1. Introduction

It has long been realized that the presence of a Reebless foliation in a compact 3-manifold $M$ reveals useful topological information about $M$. By Novikov [No65], $M$ is irreducible with infinite fundamental group. By Palmeira [Pa78], $M$ has universal cover $\mathbb{R}^3$. Building on work of Thurston and Gabai and Kazez [Ga98, GK98], Calegari [Ca] has shown that if $M$ is also atoroidal, then $\pi_1(M)$ is Gromov negatively curved. Furthermore, Thurston has proposed an approach to demonstrating geometrization for such $M$. Many 3-manifolds contain Reebless foliations, and it has often been conjectured that all closed hyperbolic 3-manifolds do. (It is our impression that for many years Hatcher provided the sole voice of dissent.) In this paper, we give the first examples of closed hyperbolic 3-manifolds which contain no Reebless foliation.

**Theorem A.** There exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain a Reebless foliation.

In particular, therefore, there exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain an Anosov flow.

In 1989, Gabai and Oertel [GO89] introduced the notion of essential lamination. Empirically, these objects seemed easier to find than Reebless foliations, but it was not known whether or not essential laminations were in fact more prevalent. See [Br93, Cl91, RS01] and [BNR] for related results. In this paper we give the first examples (again, an infinite family) of closed hyperbolic 3-manifolds which do not contain a Reebless foliation but which do contain essential laminations.

**Theorem C.** There exist infinitely many closed orientable hyperbolic 3-manifolds which contain neither a Reebless foliation nor a transversely oriented essential lamination but which do contain essential laminations.

In contrast, as discussed in Section 2, we expect that a subset of the set of examples of Theorem A will lead also to examples of closed orientable hyperbolic
3-manifolds which contain no essential lamination. Indeed, Fenley has recently announced a proof of this claim.

We establish nonexistence of Reebless foliations and transversely oriented essential laminations by proving the nonexistence of nontrivial fundamental group actions on leaf spaces. Let $\Lambda$ be any Reebless foliation in $M$. Denote its lift to the universal cover $\tilde{M}$ of $M$ by $\tilde{\Lambda}$. Then the leaf space (see Section 6) of $\tilde{\Lambda}$, $T_{\tilde{\Lambda}}$, is a second countable but not necessarily Hausdorff simply-connected 1-manifold, and the action of $\pi_1(M)$ on $\tilde{M}$ induces a nontrivial action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ by homeomorphisms ([HR57], [Pa78]; see also [Ba98], [CC]). (An action of a group $G$ on a topological space $X$ is called trivial if there is an $x \in X$ such that for all $g \in G$, $x$ and $xg$ are not separated in $X$.) We obtain Theorem A by describing an infinite family of closed hyperbolic 3-manifolds whose fundamental groups do not act nontrivially on simply-connected (second countable but not necessarily Hausdorff) 1-manifolds. More generally, we investigate group actions on $\mathbb{R}$-order trees [GK97].

Let $\Lambda$ be any essential lamination in $M$, and denote by $\tilde{\Lambda}$ its lift to $\tilde{M}$. Then the leaf space (as defined in Section 6) of $\tilde{\Lambda}$, $T_{\tilde{\Lambda}}$, is an $\mathbb{R}$-order tree, and the action of $\pi_1(M)$ on $\tilde{M}$ induces a nontrivial action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ by homeomorphisms [GO89], [GK97]. Using the same set of examples, but instead ruling out nontrivial orientation preserving actions by the fundamental groups on $\mathbb{R}$-order trees, we obtain Theorem C.

Here is a brief outline of the structure of the paper. In Section 2, we describe the family of examples of Theorems A and C. They form a subset of a family of examples proposed by Hatcher [Ha92]. In Section 3, we begin by examining the case that the simply-connected 1-manifold is $\mathbb{R}$. In Section 4, we build on a paper of Barbot [Ba98] and also the well-known work on isometric actions on real trees to investigate actions on non-Hausdorff 1-manifolds. In Section 5, we pass from simply-connected 1-manifolds to the more general world of $\mathbb{R}$-order trees. In particular, we obtain the following results.

**Corollary 5.7.** $G$ acts nontrivially on an $\mathbb{R}$-order tree, then $G$ acts nontrivially on a Hausdorff $\mathbb{R}$-order tree.

**Proposition 5.10.** If $G$ acts nontrivially on an oriented $\mathbb{R}$-order tree by orientation preserving order tree automorphisms, then $G$ acts nontrivially on a simply-connected 1-manifold (by orientation preserving homeomorphisms).

In Section 6, we recall the definition of leaf space and relate the existence of Reebless foliations (and essential laminations) to the existence of actions on simply-connected 1-manifolds ($\mathbb{R}$-order trees). In Sections 7 and 8, we prove the nonexistence of group actions for the examples. Finally, in the appendix, we make precise the notion of Denjoy blow-up for simply-connected 1-manifolds and order trees.

We note that recently Calegari and Dunfield [CD03] announced that they too can generate examples of closed hyperbolic 3-manifolds containing no taut foliation. Their approach is also via group actions but from a different viewpoint. They obtain their examples by using their result that any atoroidal 3-manifold with a taut foliation has a finite abelian cover whose fundamental group is left-orderable.

2. The examples

Once-punctured torus bundles over $S^1$,

$$M_\phi = (F \times I)/\phi$$
Now let $\Lambda_s$ be Theorem 5.3 of [GO89], $\Lambda_\rho$, $\gamma$ is a nonzero even integer, the suspension laminations extend to taut foliations exactly when they are transversely orientable (cf. [Ga97]).

Furthermore, as noted by Thurston, Fried and Ghys, these suspension laminations are essential in the closed manifolds $\hat{\mathcal{M}}$ and $\mathcal{M}$ is hyperbolic and orientable.

Recall that a slope is an isotopy class of unoriented simple closed curves in $\partial M$. Let $|\langle \zeta, \eta \rangle|$ denote the absolute value of the homological intersection number of representatives of slopes $\zeta$ and $\eta$. Note that this number is well defined even though the homology classes of $\zeta$ and $\eta$ are defined only up to $\pm 1$.

Essential surfaces in $\hat{\mathcal{M}}$ are classified in [FHS2] and [CJR]. In particular, it follows from these classifications that there are essential surfaces in $\hat{\mathcal{M}}(\rho)$ for at most finitely many $\rho$. Fixing a Riemannian metric on $F$ and choosing the corresponding pseudo-Anosov representative for $\phi$ [Th88], let $f^s$ and $f^u$ denote, respectively, the stable and unstable laminations fixed by $\phi$. Let $\gamma$ denote the isotopy class of a closed orbit of the pseudo-Anosov flow of $\phi$ restricted to $\partial M$. Choose transverse orientations for $f^s$ and $f^u$. Notice that

- $\phi : f^s \to f^s$ preserves the transverse orientation iff $\text{trace}(\phi_\gamma) > 2$ iff $|\gamma \cap \partial F| = 1$.
- $\phi : f^s \to f^s$ reverses the transverse orientation iff $\text{trace}(\phi_\gamma) < -2$ iff $|\gamma \cap \partial F| = 2$.

Now let $\Lambda^s = (f^s \times I)/\phi$ and $\Lambda^u = (f^u \times I)/\phi$ denote the suspension laminations. Notice that $\Lambda^s$ and $\Lambda^u$ are transversely oriented if and only if $\text{trace}(\phi_\gamma) > 2$. By Theorem 5.3 of [OS9], $\Lambda^s$ and $\Lambda^u$ are essential in $\hat{\mathcal{M}}(\rho)$ for all $\rho$ not isotopic to $\gamma$ when $|\gamma \cap \partial F| = 1$, and for all $\rho$ satisfying $|\langle \rho, \gamma \rangle| \geq 2$ when $|\gamma \cap \partial F| = 2$. Furthermore, as noted by Thurston, Fried and Ghys, these suspension laminations extend to taut foliations exactly when they are transversely orientable (cf. [Ga97]). Namely, when $|\gamma \cap \partial F| = 1$ and $\rho$ is not isotopic to $\gamma$, and when $|\gamma \cap \partial F| = 2$ and $\langle \rho, \gamma \rangle$ is a nonzero even integer, the suspension laminations extend to taut foliations in $\hat{\mathcal{M}}(\rho)$. Otherwise, they do not.

There exists a family of taut foliations discovered by Hatcher [Ha92]. If $|\gamma \cap \partial F| = 1$, then $\hat{\mathcal{M}}(\rho)$ contains taut foliations transverse to the pseudo-Anosov flow inherited from $\mathcal{M}$ for all slopes $\rho$ not isotopic to $\gamma$. If $|\gamma \cap \partial F| = 2$, the situation is again a little more complicated to describe. When $|\gamma \cap \partial F| = 2$, there exist
exactly two slopes, $\mu_1, \mu_2$ say, determined by the intersection number conditions $|\langle \mu_i, \partial F \rangle| = 1$ and $|\langle \mu_i, \gamma \rangle| = 2$. Fixing a basis on $\partial M$ yields a canonical identification of the set of slopes with $\mathbb{Q} \cup \{\infty\}$, which in turn embeds as a dense subset of $S^1$. We can therefore think of the boundary slopes $\{\mu_1, \mu_2\}$ as disconnecting $S^1$ into two open subintervals; denote by $\langle \mu_1, \mu_2 \rangle$ the interval which does not contain $\gamma$. Then $\hat{M}_\phi(\rho)$ contains taut foliations transverse to the pseudo-Anosov flow inherited from $M_\phi$ for all slopes $\rho$ in the interval $\langle \mu_1, \mu_2 \rangle$.

Up to minor modifications, we have just listed all essential laminations known to exist in manifolds $\hat{M}$.

**Question** (Hatcher [Ha92]). Is this list complete?

We are almost ready to describe the examples considered in this paper. First however we must fix a coordinate system on $\partial M$. As is standard, we describe a coordinate system on $\partial M$ by specifying two oriented simple closed curves, called the longitude, $\lambda$, and the meridian, $\mu$, respectively, and satisfying $\langle \lambda, \mu \rangle = 1$. Given any essential simple closed curve $\gamma$ in $T$, we define

$$\text{slope } \gamma = \langle \gamma, \lambda \rangle \langle \mu, \gamma \rangle.$$

(See, for example, [Ro77], p. 259.) Note that the slope of $\lambda$ is therefore $1/1$, the slope of $\mu$, $1/1$. We follow convention and set $\lambda = \partial F$, with the orientation inherited from $F$. When $|\langle \gamma, \partial F \rangle| = 1$, we choose $\mu = \gamma$. Otherwise, $|\langle \gamma, \partial F \rangle| = 2$ and we choose $\mu$ so that $\gamma$ has slope $\frac{1}{1}$. Let $\rho$ have slope $\frac{p}{q}$. Note that if $\text{trace}(\phi_\rho) > 2$, then $|\langle \rho, \gamma \rangle| = 2|q|$, and if $\text{trace}(\phi_\rho) < -2$, then $|\langle \rho, \gamma \rangle| = |p - 2q|$.

We can now summarize the existence results described above as follows.

1. $\hat{M}_\phi(\frac{p}{q})$ contains an essential surface for at most finitely many $\frac{p}{q}$.
2. $\hat{M}_\phi(\frac{p}{q})$ contains a taut foliation if one of the following is true:
   - $\text{trace}(\phi_\rho) > 2$ and $\frac{p}{q} \neq \frac{1}{1}$.
   - $\text{trace}(\phi_\rho) < -2$ and $\frac{p}{q} \in (-\infty, 1)$.
   - $\text{trace}(\phi_\rho) < -2$ and $p$ is even.
3. $\hat{M}_\phi(\frac{p}{q})$ contains a transversely oriented essential lamination if one of the following is true:
   - $\text{trace}(\phi_\rho) > 2$ and $\frac{p}{q} \neq \frac{1}{1}$.
   - $\text{trace}(\phi_\rho) < -2$ and $\frac{p}{q} \in (-\infty, 1)$.
4. $\hat{M}_\phi(\frac{p}{q})$ contains an essential lamination if one of the following is true:
   - $\text{trace}(\phi_\rho) > 2$ and $\frac{p}{q} \neq \frac{1}{1}$.
   - $\text{trace}(\phi_\rho) < -2$ and $|p - 2q| \geq 1$.

Next we fix a standard group presentation for $\pi_1(\hat{M}_\phi(\frac{p}{q}))$. Isotope $\mu$ as necessary so that $|\lambda \cap \mu| = 1$ and set $\{x_0\} = \lambda \cap \mu$. Let $t = [\mu] \in \pi_1(\hat{M}_\phi(\frac{p}{q}), x_0)$, and choose a basis $a, b$ for $\pi_1(F, x_0)$. Let $\phi_\ast : \pi_1(F, x_0) \to \pi_1(F, x_0)$ be the map induced by $\phi : F \to F$. Then $\pi_1(\hat{M}_\phi(\frac{p}{q}))$ has group presentation

$$\langle a, b, t | a^t = a \delta_\ast, b^t = b \delta_\ast, t^p \rangle = 1,$$
where we use the notation $g^h := h^{-1}gh$ and $[g, h] := ghg^{-1}h^{-1}$. In this paper, we pass to the subset of these examples satisfying

$$\phi_\mathbb{Z} = \begin{bmatrix} m & 1 \\ -1 & 0 \end{bmatrix}.$$  

(To view these manifolds in an alternate context, namely, as surgeries on the Whitehead link, see [HMW].) Now for integers $p, q, m$ with gcd$(p, q) = 1$, define $G(p, q, m)$ to be the group generated by $t, a$ and $b$ subject to the relations

1. \[(R1) a^t = aba^{m-1},\]
2. \[(R2) b^t = a^{-1},\]
3. \[(R3) t^p [a, b]^q = 1.\]

Note that relations (R1) and (R2) imply that $t [a, b] = [a, b] t$.

Since by Nielsen [N], $Aut^+(F) \cong \{ f \in Aut(\pi_1(F)) | f \text{ fixes } [a, b] \} \cong SL_2(\mathbb{Z})$, we conclude that $\pi_1(\tilde{M}_\phi(p, q)) \cong G(p, q, m)$.

Now let $T$ be a simply-connected (second countable but not necessarily Hausdorff) 1-manifold. Since $T$ is a simply-connected 1-manifold, it possesses exactly two orientations. Orient $T$ and let $\text{Homeo}^+(T)$ be the subgroup of $\text{Homeo}(T)$ consisting of the orientation preserving homeomorphisms of $T$.

**Convention.** Throughout this paper, we assume that all group actions on all sets are from the right. This includes the action of $\text{Homeo}(X)$ on $X$ for any space $X$.

So we are interested in continuous (right) actions of $G(p, q, m)$ on $T$, that is, homomorphisms $\Phi$ from $G(p, q, m)$ to the group $\text{Homeo}(T)$ of homeomorphisms of $T$. As noted in the introduction, we say that a subgroup $H$ of $\text{Homeo}(T)$ has a global fixed point, or that $H$ acts trivially on $T$, if there is some $x \in T$ such that $xh, x$ are nonseparated in $T$ for all $h \in H$. We will prove the following result.

**Theorem 2.1.** Suppose $m, p, q$ are integers satisfying $m < -2$, $p \geq q \geq 1$, and $(p, q) = 1$. Suppose further that both $m$ and $p$ are odd. Then the image of any homomorphism $\Phi : G(p, q, m) \to \text{Homeo}^+(T)$ has a global fixed point.

**Proof.** This is proved in Sections 3, 7 and 8. \qed

Since the commutator quotient $H_1(G(p, q, m))$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_{|m - 2|}$, the restriction to orientation preserving homeomorphisms is no restriction at all when $p$ and $m$ are both odd, as in this case $\mathbb{Z}_2$ is not a quotient of $G$. In fact, slightly more is true.

**Lemma 2.2.** Let $X$ be any oriented manifold and let

$$\Psi : G(p, q, m) \to \text{Homeo}(X)$$

be any homomorphism.

1. If $m$ is odd, then $\Psi(a), \Psi(b) \in \text{Homeo}^+(X)$.
If \( p \) is odd, then \( \Psi(t) \in \text{Homeo}^+(X) \).

Proof. Note we first that \([\text{Homeo}(X) : \text{Homeo}^+(X)] \leq 2\). So in particular, \([\text{Homeo}(X), \text{Homeo}(X)] \subseteq \text{Homeo}^+(X)\), and any even power of any homeomorphism is in \( \text{Homeo}^+(X) \). Now these two facts together with relation \((R1)\) imply that when \( m \) is odd, we have \( \Psi(b) \in \text{Homeo}^+(X) \). But relation \((R2)\) guarantees that \( \Psi(b) \in \text{Homeo}^+(X) \) if and only if \( \Psi(a) \in \text{Homeo}^+(X) \), establishing the first claim of the lemma. Now if \( p \) is odd, we have \( \Psi(t) \in \text{Homeo}^+(X) \) if and only if \( \Psi(t^p) \in \text{Homeo}^+(X) \). So relation \((R3)\) and the fact that \([a, b] \in \text{Homeo}^+(X)\) imply the second claim. \( \square \)

Thus Theorem 2.1 gives the following result.

**Corollary 2.3.** Suppose \( m, p, q \) are integers satisfying \( m < -2 \), \( p \geq q \geq 1 \), and \( (p, q) = 1 \). Suppose further that both \( m \) and \( p \) are odd. Then the image of any homomorphism \( \Phi : G(p, q, m) \to \text{Homeo}(T) \) has a global fixed point.

**Theorem A.** There exist infinitely many closed hyperbolic 3-manifolds which do not contain a Reebless foliation.

Proof. As noted in the introduction, if \( M \) contains a Reebless foliation, then \( \pi_1(M) \) acts nontrivially on a simply-connected 1-manifold \([HR57, Pa78]\). A simple proof is as follows. Since \( M \) is hyperbolic, it is taut \([Go75]\). In particular, there is a homotopically nontrivial simple closed curve transverse to every leaf. This implies that the orbit of every leaf in the universal cover contains a pair of leaves which are joined in \( T \) by an embedded interval; in particular, the orbit contains separated leaves. So the action is nontrivial.

Theorem A therefore follows immediately from Corollary 2.3 as soon as we show that the set
\[
\mathcal{M} = \left\{ \hat{M}_\rho \left( \frac{p}{q} \right) | m < -2 \text{ is odd, } p \text{ is odd, } (p, q) = 1, \frac{p}{q} \geq 1 \right\}
\]
contains infinitely many distinct hyperbolic 3-manifolds. As we noted above, Thurston’s Hyperbolic Dehn Surgery Theorem \([Th82]\) guarantees that for any given \( m \), \( \hat{M}_\rho(\rho) \) is hyperbolic for all except possibly finitely many \( \frac{p}{q} \). Furthermore, since
\[
H_1(\hat{M}_\rho(\rho)) = \mathbb{Z}/p \oplus \mathbb{Z}/(m - 2),
\]
it follows that the set \( \mathcal{M} \) is infinite. \( \square \)

**Theorem 5.10.** Any nontrivial orientation preserving action on an oriented \( \mathbb{R} \)-order tree \( T_0 \) canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold \( X \).

Proof. This is proved in Section 5. \( \square \)

From Corollary 2.3 we therefore also obtain

**Theorem B.** There exist infinitely many closed hyperbolic 3-manifolds which do not contain a transversely oriented essential lamination.

Furthermore, infinitely many of the manifolds in \( \mathcal{M} \) do contain essential laminations: the essential laminations \( \Lambda^s \) and \( \Lambda^u \).
Theorem C. There exist infinitely many closed hyperbolic 3-manifolds which contain neither a Reebless foliation nor a transversely oriented essential lamination, but which do contain essential laminations.

On the other hand, when $|p - 2q| = 1$ and $m < -2$, there are no known essential laminations in $\hat{M}_\phi(\frac{p}{q})$. In fact, restricting to these cases and adding the condition that $m$ be odd, we conjecture that there exist no nontrivial actions of such $\pi_1(\hat{M}_\phi(\frac{p}{q}))$ on $\mathbb{R}$-order trees, and hence that there are infinitely many hyperbolic 3-manifolds which contain no essential lamination. As noted in the introduction, Fenley has announced a proof of this conjecture, without the condition that $m$ be odd, for $m < -3$.

We also turn our attention to $\mathbb{R}$-covered foliations. When $T = \mathbb{R}$, it is possible to make slightly stronger statements.

Proposition 3.1. If $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$, then the image of any homomorphism $\Phi : G(p, q, m) \to \text{Homeo}^+(\mathbb{R})$ is trivial.

Corollary 2.4. If $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$, then $\hat{M}_\phi(\frac{p}{q})$ contains no transversely oriented $\mathbb{R}$-covered foliations.

Corollary 3.2. If $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$, and $m, p$ are both odd, then the image of any homomorphism $\Phi : G(p, q, m) \to \text{Homeo}(\mathbb{R})$ is trivial.

Corollary 2.5. If $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$, and both $p$ and $m$ are odd, then $\hat{M}_\phi(\frac{p}{q})$ contains no $\mathbb{R}$-covered foliations.

Finally, we note that in [RSS], we examine all $\phi_\sharp \in SL_2(\mathbb{Z})$ with odd negative trace. By working with a standard normal form for $\phi_\sharp$, we obtain the conclusions of Proposition 3.1, Corollary 3.2, Corollary 2.4, and Corollary 2.5 for this larger family. We suspect that the conclusion of Theorem 2.1 is also true. So our restriction to $\phi$ satisfying

$$
\phi_\sharp = \begin{bmatrix} m & 1 \\ -1 & 0 \end{bmatrix}
$$

seems likely to be merely a convenience. On the other hand, we have yet really to understand the condition that $m$ be odd. Notice that in contrast with the condition that $p$ be odd, which is topologically necessary (since when $p$ is even, $M$ contains a Reebless foliation), the role of $m$ odd is still unclear. The condition that $m$ be odd does seem to be important in our proof of Theorem 2.1.

Question. Is the condition that $m$ be odd necessary to rule out nontrivial group actions? If yes, is the condition that $m$ be odd necessary to rule out existence of Reebless foliations?

3. The case $T = \mathbb{R}$

As a warm-up and for those readers primarily interested in actions on $\mathbb{R}$, we first prove

Proposition 3.1. If $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$, then the image of any homomorphism $\Phi : G(p, q, m) \to \text{Homeo}^+(\mathbb{R})$ is trivial.

As noted in Section 2, the restriction to orientation preserving homeomorphisms is no restriction at all when $p$ and $m$ are both odd. Thus Proposition 3.1 gives the following result.
Corollary 3.2. If \( m < 0 \) and \( p \geq q \geq 1 \), \( (p, q) = 1 \), and \( m, p \) are both odd, then the image of any homomorphism \( \Phi : G(p, q, m) \to \text{Homeo}(\mathbb{R}) \) is trivial.

Our proof of Proposition 3.1 is by contradiction; so we assume that there is some nontrivial homomorphism \( \Phi : G(p, q, m) \to \text{Homeo}^+(\mathbb{R}) \). Note that if every representation describes an action with global fixed point, then necessarily every representation is trivial, by the following argument. Fix a nontrivial homomorphism \( \phi : G \to \text{Homeo}^+(\mathbb{R}) \) and observe that \( F := \{ x \mid x\phi(\gamma) = x \text{ for each } \gamma \in G \} \) is a closed, proper subset of \( \mathbb{R} \). Each component of the nonempty set \( \mathbb{R} \setminus F \) is homeomorphic to \( \mathbb{R} \) and is invariant under the given action. Furthermore, by construction the action on each component has no global fixed point and is orientation preserving.

So we may equivalently assume that there is some representation describing an action with no global fixed point.

Set
- \( \tau := \Phi(t) \),
- \( \alpha := \Phi(a) \),
- \( \beta := \Phi(b) \), and
- \( \gamma := \Phi([a, b]) \).

Note that \( \gamma = \alpha \beta \alpha^{-1} \beta^{-1} \). Using the relations which define \( G(p, q, m) \), we see that we have

\[
\begin{align*}
(A) & \quad \tau^{-1} \alpha \tau = \alpha \beta \alpha^{-1}, \\
(B) & \quad \tau^{-1} \beta \tau = \alpha^{-1}, \\
(C) & \quad \tau^p = \gamma^{-q}, \text{ and} \\
(D) & \quad \tau \gamma = \gamma \tau.
\end{align*}
\]

Condition (B) guarantees that the image of \( \Phi \) is generated by both \( \{\tau, \alpha\} \) and \( \{\tau, \beta\} \), and the next lemma follows.

Lemma 3.3. There is no \( x \in \mathbb{R} \) which is fixed by \( \tau \) and at least one of \( \alpha, \beta \).

Lemma 3.4. Let \( g, h \) be elements of a group \( G \) such that \( gh = hg \) and such that there exist relatively prime integers \( p, q \) with \( g^p = h^{-q} \). Then there is some \( k \in G \) such that \( g = k^q \) and \( h = k^{-p} \).

Proof. Take integers \( r, s \) with \( rp + sq = 1 \) and verify that \( k = g^r h^{-s} \) has the desired properties. \( \square \)

We conclude that there is some \( \kappa \in \text{Image}(\Phi) \) such that:

\[
\begin{align*}
(E1) & \quad \tau = \kappa^q, \\
(E2) & \quad \gamma = \kappa^{-p}.
\end{align*}
\]

Another relation which will be used repeatedly and without reference is:

\[
(F) \quad \tau^{-1} \alpha \tau = \alpha \beta \alpha^{-1} = \gamma \beta \alpha^m.
\]

And finally, we highlight the following elementary but key fact:

If \( g \) is orientation preserving and \( x < y \), then \( xg < yg \).

3.1. A general lemma for posets. Our proof of Proposition 3.1 involves examining the fixed point sets of \( \kappa \) and \( \alpha \). The following general lemma about actions of \( G(p, q, m) \) on partially ordered sets (posets) will be of use not only for certain cases in this section, but also when proving Theorem 2.1 for general \( T \). We say that a group \( G \) acts on a poset \( P \) if we have a homomorphism from \( G \) to the group
of order preserving bijections on $P$. Note that in this lemma, the existence of $k$ is guaranteed by Lemma 3.4.

**Lemma 3.5.** Let $G = G(p, q, m)$ act on a partially ordered set $P$. Assume $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$. Let $k \in G$ satisfy $k^q = t$ and $k^{-p} = [a, b]$. If some $x \in P$ satisfies either of the conditions

1. $xk = x$ and $x, xa$ are related in $P$, or
2. $xa = x$ and $x, xk$ are related in $P$,

then $x$ is fixed by every $g \in G$.

**Proof.** Say condition (1) holds, so $xk = x$. Then $xt = x[a, b] = x$. If $xa = x$, then $x$ is fixed by every element of $G$, so assume (for contradiction) that $xa \neq x$. Replacing $P$ with the poset $P^{op}$ (so $y \leq p z$ if and only if $z \leq p^{} y$) if necessary, we may (and do) assume that $xa > x$. Then

$$xba^m = x[a, b]ba^m = xt^{-1}at = xat > xt = x,$$

and since $m < 0$, we have

(3.1) \hspace{1cm} xb > xa^{-m} > x.

However, we have

$$xbt = xt^{-1}bt = x^{-1}a < x,$$

so

(3.2) \hspace{1cm} xb < xt^{-1} = x,$$

and equations (3.1) and (3.2) give the desired contradiction.

Now say condition (2) holds, so $xa = x$. As above, we may assume (for contradiction) that $xk > x$. Then

$$xt^{-1}b = xa^{-1}t^{-1} = xt^{-1}.$$

Since

$$x[a, b] = xk^{-p} \leq xk^{-q} = xt^{-1},$$

we have

$$xt^{-1}at = x[a, b]ba^m \leq xt^{-1}ba^m = xt^{-1}a^m < xa^m = x,$$

so

(3.3) \hspace{1cm} xt^{-1}a < xt^{-1}.$$

On the other hand, we have

$$xba^{-1} = xaba^{-1} = x[a, b]b \leq x^{-1}b = xt^{-1},$$

so

(3.4) \hspace{1cm} xt^{-1}a \geq xb > xt^{-1}b = xt^{-1}.$$

Now equations (3.3) and (3.4) give the desired contradiction. \qed
3.2. Completing the proof of Proposition 3.1. As noted above, a homomorphism from \( G(p, q, m) \) to \( \text{Homeo}^+(\mathbb{R}) \) determines an action of \( G(p, q, m) \) on the poset \( \mathbb{R} \) (with the usual linear order). Since all pairs \( x, y \) of elements of \( \mathbb{R} \) are related in this order, the next result follows immediately from Lemma 3.5.

**Corollary 3.6.** Assume \( p \geq q \geq 1 \), \( (p, q) = 1 \), and \( m < 0 \). If the image of \( \Phi : G(p, q, m) \to \text{Homeo}^+(\mathbb{R}) \) has no global fixed point, then \( \text{Fix}(\kappa) = \text{Fix}(\alpha) = \emptyset \).

We complete the proof of Proposition 3.1 by showing that necessarily \( \text{Fix}(\alpha) \neq \emptyset \).

Suppose, by way of contradiction, that \( \text{Fix}(\alpha) = \emptyset \). By the intermediate value theorem, either \( x_0 < x \) for all \( x \in \mathbb{R} \) or \( x_0 > x \) for all \( x \in \mathbb{R} \). We may orient \( \mathbb{R} \) so that \( x_0 > x \) for all \( x \in \mathbb{R} \). So

\[
(x^{-1})x > (x^{-1})x = x
\]

for all \( x \in \mathbb{R} \). On the other hand, \( x^m = x^{\tau x^{-1} x^{-1}} < x \), which implies \( x^m x^m x^m < x \) for all \( x \in \mathbb{R} \), and hence

\[
(x^{-1})x = x^m x^m x^m = x^m (x^{-1})^m < x
\]

for all \( x \in \mathbb{R} \). Hence, \( \text{Fix}(\alpha) = \emptyset \) is impossible and, necessarily, the image of \( \Phi \) has a global fixed point.

4. Non-Hausdorff 1-manifolds

Let \( T \) be a (path-connected and) simply-connected 1-manifold. We will assume that \( T \) is second countable but not necessarily Hausdorff. Since \( T \) is path-connected, there is a path between any two points. In general, however, unique minimal paths do not exist. Given \( x, y \in T \), we often consider instead the *geodesic spine* \([x, y] = \{ z \in T | x, y \text{ lie in distinct components of } T \setminus \{z\} \} \cup \{x, y\}\) from \( x \) to \( y \) (see p. 563, [Ba98]). Note that \([x, y]\) is the intersection of all paths from \( x \) to \( y \) (Theorem 3.6, [RS01]). Moreover, \([x, y]\) is the union of a finite number \( n \geq 1 \) of disjoint (possibly degenerate) closed intervals (Proposition 2.3, [Ba98])

\([x, y] = [x, y_1] \cup [x_2, y_2] \cup \cdots \cup [x_n, y],\)

where \( y_i \) is not separated from \( x_{i+1} \). (To obtain finiteness, it suffices to note that if \( \rho : [0, 1] \to T \) is any path from \( x \) to \( y \), then \( \rho([0, 1]) \) is compact and hence has a finite open cover by sets homeomorphic to \( \mathbb{R} \).) Let \((x, y), ([x, y]), ([x, y]) \) denote \([x, y] \setminus \{x, y\}, [x, y] \setminus \{y\} \) and \([x, y] \setminus \{x\}\), respectively. As in [Ba98], set

\[d(x, y) = n - 1.\]

Only in exceptional cases is \( d \) a metric. In general, it certainly might be true that \( x \neq y \) but \( d(x, y) = 0 \). For example, if \( T = \mathbb{R} \), then \( d \equiv 0 \). And, in general, only a modified version of the triangle inequality holds. See, for example, Figure 11. (On the other hand, replacing \( d \) by \( d + 1 \) on \( T \times T \setminus \{(x, x) : x \in T\} \) does yield a metric on \( T \).)

**Lemma 4.1.** Let \( x, y, z \in T \). Then

- [Proposition 2.5, [Ba98]]. If \( y \in [x, z] \), then \( d(x, z) = d(x, y) + d(y, z) \).
- \[d(x, z) \leq d(x, y) + d(y, z) + 1.\]
Figure 1. $d(x, z) = 1 > 0 = d(x, y) + d(y, z)$.

**Proof.** If $x$, $y$, and $z$ lie on a common geodesic spine, then $d(x, z) \leq d(x, y) + d(y, z)$. So we may assume that $x$, $y$, and $z$ satisfy one of the configurations of Figure 2 as described in Theorem 3.10, [RS01]. It is easy to verify that in the first three cases, $d(x, z) \leq d(x, y) + d(y, z)$, whereas in the fourth case, $d(x, z) \leq d(x, y) + d(y, z) + 1$ is the best possible. □

We shall call a subset $X$ of $T$ spine-connected if for all $x, y \in X$, $[[x, y]] \subset X$.

**Lemma 4.2.** Suppose $X \subset T$ is spine-connected and $X \subset Y \subset X$. Then $Y$ is spine-connected.

**Proof.** Consider distinct points $y_1, y_2 \in Y$ and let $z \in (y_1, y_2)$. Let $U_1, U_2$ be neighbourhoods of $y_1, y_2$, respectively, which are homeomorphic to $\mathbb{R}$ and which lie in $T \setminus \{z\}$. Since $U_1$ and $U_2$ are separated by $z$ with $U_1 \cap X \neq \emptyset$ and $U_2 \cap X \neq \emptyset$, necessarily $z \in X$. Hence, $[[y_1, y_2]] \subset Y$. □

Since $T$ is simply-connected, it is orientable. Orient $T$ by choosing either one of the two possible orientations.

**Definition 4.3.** (See Section 2.1, [Ba98].) For $x \in T$, let $I_x$ be an open set in $T$ containing $x$ which is homeomorphic (as an oriented manifold) to $\mathbb{R}$ (such an $I_x$ exists since $T$ is an oriented 1-manifold). Let $I_x^+$ be the set of elements of $I_x \setminus \{x\}$ which can be reached from $x$ by walking in the positive direction according to the orientation on $T$; let $I_x^- = (I_x \setminus I_x^+) \setminus \{x\}$. Since $T$ is simply-connected, $T \setminus \{x\}$ has at least two connected components, and since $T$ is a 1-manifold, $T \setminus \{x\}$ has exactly two connected components. Let $x^+$ be the component containing $I_x^+$ and let $x^-$ be the component containing $I_x^-$. We now define a partial order $\leq$ on $T$.

**Definition 4.4.** For $x, y \in T$, we say that

$$x \leq y \iff x^+ \supseteq y^+.$$

Note that for distinct elements $x, y \in T$, $y^+ \subseteq x^+$ if and only if both $y \in x^+$ and $x \in y^-$. A straightforward induction on $d(x, y)$ therefore yields the following:
Lemma 4.5. For all \( x, y \in T \), we have:

- \( x^+ \supseteq y^+ \) if and only if \( x^- \subseteq y^- \).
- \( x, y \) are comparable with respect to \( \leq \) if and only if \( d(x, y) \) is even.

Define a relation \( \sim \) on \( T \) by

- \( x \sim y \) if and only if \( x \) and \( y \) are not separated in \( T \).

Set

\[ [x] = \{ y \in T | y \sim x \} \]

If \( x \sim y \), let \( T_{\{x,y\}} \) denote the submanifold defined as follows:

- if \( x \in y^+ \) (equivalently, if \( y \in x^+ \)), set \( T_{\{x,y\}} = \bigcap_{z \sim x} z^+ \) and \( T_{\{x,y\}} = \bigcap_{z \sim y} z^- \);
- if \( x \notin y^- \) (equivalently, if \( y \notin x^- \)), set \( T_{\{x,y\}} = \bigcap_{z \sim x} z^+ \) and \( T_{\{x,y\}} = \bigcap_{z \sim y} z^- \).

The relation \( \sim \) is reflexive and symmetric, but not necessarily transitive. However, since \( T \) has countable basis, there are at most countably many points \( b \) satisfying \( a \sim b \) and \( b \sim c \) but \( a \not\sim c \). (For an example of such a point \( b \), see Figure 3. For a precise description of this phenomenon, see the appendix or [Ba98].) Hence, by blowing up these countably many points to closed nondegenerate intervals in the spirit of Denjoy [De32, Sc74] (see the appendix for details), we obtain a related simply-connected 1-manifold \( T' \) on which the relation \( \sim \) is transitive, and hence an equivalence relation. Note that all trees \( T' \) obtained in this way are homeomorphic.

Definition 4.6. Let \( T \) be a simply-connected 1-manifold. Form the quotient space

\[ T_H = T' / \sim, \]

where \( T' \) is a 1-manifold obtained from \( T \) as above. Since \( T' \) is uniquely determined up to homeomorphism, so is \( T_H \). Call \( T_H \) the Hausdorff tree associated to \( T \). Let \( p : T \to T_H \) denote the corresponding quotient map.

When proving Theorem 2.1 we will often examine subsets of \( T \) whose images in \( T_H \) are homeomorphic to subintervals of \( \mathbb{R} \). Working in \( T_H \) rather than \( T \) whenever possible allows us to avoid tedious case analyses when examining such subsets of \( T \). For example, notice that if \( [[x,y]] \) is a geodesic spine in \( T \), then \( p([[x,y]]) \subseteq T_H \) is homeomorphic to a closed interval in \( \mathbb{R} \). Other examples of such subsets are bridges, which we now define. If \( X, Y \) are disjoint, nonempty, spine-connected subsets of \( T \), the bridge from \( X \) to \( Y \) is simply the intersection of all paths in \( T \) with one endpoint in \( X \) and the other in \( Y \). Similarly, if \( X, Y \) are disjoint, nonempty connected subsets of \( T_H \), then the bridge from \( X \) to \( Y \) in \( T_H \) is the intersection of all paths in \( T_H \) with one endpoint in \( X \) and the other in \( Y \). Such a bridge in \( T_H \) is always of the form \( [x, y] \) for some \( x \in X \) and \( y \in Y \). Some possible structures at \( [x] \) of a bridge in \( T \) whose image in \( T_H \) is \([x, y]\) are illustrated in Figure 4. (The bridge near \( X \) is represented schematically by the connected vertical segment and \( X \) is represented schematically by the horizontal segments.)
Now suppose that $G$ is any group acting on $T$. Let $g \in G$. Note that if $g$ is orientation preserving (reversing), it preserves (reverses, respectively) the partial order $\leq$ on $T$. As usual, write

$$\text{Fix}(g) = \{x \in T | xg = x\}.$$  

Write

$$\text{Nonsep}(g) = \{x \in T | xg \sim x\},$$

for the set of points not separated by $g$. We shall say that $x \in T$ is a global fixed point for the action of $G$ on $T$ if $xg \sim x$ for all $g \in G$. If there exists a global fixed point, we call the action trivial. By extending linearly over the blown-up intervals, any action of $G$ on $T$ induces an action of $G$ on $T'$. Moreover, the action of $G$ on $T$ is trivial if and only if the induced action of $G$ on $T'$ is trivial.

Without loss of generality therefore, and with gain an increased simplicity of exposition, we make the following assumption throughout the rest of the paper: $\sim$ is transitive on $T$.

Define the characteristic set associated to $g$ by

$$C_g = \{x \in T | d(x, xg) \text{ is even}\}.$$  

Note that in [Ba98], Barbot calls this set the fundamental axis. We will reserve the term axis for the case $\text{Nonsep}(g) = \emptyset$.

**Lemma 4.7** (See also [Ba98, Proposition 2.7(2)]). Let $x \in T$. Then $x \in C_g$ if and only if $x$ and $xg$ are comparable with respect to the partial order $\leq$.

**Proof.** This follows immediately from Lemma 4.5. $\square$

**Proposition 4.8** ([Ba98, Proposition 2.10]). Suppose $\text{Nonsep}(g) = \emptyset$. Then $C_g \neq \emptyset$ and for any $x \in C_g$,

$$C_g = \bigcup_{n \in \mathbb{Z}} [[xg^n, xg^{n+1}]].$$

For an alternate approach to the proof of Proposition 4.8, beginning with the characterization of $C_g$ given in Corollary 4.11 see also Theorem 5.6 [RS01] or Theorem 3.4 [Fe].

**Corollary 4.9.** If $\text{Nonsep}(g) = \emptyset$, then $\text{Nonsep}(g^n) = \emptyset$, for all nonzero integers $n$.

Hence, when $\text{Nonsep}(g) = \emptyset$, $A_g := C_g$ is an axis for $g$ in the spirit of Tits-Serre (Proposition 24, Section 6.4, [Se77]). We note in passing that any fact from the theory of group actions on $\mathbb{R}$-trees which depends only on the combinatorial properties of existence of such axes still holds true in this setting. In fact, existence merely of the characteristic set $C_g$ for arbitrary $g$ is sometimes (although certainly not always) sufficient for the generalization of well-known arguments. (Good surveys
on isometric actions on real trees can be found in [Ch01, Mo92, Pa95, Sh87, Sh91]. See also [CMS74, CV96].

From Proposition 4.8 it follows that in $T_H, p(A_g) \cong \mathbb{R}$, and in $T, either A_g \cong \mathbb{R} or A_g = \bigcup_{i=1}^{\infty} \{x_i, y_i\}$, where $[x_i, y_i]$ is homeomorphic to a closed interval in $\mathbb{R}$, $[x_i, y_i] \cap [x_j, y_j] = \emptyset$ when $i \neq j, x_i \neq y_i$, and $y_i \sim x_{i+1}$ for all $i, j$. In each case, the action of $g$ on $A_g$ is conjugate to an action by translations, and there is a natural linear order $\preceq_g$ on $A_g$ satisfying $x \prec_g xg$ for all $x \in A_g$. (In general, $\preceq_g$ agrees with neither $\leq$ nor the opposite partial order $\geq$.)

Suppose $Y$ is a $g$-invariant embedded copy of $\mathbb{R}$ in $T$ on which $g$ acts freely. Then we call $Y$ a local axis for $g$. Note that if $x$ lies in a local axis for $g$, then $d(x, xg) = 0$.

Now suppose $\text{Nonsep}(g) \neq \emptyset$ and let $T_i, i \in I$, denote the path components of $T \setminus \text{Nonsep}(g)$. Notice that each $T_i$ is path-connected and open. So $T_i \cap T_j = \emptyset$ for all $i \neq j$ and hence $T_i \setminus T_i \subset \text{Nonsep}(g) \subset T_i$ for all $i$. In particular, $T_i \setminus T_i = \emptyset$ only if $\text{Nonsep}(g) = \emptyset$. Notice also that $T_i g = T_j$ for some $j \in I$. When $T_i g = T_j \neq T_i$, then since $T$ is simply-connected, $T_i \setminus T_i \neq \emptyset$ can consist of at most one, and hence exactly one, point $x \in \text{Nonsep}(g)$, and $T_i \setminus T_i$ is the point $xg$, where $xg \sim x$ and $xg \neq x$ (namely, the situation pictured in Figure 5 must hold). In this case, we call $xg \sim x$ the root of $T_i$ in $T$ (and $xg$ the root of $T_i$ in $T$). On the other hand, whenever $T_i g = T_i$, $g$ acts freely on $T_i$, and hence this local action has an axis $A_g^i \subset T_i$.

Using distance to an element of $T_i \cap \text{Nonsep}(g)$, one can check that such an $A_g^i$ is homeomorphic to $\mathbb{R}$ and hence is an example of a local axis for $g$; in fact, all local axes for $g$ arise in this way. We summarize some of these observations in the following lemma.

**Lemma 4.10.** Suppose $\text{Nonsep}(g) \neq \emptyset$. Then:

- $\text{Nonsep}(g) \subset \text{Nonsep}(g^n)$ and hence $C_g \subset C_{g^n}$, for any $n$.
- $C_g = \{x \in T | d(x, xg) = 0\}$.
- $C_g = \text{Fix}(g) \cup \{x \in T | x \text{ lies on a local axis for } g\}$.

**Corollary 4.11.** For any $g \in G$, we have

$$C_g = \{x \in T | x \in [xg^{-1}, xg]\}.$$  

**Proof.** Suppose $x \in T$ satisfies $x \in [xg^{-1}, xg]$. If $d(xg^{-1}, xg) = 0$, then $d(x, xg) = 0$ and so Lemma 4.10 guarantees that $x \in C_g$. So suppose that $d(xg^{-1}, xg) > 0$. By Lemma 4.11

$$d(xg^{-1}, xg) = d(xg^{-1}, x) + d(x, xg) = 2d(x, xg)$$

and so $x \in C_g$. By Lemma 4.10 $\text{Nonsep}(g^2) = \emptyset$, and so $C_{g^2} = A_{g^2}$. But $\text{Nonsep}(g) \subset \text{Nonsep}(g^2) = \emptyset$ and in particular, $x \in C_{g^2} = A_{g^2} = A_g = C_g$.

The reverse inclusion follows immediately from Proposition 4.8 and Lemma 4.10. \qed
Corollary 4.12. If there is some \( x \in T \) such that \( d(x, xg) \neq 0 \) is even, then \( \text{Nonsep}(g) = \emptyset \).

Corollary 4.13. Let \( g \in G \). Then both \( C_g \) and \( C_g \cup \text{Nonsep}(g) \) are spine-connected.

Proof. If \( \text{Nonsep}(g) = \emptyset \), then \( C_g \cup \text{Nonsep}(g) = A_g \) is spine-connected. So we may assume \( \text{Nonsep}(g) \neq \emptyset \). By Lemma 4.2 it suffices to prove that \( C_g \) is spine-connected.

Consider first the special case that \( x \in \text{Fix}(g) \) and \( y \in C_g \). Since \( d(x, y) = d(x, yg) = 0 \), necessarily \( x, y \) and \( yg \) are collinear in \( T \) (i.e., lie on a common geodesic spine). If \( g \) is orientation reversing, then \( [[x, y]] \subset [[y, yg]] = [y, yg] \). And if \( g \) is orientation preserving, then either \( yg \in [[x, y]] \) or \( y \in [[x, y]] \). So either \( [[x, y]]g \subset [[x, y]] \) or \( [[x, y]]g^{-1} \subset [[x, y]] \), and hence, for every \( z \in [[x, y]] \), \( d(z, zg) = 0 \). In either case, \( [[x, y]] \subset C_g \).

Consider next the special case that \( x, y \in C_g \setminus \text{Fix}(g) \). If \( x \) and \( y \) lie in a common component of \( T \) \( \setminus \text{Nonsep}(g) \), then necessarily \( x, y \) lie on a common local axis, and hence \( [[x, y]] = [x, y] \) is also contained in this common local axis (and hence in \( C_g \)). Otherwise, \( x \) and \( y \) are separated by some \( z \in \text{Nonsep}(g) \). Now \( z \in \text{Fix}(g) \) since otherwise \( d(x, xg) \) and \( d(y, yg) \) are odd. So \( [[x, y]] = [[x, z]] \cup [[z, y]] \) is contained in \( C_g \) by the first special case. \( \Box \)

It will sometimes be useful to consider an object obtained by adding one point \( \hat{x} \), called an ideal point of \( T \), to \( T \) for each \( \sim \)-equivalence class \( [x] \) in \( T \) which contains more than one point. This object, denoted by \( \hat{T} \), is called the completion of \( T \).

(Compare with Section 5 of [RS01].) We say that an ideal point \( \hat{x} \) is a source if whenever \( y, z \) are distinct elements of \( [x] \) we have \( y \in z^- \), and we say that \( \hat{x} \) is a sink if whenever \( y, z \) are distinct elements of \( [x] \) we have \( y \in z^+ \). Note that every ideal point \( \hat{x} \) is either a source or a sink. The action of any subgroup of \( \text{Homeo}(T) \) extends to an action on \( \hat{T} \) in the obvious way, that is, we set \( \hat{x}g = \hat{y} \) if \( [x][g] = [y] \).

We want to extend our partial order on \( T \) to \( \hat{T} \) so that group actions on \( \hat{T} \) obtained from orientation preserving actions on \( T \) preserve this extended partial order. For an ideal point \( \hat{x} \), we define

\[
\hat{x}^+ = \begin{cases} \bigcup_{y \in [x]} \{y \cup y^+\}, & \text{\( \hat{x} \) a source,} \\ \bigcap_{y \in [x]} y^+, & \text{\( \hat{x} \) a sink,} \end{cases}
\]

and set

\[
\hat{x}^- = T \setminus \hat{x}^+.
\]

Note that \( \hat{x}^+ \subseteq T \). It is straightforward to show that if \( H \) is any group of orientation preserving homeomorphisms of \( T \), then for \( x, y \in \hat{T} \) and \( h \in H \), we have \( x^+ \subseteq y^+ \) if and only if \( (xh)^+ \subseteq (yh)^+ \). So, we have the following result, which will allow us to invoke Lemma 3.5 more often in the proof of Theorem 2.1 than would be possible without the introduction of \( \hat{T} \).

Proposition 4.14. Define the relation \( \leq \) on \( \hat{T} \) by \( x \leq y \) if \( y^+ \subseteq x^+ \). This relation is a partial order which extends the order \( \leq \) on \( T \) defined in Definition 4.2. In addition, if \( H \leq \text{Homeo}^+(T) \), then the induced action of \( H \) on \( \hat{T} \) is order preserving.
5. $\mathbb{R}$-order trees

Both simply-connected 1-manifolds and their associated Hausdorff trees are special cases of a more general tree-like object, the $\mathbb{R}$-order tree. An order tree $T_{\text{GO89}}$ is a set $T$ together with a collection $S$ of linearly ordered subsets called segments. If $\sigma$ is a segment, then $-\sigma$ denotes the same subset with reverse order. The segments satisfy:

1. Each segment $\sigma$ has distinct least and greatest elements, which we will denote by $i(\sigma)$ and $f(\sigma)$, respectively. (We also write $\sigma = [i(\sigma), f(\sigma)]$.)
2. If $\sigma$ is a segment, so is $-\sigma$.
3. A closed nondegenerate (i.e., containing more than one element) subinterval of a segment is a segment.
4. Given $x, y \in T$, there exists a path from $x$ to $y$; namely, a sequence $\sigma_1, \ldots, \sigma_k$ of segments such that $i(\sigma_1) = x$, $f(\sigma_k) = y$, and $f(\sigma_j) = i(\sigma_{j+1})$ for all $j$.
5. Given a cyclic path $\sigma_0\sigma_1\cdots\sigma_{k-1}$ (where cyclic means $f(\sigma_{k-1}) = i(\sigma_0)$), there is a subdivision of the path $\sigma_0\sigma_1\cdots\sigma_{k-1}$ to a path $\rho_0\cdots\rho_{n-1}$ so that after cancelling all adjacent pairs of the form $(\rho)(-\rho)$, we have the empty sequence.
6. If $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2$ is a segment.

An $\mathbb{R}$-order tree $\text{GO97}$ is an order tree satisfying also:

7. Each segment is order isomorphic to a closed interval in $\mathbb{R}$.
8. $T$ is a countable union of segments.

$T$ is topologized by giving segments the order topology and then declaring a set $U \subset T$ to be open in $T$ if and only if $U \cap \sigma$ is open in $\sigma$ for every segment $\sigma$. Note that defining axiom (4) guarantees that $T$ is path-connected and that defining axiom (5) guarantees that $T$ is simply-connected.

An orientation of an order tree is a choice of a subset $S_+ \subset S$ such that

- $S_+ \cap (-S_+) = \emptyset$, where $-S_+ = \{-\sigma | \sigma \in S_+\}$.
- A closed nondegenerate subinterval of a segment in $S_+$ is in $S_+$.
- Any two elements of $T$ can be joined by a sequence $\sigma_1, \ldots, \sigma_k$ of segments in $S_+ \cup (-S_+)$ such that $f(\sigma_j) = i(\sigma_{j+1})$ for all $j$.
- If $\sigma_1, \sigma_2 \in S_+$, and $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2 \in S_+$.

Since there are no nontrivial cyclic words, orientations always exist. In contrast to the situation when $T$ is a simply-connected 1-manifold and therefore has exactly two orientations, there are generally many possible choices of orientation for an $\mathbb{R}$-order tree. Note that if $S_0$ is a collection of linearly ordered subsets of $T$ such that $S_0 \cup -S_0$ satisfies conditions (1), (4) and (5) of the definition of order tree, then there is a unique smallest set $S$ containing $S_0$ and also satisfying all six defining conditions.

As we will discuss further in Section 6 if $\Lambda$ is an essential lamination in $M$ with no isolated leaves, then its lift to the universal cover of $M$ has leaf space an $\mathbb{R}$-order tree $\text{GO89}$. If $\Lambda$ is a transversely oriented essential lamination with no isolated leaves in $M$, then its lift to the universal cover of $M$ has leaf space an oriented $\mathbb{R}$-order tree.

Now let $T$ be any simply-connected 1-manifold for which $\sim$ is transitive. Let $T_H$ denote the associated Hausdorff tree. It is easy to see that the (oriented) 1-manifold structure on $T$ induces canonical (oriented) $\mathbb{R}$-order tree structures on $T$ and $T_H$.
respectively. Choose either of the two orientations of \( T \). Let \( S_+ = \{ \sigma \subset T | \sigma \) is homeomorphic to a nondegenerate closed interval in \( \mathbb{R} \), with linear order inherited from the orientation of \( T \} \), and let \( S \) be the smallest set containing \( S_+ \) and also satisfying the axioms defining an order tree. Then \( T \) together with \( S \) is an \( \mathbb{R} \)-order tree, and \( S_+ \) is an order tree orientation for \( T \). Next, let \((S_H)_+ = \{ p(\sigma)| \sigma \in S_+ \} \), and let \( S_H \) be the smallest set containing \((S_H)_+ \) and also satisfying the axioms defining an order tree. Then \( T_H \) together with \( S_H \) is an \( \mathbb{R} \)-order tree, and \((S_H)_+ \) is an order tree orientation for \( T_H \). (The first two defining conditions of order tree orientation are clearly satisfied. The third condition follows from the finiteness of \( d(x, y) \) for every pair \( x, y \in T \); namely, from Proposition 2.3 of [Ba98]. The fourth condition follows from the fact that since \( T \) is an oriented 1-manifold, it is not possible to find distinct \( x, y \in T \) and \( \sigma_1, \sigma_2 \in S_+ \), such that \( f(\sigma_1) = x \sim y = i(\sigma_2) \).

So any orientation of \( T \) projects to an orientation of \( T_H \) as an \( \mathbb{R} \)-order tree. In contrast, not all order tree orientations of \( T_H \) lift to orientations of \( T \).

Now let \( T \) be any order tree. A function \( \phi : T \rightarrow T \) is called an \( \text{(order tree) automorphism} \) if \( \phi \) is a set bijection satisfying \( [x, y] \phi = [x \phi, y \phi] \in S \iff [x, y] \in S \).

We say that \( \phi \) is orientation preserving if \( \sigma \phi \in S_+ \iff \sigma \in S_+ \). Set \( \text{Aut}(T) = \{ \phi : T \rightarrow T | \phi \) is an order tree automorphism \} \), and set \( \text{Aut}^+(T) = \{ \phi \in \text{Aut}(T) | \phi \) is orientation preserving \} \). Let \( G \) be any group. A right action of \( G \) on \( T \) as an order tree is a mapping

\[
T \times G \rightarrow T : (x, g) \mapsto xg = x\Phi(g),
\]

for some homomorphism \( \Phi : G \rightarrow \text{Aut}(T) \). An orientation preserving action is an action satisfying \( \Phi(G) \subset \text{Aut}^+(T) \). Now consider the special case that \( T \) is a simply-connected 1-manifold with canonically induced \( \mathbb{R} \)-order tree structure. Then \( \phi \in \text{Aut}(T) \iff \phi \in \text{Homeo}(T) \). So an action of \( G \) on \( T \) as a 1-manifold is also an action of \( G \) on \( T \) as an \( \mathbb{R} \)-order tree, and an action of \( G \) on \( T \) as an \( \mathbb{R} \)-order tree (still with canonically induced \( \mathbb{R} \)-order tree structure) is also an action of \( G \) on \( T \) as a 1-manifold.

Many of the properties of simply-connected 1-manifolds hold true for or generalize to \( \mathbb{R} \)-order trees. (In particular, the notation used in this section for \( \mathbb{R} \)-order trees \( T \) is consistent with the notation used in Section 3 in the special case that \( T \) is a simply-connected 1-manifold.)

Given \( x, y \in T \), we again consider the \text{geodesic spine}

\[
[[x, y]] = \{ z \in T | x, y \text{ lie in distinct components of } T \setminus \{ z \} \} \cup \{ x, y \}
\]

from \( x \) to \( y \). Again, \( [[x, y]] \) is the intersection of all paths from \( x \) to \( y \) (Theorem 3.6, [RS01]). Moreover, \( [[x, y]] \) is the union of a finite number \( n \geq 1 \) of disjoint (possibly degenerate) closed intervals (Axiom (4))

\[
[[x, y]] = [x, y_1] \cup [x_2, y_2] \cup \cdots \cup [x_n, y],
\]

where \( y_i \) is not separated from \( x_{i+1} \). Let \( ([x, y]), [[x, y]], ((x, y)] \) denote \( [[x, y]] \setminus \{ x, y \}, [[x, y]] \setminus \{ y \} \) and \( [[x, y]] \setminus \{ x \} \), respectively. Set

\[
d(x, y) = n - 1,
\]

and note that only a modified version of the triangle inequality holds.

**Lemma 5.1.** Let \( x, y, z \in T \). Then

- If \( y \in [[x, z]] \), then \( d(x, z) = d(x, y) + d(y, z) \).
\[ d(x, z) \leq d(x, y) + d(y, z) + 1. \]

Again, we call a subset \( X \) of \( T \) spine-connected if for all \( x, y \in X \), \( [[x, y]] \subset X \).

Define a relation \( \sim \) on \( T \) by

- \( x \sim y \) if and only if \( x \) and \( y \) are not separated in \( T \).

Set

\[ [x] = \{ y \in T | y \sim x \}. \]

If \( x \sim y \), let \( T_{\{x, y\}} \) denote the component of \( T \setminus [x] \) which has both \( x \) and \( y \) as limit points. Note that alternatively, \( \sim \) can be defined in terms of segments as follows:

\[ x \sim y \text{ if and only if } \sigma_1 \cap \sigma_2 \neq \emptyset \text{ for every pair } \sigma_1, \sigma_2 \in S_+ \text{ such that } x \in \sigma_1 \setminus \{i(\sigma_1), f(\sigma_1)\} \text{ and } y \in \sigma_2 \setminus \{i(\sigma_2), f(\sigma_2)\}. \]

The relation \( \sim \) is reflexive and symmetric, but not necessarily transitive. However, as described in the appendix, there is a naturally associated \( \mathbb{R} \)-order tree \( T' \) on which \( \sim \) is transitive and hence an equivalence relation.

The notions of characteristic set, axis and local axis also generalize to the case of order trees. Define the characteristic set of \( g \) to be

\[ C_g = \{ x \in T | x \in [[x^{-1}g, xg]] \}. \]

Again we note that any fact from the theory of group actions on \( \mathbb{R} \)-trees which depends only on the combinatorial properties of existence of such characteristic sets still holds true in this setting.

**Proposition 5.2** ([RS01, Theorem 5.6]). Suppose \( \text{Nonsep}(g) = \emptyset \). Then \( C_g \neq \emptyset \), with

\[ C_g = \bigcup_i [[x^{-1}g^i, xg^i]] \]

for any \( x \in C_g \).

Hence, when \( \text{Nonsep}(g) = \emptyset \), it again makes sense (in the sense of Tits-Serre) to call \( A_g := C_g \) an axis for \( g \).

And again, by local axis for \( g \) we mean either an axis for \( g \) or, when \( \text{Nonsep}(g) \neq \emptyset \), any subset of \( T \) order isomorphic to \( \mathbb{R} \) on which \( g \) acts freely.

**Lemma 5.3.** Suppose \( \text{Nonsep}(g) \neq \emptyset \). Then

- \( \text{Nonsep}(g) \subset \text{Nonsep}(g^n) \) and hence \( C_g \subset C_{g^n} \), for any \( n \).
- \( C_g = \text{Fix}(g) \cup \{ x \in T | x \text{ lies on a local axis for } g \} \).

**Corollary 5.4.** Let \( g \in G \). Then both \( C_g \) and \( C_g \cup \text{Nonsep}(g) \) are spine-connected.

The proof of Theorem 5.6 of [RS01] also yields the following.

**Proposition 5.5.** Let \( T \) be any \( \mathbb{R} \)-order tree. Let \( x \in T \), \( g \in \text{Homeo}(T) \) with \( xg \neq x \). Suppose further that \( [[x^{-1}g, x]] \cap [[x, xg]] = \{ x \} \). For \( n \in \mathbb{Z} \) set \( I_n = [[xg^n, xg^{n+1}]] \). Then

1. If \( j, k \in \mathbb{Z} \) with \( j < k \), then

   \[ I_j \cap I_k = \begin{cases} \{ xg^k \}, & k = j + 1, \\ \emptyset , & \text{otherwise.} \end{cases} \]

2. \( \bigcup_{n \in \mathbb{Z}} I_n \) is a local axis for \( g \).
Now let $T$ be any oriented $\mathbb{R}$-order tree and let $T_H$ denote its associated (oriented) Hausdorff tree (Definition 5.3).

**Lemma 5.6.** Any nontrivial action of $G$ on $T$ canonically induces a nontrivial action of $G$ on $T_H$.

**Proof.** If $x \sim y$, then $xg \sim yg$ for any $g \in G$. So $[x]g = [xg]$ defines a homeomorphism of $T_H$. This induced action is trivial if and only if the action of $G$ on $T$ is trivial. \qed

Note that with respect to this induced action, $Fix_{T_H}(g) = p(\text{Nonsep}(g))$.

**Corollary 5.7.** If $G$ acts nontrivially on an $\mathbb{R}$-order tree, then $G$ acts nontrivially on a Hausdorff $\mathbb{R}$-order tree.

**Lemma 5.8.** Suppose $T_H$ has orientation inherited from $T$. Then any orientation preserving action on $T_H$ canonically induces an orientation preserving action on $T$. This induced action on $T$ is nontrivial if and only if the given action of $G$ on $T_H$ is nontrivial.

**Proof.** Let $C = \{x \in T| |x| > 1\}$. Since $p : T \to T_H$ restricted to $T \setminus C$ is injective (and $C$ is $G$-invariant), we may set

$$xg = p^{-1}([x]g)$$

for all $x \in T \setminus C$. Now consider any $z \in C$. Since $C$ is countable, $z \in \sigma$ for some segment $\sigma = [x, y]$ with $x, y \in T \setminus C$, and so we may set

$$zg = p^{-1}([z]g) \cap [xg, yg].$$

\qed

Next we introduce a notion of incidence for order trees. Fix an orientation on $T$ and let $x \in T$. Define an equivalence relation $\approx_f$ on the set $S(x, f) = \{\sigma \in S_1 | f(\sigma) = x\}$ by $\sigma_1 \approx_f \sigma_2$ if and only if both $f(\sigma_1) = f(\sigma_2) = x$ and $\{x\} \subseteq \sigma_1 \cap \sigma_2$. For each $\sigma \in S(x, f)$, let $r_{\sigma} = \{r \in S(x, f) | r \approx_f \sigma\}$ and call $r_{\sigma}$ an incoming ray at $x$. Let $R(x, f) = \{r_{\sigma} | \sigma \in S(x, f)\}$. Call $n_f(x) = |R(x, f)|$ the in-degree at $x$. Similarly, define an equivalence relation $\approx_o$ on the set $S(x, o) = \{\sigma \in S_+ | i(\sigma) = x\}$ by $\sigma_1 \approx_o \sigma_2$ if and only if both $i(\sigma_1) = i(\sigma_2) = x$ and $\{x\} \subseteq \sigma_1 \cap \sigma_2$. For each $\sigma \in S(x, o)$, let $r_{\sigma} = \{r \in S(x, o) | r \approx_o \sigma\}$ and call $r_{\sigma}$ an outgoing ray at $x$. Let $R(x, o) = \{r_{\sigma} | \sigma \in S(x, o)\}$. Call $n_o(x) = |R(x, o)|$ the out-degree at $x$. We say that a segment $\sigma$ is incident to $x$ if $\sigma \in S(x, o) \cup S(x, f)$, and we say that a ray $r_{\sigma}$ is incident to $x$ if $r_{\sigma} \in R(x, o) \cup R(x, f)$. Call $x \in T$ a branch point if it is not regular, and let $B$ denote the set of branch points of $T$. Note that if $B = \emptyset$, then $T$ can also be given the structure of a simply-connected 1-manifold.

Now consider any $x \in B$. If the out-degree $n_o(x) = 0$ (in-degree $n_f(x) = 0$), call $x$ a sink (respectively, source). If $n_o(x) = 1$ and $n_f(x) > 1$, call the single element $r_{\sigma} \in R(x, o)$ the distinguished ray at $x$. Symmetrically, if $n_f(x) = 1$ and $n_o(x) > 1$, call the single element $r_{\sigma} \in R(x, f)$ the distinguished ray at $x$.

**Lemma 5.9.** Let $T_0$ be an oriented $\mathbb{R}$-order tree such that at every $x \in B$, there is a distinguished ray. Then any nontrivial orientation preserving action on $T_0$ canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold $X$. 


Proposition 5.10. Any nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree $T_0$ canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold $X$.

Proof. We show that any nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree $T_0$ canonically induces a nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree $T$ such that at every $x \in B$, there is a distinguished ray. Lemma 5.9 then applies.

First, let $D$ denote the set of branch points $x \in T_0$ with both in-degree and out-degree greater than one. Let $T$ denote the linear Denjoy blow-up of $T_0$ along $D$ with respect to the orientation on $T_0$ and extend the action of $G$ to $T$ as described in Section 9.3. Let $B$ now denote the branch points of $T$. Note that if $x \in B$, then either $x$ has a distinguished ray or else it is either a sink or a source.

Finally, we introduce distinguished rays at all sinks and sources in $B$. At every sink $x \in B$, attach a set $\sigma_x$ order isomorphic to $[0, \infty)$ (so that precisely the endpoint of $\sigma_x$ is identified with $x$). Symmetrically, at every source $x \in B$, attach a set $\sigma_x$ order isomorphic to $(-\infty, 0]$ (so that precisely the endpoint of $\sigma_x$ is identified with $x$). Let the set of segments associated to this new tree be the smallest set satisfying the defining axioms and containing $S$, the segments of $T$, together with all nondegenerate subintervals of the $\sigma_x$. Extend the action of $G$ linearly over the sets $\sigma_x$. □

6. Spaces of leaves

Let $M$ be any closed 3-manifold containing an essential lamination $\Lambda$ [GO89]. By Theorem 6.1 of [GO89], the universal cover $\tilde{M}$ of $M$ is homeomorphic to $\mathbb{R}^3$. Lift $\Lambda$ to a lamination $\tilde{\Lambda}$ of $\tilde{M}$. Now define an equivalence relation $\equiv$ on $\tilde{M}$ by $x \equiv y$
if and only if either \( x, y \) lie on a common leaf of \( \tilde{\Lambda} \) or \( x, y \) both lie in the union of some complementary region with its boundary leaves.

\[
T_{\tilde{\Lambda}} = \tilde{M}/\equiv
\]

is called the leaf space of \( \tilde{\Lambda} \). Remark that when \( \Lambda \) is not a foliation and therefore has complementary regions, \( T_{\tilde{\Lambda}} \) is not really the “space of leaves” but rather a natural quotient of this space.

When \( \Lambda \) is a Reebless foliation, \( T_{\tilde{\Lambda}} \) is a second countable but not necessarily Hausdorff simply-connected 1-manifold, and the action of \( \pi_1(M) \) on \( \tilde{M} \) induces a nontrivial action of \( \pi_1(M) \) on \( T_{\tilde{\Lambda}} \) by homeomorphisms ([HR57, Pa78]; see also [Ba98, CC]).

**Proposition 6.1 ([HR57, Pa78]).** If \( M \) contains a Reebless foliation, then \( \pi_1(M) \) acts nontrivially on a simply-connected 1-manifold.

**Corollary 6.2.** If \( M \) contains a Reebless foliation \( \Lambda \) and \( \pi_1(M) \) contains no index two subgroup, then \( \Lambda \) is necessarily transversely orientable, and \( \pi_1(M) \) acts nontrivially on a simply-connected 1-manifold by orientation preserving homeomorphisms.

More generally, when \( \Lambda \) is an essential lamination with no isolated leaves, \( T_{\tilde{\Lambda}} \) is an \( \mathbb{R} \)-order tree [GO89]. Roughly speaking, segments in the \( \mathbb{R} \)-order tree arise from a family of well-chosen transversals \( \tau \) to \( \Lambda \); if \( \Lambda \) (equivalently, \( \Lambda \)) has no isolated leaves, then each \( \Lambda \cap \tau \) is a closed perfect set, and hence, by a devil’s staircase-like argument (cf. [Be99]), \( \tau/\equiv \) is order isomorphic to \( \mathbb{R} \). As remarked in [GK97], if \( M \) contains an essential lamination, then \( M \) contains an essential lamination with no isolated leaves (isolated leaves can simply be replaced by products as described in [Ga92]). On the leaf space level, this replacement of isolated leaves by products results in the Denjoy blow-up operation as defined in the appendix.

Now consider the action of \( \pi_1(M) \) on \( T_{\tilde{\Lambda}} \) induced by the action of \( \pi_1(M) \) on \( \tilde{M} \) by deck transformations. This action has no global fixed point. (See, for example, Proposition 8.1 of [RS01].) Furthermore, if \( \Lambda \) is transversely oriented, then the transverse orientation on \( \Lambda \) lifts to a transverse orientation on \( \tilde{\Lambda} \), and hence induces an orientation on \( T_{\tilde{\Lambda}} \) which is preserved by the action of \( \pi_1(M) \). (Note that if \( \Lambda \) is an essential surface, then by passing to a double cover of \( \Lambda \) as necessary, we may assume that \( \Lambda \) is transversely oriented.)

**Proposition 6.3.** If \( M \) contains an oriented essential lamination \( \Lambda \), then \( \pi_1(M) \) acts nontrivially by orientation preserving order tree automorphisms on an oriented \( \mathbb{R} \)-order tree.

7. **Case I: Nonsep(\( \kappa \)) = \emptyset**

This section is devoted to the proof of the following special case of Theorem 2.1.

**Lemma 7.1.** Let \( m < -2 \) and \( p \geq q \geq 1 \), with \( m \) and \( p \) both odd. Let \( \Phi : G(p, q, m) \to \text{Homeo}^+(T) \) be a homomorphism with the property that Nonsep(\( \kappa \)) = \emptyset. Then the image of \( \Phi \) has a global fixed point.

**Proof.** As noted in Section 4, we may assume that \( \sim \) is an equivalence relation on \( T \). In much of the following argument, we work in \( T_H \). When doing so, we often abuse notation and write \( x \) for \( [x] \). However, we are careful to remind the reader of this whenever we think confusion might otherwise arise.
Suppose $\text{Nonsep}(\kappa) = \emptyset$. Consider the action of $G(p, q, m)$ on the Hausdorff tree $T_H$. In $T_H$, $A_\kappa \approx \mathbb{R}$ and there are exactly three possibilities for $A_\kappa \cap A_\kappa \alpha$:

(a) $A_\kappa \cap A_\kappa \alpha = A_\kappa$.
(b) $A_\kappa \cap A_\kappa \alpha$ is a nonempty proper closed connected subset $I$ of $A_\kappa$.
(c) $A_\kappa \cap A_\kappa \alpha = \emptyset$.

Recall that $A_\kappa \alpha = A_{\alpha^{-1} \kappa \alpha}$ and $A_\kappa \alpha^{-1} = A_{\kappa \alpha^{-1}}$.

**7.1. Case (a).** If $A_\kappa \cap A_\kappa \alpha = A_\kappa$, then $A_\kappa \approx \mathbb{R}$ is invariant under $\text{Im} \Phi$ and hence there is a fixed point in $T_H$ by Corollary 3.2. Thus, there is a global fixed point in $T$ by Lemma 5.6.

Although unnecessary for this proof, we observe here that an element which is orientation preserving as a homeomorphism of $T$ can induce an orientation reversing homeomorphism on a copy of $\mathbb{R}$ properly embedded in $T_H$.

**7.2. Case (b).** Let $\preceq$ denote the total order on $A_\kappa$ specified by $x \preceq x\kappa$ for all $x \in A_\kappa$. (When $d(x, x\kappa) \neq 0$, this total order bears no resemblance to the partial order $\preceq$ on $T$.) With respect to this total order, let $r$ (respectively, $s$) denote the lower bound (respectively, upper bound), if it exists, of $A_\kappa \cap A_\kappa \alpha$. Otherwise, set $r = -\infty$ (respectively, $s = \infty$). Note that at least one of $r$ and $s$ is finite since we are in Case (b). For ease of exposition (namely, to avoid breaking into the three cases shown in Figure 6), we set $\pm \infty g = \pm \infty$ or $\mp \infty g = \mp \infty$, as necessary, for elements $g \in G(p, q, m)$.

Let $\preceq_\alpha$ denote a total order on $A_\kappa \alpha$ such that $\preceq$ and $\preceq_\alpha$ agree on $A_\kappa \cap A_\kappa \alpha$, and let $\preceq_\alpha^{-1}$ denote a total order on $A_\kappa \alpha^{-1}$ such that $\preceq$ and $\preceq_\alpha^{-1}$ agree on $A_\kappa \cap A_\kappa \alpha^{-1}$.

When $r = s$, choose $\preceq_\alpha$ so that $\alpha^{-1} \kappa \alpha$ is increasing with respect to $\preceq_\alpha$ on $A_\kappa \alpha$, and choose $\preceq_\alpha^{-1}$ so that $\alpha \kappa \alpha^{-1}$ is increasing with respect to $\preceq_\alpha^{-1}$ on $A_\kappa \alpha^{-1}$. Note that $\preceq_\alpha$ and $\preceq_\alpha^{-1}$ are uniquely determined.

**Lemma 7.2.** The following are equivalent:

- $r \alpha^{-1} \preceq s \alpha^{-1}$ on $A_\kappa$.
- $\alpha^{-1} \kappa \alpha$ is increasing with respect to $\preceq_\alpha$ on $A_\kappa \alpha$.
- $\alpha \kappa \alpha^{-1}$ is increasing with respect to $\preceq_\alpha^{-1}$ on $A_\kappa \alpha^{-1}$.

**Proof.** Assume $r \neq s$. The map

$$\alpha : (A_\kappa, \prec) \to (A_\kappa \alpha, \prec_\alpha)$$

must be either order preserving or order reversing. Since $(r \alpha^{-1}) \alpha = r \prec_\alpha s = (s \alpha^{-1}) \alpha$, we see that $\alpha$ is order preserving if $r \alpha^{-1} \prec s \alpha^{-1}$ and order reversing if $s \alpha^{-1} \prec r \alpha^{-1}$. Since $r \alpha^{-1}$, $s \alpha^{-1} \in A_\kappa \cap A_\kappa \alpha^{-1}$, we have

$$r \alpha^{-1} \prec_\alpha^{-1} s \alpha^{-1} \iff r \alpha^{-1} \prec s \alpha^{-1}$$
We compare \((7.1)\) \((\alpha \tau \alpha)^{-1}\) (by definition of \(\prec \)). Therefore \(\alpha^{-1}: (A_\alpha, \prec) \to (A_\alpha, \prec^{-1})\) is order preserving if \(ra^{-1} \prec sa^{-1}\) and order reversing if \(sa^{-1} \prec ra^{-1}\).

Suppose \(ra^{-1} \prec sa^{-1}\). Since \(x \prec xk\) for all \(x \in A_\alpha\), we have \(x\alpha^{-1} \prec_{\alpha^{-1}} xk\alpha^{-1} = x\alpha^{-1}(\alpha k\alpha)\) and \(x\alpha \prec_\alpha xk\alpha = x\alpha(\alpha^{-1}k\alpha)\) for all \(x \in A_\alpha\).

Symmetrically, if \(sa^{-1} \succ \alpha^{-1}\), we have \(x\alpha^{-1} \succ_{\alpha^{-1}} xk\alpha^{-1} = x\alpha^{-1}(\alpha k\alpha)\) and \(x\alpha \succ_\alpha xk\alpha = x\alpha(\alpha^{-1}k\alpha)\).

\[\square\]

Now note that by substituting \(\beta = \tau \alpha^{-1} \tau^{-1}\) into \(\alpha \beta \alpha^{-1} \beta^{-1} = \kappa^{-p}\), we obtain \((\alpha \tau \alpha^{-1})^{-1} = \kappa^{-p+q}(\alpha^{-1} \tau^{-1} \alpha)\).

Let \(\omega\) denote the element represented by the two words in \((7.1)\). Using the axes \(A_\kappa\) and \(A_{\alpha^{-1}k\alpha}\), we will derive information about the translate \(A_\kappa \omega = A_\kappa \kappa^{-p+q}(\alpha^{-1} \tau^{-1} \alpha) = A_\kappa (\alpha^{-1} \tau^{-1} \alpha)\).

Then, using instead the axes \(A_\kappa\) and \(A_{\alpha \kappa \alpha^{-1}}\), we will derive information about the translate \(A_\kappa \omega = A_\kappa (\alpha \tau \alpha^{-1})^{-1}\).

Happily, contradictions are plentiful.

We order \(A_\kappa \omega\) by setting, for \(x, y \in A_\kappa\), \(x\omega \leq_\omega y\omega\) if and only if \(x \leq y\).

Suppose first that \(ra^{-1} \leq sa^{-1}\). By Lemma 7.2 \(ra^{-1} \tau^{-1} \alpha \prec_\alpha r\) along \(A_\alpha\). We compare \(sa^{-1} \tau^{-1} \alpha\) and \(r\) with respect to \(\prec_\alpha\). It is straightforward to show that

- if \(r \prec_\alpha sa^{-1} \tau^{-1} \alpha\), then \(A_\kappa \cap A_\kappa \omega = [r, sa^{-1} \tau^{-1} \alpha]\) and the orders \(\leq\) and \(\preceq_\omega\) agree on \(A_\kappa \cap A_\kappa \omega\);
- if \(sa^{-1} \tau^{-1} \alpha \prec_\alpha r\), then \(A_\kappa \cap A_\kappa \omega = \emptyset\), with \([r, sa^{-1} \tau^{-1} \alpha]\) the bridge connecting \(A_\kappa\) and \(A_\kappa \omega\), and
- if \(r = sa^{-1} \tau^{-1} \alpha\), then both \(r, s\) are finite, \(A_\kappa \cap A_\kappa \omega = [x, r] = [x, sa^{-1} \tau^{-1} \alpha]\) for some \(x\), and either \(x = r\) or the orders \(\leq\) and \(\preceq_\omega\) disagree on \(A_\kappa \cap A_\kappa \omega\).
Lemma 7.3. Suppose that in order \( \leq \) it is straightforward to show that 

\[ x \kappa \]

are comparable with respect to \( \prec \). After noting that \( A_\kappa \cap A_\kappa \omega = (A_\kappa \cap A_\kappa \alpha \tau^{-1})^{-1} \), it is

\[ \text{Figure 8.} \]

These three possibilities are illustrated in Figure [8]. Also by Lemma 7.2, we have \( s \alpha^{-1} \prec_{\alpha^{-1}} s \tau \alpha^{-1} = s \alpha^{-1} (\alpha \tau \alpha^{-1}) \) along \( A_\kappa \alpha^{-1} \), and we compare \( r \tau \alpha^{-1} \) and \( s \alpha^{-1} \) with respect to \( \prec_{\alpha^{-1}} \). After noting that \( A_\kappa \cap A_\kappa \omega = (A_\kappa \cap A_\kappa \alpha \tau^{-1})^{-1} \), it is

\[ \text{Figure 8.} \]

\[ \text{Figure 8.} \]

straightforward to show that

- if \( r \tau \alpha^{-1} \prec_{\alpha^{-1}} s \alpha^{-1} \), then \( A_\kappa \cap A_\kappa \omega = [r \tau \alpha^{-1}, s \alpha^{-1}] \) and the orders \( \leq \) and \( \leq \omega \) agree on \( A_\kappa \cap A_\kappa \omega \),

- if \( s \alpha^{-1} \prec_{\alpha^{-1}} r \tau \alpha^{-1} \), then \( A_\kappa \cap A_\kappa \omega = \emptyset \), with \([s \alpha^{-1}, r \tau \alpha^{-1}] \) the bridge connecting \( A_\kappa \) and \( A_\kappa \omega \), and

- if \( r \tau \alpha^{-1} = s \alpha^{-1} \), then both \( r, s \) are finite. \( A_\kappa \cap A_\kappa \omega = [s \alpha^{-1}, y] = [r \tau \alpha^{-1}, y] \) for some \( y \), and either \( y = s \alpha^{-1} \) or the orders \( \leq \) and \( \leq \omega \) disagree on \( A_\kappa \cap A_\kappa \omega \).

These three possibilities are illustrated in Figure [8]. Hence, one of the following cases holds.

1. \([r, s \alpha^{-1}, y] = [r \tau \alpha^{-1}, s \alpha^{-1}] \).
2. \([r, s \alpha^{-1}, y] = [s \alpha^{-1}, r \tau \alpha^{-1}] \).
3. \([x, y] = [s \alpha^{-1}, r \tau \alpha^{-1}] \), \( r \tau \alpha^{-1} \tau \alpha = s = r \tau \), and \( r, s \) are both finite.

In case (1), at least one of \( r \tau \) and \( s \alpha^{-1} \tau^{-1} \) (is finite and) lies in \( \text{Nonsep}(\alpha) \cap A_\kappa \), and hence the following lemma yields the desired contradiction. (Considering \( \text{Nonsep}(\alpha) \) in \( T \) rather than \( \text{Fix}(\alpha) \) in \( T_H \) allows us to take advantage of the partial order \( \leq \) defined on \( T \).)

**Lemma 7.3.** Suppose that in \( T \) we have \( \text{Nonsep}(\alpha) \cap A_\kappa \neq \emptyset \).

- If \( \alpha : (A_\kappa, \prec) \to (A_\kappa, \prec_\alpha) \) is orientation preserving, then necessarily the action is trivial.
- If \( \alpha : (A_\kappa, \prec) \to (A_\kappa, \prec_\alpha) \) is orientation reversing and \( p \neq 4q \), then necessarily the action is trivial.

**Proof.** If \( x \in \text{Fix}(\alpha) \cap A_\kappa \neq \emptyset \), then \( d(x, x \kappa) \) is necessarily even, and hence \( x \) and \( x \kappa \) are comparable with respect to the partial order \( \leq \) on \( T \). Lemma 7.2 therefore applies.
So we may assume that \( Fix(\alpha) \cap A_\kappa = \emptyset \), and choose \( x \in (Nonsep(\alpha) \setminus Fix(\alpha)) \cap A_\kappa \). Consider first the possibility that \( x\alpha \in A_\kappa \) or \( x\alpha^{-1} \in A_\kappa \) (and therefore \( \alpha: (A_\kappa, \prec) \to (A_\kappa, \prec_\alpha) \) is orientation reversing). By replacing \( x \) with \( x\alpha^{-1} \) as necessary, we may assume that \( x, x\alpha \in A_\kappa \). We consider separately the cases 

\[ x \prec x\alpha \text{ and } x\alpha \prec x. \]

Note that since \( x\alpha \sim x \), we have \( x\tau^{-1}\beta \sim x\tau^{-1} \). Therefore, as illustrated in Figures 9 and 10 respectively, straightforward computations reveal that 

\[ d(x, x\alpha \beta \alpha^{-1} \beta^{-1}) = d(x\beta \alpha, x\alpha \beta) = 4(2nq), \]

where \( d(y, y\kappa) = 2n \) for all \( y \in A_\kappa \). So \( 2np = d(x, x\gamma) = 4(2nq) \), and hence \( p = 4q \), which is impossible.

Now assume that if \( z \in Nonsep(\alpha) \cap A_\kappa \), then \( z\alpha^{-1}, z\alpha \notin A_\kappa \). Consider \( [x] \cap A_\kappa \). Either \( [x] \cap A_\kappa = \{x\} \) or \( [x] \cap A_\kappa = \{x, y\} \) for some \( y \neq x \). In the first case, Lemma 3.5 applied to the ideal point determined by \( [x] \) shows that the action of \( G \) on \( T \) is trivial. In the second case, note that \( y\alpha \sim x\alpha \sim x \), but \( y\alpha \notin \{x, y\} \) by assumption. We therefore have the situation modelled in Figure 11. For the details, proceed as follows, working now in \( T_H \). We have 

\[ A_\kappa \cap A_\kappa \alpha = \{x\} = A_\kappa \cap A_\kappa \alpha^{-1}. \]

Hence,

\[ A_\kappa \cap A_\kappa \beta = A_\kappa \cap A_\kappa \tau \alpha^{-1} \tau^{-1} = A_\kappa \cap A_\kappa \alpha^{-1} \tau^{-1} = (A_\kappa \cap A_\kappa \alpha^{-1}) \tau^{-1} = \{x\tau^{-1}\}. \]

Furthermore,

\[ A_\kappa \cap A_\kappa \alpha = \{x\} \implies A_\kappa \beta \cap A_\kappa \alpha \beta = \{x\beta\}. \]

Note that \( x\beta \neq x\tau^{-1} \), since otherwise \( x\alpha = x\tau \in A_\kappa \). Therefore, by simple connectivity,

\[ A_\kappa \alpha \beta \cap A_\kappa \alpha = \emptyset, \]
whereas $A_\kappa \cap A_\kappa \beta = \{x\tau^{-1}\} \implies A_\kappa \alpha \cap A_\kappa \beta \alpha = \{x\tau^{-1}\alpha\}$.

Since $A_\kappa \gamma = A_\kappa \implies A_\kappa \alpha \beta = A_\kappa \beta \alpha$, this is impossible and the lemma is proved.

In case (3), we have $s\alpha^{-1}\tau^{-1} = r\alpha^{-1}$, so $A_\kappa \cap A_\kappa \omega = [r\alpha^{-1}, r]$. In particular, $r\alpha^{-1} \leq r$. If $r\alpha^{-1} = r$ in $T_H$, then Lemma 7.3 applies. So, we may assume that $r\alpha^{-1} \prec r$. If $s \leq s\alpha^{-1}$, then $\alpha$ determines a homeomorphism from the subinterval $[r, s]$ to $[r\alpha^{-1}, s\alpha^{-1}]$. Therefore, $\alpha$ fixes some element of $[r, s]$ and Lemma 7.3 applies again. So we may assume that $s\alpha^{-1} \prec s$. Recall that since $r\alpha^{-1} \prec s\alpha^{-1}$, the map $\alpha : (A_\kappa, \prec) \to (A_\kappa \alpha, \prec_\alpha)$ is order preserving. Thus $s \leq_\alpha s\alpha$. Now, since $r\alpha^{-1}\tau\alpha = s = r\tau$, relation (B) gives

$$r\beta\alpha = r\tau\alpha^{-1}\tau^{-1}\alpha = r.$$ 

Therefore,

$$s = r\tau \prec rr^p = r\gamma^{-1} = r\beta\alpha\beta^{-1}\alpha^{-1} = r\beta^{-1}\alpha^{-1} = r\tau\alpha\tau^{-1}\alpha^{-1} = s\alpha\tau^{-1}\alpha^{-1}.$$ 

So $s\alpha\tau^{-1} = rr^p\alpha \in A_\kappa \alpha$ with $s \prec_\alpha s\alpha \prec_\alpha s\alpha\tau^{-1}$. But this in turn gives $s\alpha = (s\alpha\tau^{-1})\tau \notin A_\kappa \alpha$, a contradiction.

In case (2), we obtain

$$r = s\alpha^{-1}\tau^{-1} \implies s = r\tau\alpha$$

(so both $r$ and $s$ are finite) and

$$s\alpha^{-1}\tau^{-1}\alpha = r\tau\alpha^{-1}\tau^{-1} \implies r\alpha = r\tau\alpha^{-1}\tau^{-1}.$$ 

Hence,

$$(r\alpha^{-1})(\alpha\tau\alpha^{-1})\tau^{-1} = r\tau\alpha^{-1}\tau^{-1} = r\alpha.$$
Next apply (7.1) to the element $rα^{-1}$:

$$\begin{align*}
rα &= (rα^{-1})(κ^{-p+q})(α^{-1})^{-1}α \\
⇒ rτα &= (rα^{-1})(κ^{-p+q}) \\
⇒ s &= (rα^{-1})(κ^{-p+q}).
\end{align*}$$

Therefore,

$$sk^{p-q} = rα^{-1}$$

Since $p \geq q$, $s < rα^{-1}$ on $A_κ$. So

$$r \preceq s \prec rα^{-1} \preceq sa^{-1}$$
on $A_κ$. Since $α : (A_κ, \prec) → (A_κα, \prec_α)$ and $α^{-1} : (A_κ, \prec) → (A_κα^{-1}, \prec_{α^{-1}})$ are order preserving, we see that

$$rα ≼_α sa_α ≼_α r ≼_α s$$
on on $A_κα$ and

$$rα^{-1} ≼_{α^{-1}} sa^{-1}_α ≼_{α^{-1}} rα^{-2} ≼_{α^{-1}} sa^{-2}$$
on on $A_κα^{-1}$. In particular, $[r, rα^{-1}] = [r, s] ∪ [s, rα^{-1}]$, with $[r, rα^{-1}]α^n ⊆ A_κα^n$ and $[r, s]α^n ∩ [r, s]α^{n+1} = ∅$, for all $n ∈ Z$. This is illustrated in Figure 12. Therefore,

$$A_α = \bigcup_{-∞}^{∞} [r, rα^{-1}]α^n$$
is a local axis or axis for $α$.

Next we investigate the orientation that $A_κ$ inherits from $T_H$. For any vertices $x, y$ both of which lie on one of the (simplicial) trees $A_κ$, $A_κα$, $A_κα^{-1}$ in $T_H$, let $f_{x,y}$ be the first edge in the simplicial path from $x$ to $y$ in the given tree.

After reversing the orientation of $T_H$ if necessary, we may assume that $f_{r, rτ}$ is positively oriented. Since $sa^{-1} = rτ$ and $τ$ preserves orientation, we see that $f_{r, rτ}$ and $f_{sa^{-1}, sa^{-1}τ}$ have the same orientation, as do $f_{r, rτ^{-1}}$ and $f_{sa^{-1}, sa^{-1}τ^{-1}}$. Since $α^{-1}$ preserves orientation, we see that $f_{r, rα}$ and $f_{rα^{-1}, r} = f_{rα^{-1}, rα^{-1}τ^{-1}}$ have the same orientation, as do both edges from each pair $f_{r, rα^{-1}} = f_{r, rτ}$ and $f_{rα^{-1}, rα^{-2}} = f_{rα^{-1}, rα^{-1}τ} f_{r, rτ^{-1}}$ and $f_{rα^{-1}, rα^{-1}τ^{-1}}$; $f_{s, sa} = f_{s, sτ}$ and $f_{sa^{-1}, s} = f_{sa^{-1}, sa^{-1}τ}$; $f_{s, sa^{-1}τ} = f_{s, sτ}$ and $f_{sa^{-1}, sa^{-2}}$; and $f_{s, sa^{-1}τα}$ and $f_{sa^{-1}, sa^{-1}τ}$. Now, using (7.2), we see that $f_{s, sτ}$ and $f_{rα^{-1}, rα^{-1}τ}$ have the same orientation, as do $f_{s, sτ}$ and $f_{rα^{-1}, rα^{-1}τ^{-1}}$. Finally, after applying $τα$ to the interval on the right side of the equality

$$[r, rα] = [sa^{-1}τ^{-1}, rτα^{-1}τ^{-1}],$$

Figure 12.
we see that $f_{r,ra}$ and $f_{s,rt} = f_{s,rt}$ have the same orientation. It follows that all of the edges under consideration have the same orientation, which we have assumed to be positive. We now see that each of the points $r, s, ra^{-1}, sa^{-1}$ in $T_H$ corresponds to a pair of (distinct) nonseparated points along $A_\kappa$ in $T$, and that the corresponding branching at $A_\kappa\alpha$ and $A_\kappa\alpha^{-1}$ is as shown in Figure 13.

So we change viewpoint and consider instead the non-Hausdorff 1-manifold $T$. Let $n \in \mathbb{N}$ such that $d(x, x\kappa) = 2n$ for all $x \in A_\kappa$. Then

$$2nq = d(r, rt) = d(r, s) + d(s, sr^{\kappa^{-q}}) + d(ra^{-1}, sa^{-1}) = 2d(r, s) + 2n(p - q)$$

$$\Rightarrow d(r, s) = n(2q - p).$$

(In particular, $2q \geq p$.) Also,

$$d(r, ra^{-1}) = d(r, s) + d(s, sr^{\kappa^{-q}}) = n(2q - p) + 2n(p - q) = np.$$ 

(Therefore, $np$ is necessarily even.) Finally, we use relation (A) from Section 3 in the form

$$\tau (\alpha r^{-1}\alpha^{-1})^\tau = (\alpha r^{-1}\alpha^{-1})^m = \alpha^{m-2},$$

by applying each of the given words to the element $r$.

Let $v = r\tau (\alpha r^{-1}\alpha^{-1}) = (r\tau\alpha)^{-1} = s_2 r^{-1} a^{-1}$. Note that since $s_1 < s_1\alpha^{-1} = r_1\tau$, we have $s_1\tau^{-1} \prec r_1$ along $A_\kappa$, and hence $s_1\tau^{-1} \prec a^{-1} \prec r_1 \alpha^{-1}$ along $A_\kappa\alpha^{-1}$. Therefore, since $v \sim s_1\tau^{-1} \alpha^{-1}$, we see that $[v, r_1\alpha^{-1}]$ is the bridge from $v$ to $A_\kappa$. So $[v^{-1}, r_1\alpha^{-1}]$ is the bridge from $v^{-1}$ to $A_\kappa$ and hence $[v^{-1}, r_2\alpha^{-1}]$ is the bridge from $v^{-1}$ to $A_\kappa\alpha^{-1}$. So

$$[ra^{m-2}, r_2\alpha^{-1}(\alpha r^{-1}\alpha^{-1})] = [v^{-1}(\alpha r^{-1}\alpha^{-1}), r_2\tau r^{-1}]$$

is the bridge from $ra^{m-2}$ to $A_\kappa\alpha^{-1}$. Since $r_2\tau r^{-1} \sim sa^{-2}$ and $ra^{m-2} \in A_\alpha$, necessarily $r_2\tau\alpha^{-1} = sa^{-2}$. (See Figure 14.)

By computing the length of the path $[v^{-1}(\alpha r^{-1}\alpha^{-1}), r^{-1}] = [ra^{m-2}, ra^{-1}]$, we obtain

$$d(ra^{m-2}, ra^{-1}) = d(ra^{m-2}, sa^{-2}) + d(sa^{-2}, ra^{-1}) = d(v^{-1}, r_2\alpha^{-1}) + d(sa^{-1}, r)$$

$$= [d(v^{-1}, r_1\alpha^{-1}\tau^{-1}) + d(r_1\alpha^{-1}\tau^{-1}, sa^{-1}\tau^{-1})$$

$$+ d(r_1^{-1}, r_2\alpha^{-1})] + d(r, \tau) = 4nq + np.$$ 

On the other hand,

$$d(ra^{m-2}, ra^{-1}) = |m - 1| d(r, ra^{-1}) = |m - 1| np = (|m| + 1)np.$$
Figure 14.

So

\[ 4nq + np = (|m| + 1)np \implies 4q = |m|p, \]

which is impossible since both \( p \) and \( m \) are odd, and hence we have reached our contradiction.

Since we have been working under the assumption that \( ra^{-1} \preceq sa^{-1} \) (and hence \( r \neq s \)). Consider first the possibility that \([r, s] \cap [sa^{-1}, ra^{-1}] \neq \emptyset\). In this case, the Intermediate Value Theorem guarantees the existence of an element \( x \in \text{Fix}(\alpha) \cap A_\kappa \subset TH \). Since \( p \neq 4q \), Lemma 7.3 therefore applies. So restrict attention to the case that \([r, s] \cap [sa^{-1}, ra^{-1}] = \emptyset\). By appealing to symmetry, we may assume that \( r \preceq s \preceq sa^{-1} \preceq ra^{-1} \). It follows that \([s, sa^{-1}] \cap [s, sa] = \{s\} \), and hence that

\[ A_\alpha = \bigcup_{n=-\infty}^{\infty}[s, sa^{-1}]\alpha^n \]

is a local axis or axis for \( \alpha \). Since \( A_\beta = A_\alpha \tau^{-1} \),

\[ A_\beta \cap A_\kappa = [s\tau^{-1}, sa^{-1}\tau^{-1}] \]

Suppose first that \( sa^{-1}\tau^{-1} \neq s \) and apply each of the words from the relation \( \tau^{-1}\alpha \tau = \alpha \beta \alpha^{-m-1} \) to \( s \). Referring to the axes \( A_\kappa, A_\alpha \) and \( A_\beta \), a straightforward computation reveals that the bridge from \( s\tau^{-1} \alpha \tau \) to \( A_\kappa \) has endpoint \( s\tau \) at \( A_\kappa \), whereas the bridge from \( sa\beta \alpha^{-m-1} \) to \( A_\kappa \) has endpoint \( sa^{-1} \) at \( A_\kappa \). We conclude that \( sa^{-1} = s\tau \).

Without loss of generality, we may assume that \( f_{s, s\tau} = f_{s, sa^{-1}} \) is positively oriented. Applying \( \tau \) and \( \alpha^{-1} \) to \( f_{s, s\tau} \), we see that \( f_{sa^{-1}, sa^{-1}\tau} \) and \( f_{sa^{-1}, sa^{-2}} \) are also positively oriented. Now applying \( \alpha \) to \( f_{sa^{-1}, sa^{-1}\tau} = f_{sa^{-1}, ra^{-1}} \) shows that \( f_{s, sa\tau^{-1}} = f_{s, r} \) is positively oriented, and applying \( \tau \) to \( f_{s, sa\tau^{-1}} \) shows that \( f_{sa^{-1}, sa^{-1}\tau^{-1}} \) is positively oriented. Finally, applying \( \alpha \) to \( f_{sa^{-1}, sa^{-1}\tau^{-1}} \) shows that \( f_{s, sa} \) is positively oriented. Hence, in \( T \) we have the situation shown in Figure 15 with

\[ d(s, sa^{-1}) = d(s, s\tau) = 2nq \geq 2. \]

Let \( u, v, w \in [s] \) be as given in Figure 15. Notice that

\[ sa\tau = (s\tau)(\tau^{-1}\alpha\tau) = (s\tau)\alpha\beta\alpha^{-m-1}. \]
Now $d(s\alpha \tau, s\tau) = d(s\alpha, s) = 2nq$, together with Lemma \ref{lem10}, guarantees that $s\alpha \tau \in (s\alpha^{-2})^-$. On the other hand, $u \neq s\tau \alpha \sim v$, and since $v \in A_\beta$, $d(v, v\beta) = 2nq$. And $d(v, v\beta) = 2nq$, together with Lemma \ref{lem10} reveals that $v\beta \in (ua)^-$. So $(s\tau \alpha \beta) \in (ua)^- \cup \{ua\}$, and hence $s\alpha \tau = (s\tau \alpha \beta) \alpha^{m-1} \in (ua)^- \cup \{ua\}^m$. Since $m \leq -3$, we have reached a contradiction.

7.3. Case (c). Suppose that $A_\kappa \cap A_\kappa \alpha = \emptyset$, and let $\rho = [r, s]$ be the bridge from $A_\kappa$ to $A_\kappa \alpha$ in $T_H$. Once again we consider

$$A_\kappa \omega = A_\kappa \cdot \kappa^{-p+q}(\alpha^{-1} \tau^{-1} \alpha) = A_\kappa (\alpha^{-1} \tau^{-1} \alpha).$$

Since $[r, s]$ is the bridge from $A_\kappa$ to $A_\kappa \alpha$, we see that $[r(\alpha^{-1} \tau^{-1} \alpha), s(\alpha^{-1} \tau^{-1} \alpha)]$ is the bridge from $A_\kappa \omega$ to $A_\kappa \alpha$. Since $s\alpha^{-1} \tau^{-1} \alpha \neq s$, we see that $[r(\alpha^{-1} \tau^{-1} \alpha), r] = [r(\alpha^{-1} \tau^{-1} \alpha), s(\alpha^{-1} \tau^{-1} \alpha)] \sqcup [s(\alpha^{-1} \tau^{-1} \alpha), r]$ is the bridge from $A_\kappa \omega$ to $A_\kappa$. Next consider $A_\kappa \omega = A_\kappa (\alpha \tau \alpha^{-1}) \tau^{-1}$. Since $[\alpha^{-1}, r \alpha^{-1}]$ is the bridge from $A_\kappa$ to $A_\kappa \alpha^{-1}$, we know that $[s\alpha^{-1} (\alpha \tau \alpha^{-1}), r \alpha^{-1} (\alpha \tau \alpha^{-1})]$ is the bridge from $A_\kappa (\alpha \tau \alpha^{-1})$ to $A_\kappa \alpha^{-1}$. Hence, $[s\tau \alpha^{-1}, s\alpha^{-1}]$ is the bridge from $A_\kappa (\alpha \tau \alpha^{-1})$ to $A_\kappa$. So $[s\tau \alpha^{-1} \tau^{-1}, s\alpha^{-1} \tau^{-1}]$ is the bridge from $A_\kappa \omega$ to $A_\kappa$. Hence, $[r(\alpha^{-1} \tau^{-1} \alpha), r] = [s\tau \alpha^{-1} \tau^{-1}, s\alpha^{-1} \tau^{-1}]$.

(See Figure \ref{fig16}) In particular, $r(\alpha^{-1} \tau^{-1} \alpha) = s\tau \alpha^{-1} \tau^{-1}$ and $s = r \tau \alpha$. Now apply

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{Figure 15.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{Figure 16.}
\end{figure}
both sides of \( \tau \) to the element \( rk^{p-q} \).

\[
\begin{align*}
  r k^{p-q} \cdot (\alpha \tau \alpha^{-1}) \tau^{-1} &= r k^{p-q} \cdot k^{-p+q} \cdot (\alpha^{-1} \tau^{-1} \alpha) \\
  &= r (\alpha^{-1} \tau^{-1} \alpha) = s t \alpha^{-1} \tau^{-1} \\
  &= (r \tau \alpha)(\tau \alpha^{-1} \tau^{-1}).
\end{align*}
\]

So \( r k^{p-q} = r \tau \Longleftrightarrow r \in Fix(\kappa^{p-2q}) \).

However, \( r \in A_\kappa \) means that no nontrivial power of \( \kappa \) fixes \( r \). Since \( p \) is odd, we have \( \kappa^{p-2q} \neq 1 \), giving the final contradiction. \( \square \)

We conclude this section by recording, for future reference, a lemma which follows easily from the above arguments restricted to the special case that \( A_\kappa \approx \mathbb{R} \). Note that the only way in which we have used the fact that \( \text{Nonsep}(\kappa) = \emptyset \) in this section is to guarantee the existence of the axis \( A_\kappa \). The fact that \( A_\kappa \) is an axis is then used to control the structure of the intersections of \( A_\kappa \) with some of its translates and that of the bridges from \( A_\kappa \) to such translates in the case of empty intersection. The arguments in this section can be easily adjusted to obtain the following result, which will be used repeatedly in the next section, where we examine the case where \( \text{Nonsep}(\kappa) \neq \emptyset \).

**Lemma 7.4.** Suppose \( Y \) is a \( \kappa \)-invariant embedded copy of \( \mathbb{R} \) in \( T \) on which \( \kappa \) acts freely. If

- \( \emptyset \neq Y \cap Y \alpha \subset [r, s] \) for some \( r, s \in Y \), or
- \( Y \cap Y \alpha = \emptyset \), and the bridge from \( Y \) to \( Y \alpha \) has the form \([ [r, s] ] \) for some \( r \sim r' \in Y \), \( s \sim s' \in Y \alpha \),

then the action of \( G \) on \( T \) has a global fixed point.

8. CASE II: \( \text{Nonsep}(\kappa) \neq \emptyset \).

In this section we prove that if \( \text{Nonsep}(\kappa) \neq \emptyset \), then necessarily the action of \( G \) on \( T \) is trivial. First we give some preliminary lemmas, whose primary import is the fact that in most cases, the argument reduces to the case that \( T = \mathbb{R} \).

**Lemma 8.1.** There is no \( x \in T \) which is nonseparated by \( \tau \) and at least one of \( \alpha, \beta \).

**Lemma 8.2.** If \( u \sim v < w \), then \( u \in w^- \), with \( u < w \) \( \iff \) \( u, w \) are comparable \( \iff \) \( w \in T_{\{u, v\}} \). Similarly, if \( u \sim v > w \), then \( u \in w^+ \), with \( u > w \) \( \iff \) \( u, w \) are comparable \( \iff \) \( w \in T_{\{u, v\}} \).

**Lemma 8.3.** If \( \text{Fix}(\tau) \cap \text{Nonsep}(\kappa) \cap C_\alpha \neq \emptyset \), then the action of \( G \) on \( T \) is trivial.

**Proof.** Let \( x \in \text{Fix}(\tau) \cap \text{Nonsep}(\kappa) \cap C_\alpha \). We modify slightly the arguments of Section 5.1. Once again, we may assume that \( x < x_\alpha \). Since \( x \in \text{Nonsep}(\kappa) \), we have \( x \gamma \sim x \).

Therefore, \( x \beta \alpha^m \sim x \gamma \beta \alpha^m = x \tau^{-1} \alpha \tau = (x \alpha) \tau > x \tau = x \) and hence, by Lemma 5.2, \( x \beta \alpha^m \in x^+ \). Since \( x \alpha^{-m} > x \), \( x \beta \in x^+ \alpha^{-m} \subset x^+ \).

On the other hand, \( x \beta \tau = x \tau^{-1} \beta \tau = x \alpha^{-1} \Rightarrow x \beta < x \tau^{-1} = x \Rightarrow x \beta \in x^- \).

\( \square \)

**Lemma 8.4.** If \( \text{Nonsep}(\kappa) \cap C_\alpha \neq \emptyset \), then the action of \( G \) on \( T \) is trivial.
Proof. Let \( x \in \text{Nonsep}(\kappa) \cap C_\alpha \). By Lemma 8.3, we may assume that \( x \sim x\tau \) but \( x \neq x\tau \). Set \( T_0 = T_{(x,x\tau)} \). Again, we may assume that \( x < x\alpha \).

If \( x\alpha^{-1} \notin T_0 \) or \( x\alpha \in T_0 \), then the ideal point \( \hat{x} \in \hat{T} \) is fixed by \( \kappa \) and related to \( \hat{x}\alpha \), and Lemma 8.3 applies.

So we may assume that \( x\alpha, x\alpha^{-1} \notin T_0 \). Since \( x < x\alpha \), either \( T_0 \subset x^− \) and \( x\alpha^{-1} \notin y^+ \) for some \( y \sim x \), \( y \neq x \), or \( T_0 \subset x^+ \) and \( x\alpha \notin y^− \) for some \( y \sim x \), \( y \neq x \). In each case, by Lemma 8.3 we may assume that \( y\alpha \neq \alpha \). In fact, by reversing the orientation on \( T \) and exchanging the roles of \( x \) and \( y \) as necessary, we may assume that the first possibility holds; namely, \( T_0 \subset x^− \) and \( x\alpha^{-1} \in y^+ \) for some \( y \sim x \), \( y \neq x \). These possibilities are illustrated in Figure 17. Notice that

\[
\{x, y\} \subset [x\alpha^{-1}, x\alpha] \text{ and so } d(x, x\alpha) = 2n > 0 \text{ for some } n \in \mathbb{N}. \quad \text{In particular, by Proposition 4.8 and Corollary 4.12, we have } \text{Nonsep}(\alpha) = \emptyset \text{ and } C_\alpha = A_\alpha.
\]

Consider first the case that \( y \neq x\tau \). Since \( x\tau^{-1}x\alpha \sim x\alpha \tau > x\tau \), we have

\[
(8.1) \quad x\tau^{-1}x\alpha \in x\tau^+ \subseteq y^−.
\]

Also, \( x\gamma \tau x\alpha^{-1} \sim x\alpha^{-1} \tau^{-1} \), and since \( x\alpha^{-1} \in y^+ \), we have

\[
(8.2) \quad x\tau^{-1}x\alpha \in (y\tau^{-1})^+ \subseteq x^-.
\]

This gives

\[
(8.3) \quad x\tau^{-1}x\alpha = x\gamma \beta \alpha^m = x\gamma \tau x\alpha^{-1} \tau^{-1} \alpha^m \in (x\alpha^{-1})^− \subseteq (x\alpha^{-1})^−.
\]

However, since \( x \in (x\alpha^{-1})^+ \), we have

\[
y^- \cap (x\alpha^{-1})^- = \emptyset,
\]

and (8.1) and (8.2) now give a contradiction.

So we may assume that \( y = x\tau \). In this case,

\[
d(x\tau \alpha, x\tau) = d(x\tau \alpha, x) = 2n - 1 < d(x\tau \alpha^{-1}, x\tau) \implies x\tau \alpha \tau \in (x\tau \alpha)^- \).
\]

Therefore, since \( x\tau^{-1}x\alpha \sim x\tau \alpha \), we have

\[
(8.4) \quad x\tau^{-1}x\alpha \in (x\tau \alpha)^+ \cup \{x\tau \alpha\}^{-1}.
\]

On the other hand, since \( d(x\alpha^{-1} \tau^{-1}, x) = 2n - 1 \), we have \( x\alpha^{-1} \tau^{-1} \in (x\alpha)^- \) and hence \( (x\gamma \tau)\alpha^{-1} \tau^{-1} \in (x\alpha)^- \cup \{x\alpha\} \). So, since \( m \leq -3 \), we have

\[
(8.5) \quad x\tau^{-1}x\alpha = x\gamma \tau x\alpha^{-1} \tau^{-1} \alpha^m \in (x\alpha^{-1})^+ \cup \{x\alpha^{-1} \alpha^{-1} \} \in (x\alpha^{-2})^+ \cup \{x\alpha^{-2}\}.
\]

Since \( y \in (x\alpha^{-1} \tau^{-1})^+ \), we have \( x\tau \alpha^{-1} = y\alpha^{-1} \in (x\alpha^{-2})^+ \), and it follows that

\[
(8.6) \quad ((x\tau \alpha^{-1})^+ \cup \{x\tau \alpha^{-1}\}) \cap ((x\alpha^{-2})^+ \cup \{x\alpha^{-2}\}) = \emptyset.
\]

Now (8.3), (8.4) and (8.5) together give a contradiction. \( \square \)
Lemma 8.5. If Nonsep(κ) ≠ ∅ and Nonsep(α) ∩ C_κ ≠ ∅, then the action of G on T is trivial.

Proof. Let x ∈ Nonsep(α) ∩ C_κ. By Lemma 4.10, either x ∈ Fix(κ) or x lies on some local axis A^i_κ ∼= ℜ (in T) for κ. By Lemma 5.11 we may assume that x lies on some local axis A^i_κ (in T). Then either x ∈ Fix(α) or the ideal point ̄x ∈ T is fixed by α and related to ̄κ. In either case, Lemma 5.5 applies.

Let \{T_i\}_{i ∈ I} denote the path components of T \ Nonsep(κ); so T \ Nonsep(κ) = \bigsqcup_{i ∈ I} T_i. Notice that for each i ∈ I we have T_iκ = T_j for some j ∈ I. Moreover, whenever T_iκ = T_i, κ acts freely on T_i with local axis A^i_κ ⊂ T_i (and since Nonsep(κ) ≠ ∅, A^i_κ ∼= ℜ).

Similarly, if Nonsep(α) ≠ ∅, let \{X_j\}_{j ∈ J} denote the set of path components of T \ Nonsep(α). Again, either X_jα = X_j, and α acts freely on X_j with local axis A^j_α, or X_jα = X_k ≠ X_j. When Nonsep(α) = ∅, we write T = X_1 and let A^i_1 denote the axis for α.

Lemma 8.6. If G acts nontrivially on T, then

- C_κ ∪ Nonsep(κ) ⊂ X_{j_0} for some j_0 ∈ J, and
- C_α ∪ Nonsep(α) ⊂ T_{ι_0} for some ι_0 ∈ I.

Proof. By Lemma 8.5 (C_κ ∪ Nonsep(κ)) ∩ Nonsep(α) = ∅. By Corollary 4.13 therefore, C_κ ∪ Nonsep(κ) ⊂ X_{j_0} for some j_0 ∈ J. A symmetric argument proves the second statement. □

Proposition 8.7. Suppose Nonsep(κ) ≠ ∅. Then the action of G on T is trivial.

Proof. Let ι_0, j_0 be as guaranteed in Lemma 8.4.

[Case 1] Suppose first that T_{ι_0κ} = T_{ι_0}. As remarked above, A^ι_0κ ∼= ℜ. By Lemma 8.6 Nonsep(κ) ∪ A^ι_0κ ⊂ X_{j_0}.

Consider first the possibility that X_{j_0α} = X_{j_0}, and hence A^ι_0α ⊂ T_{ι_0}. In fact, T_{ι_0} ∩ X_{j_0} is a subtree of T containing both A^ι_0κ and A^ι_0α. Therefore, if A^ι_0κ ∩ A^ι_0α = ∅, the bridge from A^ι_0κ to A^ι_0α lies in T_{ι_0} ∩ X_{j_0}. If either of the two potential endpoints of A^ι_0κ (respectively, A^ι_0α) exist in T, they are in Nonsep(κ) (respectively, Nonsep(α)) and hence are not elements of T_{ι_0} (respectively, X_{j_0}), and therefore cannot be on the bridge. Hence this bridge has the form [[u, v]] or [[u, v]], where u and v are not separated from points in A^ι_0κ and A^ι_0α, respectively. (See Figure 18)

\[\begin{array}{c}
\bullet & \bullet & \bullet \\
| & \downarrow & | \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}\]

\text{Figure 18.}

A^ι_0κα in this case, we see that A^ι_0κ ∩ A^ι_0κα = ∅, with the bridge from A^ι_0κ to A^ι_0κα of the form [[u, w]] for some w ∼ w' ∈ A^ι_0κα. So Lemma 7.4 reveals that the action of
$G$ on $T$ is necessarily trivial. On the other hand, if $A_{\alpha}^{j_0} \cap A_{\alpha}^{j_0} \neq \emptyset$, then Lemma 8.3 guarantees that $A_{\alpha}^{j_0} \cap A_{\alpha}^{j_0} \subset [u, v]$ for some $u, v \in A_{\alpha}^{j_0}$. (See Figure 19.) Computing $A_{\alpha}^{i_0}$ in this case, we see that one of the two conditions of Lemma 7.4 is satisfied, and so once again, the action of $G$ on $T$ must be trivial.

Next consider the possibility that $X_{j_0}^\alpha = X_{j_1}^\alpha \neq X_{j_0}$. Let $y$ and $y^\alpha$ denote the roots of $X_{j_0}$ and $X_{j_1}^\alpha$, respectively. Let $[[y, r]]$ denote the bridge from $y$ to $A_{\alpha}^{j_0}$ in $T$. By Lemma 8.3, we may assume that $r \sim r'$ for some $r' \in A_{\alpha}^{j_0}$. So $A_{\alpha}^{j_0} \cap A_{\alpha}^{j_0} = \emptyset$ with bridge $[[r, r^\alpha]]$. Again, by Lemma 7.4, the action of $G$ on $T$ has a global fixed point.

[Case 2] Finally, we assume that $T_{i_0}^\kappa = T_{i_1} \neq T_{i_0}$. Let $x$ and $x^\kappa$ denote the roots of $T_{i_0}$ and $T_{i_1}$, respectively. Set $T_0 = T_{\{x, x^\kappa\}}$. Without loss of generality, we may assume that $T_0 \subset x^+$. If $X_{j_0}^\alpha = X_{j_0}$, then $A_{\alpha}^{j_0} \subset T_{i_0}$. Therefore, since $x \in Nonsep(\kappa) \subset X_{j_0}$, the bridge from $x$ to $A_{\alpha}^{j_0}$ is of the form $[[x, r]]$ or $[[x, r]]$. Deleting $x$ from this bridge, we obtain $[[x, r]]$ (or $[(x, r)]$, respectively), which lies in $T_0 \cap X_{j_0}$. In particular, $r(\sim x)$ is nonseparated from a point in $A_{\alpha}^{j_0}$. If $X_{j_0}^\alpha = X_{j_1} \neq X_{j_0}$, let $y$ and $y^\alpha$ denote the roots of $X_{j_0}$ and $X_{j_1}$, respectively. Note that since $x \in X_{j_0}$, $y \in [x, x^\alpha]$. These two possibilities are illustrated in Figure 20. Note that in either case, $d(x, x^\alpha)$ is odd; so $d(x, x^\alpha) = 2n - 1$ for some $n \in \mathbb{N}$.

Consider first the case that $x \neq x^\tau$. This is illustrated in Figure 21. Note that since $d(x^\tau, x^\alpha) = 4n$ and $(x^\tau, x^\alpha) \subset (x^\tau) +$, we have

$$((x^\tau, x^\alpha)\alpha^{-1}r^{-1} = (x, x^\alpha \tau^{-1}r^{-1}) \subset x^+.$$

![Figure 19](image1.png)

![Figure 20](image2.png)
Therefore, since $x\alpha \in x^-$, we have $x \in [x\alpha, x\alpha\alpha^{-1}\tau^{-1}]$ and hence,

$$
\begin{align*}
  d(x\alpha, x\alpha\alpha^{-1}\tau^{-1}) &= d(x\alpha, x) + d(x, x\alpha\alpha^{-1}\tau^{-1}) \\
  &= (2n - 1) + d(x\alpha, x\alpha) \\
  &= 6n - 1.
\end{align*}
$$

On the other hand,

$$
\begin{align*}
  d(x\alpha, x\alpha\alpha^{-1}\tau^{-1}) &= d(x\alpha, x\kappa^{-p}q\alpha^{-1}\tau^{-1}) \\
  &= d(x, x\kappa^{-p}q\alpha^{-1}\tau^{-1}) \leq d(x, x\alpha^{-1}\tau^{-1}) + 1 \\
  &= d(x\alpha, x) + 1 \\
  &= 2n + 1.
\end{align*}
$$

Since $n \geq 1$, this is impossible.

Therefore, we may assume that $x = x\tau$. So $x\alpha\tau \in x^-$. Notice that $d(x, x\alpha) = d(x, x\alpha\tau)$, and either

- $x\alpha \in (x\alpha\tau)^-$, or
- $x\alpha\tau$ separates $x\alpha$ and $x$.

Consider first the case that $x\alpha \in (x\alpha\tau)^-$. Note that since $x\tau = x, x\kappa \neq x$ and $(p,q) = 1$, we have $x\gamma \neq x$. (See Figure 22.) Now $x\alpha \in (x\alpha\tau)^-$ implies that $x = (x\alpha)\alpha^{-1}\tau^{-1} \in (x\alpha\alpha^{-1}\tau^{-1})^- = (x\kappa^{-p}q\alpha^{-1}\tau^{-1})^-$

which implies in turn that

$$
x\alpha^{-1} \in (x\kappa^{-p}q\alpha^{-1}\tau^{-1})^-.
$$

This gives

$$
x\alpha^{-1} \in (x\alpha^{-1}\tau^{-1})^+.
$$

But $x\alpha \in x^-$ implies that

$$
x = (x\alpha)(\alpha^{-1}\tau^{-1}) \in (x\alpha^{-1}\tau^{-1})^-.
$$
Therefore,
\[ \{ x^{α^{-1}τ^{-1}}, x^{κ - p + qα^{-1}τ^{-1}} \} \subset [xα^{-1}, x], \]
and hence
\[ d(x, xα^{-1}) > d(x, xα^{-1}τ^{-1}) = d(x, xα^{-1}), \]
which is impossible.

Hence, necessarily, \( xατ \) separates \( xα \) and \( x \). In particular, \([x, xα]τ \subset [x, xα]\) and \( d(x, xα) = d(x, xατ) \) together imply that \( d(xα, xατ) = 0 \). If \( X_{j_{i}}α = X_{j_{l}} ≠ X_{j_{y}}, \) then \([y, yα] \subset [x, xα]\) gives \( yτ = y \). So \( y ∈ Fix(τ) ∩ Nonsep(α) \), and by Lemma 5.1.4, the action of \( G \) on \( T \) is trivial.

Therefore, we may assume that \( X_{j_{i}}α = X_{j_{y}} \). Since \( d(xα, xατ) = 0 \), we have
\[ d(xα^{-1}τ, xγα^{-1}) = d((xα)(α^{-1}τ^{-1}α^{-1}τ), (xατ)(α^{-1}τ^{-1}α^{-1}τ)) = 0. \]

Therefore,
\[ xα^{-1}τ ∈ (xγα^{-1})^- ∪ T_0α^{-1}. \]

But we also have \( d(xα^{-1}τ, x) = d(xα^{-1}, x) \), and hence \( xα^{-1}τ ∈ T_0α^{-1} \). But this means \( xα^{-1} ∈ [x, xα^{-1}τ] \), which gives \( xα^{-1}τ^{-1} ∈ [x, xα^{-1}] \). Therefore,
\[ xατ = xτ^{-1}ατ = xγβα^m = xγτα^{-1}τ^{-1}α^m \sim xα^{-1}τ^{-1}α^m ∈ [xα^{-1}τ, xα^{-1}τ^{-1}α^m]. \]

Since \( xατ ∈ [x, xα] \), this is impossible.

\[ \Box \]

9. Appendix: Denjoy blow-ups

9.1. Denjoy blow-up of a 1-manifold. We describe a well-known operation from [De32] in which countably many points in a closed subinterval of \( \mathbb{R} \) are “blown up” into nondegenerate closed subintervals so as to obtain a new closed subinterval of \( \mathbb{R} \). Topologically, it is straightforward to check that this operation is well defined and that it extends to arbitrary (not necessarily Hausdorff) 1-manifolds. For completeness, we do so here.

Let \( X \) be any oriented 1-manifold. Let \( C ⊂ X \) be countable. For every \( c ∈ C \), let \([c_1, c_2]\) denote an associated closed interval in \( \mathbb{R} \), with standard orientation satisfying \( c_1 < c_2 \). We assume that the intervals \([c_1, c_2], c ∈ C\), are pairwise disjoint and disjoint from \( X \). Let \( Y \) be the set obtained from \( X \) by replacing each \( c ∈ C \) with the corresponding interval \([c_1, c_2]\), and define a topology on \( Y \) as follows.

Let \( \{I_x|x ∈ X\} \) be any oriented basis for \( X \) satisfying \( I_x ≈ \mathbb{R} \) for every \( x ∈ X \).

For each \( x ∈ X \), set
\[ J_x = (I_x ∩ (X \setminus C)) ∪ \left( \bigcup_{c∈C ∩ I_x} [c_1, c_2] \right), \]
with linear ordering determined uniquely by the following conditions:

- If \( y < z \) for some \( y, z ∈ I_x ∩ C \) or \( y, z ∈ [c_1, c_2] \), for some \( c ∈ C \), then \( y < z \).
- If \( c < z \) for some \( z ∈ I_x ∩ C \) and for some \( c ∈ C ∩ I_x \), then \( c_2 < z \).
- If \( c > z \) for some \( z ∈ I_x ∩ C \) and for some \( c ∈ C ∩ I_x \), then \( c_1 > z \).
Then, for every $c \in C$, let $J_c^- = \{ x \in J_c | x < c \}$ and let $J_c^+ = \{ x \in J_c | c < x \}$, and for every $r \in \mathbb{Q} \cap [c_1, c_2]$, set

$$J_{r,-} = J_c^- \cup \{c_1, r\}$$

and set

$$J_{r,+} = (r, c_2] \cup J_c^+.$$

Let $B$ be any countable basis for $\bigcup_{c \in C} (c_1, c_2)$ consisting of sets homeomorphic to $\mathbb{R}$. Finally, let $T$ be the topology on $Y$ with basis

$$B \cup \{J_c | x \in X\} \cup \{J_{r,-} | r \in \mathbb{Q} \cap [c_1, c_2], \exists c \in C\} \cup \{J_{r,+} | r \in \mathbb{Q} \cap [c_1, c_2], \exists c \in C\}.$$

Note that if $X$ has countable basis consisting of sets homeomorphic to $\mathbb{R}$, then so does $Y$. So the space $(Y, T)$ is again an oriented 1-manifold. Moreover, if $X$ is simply-connected, so is $Y$. Notice that if we remove the requirement that manifolds be second countable, then we may remove the condition that $C$ be countable in this construction.

**Definition 9.1.** $(Y, T)$ is called the Denjoy blow-up of $X$ along $C$. If we begin with an action of $G$ on $X$ and extend this action linearly over the intervals $[c_1, c_2], c \in C$, we call the resulting action of $G$ on $Y$ the (canonically) induced action.

Now let

$$C = \{ x \in X | \exists y, z \in [x] \text{ such that } y \not\sim z \}.$$ 

Since $X$ has countable basis, $C$ is necessarily countable. Note that $C$ is the set of points at which $\sim$ fails to be transitive. Since for each point $c \in C$, $[c]$ splits up into two subsets on which $\sim$ is transitive, we will blow the point $c$ up into a segment $[c_1, c_2]$ which then splits the set $[c] \setminus C$ into two sets, $[c_1]$ and $[c_2]$, and $\sim$ will be transitive on each of these sets. More precisely, choose an orientation for $X$, and let $X'$ denote the Denjoy blow-up of $X$ along $C$.

**Lemma 9.2.** The relation $\sim$ is transitive on $X'$.

**Proof.** Let $C' = \{ x \in X' | \exists y, z \in [x] \text{ with } y \not\sim z \}$. We wish to show that $C' = \emptyset$.

Let $c \in C$. Consider $y, z \in [c]$ with $y \not\sim z$. Then $\exists y', z'$ such that $[[y', y]] = [[y', c]]$ and $[[y', c]] = [[z', c]]$ but $[[y', c]] \cap [[z', c]] = \emptyset$. Hence, since $X$ is a 1-manifold, any other set of the form $[[r, c]]$ must intersect either $[[y', c]]$ or $[[z', c]]$ (but of course not both) in some set $[[r', c]]$. In addition, since $X$ is an oriented 1-manifold, exactly one of $y'$ and $z'$ is in $c^+$. Hence if $w \in [c], w$ distinct from $y, z, c$, then necessarily $\exists w'$ such that $[[w', c]] = [[w', w]]$, which implies that $w \sim y$ or $w \sim z$, but not both. Moreover, if $v \in [c], v$ distinct from $y, z, c, w$, and either $w \sim y$ and $v \sim y$, or $w \sim z$ and $v \sim z$, then $w \sim v$.

Hence $[c] \setminus C$ splits into two subsets: $[c]_y = \{ b \in [c] | b \sim y \}$ and $[c]_z = \{ b \in [c] | b \sim z \}$. Moreover, notice that in $X'$, either $[c_1] = [c]_y$ and $[c_2] = [c]_z$ if $y' \in c^+$, or vice versa if $y' \in c^-$. See Figure 23. In particular, $C' \cap \{c_1, c_2 | c \in C\} = \emptyset$. But since $|x| = 1$ for all $x \in (c_1, c_2)$, for all $c \in C$, we know that $C' \subset \{c_1, c_2 | c \in C\}$. So $C' = \emptyset$. 

□
9.2. **Star Denjoy blow-up of an order tree.** We now describe a similar blow-up construction for an $\mathbb{R}$-order tree $T$ which will result in an order tree $T'$ on which the relation $\sim$ is transitive. As in the 1-manifold case, we need to replace the set

$$C = \{x \in T | \exists y, z \in [x] \text{ such that } y \not\sim z\}.$$ 

However, since $T$ may not be a 1-manifold, it is no longer the case that for each $x \in C$, $[x]$ splits up into just two sets on which $\sim$ is transitive. Instead $[x]$ splits into at most countably many such subsets, one for each $T_{\{x,y\}}$ where $y \sim x, y \neq x$. So we replace the point $x$ by a union of segments, one for each $T_{\{x,y\}}$, all identified at exactly one common endpoint into a star shape. Then if we denote the center of the star by $x$, and the segment $[x, x']$ corresponds to the tree $T_{\{x,y\}}$, we define a set of segments $S'$ for $T'$ in the obvious way so that both $x$ and $y$ are limit points of the distinguished ray of the tree $T_{\{x,y\}}$.

If $T$ is oriented, the orientation extends naturally to $T'$. If we begin with an action of $G$ on $T$, we may extend to an action on $T'$ in the natural way. This blow-up insures that the relation $\sim$ is transitive on $T'$.

**Definition 9.3.** Hence, given an $\mathbb{R}$-order tree $T$ we may define the Hausdorff tree associated to $T$ as follows. Set

$$T_H = \{[x] | x \in T'\},$$

and set

$$S_H = \{[[i(\sigma)], [f(\sigma)]]| \sigma \in S'\}.$$ 

Then $(T_H, S_H)$ is a Hausdorff $\mathbb{R}$-order tree, which we call the Hausdorff tree associated to the oriented $\mathbb{R}$-order tree $T$. Given an orientation $(S')_+$ for $T$, we say that the orientation

$$(S_H)_+ = \{[[i(\sigma)], [f(\sigma)]]| \sigma \in (S')_+\}$$

is the orientation on $T_H$ induced by, or inherited from, the orientation on $T$. Define

$$p : T \to T_H : x \mapsto [x].$$

9.3. **Linear Denjoy blow-up of an oriented order tree.** Now let $T$ be any oriented $\mathbb{R}$-order tree, with set $S$ of segments and orientation $S_+$. Occasionally it is useful to allow the Denjoy blow-up of points to intervals in a way more closely following the construction of Section 9.1. In this construction, the orientation of $T$ plays a crucial role. We proceed as follows. Let $C \subset T$ be countable. Let $< \text{ denote the partial order on } T \text{ induced by the orientation } S_+$. Again, for every $c \in C$, let $[c_1, c_2]$ denote an associated closed interval in $\mathbb{R}$, with standard orientation satisfying $c_1 < c_2$. We assume that the intervals $[c_1, c_2], c \in C$, are pairwise disjoint and disjoint from $T$. Let $Y$ be the set obtained from $T$ by replacing each $c \in C$ with the corresponding interval $[c_1, c_2]$, and put an $\mathbb{R}$-order tree structure on $Y$ as follows.
For each $\sigma \in S_+$, set
\[
\sigma' = (\sigma \cap (T \setminus \mathcal{C})) \cup \left( \bigcup_{c \in \mathcal{C} \cap \sigma} [c_1, c_2] \right),
\]
with linear ordering determined uniquely by the following conditions:

- If $x < y$ for some $x, y \in \sigma \setminus \mathcal{C}$ or $x, y \in [c_1, c_2]$, for some $c \in \mathcal{C}$, then $x < y$.
- If $c < y$ for some $y \in \sigma \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap \sigma$, then $c_2 < y$.
- If $c > y$ for some $y \in \sigma \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap \sigma$, then $c_1 > y$.

Let $(\mathcal{S}')_+$ be the smallest orientation on $Y$ containing $\{\sigma' | \sigma \in S_+\}$. Let $S'$ be the smallest set satisfying the defining axioms of $\mathbb{R}$-order tree and also containing $(\mathcal{S}')_+$. Then $(T', S')$ with orientation $(\mathcal{S}')_+$ is an oriented $\mathbb{R}$-order tree. Notice that if we remove the requirement that $T$ be second countable, then we may remove the condition that $\mathcal{C}$ be countable in this construction.

**Definition 9.4.** $(T', S')$ is called the linear Denjoy blow-up of $T$ along $\mathcal{C}$ with respect to the orientation $S_+$. If we begin with an action of $G$ on $T$ and extend this action linearly over the intervals $[c_1, c_2], c \in \mathcal{C}$, we call the resulting action of $G$ on $Y$ the (canonically) induced action.

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**References**


**Department of Mathematics, Washington University, St Louis, Missouri 63130**

*E-mail address: roberts@math.wustl.edu*

**Department of Mathematics, Washington University, St Louis, Missouri 63130**

*E-mail address: shareshi@math.wustl.edu*

**Department of Mathematics, Trinity College, Hartford, Connecticut 06106**

*E-mail address: Melanie.Stein@mail.cc.trincoll.edu*