ON THE EQUATION $\text{div} \, Y = f$ AND APPLICATION TO CONTROL OF PHASES

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1. Introduction

The purpose of this paper is to present new results concerning the equation

$$\text{div} \, Y = f \quad \text{on } \mathbb{T}^d,$$

i.e., we work on $\mathbb{R}^d$ with $2\pi$-periodic functions in all variables. In what follows we will always assume that $d \geq 2$ and that

$$\int_Q f = 0$$

where $Q = (0, 2\pi)^d$. The notations $L^p, W^{1,p}$, etc. refer to $L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d)$, etc. or to $2\pi$-periodic functions in $L^p_{\text{loc}}(\mathbb{R}^d), W^{1,p}_{\text{loc}}(\mathbb{R}^d)$, etc. We denote by $L^p_\#$ the space of functions in $L^p$ satisfying (1.2).

Clearly, (1.1) is an underdetermined problem which admits many solutions. A standard way of tackling (1.1) is to look for a vector field $Y$ satisfying the additional condition

$$\text{curl} \, Y = 0,$$

i.e., one looks for a special $Y$ of the form

$$Y = \text{grad} \, u.$$

Equation (1.1) then becomes

$$\Delta u = f$$

and the standard $L^p$-regularity theory yields a solution $u \in W^{2,p}$ when $f \in L^p_\#, 1 < p < \infty$. Consequently (1.1) has a solution $Y \in W^{1,p}$ for every $f \in L^p_\#, 1 < p < \infty$.

More precisely, the operator $\text{div} : W^{1,p} \rightarrow L^p_\#$ admits a right inverse which is a bounded linear operator $K : L^p_\# \rightarrow W^{1,p}$. Strictly speaking, we should write $Y \in (W^{1,p})^d (= d\text{-fold copy of } W^{1,p})$, $\text{div} : (W^{1,p})^d \rightarrow L^p$, etc. But we will often omit the superscript $d$ to alleviate notation.

Three limiting cases are of interest:

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Case 1: $p = 1$. It is well known that when $f \in L^1$ equation (1.3) does not necessarily admit a solution $u \in W^{2,1}$. However, one might still hope to have some solution $Y$ of (1.1) in $W^{1,1}$ or at least in $BV$. This is not true: for some $f$’s in $L^1$, equation (1.1) has no solution in $BV$ and not even in $L^{d/(d-1)}$; see Section 2.1.

Case 2: $p = \infty$. It is well known that when $f \in L^\infty$ equation (1.3) does not necessarily admit a solution $u \in W^{2,\infty}$. However, one might hope to find a solution $Y$ of (1.1) in $W^{1,\infty}$. This is not true: McMullen [13] has shown that for some $f$’s in $L^\infty$ (even continuous $f$) equation (1.1) has no solution in $W^{1,\infty}$. This is proved using a duality argument and a “non-estimate” of Ornstein [16]; see Section 2.2.

Case 3: $p = d$. This is the heart of our work. For every $f \in L^d_{\#}$, equation (1.3) admits a solution $u \in W^{2,d}$ and thus equation (1.1) admits a solution $Y = \text{grad } u \in W^{1,d}$. Since $W^{1,d}$ is not contained in $L^\infty$ (this is a limiting case for the Sobolev imbedding), we cannot assert that this $Y$ belongs to $L^\infty$. In fact, we give in Section 3 (Remark 7) an explicit $f \in L^d$ such that the corresponding $Y = \text{grad } u$ does not belong to $L^\infty$. However one might still hope that given any $f \in L^d_{\#}$ there is some $Y \in L^\infty$ solving (1.1). This is indeed true:

**Proposition 1.** Given any $f \in L^d_{\#}$ there exists some $Y \in L^\infty$ solving (1.1) (in the sense of distributions) with

$$
\|Y\|_{L^\infty} \leq C(d)\|f\|_{L^d}.
$$

**Remark 1.** A more precise statement established in the course of the proof says that there exists $Y \in C^0$ satisfying (1.1) and (1.4).

The proof of Proposition 1 is quite elementary; see Section 3. It relies on the Sobolev-Nirenberg imbedding $W^{1,1} \subset L^{d/(d-1)}$ (and even $BV \subset L^{d/(d-1)}$) combined with duality, i.e., Hahn-Banach. As a consequence, the argument is not constructive, and $Y$ is not obtained as above via a bounded linear operator acting on $f$. In fact, surprisingly, the operator div has no bounded right inverse in this setting:

**Proposition 2.** There exists no bounded linear operator $K$: $L^d_{\#} \to L^\infty$ such that $\text{div } Kf = f \ \forall f \in L^d_{\#}$ (in the sense of distributions).

**Remark 2.** Another way of formulating Proposition 2 is to say that the subspace $\{Y \in L^\infty; \text{div } Y = 0\}$ admits no complement in the space $\{Y \in L^\infty; \text{div } Y \in L^d\}$ equipped with its natural norm. Alternatively, the closed subspace $\{\text{grad } u; u \in W^{1,1}\}$ has no complement in $L^1$; see Section 3.

To summarize: for every $f \in L^d_{\#}$, equation (1.1) admits

a) a solution $Y_1 \in W^{1,d}$,

b) a solution $Y_2 \in L^\infty$.

A natural question is whether there exists a solution $Y$ of (1.1) in $L^\infty \cap W^{1,d}$. This is indeed one of our main results.

**Theorem 1.** For every $f \in L^d_{\#}$ there exists a solution $Y \in L^\infty \cap W^{1,d}$ of (1.1) satisfying

$$
\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \leq C(d)\|f\|_{L^d}.
$$

Despite the simplicity of this statement the argument is rather involved and a simpler proof would be desirable.

We will present two techniques to tackle Theorem 1.
First proof of Theorem 1 when \( d = 2 \) (see Section 4). It relies on Hahn-Banach (via duality) and thus it is not constructive. But it is rather elementary; the main ingredient is the new estimate (1.6) which is established by \( L^2 \)-Fourier methods.

Lemma 1. On \( T^2 \) we have

\[
\|u - \int u\|_{L^2} \leq C \|\text{grad } u\|_{L^{1+H^{-1}}}, \quad \forall u \in L^2,
\]

for some absolute constant \( C \).

The main difficulty, in proving (1.6), stems from the fact that if we decompose

\[
\text{grad } u = h_1 + h_2
\]

with \( h_1 \in L^1 \) and \( h_2 \in H^{-1} \), then \( h_1 \) and \( h_2 \) need not be gradients themselves; it is only their sum which is a gradient.

The analogue of Lemma 1 for \( d > 2 \) is the estimate on \( T^d \),

\[
\|u - \int u\|_{L^{d/(d-1)}} \leq C(d) \|\text{grad } u\|_{L^{1+W^{-1},d/(d-1)}}.
\]

We have no direct proof of (1.7). But it can be deduced by duality from the statement of Theorem 1 (and thus from the second proof presented in Section 7).

Second proof of Theorem 1, valid for all \( d \geq 2 \) (see Sections 5 and 6). We exhibit via a constructive (nonlinear) argument some explicit \( Y \in W^{1,d} \cap L^\infty \) satisfying (1.1) and (1.5). The argument for \( d = 2 \) is simpler and we start with this case for expository reasons.

One should observe a certain analogy with the Fefferman-Stein [10] decomposition of BMO-functions and Uchiyama’s [21] constructive proof. Indeed, returning to equation (1.1) and defining \( F \) by \( \hat{\xi} F(\xi) = \hat{f}(\xi) \), we obtain that \( F \in W^{1,d} \subset BMO \) and (1.1) becomes

\[
F = \sum_{j=1}^d R_j Y_j
\]

with \( R_j = j^{th} \) Riesz transform \( (R_j \hat{\psi}(\xi) = \hat{\psi}(\xi) \xi_j) \), \( Y = (Y_1, \ldots, Y_d) \).

The statement of Theorem 1 is that (1.8) has a solution \( Y \in L^\infty \cap W^{1,d} \). Recall that according to Fefferman-Stein [10] any \( F \in BMO \) has a decomposition of the form

\[
F = Y_0 + \sum_{j=1}^d R_j Y_j \quad \text{with } Y_0, Y_1, \ldots, Y_d \in L^\infty.
\]

The proof of this decomposition is again by duality and nonconstructive. The later constructive approach from Uchiyama [21] gives a different proof of (1.9). If we assume moreover that \( F \in W^{1,d} \), Uchiyama’s argument gives that (1.9) has a solution \( Y_0, Y_1, \ldots, Y_d \in L^\infty \cap W^{1,d} \). The new result in this paper shows that, in fact, for \( F \in W^{1,d} \), the \( Y_0 \)-component is unnecessary and (1.8) holds for some \( Y_1, \ldots, Y_d \in L^\infty \cap W^{1,d} \).

It should be mentioned that to achieve our decomposition we do use significantly different methods from Uchiyama. This raises the question what are the function
spaces $X$, $W^{1,d} \subset X \subset BMO$, such that every $F \in X$ has a decomposition

\begin{equation}
F = \sum_{j=1}^{d} R_j Y_j
\end{equation}

where $Y_j \in L^\infty$ or (assuming the Riesz transforms bounded on $X$) the stronger property $Y_j \in L^\infty \cap X$.

**Remark 3.** Using Theorem 1 we will prove (in Sections 4 and 6) that a slightly stronger conclusion holds:

**Theorem 1'.** For every $f \in L^d_#$ there exists a solution $Y \in C^0 \cap W^{1,d}$ of (1.1) satisfying (1.5).

The original motivation for studying (1.1) comes from the following question about lifting discussed in Bourgain-Brezis-Mironescu [3], [4], [5]. Consider the equation

$$ g = e^{i\varphi} \quad \text{on } \mathbb{T}^d $$

where $\varphi$ is a smooth real-valued function.

**Question.** Assuming $g$ is controlled in $H^{1/2}$, what kind of estimate can we deduce for $\varphi$?

Here is a first easy consequence of Theorem 1.

**Corollary 1.** We have

\begin{equation}
\| \varphi - \int \varphi \|_{L^d/(d-1)} \leq C(d)(1 + \|g\|_{H^{1/2}})\|g\|_{H^{1/2}}.
\end{equation}

**Proof.** Write

$$ \text{grad } g = ie^{i\varphi} \text{ grad } \varphi $$

and thus

\begin{equation}
\text{grad } \varphi = -i\bar{g} \text{ (grad } g)\).
\end{equation}

Multiplying by $Y$ gives

\begin{equation}
\int_Q \varphi \text{ div } Y = \int_Q i\bar{g}Y \cdot \text{ grad } g.
\end{equation}

Given $f \in L^d$ we obtain from Theorem 1 some $Y$ satisfying (1.1) (with $f$ replaced by $f - \int f$) and (1.5). Thus we have

\begin{equation}
|\int (\varphi - \int \varphi) f| \leq \|g\|_{H^{1/2}}(\|\bar{g}Y\|_{H^{1/2}}).
\end{equation}

But

\begin{equation}
\|\bar{g}Y\|_{H^{1/2}} \leq \|g\|_{H^{1/2}}\|Y\|_{L^\infty} + \|g\|_{L^\infty}\|Y\|_{H^{1/2}}
\end{equation}

(by (1.5)) \leq C(\|g\|_{H^{1/2}}\|f\|_{L^d} + \|f\|_{L^d})

where we have used the obvious fact that $\|Y\|_{H^{1/2}} \leq C\|Y\|_{W^{1,d}}$. Combining (1.14) and (1.15) yields (1.11).

**Remark 4.** Estimate (1.11) cannot be improved, replacing the norm $\| \|_{L^d/(d-1)}$ by $\| \|_{L^p}$, $p > d/(d-1)$. This may be seen by choosing $g = e^{i\varphi}$ with $\varphi(x) = (|x|^2 + \varepsilon^2)^{-\alpha/2}$ with $\alpha < d-1$, $\alpha$ close to $(d-1)$ and $\varepsilon$ close to 0 (the same example has already been used in Bourgain-Brezis-Mironescu [3], Lemma 5). There is however a better estimate than (1.11), namely
Theorem 4. Let \( \varphi \) be a smooth real-valued function on \( \mathbb{T}^d \) and set \( g = e^{i\varphi} \), then
\[
\| \varphi \|_{H^{1/2 + 1} + W^{1,1}} \leq C(d)(1 + \| g \|_{H^{1/2}}) \| g \|_{H^{1/2}}.
\]

Theorem 4 has been announced in Bourgain-Brezis-Mironescu [4] (Theorem 3) and is proved in Section 8. Our proof of Theorem 4 is a direct estimate based on paraproducts. In view of the preceding argument one may wonder whether Theorem 4 can be proved by solving a divergence equation. After duality the required statement would be
\[
\| u - f u \|_{H^{1/2 + 1}} \leq C \| \text{grad} u \|_{H^{-1/2 + 1}}
\]
but we do not know whether (1.16) holds.

We now turn to the question of coupling equation (1.1) with the Dirichlet condition
\[
(1.17) \quad Y = 0 \quad \text{on } \partial Q.
\]

This question was addressed (in various forms) by a few authors; see e.g. Arnold–Scott–Vogelius [2], Duvaut–Lions [9] (Theorem 3.2), X. Wang [22], Temam [20] (Proposition 1.2(ii) and Lemma 2.4) and the references therein to Magenes–Stampacchia [12] and Necas [14]. Our aim is to establish the analogue of Theorem 1’ under the Dirichlet condition. We start with the following known fact (see e.g. Arnold–Scott–Vogelius [2] for \( d = 2 \)).

Theorem 2. Given \( f \in L^p_\#(Q), 1 < p < \infty \), there exists some \( Y \in W^{1,p}_0(Q) \) satisfying (1.1) with
\[
(1.18) \quad \| Y \|_{W^{1,p}} \leq C(p) \| f \|_{L^p}.
\]

Moreover \( Y \) can be chosen, depending linearly on \( f \).

The operator and the estimate do not depend on \( p \) assuming we stay away from the end points.

For the convenience of the reader we include a new proof; our technique is extremely elementary and can be adapted to establish, for the limiting case \( p = \infty \),

Theorem 3. Given \( f \in L^d_\#(Q) \) there exists some \( Y \in C^0(\overline{Q}) \cap W^{1,d}_0(Q) \) satisfying (1.1) with
\[
\| Y \|_{L^\infty} + \| Y \|_{W^{1,d}} \leq C \| f \|_{L^d}.
\]

Theorem 3 is stronger than Theorem 1’. However it will be deduced from Theorem 1’. There are variants of Theorems 2 and 3 when \( Q \) is replaced by a Lipschitz domain in \( \mathbb{R}^d \) (see Section 7.2).

The plan of the paper is the following:
1. Introduction.
2. The cases \( f \in L^p \) with \( p = 1 \) and \( p = \infty \).
3. Proofs of Propositions 1 and 2 and related questions.
4. Proof of Theorem 1 when \( d = 2 \) via duality.
5. Proof of Theorem 1 when \( d = 2 \) (explicit construction).
6. Proof of Theorem 1 when \( d > 2 \) (explicit construction).
7. The equation \( \text{div} Y = f \) with Dirichlet condition. Proof of Theorems 2 and 3.
8. Estimation of the phase in \( H^{1/2} + W^{1,1} \). Proof of Theorem 4.
2. The cases $f \in L^p$ with $p = 1$ and $p = \infty$

We consider here equation (1.1) with $f \in L^p_\#$ and ask whether there exists a solution $Y \in W^{1,p}$ of (1.1) when $p = 1$ and $p = \infty$. As we have already mentioned in the Introduction the answer is negative. Here is the proof.

2.1. The case $p = 1$. Assume by contradiction that for every $f \in L^1_\#$ there is some $Y \in W^{1,1}$ satisfying (1.1). It follows that the linear operator

$$Tu = \text{div } u$$

from $E = W^{1,1}$ into $F = L^1_\#$ is bounded and surjective. By the open mapping principle there is a constant $C$ such that for every $f \in F$ there exists a solution $Y \in E$ of (1.1) satisfying

$$\|Y\|_{W^{1,1}} \leq C \|f\|_{L^1}.$$

We now use a duality argument which occurs frequently in the rest of the paper. We will deduce that $W^{1,1} \subset L^\infty$ with continuous injection, and since this is false, we infer that for some $f$’s in $F$ there is no $Y \in W^{1,1}$ satisfying (1.1).

Let $u \in W^{1,d}$ and set

$$\text{grad } u = h \in L^d.$$  \hfill (2.1)

Given any $f \in L^1$, let $Y \in W^{1,1}$ be such that

$$\text{div } Y = f - \int f$$

and

$$\|Y\|_{W^{1,1}} \leq C \|f - \int f\|_{L^1}.$$

Taking the scalar product of (2.1) with $Y$ and integrating yields

$$\int_Q (u - \int_Q u) f = - \int_Q h Y.$$

Consequently

$$\int_Q (u - \int_Q u) f \leq \|h\|_{L^d} \|Y\|_{L^{d/(d-1)}}. \hfill (2.2)$$

By the Sobolev-Nirenberg imbedding we have $W^{1,1} \subset L^{d/(d-1)}$ and thus

$$\|Y\|_{L^{d/(d-1)}} \leq C \|Y\|_{W^{1,1}} \leq C \|f\|_{L^1}. \hfill (2.3)$$

Combining (2.2) and (2.3) we deduce that $(u - \int_Q u) \in L^\infty$ with

$$\|u - \int_Q u\|_{L^\infty} \leq C \|\text{grad } u\|_{L^d}.$$  Impossible.

Remark 5. The same argument shows that equation (1.1) with $f \in L^1_\#$ need not have a solution $Y$ in the sense of distributions with $Y \in L^{d/(d-1)}$. (Note, however, that the solution $Y$ given via (1.3) belongs to $L^p$, $\forall p < d/(d - 1)$, and even to weak-$L^{d/(d-1)}$). It suffices to follow the above argument with $E = W^{1,1}$ replaced by

$$\tilde{E} = \{Y \in L^{d/(d-1)}; \text{ div } Y \in L^1\}$$

equipped with its natural norm.
2.2. The case $p = \infty$. This case has been settled negatively by McMullen [13] (the interest in this kind of problem grew out of the study of the equation $\det(\nabla \varphi) = f$ with $\varphi$ bi-Lipschitz and also from a question of Gromov [11] on separated nets; see Dacorogna-Moser [18], Ye [23], Rivière-Ye [17], [18], Burago-Kleiner [7]).

For the convenience of the reader we sketch a proof when $d = 2$, which is essentially similar to the one of McMullen [13]. We argue by contradiction as above. Then, for every $f \in L^\infty$ there is a $Y \in W^{1,\infty}$ satisfying

$$\text{div} \ Y = f$$

and

$$\|Y\|_{W^{1,\infty}} \leq C \|f\|_{L^\infty}.$$  

Let $\psi$ be a smooth function on $T^2$ and set $g = \psi x_1x_2$. Write

$$\int g_{x_1} Y_1 + g_{x_2} Y_2 = - \int g f = - \int \psi x_1 x_1 Y_{x_1} + \psi x_2 x_2 Y_{x_2}.$$  

Consequently

$$\left| \int g f \right| \leq C (\|\psi x_1 x_1\|_{L^1} + \|\psi x_2 x_2\|_{L^1}) \|f\|_{L^\infty}$$

and thus

$$\|g\|_{L^1} = \|\psi x_1 x_2\|_{L^1} \leq C (\|\psi x_1 x_1\|_{L^1} + \|\psi x_2 x_2\|_{L^1}).$$

This contradicts a celebrated “non-inequality” of Ornstein [16] and completes the proof.

Remark 6. The same argument shows that equation (1.1) with $f \in C^0$ and $\int f = 0$ need not have a solution $Y \in W^{1,\infty}$.

3. Proofs of Propositions 1 and 2 and related questions

Proof of Proposition 1. Recall the Sobolev-Nirenberg imbedding $W^{1,1} \subset L^{d/(d-1)}$ and, more generally, $BV \subset L^{d/(d-1)}$ with

$$(3.1) \quad \|u - f u\|_{L^{d/(d-1)}} \leq C(d) \|\text{grad} u\|_\mathcal{M} \quad \forall u \in BV,$$

where $\mathcal{M}$ denotes the space of measures. Set

$$E = C^0, \quad F = L^d_\#$$

and consider the unbounded linear operator $A = D(A) \subset E \rightarrow F$, defined by

$$D(A) = \{Y \in E; \text{div} Y \in L^d\}, \quad AY = \text{div} Y,$$

so that $A$ is densely defined and has closed graph. Clearly we have

$$E^* = \mathcal{M}, \quad F^* = L^{d/(d-1)}_\#,$$

$$D(A^*) = F^* \cap BV, \quad A^* u = \text{grad} u.$$  

By (3.1) we have

$$\|u\|_{F^*} \leq C(d) \|A^* u\|_{F^*} \quad \forall u \in D(A^*).$$

It follows from the closed-range theorem (see e.g. Brezis [6], Section II.7) that $A$ is surjective. More precisely, we claim that for any $f \in F$ there is some $Y \in E$ satisfying (1.1) and

$$\|Y\|_{L^\infty} \leq 2C(d) \|f\|_{L^d},$$

where $C(d)$ is the constant in (3.1). \hfill $\square$
Indeed, let \( f \in F \) with \( \| f \|_{L^d} = 1 \) and consider the two convex sets
\[
B = \{ Y \in E; \| Y \|_E < 2C(d) \}
\]
and
\[
L = \{ Y \in E; \text{div } Y = f \}.
\]
We have to prove that \( B \cap L \neq \emptyset \). Suppose not, and \( B \cap L = \emptyset \). Then, by Hahn-Banach there exists \( \mu \in E^*, \mu \neq 0 \), and \( \alpha \in \mathbb{R} \) such that
\[
\langle \mu, Y \rangle \leq \alpha \quad \forall Y \in B
\]
and
\[
\langle \mu, Y \rangle \geq \alpha \quad \forall Y \in L.
\]
From (3.2) we have \( \| \mu \| \leq \alpha / 2C(d) \) and from (3.3) we deduce, in particular, that \( \langle \mu, Z \rangle = 0 \quad \forall Z \in N(A) \). It follows that \( \mu \in N(A)^\perp = R(A^*) \). Hence there exists some \( u \in F^* \cap BV \) such that \( \text{grad } u = \mu \). Applying (3.1) we see that
\[
\| u \|_{L^{d/(d-1)}} \leq C(d) \| \mu \| \leq \alpha / 2.
\]
On the other hand, by (3.3), \( \forall Y \in L \),
\[
\alpha \leq \langle \mu, Y \rangle = \langle \text{grad } u, Y \rangle = -\int u \text{div } Y = -\int uf \leq \| u \|_{L^{d/(d-1)}} \leq \alpha / 2.
\]
This is impossible since \( \alpha > 0 \) (because \( \mu \neq 0 \)).

**Remark 7.** The special solution of (1.1) given by \( Y = \text{grad } u \), where \( u \) is the solution of (1.3), belongs to \( W^{1,d} \) when \( f \in L^d \); however, in general, it does not belong to \( L^\infty \). Here is an example due to L. Nirenberg. Using \((x_1, x_2, \ldots, x_d)\) as coordinates in \( \mathbb{R}^d \) consider the function
\[
u = x_1 |\log r|^\alpha \zeta
\]
where \( \zeta \) is a smooth cut-off function with support near 0 and \( 0 < \alpha < (d-1)/d \).
Note that \( Y = \text{grad } u \) does not belong to \( L^\infty \) while
\[
|\Delta u| \leq \frac{C}{r} |\log r|^{\alpha - 1},
\]
so that \( \Delta u \in L^d \).

We now turn to the proof of Proposition 2, i.e., the non-existence of a bounded right inverse \( K : L^d_\# \to L^\infty \) for the operator \( \text{div} \). We present two proofs. The first is the simplest: after a standard averaging trick we obtain a bounded multiplier \( L^d \to L^\infty \) and we reach a contradiction by a direct summability consideration. The second proof is related to Remark 2: the existence of \( K \) would yield a factorization of the identity map \( I : W^{1,1} \to L^{d/(d-1)} \) through the Banach space \( L^1 \); however no such factorization exists by a general argument from the geometry of Banach spaces.

**First proof of Proposition 2.** Assume \( K : L^d_\# \to L^\infty \) is a bounded operator satisfying \( \text{div } K = I \) on \( L^d_\# \). Then the averaged operator
\[
\bar{K} = \int_{T^d} \tau_x K \tau_x dx,
\]
where \( \tau_x f(y) = f(y + x) \), still satisfies
\[
\text{div } \bar{K} = I \quad \text{on } L^d.
\]
On the other hand, $\tilde{K}$ is clearly a multiplier

$$\tilde{K}(e^{in \cdot x}) = (\lambda_1(n), \lambda_2(n), \ldots, \lambda_d(n))e^{in \cdot x}$$

which is bounded from $L^d$ into $L^\infty$ and hence from $L^1$ into $L^{d'}$ where $d' = d/(d-1)$. By (3.5) we have

$$\sum_{j=1}^{d} n_j \lambda_j(n) = 1 \quad \forall n \in \mathbb{Z}^d$$

so that

$$|\lambda(n)|^2 = \sum_{j=1}^{d} |\lambda_j(n)|^2 \geq \frac{1}{|n|^2} \quad \forall n.$$

Consider the multiplier

$$M(e^{in \cdot x}) = \frac{1}{|n|^{d-1}}e^{in \cdot x}, \quad n \neq 0.$$ 

Then $M$ is bounded from $L^{d'}$ into $L^2$. Hence $M\tilde{K}$ is a bounded multiplier from $L^1$ into $L^2$. Thus

$$\sum_{n \in \mathbb{Z}^d \setminus n \neq 0} |\lambda_j(n)|^2 < \infty, \quad \forall j.$$ 

Summing over $j = 1, 2, \ldots, d$, and using (3.6) we deduce

$$\sum_{n \in \mathbb{Z}^d \setminus n \neq 0} \frac{1}{|n|^d} < \infty.$$ 

A contradiction. \qed

*Second proof of Proposition 2.* Assuming the existence of $K : L^d_{\#} \to L^\infty$ we obtain a factorization of the identity map $I : W^{1,1} \to L^{d'}$ as

$$I = K^* \circ \text{grad}$$

which, in particular, gives a factorization of $I$ through the Banach space $L^1$. We claim that there in no such factorization, as a consequence of Grothendieck’s theorem on absolutely summing operators. Both the result and the method are well known and we briefly recall them (see Wojtaszczyk [22] for details). First take $d = 2$. Then $I : W^{1,1} \to L^2$ and we consider the operator $I \circ D$ where $D : L^2 \to W^{1,1}$ is defined by

$$D(e^{in \cdot x}) = \frac{1}{\sqrt{1 + |n|^2}}e^{in \cdot x}.$$ 

Thus $D$ is clearly bounded as an operator into $H^1$, hence into $W^{1,1}$. Since $I$ is assumed to factor through $L^1$, so does $I \circ D$:
Next, recall Grothendieck’s theorem that any bounded operator $B : L^1 \to L^2$ is 1-summing, i.e.,
\[
\pi_1(B) = \sup \left\{ \sum \|Bx_i\| ; (x_i) \subset L^1 \text{ and } \max_{x^* \in L^{\infty}} \|x^*\| \leq 1 \sum |\langle x_i, x^* \rangle| \leq K_G \|B\| , \right\}
\]
where $K_G$ is Grothendieck’s constant.

From the usual ideal properties, we obtain
\[
\left( \sum_{n \in \mathbb{Z}^2} \frac{1}{1 + |n|^2} \right)^{1/2} = \|I \circ D\|_{HS} = \pi_2(I \circ D) \leq \pi_1(I \circ D) \leq \pi_1(B \circ A) \leq \|A\| \pi_1(B) \leq K_G \|A\| \|B\| < \infty,
\]
which in an obvious contradiction.

For $d > 2$, we have $I : W^{1,1} \to L^d$ and we consider the multiplication operator $M : L^d \to L^2$ given by $M(e^{in \cdot x}) = (1 + |n|) \frac{1}{2} e^{in \cdot x}$. Hence, considering now $M \circ I \circ D : L^2 \to L^2$ factoring through $L^1$, we obtain a contradiction again:
\[
\left( \sum_{n \in \mathbb{Z}^2} \frac{1}{(1 + |n|)^{d-2}(1 + |n|^2)} \right)^{1/2} = \|M \circ I \circ D\|_{HS} = \pi_2(M \circ I \circ D) \leq \pi_1(M \circ I \circ D) < \infty.
\]

Proof of Remark 2. Consider the Banach space
\[ E = \{ Y \in L^\infty ; \text{ div } Y \in L^d \} \]
equipped with its natural norm $\|Y\|_{L^\infty} + \| \text{ div } Y\|_{L^d}$. Then
\[ N = \{ Y \in L^\infty ; \text{ div } Y = 0 \}
\]
is a closed subspace of $E$ which admits no complement in $E$. Indeed, set
\[ F = L^d_\# \]
and consider the bounded linear operator $T : E \to F$ defined by $TY = \text{ div } Y$. By Proposition 1, $T$ is surjective. If $N = N(T)$ admits a complement in $E$, then $T$ has a bounded right inverse, i.e., an operator $S : F \to E$ such that
\[ \text{ div } (Sf) = f \quad \forall f \in F \]
(see e.g. Brezis [6], Théorème II.10). But this is impossible by Proposition 2.

Similarly, the subspace
\[ R = \{ \text{ grad } u ; u \in W^{1,1} \} \]
of $L^1$ is closed and admits no complement in $L^1$. Indeed, consider the spaces
\[ E = \{ u \in W^{1,1} ; \int u = 0 \} ; F = L^1 \]
and the operator $T = \text{ grad }$, a bounded linear injective operator from $E$ into $F$. If $R = R(T)$ admits a complement in $F$, then
$T$ has a bounded left inverse $S : F \to E$ (see e.g. Brezis [6], Théorème II.11). In particular, $S : F \to L^d/(d-1)$ satisfies

$$S(\text{grad } u) = u, \quad \forall u \in W^{1,1} \text{ with } \int u = 0.$$ 

Then $S^* : L^d_# \to L^\infty$ satisfies

$$\text{div } (S^* f) = f, \quad \forall f \in L^d_#,$$

and this is again impossible by Proposition 2.

\[\square\]

4. Proof of Theorem 1 when $d = 2$ via duality

We now return to the periodic setting and we will prove the slightly stronger form of Theorem 1, Theorem $1'$ (for $d = 2$).

**Theorem $1'$ (for $d = 2$).** For every $f \in L^2_#$ there exists a solution $Y \in C^0 \cap H^1$ of (1.1) with

$$\|Y\|_{L^\infty} + \|Y\|_{H^1} \leq C\|f\|_{L^2}$$

for some absolute constant $C$.

Theorem $1'$ is proved by duality from

**Lemma 2.** On $\mathbb{T}^2$ we have

$$\|u - f u\|_{L^2} \leq C\|\text{grad } u\|_{L^1 + H^{-1}}, \quad \forall u \in L^2$$

where $C$ is an absolute constant.

Assuming the lemma we turn to the

**Proof of Theorem $1'$.** First observe that

$$L^1 + H^{-1} \subset \mathcal{M} + H^{-1}$$

and that

$$\|\cdots\|_{L^1 + H^{-1}} = \|\cdots\|_{\mathcal{M} + H^{-1}} \text{ on } L^1 + H^{-1}$$

(this may be easily seen using regularization by convolution).

Let $E = C^0 \cap H^1, F = L^2_#$ and consider the bounded operator $T : E \to F$ defined by $TY = \text{div } Y$. Clearly, $T^* : F^* = F \to E^* = \mathcal{M} + H^{-1}$ is given by $T^* u = \text{grad } u$. By Lemma 2 we have

$$\|u\|_{F^*} \leq C\|T^* u\|_{E^*}, \quad \forall u \in F^*,$$

and therefore $T$ is surjective from $E$ onto $F$. Estimate (4.1) follows from the open mapping principle or one could argue directly using (4.2) and Hahn-Banach as in the proof of Proposition 1. \[\square\]

**Proof of Lemma 2.** Assume

$$u \in L^2_#,$$

$$\partial_x u = F_1 + h_1, \quad \partial_y u = F_2 + h_2$$

and

$$\|F_1\|_{L^1} + \|F_2\|_{L^1} + \|h_1\|_{H^{-1}} + \|h_2\|_{H^{-1}} \leq 1.$$
We have to prove that
\[
(4.7) \quad \|u\|_{L^2} \leq C.
\]

The main ingredient is

**Lemma 3.** Under assumptions (4.4)–(4.6) we have
\[
(4.8) \quad \sum_{n_1, n_2 \in \mathbb{Z}} \frac{n_1^2 n_2^2}{(n_1^2 + n_2^2)^2} |\hat{u}(n_1, n_2)|^2 \leq C(\|u\|_{L^2} + 1).
\]

Assuming Lemma 3 we may now complete the proof of Lemma 2. Define
\[
(4.9) \quad u'(x', y') = u(x' + y', x' - y') = \sum_{n_1, n_2} \hat{u}(n_1, n_2) e^{i(n_1 + n_2)x' + (n_1 - n_2)y'}
\]
so that
\[
(4.10) \quad \hat{u}'(n_1 + n_2, n_1 - n_2) = \hat{u}(n_1, n_2)
\]
and
\[
\partial_x u'(x', y') = \partial_x u(x' + y', x' - y') + \partial_y u(x' + y', x' - y') = (F_1 + F_2)(x' + y', x' - y') + (h_1 + h_2)(x' + y', x' - y') \in L^1 + H^{-1}
\]
and similarly for \(\partial_y u'\).

From (4.8) and (4.10) we obtain
\[
(4.11) \quad \sum_{n_1, n_2} \frac{(n_1 + n_2)^2(n_1 - n_2)^2}{4(n_1^2 + n_2^2)^2} |\hat{u}(n_1, n_2)|^2 = \sum_{n_1, n_2} \frac{(n_1')^2(n_2')^2}{((n_1')^2 + (n_2')^2)^2} |\hat{u}'(n_1', n_2')|^2
\]
\[
\leq C(\|u'\|_{L^2} + 1) = C(\|u\|_{L^2} + 1).
\]

Addition of (4.8) and (4.11) implies that
\[
\|u\|_{L^2}^2 = \sum_{n_1, n_2} |\hat{u}(n_1, n_2)|^2 \leq C(\|u\|_{L^2} + 1)
\]
and the desired estimate (4.7) follows.

We now turn to the

**Proof of Lemma 3.** We have
\[
\sum_{n \neq 0} \frac{n_1^2 n_2^2}{(n_1^2 + n_2^2)^2} |\hat{u}(n)|^2 = \frac{1}{i} \sum_{n_1, n_2} \frac{n_1 n_2^2}{(n_1^2 + n_2^2)^2} \hat{\partial_x u}(n) \hat{u}(-n)
\]
\[
\quad \quad \text{by (4.5)} = \frac{1}{i} \sum_{n_1, n_2} \frac{n_1 n_2^2}{(n_1^2 + n_2^2)^2} \hat{F}_1(n) \hat{u}(-n) + \frac{1}{i} \sum_{n_1, n_2} \frac{n_1 n_2^2}{(n_1^2 + n_2^2)^2} \hat{h}_1(n) \hat{u}(-n)
\]
\[
= (4.12) + (4.13).
\]

Estimate
\[
(4.14) \quad |(4.13)| \leq \sum_{n_1, n_2} \frac{\hat{h}_1(n)}{\sqrt{n_1^2 + n_2^2}} \hat{u}(-n) \leq \|h_1\|_{H^{-1}} \|u\|_{L^2}.
\]
Write
\begin{equation}
(4.12) \quad \sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \hat{F}_1(n) \hat{\partial_y \hat{u}}(-n) \\
= \sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \hat{F}_1(n) \hat{F}_2(-n) + \sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \hat{F}_1(n) \hat{h}_2(-n) \\
= (4.15) + (4.16).
\end{equation}

Estimate
\begin{equation}
(4.17) \quad |(4.16)| \leq \sum \frac{|n_1| |n_2|}{(n_1^2 + n_2^2)^2} (|\hat{\partial_x \hat{u}}(n)| + |\hat{h}_1(n)|) |\hat{h}_2(-n)| \\
\leq \sum \frac{n_1^2 |n_2|}{(n_1^2 + n_2^2)^2} |\hat{u}(n)| |\hat{h}_2(-n)| + \sum \frac{|\hat{h}_1(n)|}{\sqrt{n_1^2 + n_2^2}} |\hat{h}_2(-n)| \\
\leq \|f\|_{L^2} \|h_2\|_{H^{-1}} + \|h_1\|_{H^{-1}} \|h_2\|_{H^{-1}}.
\end{equation}

\begin{flushright}
□
\end{flushright}

Estimation of (4.15). This is the key point. Since \(\|F_1\|_{L^1} \leq 1, \|F_2\|_{L^1} \leq 1\), it suffices (by convexity) to replace \(\hat{F}_1(n)\) by
\begin{equation}
(4.18) \quad \hat{F}_1(n) = e^{in\cdot a}, \quad \hat{F}_2(n) = e^{in\cdot b}
\end{equation}
for some \(a, b \in \mathbb{T}^2\) (this amounts to replacing \(F_1, F_2\) by the Dirac measures \(\delta_a, \delta_b\), respectively).

Thus we obtain
\begin{equation}
(4.19) \quad \sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \hat{F}_1(n) \hat{F}_2(-n) = \sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} e^{i[n_1(a_1 - b_1) + n_2(a_2 - b_2)]} \\
= -\sum \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \sin n_1(a_1 - b_1) \sin n_2(a_2 - b_2)
\end{equation}
by parity considerations.

Claim. For all \(\theta_1, \theta_2 \in \mathbb{T}\)
\begin{equation}
(4.20) \quad \left| \sum_{n_1, n_2} \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \sin n_1 \theta_1 \sin n_2 \theta_2 \right| \leq C.
\end{equation}

From the claim, we conclude that \(|(4.15)|, |(4.19)| \leq C\) and, recalling also (4.14), (4.17), inequality (4.8) follows.

Proof of the Claim. Splitting \(Z\) in dyadic intervals, we obtain
\begin{equation}
(4.21) \quad \sum_{k_1, k_2 \geq 0} \left| \sum_{n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2}} \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \sin n_1 \theta_1 \sin n_2 \theta_2 \right|.
\end{equation}

Recall the inequality
\begin{equation}
(4.22) \quad \left| \sum_{n \in I} \sin n\theta \right| \lesssim 4^k |\theta| \wedge \frac{1}{|\theta|}
\end{equation}
if \(\theta \in \mathbb{T}\) and \(I \subseteq [2^{k-1}, 2^k]\) is an interval (where \(\wedge\) denotes min).
From (4.22), assuming \( k_1 \geq k_2 \), we have

\[
\left| \sum_{n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2}} \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \sin n_1 \theta_1 \sin n_2 \theta_2 \right| \leq \left( 4^{k_1} |\theta_1| \wedge \frac{1}{|\theta_1|} \right) \left( 4^{k_2} |\theta_2| \wedge \frac{1}{|\theta_2|} \right) \left\| \left\{ \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \right\} \right\|_{\ell^\infty(n_1 \sim 2^{k_1}) \hat{\otimes} \ell^\infty(n_2 \sim 2^{k_2})}
\]

where \( \ell^\infty(I) \hat{\otimes} \ell^\infty(J) \) denotes the usual projective tensor product. Thus the last factor in (4.23) may be bounded by

\[
\left\| \frac{\partial^2}{\partial n_1 \partial n_2} \frac{n_1 n_2}{(n_1^2 + n_2^2)^2} \right\|_{\ell^1(n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2})} \leq C \left\| \frac{1}{(n_1^2 + n_2^2)^2} \right\|_{\ell^1(n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2})} \leq C \frac{2^{k_2}}{8^{k_1}}.
\]

Substitution of (4.23), (4.24) in (4.21) gives the bound

\[
(4.20), (4.21) \leq C \sum_{k_1 \geq k_2 \geq 0} 4^{k_2-k_1} \left( 2^{k_1} |\theta_1| \wedge \frac{1}{2^{k_1} |\theta_1|} \right) \left( 2^{k_2} |\theta_2| \wedge \frac{1}{2^{k_2} |\theta_2|} \right)
\]

\[
\lesssim C \prod_{i=1}^2 \left[ \sum_{k \in \mathbb{Z}_+} \left( 2^k |\theta_i| \wedge \frac{1}{2^k |\theta_i|} \right) \right] \leq C.
\]

This completes the proof of the Claim and of Theorem 1' for \( d = 2 \).

5. Proof of Theorem 1 when \( d = 2 \) (explicit construction)

Our aim is to construct \( Y \in L^\infty \cap H^1 \) such that

\[
\text{div} Y = f \in L^2_\#, (T^2).
\]

Write

\[
\mathbb{Z}^2 = \bigcup_{j \geq 0} (\Lambda_j^1 \cup \Lambda_j^2)
\]

where

\[
\Lambda_j^1 = [2^{j-1} < |n_1| \leq 2^j, |n_2| \leq 2^j]
\]

\[
\Lambda_j^2 = [2^j < |n_2| \leq 2^{j+1}, |n_1| \leq 2^j].
\]

Let

\[
\Lambda^\alpha = \bigcup_j \Lambda_j^\alpha \quad (\alpha = 1, 2).
\]

Decompose

\[
f = f^1 + f^2 \text{ where } f^\alpha = P_{\Lambda^\alpha} f = \sum_{n \in \Lambda^\alpha} \tilde{f}(n)e^{in \cdot x}.
\]
**Claim.** Let $\delta > 0$ be small enough and $\|f\|_2 \leq \delta$. Then there are $Y_1, Y_2$ such that

\[(5.2) \quad \|Y_\alpha\|_{L^\infty \cap H^1} \leq 1\]

and

\[(5.3) \quad \|\partial_\alpha Y_\alpha - f^\alpha\|_2 \leq \delta^{4/3} \quad (\alpha = 1, 2).\]

Thus if $\|f\|_2 = \delta$, then

\[\|f - \partial_1 Y_1 - \partial_2 Y_2\|_2 \leq \delta^{1/3}\|f\|_2\]

and iteration of this gives (5.1).

The construction of $Y_1, Y_2$ is explicit but nonlinear (see Proposition 2).

Take $\alpha = 1$ and denote $f^1$ by $f$, $\Lambda_1^j$ by $\Lambda_j$.

Define

\[f_j = P_{\Lambda_j} f,\]

\[c_j = \|f_j\|_2,\]

\[F_j = D_x^{-1} f_j = \sum_{n_1} \frac{1}{n_1} \hat{f_j}(n)e^{in\cdot x}.\]

Hence

\[\left( \sum c_j^2 \right)^{1/2} = \|f\|_2,\]

\[(5.4) \quad \|F_j\|_\infty \leq \sum_{n \in \Lambda_j} \frac{1}{n_1} |\hat{f}(n)| \lesssim 2^{-j}|\Lambda_j|^{1/2}\|f_j\|_2 \lesssim c_j.\]

Fix $\varepsilon > 0$ a small constant and partition

\[\Lambda_j = \bigcup_{r < \frac{1}{4} + 1} \Lambda_{j,r}\]

in stripes $\Lambda_{j,r}$ such that

\[(5.5) \quad |\text{Proj}_{n_1} \Lambda_{j,r}| \sim \varepsilon 2^j.\]

Define first

\[(5.6) \quad \tilde{F}_j(x) = \sum_r \left| \sum_{n \in \Lambda_{j,r}} \frac{1}{n_1} \hat{f_j}(n)e^{in\cdot x} \right|.\]

Thus

\[(5.7) \quad |F_j(x)| \leq |\tilde{F}_j(x)| \lesssim c_j.\]
From Cauchy-Schwarz

\[ \| \tilde{F}_j \|_2 \leq \varepsilon^{-1/2} \| F_j \|_2 \lesssim \varepsilon^{-1/2} 2^{-j} c_j. \]  

(5.8)

Observe that if \( \text{Proj}_{\Lambda_{j,r}} = [a_r, b_r] \), \( b_r - a_r \sim \varepsilon 2^j \), then

\[ |\partial_1 \tilde{F}_j| \leq \sum_r \left| \sum_{n \in \Lambda_{j,r}} \frac{n_1}{n_1} \tilde{f}_j(n) e^{i n_1 x} \right| \]

where

\[ \left| \frac{n_1}{n_1} \right| < \varepsilon. \]

Therefore

\[ \| \partial_1 \tilde{F}_j \|_2 \lesssim \sum_r \varepsilon \| P_{\Lambda_{j,r}} f \|_2 \lesssim \varepsilon^{1/2} \| P_{\Lambda_j} f \|_2 = \varepsilon^{1/2} c_j \]  

(5.9)

(this is the purpose of the construction of \( \tilde{F}_j \)).

We also need to make an appropriate localization of the Fourier transform of \( \tilde{F}_j \).

Denote

\[ K_N(y) = \sum_{|n| < N} \frac{N - |n|}{N} e^{i n y}, \]

the usual F\'ejer kernel on \( T \). It is easy to see that if

\[ P(y) = \sum_{|n| < N} \hat{P}(n) e^{i n y} \]

is a trigonometric polynomial, then

\[ |P| \leq 3(|P| * K_N). \]  

(5.10)

Using this fact in the variables \( x_1, x_2 \), we see that

\[ |F_j| \leq \tilde{F}_j \leq G_j \]  

(5.11)

denoting

\[ G_j = 9 \tilde{F}_j * (K_{N_1} \otimes K_{N_2}) \]

(5.12)

where each \( \Delta_{j,r} \) is an \( N_1 \times N_2 \) rectangle, \( N_1 \sim \varepsilon 2^j, N_2 \sim 2^j \).

Thus, by construction

\[ \text{supp} \hat{G}_j \subset [-N_1, N_1] \times [-N_2, N_2] \subset [|n| \leq 2^j] \]  

(5.13)

and inequalities (5.7), (5.8), (5.9) remain preserved.

Therefore,

\[ \| G_j \|_\infty \leq 9 \| \tilde{F}_j \|_\infty \lesssim c_j \quad (0 < \delta < 1), \]  

(5.14)

\[ \| G_j \|_2 \lesssim \varepsilon^{-1/2} 2^{-j} c_j, \]  

(5.15)

\[ \| \partial_1 G_j \|_2 \lesssim \varepsilon^{1/2} c_j, \]  

(5.16)

\[ \| \nabla G_j \|_2 \lesssim \varepsilon^{-1/2} c_j. \]  

(5.17)

Assume that \( \{ f_j | j \leq K \} \) is a finite sequence (which is no restriction).
Define
\[ Y_1 = F_K + F_{K-1}(1 - G_K) + F_{K-2}(1 - G_{K-1})(1 - G_K) + \cdots \]
\[ = \sum_{j\leq K} F_j \prod_{k>j} (1 - G_k). \]

(5.18)

Thus from (5.11)
\[ |Y_1| \leq |F_K| + (1 - |F_K|)|F_{K-1}| + (1 - |F_K|)(1 - |F_{K-1}|)|F_{K-2}| + \cdots \leq 1. \]

One may also rewrite (5.18) as
\[ Y_1 = \sum F_j - \sum G_j H_j \]

(5.19)

with
\[ H_j = F_{j-1} + F_{j-2}(1 - G_{j-1}) + F_{j-3}(1 - G_{j-2})(1 - G_{j-1}) + \cdots \]
\[ = \sum_{k<j} F_k \prod_{k<k'<j} (1 - G_{k'}). \]

(5.20)

Clearly
\[ |H_j| < 1. \]

By construction
\[ \partial_t Y_1 = \sum f_j - \sum \partial_t (G_j H_j). \]

(5.21)

Next, we estimate the second term in (5.21) that will appear as an error term.

Observe that since supp $\hat{F}_j \subset [|n| \sim 2^j]$ and (5.13), also
\[ \text{supp } \hat{H}_j \subset [|n| \lesssim 2^j]. \]

(5.22)

Denote $P_k$ Fourier projection operators on $[|n| \sim 2^k]$ such that $Id = \sum_{k \geq 0} P_k$.

From the preceding, we may thus ensure that
\[ G_j H_j = \sum_{k \leq j} P_k(G_j H_j). \]

(5.23)

Estimate then
\[ \left\| \sum_j \partial_t (G_j H_j) \right\|_2 \leq \sum_{s \geq 0} \left( \sum_j \|\partial_t P_{j-s}(G_j H_j)\|_2^2 \right)^{1/2} \]

(5.24)

(since for fixed $s$, the $P_{j-s}$ have disjoint ranges).

Returning to the parameter $0 < \varepsilon < 1$ introduced earlier, write
\[ \varepsilon = 2^{-s_*} \quad (s_* > 0) \]

(5.25)

and estimate (5.24) in the ranges
\[ s > s_* \]

(5.26)
\[ 0 \leq s \leq s_* \]

(5.27)
Contribution of (5.26). Since $|H_j| \leq 1$ and (5.15),
\[
\|\partial_1 P_{j-s}(G_j H_j)\|_2 \lesssim 2^{j-s} \|G_j H_j\|_2 \leq 2^{j-s} \|G_j\|_2 \leq \varepsilon^{-1/2} 2^{-s} c_j.
\]
Substitution in (5.24) gives the contribution
\[
(5.29) \sum_{s \geq s_*} 2^{-s} \varepsilon^{-1/2} \left( \sum c_j^2 \right)^{1/2} < 2^{-s_*} \varepsilon^{-1/2} \|f\|_2 < \varepsilon^{1/2} \|f\|_2.
\]
Contribution of (5.27). Estimate now
\[
\|\partial_1 P_{j-s}(G_j H_j)\|_2 \leq \|\partial_1 (G_j H_j)\|_2 \leq \|\partial_1 G_j\|_2 + \|G_j \partial_1 H_j\|_2 \leq \varepsilon^{1/2} c_j + \|G_j \partial_1 H_j\|_2
\]
using (5.16).

Recalling definition (5.20) of $H_j$, one easily verifies that
\[
(5.31) \quad \|\nabla H_j\| \leq \sum_{k<j} (|\nabla F_k| + |\nabla G_k|).
\]
Hence
\[
(5.32) \quad \|\nabla H_j\|_{\infty} \leq \sum_{k<j} 2^k c_k
\]
and from (5.15)
\[
(5.33) \quad \|G_j \partial_1 H_j\|_2 \leq \varepsilon^{-1/2} c_j \left( \sum_{k<j} 2^{-(j-k)} c_k \right).
\]
Substitution of (5.30), (5.33) in (5.24) gives the following bound on the contribution of (5.27):
\[
(5.34) \quad s_* \varepsilon^{1/2} \left( \sum c_j^2 \right)^{1/2} + s_* \varepsilon^{-1/2} \left( \sum c_j^2 \left( \sum_{k<j} 2^{-(j-k)} c_k \right)^2 \right)^{1/2} \leq \left( \log \frac{2}{\varepsilon} \right) \varepsilon^{1/2} \|f\|_2 + \left( \log \frac{1}{\varepsilon} \right) \varepsilon^{-1/2} \|f\|_2^2.
\]
Consequently, from (5.21), (5.29), (5.34),
\[
(5.35) \quad \|f - \partial_1 Y_1\|_2 = \left\| \sum_{j} \partial_1 (G_j H_j) \right\|_2 \leq \log \frac{2}{\varepsilon} \varepsilon^{1/2} \|f\|_2 + \varepsilon^{-1/2} \|f\|_2^2.
\]
Under the assumption $\|f\|_2 \leq \delta$, letting $\varepsilon = \delta$ in (5.35), we obtain thus
\[
(5.36) \quad \|f - \partial_1 Y_1\|_2 \leq \delta \frac{2}{\delta} \leq \delta \frac{2}{\delta} = \delta \frac{2}{\delta}
\]
which is (5.3).

It remains to estimate $\|Y_1\|_{H^1} = \|\nabla Y_1\|_2$.

By (5.19)
\[
(5.37) \quad \|\nabla Y_1\|_2 \leq \left\| \sum \nabla F_j \right\|_2 + \left\| \sum \nabla (G_j H_j) \right\|_2.
\]
From the definition of $F_j$ and since $\text{supp} \, \hat{F}_j \subset \Lambda^1_j$, it follows that

\begin{equation}
\left\| \sum_j \nabla F_j \right\|_2 \sim \left( \sum_j \| f_j \|_2^2 \right)^{1/2} = \| f \|_2.
\end{equation}

Estimate the second term in (5.37) as in (5.24),

\begin{equation}
\left\| \sum_j \nabla (G_j H_j) \right\|_2 \leq \sum_{s \geq 0} \left( \sum_j \| \nabla P_{j-s} (G_j H_j) \|_2^2 \right)^{1/2}
\end{equation}

and

\begin{equation}
\| \nabla P_{j-s} (G_j H_j) \|_2 \lesssim 2^{j-s} \| G_j H_j \|_2 \leq \varepsilon^{-1/2} 2^{-s} c_j.
\end{equation}

Thus

\begin{equation}
(5.39) \leq \varepsilon^{-1/2} \sum_{s \geq 0} 2^{-s} \left( \sum_j c_j^2 \right)^{1/2} \leq \varepsilon^{-1/2} \| f \|_2
\end{equation}

and

\begin{equation}
\| \nabla Y_1 \|_2 \leq \delta^{-1/2} \| f \|_2 \leq \delta^{1/2}.
\end{equation}

Since $\| Y_1 \|_\infty \lesssim 1$, this establishes (5.2).

This proves the Claim and completes the proof of Theorem 1 for $d = 2$.

6. Proof of Theorem 1 when $d > 2$ (explicit construction)

Let $f \in L^d_{\#}(\mathbb{T}^d)$. Our aim is to construct a solution $Y$ of $\text{div} \, Y = f$ satisfying

\begin{align}
\| Y \|_\infty &\leq C \| f \|_d, \\
\| \nabla Y \|_d &\leq C \| f \|_d.
\end{align}

We do this by standard modification of the previous $L^2$-argument with the Littlewood-Paley square function theory as main additional ingredient. Consider again a partition

\[ \mathbb{Z}^d = \bigcup_{j \geq 0} (\Lambda^1_j \cup \cdots \cup \Lambda^d_j) \]

of disjoint $d$-rectangles $\Lambda^\alpha_j$ of side length $\sim 2^j$.

We formulate the analogue of the Claim with $Y_\alpha$ satisfying bounds (6.1), (6.2).

Letting $\alpha = 1$, $f = f^1$, define again

\begin{equation}
F_j = D^{-1}_{x_1} f_j
\end{equation}

satisfying

\begin{equation}
\| F_j \|_\infty \lesssim (2^{j/2})^d \| f_j \|_d = 2^j \| D^{-1}_{x_1} f_j \|_d \sim \| f_j \|_d \equiv c_j.
\end{equation}
Define $\tilde{F}_j$ and $G_j$ as in (5.6), (5.12). Thus (5.11), (5.13) hold. Also

$$
\|G_j\|_\infty \lesssim \|\tilde{F}_j\|_\infty \leq \varepsilon^{-1/d'} \left( \sum_{r<\varepsilon^{-1/d}} \left\| \sum_{n\in A_{j,r}} \frac{1}{n_1} \hat{f}_j(n) e^{inx} \right\|_\infty^d \right)^{1/d}
$$

$$
\leq \varepsilon^{-1/d'} \left( \sum_{r<\varepsilon^{-1/d}} \left( 2^{j-d} (\varepsilon 2^j)^{d/2} \left\| \sum_{n\in A_{j,r}} \frac{1}{n_1} \hat{f}_j(n) e^{inx} \right\|_d \right)^{d/2} \right)^{1/d}
$$

$$
\lesssim \varepsilon^{-1/d'+1/d} \left( \sum_{r<\varepsilon^{-1/d}} \left\| \sum_{n\in A_{j,r}} \hat{f}_j(n) e^{inx} \right\|_d \right)^{1/2}
$$

(6.5)

$$
\lesssim \varepsilon^{\frac{d}{d'}-1} \|f_j\|_d = \varepsilon^{\frac{d}{d'}-1} c_j \leq \varepsilon^{\frac{d}{d'}-1} \delta.
$$

(We assume that $\delta$ is small enough compared with $\varepsilon$ to ensure, in particular, that $\varepsilon^{\frac{d}{d'}-1} \delta \ll 1$.)

Repeat the construction from Section 5. In place of estimate (5.24) we now have

$$
\left\| \sum_j \partial_1 (G_j H_j) \right\|_d \leq \sum_{s\geq 0} \left\| \sum_j |\partial_1 P_{j-s} (G_j H_j)|^2 \right\|_d^{1/2}
$$

(6.6)

and distinguish between the cases (5.26), (5.27).

**Contribution of (5.26).** Estimate

$$
\left\| \left( \sum_j (\nabla P_{j-s} (G_j H_j))^2 \right)^{1/2} \right\|_d
$$

$$
\lesssim \left\| \left( \sum_j 4^{j-s} |P_{j-s} (G_j H_j)|^2 \right)^{1/2} \right\|_d
$$

$$
\lesssim 2^{-s} \left\| \left( \sum_j 4^j |G_j H_j|^2 \right)^{1/2} \right\|_d
$$

(6.7)

$$
\lesssim 2^{-s} \left\| \left( \sum_j 4^j (\tilde{F}_j * K_j)^2 \right)^{1/2} \right\|_d
$$

(6.8)

where $K_j$ is a product of Fejér kernels

$$
K_{N_1} \otimes K_{N_2} \otimes \cdots \otimes K_{N_d}, \quad N_1 \sim \varepsilon 2^j, \quad \text{and} \quad N_2, \ldots, N_d \sim 2^j.
$$

Again from standard square function inequalities

$$
(6.7) \lesssim 2^{-s} \left\| \left( \sum_j 4^j (\tilde{F}_j)^2 \right)^{1/2} \right\|_d.
$$

Recalling the definition of $\tilde{F}_j$, estimate

$$
(\tilde{F}_j)^2 \leq \varepsilon^{-1} \sum_{s \leq \varepsilon^{-1}} \left( \sum_{n \in A_{j,s}} \frac{1}{n_1} \hat{f}(n) e^{inx} \right)^2.
$$

(6.9)
Substituting in (6.8), this gives

\[
\varepsilon^{-1/2} 2^{-s} \left\| \left( \sum_j \sum_{r<\varepsilon^{-1}} \left| \sum_{n \in \Lambda_{j,r}^1} \frac{2^j}{n_1} \hat{f}(n)e^{inx} \right|^2 \right)^{1/2} \right\|_d
\]

(6.10)

\[
\lesssim \varepsilon^{-1/2} 2^{-s} \left\| \left( \sum_j \sum_{r<\varepsilon^{-1}} \left| \sum_{n \in \Lambda_{j,r}^1} \hat{f}(n)e^{inx} \right|^2 \right)^{1/2} \right\|_d.
\]

We use here the fact that \(|n_1| \sim |n| \sim 2^j\) for \(n \in \Lambda_j^1\).

Recall also the definition of \(\Lambda_{j,r}^1\) obtained by partitioning the \(n_1\)-variable in intervals of size \(\varepsilon 2^j\).

At this stage, we use the following (1-variable) inequality due to Rubio de Francia [19], which generalizes the Littlewood-Paley inequality to arbitrary intervals.

**Proposition 3.** Let \(\{I_\alpha\}\) be disjoint intervals in \(\mathbb{Z}\) and

\[P_{I_\alpha} f = \sum_{n \in I_\alpha} \hat{f}(n)e^{inx}\]

the corresponding Fourier projection.

Then, for \(2 \leq d < \infty\), there is the (one-sided) inequality

\[
\left\| \left( \sum_{I_\alpha} |P_{I_\alpha} f|^2 \right)^{1/2} \right\|_d \leq C \|f\|_d.
\]

(6.11)

Since \(\{|\text{Proj}_{n_1} \Lambda_{j,r}^1\}\}\) are disjoint intervals in \(\mathbb{Z}\), application of (6.11) in the \(x_1\)-variable implies that

\[
(6.6) \lesssim \varepsilon^{-1/2} 2^{-s} \|f\|_d.
\]

Summation of (6.12) for \(s \geq s_*\) gives then

\[
(5.26) \text{-} \text{contribution} \leq \varepsilon^{1/2} \|f\|_d.
\]

**Remark 8.** We used the general Proposition 3 for convenience; the present case could in fact be treated by more elementary means.

**Contribution of (5.27).** Estimate

\[
\left\| \left( \sum_j |\partial_1 P_{j-s}(G_j H_j)|^2 \right)^{1/2} \right\|_d \leq \left\| \left( \sum_j |\partial_1 (G_j H_j)|^2 \right)^{1/2} \right\|_d
\]

\[
\leq \left\| \left( \sum_j |\partial_1 G_j|^2 \right)^{1/2} \right\|_d + \left\| \left( \sum_j |G_j (\partial_1 H_j)|^2 \right)^{1/2} \right\|_d = (6.14) + (6.15).
\]

Estimate (6.14) by

\[
(6.16) \quad \left\| \left( \sum_j |\partial_1 \tilde{F_j}|^2 \right)^{1/2} \right\|_d.
\]
We have that

\[ |\partial_1 \tilde{F}_j| \leq \sum_{r < \varepsilon^{-1}} \left| \sum_{n \in \Lambda_j^r} \frac{n_1 - a_{j,r}}{n_1} \tilde{f}(n)e^{inx} \right| \]

\[ \leq \varepsilon^{-1/2} \left( \sum_{r < \varepsilon^{-1}} \left| \sum_{n \in \Lambda_j^r} \frac{n_1 - a_{j,r}}{n_1} \tilde{f}(n)e^{inx} \right|^2 \right)^{1/2} \]

where \( \text{Proj}_{\Lambda_j^r} = [a_{j,r}, b_{j,r}] \), \( b_{j,r} - a_{j,r} \sim \varepsilon). 

Thus \( |n_1 - a_{j,r} n_1| \leq \varepsilon. \)

We get therefore

(6.16) \( \leq \varepsilon^{-1/2} \cdot \varepsilon \left( \sum_j \sum_{r < \varepsilon^{-1}} \left| \sum_{n \in \Lambda_j^r} \tilde{f}(n)e^{inx} \right|^2 \right)^{1/2} \|d \]

\( \leq \varepsilon^{1/2} \|f\|_d. \)

To estimate (6.15), use again inequality (5.31), together with (6.4), (6.5). Thus

(6.18) \( \| \nabla H_j \|_\infty \leq \varepsilon^{2/3} \cdot (\sum_{k < j} 2^k c_k) < \varepsilon^{2/3} \cdot 2^j \|f\|_d. \)

Hence

(6.15) \( \leq \varepsilon^{2/3} \|f\|_d \left( \sum_j (2^j \tilde{F}_j)^2 \right)^{1/2} \|d \]

\( \leq \varepsilon^{2/3} \|f\|_d^2 \)

applying again the (6.8)-bound using Proposition 3.

Thus the (5.27)-contribution is

(6.20) \( \leq \varepsilon^{1/2} \log \left( \frac{1}{\varepsilon} \|f\|_d + \varepsilon^{2/3} \log \left( \frac{1}{\varepsilon} \|f\|_d \right) \right) \)

Collecting estimates (6.13), (6.20), it follows that

(6.21) \( \|f - \partial_1 Y\|_d = \left\| \sum_j \partial_1 (G_j H_j) \right\|_d \)

\( \leq \varepsilon^{1/2} \log \left( \frac{1}{\varepsilon} \|f\|_d + \varepsilon^{2/3} \log \left( \frac{1}{\varepsilon} \|f\|_d \right) \right) \)

which is the analogue of (5.35). Assuming \( \|f\|_d = \delta \), take \( \varepsilon = \delta^{1/2} \) to obtain

(6.22) \( \|f - \partial_1 Y\|_d \leq \delta^{3/5} \|f\|_d. \)

It remains to estimate

(6.23) \( \| \nabla Y \|_d \leq \left\| \sum_j \nabla F_j \right\|_d + \left\| \sum_j \nabla (G_j H_j) \right\|_d = (6.23) + (6.24). \)

We have

(6.23) \( \sim \left\| \left( \sum_j |\nabla F_j|^2 \right)^{1/2} \right\|_d \sim \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_d \lesssim \|f\|_d. \)
Estimate (6.24) as
\[
\left\| \sum_{s \geq 0} \left( \sum_j |\nabla P_{j-s}(G_j H_j)|^2 \right)^{1/2} \right\|_d \lesssim \varepsilon^{-1/2} \|f\|_d
\]
using (6.7)-(6.12).

This completes the argument.

We conclude this section with a proof of Theorem 1 when \( d > 2 \). The argument is somewhat bizarre: one uses duality twice! First, from Theorem 1 we easily deduce the estimate on \( T_d \)
\[
\|u - f\|_{L^d/(d-1)} \leq C(d) \| \text{grad } u \|_{L^1+W^{-1,d/(d-1)}}, \forall u \in L^d/(d-1).
\]
Next, we argue as in the beginning of Section 4. Observe that \( L^1 + W^{-1,d/(d-1)} \subset M + H^{-1} \) and that
\[
\|\cdots\|_{L^1+W^{-1,d/(d-1)}} = \|\cdots\|_{M+W^{-1,d/(d-1)}} \quad \text{on } L^1 + W^{-1,d/(d-1)}
\]
(this may be easily seen using regularization by convolution).

Let \( E = C^0 \cap W^1,d \), \( F = L^d \# \) and consider the bounded operator \( T : E \to F \) defined by \( TY = \text{div } Y \). Clearly \( T^* : F^* \to E^* = M + W^{-d,d/(d-1)} \) is given by \( T^*u = \text{grad } u \). By (6.26) and (6.27) we obtain
\[
\|u\|_{F^*} \leq C\|T^*u\|_{E^*} \quad \forall u \in F^*
\]
and therefore \( T \) is surjective from \( E \) onto \( F \). Applying the open mapping principle (or use Hahn-Banach as in the proof of Proposition 1), we see that for every \( f \in F \) there is some \( Y \in E \) satisfying \( TY = f \) and \( \|Y\|_E \leq C\|f\|_F \).

\textbf{Remark 9.} Alternatively, one may approximate \( f \in L^d_\#(\mathbb{T}^d) \) by trigonometric polynomials. If \( f \) is a trigonometric polynomial, we may clearly obtain \( Y \) as a trigonometric polynomial (after convolution). A standard limit procedure permits then to complete the argument.

\section{The equation \( \text{div } Y = f \) with Dirichlet condition. Proof of Theorems 2 and 3}

So far we have studied problem (1.1) coupled with a periodic condition. We consider here problem (1.1) coupled with a Dirichlet condition. Usually one associates with (1.1) the “partial” Dirichlet condition
\[
Y \cdot n = 0 \quad \text{on } \partial Q \tag{7.1}
\]
\((n \text{ is normal to } \partial Q)\). It is quite standard that for every \( f \in L^p_\#, 1 < p < \infty \), there is some \( Y \in W^{1,p} \) satisfying (1.1), (7.1) and
\[
\|Y\|_{W^{1,p}} \leq C\|f\|_{L^p}.
\]
Indeed, one may look for a \textit{special} \( Y \) of the form \( Y = \text{grad } u \) and one is led to the Neumann problem
\[
\begin{cases}
\Delta u = f & \text{in } Q, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial Q,
\end{cases}
\]
which admits a solution \( u \in W^{2,p} \) such that
\[
\|u\|_{W^{2,p}} \leq C\|f\|_{L^p}.
\]

It is also possible to couple problem (1.1) with the full Dirichlet condition
\[(7.3)\]
\( Y = 0 \) on \( \partial Q \).

For simplicity we investigate first the case where the domain is a cube and then the case of a Lipschitz bounded domain.

7.1. The case of a cube. Let \( Q = (0,1)^d \). Here is the first result:

**Theorem 2.** Given \( f \in L^p_\#(Q), 1 < p < \infty \), there exists some \( Y \in W^{1,p}_0(Q) \) solving (1.1) with
\[
\|Y\|_{W^{1,p}(Q)} \leq C(p,d)\|f\|_{L^p}.
\]

where we use the standard notation
\[
W^{1,p}_0(Q) = \{Y \in W^{1,p}(Q); Y = 0 \text{ on } \partial Q\}.
\]

Moreover \( Y \) can be chosen, depending linearly on \( f \).

We will make use of the following lemma (which is a special case of Theorem 2).

**Lemma 4.** Given \( f \in W^{1,p}_0(Q), 1 < p < \infty \), with \( \int f = 0 \), there exists \( Y \in W^{1,p}_0(Q) \), such that
\[
\text{div } Y = f
\]

and
\[(7.4)\]
\[\|Y\|_{W^{1,p}(Q)} \leq C(d)\|f\|_{W^{1,p}(Q)}.
\]

Moreover \( Y \) can be chosen, depending linearly on \( f \).

**Proof.** Following a known construction (see Adams [1], p. 58 and Nirenberg [15]), we construct \( Y \) by induction on the dimension \( d \). The assertion is obvious for \( d = 1 \).

Assume that it holds in dimension \( (d - 1) \). Let \( f \in W^{1,p}_0(Q_d) \), where \( Q_d = (0,1)^d \), with \( \int_{Q_d} f = 0 \).

Set
\[
g(x') = \int_0^1 f(x', t) dt, \text{ where } x' = (x_1, \ldots, x_{d-1}) \in Q_{d-1}.
\]

Clearly, \( g \in W^{1,p}_0(Q_{d-1}) \) with
\[
\|g\|_{W^{1,p}(Q_{d-1})} \leq C\|f\|_{W^{1,p}(Q_d)}
\]
and also \( \int_{Q_{d-1}} g = 0 \). By the induction assumption there is some \( Z \in W^{1,p}_0(Q_{d-1}) \) such that
\[(7.5)\]
\[\text{div}_x Z = g \text{ on } Q_{d-1}
\]
and
\[
\|Z\|_{W^{1,p}(Q_{d-1})} \leq C\|g\|_{W^{1,p}(Q_{d-1})} \leq C\|f\|_{W^{1,p}(Q_d)}.
\]

Fix a function \( \zeta \in C_0^\infty(0,1) \) such that
\[(7.6)\]
\[\int_0^1 \zeta(t) dt = 1.
\]

For \( x = (x', x_d) \in Q_d \) set
\[
h(x) = \int_0^{x_d} (f(x', t) - \zeta(t)g(x')) dt.
\]
It is easy to see (using (7.6)) that \( h \in W^{1,p}_0(Q_d) \) and
\[
\|h\|_{W^{1,p}(Q_d)} \leq C\|f\|_{W^{1,p}(Q_d)}.
\]
Moreover
\[
\frac{\partial h}{\partial x_d}(x) = f(x) - \zeta(x_d)g(x').
\]
Combining this with (7.5) yields
\[
f(x) = \text{div}_x' (\zeta(x_d)Z(x')) + \frac{\partial h}{\partial x_d}
\]
i.e., the conclusion holds with
\[
Y(x) = (\zeta(x_d)Z(x'), h(x)).
\]
\[\Box\]

**Proof of Theorem 2.** For simplicity we assume that \( d = 2 \); the argument is similar for \( d > 2 \).

Let
\[
Q = \{(x, y) \in \mathbb{R}^2; \ 0 < x < 1, 0 < y < 1\}.
\]

Given \( f \in L^p_m(Q), 1 < p < \infty \), we will construct a solution \( Y \in W^{1,p}_0(Q) \) of (1.1); moreover
\[
\|Y\|_{W^{1,p}} \leq C_p\|f\|_{L^p}
\]
and \( Y \) depends linearly on \( f \). This is done in three steps. \[\Box\]

**Step 1.** Construct a solution \( Y \in W^{1,p}(Q) \) of (1.1) satisfying (7.7) and
\[
(7.8) \quad Y = 0 \text{ on the edge } \{(x, 0); 0 < x < 1\}.
\]

**Proof.** Set
\[
\tilde{Q} = \{(x, y); 0 < x < 1, -2 < y < 1\}
\]
and
\[
(7.9) \quad \tilde{f} = \begin{cases} f & \text{in } Q, \\ 0 & \text{in } \tilde{Q} \backslash Q. \end{cases}
\]

Let \( Z \in W^{1,p}(\tilde{Q}) \) be the solution of
\[
(7.10) \quad \text{div } \tilde{Z} = \tilde{f} \quad \text{in } \tilde{Q}
\]

obtained via (7.2) (or via periodic conditions on \( \tilde{Q} \)).

The heart of the matter is the following construction. Write \( Z = (Z_1, Z_2) \) and define \( Y = (Y_1, Y_2) \) in \( Q \), where
\[
(7.11) \quad Y_1(x, y) = Z_1(x, y) + 3Z_1(x, -y) - 4Z_1(x, -2y),
\]
\[
Y_2(x, y) = Z_2(x, y) - 3Z_2(x, -y) + 2Z_2(x, -2y).
\]

(This type of “reflection” is reminiscent of standard extension techniques in \( W^{m,p} \), \( m \geq 2 \); see e.g. Adams [1].)

It is easy to see using (7.9), (7.10) and (7.11) that
\[
\text{div } Y = f \quad \text{in } Q
\]
while (7.8) is clear from the definition of \( Y \).
It is important (for the next step) to observe that if we had started with the additional information
\[ Z = 0 \text{ on the edge } \{(0, y); -2 < y < 1\} \text{ of } \hat{Q}, \]
then we could infer that \( Y \) also vanishes on the edge \( \{(0, y); 0 < y < 1\} \) of \( Q \). □

**Step 2.** Construct a solution \( Y \in W^{1,p}(Q) \) of (1.1) satisfying (7.7) and (7.12)
\[ Y = 0 \text{ on the 2 adjacent edges } \{(x, 0); 0 < x < 1\} \text{ and } \{(0, y); 0 < y < 1\}. \]

**Proof.** Set
\[ \hat{Q} = \{(x, y); -2 < x < 1, 0 < y < 1\} \]
and
\[ \hat{f} = \begin{cases} f & \text{in } Q, \\ 0 & \text{in } Q \setminus Q. \end{cases} \]
From Step 1 applied to \( \hat{f} \) in \( \hat{Q} \) we obtain a solution \( \hat{Z} \) of
\[ \text{div } \hat{Z} = \hat{f} \text{ in } \hat{Q} \]
such that
\[ \hat{Z} = 0 \text{ on the edge } \{(x, 0); -2 < x < 1\} \text{ of } \hat{Q}. \]
Starting with \( \hat{Z} \) (instead of \( Z \)) we repeat the construction of Step 1 changing the roles of \( x \) and \( y \). We thus obtain a \( Y \in W^{1,p}(Q) \) satisfying (1.1) in \( Q \), (7.7) and (7.12). □

**Step 3.** Proof of Theorem 2 completed.

Consider a smooth partition of unity \( (\theta_i) \), \( i = 1, 2, 3, 4 \), subordinate to the covering of \( Q \) consisting of the 4 discs of radius 1 centered at the 4 vertices. Let \( Y_i \in W^{1,p}(Q) \) be the solution constructed in Step 2 relative to each vertex.

Set
\[ Z = \sum_{i=1}^{4} \theta_i Y_i. \]
It is easy to see from this construction that \( \theta_i Y_i \in W^{1,p}_0(Q) \), \( \forall i \) and thus \( Z \in W^{1,p}_0(Q) \). Moreover
\[ \text{div } Z = f + \sum_i \nabla \theta_i \cdot Y_i \]
and \( \sum_i \nabla \theta_i \cdot Y_i \in W^{1,p}_0(Q) \). By Lemma 4 we may construct \( X \in W^{1,p}_0(Q) \) satisfying
\[ \text{div } X = \sum_i \nabla \theta_i \cdot Y_i \]
and \( Y = Z - X \) has all the desired properties in Theorem 2.

Next we have a variant of Theorem 1′ for the full Dirichlet condition.

**Theorem 3.** Given \( f \in L^d_\#(Q) \) there exists some \( Y \in C^0(\overline{Q}) \cap W^{1,d}_0(Q) \) satisfying
(1.1) with
\[ \|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \leq C\|f\|_{L^d}. \]

**Remark 10.** Clearly, Theorem 3 implies Theorem 1′ since the function \( Y \) extended by periodicity belongs to \( C^0(\mathbb{T}^d) \cap W^{1,d}(\mathbb{T}^d) \) and satisfies (1.1) on \( \mathbb{T}^d \). However its proof relies heavily on Theorem 1′.
Proof of Theorem 3. Follow the same strategy as in the proof of Theorem 2. The only difference is that in Step 1 use Theorem 1' to obtain $Z$ (instead of taking the special $Z$ in the form of a gradient). Of course the dependence of $Y$ on $f$ is not linear anymore.

In Step 3 rely on the following variant of Lemma 4 (with an identical proof).

Lemma 4'. Given $f \in C^0(\bar{Q}) \cap W^{1,p}_0(Q)$, $1 < p < \infty$, with $\int f = 0$, there exists $Y \in C^0(\bar{Q}) \cap W^{1,p}_0(Q)$ such that

$$\text{div} \ Y = f$$

and

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,p}} \leq C(\|f\|_{L^\infty} + \|f\|_{W^{1,p}}).$$

7.2. The case of Lipschitz domains. Let $\Omega$ be a Lipschitz, connected, bounded domain in $\mathbb{R}^d$. Recall that $\Omega$ is Lipschitz if there is a $\delta > 0$ such that for every point $p \in \partial \Omega$, $\partial \Omega \cap B_\delta(p)$ is the graph of a Lipschitz function (in an appropriate coordinate system varying with $p$).

We have the following variants of Theorems 2 and 3.

Theorem 2'. Given any $f \in L^p_\#(\Omega)$, $1 < p < \infty$, there exists some $Y \in W^{1,p}_0(\Omega)$ solving (1.1) with

$$\|Y\|_{W^{1,p}} \leq C(p, \Omega) \|f\|_{L^p}.$$  (7.13)

Moreover $Y$ can be chosen, depending linearly on $f$.

Theorem 3'. For every $f \in L^d_\#(\Omega)$ there exists some $Y \in C^0(\bar{\Omega}) \cap W^{1,d}_0(\Omega)$ solving (1.1) with

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \leq C(p, \Omega) \|f\|_{L^d}.$$  (7.14)

The heart of the argument (for both theorems) is the following.

Lemma 5. There is a bounded operator $S : L^p(\Omega) \to W^{1,p}_0(\Omega)$ such that

$$f - \text{div} \ Sf \in W^{1,p}_0(\Omega) \quad \forall f \in L^p$$

and

$$\|f - \text{div} \ Sf\|_{W^{1,p}} \leq C \|f\|_{L^p}.$$  (7.15)

The variant needed for the proof of Theorem 3' is

Lemma 5'. There is a nonlinear map $S : L^d(\Omega) \to C^0(\bar{\Omega}) \cap W^{1,d}_0(\Omega)$ such that

$$\|Sf\|_{L^\infty} + \|Sf\|_{W^{1,d}} \leq C \|f\|_{L^d}$$

and

$$\|f - \text{div} \ Sf\|_{W^{1,d}} \leq C \|f\|_{L^d}.$$  (7.16)

The proof of Lemma 5 relies on the following construction. Let $Q'$ be a cube of side $\delta$ in $\mathbb{R}^{d-1}$ and set

$$U = \{(x', y) \in Q' \times \mathbb{R}; \psi(x') < y < \psi(x') + \delta\}$$

where $\psi \in \text{Lip}(Q')$.  

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Lemma 6. Assume
\[(7.18) \quad \| \nabla \psi \|_{L^\infty(Q')} \leq \varepsilon_0(d) \text{ sufficiently small (depending only on } d).\]
Then, given any \(g \in L^p(U)\) there is some \(Z \in W^{1,p}(U)\) satisfying
\[(7.19) \quad \text{div } Z = g \text{ in } U,\]
\[(7.20) \quad Z = 0 \text{ on } \{ y = \psi(x') ; \ x' \in Q' \} \text{ and on the lateral boundary of } U,\]
with
\[\| Z \|_{W^{1,p}(U)} \leq C(p, d) \| g \|_{L^p(U)}.\]
Moreover \(Z\) can be chosen to depend linearly on \(g\).

Proof. For \(x' \in Q'\) and \(0 < y < \delta\) set
\[
\tilde{g}(x', y) = g(x', y + \psi(x')).
\]
Note that \(\| \tilde{g} \|_{L^p(Q)} = \| g \|_{L^p(U)}\)
where \(Q = Q' \times (0, \delta)\).

By Theorem 2 there exists \(\tilde{Z} \in W^{1,p}(Q)\) such that
\[
\begin{cases}
\text{div } \tilde{Z} = \tilde{g} & \text{in } Q, \\
\tilde{Z} = 0 & \text{on } \{(x', 0); \ x' \in Q'\} \cup (\partial Q' \times (0, \delta))
\end{cases}
\]
with
\[(7.21) \quad \| \tilde{Z} \|_{W^{1,p}(Q)} \leq C(d) \| \tilde{g} \|_{L^p(Q)}.
\]

Note that here \(\int \tilde{g} = 0\) is not required since we may consider in \(\hat{Q} = Q' \times (0, 2\delta)\) the function
\[
\hat{g}(x', y) = \begin{cases}
\tilde{g}(x', y) & \text{for } x' \in Q' \text{ and } 0 < y < \delta, \\
\tilde{g}(x', y - \delta) & \text{for } x' \in Q \text{ and } \delta < y < 2\delta,
\end{cases}
\]
and then solve (using Theorem 2)
\[
\begin{cases}
\text{div } \hat{Z} = \hat{g} & \text{in } \hat{Q}, \\
\hat{Z} = Q & \text{on } \partial \hat{Q},
\end{cases}
\]
with
\[(7.22) \quad \| \hat{Z} \|_{W^{1,p}(\hat{Q})} \leq C(d) \| \hat{g} \|_{L^p(Q)}.
\]
The restriction \(\hat{Z}\) of \(\hat{Z}\) to \(Q' \times (0, \delta)\) satisfies the desired properties.

Also, it is clear by scaling that the constant in (7.21) is independent of \(\delta\).
Returning to \((x', y) \in U,\) set
\[
Z(x', y) = \hat{Z}(x', y - \psi(x')); \]
it is easy to see, using (7.18) and (7.21), that
\[
\| \text{div } Z - g \|_{L^p(U)} \leq C(d) \varepsilon_0 \| g \|_{L^p(U)}
\]
and
\[
\| Z \|_{W^{1,p}(U)} \leq C(d)(1 + \varepsilon_0) \| g \|_{L^p(U)}.
\]
Choosing \(\varepsilon_0\) such that \(C(d)\varepsilon_0 < 1\) and iterating this construction yields the lemma.
\[\square\]

The variant necessary for Theorem 3’ is
Lemma 6'. Assume (7.18). Then given \(g \in L^2(U)\) there is some \(Z \in C^0(\overline{U}) \cap W^{1,p}(U)\) satisfying (7.19), (7.20) and
\[
\|Z\|_{L^\infty(U)} + \|Z\|_{W^{1,p}(U)} \leq C(d)\|g\|_{L^p(U)}.
\]

Next, we remove the smallness condition (7.18) on the Lipschitz constant of \(\psi\).

Lemma 7. With the same notation as in Lemma 6, assume only that \(\psi \in \text{Lip}(Q')\). Then, given any \(g \in L^p(U)\), there is some \(Z \in W^{1,p}(U)\) satisfying (7.19), (7.20) and
\[
\|Z\|_{W^{1,p}(U)} \leq C(p,d,\|\nabla \psi\|_{L^\infty(Q')}\|g\|_{L^p(U)}).
\]
Moreover \(Z\) can be chosen to depend linearly on \(g\).

Proof. Consider the dilation \(x' \mapsto \tilde{x}' = N x'\) (only in \(x'\), not in the full \(x\)-variable). Set \(\tilde{Q}' = NQ'\) and define on \(\tilde{Q}'\) the function \(\tilde{\psi}(\tilde{x}') = \psi(\tilde{x}'/N)\).

Fix an integer \(N\) sufficiently large so that
\[
\|\nabla \tilde{\psi}\|_{L^\infty(\tilde{Q}')} = \frac{1}{N}\|\nabla \psi\|_{L^\infty(Q')} \leq \varepsilon_0(d)
\]
where \(\varepsilon_0(d)\) comes from (7.18).

Set
\[
\tilde{g}(\tilde{x}',y) = g\left(\frac{\tilde{x}'}{N},y\right).
\]

Divide the cube \(\tilde{Q}'\) (of side \(N\delta\)) into \(N^{d-1}\) cubes of side \(\delta\) and apply, in each of them, Lemma 6 to \(\tilde{\psi}\) and \(\tilde{g}\). By gluing the corresponding solutions (this is possible because all these solutions vanish on the lateral boundaries of their domains), we obtain some \(\tilde{Z}(\tilde{x}',y) \in W^{1,p}(\tilde{U})\) satisfying
\[
\begin{cases}
\text{div}_{\tilde{x}',y} \tilde{Z} = \tilde{g} & \text{in } \tilde{U} = \{(\tilde{x}',y) \in \tilde{Q}' \times \mathbb{R}; \, \tilde{\psi}(\tilde{x}') < y < \tilde{\psi}(\tilde{x}') + \delta\}, \\
\tilde{Z} = 0 & \text{on } \{y = \tilde{\psi}(\tilde{x}'); \, \tilde{x}' \in \tilde{Q}'\},
\end{cases}
\]
and the corresponding \(W^{1,p}\)-estimate for \(\tilde{Z}\).

We now return to the variables \((x',y) \in U\). Write the components of \(\tilde{Z}\) as
\[
\tilde{Z} = (\tilde{Z}', \tilde{Z}_d)
\]
and set
\[
Z(x',y) = \left(\frac{1}{N}\tilde{Z}'(Nx',y), \tilde{Z}_d(Nx',y)\right).
\]
It is easy to check that \(Z\) satisfies all the required properties. \(\square\)

The variant necessary for Theorem 3' is

Lemma 7'. With the same notation as in Lemma 6, assume only that \(\psi \in \text{Lip}(Q')\). Then, given any \(g \in L^2(U)\), there is some \(Z \in C^0(\overline{U}) \cap W^{1,p}(U)\) satisfying (7.19), (7.20) and
\[
\|Z\|_{L^\infty(U)} + \|Z\|_{W^{1,p}(U)} \leq C(d,\|\nabla \psi\|_{L^\infty(Q')}\|g\|_{L^p(U)}).
\]

We now return to the
Proof of Lemma 5. Consider a finite covering of \( \partial \Omega \) by a collection of cubes \( Q_i, i = 1, \ldots, k \), of side \( \delta \) such that in each \( Q_i \), \( \partial \Omega \cap Q_i \) admits a Lipschitz parametrization \( \psi_i \). To this covering we associate functions \( \theta_0, \theta_1, \ldots, \theta_k \) such that
\[
\theta_0 + \sum_{i=1}^k \theta_i = 1 \quad \text{on } \Omega,
\]
\[
\theta_0 \in C_0^\infty(\Omega) \text{ and } \theta_i \in C_0^\infty(Q_i) \text{ for } i = 1, \ldots, k.
\]
Given \( g \in L^p(\Omega) \) solve, using Lemma 7, for \( i = 1, 2, \ldots, k \),
\[
\begin{cases}
\text{div } Z_i = g & \text{in } U_i, \\
Z_i = 0 & \text{on } \partial \Omega \cap Q_i.
\end{cases}
\]
Next solve
\[
\text{div } Z_0 = g \quad \text{in } \Omega,
\]
for example \( Z_0 = \text{grad}(\Delta)^{-1} \) where \( \Delta^{-1} \) is used with zero Dirichlet condition on \( \partial \Omega \).

Note that
\[
Z = \sum_{i=0}^k \theta_i Z_i \in W_0^{1,p}
\]
and
\[
\text{div } Z = g + \sum_{i=0}^k \nabla \theta_i \cdot Z_i.
\]
All the conclusions of Lemma 5 hold with
\[
Sg = Z.
\]

\[\square\]

Proof of Lemma 5’. We make the same construction as above, using Lemma 7’ in place of Lemma 7 and Theorem 2 to solve \( \text{div } Z_0 = g \) in any large cube containing \( \Omega \).

Theorem 2’ is an immediate consequence of Lemma 5 and the following general functional analysis argument applied with \( E = W_0^{1,p}, F = L^p_\# \) and \( T = \text{div} \). (Note that \( T^* = \text{grad} \) is injective on \( F^* = L^q_\# \), since \( \Omega \) is connected.)

Lemma 8. Let \( E, F \) be two Banach spaces and let \( T \) be a bounded operator from \( E \) into \( F \). Assume
\[
(7.22) \quad N(T^*) = \{0\}.
\]

\[
(7.23) \quad \begin{cases}
\text{There is a bounded operator } S \text{ from } F \text{ into } E \text{ and } \\
a \text{compact operator } K \text{ from } F \text{ into itself such that } \\
T \circ S = I + K.
\end{cases}
\]

Then \( T \) admits a right inverse.

Proof. First we note that \( T \) is onto. Indeed, in view of (7.22) it suffices to show that \( T \) (or equivalently \( T^* \)) has closed range. This is an obvious consequence of the inequality
\[
\|f\| \leq C\|T^*f\| + \|K^*f\| \quad \forall f \in F^*
\]
(which follows from (7.23)).
Next, let $X$ be a complementing subspace for $N(I + K)$ in $F$ and set $Y = R(I + K)$. Since $u = (I + K)|_X$ is an isomorphism onto $Y$, its inverse $u^{-1} : Y \to X \subset F$ satisfies

$$ (I + K) \circ u^{-1} = I \text{ on } Y. $$

Let $Q$ be a projector from $F$ onto $Y$; since $R(I - Q)$ is finite dimensional, we may choose a base $(e_\alpha)$ of $R(I - Q)$ and write

$$ f = Qf + \sum_\alpha \langle e_\alpha^*, f \rangle e_\alpha \quad \forall f \in F, $$

for some $e_\alpha^*$'s in $F^*$. Since we showed that $T$ is onto, one has, for each $\alpha$, some $\bar{e}_\alpha \in E$ satisfying

$$ (7.26) \quad T \bar{e}_\alpha = e_\alpha \quad \forall \alpha. $$

Consider the operator $S_1 : F \to E$ defined for every $f \in F$, by

$$ S_1f = S \circ u^{-1} \circ Qf + \sum_\alpha \langle e_\alpha^*, f \rangle \bar{e}_\alpha. $$

Using (7.24), (7.25) and (7.26) we see that

$$ T \circ S_1f = (I + K) \circ u^{-1} \circ Qf + \sum_\alpha \langle e_\alpha^*, f \rangle e_\alpha $$

$$ = Qf + \sum_\alpha \langle e_\alpha^*, f \rangle e_\alpha = f $$

for every $f \in F$. Thus $S_1$ is a right inverse for $T$. \hfill \Box

**Proof of Theorem 3'.** Given $f \in L^d$ write, using Lemma 5',

$$ f = \text{div} Y_1 + R $$

with $Y_1 \in C^0(\bar{\Omega}) \cap W^{1,d}_0(\Omega)$ and $R \in W^{1,d}_0(\Omega)$ (and the corresponding estimates).

If $\int f = 0$, then $\int R = 0$ and we may apply Theorem 2' in any $L^p$ (since $W^{1,d} \subset L^p$, $\forall p < \infty$). In particular, if we choose $p > d$, we obtain $Y_2 \in W^{1,p}_0(\Omega)$ such that

$$ R = \text{div} Y_2. $$

By the Sobolev imbedding, $Y_2 \in C^0(\bar{\Omega})$ and $Y = Y_1 + Y_2$ satisfies all the required properties. \hfill \Box

**8. Estimation of the phase in $H^{1/2} + W^{1,1}$. Proof of Theorem 4**

We return in this last section to the question discussed in the Introduction concerning the control of the phase $\varphi$ in terms of $\|e^{i\varphi}\|_{H^{1/2}}$.

Let $\varphi$ be a smooth real-valued function on $\mathbb{T}^d$ and set $g = e^{i\varphi}$. The main result is the estimate

$$ (8.1) \quad \|\varphi\|_{H^{1/2} + W^{1,1}} \leq C(d)(1 + \|g\|_{H^{1/2}})\|g\|_{H^{1/2}}. $$

Write $g$ as a Fourier series

$$ g = \sum_{\xi \in \mathbb{Z}^d} \hat{g}(\xi)e^{ix\xi}. $$
The $H^{1/2}$-component in the decomposition of $\varphi$ will be obtained as a paraproduct of $g$ and $\bar{g}$.

\begin{equation}
P = \sum_k \left[ \sum_{\xi_2} \lambda_k(|\xi_2|) \bar{g}(\xi_2) e^{-i\mathbf{x}\cdot \xi_2} \right] \left[ \sum_{2^k \leq |\xi_1| < 2^{k+1}} \hat{g}(\xi_1) e^{ix\xi_1} \right],
\end{equation}

where for each $k$ we let $0 \leq \lambda_k \leq 1$ be a smooth function on $\mathbb{R}^+$:

\begin{center}
\begin{tikzpicture}
  \draw[->] (2^k-2) -- (2^k+1); % Intended to be a line
  \draw (2^k) -- (2^k-2); % Intended to be a line
  \node at (2^k+1) {$1$}; % Intended to be a node
  \node at (2^k-2) {$1/2$}; % Intended to be a node
\end{tikzpicture}
\end{center}

We claim that
\begin{equation}
\|P\|_{H^{1/2}} \leq C \|g\|_{\infty} \|g\|_{H^{1/2}}
\end{equation}
and
\begin{equation}
\|\varphi - \frac{1}{i} P\|_{W^{1,1}} \leq C \|g\|_{H^{1/2}}^2.
\end{equation}

**Proof of (8.3).** This is totally obvious from the construction
\begin{equation}
\|P\|_{H^{1/2}}^2 \sim \sum_k 2^k \left[ \sum_{\xi_2} \lambda_k(|\xi_2|) \bar{g}(\xi_2) e^{-i\mathbf{x}\cdot \xi_2} \right] \left[ \sum_{2^k \leq |\xi_1| < 2^{k+1}} \hat{g}(\xi_1) e^{ix\xi_1} \right] \leq \sum_k 2^k \left[ \sum_{\xi_2} \lambda_k(|\xi_2|) |\bar{g}(\xi_2)| |\hat{g}(\xi_1)| e^{-i\mathbf{x}\cdot \xi_2} \right] \left[ \sum_{|\xi_1| \sim 2^k} |\hat{g}(\xi)|^2 \right] \leq C \|g\|_{\infty}^2 \|g\|_{H^{1/2}}^2.
\end{equation}

**Proof of (8.4).** We estimate for instance
\begin{equation}
\|\partial_1 \varphi - \frac{1}{i} \partial_1 P\|_{L^1}.
\end{equation}

Thus, letting $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{Z}^d$,
\begin{equation}
\partial_1 \varphi = \frac{1}{i} \partial_1 \bar{g} \cdot g = \sum_{\xi_1, \xi_2 \in \mathbb{Z}^d} \xi_1^1 \bar{g}(\xi_1) \bar{g}(\xi_2) e^{ix\cdot (\xi_1 - \xi_2)}
\end{equation}
and by (8.2)
\begin{equation}
\frac{1}{i} \partial_1 P = \sum_k \sum_{2^k \leq |\xi_1| < 2^{k+1}, \xi_2} (\xi^1_1 - \xi^1_2) \lambda_k(|\xi_2|) \bar{g}(\xi_1) \bar{g}(\xi_2) e^{ix\cdot (\xi_1 - \xi_2)},
\end{equation}
\begin{equation}
\partial_1 \varphi - \frac{1}{i} \partial_1 P = \sum_k \sum_{2^k \leq |\xi_1| < 2^{k+1}, \xi_2} m_k(\xi_1, \xi_2) \bar{g}(\xi_1) \bar{g}(\xi_2) e^{ix\cdot (\xi_1 - \xi_2)},
\end{equation}
where by definition of $\lambda_k$
\begin{equation}
m_k(\xi_1, \xi_2) = \xi^1_1 - \lambda_k(|\xi_2|)(\xi^1_1 - \xi^1_2) = \begin{cases} \xi^1_1 & \text{if } |\xi_2| \leq 2^{k-2}, \\ \xi^1_1 & \text{if } |\xi_2| \geq 2^{k-1}. \end{cases}
\end{equation}
Estimate
\[
\|\partial_1 \varphi - \frac{1}{i} \partial_1 P\|_1 \leq \sum_{k_1, k_2} m_{k_1}(\xi_1, \xi_2) \left| g(\xi_1) \overline{g(\xi_2)} e^{ix(\xi_1 - \xi_2)} \right|_1.
\]

Distinguish the contributions of
\[
\sum_{k_1 \sim k_2} + \sum_{k_1 < k_2 - 4} + \sum_{k_1 > k_2 + 4} = (8.12) + (8.13) + (8.14).
\]

Clearly \(2^{-k} m_k(\xi_1, \xi_2)\) restricted to \([|\xi_1| \sim 2^k] \times [|\xi_2| \sim 2^k]\) is a smooth multiplier satisfying the usual derivative bounds. Therefore
\[
(8.15) \quad (8.12) \leq C \sum_k 2^{\frac{k_1}{2}} \left( \sum_{|\xi_1| \sim 2^k} \left| \frac{g(\xi_1)}{g(\xi_2)} e^{ix(\xi_1 - \xi_2)} \right|_2 \right) \sim \|g\|_{H^{1/2}}^2.
\]

If \(k_1 < k_2 - 4\), then \(|\xi_2| > 2^{k_1}\) and \(m_{k_1}(\xi_1, \xi_2) = \xi_1^1\) by (8.10). Therefore
\[
(8.13) = \sum_{k_1 < k_2 - 4} \left( \sum_{|\xi_1| \sim 2^k} \left| \frac{g(\xi_1)}{g(\xi_2)} e^{ix(\xi_1 - \xi_2)} \right|_2 \right) \sim \|g\|_{H^{1/2}}^2.
\]

If \(k_1 > k_2 + 4\), then \(|\xi_2| < 2^{k_1 - 2}\) and \(m_{k_1}(\xi_1, \xi_2) = \xi_2^1\) and the bound on (8.14) is similar.

\[
\text{REFERENCES}
\]

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