + u_2 + u_3 + u_1q^{-a}$, where $0 < a < \frac{1}{2}$;
$S_3$; $a_v = u_1q^{a_1} + u_2q^{a_2} + u_1q^{-a_1} + u_2q^{-a_2}$, where $0 < a_2 < a_1 < \frac{1}{2}$.

**Lemma 4.1.4.** *Given $\epsilon > 0$, the set of places $a > \epsilon$ in $S_2$ has density zero.*

*Proof.* Just note that since $q^{-a} < 1$, $|a_v| > q^a - 3$. Use Lemma 4.1.3. □

Now we have

**Lemma 4.1.5.** *In (4.1), if $r_i \neq 0$, then $n_i = 1$.*

*Proof.* If $\pi_v$ is unramified, the Hecke conjugacy class of $\wedge^2\pi_v$ is given by one of the following forms:

$S_1 : diag(u_1u_2q^{2a}, u_1u_2, u_1^2, u_2^2, u_1u_2, u_1u_2q^{-2a})$,
$S_2 : diag(u_1u_2q^a, u_1u_3q^a, u_1^2, u_2u_3, u_1u_2q^{-a}, u_1u_3q^{-a})$,

$S_3 : diag(u_1u_2q^{a_1+a_2}, u_1u_2q^{a_1-a_2}, u_1^2, u_2^2, u_1u_2q^{-a_1+a_2}, u_1u_2q^{-a_1-a_2})$,
$S_0 : diag(u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3u_4)$,

where the $u_i$'s are complex numbers with absolute value one and $\pi_v$ is tempered for $v \in S_0$.

Suppose $r_1 \neq 0$. We will show that $n_1 = 1$:

Suppose $n_1 = 5$. Then $n_2 = 1$. By checking case by case, we see that the Hecke conjugacy class of $\Pi_v$ can never be of the above form.

Suppose $n_1 = 4, n_2 = 2$. Then $r_2 = -2r_1$. By [Ra2, Theorem A], $\tau_{2v}$ is tempered for a set $T$ of lower density at least $\frac{9}{10}$. Since the Hecke conjugacy class of $\Pi_v$ should be one of the above forms, they should be, for $v \in T$, of the form in $S_2$ above:

$$diag(u_1u_2q^{2r_1}, u_1u_3q^{2r_1}, u_1^2, u_2u_3, u_1u_2q^{-2r_1}, u_1u_3q^{-2r_1}).$$

In this case $\Pi_v = \wedge^2\pi_v$, where the Hecke conjugacy class of $\pi_v$ is given by

$$diag(u_1q^{2r_1}, u_2, u_3, u_1q^{-2r_1}).$$

Note that $r_1$ is fixed and the Hecke conjugacy class of $\pi_v$ is given by the above form for all $v \in T$. This contradicts Lemma 4.1.4. The same proof works for $n_1 = 4, n_2 = n_3 = 1$.

Suppose $n_1 = 3$. Since cuspidal representations of $GL_2, GL_3$ satisfy the weak Ramanujan property, by taking $\epsilon < |r_1|$, we can see that the Hecke conjugacy class of $\Pi_v$ can never be of the above form for $\wedge^2\pi_v$.

Suppose $n_1 = 2$. By [Ra2, Theorem A], $\tau_{1v}$ is tempered for a set $T$ of lower density at least $\frac{9}{10}$. Then we see that the Hecke conjugacy class of $\Pi_v$ should be of the form

$$diag(u_1u_2q^{r_1}, u_1u_3q^{r_1}, u_1^2, u_2u_3, u_1u_2q^{-r_1}, u_1u_3q^{-r_1}).$$

In this case $\Pi_v = \wedge^2\pi_v$, where the Hecke conjugacy class of $\pi_v$ is given by

$$diag(u_1q^{r_1}, u_2, u_3, u_1q^{-r_1}).$$

Note that $r_1$ is fixed and the Hecke conjugacy class of $\pi_v$ is given by the above form for all $v \in T$. This contradicts Lemma 4.1.4.

Hence if $r_1 \neq 0$, $n_1 = 1$. The same is true for $i > 1$. □

**Proposition 4.1.6.** *In* (4.1), $r_1 = \cdots = r_k = 0$.

*Proof.* Suppose not all of the $r_i$'s are zero. Suppose $r_1 < 0$ is smallest. Then by Lemma 4.1.5, $n_1 = 1$ and

$$L_S(s, \tau_1^{-1} \times \Pi) = L_S(s, \pi, \wedge^2\rho_4 \otimes \tau_1^{-1}) = \prod_{i=1}^{k} L_S(s + r_i, \tau_1^{-1} \times \tau_i).$$

Here $L_S(s+r_1, \tau_1^{-1} \times \tau_1)$ has a pole at $s = 1 - r_1$ and $L(s+r_i, \tau_1^{-1} \times \tau_i)$ has no zero at $s = 1 - r_1 > 1$ for $i = 2, ..., k$. Hence $L_S(s, \tau_1^{-1} \times \Pi)$ has a pole at $s = 1 - r_1$. The same is true for $L_S(s, \pi, \wedge^2\rho_4 \otimes \tau_1^{-1})$. This contradicts Proposition 3.9. □

This finishes the proof of Theorem 4.1.1.

**4.2 Lift in the general case.** In this subsection, since we are dealing with various fields, we denote the ring of adeles of $F$ by $\mathbb{A}_F$. We start with

**Theorem 4.2.1** ([He1])**.** *Let $\pi_v$ be a supercuspidal representation of $GL_n(F_v)$. Then there exists a finite sequence of fields $E_0 = F_v \subset E_1 \subset \cdots \subset E_r$, with $E_{i+1}$ finite cyclic of prime degree over $E_i$, such that the representation $\Pi_v$ of $GL_n(E_r)$ obtained from $\pi_v$ by successive base changes from $E_i$ to $E_{i+1}$ is no longer cuspidal. In fact, we can choose $E_r$ to be Galois over $F_v$ and $\Pi_v$ to be unramified principal series $Ind\,\chi \otimes \cdots \otimes \chi$, where $\chi$ is an unramified character of $E_r^\times$. We define $l(\pi_v)$ to be the minimal length $r$ of $E_r/F_v$ such that the base change $(\pi_v)_{E_r}$ is in the principal series.*

**Lemma 4.2.2** ([Ra1, Lemma 3.6.2])**.** *Let $\pi$ be a cuspidal representation of $GL_n(\mathbb{A}_F)$. Then there exist at most a finite number of grössencharacters $\chi$ such that*

$$\pi \simeq \pi \otimes \chi.$$

Our goal is to prove the following main theorem.

**Theorem 4.2.3.** *Let $\pi$ be a cuspidal representation of $GL_4(\mathbb{A}_F)$. Then there exists a weak exterior square lift $\Pi$ of $GL_6(\mathbb{A}_F)$. It is of the form $\tau_1 \boxplus \cdots \boxplus \tau_k$ in the notation of* [J-S3]*, where $\tau_i$ is a (unitary) cuspidal representation of $GL_{n_i}(\mathbb{A}_F)$.*

*Proof.* We follow [Ra1] closely. We thank Prof. Ramakrishnan for his help. Let $S$ be a finite set of finite places such that $\pi_v$ is supercuspidal for $v \in S$. For each $v \in S$, let $l(\pi_v)$ be as in Theorem 4.2.1, i.e., the minimal degree of all the solvable extensions $E(v)/F_v$ for which the base changes $(\pi_v)_{E(v)}$ are in the principal series. Let $l(\pi)$ be the maximum of $\{l(\pi_v) | v \in S\}$, and let $S'$ be the subset of $S$ where this maximum is attained. Further, for each $v \in S'$, let $p(v)$ denote the maximum over all $E(v)$, of the degree, required to be a prime or 1, of the largest cyclic extension $K(v)$ of $F_v$, contained in $E(v)$. Let $p = p(\pi)$ be the maximum of $p(v)$ over all $v \in S'$, and let $S''$ denote the subset of $S'$ where $p(v) = p$ (and $l(\pi_v) = l(\pi)$). Note that $p$ is a prime unless $\pi$ is good over $F$, i.e., has no supercuspidal components, in which case $p = 1$.

Now set $r(\pi) = (l(\pi), p(\pi))$. We will order these pairs as follows: $(l, p) < (l', p')$ if either $l < l'$, or $l = l'$ and $p < p'$. If $r = r(\pi) = (0, 1)$, we are done. So we will assume that $r > (0, 1)$ and assume by induction that the theorem is proved (over all number fields $K$) for all cuspidal representations $\pi$ of $GL_4(\mathbb{A}_K)$ with $r(\pi) < r$.

Fix, at every place $v \in S''$, a character $\chi_v$ of $F_v^\times$, given by the class field theory for the cyclic extension $K(v)/F_v$ of degree $p$. Enumerate the set of finite places where $\pi$ is unramified as $\{v_1, v_2, ...\}$.

Fix an index $j \geq 1$, and let $S(j) = \{v_j\} \cup S''$. Let $\chi_{v_j}$ denote the trivial character of $F_{v_j}^\times$.

Now by the Grunwald-Wang theorem (see [A-T, Chap. 10, Theorem 5]), we can find a grössencharacter $\chi(j)$ of order $p$ whose local restrictions are given by $\chi_v$ for every $v \in S(j)$.

Let $K_j$ be the $p$-extension of $F$ attached to $\chi(j)$ by the class field theory. Note that for each $j \geq 1$, $v_j$ splits completely in $K_j$, but every place $v \in S''$ is either inert or ramifies in $K_j$. By throwing away finitely many indices, we can assume that the $K_j$'s are all different: This is because one cannot choose a finite number of $p$-extensions of $F$ such that every $v_j$ splits in one of them; put another way, given any finite number of $p$-extensions of $F$, the Tchebotarev density theorem states that the set of primes which are inert in each of these finite sets of $p$-extensions will have positive density. On the other hand, the set $\{v_j\}$ has density 1.

Let $\pi_{K_j}$ be the base change of $\pi$ to $K_j$ for each $j$. So by construction, for every $j \geq 1$, $r(\pi_{K_j}) < r$.

Thus, by induction, Theorem 4.2.3 holds for $\pi_{K_j}$ for each $j$. Note that if the automorphic representation $\pi_{K_j}$ is not cuspidal for some $j$, then $p = 2$ and $\pi \simeq \pi \otimes \eta$, where $\eta$ is the quadratic character of $F$ attached to the quadratic extension $K_j/F$ (see Proposition 2.3.1 of [Ra1]). Hence by Lemma 4.2.2, $\pi_{K_j}$ is cuspidal for almost all $j$, and, by throwing away finitely many indices, we can assume that $\pi_{K_j}$ is cuspidal for all $j$. Let $\Pi_j$ be a weak exterior square lift of $\pi_{K_j}$.

Recall the following descent criterion in [Ra1].

**Proposition 4.2.4.** *Fix $n, p \in \mathbb{N}$ with $p$ prime. Let $F$ be a number field, let $\{K_j | j \in \mathbb{N}\}$ be a family of cyclic extensions of $F$ with $[K_j : F] = p$, and for each $j \in \mathbb{N}$, let $\pi_j$ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_{K_j})$. Suppose that, given $j \in \mathbb{N}$,*

$$(\mathrm{DC}) \qquad (\pi_j)_{K_j K_r} \simeq (\pi_r)_{K_j K_r},$$

*for almost all $r \in \mathbb{N}$. Then there exists a unique cuspidal automorphic representation $\pi$ of $GL_n(\mathbb{A}_F)$ such that*

$$(\pi)_{K_j} \simeq \pi_j,$$

*for all but a finite number of $j$.*

*Remark 4.2.* In [Ra1], it is stated that (DC) holds for all $j, r \in \mathbb{N}$. However, the proof shows our condition suffices.

Appendix 1 extends the above proposition to isobaric automorphic representations, i.e., automorphic representations induced from cuspidal representations.

**Proposition 4.2.5** (Appendix 1)**.** *The result in the above proposition holds when the $\pi_j$'s are isobaric automorphic representations.*

*Proof of Theorem 4.2.3 (contd.).* Now we fix a pair $(j, r)$ of indices and consider the descent criterion (DC). Let $w$ be a finite place where $((\Pi_j)_{K_j K_r})_w$, $((\Pi_r)_{K_j K_r})_w$ and $K_j K_r$ are all unramified. Then, by construction, both of these local representations correspond to the restriction (to the Weil group of $(K_j K_r)_w$) of $\wedge^2 \phi_v$, where $v$ signifies the place of $F$ below $w$. (Recall that $\phi_v$ is associated to $\pi_v$.) Then

$$((\Pi_j)_{K_j K_r})_w \simeq ((\Pi_r)_{K_j K_r})_w.$$

Hence the strong multiplicity one theorem gives (DC). Thus by applying Proposition 4.2.5, we obtain a unique automorphic descent $\Pi$ on $GL_6(\mathbb{A}_F)$ such that, for all but a finite number of indices,

$$\Pi_{K_j} \simeq \Pi_j.$$

Finally, by construction, each (unramified) finite place $v_j$ splits completely in $K_j$; let $w_j$ be a divisor of $v_j$ in $K_j$. Let $\sigma_v$ be a discrete series of $GL_m(F_v)$, $m = 1, 2, 3, 4$. Then by the definition of base change, for almost all $j$,

$$\begin{aligned} L(s, \sigma_{v_j} \times \Pi_{v_j}) &= L(s, (\sigma_{K_j})_{w_j} \times (\Pi_j)_{w_j}) = L(s, (\sigma_{K_j})_{w_j} \otimes (\Pi_j)_{w_j}, \rho_m \otimes \wedge^2 \rho_4) \\ &= L(s, \sigma_{v_j} \otimes \pi_{v_j}, \rho_m \otimes \wedge^2 \rho_4). \end{aligned}$$

Similarly for the $\epsilon$-factors. Thus $\Pi$ is a weak exterior square lift of $\pi$. This finishes the proof of Theorem 4.2.3. $\square$

## 5. Exterior square lift; strong lift

**5.1 Functorial lift from $GL_2 \times GL_2$ to $GL_4$.** We give a new proof of the existence of the functorial product, corresponding to the tensor product map $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$. It is originally due to Ramakrishnan [Ra1]. However, we give a proof based entirely on the Langlands-Shahidi method. Also we need this in the proof of Corollary 5.1.6

More precisely, let $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$ be the map given by the tensor product. Let $\pi_1, \pi_2$ be cuspidal representations of $GL_2(\mathbb{A})$. Let $\phi_{iv} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$ be the parametrization of $\pi_{iv}$ for $i = 1, 2$. Then we obtain a map $\phi_{1v} \otimes \phi_{2v} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$. Let $\pi_{1v} \boxtimes \pi_{2v}$ be the irreducible admissible representation of $GL_4(F_v)$ attached to $\phi_{1v} \otimes \phi_{2v}$ by the local Langlands correspondence [H-T], [He2], [La4]. Let $\pi_1 \boxtimes \pi_2 = \bigotimes_v (\pi_{1v} \boxtimes \pi_{2v})$. Ramakrishnan [Ra1] showed that $\pi_1 \boxtimes \pi_2$ is an automorphic representation of $GL_4(\mathbb{A})$, as predicted by Langlands' functoriality.

In this section, we prove the functoriality of such a tensor product entirely by the Langlands-Shahidi method. Note that all the necessary analytic properties of the triple product $L$-functions $L(s, \sigma \times \pi_1 \times \pi_2)$ were proved in [Ki-Sh], where $\pi_1, \pi_2, \sigma$ are cuspidal representations of $GL_2(\mathbb{A})$. We follow Section 4.1 closely. Let $T$ be a set of places where $\pi_{1v}, \pi_{2v}$ are both supercuspidal representations. First we show

**Lemma 5.1.1.** *If $v \notin T$, then for all irreducible, generic representations $\sigma_v$ of $GL_m(F_v)$, $m = 1, 2$,*

$$\gamma(s, \sigma_v \times \pi_{1v} \times \pi_{2v}, \psi_v) = \gamma(s, \sigma_v \times (\pi_{1v} \boxtimes \pi_{2v}), \psi_v),$$
$$L(s, \sigma_v \times \pi_{1v} \times \pi_{2v}) = L(s, \sigma_v \times (\pi_{1v} \boxtimes \pi_{2v})).$$

*Proof.* By assumption, in the multiplicativity of $\gamma$-factors and $L$-factors (cf. Propositions 2.4 and 2.5), all rank-one $\gamma$- and $L$-factors are for $GL_k \times GL_l$ and, in that case, Shahidi (Proposition 2.3) has shown that his $\gamma$-factors are Artin factors. Hence by Proposition 2.6, the left-hand sides are Artin factors. Thus we have the equalities. $\square$

Now let $S = T$ if $T$ is not empty. If $T$ is empty, then let $S = \{v\}$, where $v$ is any finite place. Note that for $\sigma \in \mathcal{T}^S(m)$, $\sigma_v$ is in the principal series for $v \in S$. Hence, in the multiplicativity of $\gamma$-factors and $L$-factors (cf. Propositions 2.4 and 2.5), all rank-one $\gamma$- and $L$-factors are for $GL_k \times GL_l$, namely, the product of the form $\gamma(s, \pi_{1v} \times (\pi_{2v} \otimes \chi_v), \psi_v)$ and $L(s, \pi_{1v} \times (\pi_{2v} \otimes \chi_v))$, resp. In that case, Shahidi (Proposition 2.3) has shown that his $\gamma$-factors are Artin factors. Hence the equalities in Lemma 5.1.1 hold. We apply the converse theorem (Theorem 2.1) to $\pi_1 \boxtimes \pi_2 = \bigotimes_v (\pi_{1v} \boxtimes \pi_{2v})$ with $S$, and obtain an automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $GL_4(\mathbb{A})$ such that $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$ for all $v \notin S$.

**Proposition 5.1.2.** $\Pi$ *is of the form*

$$\Pi = \tau_1 \boxplus \cdots \boxplus \tau_k,$$

*in the notation of* [J-S3], *where $\tau_i$ is a (unitary) cuspidal representation of* $GL_{n_i}(\mathbb{A})$.

*Proof.* By the classification of automorphic representations of $GL_n(\mathbb{A})$ [J-S3], $\Pi$ is equivalent to a subquotient of

$$Ind\,|det|^{r_1}\tau_1 \otimes \cdots \otimes |det|^{r_k}\tau_k,$$

where $\tau_i$ is a unitary cuspidal representation of $GL_{n_i}(\mathbb{A})$ and $r_i \in \mathbb{R}$. Note that for almost all $v$, $\Pi_v$ is the unique unramified subquotient of $\Xi_v = Ind\,|det|^{r_1}\tau_{1v} \otimes \cdots \otimes |det|^{r_k}\tau_{kv}$. The Hecke conjugacy class of $\Pi_v$ is that of $\Xi_v$.

Let $\pi_{1v}, \pi_{2v}$ be unramified local components with the Hecke conjugacy classes given by $diag(\alpha_{1v}, \beta_{1v}), diag(\alpha_{2v}, \beta_{2v})$, resp. Then the Hecke conjugacy class of $\Pi_v$ is given by

$$diag(\alpha_{1v}\alpha_{2v}, \alpha_{1v}\beta_{2v}, \alpha_{2v}\beta_{1v}, \alpha_{2v}\beta_{2v}).$$

By Proposition 3.7, $\pi_1, \pi_2$ satisfy the weak Ramanujan property, and so does $\Pi$. We can show $r_1 = \cdots = r_k = 0$ in the same way as in Proposition 4.1.2. $\square$

**Proposition 5.1.3.** *Suppose $v \in T$, i.e., $\pi_{1v}, \pi_{2v}$ are both supercuspidal representations. Then there exists an irreducible admissible representation $\Pi_v$ which is a local lift of $\pi_{1v} \otimes \pi_{2v}$, in the sense that*

$$\gamma(s, \sigma_v \times \pi_{1v} \times \pi_{2v}, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$
$$L(s, \sigma_v \times \pi_{1v} \times \pi_{2v}) = L(s, \sigma_v \times \Pi_v),$$

*for all generic irreducible representations $\sigma_v$ of $GL_m(F_v)$, $m = 1, 2$. Moreover, $\Pi_v$ is tempered.*

*Proof.* Let $\pi_{1v} \otimes \pi_{2v}$ be a supercuspidal representation of $GL_2(F_v) \times GL_2(F_v)$. Let $\pi_1 \otimes \pi_2 = \bigotimes_w (\pi_{1w} \otimes \pi_{2w})$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}) \times GL_2(\mathbb{A})$ such that $\pi_{1w} \otimes \pi_{2w}$ is unramified for all $w < \infty$ and $w \neq v$ (Proposition 5.1 of [Sh1]).

Let $\Pi$ be a weak lift of $\pi_1 \otimes \pi_2$ as in Proposition 5.1.2 such that $\Pi_w \simeq \pi_{1w} \boxtimes \pi_{2w}$ for $w \neq v$. (We use a similar definition of weak lift as in Definition 2.2.) We note that $\Pi_v$ is irreducible, unitary, and generic.

Claim: $\Pi_v$ is a local lift of $\pi_{1v} \otimes \pi_{2v}$.

By the multiplicativity of $\gamma$- and $L$-factors (cf. Propositions 2.4 and 2.5), it is enough to show this claim for discrete series $\sigma_v$. Then we can find a cuspidal representation $\sigma$ whose local component at $v$ is $\sigma_v$ [Ro].

Consider the two $L$-functions $L(s, \sigma \times \pi_1 \times \pi_2)$ and $L(s, \sigma \times \Pi)$. Both have the functional equations:

$$L(s, \sigma \times \pi_1 \times \pi_2) = \epsilon(s, \sigma \times \pi_1 \times \pi_2) L(1 - s, \tilde{\sigma} \times \tilde{\pi}_1 \times \tilde{\pi}_2),$$
$$L(s, \sigma \times \Pi) = \epsilon(s, \sigma \times \Pi) L(1 - s, \tilde{\sigma} \times \tilde{\Pi}).$$

Since $\pi_{1w}, \pi_{2w}, \Pi_w$ are unramified for $w \neq v$, $w < \infty$, it follows that $\Pi_w$ is the lift of $\pi_{1w} \otimes \pi_{2w}$ for all $w \neq v$. Hence

$$L(s, \sigma_w \times \pi_{1w} \times \pi_{2w}) = L(s, \sigma_w \times \Pi_w), \quad \epsilon(s, \sigma_w \times \pi_{1w} \times \pi_{2w}, \psi_w) = \epsilon(s, \sigma_w \times \Pi_w, \psi_w),$$

for all $w \neq v$. The functional equations above can be written in the form

$$\gamma(s, \sigma_v \times \pi_{1v} \times \pi_{2v}, \psi_v) = \prod_{w \neq v} \frac{L(s, \sigma_w \times \pi_{1w} \times \pi_{2w})}{\epsilon(s, \sigma_w \times \pi_{1w} \times \pi_{2w}, \psi_w) L(1 - s, \tilde{\sigma}_w \times \tilde{\pi}_{1w} \tilde{\pi}_{2w})},$$
$$\gamma(s, \sigma_v \times \Pi_v, \psi_v) = \prod_{w \neq v} \frac{L(s, \sigma_w \times \Pi_w)}{\epsilon(s, \sigma_w \times \Pi_w, \psi_w) L(1 - s, \tilde{\sigma}_w \times \tilde{\Pi}_w)}.$$

Hence
$$\gamma(s,\sigma_v\times\pi_{1v}\times\pi_{2v},\psi_v)=\gamma(s,\sigma_v\times\Pi_v,\psi_v).$$

In order to show that the equality of $\gamma$-factors implies the equality of $L$-factors, we need a little care, since we do not know, a priori, that $\Pi_v$ is tempered. As we remarked earlier, $\Pi_v$ is irreducible, unitary, and generic. Hence it is of the form $Ind\,\tau_1|\det|^{s_1}\otimes\cdots\otimes\tau_\ell|\det|^{s_\ell}\otimes\tau_{\ell+1}\otimes\cdots\otimes\tau_{\ell+u}\otimes\tau_\ell|\det|^{-s_\ell}\otimes\cdots\otimes\tau_1|\det|^{-s_1}$, where the $\tau_i$'s are discrete series representations of smaller $GL$'s and $0<s_l\le\cdots\le s_1<\frac{1}{2}$ (cf. [Ta]).

For $\sigma_v$ in the discrete series of $GL_n(F_v)$, $n=1,2$, the $L$-function $L(s,\sigma_v\times\Pi_v)$ is equal to
$$\prod_{k=1}^{\ell}L(s-s_k,\sigma_v\times\tau_k)L(s+s_k,\sigma_v\times\tau_k)\prod_{j=1}^{u}L(s,\sigma_v\times\tau_{\ell+j}).$$
Using the strict inequalities $0<s_k<1/2$ and the holomorphy of each $L(s,\sigma_v\times\tau_k)$ for $Re\,s>0$, it is easy to see that as a function of $q_v^{-s}$, $L(s,\sigma_v\times\Pi_v)^{-1}$ has the same zeros as $\gamma(s,\sigma_v\times\Pi_v,\psi_v)$ and therefore the equality
$$L(s,\sigma_v\times\pi_{1v}\times\pi_{2v})=L(s,\sigma_v\times\Pi_v)$$
follows from the equality of $\gamma$-factors, since $\pi_{iv}$ and $\sigma_v$ are tempered (cf. Section 7 of [Sh1]).

The temperedness of $\Pi_v$ follows easily from the above equality of $L$-factors by comparing poles of both sides. More precisely, let $\Pi_v$ be of the above form, and suppose $s_l>0$. Then take $\sigma_v=\tilde{\tau}_l$:
$$L(s,\tilde{\tau}_l\times\pi_{1v}\times\pi_{2v})=\prod_{k=1}^{\ell}L(s-s_k,\tilde{\tau}_l\times\tau_k)L(s+s_k,\tilde{\tau}_l\times\tau_k)\prod_{j=1}^{u}L(s,\tilde{\tau}_l\times\tau_{\ell+j}).$$
The left-hand side has no poles for $Re(s)>0$ ([Sh1, Proposition 7.2], see also [Ki-Sh, Proposition 3.2]); but the right-hand side has a pole at $Re(s)=s_l>0$. □

**Proposition 5.1.4.** *For $v\in T$,*
$$\gamma(s,\sigma_v\times\Pi_v,\psi_v)=\gamma(s,\sigma_v\times(\pi_{1v}\boxtimes\pi_{2v}),\psi_v)=\gamma(s,\sigma_v\times\pi_{1v}\times\pi_{2v},\psi_v),$$
*for any generic representation $\sigma_v$ of $GL_m(F_v)$, $m=1,2$.*

*Remark.* By the local converse theorem due to Chen [Ch] (cf. [Co-PS1]), the above equality implies that $\Pi_v\simeq\pi_{1v}\boxtimes\pi_{2v}$ for $v\in T$. However, we do not need the local converse theorem. The equivalence will be a consequence of Proposition 5.1.5.

*Proof.* We follow [Ra1, Proposition 4.3.1]. By the multiplicativity of $\gamma$-factors, we only need to show that
$$\gamma(s,\sigma_v\times\Pi_v,\psi_v)=\gamma(s,\sigma_v\times(\pi_{1v}\boxtimes\pi_{2v}),\psi_v),$$
for any supercuspidal representation $\sigma_v$ of $GL_m(F_v)$, $m=1,2$. We show this for the case $m=2$. Since we need a local-global argument, in order to avoid confusion, we use the following setup: Let $k$ be a non-archimedean local field of characteristic zero. Let $\eta_i$, $i=1,2,3$, be supercuspidal representations of $GL_2(k)$ with corresponding parametrization $\tau_i: W_k\longrightarrow GL_2(\mathbb{C})$. Since any representation $\tau$ of $W_k$ is of the form $\tau'\otimes|\ |^u$, where $\tau'$ is a representation of $Gal(\bar{k}/k)$, we can think of $\tau_i$ as a representation of $Gal(\bar{k}/k)$, i.e., $\tau_i: Gal(\bar{k}/k)\longrightarrow GL_2(\mathbb{C})$. Note that $\tau_i$ has a solvable image, i.e., a representation of icosahedral type does not occur over

a local field (see, for example, [G-L, p. 121]). As in [Ra1, Proposition 4.3.1], we can find a number field $F$ with $k = F_v$ and irreducible 2-dimensional representations $\sigma_i$ of $Gal(\bar{F}/F)$ with solvable image such that $\sigma_{iv} = \tau_i$. The global Langlands correspondence is available for those representations with solvable image [La3], [Tu], and hence we can find corresponding cuspidal representations $\pi_i$ of $GL_2(\mathbb{A}_F)$ such that $\pi_{iv} = \eta_i$. We compare the functional equations of $L(s, \pi_1 \times \pi_2 \times \pi_3)$ and $L(s, \sigma_1 \otimes \sigma_2 \otimes \sigma_3)$. Even though we do not know the holomorphy of $L(s, \sigma_1 \otimes \sigma_2 \otimes \sigma_3)$, the functional equation is known and it suffices for our purpose. Since $L(s, \pi_{1w} \times \pi_{2w} \times \pi_{3w}) = L(s, \sigma_{1w} \otimes \sigma_{2w} \otimes \sigma_{3w})$ for unramified places, we have an equality

$$\prod_{u \in S} \gamma(s, \pi_{1u} \times \pi_{2u} \times \pi_{3u}, \psi_u) = \prod_{u \in S} \gamma(s, \sigma_{1u} \otimes \sigma_{2u} \otimes \sigma_{3u}, \psi_u),$$

where $S$ is a finite set of finite places containing $T$ and the $\pi_{iw}$'s are unramified for $w \notin S$. Now we use the idea of using highly ramified characters (see, for example, [He3, Theorem 4.1]). Note that by Lemma 5.1.1 and Proposition 5.1.3, there exists a representation $\Pi_u$ such that

$$\gamma(s, \pi_{1u} \times \pi_{2u} \times \pi_{3u}, \psi_u) = \gamma(s, \Pi_u \times \pi_{3u}, \psi_u),$$

for each $u \in S$. Also we have

$$\gamma(s, \sigma_{1u} \otimes \sigma_{2u} \otimes \sigma_{3u}, \psi_u) = \gamma(s, (\pi_{1u} \boxtimes \pi_{2u}) \times \pi_{3u}, \psi_u).$$

Hence by [J-S2], for every highly ramified character $\chi$, $\gamma(s, \Pi_u \times (\pi_{3u} \otimes \chi), \psi_u)$ and $\gamma(s, (\pi_{1u} \boxtimes \pi_{2u}) \times (\pi_{3u} \otimes \chi), \psi_u)$ are independent of the $\pi_{iv}$'s. Namely, for every highly ramified character $\chi$,

$$\gamma(s, \pi_{1u} \times \pi_{2u} \times (\pi_{3u} \otimes \chi), \psi_u) = \gamma(s, \sigma_{1u} \otimes \sigma_{2u} \otimes (\sigma_{3u} \otimes \chi), \psi_u).$$

Now choose a grössencharacter $\mu$ which is highly ramified at all the ramified places except $v$, in which place it is trivial. Comparing the functional equations of $L(s, \pi_1 \times \pi_2 \times (\pi_3 \otimes \mu))$ and $L(s, \sigma_1 \otimes \sigma_2 \otimes (\sigma_3 \otimes \mu))$, we obtain

$$\gamma(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}, \psi_v) = \gamma(s, \sigma_{1v} \otimes \sigma_{2v} \otimes \sigma_{3v}, \psi_v).$$

□

The temperedness of $\Pi_v$ would follow also from Proposition 5.1.4, by noting that if $\phi_{iv} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$ is the parametrization of $\pi_{iv}$, $i = 1, 2$, then $\pi_{iv}$ is tempered if and only if the image $\phi_{iv}(W_{F_v})$ is bounded (see, for example, [Ku, Lemma 5.2.1]). In that case, it is obvious that $(\phi_{1v} \otimes \phi_{2v})(W_{F_v})$ is bounded. Hence $\pi_{1v} \boxtimes \pi_{2v}$ is tempered.

**Proposition 5.1.5.** *Let $\pi_1, \pi_2$ be two cuspidal representations of $GL_2(\mathbb{A})$. Then $\pi_1 \boxtimes \pi_2$ is an automorphic representation of $GL_4(\mathbb{A})$. It is of the form $\tau_1 \boxplus \cdots \boxplus \tau_k$, where the $\tau_i$'s are cuspidal representations of $GL_{n_i}(\mathbb{A})$.*

*Proof.* Pick two finite places $v_1, v_2$, where $\pi_{iv_1}, \pi_{iv_2}$, $i = 1, 2$, are unramified. Let $S_i = \{v_i\}$, $i = 1, 2$. We apply the converse theorem twice to $\pi_1 \boxtimes \pi_2 = \bigotimes_v \pi_{1v} \boxtimes \pi_{2v}$ with $S_1$ and $S_2$, and find two automorphic representations $\Pi_1, \Pi_2$ of $GL_4(\mathbb{A})$ such that $\Pi_{1v} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \neq v_1$, and $\Pi_{2v} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \neq v_2$. Hence $\Pi_{1v} \simeq \Pi_{2v}$ for all $v \neq v_1, v_2$. Note that $\Pi_1, \Pi_2$ are of the form $\tau_1 \boxplus \cdots \boxplus \tau_k$, where the $\tau_i$'s are (unitary) cuspidal representations of $GL$ by Proposition 5.1.2. By the strong multiplicity one theorem [J-S3], $\Pi_1 \simeq \Pi_2$, in particular, $\Pi_{1v_i} \simeq \Pi_{2v_i} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for all $v$. □

**Corollary 5.1.6.** *Let $\pi_{1v}, \pi_{2v}$ be supercuspidal representations of $GL_2(F_v)$. Let $\sigma_v$ be a supercuspidal representation of $GL_n(F_v)$. Then*

$$\gamma(s, \sigma_v \times \pi_{1v} \times \pi_{2v}, \psi_v) = \gamma(s, \sigma_v \times (\pi_{1v} \boxtimes \pi_{2v}), \psi_v).$$

*Proof.* Consider the $D_{n+1} - 2$ case in [Sh1]. Then we obtain the triple $L$-function $L(s, \sigma_v \times \pi_{1v} \times \pi_{2v})$. Let $\sigma, \pi_1, \pi_2$ be cuspidal representations of $GL_n(\mathbb{A}_F), GL_2(\mathbb{A}_F)$, $GL_2(\mathbb{A}_F)$, resp., whose local components at $v$ are $\sigma_v, \pi_{1v}, \pi_{2v}$ and unramified for all other finite places. Let $\Pi = \pi_1 \boxtimes \pi_2$. Consider two $L$-functions $L(s, \sigma \times \pi_1 \times \pi_2)$ and $L(s, \sigma \times \Pi)$. Comparing the functional equations as in Proposition 5.1.3, we have the equality

$$\gamma(s, \sigma_v \times \pi_{1v} \times \pi_{2v}, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v).$$

□

Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ with central character $\omega_\pi$. By the local Langlands correspondence, $Sym^2(\pi_v)$ is well defined for all $v$. Let $Sym^2(\pi) = \bigotimes_v Sym^2(\pi_v)$. Gelbart and Jacquet [Ge-J] proved that $Sym^2(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$. Here we can prove it as a corollary to Proposition 5.1.5.

**Corollary 5.1.7** ([Ge-J])**.** *$\pi \boxtimes \pi = Sym^2(\pi) \boxplus \omega_\pi$. Hence $Sym^2(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$. It is cuspidal if and only if it is not monomial.*

*Proof.* By Proposition 5.1.5, $\pi \boxtimes \pi$ is an automorphic representation of $GL_4(\mathbb{A})$. But $L(s, \pi \times (\pi \otimes \omega_\pi^{-1}))$ has a pole at $s = 1$. Hence $\pi \boxtimes \pi = \sigma \boxplus \omega_\pi$ for some automorphic representation $\sigma$ of $GL_3(\mathbb{A})$. It is easy to see that $\sigma_v \simeq Sym^2(\pi_v)$ for all $v$. Hence our result follows.

In order to prove the second assertion, we use the identity

$$L(s, \pi \times (\pi \otimes \chi)) = L(s, Sym^2(\pi) \otimes \chi) L(s, \omega_\pi \chi),$$

where $\chi$ is a grössencharacter. Note that $L(s, \pi \times (\pi \otimes \chi))$ has a pole at $s = 1$ if and only if $\pi \otimes \chi \simeq \tilde{\pi}$, namely, $\pi \otimes (\omega_\pi \chi) \simeq \pi$. Hence $L(s, Sym^2(\pi) \otimes \chi)$ has a pole at $s = 1$ if and only if $\pi \otimes (\omega_\pi \chi) \simeq \pi$ and $\omega_\pi \chi \neq 1$, namely, $\pi$ is monomial. □

**5.2 Local lifts from $GL_4$ to $GL_6$.** Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. In this section, we construct a local exterior square lift $\Pi_v$ for each $\pi_v$ in the sense of Definition 2.2, i.e.,

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4) = L(s, \sigma_v \times \Pi_v),$$

for all generic irreducible representations $\sigma_v$ of $GL_m(F_v)$, $1 \leq m \leq 4$.

First we show, by extending Proposition 4.2, that if $\pi_v$ is not supercuspidal, then $\wedge^2 \pi_v$ is the local exterior square lift of $\pi_v$ in the above sense. Namely,

**Lemma 5.2.1.** *Suppose $\pi_v$ is not supercuspidal. Then*

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times (\wedge^2 \pi_v), \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4) = L(s, \sigma_v \times (\wedge^2 \pi_v)),$$

*for all generic irreducible representations $\sigma_v$ of $GL_m(F_v)$, $1 \leq m \leq 4$.*

*Proof.* By assumption, in the multiplicativity of $\gamma$- and $L$-factors (cf. Propositions 2.4 and 2.5), all rank-one $\gamma$- and $L$-factors are either for $GL_k \times GL_l$, or for $D_n - 2$, $n = 4, 5, 6$. For $GL_k \times GL_l$, Shahidi (Proposition 2.3) showed that his $\gamma$-factors are Artin factors and the $D_n - 2$ case follows from Corollary 5.1.6. Our result follows from Proposition 2.6. □

As an example of a local lift, we show

**Lemma 5.2.2.** *Suppose $\pi_v$ is a discrete series, given as the unique subrepresentation of $Ind\,|det|^{\frac{1}{2}}\rho \otimes |det|^{-\frac{1}{2}}\rho$, where $\rho$ is a supercuspidal representation of $GL_2(F_v)$. Then the lift $\wedge^2\pi_v$ is given by*

$$\wedge^2\pi_v = Sym^2\rho \boxplus \sigma,$$

*where $Sym^2\rho$ is the symmetric square lift of $\rho$ and $\sigma$ is a discrete series of $GL_3(F_v)$, given as the unique subrepresentation of $Ind\,|\ |\omega_\rho \otimes \omega_\rho \otimes |\ |^{-1}\omega_\rho$.*

*Proof.* Note the identity $\wedge^2(\tau_1 \boxplus \tau_2) = (\tau_1 \boxtimes \tau_2) \boxplus \omega_{\tau_1} \boxplus \omega_{\tau_2}$ for irreducible representations $\tau_1, \tau_2$ of $GL_2(F_v)$. Hence $\wedge^2\pi_v$ is a subrepresentation of $(\rho \boxtimes \rho) \boxplus |\ |\omega_\rho \boxplus |\ |^{-1}\omega_\rho$. Note that $\rho \boxtimes \rho = Sym^2\rho \boxplus \omega_\rho$. By [Sh5, Proposition 8.1],

$$L(s, \pi_v, \wedge^2) = L(s, \rho, Sym^2)L(s+1, \omega_\rho),$$

and note that $L(s, \sigma) = L(s+1, \omega_\rho)$. Hence our result follows. □

We now show that a supercuspidal representation of $GL_4(F_v)$ has a local exterior square lift to $GL_6(F_v)$. Let $\pi_v$ be a supercuspidal representation of $GL_4(F_v)$. Let $\pi = \bigotimes_w \pi_w$ be a cuspidal automorphic representation of $GL_4(\mathbb{A})$ such that $\pi_w$ is unramified for all $w < \infty$ and $w \neq v$ (Proposition 5.1 of [Sh1]).

By Theorem 4.2.3, there exists a weak exterior square lift $\Pi$ of $\pi$ such that $\Pi_w \simeq \wedge^2\pi_w$ for $w \notin S'$, where $S'$ is a finite set of finite places, containing $v$. We remark that $\Pi_v$ is irreducible, unitary, and generic.

**Proposition 5.2.3.** *$\Pi_v$ is a local exterior square lift of $\pi_v$. Moreover, $\Pi_v$ is tempered.*

*Proof.* By the multiplicativity of $\gamma$- and $L$-factors (cf. Propositions 2.4 and 2.5), it is enough to show the identities in Definition 2.2 for discrete series $\sigma_v$. Then we can find a cuspidal representation $\sigma$ whose local component at $v$ is $\sigma_v$ [Ro].

Consider the two $L$-functions $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4)$ and $L(s, \sigma \times \Pi)$. Both have the functional equations:

$$L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4) = \epsilon(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4)L(1-s, \tilde{\sigma} \otimes \tilde{\pi}, \rho_m \otimes \wedge^2\rho_4),$$
$$L(s, \sigma \times \Pi) = \epsilon(s, \sigma \times \Pi)L(1-s, \tilde{\sigma} \times \tilde{\Pi}).$$

Since

$$L(s, \sigma_w \otimes \pi_w, \rho_m \otimes \wedge^2\rho_4) = L(s, \sigma_w \times \Pi_w),$$
$$\epsilon(s, \sigma_w \otimes \pi_w, \rho_m \otimes \wedge^2\rho_4, \psi_w) = \epsilon(s, \sigma_w \times \Pi_w, \psi_w),$$

for all $w \notin S'$, we have

$$\prod_{w \in S'} \gamma(s, \sigma_w \otimes \pi_w, \rho_m \otimes \wedge^2\rho_4, \psi_w) = \prod_{w \in S'} \gamma(s, \sigma_w \times \Pi_w, \psi_w).$$

If $w \in S'$, $w \neq v$, $\pi_w$ is unramified. Hence $\gamma(s, \sigma_w \otimes \pi_w, \rho_m \otimes \wedge^2 \rho_4, \psi_w)$ is a product of $\gamma$-factors for $GL_k \times GL_l$. Hence by the stability of $\gamma$-factors [J-S2], for every highly ramified character $\chi$,

$$\gamma(s, (\sigma_w \otimes \chi) \otimes \pi_w, \rho_m \otimes \wedge^2 \rho_4, \psi_w) = \gamma(s, (\sigma_w \otimes \chi) \times \Pi_w, \psi_w).$$

Hence by using a grössencharacter which is highly ramified at all the places in $S'$ except $v$, in which place it is trivial, we obtain (see the proof of Proposition 5.1.4)

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v).$$

We proceed exactly in the same way as in the proof of Proposition 5.1.3 to show that the equality of $\gamma$-factors implies the equality of $L$-factors.

The temperedness of $\Pi_v$ follows from the equality of $L$-factors as in Proposition 5.1.3, by noting that the holomorphy of $L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4)$ for $Re(s) > 0$ when $\sigma_v$ is tempered is proved in [As]. $\square$

Proposition 5.2.3 does not imply that $\Pi_v \simeq \wedge^2 \pi_v$. Let $T$ be the set of places where $v|2,3$ and $\pi_v$ is a supercuspidal representation. Then we can prove

**Proposition 5.2.4.** *If $v \notin T$,*

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$

*for any supercuspidal representation $\sigma_v$ of $GL_m(F_v)$, $m = 1, 2, 3, 4$.*

*Remark.* By the local converse theorem due to Chen [Ch], the above equality implies that $\Pi_v \simeq \wedge^2 \pi_v$ for $v \notin T$. However we do not need it. The equivalence will be a consequence of Theorem 5.3.1.

*Proof.* Suppose $v \nmid 2, 3$ and $\pi_v$ is a supercuspidal representation. Since we need the local-global argument, in order to avoid confusion, we use the following setup: Let $k$ be a non-archimedean local field of characteristic zero, and let $\eta_1, \eta_2$ be supercuspidal representations of $GL_m(k), m = 1, 2, 3, 4$, $GL_4(k)$, resp. Then since $v \nmid 2, 3$, $\eta_1, \eta_2$ are induced, i.e., $\eta_1$ corresponds to $\tau_1 = Ind(W_k, W_{K_1}, \mu_1)$, where $K_1/k$ is an extension of degree $m$ (not necessarily Galois) and $\mu_1$ is a character of $K_1^\times$, and $\eta_2$ corresponds to $\tau_2 = Ind(W_k, W_{K_2}, \mu_2)$, where $K_2/k$ is an extension of degree 4 (not necessarily Galois) and $\mu_2$ is a character of $K_2^\times$. Then we need to prove

$$\gamma(s, \eta_1 \otimes \eta_2, \rho_m \otimes \wedge^2 \rho_4, \psi) = \gamma(s, \tau_1 \otimes \tau_2, \rho_m \otimes \wedge^2 \rho_4, \psi).$$

By [H, Section 4] (see [He2, p. 449]), we can find extensions of number fields $E_1/F$ and $E_2/F$, and grössencharacters $\chi_i$ of $E_i$, $i = 1, 2$, such that

(1) $F_v = k$, $E_{1w} = K_1$, $E_{2w} = K_2$, $w|v$, and $\chi_{1w} = \mu_1, \chi_{2w} = \mu_2$, and
(2) there exist cuspidal representations $\pi_1, \pi_2$ of $GL_m(\mathbb{A}_F), GL_4(\mathbb{A}_F)$, corresponding to $\tau_1, \tau_2$, resp., with $\pi_{1v} = \eta_1, \pi_{2v} = \eta_2$.

Now we proceed exactly as in the proof of Proposition 5.1.4: By comparing the functional equations of $L(s, \pi_1 \otimes \pi_2, \rho_m \otimes \wedge^2 \rho_4)$ and $L(s, \sigma_1 \otimes \sigma_2, \rho_m \otimes \wedge^2 \rho_4)$, and using a grössencharacter which is highly ramified at all the ramified places except $v$, in which place it is trivial, we obtain

$$\gamma(s, \pi_{1v} \otimes \pi_{2v}, \rho_m \otimes \wedge^2 \rho_4 \psi_v) = \gamma(s, \sigma_{1v} \otimes \sigma_{2v}, \rho_m \otimes \wedge^2 \rho_4, \psi_v).$$

$\square$

By arguing as before (right after Proposition 5.1.4), Proposition 5.2.4 also implies that if $\pi_v$ is tempered, then so is $\Pi_v$ for $v \notin T$.

*Remark 5.1.* If $v|3$, any supercuspidal representation of $GL_4(F_v)$ is induced. However, we need to twist by supercuspidal representations of $GL_3(F_v)$. There are supercuspidal representations of $GL_3(F_v)$ which are not induced if $v|3$. Let $k$ be a non-archimedean local field with residual characteristic 3. Let $\eta$ be a supercuspidal representation of $GL_3(k)$ with a parametrization $\tau : Gal(\bar{k}/k) \longrightarrow GL_3(\mathbb{C})$. Then surely we can find a number field $F$ with $F_v = k$ and a global irreducible representation $\sigma : Gal(\bar{F}/F) \longrightarrow GL_3(\mathbb{C})$ such that $\sigma_v = \tau$. If we can find a cuspidal representation $\pi = \bigotimes_v \pi_v$ of $GL_3(\mathbb{A}_F)$ which corresponds to $\sigma$ such that $\pi_v = \eta$, then our proof above goes through.

**5.3 Strong exterior square lift from $GL_4$ to $GL_6$.** Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. Let $T$ be the set of places where $v|2,3$ and $\pi_v$ is a supercuspidal representation. Then, in Section 5.2, we constructed a local lift $\Pi_v$ for each $\pi_v$ such that $\Pi_v \simeq \wedge^2\pi_v$ for $v \notin T$ if we apply the local converse theorem (as remarked before, we do not need the local converse theorem). Let $\Pi' = \bigotimes_v \Pi'_v$, where $\Pi'_v = \Pi_v$ if $v \in T$ and $\Pi'_v = \wedge^2\pi_v$ if $v \notin T$. It is an irreducible admissible representation of $GL_6(\mathbb{A})$. We prove

**Theorem 5.3.1.** $\Pi'$ *is an automorphic representation of* $GL_6(\mathbb{A})$*, i.e.,* $\Pi'$ *is the strong exterior square lift of* $\pi$*. It is of the form* $\Pi' = \sigma_1 \boxplus \cdots \boxplus \sigma_k$ *in the notation of* [J-S3]*, where the* $\sigma_i$*'s are (unitary) cuspidal representations of* $GL_{n_i}(\mathbb{A})$*.*

*Proof.* Pick two finite places $v_1, v_2$, where $\pi_{v_1}, \pi_{v_2}$ are unramified. Let $S_i = \{v_i\}$, $i = 1, 2$. We apply the converse theorem (Theorem 2.1) to $\Pi' = \bigotimes_v \Pi'_v$ with $S_1$ and $S_2$ and find two automorphic representations $\Pi_1, \Pi_2$ of $GL_6(\mathbb{A})$ such that $\Pi_{1v} \simeq \Pi'_v$ for $v \neq v_1$, and $\Pi_{2v} \simeq \Pi'_v$ for $v \neq v_2$. Hence $\Pi_{1v} \simeq \Pi_{2v}$ for all $v \neq v_1, v_2$. By Theorem 4.2.3, $\Pi_1$ and $\Pi_2$ are of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_k$, where the $\sigma_i$'s are (unitary) cuspidal representations of $GL$. By the strong multiplicity one theorem, $\Pi_1 \simeq \Pi_2$, in particular, $\Pi_{1v_i} \simeq \Pi_{2v_i} \simeq \Pi'_{v_i}$ for $i = 1, 2$. □

## 6. Some applications

**Proposition 6.1.** *Let* $\sigma$ *be a cuspidal representation of* $GL_m(\mathbb{A})$*, and let* $\pi$ *be a cuspidal representation of* $GL_4(\mathbb{A})$*. Then* $L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4)$ *is holomorphic except possibly at* $s = 0, 1$*. If* $m > 6$*, it is entire. In particular, the exterior square* $L$*-function* $L(s, \pi, \wedge^2)$ *is holomorphic except possibly at* $s = 0, 1$*.*

*Proof.* Let $\Pi$ be the strong exterior square lift of $\pi$ in Theorem 5.3.1. It is given by $\tau_1 \boxplus \cdots \boxplus \tau_k$, where the $\tau_i$'s are cuspidal representations of $GL_{n_i}(\mathbb{A})$. Then

$$L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4) = L(s, \sigma \times \Pi) = \prod_{i=1}^{k} L(s, \sigma \times \tau_i).$$

Our result follows easily from the well-known property of the Rankin-Selberg $L$-functions of $GL_a \times GL_b$. □

**Proposition 6.2.** *Let* $\pi$ *be a cuspidal representation of* $GL_4(\mathbb{A})$*. Then the exterior square* $L$*-function* $L(s, \pi, \wedge^2)$ *and the symmetric square* $L$*-function* $L(s, \pi, Sym^2)$ *are both absolutely convergent for* $Re(s) > 1$*.*

*Proof.* Let $\Pi$ be the strong exterior square lift of $\pi$ as in Theorem 5.3.1. It is given by $\tau_1 \boxplus \cdots \boxplus \tau_k$, where the $\tau_i$'s are cuspidal representations of $GL_{n_i}$. Then

$$L(s, \pi, \wedge^2) = \prod_{i=1}^{k} L(s, \tau_i).$$

Our result follows easily from the well-known property of $L$-functions of $GL_n$. The result on the symmetric square $L$-functions follows immediately from the following identity and the absolute convergence of $L(s, \pi \times \pi)$ for $Re(s) > 1$:

$$L(s, \pi \times \pi) = L(s, \pi, \wedge^2) L(s, \pi, Sym^2).$$

$\square$

**Proposition 6.3.** *Let $\pi$ be a cuspidal representation of $GL_4(\mathbb{A})$. Then $\pi$ satisfies the weak Ramanujan property.*

*Proof.* Recall from the paragraph after Lemma 4.1.3 that the trace of a non-tempered unramified component has one of the following three forms (here the $u_i$'s are complex numbers with absolute value one):

$S_1$; $a_v = u_1 q^a + u_2 q^a + u_1 q^{-a} + u_2 q^{-a}$, where $0 < a < \frac{1}{2}$;
$S_2$; $a_v = u_1 q^a + u_2 + u_3 + u_1 q^{-a}$, where $0 < a < \frac{1}{2}$;
$S_3$; $a_v = u_1 q^{a_1} + u_2 q^{a_2} + u_1 q^{-a_1} + u_2 q^{-a_2}$, where $0 < a_2 < a_1 < \frac{1}{2}$.

Fix $\epsilon > 0$. Then inside $S_2$, the set of places where $|a_v| > q^\epsilon$ has density zero. It means the set of places where $a > \epsilon$ has density zero.

Suppose $S_1$ has a subset $S'$ of positive density where $q^a > q^\epsilon$ for $v \in S'$. Then consider the lift $\wedge^2 \pi$. For $v \in S_1$, the trace of $\wedge^2 \pi_v$ has the form

$$b_v = u_1 u_2 q^{2a} + u_1^2 + u_1 u_2 + u_2^2 + u_1 u_2 q^{-2a}.$$

Then $|b_v| > q^\epsilon$ for $v \in S'$. This is a contradiction to [Ra2, Lemma 3.1].

Suppose $S_3$ has a subset $S'$ of positive density where $q^{a_1} > q^\epsilon$ for $v \in S'$. Then consider the lift $\wedge^2 \pi$. For $v \in S_3$, the trace of $\wedge^2 \pi_v$ has the form

$$b_v = u_1 u_2 q^{a_1 + a_2} + u_1 u_2 q^{a_1 - a_2} + u_1^2 + u_2^2 + u_1 u_2 q^{-a_1 + a_2} + u_1 u_2 q^{-a_1 - a_2}.$$

Then $|b_v| > q^\epsilon$ for $v \in S'$. This again contradicts Lemma 3.1 of [Ra2]. $\square$

## 7. Symmetric fourth lift of $GL_2$

Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the $m$th symmetric power representation of $GL_2(\mathbb{C})$ on the space of symmetric tensors of rank $m$. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ with central character $\omega_\pi$. By the local Langlands correspondence [H-T], [He2], [La4], $Sym^m(\pi_v)$ is a well-defined representation of $GL_{m+1}(F_v)$ for all $v$. Let $\rho_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$ be the parametrization of $\pi_v$. Then we have a map $Sym^m(\rho_v) : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$. Then $Sym^m(\pi_v)$ is the representation of $GL_{m+1}(F_v)$, corresponding to $Sym^m(\rho_v)$. Let $Sym^m(\pi) = \bigotimes_v Sym^m(\pi_v)$. It is an irreducible admissible representation of $GL_{m+1}(\mathbb{A})$. Langlands' functoriality predicts that $Sym^m(\pi) = \bigotimes_v Sym^m(\pi_v)$ is an automorphic representation of $GL_{m+1}(\mathbb{A})$. It is convenient to introduce $A^m(\pi) = Sym^m(\pi) \otimes \omega_\pi^{-1}$ (Shahidi called it $Ad^m(\pi)$). If $m = 2$, $A^2(\pi) = Ad(\pi)$ and it is the well-known Gelbart-Jacquet lift. If $m = 3$, recall

**Theorem 7.1** ([Ki-Sh2]). *Let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A})$. Then the symmetric cube $Sym^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$. It is cuspidal unless either $\pi$ is a monomial representation or $Ad(\pi) \simeq Ad(\pi) \otimes \eta$, for a non-trivial grössencharacter $\eta$.*

We are concerned with $m = 4$. We prove that $A^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$, using the exterior square lift from $GL_4$ to $GL_6$. More precisely, we show that $\wedge^2(A^3(\pi)) = A^4(\pi) \boxplus \omega_\pi$.

Let $\pi_v$ be an unramified component, and let the Hecke conjugacy class of $\pi_v$ be given by $diag(\alpha_v, \beta_v)$. Then by direct calculation, we see that the Hecke conjugacy class of $\wedge^2(A^3(\pi_v))$ is given by

$$diag(\alpha_v^3\beta_v^{-1}, \alpha_v^2, \alpha_v\beta_v, \beta_v^2, \alpha_v^{-1}\beta_v^3, \alpha_v\beta_v).$$

Note that $\omega_{\pi_v} = \alpha_v\beta_v$ and the Hecke conjugacy class of $A^4(\pi_v)$ is given by

$$diag(\alpha_v^3\beta_v^{-1}, \alpha_v^2, \alpha_v\beta_v, \beta_v^2, \alpha_v^{-1}\beta_v^3).$$

We divide into three cases.

**7.1 $\pi$ is a monomial cuspidal representation.** In this case, $\pi \otimes \eta \simeq \pi$ for a non-trivial grössencharacter $\eta$. Then $\eta^2 = 1$ and $\eta$ determines a quadratic extension $E/F$. According to [L-La], there is a grössencharacter $\chi$ of $E$ such that $\pi = \pi(\chi)$, where $\pi(\chi)$ is the automorphic representation whose local factor at $v$ is the one attached to the representation of the local Weil group induced from $\chi_v$. Let $\chi'$ be the conjugate of $\chi$ by the action of the non-trivial element of the Galois group. The Gelbart-Jacquet lift (adjoint square) of $\pi$ is given by

$$Ad(\pi) = \pi(\chi{\chi'}^{-1}) \boxplus \eta.$$

Case 1. $\chi{\chi'}^{-1}$ factors through the norm, i.e., $\chi{\chi'}^{-1} = \mu \circ N_{E/F}$ for a grössencharacter $\mu$ of $F$. Then $\pi(\chi{\chi'}^{-1})$ is not cuspidal. In fact, $\pi(\chi{\chi'}^{-1}) = \mu \boxplus \mu\eta$. In this case, $A^3(\pi) = (\mu \otimes \pi) \boxplus (\mu \otimes \pi)$ and

$$A^4(\pi) = (\pi \boxtimes \pi) \boxplus \omega_\pi = \omega_\pi \boxplus \omega_\pi \boxplus \mu\omega_\pi \boxplus \eta\omega_\pi \boxplus \mu\eta\omega_\pi.$$

We used the fact that $\eta, \mu$ are quadratic grössencharacters.

Case 2. $\chi{\chi'}^{-1}$ does not factor through the norm. In this case, $\pi(\chi{\chi'}^{-1})$ is a cuspidal representation. Then $A^3(\pi) = \pi(\chi^2{\chi'}^{-1}) \boxplus \pi$ (note here that $\chi^2{\chi'}^{-1}$ can factor through the norm, and in that case $\pi(\chi^2{\chi'}^{-1})$ is not cuspidal) and

$$A^4(\pi) = (\pi(\chi^2{\chi'}^{-1}) \boxtimes \pi) \boxplus \omega_\pi = \pi(\chi^3{\chi'}^{-1}) \boxplus \pi(\chi^2) \boxplus \omega_\pi.$$

**7.2 $A^3(\pi)$ is not cuspidal.** This is the case when there exists a non-trivial grössencharacter $\eta$ such that $Ad(\pi) \simeq Ad(\pi) \otimes \eta$. Note that $\eta^3 = 1$. Then by [Ki-Sh2], $A^3(\pi) = (\pi \otimes \eta) \boxplus (\pi \otimes \eta^2)$. Hence

$$\wedge^2(A^3\pi) = Sym^2(\pi) \boxplus \omega_\pi \boxplus \omega_\pi\eta \boxplus \omega_\pi\eta^2.$$

So

$$A^4(\pi) = Sym^2(\pi) \boxplus \omega_\pi\eta \boxplus \omega_\pi\eta^2.$$

**7.3** $A^3(\pi)$ **is cuspidal.** This is the case when $\pi$ is not monomial and $Ad(\pi) \not\simeq Ad(\pi) \otimes \eta$ for any non-trivial grössencharacter $\eta$.

Consider $\tau = A^3(\pi)$ and its strong exterior square lift $\Pi(\tau)$ as in Theorem 5.3.1. It is an automorphic representation of $GL_6(\mathbb{A})$, unitarily induced from cuspidal representations of $GL_{n_i}(\mathbb{A})$, and $\Pi(\tau)_v \simeq \wedge^2\tau_v$ unless $v|2,3$ and $\tau_v$ is a supercuspidal representation.

**Theorem 7.3.1.** *Let $\chi$ be a grössencharacter. Let $S$ be a finite set of places, including all archimedean places such that $\pi_v, \chi_v$ are all unramified for $v \notin S$. Then $L_S(s, \chi \otimes \Pi(\tau)) = L_S(s, \tau, \wedge^2 \otimes \chi)$ has a pole at $s = 1$ if and only if $\chi = \omega_\pi^{-1}$.*

*Proof.* Consider the equality

$$\begin{aligned} &L_S(s, Ad(\pi) \times (Ad(\pi) \otimes (\omega_\pi\chi))) \\ &\quad = L_S(s, \omega_\pi\chi) L_S(s, Ad(\pi) \otimes (\omega_\pi\chi)) L_S(s, \pi, Sym^4 \otimes (\omega_\pi^{-1}\chi)) \\ &\quad = L_S(s, Ad(\pi) \otimes (\omega_\pi\chi)) L_S(s, \tau, \wedge^2 \otimes \chi). \end{aligned}$$

Note that $L_S(s, Ad(\pi) \otimes (\omega_\pi\chi))$ has no zero and no pole at $s = 1$. Therefore $L_S(s, Ad(\pi) \times (Ad(\pi) \otimes (\omega_\pi\chi)))$ has a pole at $s = 1$ if and only if $L_S(s, \tau, \wedge^2 \otimes \chi)$ has a pole at $s = 1$.

Since $Ad(\pi) \not\simeq Ad(\pi) \otimes \eta$ for any non-trivial grössencharacter $\eta$, it follows that $L_S(s, Ad(\pi) \times (Ad(\pi) \otimes (\omega_\pi\chi)))$ has a pole at $s = 1$ if and only if $\omega_\pi\chi = 1$. □

Hence we have $\Pi(\tau) = \Pi \boxplus \omega_\pi$, where $\Pi$ is an automorphic representation of $GL_5(\mathbb{A})$. We have $\Pi(\tau)_v \simeq \wedge^2(A^3(\pi_v)) = A^4(\pi_v) \boxplus \omega_{\pi_v}$ for $v \notin T$, where $T$ is the set of places such that $v|2,3$ and $A^3(\pi_v)$ is a supercuspidal representation. Hence $\Pi_v \simeq A^4(\pi_v)$ for $v \notin T$.

**Theorem 7.3.2.** *For all $v$, $\Pi_v \simeq A^4(\pi_v)$. Hence $A^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$. It is either cuspidal or unitarily induced from cuspidal representations of $GL_2(\mathbb{A})$ and $GL_3(\mathbb{A})$.*

*Proof.* If $v \nmid 2$, it is well known (see, for example, [G-L]) that any supercuspidal representation $\pi_v$ of $GL_2(F_v)$ is monomial, i.e., it corresponds to $Ind(W_{F_v}, W_K, \mu)$, where $K/F_v$ is quadratic and $\mu$ is a character of $K^\times$. Hence $Ad(\pi_v)$ is not supercuspidal. Therefore, if $A^3(\pi_v)$ is supercuspidal, then $v|2$ and $\pi_v$ is an extraordinary supercuspidal representation.

By the local converse theorem due to Chen [Ch] (cf. [Co-PS1]), we need to show that, for every supercuspidal representation $\sigma_v$ of $GL_m(F_v)$, $m = 1, 2, 3$,

$$\gamma(s, \sigma_v \times \Pi_v, \psi_v) = \gamma(s, \sigma_v \times A^4(\pi_v), \psi_v).$$

We follow the proof of Proposition 5.1.4. As before, we use the following setup: Let $k$ be a non-archimedean local field of characteristic zero. Let $\eta_1, \eta_2$ be supercuspidal representations of $GL_m(k), GL_2(k)$ with corresponding parametrizations $\tau_1 : W_k \longrightarrow GL_m(\mathbb{C}), \tau_2 : W_k \longrightarrow GL_2(\mathbb{C})$, resp. We can think of $\tau_i$ as a representation of $Gal(\bar{k}/k)$. Since $\wedge^2(A^3(\tau_2)) \simeq A^4(\tau_2) \oplus det(\tau_2)$ and $det(\tau_2)$ corresponds to $\omega_{\eta_2}$, we need to show that

$$\gamma(s, \eta_1 \otimes A^3(\eta_2), \rho_m \otimes \wedge^2\rho_4, \psi) = \gamma(s, \tau_1 \otimes A^3(\tau_2), \rho_m \otimes \wedge^2\rho_4, \psi),$$

for $m = 1, 2, 3$. Since $m = 1$ is easy, we deal with $m = 2, 3$. First, $m = 2$. By appealing to [P-Ra, Lemma 3, Section 4], we can find a number field $F$ with $k = F_v$ and irreducible 2-dimensional representations $\sigma_i$ of $Gal(\bar{F}/F)$ with solvable image such that $\sigma_{iv} = \tau_i$ and $\sigma_{iu}$ are unramified for $u|2, u \neq v$. The global Langlands correspondence is available for representations with solvable image [La3], [Tu], and hence we can find corresponding cuspidal representations $\pi_i$ of $GL_2(\mathbb{A}_F)$ such that $\pi_{iv} = \eta_i$. We compare the functional equations for $L(s, \pi_1 \otimes A^3(\pi_2), \rho_2 \otimes \wedge^2\rho_4)$ and $L(s, \sigma_1 \otimes A^3(\sigma_2), \rho_2 \otimes \wedge^2\rho_4)$. Even though we do not know the holomorphy of $L(s, \sigma_1 \otimes A^3(\sigma_2), \rho_2 \otimes \wedge^2\rho_4)$, the functional equation is known and it suffices for our purpose. Since $L(s, \pi_{1w} \otimes A^3(\pi_{2w}), \rho_2 \otimes \wedge^2\rho_4) = L(s, \sigma_{1w} \otimes A^3(\sigma_{2w}), \rho_2 \otimes \wedge^2\rho_4)$ for unramified places, we have an equality

$$\prod_{u \in S} \gamma(s, \pi_{1u} \otimes A^3(\pi_{2u}), \rho_2 \otimes \wedge^2\rho_4, \psi_u) = \prod_{u \in S} \gamma(s, \sigma_{1u} \otimes A^3(\sigma_{2u}), \rho_2 \otimes \wedge^2\rho_4, \psi_u).$$

Note that the $\pi_{iu}$'s are unramified if $u|2, u \neq v$. Also if $u \in S, u \nmid 2$, then $A^3(\pi_{2u})$ is not supercuspidal. Therefore, if $u \in S, u \neq v$, then $A^3(\pi_{2u})$ is not supercuspidal. Hence by Lemma 5.2.1,

$$\gamma(s, \pi_{1u} \otimes A^3(\pi_{2u}), \rho_2 \otimes \wedge^2\rho_4, \psi_u) = \gamma(s, \sigma_{1u} \otimes A^3(\sigma_{2u}), \rho_2 \otimes \wedge^2\rho_4, \psi_u),$$

for each $u \in S, u \neq v$. Therefore,

$$\gamma(s, \pi_{1v} \otimes A^3(\pi_{2v}), \rho_2 \otimes \wedge^2\rho_4, \psi_v) = \gamma(s, \sigma_{1v} \otimes A^3(\sigma_{2v}), \rho_2 \otimes \wedge^2\rho_4, \psi_v).$$

Second, $m = 3$. Since $v|2$, $\eta_1$ is induced from a character, i.e., it corresponds to $\tau_1 = Ind(W_k, W_K, \mu)$, where $K/k$ is a cubic extension (not necessarily Galois extension) and $\mu$ is a character of $K^\times$. We choose a cubic extension of number fields $E/F$ such that $F_v = k, E_w = K, w|v$ and choose a grössencharacter $\chi$ of $E$ such that $\chi_w = \mu$. Let $\pi_1$ be a cuspidal automorphic representation of $GL_3(\mathbb{A}_F)$ corresponding to $\sigma_1 = Ind(W_F, W_E, \chi)$ (see [J-PS-S2] for the existence). Then in the same way as above, we have

$$\gamma(s, \pi_{1v} \otimes A^3(\pi_{2v}), \rho_2 \otimes \wedge^2\rho_4, \psi_v) = \gamma(s, \sigma_{1v} \otimes A^3(\sigma_{2v}), \rho_2 \otimes \wedge^2\rho_4, \psi_v).$$

Hence $\Pi \simeq A^4(\pi)$ and $A^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A}_F)$. By Theorem 7.3.1, it is either cuspidal, or unitarily induced from cuspidal representations of $GL_2(\mathbb{A})$ and $GL_3(\mathbb{A})$. $\square$

*Remark 7.1.* Suppose $\tau = A^3(\pi)$ is cuspidal. Then by Theorem 7.3.2, $A^4(\pi)$ is not cuspidal if and only if $L(s, \sigma \otimes \tau, \rho_2 \otimes \wedge^2\rho_4)$ has a pole at $s = 1$ for a cuspidal representation $\sigma$ of $GL_2(\mathbb{A})$. In a forthcoming paper [Ki-Sh3], we show that this happens if and only if there exists a non-trivial quadratic character $\eta$ such that $\tau \simeq \tau \otimes \eta$, or equivalently, there exists a non-trivial grössencharacter $\chi$ of $E$ such that $(Ad(\pi))_E \simeq (Ad(\pi))_E \otimes \chi$, where $E/F$ is the quadratic extension, determined by $\eta$. In this case, $A^4(\pi) = \sigma_1 \boxplus \sigma_2$, where $\sigma_1 = \pi(\chi^{-1}) \otimes \omega_\pi$ and $\sigma_2 = Ad(\pi) \otimes (\omega_\pi \eta)$.

**Corollary 7.3.3.** *Let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A})$, and let $\pi_v$ be a spherical local component (finite or infinite) given by $\pi_v = Ind(|\ |_v^{s_{1v}} \otimes |\ |_v^{s_{2v}})$. Then*

$$|Re(s_{iv})| \leq \frac{3}{26}.$$

*If $F = \mathbb{Q}$, $v = \infty$, this signifies*

$$\lambda_1 = \frac{1}{4}(1 - s^2) \geq \frac{40}{169} \approx 0.237,$$

*where $s = 2Re(s_{1v}) = -2Re(s_{2v})$ and $\lambda_1$ is the first positive eigenvalue of the Laplace operator on the corresponding hyperbolic space.*

*Proof.* The worst case is when $A^4(\pi)$ is a cuspidal representation of $GL_5(\mathbb{A})$. Suppose $\pi_v$ is a non-tempered representation given by $\pi(\mu|\ |^r, \mu|\ |^{-r})$, where $\mu$ is a unitary character of $F_v^\times$ and $0 < r < \frac{1}{2}$. We apply the result of Luo-Rudnick-Sarnak [Lu-R-Sa] to $A^4(\pi)$: It states that if $\Pi = \bigotimes_v \Pi_v$ is a cuspidal representation of $GL_n(\mathbb{A})$, and if $\Pi_v$ is the spherical component given by $Ind_{B(F_v)}^{GL_n(F_v)} |\ |^{t_{1v}} \otimes \cdots \otimes |\ |^{t_{nv}}$, $t_{iv} \in \mathbb{C}$, then $|Re(t_{iv})| \leq \frac{1}{2} - \frac{1}{n^2+1}$. In our case, $n = 5$, and we have

$$4r \leq \frac{1}{2} - \frac{1}{5^2+1} = \frac{12}{26}.$$

$\square$

**Corollary 7.3.4.** *Let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A})$. Then the 4th symmetric power L-function $L(s, \pi, Sym^4)$ is holomorphic except possibly for $s = 0, 1$. It has a pole at $s = 1$ if and only if $\pi$ is monomial or $\pi$ is of the tetrahedral type, namely, $\pi$ is not monomial and $Sym^2(\pi) \simeq Sym^2(\pi) \otimes \eta$ for $\eta \neq 1$.*

*Remark 7.2.* We can give a simpler proof of the functoriality of $\wedge^2(A^3(\pi))$, and hence that of $A^4(\pi)$, without

(1) Section 4.1 about comparison of Hecke conjugacy classes, and
(2) Ramakrishnan's idea of descent using the base change method (Section 4.2) and hence Appendix 1.

They are needed for the general case of the functoriality of the exterior square of $GL_4$. The reason is that first $A^3(\pi)$ satisfies the weak Ramanujan property, and hence we can just use Proposition 4.1.2. Secondly the reason we needed the base change method was that we could not verify Proposition 4.2 in the case of supercuspidal representations. But we now have a direct proof of the equality of $\gamma$-functions by Theorem 7.3.2. Recall from Proposition 4.2 that we only need the equality for $m = 1$. Since this is very crucial, we give an argument: Let $k$ be a non-archimedean local field of characteristic zero. Let $\eta$ be supercuspidal representations of $GL_2(k)$ with the corresponding parametrization $\tau : W_k \longrightarrow GL_2(\mathbb{C})$. We can think of $\tau$ as a representation of $Gal(\bar{k}/k)$. We need to show that

$$\gamma(s, A^3(\eta), \wedge^2\rho_4 \otimes \chi, \psi) = \gamma(s, A^3(\tau), \wedge^2\rho_4 \otimes \chi, \psi),$$

for any character $\chi$ of $k^\times$, which we identify as a character of $Gal(\bar{k}/k)$. By appealing to [P-Ra, Lemma 3, Section 4], we can find a number field $F$ with $k = F_v$ and irreducible 2-dimensional representations $\sigma$ of $Gal(\bar{F}/F)$ with solvable image such that $\sigma_v = \tau$ and $\sigma_u$ is unramified for $u|2, u \neq v$. Let $\pi$ be the cuspidal representation of $GL_2(\mathbb{A}_F)$ such that $\pi_v = \eta$, given by the global Langlands correspondence. Take a grössencharacter $\mu$ such that $\mu_v = \chi$. By comparing the functional equations of $L(s, A^3(\pi), \wedge^2\rho_4 \otimes \mu)$ and $L(s, A^3(\sigma), \wedge^2\rho_4 \otimes \mu)$, we obtain the equality, by noting that if $u|2, u \neq v$, $\pi_u$ is unramified.

Hence we can apply the converse theorem (Theorem 2.1) to $A^3(\pi)$ as in Section 4.1 and obtain a weak lift, and follow Section 5.2 to obtain the strong lift.

## Appendix 1:
## A descent criterion for isobaric representations

By Dinakar Ramakrishnan

The object here is to prove the following extension (from cuspidal) to *isobaric* automorphic representations of Proposition 3.6.1 of [Ra], which was itself an extension to $GL(n)$ of Proposition 4.2 (for $GL(2)$) in [BR]. The argument is essentially the same as in [Ra], but requires some delicate bookkeeping.

**Proposition.** *Fix $n, p \in \mathbb{N}$ with $p$ prime. Let $F$ be a number field, let $\{K_j \mid j \in \mathbb{N}\}$ be an infinite family of cyclic extensions of $F$ with $[K_j : F] = p$, and for each $j \in \mathbb{N}$, let $\pi_j$ be an isobaric automorphic representation of $GL(n, \mathbb{A}_{K_j})$. Suppose that, for all $j, r \in \mathbb{N}$, the base changes of $\pi_j, \pi_r$ to the compositum $K_jK_r$ satisfy*

$$\text{(DC)} \qquad (\pi_j)_{K_jK_r} \simeq (\pi_r)_{K_jK_r}.$$

*Then there exists a unique isobaric automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$ such that*

$$(\pi)_{K_j} \simeq \pi_j,$$

*for all but a finite number of $j$.*

*Proof.* Recall that the set *Isob* of isobaric automorphic representations of $GL(n, \mathbb{A}_F)$ for all $n \geq 1$ admits a sum operation $\boxplus$, called the *isobaric sum*, such that

$$L(s, \pi \boxplus \pi') = L(s, \pi)L(s, \pi'), \, \forall\, \pi, \pi' \in \mathit{Isob}.$$

Moreover, given any isobaric automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$ there exist cuspidal automorphic representations $\pi^1, \ldots, \pi^d$ of $GL(n_1, \mathbb{A}_F)$, ..., $GL(n_d, \mathbb{A}_F)$, with $n = n_1 + \cdots + n_d$, such that

$$(1) \qquad \pi \simeq \pi^1 \boxplus \cdots \boxplus \pi^d.$$

Here the cuspidal datum $(\pi^1, \ldots, \pi^d)$ is unique up to (isomorphism and) permutation. We will say that $\pi$ is of *width* $d$. For the basic properties of isobaric representations see [La] and [JS].

Given any isobaric automorphic representation $\pi$ of width $d$ in the form (1) and any $d$-tuple $\chi := (\chi^1, \ldots, \chi^d)$ of idele class characters of $F$, we define the $\chi$-twist of $\pi$ to be

$$(2) \qquad \pi[\chi] := (\pi^1 \otimes \chi^1) \boxplus \cdots \boxplus (\pi^d \otimes \chi^d).$$

If an isobaric automorphic representation $\pi'$ is isomorphic to $\pi[\chi]$ for some $\chi$, we will say that $\pi'$ is a twist of $\pi$. Moreover, if $\mu$ is an idele class character of $F$ and if $m = (m(1), \ldots, m(d))$ is a $d$-tuple of integers, we will set

$$\mu^m := (\mu^{m(1)}, \ldots, \mu^{m(d)}).$$

Now we need the following

**Lemma.** *Let $\pi = \pi^1 \boxplus \cdots \boxplus \pi^d$ be an isobaric automorphic representation of $GL(n, \mathbb{A}_F)$, where $\pi^1, \ldots, \pi^d$ are cuspidal automorphic representations of $GL(n_1, \mathbb{A}_F)$, ..., $GL(n_d, \mathbb{A}_F)$, $n = n_1 + \cdots + n_d$. Then there exist at most a finite number of $d$-tuples $\chi = (\chi^1, \ldots, \chi^d)$ of idele class characters such that*

$$\pi \simeq \pi[\chi].$$

*Proof of the Lemma.* By the uniqueness of the isobaric sum decomposition of $\pi$ into cuspidals, there must be a permutation $\sigma$ in $S_d$ such that we have, for each $i \leq d$, an isomorphism

$$\pi^i \simeq \pi^{\sigma(i)} \otimes \chi_{\sigma(i)}.$$

We must necessarily have $n_i = n_{\sigma(i)}$ for each $i$. So the Lemma is a consequence of the following

**Sublemma.** *Let $\eta, \eta'$ be cuspidal automorphic representations of $GL(m, \mathbb{A}_F)$. Then the set $X$ of idele class characters $\mu$ such that*

$$\eta \simeq \eta' \otimes \mu$$

*is finite.*

*Proof of the Sublemma.* We may assume that $X$ is non-empty, as there is nothing to prove otherwise. Pick, and fix, any member, call it $\nu$, of $X$. Put

$$Y = \{\mu\nu^{-1} \mid \mu \in X\}.$$

Since $X$ and $Y$ have the same cardinality, it suffices to prove that $Y$ is finite. We *claim* that for any $\chi$ in $Y$,

$$\eta \simeq \eta \otimes \chi.$$

Indeed, if $\chi = \mu\nu^{-1}$ with $\mu \in X$, we have

$$\eta \simeq \eta' \otimes \mu \simeq (\eta' \otimes \nu) \otimes (\mu\nu^{-1}) \simeq \eta \otimes \chi,$$

whence the claim.

Now the set $Y$, which parametrizes the self-twists of $\eta$, is finite by Lemma 3.6.2 of [Ra], and hence the Sublemma is proved; so is the Lemma. $\square$

*Proof of the Proposition (contd.).* For each $j$, let $\theta_j$ be a generator of $\mathrm{Gal}(K_j/F)$, and let $\delta_j$ be a character of $F$ cutting out $K_j$ (by class field theory). Note that, for each $i \geq 1$, the pull back to $K_i$ of $\delta_j$ by the norm map $N_i$ from $K_i$ to $F$ cuts out the compositum $K_iK_j$.

We will write, for each $j$,

$$\pi_j \simeq \boxplus_{k=1}^{d(j)} \pi_j^k, \tag{3}$$

with each $\pi_j^k$ a cuspidal automorphic representation of $\mathrm{GL}(n_k(j), \mathbb{A}_F)$, with $n = \sum_{k=1}^{d(j)} n_k(j)$.

We claim that

$$\pi_j \circ \theta_j \simeq \pi_j \quad (\forall j). \tag{4}$$

For all $j, r \geq 1$, let $\theta_{j,r}$ denote the automorphism of $K_jK_r$ such that (i) $\theta_{j,r}|_{K_j} = \theta_j$, and (ii) $\theta_{j,r}|_{K_r} = 1$ (where 1 denotes the identity automorphism). It is easy to see that the base change of $\pi_j \circ \theta_j$ to $K_jK_r$ is simply $(\pi_j)_{K_jK_r} \circ \theta_{j,r}$. (For the basic results on base change, see [AC]; for a quick summary see Proposition 2.3.1 of [Ra].) Applying (DC), we then have

$$(\pi_j \circ \theta_j)_{K_jK_r} \simeq (\pi_r)_{K_jK_r} \circ \theta_{j,r} \simeq (\pi_r)_{K_jK_r} \simeq (\pi_j)_{K_jK_r},$$

since $\theta_{j,r}$ is trivial on $K_r$. Since $K_jK_r$ is a cyclic extension of $K_j$ of prime degree, we must have by Arthur-Clozel,

$$\pi_j \circ \theta_j \simeq \pi_j[(\delta_r \circ N_j)^{m_r}], \tag{5}$$

for some $d(j)$-tuple $m_r = (m_r(1), \dots, m_r(d(j)))$ of integers in $\{0, 1, \dots, p-1\}$. For every fixed $r \geq 1$, and for all $k \neq r$, we then have the self-twist identity

$$\pi_j \simeq \pi_j[(\delta_r \circ N_j)^{m_r}][(\delta_k \circ N_j)^{-m_k}].$$

Note that $\delta_r \circ N_j$ and $\delta_k \circ N_j$ must be distinct unless their ratio is a power of $\delta_j$. So the Lemma above forces $m_r$ to be the zero vector for all but a finite number of $r$. The claimed identity now follows by taking $r$ to be outside this exceptional finite set.

As a result, by applying base change ([AC]; Proposition 2.3.1 of [Ra]) once again, we see that there exists, for each $j \geq 1$, an isobaric automorphic representation of $GL(n, \mathbb{A}_F)$,

$$\pi(j) = \boxplus_{k=1}^{b(j)} \pi(j)^k,$$

with each $\pi(j)^k$ a cuspidal automorphic representation of $GL(N_k(j), \mathbb{A}_F)$ and

$$n = \sum_{k=1}^{b(j)} N_k(j),$$

such that

$$\pi_j \simeq (\pi(j))_{K_j}. \tag{6}$$

Such a $\pi(j)$ is of course unique only up to replacing it by $\pi(j)[\delta_j^a]$ for some $d(j)$-tuple $a = (a_1, \dots, a_{b(j)})$ of integers in $\{0, 1, \dots, p-1\}$. Clearly we have

$$b(j) \leq d(j),$$

but equality need not hold.

It is important to note that, for any $r \neq j$, we have the following compatibility for base change in (cyclic) stages:

$$((\pi(j))_{K_j})_{K_jK_r} \simeq ((\pi(j))_{K_r})_{K_jK_r}. \tag{7}$$

We see this as follows. Let $v$ be a finite place of $K_jK_r$ which is unramified for the data. Denote by $u$ (resp. $w$, resp. $w'$) the place of $F$ (resp. $K_j$, resp. $K_r$) below $v$. If $\sigma_u$ denotes the representation of $W'_{F_u}$ associated to $\pi(j)_u$, then

$$\mathrm{res}^{(K_j)_w}_{(K_jK_r)_v}(\mathrm{res}^{F_u}_{(K_j)_w}(\sigma_u)) \simeq \mathrm{res}^{(K_r)_{w'}}_{(K_jK_r)_v}(\mathrm{res}^{F_u}_{(K_r)'_w}(\sigma_u)).$$

Then (2.3.0) of [Ra] implies the local identity (for all such $v$)

$$((\pi(j)_u)_{(K_j)_w})_{(K_jK_r)_v} \simeq ((\pi(j)_u)_{(K_r)'_w})_{(K_jK_r)_v}.$$

The global isomorphism (7) follows by the strong multiplicity one theorem for isobaric automorphic representations ([JS]).

We can then rewrite (DC) as saying, for all $j, r \geq 1$,

$$((\pi(j))_{K_j})_{K_jK_r} \simeq ((\pi(r))_{K_j})_{K_jK_r}. \tag{8}$$

Consequently we must have, after renumbering, an equality of partitions $(\forall\,(r,j))$:

$$(N_1(j), \dots, N_{b(j)}(j)) = (N_1(r), \dots, N_{b(r)}(r))$$

of $n$. In particular, we have

$$b := b(j) = b(r) \quad \text{and} \quad N_k := N_k(j) = N_k(r). \tag{9}$$

Moreover,

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j}[(\delta_r \circ N_j)^{m(r,j)}], \tag{10}$$

for some $b$-tuple $m(r,j) = (m(r,j)_1, \ldots, m(r,j)_b)$ of integers. We can replace $\pi(r)$ by $\pi(r)[\delta_r^{-m(r,j)}]$ and get

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j}. \tag{11}$$

Then, by replacing $\pi(j)$ by a twist by $\delta_j^a$ for a $b$-tuple $a$ of integers, we can arrange to have $\pi(j)$ and $\pi(r)$ be isomorphic. In sum, we have produced, for *every pair* $(j,r)$, a *common descent*, say $\pi(j,r)$, of $\pi_j, \pi_r$, i.e., with

$$\pi(j,r)_{K_j} \simeq \pi_j \quad \text{and} \quad \pi(j,r)_{K_r} \simeq \pi_r. \tag{12}$$

Fix non-zero vectors $a, c$ in $(\mathbb{Z}/p)^b$, and consider the possible isomorphism

$$\pi(j,r) \simeq \pi(j,r)[\delta_j^a][\delta_r^{-c}]. \tag{13}$$

We claim that this cannot happen outside a finite set $S_{a,c}$ of pairs $(j,r)$. To see this fix a pair $(i,\ell)$ and consider the relationship of $\pi(i,\ell)$ to $\pi(j,r)$. Since $\pi(i,\ell)$ and $\pi(j,\ell)$ have the same base change to $K_\ell$, they must differ by twisting by a $b$-tuple power of $\delta_\ell$. Similarly, $\pi(j,\ell)$ and $\pi(j,r)$ differ by a twist as they have the same base change to $K_r$. Put together, this shows that $\pi(i,\ell)$ and $\pi(j,r)$ are twists of each other. Then (13) would imply that

$$\pi(i,\ell) \simeq \pi(i,\ell)[\delta_j^a][\delta_r^{-c}] \simeq \pi(i,\ell)[\chi_{a,-c}], \tag{14}$$

where

$$\chi_{a,-c} = (\delta_j^{a_1}\delta_r^{-c_1}, \ldots, \delta_j^{a_b}\delta_r^{-c_b}).$$

The claim now follows since, by the Lemma above, $\pi(i,\ell)$ admits only a finite number of self-twists, and since the $b$-tuples $\chi_{a,-c}$ are all distinct for distinct pairs $(j,r)$ (as $a, c$ are fixed).

Now choose a pair $(j,r)$ *not belonging* to $S_{a,c}$ for *any* pair $(a,c)$ of non-zero vectors in $(\mathbb{Z}/p)^b$, and set

$$\pi = \pi(j,r). \tag{15}$$

We assert that for all but a finite number of indices $m$,

$$\pi_{K_m} \simeq \pi_m. \tag{16}$$

It suffices to show that, for any large enough $m$, $\pi = \pi(j,r)$ is isomorphic to either $\pi(j,m)$ or $\pi(m,r)$. Suppose neither is satisfied. Then there exist non-zero vectors $a, c$ in $(\mathbb{Z}/p)^b$ such that

$$\pi(j,m) \simeq \pi(j,r)[\delta_j^a] \quad \text{and} \quad \pi(m,r) \simeq \pi(j,r)[\delta_r^c].$$

We also have $\pi(j,m) \simeq \pi(m,r)[\delta_m^e]$, for some vector $e$ in $(\mathbb{Z}/p)^b$. Putting these together, we get the self-twisting identity

$$\pi(j,r) \simeq \pi(j,r)[\delta_j^a][\delta_r^{-c}][\delta_m^{-e}]. \tag{17}$$

By our choice of $(j,r)$, $e$ cannot be the zero vector. But for each non-zero $e$, the set of indices $m$ for which such an identity can hold is finite, again by the Lemma. Hence we get a contradiction for large enough $m$, which implies that $a$ or $c$ should be 0, giving the requisite contradiction. Thus $\pi = \pi(j,r)$ must be isomorphic to either $\pi(j,m)$ or $\pi(m,r)$ for large enough $m$. Since we have, by (12),

$$\pi(j,m)_{K_m} \simeq \pi_m \simeq \pi(m,r)_{K_m},$$

the Proposition is now proved. $\square$

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## Appendix 2:
## Refined estimates towards the Ramanujan and Selberg conjectures

By Henry H. Kim and Peter Sarnak

In this appendix we apply the main results of [Ki3] concerning the symmetric fourth power of a $GL_2$ cusp form together with the methods developed in [D-I] and [L-R-S] to obtain slight improvements of the known bounds towards the Ramanujan conjectures. While the main results of [Ki3] concern automorphic forms over a general number field, the techniques in [D-I] and [L-R-S] are special to $\mathbb{Q}$ and hence so are the results below.

Let $\pi$ be an automorphic cusp form on $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_\mathbb{Q})$ and denote by $L(s,\pi,Sym^2)$ its symmetric square $L$-function. For $p$ a prime at which $\pi_p$ is unramified, let $diag(\alpha_{1,p},...,\alpha_{n,p})$ be the corresponding Satake parameter and similarly let $diag(\mu_{1,\infty},...,\mu_{n,\infty})$ be the Satake parameter for $\pi_\infty$ (assuming the latter is unramified). These are normalized so that the Ramanujan conjectures assert that $|\alpha_{j,p}|=1$ and $Re(\mu_{j,\infty})=0$.

**Proposition 1.** *Let $\pi$ be as above and assume that the series*

$$L(s,\pi,Sym^2) := \sum_{n=1}^{\infty} a(n)n^{-s}$$

*converges absolutely for $Re(s)>1$. Then for $p<\infty$ at which $\pi_p$ is unramified, we have*

$$|\log_p|\alpha_{j,p}|| \le \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1},$$

*while if $\pi_\infty$ is unramified, we have*

$$|Re(\mu_{j,\infty})| \le \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1}.$$

*Remarks.* (1) This should be compared with the general number field bounds of $\frac{1}{2}-\frac{1}{n^2+1}$ established in [L-R-S2].

(2) The condition of absolute convergence is in fact satisfied for $n\le 4$. Hence for $n=3$ or 4, Proposition 1 gives the sharpest known bounds towards Ramanujan (over $\mathbb{Q}$). For $n=2$ or 3, it is easy to see that the series converges absolutely. For $n=3$, as is shown in [R-S], this follows from the unitarity of $\pi_p$ and the well-known fact that the Rankin-Selberg $L$-function $L(s,\pi\times\tilde{\pi})$, whose coefficients

are non-negative, is absolutely convergent in $Re(s) > 1$. For $n = 4$, the absolute convergence is proved in Proposition 6.2 of [Ki3].

Our main application is for the case $n = 5$. Given a cusp form $\pi$ on $GL_2$, let $\Pi = Sym^4(\pi)$. According to the results in Section 7 of [Ki3], $\Pi$ is an automorphic form on $GL_5$. If it is not a cusp form, then as in [Ki-Sh] we may establish even sharper bounds for $\alpha_{j,p}$, $j = 1, 2$, than the ones below (precisely with $\frac{7}{64}$ replaced by $\frac{1}{10}$). So we assume that $\Pi$ is a cusp form. Now $\Pi = Sym^4(\pi)$, so it is easily seen that since $L(s, \Pi \times \tilde{\Pi})$ is absolutely convergent, so is $L(s, \Pi, Sym^2)$. Applying Proposition 1 to $\Pi$ together with the relationship: the Satake parameters of $\Pi_p$ are

$$diag(\alpha_{1,p}^4, \alpha_{1,p}^3\alpha_{2,p}, \alpha_{1,p}^2\alpha_{2,p}^2, \alpha_{1,p}\alpha_{2,p}^3, \alpha_{2,p}^4),$$

leads to:

**Proposition 2.** *Let $\pi$ be an automorphic cusp form on $GL_2/\mathbb{Q}$. If $\pi$ is unramified at $p$, then*

$$|\log_p |\alpha_{j,p}|| \leq \frac{7}{64}, \quad j = 1, 2.$$

*If $\pi_\infty$ is unramified, then*

$$|Re(\mu_{j,\infty})| \leq \frac{7}{64}, \quad j = 1, 2.$$

These give slight improvements of the recent bound of $\frac{1}{9}$ due to [Ki-Sh].

We can express the bounds for $\pi_\infty$ in terms of eigenvalues of the Laplacian (cf. [Se]). Let $\lambda_1(\Gamma)$ be the smallest (non-zero) eigenvalue of the Laplacian on $\Gamma\backslash\mathbb{H}$, where $\Gamma$ is a congruence subgroup of $SL_2(\mathbb{Z})$. Then

$$\lambda_1(\Gamma) \geq \frac{975}{4096} \approx 0.238....$$

We turn to the proof of Proposition 1. We need some facts concerning the analytic properties of $L(s, \pi, Sym^2)$ and its twists. Here $\pi$ is a cusp form on $GL_n$.

**Proposition 3.** *If $\pi$ is not self-contragredient, then the completed $L$-function (that is, the degree $\frac{n(n+1)}{2}$ Euler product over all places including the archimedean ones) $\Lambda(s, \pi, Sym^2)$ is entire and satisfies a functional equation*

$$\Lambda(s, \pi, Sym^2) = \epsilon(s, \pi, Sym^2)\Lambda(1 - s, \tilde{\pi}, Sym^2).$$

*Proof.* The functional equation is due to [Sh2]. The holomorphy is due to [Ki1]. However, we sketch the proof here. The symmetric square $L$-functions arise by considering $M = GL_n \subset G = SO_{2n+1}$. Let $I(s, \pi) = Ind_M^G \pi |det|^{\frac{s}{2}}$ be the induced representation attached to $(M, \pi)$, and let $E(s, \pi, f_s)$ be the Eisenstein series attached to $f_s \in I(s, \pi)$. Then the constant term of the Eisenstein series is given by

$$f_s + M(s, \pi, w_0)f_s,$$

where $M(s, \pi, w_0)$ is the global intertwining operator and we can write it as $M(s, \pi, w_0) = \bigotimes_v A(s, \pi_v, w_0)$. We can normalize the local intertwining operator ($N(s, \pi_v, w_0)$ is equal to 1 for all but finitely many $v$)

$$A(s, \pi_v, w_0) = \frac{L(s, \pi_v, Sym^2)}{L(s + 1, \pi_v, Sym^2)\epsilon(s, \pi_v, Sym^2)} N(s, \pi_v, w_0).$$

Hence

$$M(s,\pi,w_0)=\frac{\Lambda(s,\pi,Sym^2)}{\Lambda(s+1,\pi,Sym^2)\epsilon(s,\pi,Sym^2)}\otimes_v N(s,\pi_v,w_0).$$

We showed [Ki1] that in the case of $GL_n\subset SO_{2n}$, for each $v$, $N(s,\pi_v,w_0)$ is holomorphic and non-zero as an operator for $Re(s)\geq\frac{1}{2}$ (actually, for $Re(s)\geq 0$). The case of $GL_n\subset SO_{2n+1}$ is exactly the same. Since $w_0(\pi)=\tilde{\pi}$, by Langlands' lemma ([Ki1, Proposition 2.1]), if $\pi$ is not self-contragredient, $M(s,\pi,w_0)$ is holomorphic for $Re(s)>0$. Hence $\frac{\Lambda(s,\pi,Sym^2)}{\Lambda(s+1,\pi,Sym^2)}$ is holomorphic for $Re(s)\geq\frac{1}{2}$. Now starting at $Re(s)>N$, where $\Lambda(s,\pi,Sym^2)$ is absolutely convergent, and moving to the left, we have that $\Lambda(s,\pi,Sym^2)$ is holomorphic for $Re(s)\geq\frac{1}{2}$. Our result follows from the functional equation. □

Let $\chi$ be a Dirichlet character of conductor $q$ which we take to be prime and large. We have

$$L(s,\pi\otimes\chi,Sym^2)=L(s,\pi,Sym^2\otimes\chi^2). \tag{1}$$

Hence as long as $\chi$ is not one of at most two characters mod $q$, $\pi\otimes\chi$ is not self-contragredient, and we may apply Proposition 3.

For the analysis that follows, $\pi$ is fixed and $q\to\infty$, the dependence of a functional equation of $L(s,\pi,Sym^2\otimes\chi^2)$ on $\chi$ can be determined explicitly as in [L-R-S] (note too that the set of twists, i.e., by $\chi^2$, also coincides with the twists used there). In fact since $\chi^2(-1)=1$, the archimedean factor satisfies

$$L_\infty(s,\pi,Sym^2\otimes\chi^2)=L_\infty(s,\pi,Sym^2). \tag{2}$$

The $\epsilon$-factor takes the form

$$\epsilon(s,\pi,Sym^2\otimes\chi^2)=N_\pi^s\chi^2(l_\pi)(W(\chi^2))^{\frac{n(n+1)}{2}}q^{\frac{n(n+1)}{2}(\frac{1}{2}-s)}, \tag{3}$$

where $W(\chi^2)$ is the "sign"of the Gauss sum ($|W(\chi^2)|=1$) and $N_\pi$ and $l_\pi$ are integers depending only on $\pi$.

We proceed first with the proof of Proposition 1 for $p$ finite. We follow the method in [D-I] closely; see also [BDHI]. Fix a smooth function $F$ supported in $(\frac{1}{2},2)$ with $F(1)=1$. For $l$ a large integer and $q$ a prime, $q\nmid l$, consider

$$S=\sum_{\chi \bmod q}\overline{\chi}^2(l)\sum_m a(m)\chi^2(m)F(\frac{m}{l}). \tag{4}$$

Inverting the order of summation gives

$$S=(q-1)\sum_{m^2\equiv l^2(q)}a(m)F(\frac{m}{l}). \tag{5}$$

Here $S$ can also be analyzed by appealing to $L(s,\pi,Sym^2\otimes\chi^2)$ and its functional equation. For what follows we ignore the $\chi$'s for which $\pi\otimes\chi$ is self-contragredient. Their contribution to $S$ is negligible for our purposes. Set

$$S_\chi=\sum_m a(m)\chi^2(m)F(\frac{m}{l}). \tag{6}$$

This can be expressed as

$$S_\chi=\frac{1}{2\pi i}\int_{Re(s)=2}\tilde{F}(s)l^sL(s,\pi,Sym^2\otimes\chi^2)\,ds, \tag{7}$$

where $\tilde{F}(s)$ is the entire function of rapid decrease in $|t|$ ($s=\sigma+it$) given by

$$\tilde{F}(s)=\int_0^\infty F(x)x^{-s}\,dx. \tag{8}$$

In (7) we shift the contour to $Re(s)=-2$ and applying the functional equation yields

$$\begin{aligned} S_\chi=\frac{1}{2\pi i}\int_{Re(s)=-2} &\tilde{F}(s)l^s L(1-s,\tilde{\pi},Sym^2\otimes\bar{\chi}^2)\epsilon(s,\pi,Sym^2\otimes\chi^2)\\ &L_\infty(1-s,\tilde{\pi},Sym^2)/L_\infty(s,\pi,Sym^2)\,ds. \end{aligned} \tag{9}$$

Replacing $s$ with $-s$ and using (3) gives
(10)

$$S_\chi=\frac{\chi^2(l_\pi)W(\chi^2)^{\frac{n(n+1)}{2}}}{2\pi i}\int_{Re(s)=2} H(s)(lN_\pi)^s q^{\frac{n(n+1)}{2}(s+\frac{1}{2})}L(1+s,\tilde{\pi},Sym^2\otimes\overline{\chi}^2)\,ds,$$

where

$$H(s)=\tilde{F}(-s)L_\infty(1+s,\tilde{\pi},Sym^2)/L_\infty(-s,\pi,Sym^2). \tag{11}$$

By the local bounds on $\mu_{j,\infty}$ of [J-S], $H(s)$ is analytic in $Re(s)>0$ and is of rapid decrease as $|t|\to\infty$. Hence if $F_1(x)$ is given by

$$F_1(x)=\frac{1}{2\pi i}\int_{Re(s)=2} H(s)x^{-s}\,ds, \tag{12}$$

then $F_1(x)$ is bounded on $[0,\infty)$ and rapidly decreasing as $x\to\infty$. Expanding $L(1+s,\tilde{\pi},Sym^2\otimes\overline{\chi}^2)$ in (10) yields

$$S_\chi=q^{\frac{n(n+1)}{4}}\chi^2(l_\pi)W(\chi^2)^{\frac{n(n+1)}{2}}\sum_{m=1}^\infty\frac{\bar{a}(m)\overline{\chi}^2(m)}{m}F_1\left(\frac{N_\pi lm}{q^{\frac{n(n+1)}{2}}}\right). \tag{13}$$

Hence
(14)

$$S=\sum_\chi \overline{\chi}^2(l)S_\chi=q^{\frac{n(n+1)}{4}}\sum_m \frac{\bar{a}(m)}{m}F_1\left(\frac{N_\pi lm}{q^{\frac{n(n+1)}{2}}}\right)\sum_\chi \chi^2(l_\pi)(W(\chi^2))^{\frac{n(n+1)}{2}}\overline{\chi}(m).$$

By Deligne's estimates [De] for hyper Kloosterman sums, the sum over $\chi$ is $O(q^{\frac{1}{2}})$. Hence

$$|S|<<q^{\frac{1}{2}+\frac{n(n+1)}{4}}\sum_m\left|\frac{a(m)}{m}F_1\left(\frac{N_\pi lm}{q^{\frac{n(n+1)}{2}}}\right)\right|. \tag{15}$$

Using the absolute convergence assumption gives that for any $\epsilon>0$

$$|S|\underset{\epsilon}{<<}q^{\frac{1}{2}+\frac{n(n+1)}{4}+\epsilon}. \tag{16}$$

Combining this with (5) gives

$$(q-1)a(l)+(q-1)\sum_{\substack{m^2\equiv l^2(q)\\ m\neq l}} a(m)F_1(\frac{m}{l})\underset{\epsilon}{<<}q^{\frac{1}{2}+\frac{n(n+1)}{4}+\epsilon}.$$

Summing this over primes $q$, $Q \leq q \leq 2Q$, gives, for $\epsilon > 0$,

$$
\begin{aligned}
a(l)Q^{2-\epsilon} &\underset{\epsilon}{<<} Q^{\frac{3}{2}+\frac{n(n+1)}{4}+\epsilon} + Q \sum_{Q \leq q \leq 2Q} \sum_{\substack{m^2 \equiv l^2 (q) \\ m \neq l}} |a(m)||F_1(\frac{m}{l})| \\
&<< Q^{\frac{3}{2}+\frac{n(n+1)}{4}+\epsilon} + Ql^{\epsilon} \sum_m |a(m)||F(\frac{m}{l})| << Q^{\frac{3}{2}+\frac{n(n+1)}{4}+\epsilon} + l^{1+\epsilon}Q^{1+\epsilon}.
\end{aligned} \tag{17}
$$

Hence

$$
|a(l)| \underset{\epsilon}{<<} Q^{-\frac{1}{2}+\frac{n(n+1)}{4}+\epsilon} + \frac{l^{1+\epsilon}}{Q}. \tag{18}
$$

Choosing $Q = l^{\frac{1}{\frac{1}{2}+\frac{n(n+1)}{4}}}$ gives

$$
|a(l)| \underset{\epsilon}{<<} l^{1-\frac{1}{\frac{1}{2}+\frac{n(n+1)}{4}}+\epsilon}. \tag{19}
$$

Let $p$ be as in Proposition 1. We have

$$
\prod_{1 \leq i \leq j \leq n} (1 - \alpha_{i,p}\alpha_{j,p}X)^{-1} = \sum_{\nu=0}^{\infty} a(p^{\nu})X^{\nu} := R(X). \tag{20}
$$

According to (19) with $l = p^{\nu}$, we see from the series definition of $R(X)$, that $R(X)$ is analytic for $|X| < p^{-\left(1-\frac{1}{\frac{1}{2}+\frac{n(n+1)}{4}}\right)}$. Hence from the factorization in (20) we have for any $1 \leq i \leq j \leq n$,

$$
|\alpha_{i,p}\alpha_{j,p}| \leq p^{1-\frac{1}{\frac{1}{2}+\frac{n(n+1)}{4}}}. \tag{21}
$$

Taking $i = j$ yields

$$
|\alpha_{i,p}| \leq p^{\frac{1}{2}-\frac{1}{1+\frac{n(n+1)}{2}}}. \tag{22}
$$

Finally $\pi_p$ being unitary ensures that $\{\alpha_{j,p}\}_{j=1}^n = \{\overline{\alpha_{j,p}^{-1}}\}_{j=1}^n$. Hence (22) implies that for $1 \leq i \leq n$,

$$
p^{-\frac{1}{2}+\frac{1}{1+\frac{n(n+1)}{2}}} \leq |\alpha_{i,p}| \leq p^{\frac{1}{2}-\frac{1}{1+\frac{n(n+1)}{2}}}. \tag{23}
$$

This completes the proof of Proposition 1 for $p < \infty$.

We turn to the archimedean case in Proposition 1. Thus $\pi$ is unramified at infinity. The local $L$-factor of $\Lambda(s, \pi, Sym^2 \otimes \chi^2)$ takes the form

$$
L_{\infty}(s, \pi, Sym^2 \otimes \chi^2) = L_{\infty}(s, \pi, Sym^2) = \prod_{1 \leq i \leq j \leq n} \Gamma\left(\frac{s - (\mu_{i,\infty} + \mu_{j,\infty})}{2}\right). \tag{24}
$$

We now proceed exactly as in [L-R-S]. From the global analytic properties of $\Lambda(s, \pi, Sym^2 \otimes \chi^2)$ (again we ignore the two possible $\chi$'s mod $q$ for which $\pi \otimes \chi$ might be self-contragredient), we conclude that if for some $1 \leq i \leq j \leq n$, we set

$$
\beta_0 = \mu_{i,\infty} + \mu_{j,\infty}, \tag{25}
$$

then for any $\chi$,

$$
L(\beta_0, \pi, Sym^2 \otimes \chi^2) = 0. \tag{26}
$$

Now following [L-R-S] working with $L(s,\pi,Sym^2\otimes\chi^2)$ instead of $L(s,\pi\times(\tilde{\pi}\otimes\chi))$ and using the absolute convergence assumption of Proposition 1, we obtain:

For any $\beta$ with $0<Re(\beta)<1$ and any $\epsilon>0$, we have for $Q$ large

$$\sum_{Q\le q\le 2Q}\sum_{\chi(q)} L(\beta,\pi,Sym^2\otimes\chi^2)=\sum_{Q\le q\le 2Q} q+O_{\beta,\epsilon}(Q^{1+(\frac{\frac{n(n+1)}{2}+1}{2})(1-Re(\beta))}). \tag{27}$$

Hence if $Re(\beta)>1-\frac{2}{1+\frac{n(n+1)}{2}}$ we conclude that the first term on the right-hand side of (27) dominates the error term. In particular in this circumstance, the left-hand side of (27) is not zero. In particular, $L(\beta,\pi,Sym^2\otimes\chi^2)\neq 0$ for some (in fact many) $\chi$. Together with (26), this implies that for $\beta_0$ in (25),

$$Re(\beta_0)<1-\frac{2}{1+\frac{n(n+1)}{2}}. \tag{28}$$

In particular if $\beta_0=2\mu_{j,\infty}$, $1\le j\le n$, this gives

$$Re(\mu_{j,\infty})\le\frac{1}{2}-\frac{1}{1+\frac{n(n+1)}{2}}. \tag{29}$$

Again the unitarity of $\pi_\infty$ then ensures that for $1\le j\le n$,

$$|Re(\mu_{j,\infty})|\le\frac{1}{2}-\frac{1}{1+\frac{n(n+1)}{2}}. \tag{30}$$

This completes the proof of the case $p=\infty$ in Proposition 1.

To end we remark that the reason we don't know how to extend Proposition 1 to the general number field is that the presence of units potentially restricts the set of ray class characters $\chi$ (which have to be trivial on the units). In [L-R-S2] special lacunary conductors $q$ are used which suffice when dealing with the Rankin-Selberg $L$-functions $L(s,\pi\times\tilde{\pi})$ whose coefficients are non-negative. Since the conductor of $L(s,\pi\times(\tilde{\pi}\otimes\chi))$ is $q^{n^2}$ in place of $q^{\frac{n(n+1)}{2}}$ for the twists of the symmetric square $L$-functions, one gets in general the weaker bound of $\frac{1}{2}-\frac{1}{1+n^2}$ in Proposition 1.

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