

## CONFORMAL RESTRICTION: THE CHORDAL CASE

GREGORY LAWLER, ODED SCHRAMM, AND WENDELIN WERNER

### 1. INTRODUCTION

Conformal field theory has been extremely successful in predicting the exact values of critical exponents describing the behavior of two-dimensional systems from statistical physics. In particular, in the fundamental papers [5, 6], which were used and extended to the case of the “surface geometry” in [9], it is argued that there is a close relationship between critical planar systems and some families of conformally invariant fields. This gave rise to intense activity both in the theoretical physics community (predictions on the exact value of various exponents or quantities) and in the mathematical community (the study of highest-weight representations of certain Lie algebras). However, on the mathematical level, the explicit relation between the two-dimensional systems and these fields remained rather mysterious.

More recently, a one-parameter family of random processes called *stochastic Loewner evolution*, or *SLE*, was introduced [44]. The  $SLE_\kappa$  process is obtained by solving Loewner’s differential equation with driving term  $B(\kappa t)$ , where  $B$  is one-dimensional Brownian motion,  $\kappa > 0$ . The SLE processes are continuous, conformally invariant scaling limits of various discrete curves arising in the context of two-dimensional systems. In particular, for the models studied by physicists for which conformal field theory (CFT) has been applied and for which exponents have been predicted, it is believed that SLE arises in some way in the scaling limit. This has been proved for site-percolation on the triangular lattice [46], loop-erased random walks [29] and the uniform spanning tree Peano path [29] (a.k.a. the Hamiltonian path on the Manhattan lattice). Other models for which this is believed include the Ising model, the random cluster (or Potts) models with  $q \leq 4$ , and the self-avoiding walk.

In a series of papers [23, 24, 25, 26], the authors derived various properties of the stochastic Loewner evolution  $SLE_6$  and used them to compute the “intersection exponents” for planar Brownian paths. This program was based on the earlier realization [32] that any conformally invariant process satisfying a certain restriction property has crossing or intersection exponents that are intimately related to these Brownian intersection exponents. In particular, [32] predicted a strong relation between planar Brownian motion, self-avoiding walks, and critical percolation. As the boundary of  $SLE_6$  is conformally invariant, satisfies restriction, and can be well understood, computations of its exponents yielded the Brownian intersection exponents (in particular, exponents that had been predicted by Duplantier-Kwon

---

Received by the editors October 9, 2002.

2000 *Mathematics Subject Classification*. Primary 60K35, 82B27, 60J69, 30C99.

*Key words and phrases*. Conformal invariance, restriction property, random fractals, SLE.

©2003 American Mathematical Society

[15, 14], disconnection exponents, and Mandelbrot's conjecture [34] that the Hausdorff dimension of the boundary of planar Brownian motion is  $4/3$ ). Similarly, the determination of the critical exponents for  $\text{SLE}_6$  in [23, 24, 25] combined with Smirnov's [46] proof of conformal invariance for critical percolation on the triangular lattice (along with Kesten's hyperscaling relations) facilitated proofs of several fundamental properties of critical percolation [47, 28, 45], some of which had been predicted in the theoretical physics literature, e.g., [37, 35, 36, 38, 43].

The main goal of the present paper is to investigate more deeply the restriction property that was instrumental in relating  $\text{SLE}_6$  to Brownian motion. One of our initial motivations was also to understand the scaling limit and exponents of the two-dimensional self-avoiding walk. Another motivation was to reach a clean understanding of the relation between SLE and conformal field theory. Consequences of the present paper in this direction are the subject of [18, 19]. See also [2, 3] for aspects of SLE from a CFT perspective.

Let us now briefly describe the conformal restriction property which we study in the present paper: Consider a simply connected domain in the complex plane  $\mathbb{C}$ , say the upper half-plane  $\mathbb{H} := \{x + iy : y > 0\}$ . Suppose that two boundary points are given, say 0 and  $\infty$ . We are going to study closed random subsets  $K$  of  $\mathbb{H}$  such that:

- $\overline{K} \cap \mathbb{R} = \{0\}$ ,  $K$  is unbounded and  $\mathbb{H} \setminus K$  has two connected components.
- For all simply connected subsets  $H$  of  $\mathbb{H}$  such that  $\mathbb{H} \setminus H$  is bounded and bounded away from the origin, the law of  $K$  conditioned on  $K \subset H$  is equal to the law of  $\Phi(K)$ , where  $\Phi$  is a conformal map from  $\mathbb{H}$  onto  $H$  that preserves the boundary points 0 and  $\infty$ .

The law of such a set  $K$  is called a (chordal) restriction measure. It turns out that there exists only a one-parameter family  $\mathbf{P}_\alpha$  of such probability measures, where  $\alpha$  is a positive number, and that

$$(1.1) \quad \mathbf{P}_\alpha[K \subset H] = \Phi'(0)^\alpha$$

when  $\Phi : H \rightarrow \mathbb{H}$  is chosen in such a way that  $\Phi(z)/z \rightarrow 1$  as  $z \rightarrow \infty$ . The measure  $\mathbf{P}_1$  can be constructed easily by filling in the closed loops of a Brownian excursion in  $\mathbb{H}$ , i.e., a Brownian motion started from 0 and conditioned to stay in  $\mathbb{H}$  for all positive times.

Some of the main results of this paper can be summarized as follows:

- (1) The restriction measure  $\mathbf{P}_\alpha$  exists if and only if  $\alpha \geq 5/8$ .
- (2) The only measure  $\mathbf{P}_\alpha$  that is supported on simple curves is  $\mathbf{P}_{5/8}$ . It is the law of chordal  $\text{SLE}_{8/3}$ .
- (3) The measures  $\mathbf{P}_\alpha$  for  $\alpha > 5/8$  can be constructed by adding to the chordal  $\text{SLE}_\kappa$  curve certain Brownian bubbles with intensity  $\lambda$ , where  $\alpha$ ,  $\lambda$  and  $\kappa$  are related by

$$\alpha(\kappa) = \frac{6 - \kappa}{2\kappa}, \quad \lambda(\kappa) = \frac{(8 - 3\kappa)(6 - \kappa)}{2\kappa}.$$

- (4) For all  $\alpha \geq 5/8$ , the dimension of the boundary of  $K$  defined under  $\mathbf{P}_\alpha$  is almost surely  $4/3$  and locally "looks like" an  $\text{SLE}_{8/3}$  curve. In particular, the Brownian frontier (i.e., the outer boundary of the Brownian path) looks like a symmetric curve.

As pointed out in [30], this gives strong support to the conjecture that chordal  $\text{SLE}_{8/3}$  is the scaling limit of the infinite self-avoiding walk in the upper half-plane

and allows one to recover (modulo this conjecture) the critical exponents that had been predicted in the theoretical physics literature (e.g., [36, 16]). This conjecture has recently been tested [21, 22] by Monte Carlo methods. Let us also mention (but this will not be the subject of the present paper; see [18, 19]) that in conformal field theory language,  $-\lambda(\kappa)$  is the central charge of the Virasoro algebra associated to the discrete models (that correspond to  $\text{SLE}_\kappa$ ) and that  $\alpha$  is the corresponding highest-weight (for a degenerate representation at level 2).

To avoid confusion, let us point out that  $\text{SLE}_6$  is not a chordal restriction measure as defined above. However, it satisfies *locality*, which implies a different form of restriction. We give below a proof of locality for  $\text{SLE}_6$ , which is significantly simpler than the original proof appearing in [23].

We will also study a slightly different restriction property, which we call *right-sided restriction*. The measures satisfying right-sided restriction similarly form a 1-parameter collection  $\mathbf{P}_\alpha^+$ ,  $\alpha > 0$ . We present several constructions of the measures  $\mathbf{P}_\alpha^+$ . First, when  $\alpha \geq 5/8$ , these can be obtained from the measures  $\mathbf{P}_\alpha$  (basically, by keeping only the right-side boundary). When  $\alpha \in (0, 1)$ , the measure  $\mathbf{P}_\alpha^+$  can also be obtained from an appropriately reflected Brownian excursion. It follows that one can reflect a Brownian excursion off a ray in such a way that its boundary will have precisely the law of chordal  $\text{SLE}_{8/3}$ . A third construction of  $\mathbf{P}_\alpha^+$  (valid for all  $\alpha > 0$ ) is given by a process we call  $\text{SLE}(8/3, \rho)$ . The process  $\text{SLE}(\kappa, \rho)$  is a variant of SLE where a drift is added to the driving function. In fact, it is just Loewner's evolution driven by a Bessel-type process.

The word *chordal* refers to connected sets joining two boundary points of a domain. There is an analogous *radial* theory, which investigates sets joining an interior point to the boundary of the domain. This will be the subject of a forthcoming paper [31].

We now briefly describe how this paper is organized. In the preliminary section, we give some definitions and notation and we derive some simple facts that will be used throughout the paper. In Section 3, we study the family of chordal restriction measures, and we show (1.1). Section 4 is devoted to the Brownian excursions. We define these measures and use a result of B. Virág (see [49]) to show that the filling of such a Brownian excursion has the law  $\mathbf{P}_1$ .

The key to several of the results of the present paper is the study of the distortion of SLE under conformal maps, for instance, the evolution of the image of the SLE path under the mapping  $\Phi$  (as long as the SLE path remains in  $H$ ), which is the subject of Section 5. This study can be considered as a cleaner and more advanced treatment of similar questions addressed in [23]. In particular, we obtain a new short proof of the locality property for  $\text{SLE}_6$ , which was essential in the papers [23, 24, 25].

The SLE distortion behaviour is then also used in Section 6 to prove that the law of chordal  $\text{SLE}_{8/3}$  is  $\mathbf{P}_{5/8}$  and is also instrumental in Section 7, where we show that all measures  $\mathbf{P}_\alpha$  for  $\alpha > 5/8$  can be constructed by adding a Poisson cloud of bubbles to SLE curves.

The longer Section 8 is devoted to the one-sided restriction measures  $\mathbf{P}_\alpha^+$ . As described above, we exhibit various constructions of these measures and show as a by-product of this description that the two-sided measures  $\mathbf{P}_\alpha$  do not exist for  $\alpha < 5/8$ .

A recurring theme in the paper is the principle that the law  $\mathbf{P}$  of a random set  $K$  can often be characterized and understood through the function  $A \mapsto \mathbf{P}[K \cap A \neq \emptyset]$  on an appropriate collection of sets  $A$ . In Section 9 we use this to show that the outer boundary of chordal  $\text{SLE}_6$  is the same as the outer boundary (frontier) of appropriately reflected Brownian motion and the outer boundary of full-plane  $\text{SLE}_6$  stopped on hitting the unit circle is the same as the outer boundary of Brownian motion stopped on hitting the unit circle.

We conclude the paper with some remarks and pointers to papers in preparation.

## 2. PRELIMINARIES

In this section some definitions and notation will be given and some basic facts will be recalled.

**Important domains.** The upper half plane  $\{x + iy : x \in \mathbb{R}, y > 0\}$  is denoted by  $\mathbb{H}$ , the complex plane by  $\mathbb{C}$ , the extended complex plane by  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  and the unit disk by  $\mathbb{U}$ .

**Bounded hulls.** Let  $\mathcal{Q}$  be the set of all bounded  $A \subset \overline{\mathbb{H}}$  such that  $A = \overline{A \cap \mathbb{H}}$  and  $\mathbb{H} \setminus A$  is (connected and) simply connected. We call such an  $A$  a *bounded hull*.

**The normalized conformal maps  $g_A$ .** For each  $A \in \mathcal{Q}$ , there is a unique conformal transformation  $g_A : \mathbb{H} \setminus A \rightarrow \mathbb{H}$  with  $g_A(z) - z \rightarrow 0$  as  $z \rightarrow \infty$ . We can then define (as in [23])

$$(2.1) \quad a(A) := \lim_{z \rightarrow \infty} z(g_A(z) - z).$$

First note that  $a(A)$  is real, because  $g_A(z) - z$  has a power series expansion in  $1/z$  near  $\infty$  and is real on the real line in a neighborhood of  $\infty$ . Also note that

$$(2.2) \quad a(A) = \lim_{y \rightarrow \infty} y H(iy),$$

where  $H(z) = \text{Im}(z - g(z))$  is the bounded harmonic function on  $\mathbb{H} \setminus A$  with boundary values  $\text{Im } z$ . Hence,  $a(A) \geq 0$ , and  $a(A)$  can be thought of as a measure of the size of  $A$  as seen from infinity. We will call  $a(A)$  the *half-plane capacity* of  $A$  (from infinity). The useful scaling rule for  $a(A)$ ,

$$(2.3) \quad a(\lambda A) = \lambda^2 a(A)$$

is easily verified directly. Since  $\text{Im } g_A(z) - \text{Im } z$  is harmonic, bounded, and has nonpositive boundary values,  $\text{Im } g_A(z) \leq \text{Im } z$ . Consequently,

$$(2.4) \quad 0 < g'_A(x) \leq 1, \quad x \in \mathbb{R} \setminus A.$$

(In fact,  $g'_A(x)$  can be viewed as the probability of an event; see Proposition 4.1.)

**\*-hulls.** Let  $\mathcal{Q}^*$  be the set of  $A \in \mathcal{Q}$  with  $0 \notin A$ . We call such an  $A$  a *\*-hull*. If  $A \in \mathcal{Q}^*$ , then  $H = \mathbb{H} \setminus A$  is as the  $H$  in the introduction.

**The normalized conformal maps  $\Phi_A$ .** For  $A \in \mathcal{Q}^*$ , we define  $\Phi_A(z) = g_A(z) - g_A(0)$ , which is the unique conformal transformation  $\Phi$  of  $\mathbb{H} \setminus A$  onto  $\mathbb{H}$  fixing 0 and  $\infty$  with  $\Phi(z)/z \rightarrow 1$  as  $z \rightarrow \infty$ .

**Semigroups.** Let  $\mathcal{A}$  be the set of all conformal transformations  $\Phi : \mathbb{H} \setminus A \rightarrow \mathbb{H}$  with  $\Phi(0) = 0$  and  $\Phi(\infty) = \infty$ , where  $A \in \mathcal{Q}^*$ . That is,  $\mathcal{A} = \{\lambda \Phi_A : \lambda > 0, A \in \mathcal{Q}^*\}$ . Also let  $\mathcal{A}_1 = \{\Phi_A : A \in \mathcal{Q}^*\}$ . Note that  $\mathcal{A}$  and  $\mathcal{A}_1$  are both semigroups under composition. (Of course, the domain of  $\Phi_1 \circ \Phi_2$  is  $\Phi_2^{-1}(H_1)$  if  $H_1$  is the domain of  $\Phi_1$ .) We can consider  $\mathcal{Q}^*$  as a semigroup with the product  $\cdot$ , where  $A \cdot A'$  is defined by  $\Phi_{A \cdot A'} = \Phi_A \circ \Phi_{A'}$ . Note that

$$(2.5) \quad a(A \cdot A') = a(A) + a(A').$$

As  $a(A) \geq 0$ , this implies that  $a(A)$  is monotone in  $A$ .

**$\pm$ -hulls.** Let  $\mathcal{Q}_+$  be the set of  $A \in \mathcal{Q}^*$  with  $A \cap \mathbb{R} \subset (0, \infty)$ . Let  $\sigma$  denote the orthogonal reflection about the imaginary axis, and let  $\mathcal{Q}_- = \{\sigma(A) : A \in \mathcal{Q}_+\}$  be the set of  $A \in \mathcal{Q}^*$  with  $A \cap \mathbb{R} \subset (-\infty, 0)$ . If  $A \in \mathcal{Q}^*$ , then we can find unique  $A_1, A_3 \in \mathcal{Q}_+$  and  $A_2, A_4 \in \mathcal{Q}_-$  such that  $A = A_1 \cdot A_2 = A_4 \cdot A_3$ . Note that  $\mathcal{Q}_+, \mathcal{Q}_-$  are semigroups.

**Smooth hulls.** We will call  $A \in \mathcal{Q}$  a smooth hull if there is a smooth curve  $\gamma : [0, 1] \rightarrow \mathbb{C}$  with  $\gamma(0), \gamma(1) \in \mathbb{R}$ ,  $\gamma(0, 1) \subset \mathbb{H}$ ,  $\gamma(0, 1)$  has no self-intersections, and  $\mathbb{H} \cap \partial A = \gamma(0, 1)$ . Any smooth hull in  $\mathcal{Q}^*$  is in  $\mathcal{Q}_+ \cup \mathcal{Q}_-$ .

**Fillings.** If  $A \subset \overline{\mathbb{H}}$  is closed, let  $\mathcal{F}_{\mathbb{H}}(A)$  denote the set of all  $z \in \overline{\mathbb{H}}$  such that any path from  $z$  to  $\infty$  in  $\overline{\mathbb{H}} \cup \{\infty\}$  meets  $A$ . In other words,  $\mathcal{F}_{\mathbb{H}}(A)$  is the union of  $A$  and all the bounded connected components of  $\overline{\mathbb{H}} \setminus A$ . Similarly,  $\mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(A)$  denotes the union of  $A$  with the connected components of  $\overline{\mathbb{H}} \setminus A$  which do not intersect  $\mathbb{R}$  and  $\mathcal{F}_{\mathbb{H}}^{\mathbb{R}+}(A)$  denotes the union of  $A$  with the connected components of  $\overline{\mathbb{H}} \setminus A$  which do not intersect  $[0, \infty)$ . Also, for closed  $A \subset \mathbb{C}$ ,  $\mathcal{F}_{\mathbb{C}}(A)$  denotes the union of  $A$  with the bounded connected components of  $\mathbb{C} \setminus A$ .

**Approximation.** We will sometimes want to approximate  $A \in \mathcal{Q}$  by smooth hulls. The idea of approximating general domains by smooth hulls is standard (see, e.g., [17, Theorem 3.2]).

**Lemma 2.1.** *Suppose  $A \in \mathcal{Q}_+$ . Then there exists a decreasing sequence of smooth hulls  $(A_n)_{n \geq 1}$  such that  $A = \bigcap_{n=1}^{\infty} A_n$  and the increasing sequence  $\Phi'_{A_n}(0)$  converges to  $\Phi'_A(0)$ .*

*Proof.* The existence of the sequence  $A_n$  can be obtained by various means, for example, by considering the image under  $\Phi_A^{-1}$  of appropriately chosen paths. The monotonicity of  $\Phi'_{A_n}(0)$  follows immediately from the monotonicity of  $A_n$  and (2.4). The convergence is immediate by elementary properties of conformal maps, since  $\Phi_{A_n}$  converges locally uniformly to  $\Phi_A$  on  $\mathbb{H} \setminus A$ .  $\square$

**Covariant measures.** Our aim in the present paper is to study measures on subsets of  $\mathbb{H}$ . In order to simplify further definitions, we give a general definition that can be applied in various settings.

Suppose that  $\mu$  is a measure on a measurable space  $\Omega$  whose elements are subsets of a domain  $D$ . Suppose that  $\Gamma$  is a set of conformal transformations from subdomains  $D' \subset D$  onto  $D$  that is closed under composition. We say that  $\mu$  is *covariant under  $\Gamma$*  (or  $\Gamma$ -*covariant*) if for all  $\varphi \in \Gamma$ , the measure  $\mu$  restricted to the set  $\varphi^{-1}(\Omega) := \{\varphi^{-1}(K) : K \subset D\}$  is equal to a constant  $F_\varphi$  times the image measure  $\mu \circ \varphi^{-1}$ .

If  $\mu$  is a finite  $\Gamma$ -covariant measure, then  $F_\varphi = \mu[\varphi^{-1}(\Omega)]/\mu[\Omega]$ . Note that a probability measure  $\mathbf{P}$  on  $\Omega$  is  $\Gamma$ -covariant if and only if for all  $\varphi \in \Gamma$  with  $F_\varphi = \mathbf{P}[\varphi^{-1}(\Omega)] > 0$ , the conditional law of  $\mathbf{P}$  on  $\varphi^{-1}(\Omega)$  is equal to  $\mathbf{P} \circ \varphi^{-1}$ .

Also note that if  $\mu$  is covariant under  $\Gamma$ , then  $F_{\varphi \circ \psi} = F_\varphi F_\psi$  for all  $\varphi, \psi \in \Gamma$ , because the image measure of  $\mu$  under  $\varphi^{-1}$  is  $F_\varphi^{-1} \mu$  restricted to  $\varphi^{-1}(\Omega)$ , so that the image under  $\psi^{-1}$  of this measure is  $F_\varphi^{-1} F_\psi^{-1} \mu$  restricted to  $\psi^{-1} \circ \varphi^{-1}(\Omega)$ . Hence, the mapping  $F : \varphi \mapsto F_\varphi$  is a semigroup homomorphism from  $\Gamma$  into the commutative multiplicative semigroup  $[0, \infty)$ . When  $\mu$  is a probability measure, this mapping is into  $[0, 1]$ .

We say that a measure  $\mu$  is  $\Gamma$ -invariant if it is  $\Gamma$ -covariant with  $F_\varphi \equiv 1$ .

**Chordal Loewner chains.** Throughout this paper, we will make use of chordal Loewner chains. Let us very briefly recall their definition (see [23] for details). Suppose that  $W = (W_t, t \geq 0)$  is a real-valued continuous function. Define for each  $z \in \overline{\mathbb{H}}$ , the solution  $g_t(z)$  of the initial value problem

$$(2.6) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

For each  $z \in \overline{\mathbb{H}}$  there is a time  $\tau = \tau(z) \in [0, \infty]$  such that the solution  $g_t(z)$  exists for  $t \in [0, \tau]$  and  $\lim_{t \nearrow \tau} g_t(z) = W_\tau$  if  $\tau < \infty$ . The evolving hull of the Loewner evolution is defined as  $K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$ ,  $t \geq 0$ . It is not hard to check that  $K_t \in \mathcal{Q}$ .

Then, it is easy to see that  $g_t$  is the unique conformal map from  $\mathbb{H} \setminus K_t$  onto  $\mathbb{H}$  such that  $g_t(z) = z + o(1)$  when  $z \rightarrow \infty$ . Also,  $a(K_t) = 2t$ . When the function  $W$  is chosen to be  $W_t = \sqrt{\kappa} B_t$ , where  $B$  is a standard one-dimensional Brownian motion, then the corresponding random Loewner chain is chordal SLE $_\kappa$  (SLE stands for stochastic Loewner evolution).

### 3. TWO-SIDED RESTRICTION

In this section we will be studying certain probability measures on a collection  $\Omega$  of subsets of  $\mathbb{H}$ . We start by defining  $\Omega$ .

**Definition 3.1.** Let  $\Omega$  be the collection of relatively closed subsets  $K$  of  $\mathbb{H}$  such that

- (1)  $K$  is connected,  $\overline{K} \cap \mathbb{R} = \{0\}$  and  $K$  is unbounded.
- (2)  $\mathbb{C} \setminus \overline{K}$  is connected.

A simple example of a set  $K \in \Omega$  is a simple curve  $\gamma$  from 0 to infinity in the upper half-plane. If  $\gamma$  is just a curve from zero to infinity in the upper half-plane with double-points, then one can take  $K = \mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(\gamma) \in \Omega$ , which is the set obtained by filling in the loops created by  $\gamma$ .

We endow  $\Omega$  with the  $\sigma$ -field generated by the events  $\{K \in \Omega : K \cap A = \emptyset\}$ , where  $A \in \mathcal{Q}^*$ . It is easy to check that this family of events is closed under finite intersection, so that a probability measure on  $\Omega$  is characterized by the values of  $\mathbf{P}[K \cap A = \emptyset]$  for  $A \in \mathcal{Q}^*$ . Thus,

**Lemma 3.2.** *Let  $\mathbf{P}$  and  $\mathbf{P}'$  be two probability measures on  $\Omega$ . If  $\mathbf{P}[K \cap A = \emptyset] = \mathbf{P}'[K \cap A = \emptyset]$  holds for every  $A \in \mathcal{Q}^*$ , then  $\mathbf{P} = \mathbf{P}'$ .*

It is worthwhile to note that the  $\sigma$ -field on  $\Omega$  is the same as the Borel  $\sigma$ -field induced by the Hausdorff metric on closed subsets of  $\overline{\mathbb{H}} \cup \{\infty\}$ .

**Proposition 3.3.** *Let  $\Gamma$  be the semigroup of dilations,  $z \mapsto \lambda z, \lambda > 0$ . For any probability measure  $\mathbf{P}$  on  $\Omega$ , the following four statements are equivalent:*

- (1)  $\mathbf{P}$  is  $\Gamma$ -invariant and  $\mathcal{A}_1$ -covariant.
- (2)  $\mathbf{P}$  is  $\mathcal{A}$ -covariant.
- (3) There exists an  $\alpha > 0$  such that for all  $A \in \mathcal{Q}^*$ ,

$$\mathbf{P}[K \cap A = \emptyset] = \Phi'_A(0)^\alpha.$$

- (4) There exists an  $\alpha > 0$  such that for all smooth hulls  $A \in \mathcal{Q}^*$ ,

$$\mathbf{P}[K \cap A = \emptyset] = \Phi'_A(0)^\alpha.$$

Moreover, for each fixed  $\alpha > 0$ , there exists at most one probability measure  $\mathbf{P}_\alpha$  satisfying these conditions.

**Definition 3.4.** If the measure  $\mathbf{P}_\alpha$  exists, we call it the *two-sided restriction measure with exponent  $\alpha$* .

*Proof.* Lemma 3.2 shows that a measure satisfying (3) is unique. A probability measure is  $\Gamma$ -covariant if and only if it is  $\Gamma$ -invariant. Therefore, (1) and (2) are equivalent. As noted above, any  $A \in \mathcal{Q}^*$  can be written as  $A_+ \cdot A_-$  with  $A_\pm \in \mathcal{Q}_\pm$ . Using this and Lemma 2.1, we may deduce that conditions (3) and (4) are also equivalent. Since  $\Phi'_{\lambda A}(0) = \Phi'_A(0)$  for  $A \in \mathcal{Q}^*, \lambda > 0$ , (3) together with Lemma 3.2 implies that  $\mathbf{P}$  is  $\Gamma$ -invariant. Because  $\Phi'_{A_1 \cdot A_2}(0) = \Phi'_{A_1}(0) \Phi'_{A_2}(0)$ , (3) also implies that for all  $A_1, A_2 \in \mathcal{Q}^*$ ,

$$\mathbf{P}[K \cap (A_1 \cdot A_2) = \emptyset] = \mathbf{P}[K \cap A_1 = \emptyset] \mathbf{P}[K \cap A_2 = \emptyset],$$

which implies (1). Hence, it suffices to show that (1) implies (4).

Suppose (1) holds. Define the homomorphism  $F$  of  $\mathcal{Q}^*$  onto the multiplicative semigroup  $(0, 1]$  by  $F(A) = \mathbf{P}[K \cap A = \emptyset]$ . We also write  $F(\Phi_A)$  for  $F(A)$ . Let  $G_t(z)$  be the solution of the initial value problem

$$(3.1) \quad \partial_t G_t(z) = \frac{2G_t(z)}{G_t(z) - 1}, \quad G_0(z) = z,$$

for  $z \in \overline{\mathbb{H}}$ . Note that this function can equivalently be defined as  $G_t(z) = g_t(z) - g_t(0) = g_t(z) + 2t$ , where  $(g_t)$  is the chordal Loewner chain driven by the function  $W_t = 1 - 2t$ . Hence,  $G_t$  is the unique conformal map from  $\mathbb{H} \setminus K_t$  onto  $\mathbb{H}$  such that  $G_t(0) = 0$  and  $G_t(z)/z \rightarrow 1$  when  $z \rightarrow \infty$ . (Here,  $K_t$  is the evolving hull of  $g_t$ .) Also, and this is why we focus on these functions  $G_t$ , one has  $G_t \circ G_s = G_{t+s}$  in  $\mathbb{H} \setminus K_{t+s}$ , for all  $s, t \geq 0$ . Since  $F$  is a homomorphism, this implies that  $F(G_t) = \exp(-2\alpha t)$  for some constant  $\alpha \geq 0$  and all  $t \geq 0$ , or that  $F(G_t) = 0$  for all  $t > 0$ . However, the latter possibility would imply that  $K \cap K_t \neq \emptyset$  a.s., for all  $t > 0$ . Since  $\bigcap_{t>0} K_t = \{1\}$  and  $1 \notin \overline{K}$ , this is ruled out. Hence,  $F(G_t) = \exp(-2\alpha t)$ ,  $t \geq 0$ . Differentiating (3.1) with respect to  $z$  gives  $G'_t(0) = \exp(-2t)$ . Thus  $F(G_t) = G'_t(0)^\alpha$ .

Now, set  $G_t^\lambda(z) = \lambda G_t(\lambda^{-1}z)$ ,  $\lambda > 0$ . Then  $G_t^\lambda : \mathbb{H} \setminus \lambda K_t \rightarrow \mathbb{H}$  is a suitably normalized conformal map. By our assumption of scale invariance of the law of  $K$ , we have  $F(G_t^\lambda) = F(G_t) = G'_t(0)^\alpha = (G_t^\lambda)'(0)^\alpha$ . We may therefore conclude that  $F(\Phi_A) = \Phi'_A(0)^\alpha$  for every  $A$  in the semigroup  $\mathcal{A}_0$  generated by  $\{\lambda K_t : t \geq 0, \lambda > 0\}$ . To deduce that

$$(3.2) \quad \forall A \in \mathcal{Q}_+, \quad F(\Phi_A) = \Phi'_A(0)^\alpha,$$

we rely on the following lemma:

**Lemma 3.5.** *There exists a topology on  $\mathcal{Q}_+$  for which  $\mathcal{A}_0$  is dense,  $F$  is continuous, and  $\Phi_A \mapsto \Phi'_A(0)$  is continuous.*

*Proof of Lemma 3.5.* Given  $A \in \mathcal{Q}_+$  and a sequence  $\{A_n\} \subset \mathcal{Q}_+$ , we say that  $A_n \in \mathcal{Q}_+$  converges to  $A \in \mathcal{Q}_+$  if  $\Phi_{A_n}$  converges to  $\Phi_A$  uniformly on compact subsets of  $\overline{\mathbb{H}} \setminus A$  and  $\bigcup_n A_n$  is bounded away from 0 and  $\infty$ . (This is very closely related to what is known as the Carathéodory topology.)

Now assume that  $A_n \rightarrow A$ , where  $A_n, A \in \mathcal{Q}_+$ . It is immediate that  $\Phi'_{A_n}(0) \rightarrow \Phi'_A(0)$ , by Cauchy’s derivative formula (the maps may be extended to a neighborhood of 0 by Schwarz reflection in the real line). Set  $A_n^+ = \Phi_{A_n}(A \setminus A_n)$  and  $A_n^- = \Phi_A(A_n \setminus A)$ . We claim that there is a constant  $\delta > 0$  and a sequence  $\delta_n \rightarrow 0$  such that

$$(3.3) \quad A_n^+ \cup A_n^- \subset \{x + iy : x \in [\delta, 1/\delta], y \leq \delta_n\}.$$

Indeed, since the map  $\Phi_{A_n} \circ \Phi_A^{-1}$  converges to the identity, locally uniformly in  $\mathbb{H}$ , it follows (e.g., from the argument principle) that for every compact set  $S \subset \mathbb{H}$  for all sufficiently large  $n$ ,  $S$  is contained in the image of  $\Phi_{A_n} \circ \Phi_A^{-1}$ , which means that  $A_n^+ \cap S = \emptyset$ . Similarly,  $\Phi_A^{-1} \circ \Phi_{A_n}$  converges locally uniformly in  $\overline{\mathbb{H}} \setminus A$  to the identity map, and this implies that  $A_n^- \cap S = \emptyset$  for all sufficiently large  $n$ . It is easy to verify that  $A_n^+ \cup A_n^-$  is bounded and bounded away from 0. Consequently, we have (3.3) for some fixed  $\delta > 0$  and some sequence  $\delta_n \rightarrow 0$ .

Suppose  $\limsup_{n \rightarrow \infty} F(A_n) > F(A)$ . Then  $\limsup_{n \rightarrow \infty} \mathbf{P}[K \cap A_n \neq \emptyset = K \cap A] > 0$ . By mapping over with  $\Phi_A$  and using (1), it then follows that there is some  $\epsilon > 0$  such that for infinitely many  $n$ ,  $\mathbf{P}[K \cap A_n^-] > \epsilon$ . Therefore, with positive probability,  $K$  intersects infinitely many  $A_n^-$ . Since  $K$  is closed and (3.3) holds, this would then imply that  $\mathbf{P}[K \cap [\delta, 1/\delta]] > 0$ , a contradiction. Thus  $\limsup_{n \rightarrow \infty} F(A_n) \leq F(A)$ . A similar argument also shows that  $\liminf_{n \rightarrow \infty} F(A_n) \geq F(A)$ , and so  $\lim_{n \rightarrow \infty} F(A_n) = F(A)$ , and the continuity of  $F$  is verified.

To complete the proof of the lemma, we now show that  $\mathcal{A}_0$  is dense in  $\mathcal{Q}_+$ . Let  $A \in \mathcal{Q}_+$ . Set  $A' := A \cup [x_0, x_1]$ , where  $x_0 := \inf(A \cap \mathbb{R})$  and  $x_1 := \sup(A \cap \mathbb{R})$ . For  $\delta > 0$ ,  $\delta < \Phi_A(x_0)/2$ , let  $D_\delta$  be the set of points in  $\mathbb{H}$  with distance at most  $\delta$  from  $[\Phi_A(x_0), \Phi_A(x_1)]$ . Let  $E_\delta$  denote the closure of  $A \cup \Phi_A^{-1}(D_\delta)$ . It is clear that  $E_\delta \rightarrow A$  as  $\delta \rightarrow 0+$  in the topology considered above. It thus suffices to approximate  $E_\delta$ . Note that  $\overline{\partial E_\delta \cap \mathbb{H}}$  is a simple path, say  $\beta : [0, s] \rightarrow \overline{\mathbb{H}}$  with  $\beta(0), \beta(s) \in \mathbb{R}$ . We may assume that  $\beta$  is parametrized by half-plane capacity from  $\infty$ , so that  $a(\beta[0, t]) = 2t$ ,  $t \in [0, s]$ . Set  $g_t := g_{\beta[0, t]}$ ,  $\Phi_t := \Phi_{\beta[0, t]} = g_t - g_t(0)$ ,  $U_t := g_t(\beta(t))$ ,  $\tilde{U}_t := U_t - g_t(0) = \Phi_t(\beta(t))$ ,  $t \in [0, s]$ . By the chordal version of Loewner’s theorem, we have

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}.$$

Thus,

$$(3.4) \quad \partial_t \Phi_t(z) = \frac{2}{\Phi_t(z) - \tilde{U}_t} + \frac{2}{\tilde{U}_t} = \frac{2\Phi_t(z)}{(\Phi_t(z) - \tilde{U}_t)\tilde{U}_t}, \quad \Phi_0(z) = z.$$

Since  $\tilde{U}_t$  is continuous and positive, there is a sequence of piecewise constant functions  $\tilde{U}_t^{(n)} : [0, s] \rightarrow (0, \infty)$  such that  $\sup\{|\tilde{U}_t^{(n)} - \tilde{U}_t| : t \in [0, s]\} \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\Phi_t^{(n)}$  be the solution of (3.4) with  $\tilde{U}_t^{(n)}$  replacing  $\tilde{U}_t$ . Then, clearly,



$\Phi_s^{(n)}(z) \rightarrow \Phi_s(z) = \Phi_{E_\delta}$  locally uniformly in  $\overline{\mathbb{H}} \setminus E_\delta$ . Note that the solution of (3.4) with  $\tilde{U}_t$  constant is of the form  $G_t^\lambda$ , where  $\lambda = \tilde{U}_0$  and  $t'$  is some function of  $t$  and  $\tilde{U}_0$ . It follows that  $\Phi_s^{(n)}$  is in the semigroup generated by  $\{G_t^\lambda : \lambda > 0, t \geq 0\}$ . Hence,  $\mathcal{A}_0$  is dense in  $\mathcal{Q}_+$  and Lemma 3.5 is established.  $\square$

**End of the proof of Proposition 3.3.** Clearly, the lemma implies (3.2). By symmetry, there is a constant  $\alpha_-$  such that  $F(A) = \Phi'_A(0)^{\alpha_-}$  holds for every  $A \in \mathcal{Q}_-$ . To verify that  $\alpha_- = \alpha$ , let  $\epsilon$  be small, let  $A_+ = \{e^{i\theta} : \theta \in [0, \pi/2 - \epsilon]\}$ ,  $A_- = \{e^{i\theta} : \theta \in [\pi/2 + \epsilon, \pi]\}$  and  $A = A_+ \cup A_-$ . Set  $A_+^* = \Phi_{A_-}(A_+) \in \mathcal{Q}_+$  and  $A_-^* = \Phi_{A_+}(A_-) \in \mathcal{Q}_-$ . Note that  $\Phi'_{A_+}(0)$  and  $\Phi'_{A_-}(0)$  are bounded away from zero, but  $\lim_{\epsilon \searrow 0} \Phi'_A(0) = 0$ . As

$$(3.5) \quad \Phi_{A_-^*} \circ \Phi_{A_+} = \Phi_A = \Phi_{A_+^*} \circ \Phi_{A_-} ,$$

we have  $\Phi'_{A_-^*}(0) \rightarrow 0$  when  $\epsilon \rightarrow 0$ . By applying  $F$  to (3.5) we get

$$\Phi'_{A_-^*}(0)^{\alpha_-} \Phi'_{A_+}(0)^\alpha = \Phi'_{A_+^*}(0)^\alpha \Phi'_{A_-}(0)^{\alpha_-} .$$

As  $\Phi'_{A_-^*}(0) = \Phi'_{A_+^*}(0)$  (by symmetry), this means that  $\Phi'_{A_-^*}(0)^{\alpha-\alpha_-}$  stays bounded and bounded away from zero as  $\epsilon \searrow 0$ , which gives  $\alpha_- = \alpha$ . Since every  $A \in \mathcal{Q}^*$  can be written as  $A_+ \cdot A_-$ , this establishes (4) with  $\alpha \geq 0$ . The case  $\alpha = 0$  clearly implies  $K = \emptyset$  a.s., which is not permitted. This completes the proof.  $\square$

Let us now conclude this section with some simple remarks:

*Remark 3.6.* If  $K_1, \dots, K_n$  are independent sets with respective laws  $\mathbf{P}_{\alpha_1}, \dots, \mathbf{P}_{\alpha_n}$ , then the law of the filling  $K := \mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(K_1 \cup \dots \cup K_n)$  of the union of the  $K_j$ 's is  $\mathbf{P}_\alpha$  with  $\alpha = \alpha_1 + \dots + \alpha_n$  because

$$\mathbf{P}[K \subset \Phi^{-1}(\mathbb{H})] = \prod_{j=1}^n \mathbf{P}[K_j \subset \Phi^{-1}(\mathbb{H})] = \Phi'(0)^{\alpha_1 + \dots + \alpha_n} .$$

*Remark 3.7.* When  $\alpha < 1/2$ , the measure  $\mathbf{P}_\alpha$  does not exist. To see this, suppose it did. Since it is unique, it is invariant under the symmetry  $\sigma : x + iy \mapsto -x + iy$ . Let  $A = \{e^{i\theta} : \theta \in [0, \pi/2]\}$ . Since  $K$  is almost surely connected and joins 0 to infinity, it meets either  $A$  or  $\sigma(A)$ . Hence, symmetry implies that

$$\Phi'_A(0)^\alpha = \mathbf{P}[K \cap A = \emptyset] \leq 1/2 .$$

On the other hand, one can calculate directly  $\Phi'_A(0) = 1/4$ , and hence  $\alpha \geq 1/2$ . We will show later in the paper (Corollary 8.6) that  $\mathbf{P}_\alpha$  only exists for  $\alpha \geq 5/8$ .

*Remark 3.8.* We have chosen to study subsets of the upper half-plane with the two special boundary points 0 and  $\infty$ , but our analysis clearly applies to any simply connected domain  $O \neq \mathbb{C}$  with two distinguished boundary points  $a$  and  $b$ ,  $a \neq b$ . (We need to assume that the boundary of  $O$  is sufficiently nice near  $a$  and  $b$ . Otherwise, one needs to discuss prime ends in place of the distinguished points.) For instance, if  $\partial O$  is smooth in the neighborhood of  $a$  and  $b$ , then for a conformal map  $\Phi$  from a subset  $O'$  of  $O$  onto  $O$ , we get

$$\mathbf{P}[K \cap (O \setminus O') = \emptyset] = (\Phi'(a)\Phi'(b))^\alpha ,$$

where  $\mathbf{P}$  denotes the image of  $\mathbf{P}_\alpha$  under a conformal map from  $\mathbb{H}$  to  $O$  that takes  $a$  to 0 and  $b$  to  $\infty$ .

*Remark 3.9.* The proof actually shows that weaker assumptions on  $\Omega$  are sufficient for the proposition. Define  $\Omega^b$  just as  $\Omega$  was defined, except that condition (1) is replaced by the requirements that  $K \neq \emptyset$  and  $\overline{K} \cap \mathbb{R} \subset \{0\}$ . Then Proposition 3.3 holds with  $\Omega^b$  in place of  $\Omega$ , and any probability measure on  $\Omega^b$  satisfying any one of conditions (1)–(4) of the proposition is supported on  $\Omega$ . To see this, suppose that  $\mathbf{P}$  is a probability measure on  $\Omega^b$ . The proofs of the implications (1)  $\Leftrightarrow$  (2)  $\Rightarrow$  (3)  $\Leftrightarrow$  (4) of Proposition 3.3 are valid for  $\mathbf{P}$  without modification. Now suppose that  $\mathbf{P}$  satisfies condition (3) of the proposition. If  $\mathbf{P}[K \text{ connected}] < 1$ , then there is a smooth simple path  $\gamma : [0, 1] \rightarrow \overline{\mathbb{H}} \setminus \{0\}$  such that  $\gamma[0, 1] \cap \mathbb{R} = \{\gamma(0), \gamma(1)\}$  and with positive  $\mathbf{P}$ -probability  $K \cap \gamma[0, 1] = \emptyset$  and  $\gamma[0, 1]$  separates  $K$  in  $\overline{\mathbb{H}}$ . (This follows because there is a countable collection of smooth candidate paths, which is dense, in the appropriate sense.) Let  $A$  be the hull satisfying  $\mathbb{H} \cap \partial A = \gamma(0, 1)$ . If  $0 \in A$ , then  $\lim_{s \nearrow 1} \Phi'_{\gamma[0, s]}(0) = 0$ , implying  $\mathbf{P}[K \cap \partial A = \emptyset] = 0$ , contradicting our assumptions. If  $0 \notin A$ , then  $\lim_{s \nearrow 1} \Phi'_{\gamma[0, s]}(0) = \Phi'_A(0)$ , contradicting  $\mathbf{P}[K \cap A \neq \emptyset, K \cap \partial A = \emptyset] > 0$ . Hence,  $\mathbf{P}[K \text{ connected}] = 1$ . Similar arguments show that a.s.  $0 \in \overline{K}$  and  $K$  is unbounded. Thus,  $\mathbf{P}[\Omega] = 1$ . Using this fact, Lemma 3.2 may be applied, giving the remaining implication (3)  $\Rightarrow$  (2).

#### 4. BROWNIAN EXCURSIONS

An important example of a restriction measure is given by the law of the Brownian excursion from 0 to infinity in  $\mathbb{H}$ . Loosely speaking, this is simply planar Brownian motion started from the origin and conditioned to stay in  $\mathbb{H}$  at all positive times. It is closely related to the “complete conformal invariance” of (slightly different) measures on Brownian excursions in [32, 27].

Let  $X$  be a standard one-dimensional Brownian motion and  $Y$  an independent three-dimensional Bessel process (see, e.g., [41] for background on three-dimensional Bessel processes, its relation to Brownian motion conditioned to stay positive and stochastic differential equations). Let us briefly recall that a three-dimensional Bessel process is the modulus (Euclidean norm) of a three-dimensional Brownian motion and that it can be defined as the solution to the stochastic differential equation  $dY_t = dw_t + dt/Y_t$ , where  $w$  is standard Brownian motion in  $\mathbb{R}$ . It is very easy to see that  $(1/Y_t, t \geq t_0)$  is a local martingale for all  $t_0 > 0$  and that if  $T_r$  denotes the hitting time of  $r$  by  $Y$ , then the law of  $(Y_{T_r+t}, t < T_R - T_r)$  is identical to that of a Brownian motion started from  $r$  and conditioned to hit  $R$  before 0 (if  $0 < r < R$ ). Note that almost surely  $\lim_{t \rightarrow \infty} Y_t = \infty$ .

The Brownian excursion can be defined as  $B_t = X_t + iY_t$ . In other words,  $B$  has the same law as the solution to the following stochastic differential equation:

$$(4.1) \quad dB_t = dW_t + i \frac{1}{\operatorname{Im}(B_t)} dt$$

with  $B_0 = z = x + iy$ , where  $W$  is a complex-valued Brownian motion,  $x \in \mathbb{R}$  and  $y \geq 0$ . Note that  $B$  is a strong Markov process and that  $B(0, \infty) \subset \mathbb{H}$  almost surely. Assume that  $y = 0$ , and let  $T_r$  denote the hitting time of the line  $\mathcal{I}_r := \mathbb{R} + ir$  by  $B$  (i.e., the hitting time of  $r$  by  $Y$ ). Let  $S$  denote a random variable with the same law as  $B_{T_1}$ . Scaling shows immediately that for all  $0 < r < R$ , the law of  $B[T_r, T_R]$  is the law of a Brownian motion started with the same law as  $rS$ , stopped at its first hitting of  $\mathcal{I}_R$ , and conditioned to stay in  $\mathbb{H}$  up to that time. Note that the probability of this event is  $r/R$ .

The next proposition, which is due to Bálint Virág [49], implies that the law of the filling  $\mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(B)$  of the path of a Brownian excursion  $B$  from 0 in  $\mathbb{H}$  is  $\mathbf{P}_1$ .

**Proposition 4.1.** *For all  $A \in \mathcal{Q}^*$ ,  $\mathbf{P}[B[0, \infty) \cap A = \emptyset] = \Phi'_A(0)$ .*

For completeness, a proof is included.

*Proof.* Let  $\Phi = \Phi_A$ . Suppose that  $W$  is a planar Brownian motion and  $Z$  is a Brownian excursion in  $\mathbb{H}$ , both starting at  $z \in \mathbb{H} \setminus A$ . When  $\text{Im}(z) \rightarrow \infty$ ,  $\text{Im}(\Phi^{-1}(z)) = \text{Im}(z) + o(1)$ . Hence, with a large probability (when  $R$  is large), a Brownian motion started from  $z \in \mathcal{I}_R$  (resp.,  $z \in \Phi^{-1}(\mathcal{I}_R)$ ) will hit  $\Phi^{-1}(\mathcal{I}_R)$  (resp.,  $\mathcal{I}_R$ ) before  $\mathbb{R}$ . The strong Markov property of planar Brownian motion therefore shows that when  $R \rightarrow \infty$ ,

$$\mathbf{P}[W \text{ hits } \mathcal{I}_R \text{ before } A \cup \mathbb{R}] \sim \mathbf{P}[W \text{ hits } \Phi^{-1}(\mathcal{I}_R) \text{ before } A \cup \mathbb{R}].$$

But since  $\Phi \circ W$  is a time-changed Brownian motion, and  $\Phi : \mathbb{H} \setminus A \rightarrow \mathbb{H}$ , the right hand is equal to the probability that a Brownian motion started from  $\Phi(z)$  hits  $\mathcal{I}_R$  before  $\mathbb{R}$ , namely,  $\text{Im}(\Phi(z))/R$ . Hence,

$$\mathbf{P}[Z \text{ hits } \mathcal{I}_R \text{ before } A] = \frac{\mathbf{P}[W \text{ hits } \mathcal{I}_R \text{ before } A \cup \mathbb{R}]}{\mathbf{P}[W \text{ hits } \mathcal{I}_R \text{ before } \mathbb{R}]} = \frac{\text{Im}[\Phi(z)]}{\text{Im}(z)} + o(1)$$

when  $R \rightarrow \infty$ . In the limit  $R \rightarrow \infty$ , we get

$$(4.2) \quad \mathbf{P}[Z \subset \mathbb{H} \setminus A] = \frac{\text{Im}[\Phi(z)]}{\text{Im}(z)} = \frac{\text{Im}[g_A(z)]}{\text{Im}(z)}.$$

When  $z \rightarrow 0$ ,  $\Phi(z) = z\Phi'(0) + O(|z|^2)$  so that

$$\mathbf{P}[B[0, \infty) \cap A = \emptyset] = \lim_{s \rightarrow 0} \mathbf{P}[B[s, \infty) \subset \mathbb{H} \setminus A] = \lim_{s \rightarrow 0} \mathbf{E} \left[ \frac{\text{Im}(\Phi(B_s))}{\text{Im}(B_s)} \right] = \Phi'(0)$$

(one can use dominated convergence here, since  $\text{Im}(\Phi(z)) \leq \text{Im}(z)$  for all  $z$ ).  $\square$

We have just proved that the two-sided restriction measure  $\mathbf{P}_1$  exists. By filling unions of  $n$  independent excursions, one constructs the probability measures  $\mathbf{P}_n$

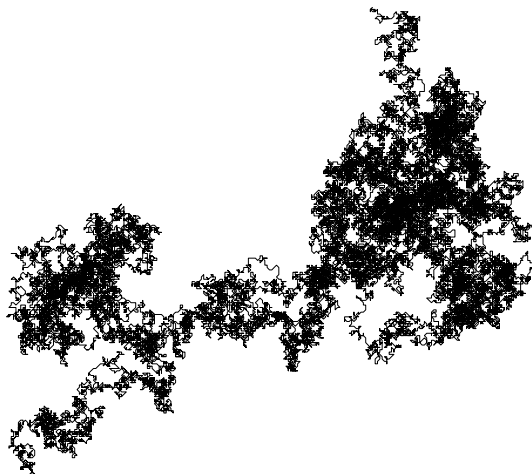


FIGURE 4.1. A sample of the beginning of a Brownian excursion.

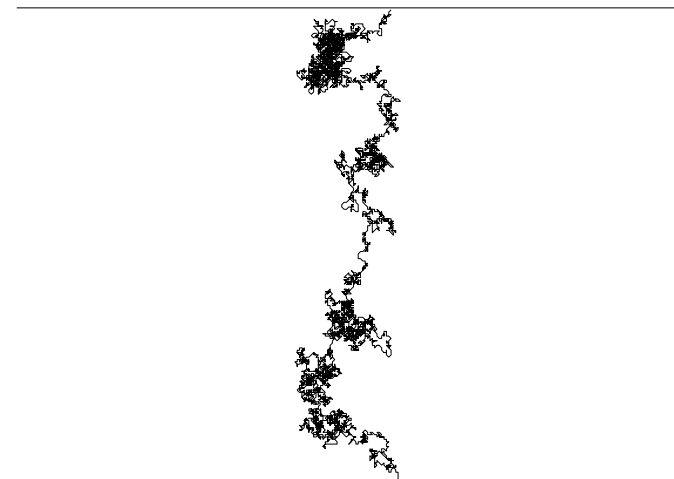


FIGURE 4.2. A sample of a Brownian excursion from 0 to  $i$  in  $\mathbb{R} \times [0, 1]$ .

for all integers  $n \geq 1$  which therefore also exist. It follows (using the fact that the dimension of the boundary of the filling of a Brownian excursion is  $4/3$  [26, 4]) that for any positive integer  $n$ , the Hausdorff dimension of the boundary of  $K$  defined under  $\mathbf{P}_n$  is almost surely  $4/3$ .

We have already mentioned that the choice of the domain  $\mathbb{H}$  and of the boundary points 0 and  $\infty$  was somewhat arbitrary. In another simply connected open domain  $O \neq \mathbb{C}$  with two invariant boundary points  $a$  and  $b$  (via a given conformal map  $\psi$  from  $\mathbb{H}$  onto  $O$ ), the Brownian excursion from  $a = \psi(0)$  to  $b = \psi(\infty)$  is the solution (up to time-change) started from  $a$  of

$$dB_t = d\beta_t + \nabla\varphi(B_t)/\varphi(B_t)dt$$

where  $\varphi = \text{Im } \psi^{-1}$  and  $\beta$  denotes planar Brownian motion.

Using almost the same proof as in Proposition 4.1 (but keeping track of the law of the path), one can prove the following:

**Lemma 4.2.** *Suppose that  $A \in \mathcal{Q}^*$  and that  $B$  is a Brownian excursion in  $\mathbb{H}$  starting at 0. Then the conditional law of  $(\Phi_A(B(t)), t \geq 0)$  given  $B \cap A = \emptyset$  is the same as a time change of  $B$ .*

Finally, let us mention the following result that will be useful later on.

**Lemma 4.3.** *Let  $\mathbf{P}^{x+iy}$  denote the law of a Brownian excursion  $B$  starting at  $x + iy \in \overline{\mathbb{H}}$ . Then for every  $A \in \mathcal{Q}^*$ ,*

$$\lim_{y \rightarrow \infty} y \mathbf{P}^{x+iy} [B[0, \infty) \cap A \neq \emptyset] = a(A),$$

and

$$\lim_{y \rightarrow \infty} y \int_{-\infty}^{\infty} \mathbf{P}^{x+iy} [B[0, \infty) \cap A \neq \emptyset] dx = \pi a(A),$$

where  $a(A)$  is as in (2.1).

*Proof.* By (4.2) and the normalization of  $g_A$  near infinity,

$$\begin{aligned} \mathbf{P}^{x+iy}[B[0, \infty) \cap A \neq \emptyset] &= 1 - \frac{\operatorname{Im}[g_A(x + iy)]}{y} \\ &= \frac{a(A)y}{x^2 + y^2} + O\left(\frac{1}{x^2 + y^2}\right), \end{aligned}$$

and the lemma readily follows.  $\square$

Using Cauchy’s Theorem, for example, it is easy to see that the second statement of the lemma may be strengthened to

$$(4.3) \quad y \int_{-\infty}^{\infty} \mathbf{P}^{x+iy}[B[0, \infty) \cap A \neq \emptyset] dx = \pi a(A), \quad y > \sup\{\operatorname{Im} z : z \in A\}.$$

### 5. CONFORMAL IMAGE OF CHORDAL SLE

Let  $W : [0, \infty) \rightarrow \mathbb{R}$  be continuous with  $W_0 = 0$ , and let  $(g_t)$  be the (chordal) Loewner chain driven by  $W$  satisfying (2.6). It is easy to verify by differentiation and (2.6) that the inverse map  $f_t(z) = g_t^{-1}(z)$  satisfies

$$\partial_t f_t(z) = -\frac{2f_t'(z)}{z - W_t}, \quad f_0(z) = z.$$

Suppose that  $A \in \mathcal{Q}^*$  is fixed, and let  $G = g_A$  and  $T = T_A = \inf\{t : K_t \cap A \neq \emptyset\}$ . For  $t < T$ , let  $A_t = g_t(A)$ ,  $\tilde{K}_t = G(K_t)$  and  $\tilde{g}_t = g_{\tilde{K}_t}$ . See Figure 5.1. Then  $\tilde{g}_t$  has an expansion

$$\tilde{g}_t(z) = z + \frac{a(t)}{z} + o(z^{-1}), \quad z \rightarrow \infty,$$

where the coefficient  $a(t)$  depends on  $G$  and  $W_t$ .

Note that  $\tilde{g}_t$  satisfies the Loewner equation

$$\partial_t \tilde{g}_t(z) = \frac{\partial_t a(t)}{\tilde{g}_t(z) - \tilde{W}_t}, \quad \tilde{g}_0(z) = z,$$

where

$$\tilde{W}_t := h_t(W_t), \quad h_t := \tilde{g}_t \circ G \circ g_t^{-1} = g_{A_t}.$$

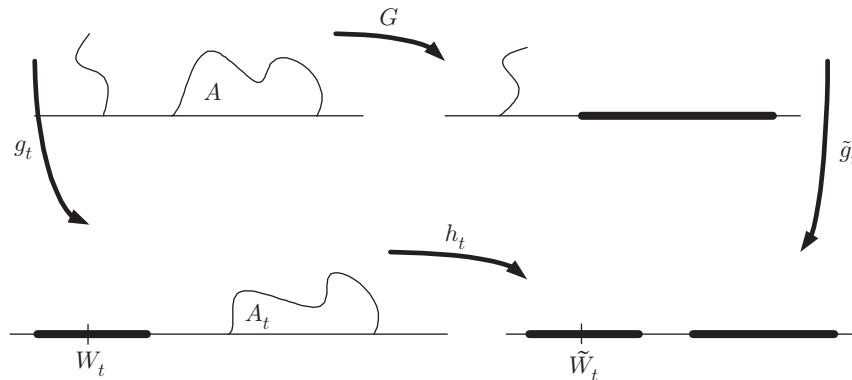


FIGURE 5.1. The various maps.

(This follows from the proof of Loewner’s theorem, because  $\tilde{g}_t(\tilde{K}_{t+\delta} \setminus \tilde{K}_t)$  lies in a small neighborhood of  $\tilde{W}_t$  when  $\delta > 0$  is small. Also see [23, (2.6)].) The identity (2.5) gives  $a(g_t(K_{t+\Delta t} \setminus K_t)) = 2\Delta t$ . The image of  $\tilde{K}_{t+\Delta t} \setminus \tilde{K}_t$  under  $\tilde{g}_t$  is  $h_t(g_t(K_{t+\Delta t} \setminus K_t))$ . The scaling rule (2.3) of  $a$  tells us that as  $\Delta t \rightarrow 0+$ , the half-plane capacity of  $h_t(g_t(K_{t+\Delta t} \setminus K_t))$  is asymptotic to  $h'_t(W_t)^2 \cdot 2\Delta t$ . (The higher order derivatives of  $h_t$  can be ignored, as follows from (2.2). Also see [23, (2.7)].) Hence,

$$(5.1) \quad \partial_t a(t) = 2h'_t(W_t)^2.$$

Using the chain rule, we get

$$(5.2) \quad [\partial_t h_t](z) = \frac{2 h'_t(W_t)^2}{h_t(z) - \tilde{W}_t} - \frac{2 h'_t(z)}{z - W_t}.$$

This formula is valid for  $z \in \mathbb{H} \setminus g_t(A)$  as well as for  $z$  in a punctured neighborhood of  $W_t$  in  $\mathbb{R}$ . In fact, it is also valid at  $W_t$  with

$$[\partial_t h_t](W_t) = \lim_{z \rightarrow W_t} \left( \frac{2 h'_t(W_t)^2}{h_t(z) - \tilde{W}_t} - \frac{2 h'_t(z)}{z - W_t} \right) = -3h''_t(W_t).$$

Computations of a similar nature appear (in a deterministic setting) in [11]. Differentiating (5.2) with respect to  $z$  gives the equation

$$[\partial_t h'_t](z) = -\frac{2 h'_t(W_t)^2 h'_t(z)}{(h_t(z) - \tilde{W}_t)^2} + \frac{2 h'_t(z)}{(z - W_t)^2} - \frac{2 h''_t(z)}{z - W_t}.$$

Therefore, at  $z = W_t$ ,

$$[\partial_t h'_t](W_t) = \lim_{z \rightarrow W_t} \partial_t h'_t(z) = \frac{h''_t(W_t)^2}{2h'_t(W_t)} - \frac{4h'''_t(W_t)}{3}.$$

Higher derivatives with respect to  $z$  can be handled similarly.

Now suppose that  $(B_t, t \geq 0)$  is a standard one-dimensional Brownian motion and that  $(W_t, t \geq 0)$  is a (one-dimensional) semimartingale satisfying  $W_0 = 0$  and

$$dW_t = b_t dt + \sqrt{\kappa} dB_t$$

for some measurable process  $b_t$  adapted to the filtration of  $B_t$  which satisfies  $\int_0^t |b_s| ds < \infty$  a.s. for every  $t > 0$ .

Itô’s formula shows that  $\tilde{W}_t = h_t(W_t)$ ,  $t < T$ , is a semimartingale with

$$(5.3) \quad d\tilde{W}_t = d[h_t(W_t)] = h'_t(W_t) dW_t + ((\kappa/2) - 3)h''_t(W_t) dt.$$

Here, we need a generalized Itô’s formula since the function  $h_t$  is random (see, e.g., exercise (IV.3.12) in [41]). However, since  $h'_t(z)$  is  $C^1$  in  $t$ , no extra terms appear. Similarly,

$$d[h'_t(W_t)] = h''_t(W_t) dW_t + \left( \frac{h''_t(W_t)^2}{2h'_t(W_t)} + \left( \frac{\kappa}{2} - \frac{4}{3} \right) h'''_t(W_t) \right) dt.$$

Let  $\alpha > 0$  and let  $Y_t^0 = h'_t(W_t)^\alpha$ . Then yet another application of Itô’s formula gives

$$(5.4) \quad \frac{dY_t^0}{\alpha Y_t^0} = \frac{h''_t(W_t)}{h'_t(W_t)} dW_t + \left( \frac{(\alpha - 1)\kappa + 1}{2} \frac{h''_t(W_t)^2}{h'_t(W_t)^2} + \left( \frac{\kappa}{2} - \frac{4}{3} \right) \frac{h'''_t(W_t)}{h'_t(W_t)} \right) dt.$$

These computations imply readily the following results:

**Proposition 5.1.** *Let  $b_t = 0$  (i.e.,  $W_t = \sqrt{\kappa}B_t$ ) and fix  $A \in \mathcal{Q}^*$ .*

- (1)  $\tilde{W}_t$  is a local martingale if and only if  $\kappa = 6$ .
- (2) Suppose that  $\kappa = 6$  and let  $T' := \inf\{t : K_t \cap \Phi_A(\partial A) \neq \emptyset\}$ . Then  $(\Phi_A(K_t), t < T)$  has the same law as a time change of  $(K_t, t < T')$ .

Claim (2) is basically the “locality property”, which is central to the papers [23, 24, 25]. It has been proven in [23] using a somewhat different, longer and more technical proof. (See [23] for a more complete discussion of this important property.) Further consequences of this locality result are discussed in §9.

*Proof.* Statement (1) is clear from (5.3). To prove (2), set  $\kappa = 6$  and define  $\tau(s) := \inf\{t : a(t) \geq 2s\}$ ,  $\tilde{W}_s := \tilde{W}_{\tau(s)}$  and  $\hat{g}_s := \tilde{g}_{\tau(s)}$ . Then by (5.1)  $\partial_s \tau(s) = 2/\partial_\tau a(\tau) = h'_t(W_t)^{-2}$ . Hence, (5.3) shows that  $\tilde{W}_s/\sqrt{6}$ ,  $s < \tau^{-1}(T)$ , is stopped Brownian motion. Statement (2) now follows since

$$\partial_s \hat{g}_s(z) = \frac{2}{\hat{g}_s(z) - \tilde{W}_s}.$$

□

**Proposition 5.2.** *Suppose  $b_t = 0$  (i.e.,  $W_t = \sqrt{\kappa}B_t$ ),  $\alpha > 0$ ,  $\kappa > 0$ . The process  $Y_t^0 = h'_t(W_t)^\alpha$ ,  $t < T$ , is a local martingale for all  $A \in \mathcal{Q}^*$  if and only if  $\kappa = 8/3$  and  $\alpha = 5/8$ .*

*Proof.* Immediate from (5.4). □

The next section will be devoted to consequences of this property of  $SLE_{8/3}$ .

Before stating a useful generalization of Proposition 5.2, we recall a few basic facts about the Schwarzian derivative,

$$Sf(z) := \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

An essential property of the Schwarzian derivative is that  $Sm = 0$  when  $m$  is a Möbius transformation,  $m(z) = (az + b)/(cz + d)$ ,  $ad - bc \neq 0$ . An easy direct calculation shows that  $Sf(0) = -6a(A)$  when  $f(z) = g_A(-1/z)$ . Since  $S(m \circ f) = S(f)$  for Möbius transformations  $m$ , it follows that

$$(5.5) \quad Sg_A(0) = -6a(\{-z^{-1} : z \in A\}).$$

Consequently,  $Sg_A \leq 0$  on  $\mathbb{R} \setminus A$ . (In fact, in subsection 7.1 we show that  $-Sg_A(z)/6$  is a hitting measure for Brownian bubbles.) If  $\alpha > 0, \lambda \in \mathbb{R}$ , let

$$Y_t^\lambda := h'_t(W_t)^\alpha \exp\left(\lambda \int_0^t \frac{Sh_s(W_s)}{6} ds\right).$$

**Proposition 5.3.** *If  $W_t = \sqrt{\kappa}B_t$ ,*

$$(5.6) \quad \alpha = \frac{6 - \kappa}{2\kappa}$$

and

$$(5.7) \quad \lambda = \frac{(8 - 3\kappa)(6 - \kappa)}{2\kappa},$$

then  $Y_t^\lambda$ ,  $t < T$ , is a local martingale. If  $\kappa \leq 8/3$ , then  $Y_t^\lambda$  is a bounded martingale (in fact,  $0 \leq Y_t^\lambda \leq 1$ ).

*Proof.* The local martingale property follows immediately from Itô’s formula and (5.4). The bound on  $Y_t^\lambda$  follows from  $Sh_t(W_t) \leq 0$  and (2.4). □

6. RESTRICTION PROPERTY FOR  $SLE_{8/3}$

We now discuss some consequences of Proposition 5.2 for  $SLE_{8/3}$ . In particular, we establish the following theorem.

**Theorem 6.1** (Restriction). *Let  $\gamma$  be the  $SLE_{8/3}$  path starting at the origin and  $A \in \mathcal{Q}^*$ . Then*

$$\mathbf{P}[\gamma[0, \infty) \cap A = \emptyset] = \Phi'_A(0)^{5/8}.$$

The law of  $\gamma(0, \infty)$  is therefore  $\mathbf{P}_{5/8}$ .

Roughly, we will need to show that as  $t \nearrow T$  the  $Y_t^0$  of Proposition 5.2 converges to 0 or 1, respectively, if  $\gamma$  hits  $A$  or not. Some simple deterministic lemmas below will help us establish this.

**Lemma 6.2.** *Let  $A \in \mathcal{Q}_+$ , let  $W : [0, \infty) \rightarrow \mathbb{R}$  be continuous, and let  $g_t$  be the corresponding solution of (2.6). Let  $K_t$  be the associated growing hull, as defined in Section 2, and suppose that  $\bigcup_{t>0} K_t \cap A = \emptyset$ . Let  $T(r) := \sup\{t \geq 0 : K_t \subset r\mathbb{U}\}$  and  $A_t = g_t(A)$ . Then*

$$\lim_{r \rightarrow \infty} g'_{A_{T(r)}}(W_{T(r)}) = 1.$$

*Proof.* Set  $T := T(r)$ ,  $W := W_T$ ,  $a_0 := \inf(A \cap \mathbb{R})$ ,  $a_1 := \sup(A \cap \mathbb{R})$ ,  $A' := A \cup [a_0, a_1]$  and  $\tilde{A} := g_T(A')$ . Let  $r' = \sup\{|z| : z \in K_T\}$  (actually,  $r' = r$ , but we do not need this fact), and let  $z_0 \in K_T$  be such that  $|z_0| = r'$ . Set  $\beta(s) := z_0 + (1-s)i$ ,  $s \in [0, 1]$ . Then the limit  $w := \lim_{s \nearrow 1} g_T \circ \beta(s)$  exists. (This is because the image of the conformal map  $g_T$  is a smooth domain, i.e.,  $\mathbb{H}$ . See, e.g., [40, Proposition 2.14].) Moreover, since  $\beta(1) = z_0$ , we must have  $w = W$ . Otherwise, one easily gets a contradiction to  $z_0 \in K_T \setminus \bigcup_{t < T} K_t$ . The extremal length (see [1] for the definition and basic properties of extremal length) from  $A'$  to the circle  $|z| = r' \geq r$  goes to infinity with  $r$ . By monotonicity and conformal invariance of extremal length, the extremal length in  $\mathbb{H}$  between  $\tilde{A}$  and  $(-\infty, W]$  goes to infinity as well. Because  $\tilde{A}$  is connected, this implies that  $\text{diam}(\tilde{A}) / \inf\{|W - z| : z \in \tilde{A}\}$  goes to zero. Since  $g'_B(W)$  is invariant under scaling  $B$  about  $W$ , under translating  $B$  and  $W$ , and is monotone decreasing in  $B$ , this means that when  $r$  is large,  $g'_A(W)$  is at least  $g'_B(0)$ , where  $B = \{z \in \mathbb{H} : |z - 1| \leq \epsilon\}$  and  $\epsilon > 0$  is arbitrarily small. The lemma follows.  $\square$

**Lemma 6.3.** *Let  $W : [0, \infty) \rightarrow \mathbb{R}$  be continuous, let  $g_t$  be the corresponding solution of (2.6), and let  $K_t$  be the associated growing hull, as defined in Section 2. Let  $A \in \mathcal{Q}^*$  be a smooth hull. Suppose that  $T := \inf\{t \geq 0 : K_t \cap A \neq \emptyset\} < \infty$  and  $K_T \cap A \cap \mathbb{R} = \emptyset$ . Set  $A_t := g_t(A)$ ,  $t < T$ . Then*

$$\lim_{t \nearrow T} \Phi'_{A_t}(W_t) = 0.$$

*Proof.* We first argue the rather obvious fact that  $K_T \cap A \subset \partial A$ . Note that  $\lim_{t \searrow 0} \text{diam}(K_t) = 0$  follows from the continuity of  $W_t$  at  $t = 0$ . This implies that for every  $z \in \mathbb{H}$  the harmonic measure of  $K_t$  from  $z$  in  $\mathbb{H}$  goes to 0 when  $t \searrow 0$ . For  $s > 0$ , the hull evolution  $t \mapsto \overline{g_s(K_{t+s} \setminus K_s)}$  is driven by  $t \mapsto W_{s+t}$ . By conformal invariance of harmonic measure, since  $W_t$  is uniformly continuous in  $[0, T]$ , for every  $z \notin K_T$  the harmonic measure of  $K_{t+s} \setminus K_s$  in  $\mathbb{H} \setminus K_s$  goes to zero uniformly as  $t \searrow 0$  while  $s \in [0, T]$ . In particular, the harmonic measure of  $K_T \setminus \bigcup_{t < T} K_t$  in  $\mathbb{H} \setminus \overline{\bigcup_{t < T} K_t}$  is zero from any  $z$  in the latter open set. Since  $K_T$  does not contain  $\partial A$ , we may apply this from a point in  $\partial A \setminus K_T$  and conclude that  $K_T \cap A \subset \partial A$ .



Let  $z_0$  be some point in  $\partial A \cap K_T$ . Let  $\beta : [0, 1] \rightarrow \mathbb{H}$  be a smooth path such that  $\beta[0, 1]$  is contained in the interior of  $A$ ,  $\beta(1) = z_0$ , and  $\beta'(1)$  is orthogonal to  $\partial A$  at  $z_0$ . By smoothness of  $\partial A \cap \mathbb{H}$ , there is some small disk  $D \subset \mathbb{H}$  with center  $z_0$  such that  $\partial D \cap \partial A$  consists of exactly two points. Let  $\sigma_1$  be the arc of  $\partial A$  from  $z_0$  to  $\partial D$  that goes away from  $z_0$  in the direction  $i\beta'(1)$ , and let  $\sigma_2$  be the other arc of  $\partial A$  from  $z_0$  to  $\partial D$ .

We claim that

$$\hat{\beta}(x) := g_T \circ \beta(x) - W_T, \quad x \in [0, 1),$$

is a path which is contained in a sector  $|\operatorname{Re} z| \leq c \operatorname{Im} z$  for some  $c$ . As in the proof of Lemma 6.2, we know that  $\lim_{x \nearrow 1} \hat{\beta}(x) = W_T$ . Suppose that we start a two-dimensional Brownian motion from  $\beta(x)$ ,  $x \in [0, 1)$ , and stop when we hit  $K_T \cup \mathbb{R}$ . Then there is probability bounded from below that we hit this set to the “right” of  $\beta[0, 1]$ , because the Brownian motion has probability bounded from below to first hit  $\partial D \cup \sigma_1$  on  $\sigma_1$  and from the side of  $\sigma_1$  not in the interior of  $A$ . (The careful reader might want to draw a little figure here.) By conformal invariance, this shows that Brownian motion started from  $\hat{\beta}(x)$  has probability bounded from below to first hit  $\mathbb{R}$  in  $(0, \infty)$ . Consequently,  $\operatorname{Re} \hat{\beta}(x) \geq -c \operatorname{Im} \hat{\beta}(x)$  for some constant  $c$  independent of  $x$ . The symmetric argument also shows  $\operatorname{Re} \hat{\beta}(x) \leq c \operatorname{Im} \hat{\beta}(x)$ , for some  $c$ , and the claim is established.

Since

$$\hat{\beta}(x) = \lim_{t \nearrow T} g_t \circ \beta(x) - W_t, \quad x \in [0, 1),$$

it follows that when  $t$  tends to  $T$ , a Brownian motion excursion from 0 to  $\infty$  in  $\mathbb{H}$  will hit  $\hat{\beta}$  before exiting  $g_t(D \setminus K_t) - W_t$  with probability tending to 1. This implies that this Brownian motion excursion will hit  $A_t - W_t$  with probability tending to 1. By Proposition 4.1, this probability is the same as  $1 - \Phi'_{A_t}(W_t)$ .  $\square$

*Proof of Theorem 6.1.* By Proposition 3.3, it suffices to consider the case where  $A$  is a smooth hull in  $\mathcal{Q}_+ \cup \mathcal{Q}_-$ . By symmetry, we may take  $A \in \mathcal{Q}_+$ . Proposition 5.2 shows that  $Y_t^0 = h'_t(W_t)^{5/8}$  is a bounded continuous local martingale. By the martingale convergence theorem, the a.s. limit  $Y_T^0 := \lim_{t \nearrow T} Y_t^0$  exists and  $Y_0^0 = \mathbf{E}[Y_T^0]$ , where  $T = \sup\{t : \gamma[0, t] \cap A = \emptyset\}$ . Lemmas 6.2 and 6.3 show that  $Y_T^0 = 1_{T=\infty}$  a.s. This proves the theorem.  $\square$

Combining this with the results of Section 3 shows the following identity in law:

**Corollary 6.4.** *The filling of the union of 8 independent chordal  $SLE_{8/3}$ 's has the same law as the filling of the union of 5 independent Brownian excursions from 0 to infinity. In both cases, the law is  $\mathbf{P}_5$ , the two-sided restriction measure with exponent 5.*

Theorem 6.1 suggests that  $SLE_{8/3}$  should be the limit as the lattice mesh goes to zero of the self-avoiding walk. See [30] for a discussion of these conjectures. Also, we know [42] that  $SLE_{8/3}$  is a simple curve. This suggests that  $\alpha = 5/8$  is the smallest possible value for which  $\mathbf{P}_\alpha$  exists. We shall see later (Corollary 8.6) that this is indeed the case.

A similar proof to that of Theorem 6.1, using Proposition 5.3, gives the following important generalization.

**Theorem 6.5.** *Suppose  $0 \leq \kappa \leq 8/3$  and let  $\alpha$  and  $\lambda$  be as in (5.6) and (5.7). If  $W_t = \sqrt{\kappa}B_t$  and  $A \in \mathcal{Q}^*$ , then*

$$(6.1) \quad \Phi'_A(0)^\alpha = \mathbf{E} \left[ 1_{\{\gamma_{[0,\infty)} \cap A = \emptyset\}} \exp\left(\lambda \int_0^\infty \frac{Sh_s(W_s)}{6} ds\right) \right].$$

The following section will provide a more concrete meaning to the right-hand side and will use the theorem to construct the measures  $\mathbf{P}_\alpha$ ,  $\alpha \geq 5/8$ .

### 7. BUBBLES

**7.1. Brownian bubbles.** We now define the measure  $\nu$  of *Brownian bubbles hanging at infinity*. This is a  $\sigma$ -finite but infinite measure on unbounded closed connected sets  $K \subset \mathbb{H}$  such that  $\mathbb{C} \setminus K$  is connected. The definition of  $\nu$  is rather simple. For  $z \in \overline{\mathbb{H}}$ , let  $\mathbf{P}^z$  denote the law of the Brownian excursion  $Z_t$  started at  $z$ , as discussed in Section 4. Let  $\hat{\mathbf{P}}^z$  denote the law of the filling of  $Z$ ,  $\mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(Z[0, \infty))$ . Set

$$(7.1) \quad \nu := \frac{1}{\pi} \lim_{y \rightarrow \infty} y \int_{\mathbb{R}} \hat{\mathbf{P}}^{x+iy} dx.$$

In other words, one considers the limit when  $y \rightarrow \infty$  of the (infinite) measure obtained by filling a Brownian excursion that is started on the line  $\mathcal{I}_y$ , where the initial point is chosen according to  $y/\pi$  times the Lebesgue measure on  $\mathcal{I}_y$ . The existence of the limit is easily justified, as follows. Set  $\sigma_y = \inf\{t : \text{Im } Z_t = y\}$ . Recall that for  $y' > y$ ,  $\mathbf{P}^{x+iy'}[\sigma_y < \infty] = y/y'$ . On the event  $\sigma_y < \infty$ , let  $Z^y$  be the path  $Z_t^y = Z_{\sigma_y+t}$ ,  $t \geq 0$ . By the strong Markov property it therefore follows that the image of the measure  $1_{\sigma_y < \infty} \int \mathbf{P}^{x+iy'} dx$  under the map  $Z_t \mapsto Z_t^y$  is precisely  $(y/y') \int \mathbf{P}^{x+iy} dx$ . The existence of the limit (7.1) readily follows.

Suppose that  $A \in \mathcal{Q}$ . We have by Lemma 4.3

$$(7.2) \quad \nu[K \cap A \neq \emptyset] = \frac{1}{\pi} \lim_{y \rightarrow \infty} y \int_{\mathbb{R}} \mathbf{P}^{x+iy}[Z \cap A \neq \emptyset] dx = a(A).$$

This can be used to give an alternative proof of the existence of the limit in the definition of  $\nu$ .

Let  $A$  and  $A'$  be in  $\mathcal{Q}$ . Define  $A'' = A \cup g_A^{-1}(A')$ . Then  $A'' \in \mathcal{Q}$  and  $a(A'') = a(A) + a(A')$  by (2.5) and invariance of  $a(A)$  under real translations. Hence,

$$\begin{aligned} \nu[g_A(K) \cap A' \neq \emptyset \text{ and } K \cap A = \emptyset] &= \nu[K \cap A'' \neq \emptyset \text{ and } K \cap A = \emptyset] \\ &= \nu[K \cap A'' \neq \emptyset] - \nu[K \cap A \neq \emptyset] \\ &= a(A'') - a(A) = a(A'). \end{aligned}$$

Therefore, the image of  $1_{K \cap A = \emptyset} \nu$  under  $g_A$  is  $\nu$ . In the terminology of Section 2, this says that  $\nu$  is invariant under the semigroup  $\{g_A : A \in \mathcal{Q}\}$ .

Now define the measure  $\mu$  on Brownian bubbles at 0 as the image of  $\nu$  under the inversion  $z \mapsto -1/z$ . It is a measure on the set of bounded  $K \subset \mathbb{H}$  with  $\overline{K} = K \cup \{0\}$ . By (5.5) we have for  $A \in \mathcal{Q}^*$

$$(7.3) \quad \mu[K \cap A \neq \emptyset] = -\frac{Sg_A(0)}{6}.$$

We may think of  $\mu$  as a measure on the space  $\Omega_b$  of connected bounded sets  $K \subset \mathbb{H}$  such that  $\overline{K} = K \cup \{0\}$  and  $\mathbb{C} \setminus \overline{K}$  is connected.

If  $\Gamma$  denotes the semigroup of dilations  $r_\lambda$ ,  $r_\lambda(z) = \lambda z$ , then it is easy to see from (2.3) that  $\nu$  and  $\mu$  are  $\Gamma$ -covariant; in fact,  $r_\lambda \circ \nu = \lambda^{-2} \nu$ ,  $r_\lambda \circ \mu = \lambda^2 \nu$ .

There are a number of alternative equivalent ways to define the measures  $\nu$  and  $\mu$  and to derive their properties:

- Define a measure on two-sided excursion in  $\mathbb{H}$ ,  $Z_t$ , starting at the origin by

$$Z_t = \begin{cases} Z_{-t}^1, & -\infty < t \leq 0, \\ Z_t^2, & 0 \leq t < \infty, \end{cases}$$

where  $Z^1, Z^2$  are independent excursions in  $\mathbb{H}$  starting at 0. The measure  $\nu$  is obtained by choosing a point  $z \in \mathbb{H}$  according to two-dimensional Lebesgue measure and letting  $B_t = Z_t + z$ . (This definition gives  $\nu$  as a measure on paths, rather than fillings. The parametrization of the paths is chosen so that at time 0 they attain their minimal imaginary part.)

- Define on different spaces a one-dimensional Brownian excursion  $(e_t, 0 \leq t \leq T)$  (defined under Itô’s excursion measure  $n$ ) and a Brownian bridge  $(b_t, 0 \leq t \leq 1)$  with  $b_0 = b_1 = 0$  (defined under a probability measure  $P$ ). Recall that Itô’s excursion measure is obtained as the limit when  $\epsilon \rightarrow 0$  of  $\epsilon^{-1}$  times the probability measure defining a Brownian motion started from  $\epsilon$  and killed at its first hitting time of 0,  $T$ . We then define the process

$$Z(t) = T^{1/2} b(t/T) + i e_t, \quad t \in [0, T].$$

If  $\Upsilon$  denotes the map  $(b, e) \mapsto \mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(Z)$ , then  $\mu = c \Upsilon(T^{-1/2} n \otimes P)$  for some constant  $c$ . The factor  $T^{-1/2}$  is needed in order for  $\mu$  to scale properly under the dilations  $r_\lambda$ .

- We can also relate Brownian bubbles to Brownian excursions in  $\mathbb{H}$ . Given  $\epsilon > 0$ , let  $m_\epsilon : \mathbb{H} \rightarrow \mathbb{H}$  denote an arbitrary Möbius transformation fixing 0 and satisfying  $m_\epsilon(\infty) = \epsilon$ . Then  $\mu$  is (the filling of) the limit when  $\epsilon \rightarrow 0$  of  $\epsilon^{-2}$  times the  $m_\epsilon$ -image of the law of a Brownian excursion from 0 to  $\infty$  in  $\mathbb{H}$ .

*Remark 7.1.* The previous description can be combined with the fact that eight  $\text{SLE}_{8/3}$ ’s are equivalent to five Brownian excursions (i.e., more precisely, Corollary 6.4) to describe the measure on Brownian bubbles using  $\text{SLE}_{8/3}$ . Actually, since we only focus on the hull of the Brownian bubbles, the description of its frontier in terms of  $\text{SLE}_{8/3}$  is natural. The probability that an  $\text{SLE}_{8/3}$  (or a Brownian excursion) in  $\mathbb{H}$  from 0 to  $\epsilon$  hits the circle of radius  $\delta$  around zero decays like a constant times  $\epsilon^2$  when  $\epsilon$  goes to zero. Hence, if we condition the union of eight  $\text{SLE}_{8/3}$ ’s (resp., five Brownian excursions) from 0 to  $\epsilon$  to intersect this circle and take the limit when  $\epsilon \rightarrow 0$ , we obtain exactly the same outer boundary (we know from the previous description that this limit exists) as if we condition just one  $\text{SLE}_{8/3}$  or one Brownian excursion, since with high probability only one of them will hit the circle. Hence, we get that  $\mu$  is the filling of the limit when  $\epsilon \rightarrow 0+$  of  $8\epsilon^{-2}/5$  times the law of chordal  $\text{SLE}_{8/3}$  in  $\mathbb{H}$  from 0 to  $\epsilon$ . Hence, the hull of a Brownian bubble is also an “ $\text{SLE}_{8/3}$ -bubble”.

*Remark 7.2.* Let  $t > 0$  and let  $X_t$  be a sample from the Poisson point process with mean (intensity)  $t\nu$ .<sup>1</sup> Let  $U_t$  be the filling of the union of bubbles in  $X_t$ ,

---

<sup>1</sup>This means that  $X$  is a countable random set of bubbles such that whenever  $D_1, D_2, \dots, D_k$  are disjoint measurable sets of bubbles, the random variables  $|X \cap D_j|, j = 1, \dots, k$ , are independent and  $\mathbf{E}[|X \cap D_j|] = \nu[D_j]$ .

$U_t = \mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(\bigcup X_t)$ . By the properties of Poisson point processes, for all  $A \in \mathcal{Q}$

$$\mathbf{P}[U_t \cap A = \emptyset] = \exp(-t\nu[\{K : K \cap A \neq \emptyset\}]) = \exp(-ta(A)).$$

But for  $A, A' \in \mathcal{Q}$  we have  $a(A \cdot A') = a(A) + a(A')$ , by (2.5). Hence, the law of  $U_t$  is covariant under the semigroup  $\{\Phi_A : A \in \mathcal{Q}\}$ . However, it is not scale-invariant, because the image of  $\nu$  under  $z \mapsto \lambda z$  is  $\lambda^{-2}\nu$ . Thus, the distribution of  $U_t$  under the map  $z \mapsto \lambda z$  is the same as that of  $U_{\lambda^{-2}t}$ . This shows that the assumption of  $\Gamma$ -invariance in statement (1) of Proposition 3.3 is important.

**7.2. Adding a Poisson cloud of bubbles to SLE.** We are now ready to give a rather concrete interpretation of the right-hand side of (6.1) and thereby construct the measures  $\mathbf{P}_\alpha$  for  $\alpha \geq 5/8$ .

Suppose that  $\kappa \leq 8/3$  and, as in Section 5, let

$$\alpha = \alpha_\kappa = \frac{6 - \kappa}{2\kappa}, \quad \lambda = \lambda_\kappa = \frac{(8 - 3\kappa)(6 - \kappa)}{2\kappa}.$$

Consider a Poisson point process  $X$  on  $\Omega_b \times [0, \infty)$  with mean (intensity)  $\lambda \mu \times dt$ , where  $dt$  is Lebesgue measure (see footnote 1 for the definition). As before, let  $\gamma$  denote the SLE $_\kappa$  path,  $g_t$  the corresponding conformal maps, and  $W_t$  the Loewner driving process. We take  $\gamma$  to be independent from  $X$ . Since  $\kappa \leq 8/3$ , we know from [42] that  $\gamma$  is a simple curve. Let

$$\hat{X} := \{g_t^{-1}(K + W_t) : (K, t) \in X, t \in [0, \infty)\},$$

and let  $\Xi$  be the filling of the union of elements of  $\hat{X}$  and  $\gamma$ ,

$$\Xi = \Xi(\kappa) := \mathcal{F}_{\mathbb{H}}^{\mathbb{R}}(\gamma(0, \infty) \cup \bigcup \hat{X}).$$

Let  $A \in \mathcal{Q}^*$ , and let  $h_t$  be the normalized conformal map from  $\mathbb{H} \setminus g_t(A)$  onto  $\mathbb{H}$  as in Section 5. By (7.3), for any  $t > 0$  on the event  $\gamma[0, t] \cap A = \emptyset$ ,

$$\begin{aligned} & \mathbf{P}[\{K : g_t^{-1}(K + W_t) \cap A \neq \emptyset\} \mid g_t] \\ &= \mathbf{P}[\{K : (K + W_t) \cap g_t(A) \neq \emptyset\} \mid g_t] \\ &= -Sh_t(W_t)/6, \end{aligned}$$

where  $K$  is independent from  $\gamma$  and has law  $\mu$ . Consequently, on the event  $\gamma[0, \infty) \cap A = \emptyset$ ,

$$\mathbf{P}[\Xi \cap A = \emptyset \mid \gamma] = \exp\left(\lambda \int_0^\infty \frac{Sh_t(W_t)}{6} dt\right).$$

By taking expectation and applying Theorem 6.5, we get

$$(7.4) \quad \mathbf{P}[\Xi \cap A = \emptyset] = \Phi'_A(0)^\alpha,$$

which almost proves,

**Theorem 7.3.** *For any  $\kappa \in [0, 8/3]$ , the law of  $\Xi(\kappa)$  is  $\mathbf{P}_{\alpha_\kappa}$ .*

*Proof.* Given the discussion above, all that remains is to show that  $\overline{\Xi} = \Xi \cup \{0\}$ . Let  $D := \{z \in \overline{\mathbb{H}} : |z - x_0| \leq \epsilon\}$ , where  $0 < \epsilon < 1$  and  $x_0 \in [1, 2]$ . Then  $1 - \Phi'_D(0) = O(\epsilon^2)$ . Consequently,  $\mathbf{P}[\text{dist}(\Xi, [1, 2]) < \epsilon] = O(\epsilon)$ . Thus, a.s.,  $\overline{\Xi} \cap [1, 2] = \emptyset$ . By scaling, it follows that  $\overline{\Xi} \cap (\mathbb{R} \setminus \{0\}) = \emptyset$  a.s. Let  $X_{t_0}^{t_1}$  denote the set of pairs  $(K, t) \in X$  with  $t \in [t_0, t_1]$ . Since the  $\mu$ -measure of the set of bubbles of diameter larger than  $\epsilon$  is finite, a.s., for every  $t_1 \in [0, \infty)$  the set of  $(K, t) \in X_0^{t_1}$  such that  $K$  has diameter at least  $\epsilon$  is finite. Therefore, the set

$\gamma[0, t^1] \cup \bigcup \{g_t^{-1}(K + W_t) : (K, t) \in X_0^{t^1}\}$  is closed a.s. when  $t_1 < \infty$ . To show that  $\Xi \cup \{0\}$  is closed, it therefore suffices to prove that  $\bigcap_{s>0} \Xi_s = \emptyset$ , where  $\Xi_s := \gamma(s, \infty) \cup \bigcup \{g_t^{-1}(K + W_t) : (K, t) \in X_s^\infty\}$ . Let  $T(R)$  denote the first time  $t$  such that  $|\gamma(t)| = R$ , and let  $A = \{z \in \mathbb{H} : |z| \leq 1\}$ . Let  $A_+$  denote the set of points in  $A$  which are to the right of  $\gamma[0, T(R)]$  or on  $\gamma$  (i.e., the intersection of  $A$  with the closure of the domain bounded by  $\gamma[0, T(R)] \cup [0, R]$  and an arc of the semicircle  $\{z \in \mathbb{H} : |z| = R\}$ ). The proof of Lemma 6.2 gives  $\Phi'_{g_{T(R)}(A_+)}(W_{T(R)}) \rightarrow 1$  as  $R \rightarrow \infty$ . The stationarity property of SLE with equation (7.4) imply

$$\mathbf{P}[\Xi_{T(R)} \cap A_+ = \emptyset \mid \gamma[0, T(R)]] = \Phi'_{g_{T(R)}(A_+)}(W_{T(R)})^\alpha \rightarrow 1.$$

A symmetric argument shows that this holds with  $A$  in place of  $A_+$ . This implies that a.s.  $\bigcap_{s>0} \Xi_s$  is disjoint from the disk  $|z| < 1$ . Scale invariance now gives  $\bigcap_{s>0} \Xi_s = \emptyset$  a.s. and completes the proof.  $\square$

The theorem shows that for all  $\alpha > 5/8$ , the measure  $\mathbf{P}_\alpha$  exists and can be constructed by adding bubbles with appropriate intensity to  $\text{SLE}_\kappa$  with  $\kappa = 6/(2\alpha + 1)$ . The frontier of the set defined under  $\mathbf{P}_\alpha$  has Hausdorff dimension  $4/3$  (because of the Brownian bubbles). For instance, for integer  $\alpha$ , this shows that  $\text{SLE}_\kappa$  can be coupled with the union of  $n$  independent excursions so as to be a subset of their filling.

Note that  $\lim_{\kappa \rightarrow 0+} \lambda_\kappa = \infty$ , while  $\lambda_{8/3} = 0$ . Also observe that  $\alpha_2 = 1$ , so that adding Brownian bubbles to  $\text{SLE}_2$  with appropriate density gives the measure on hulls of Brownian excursions. This is not surprising since  $\text{SLE}_2$  is the scaling limit of loop-erased simple random walks as proved in [29].

In [33] it is shown that there is a natural Poisson point process  $L$  of sets in  $\mathbb{H}$ , independent from  $\gamma$ , such that  $\Xi$  can also be described as the (filling of) the union of  $\gamma$  with those sets in  $L$  which meet  $\gamma$ .

### 8. ONE-SIDED RESTRICTION

**8.1. Framework.** Recall the definition of  $\mathcal{Q}_+$  from Section 2. Set  $\mathcal{A}_+ = \{\Phi_A : A \in \mathcal{Q}_+\}$ . Let  $\Omega_+$  denote the set of all closed connected sets  $K \subset \mathbb{H}$  such that  $K \cap \mathbb{R} = (-\infty, 0]$  and  $\mathbb{H} \setminus K$  is connected. We endow  $\Omega_+$  with the  $\sigma$ -field generated by the family of events  $\{K \cap A = \emptyset\}$ , where  $A \in \mathcal{Q}_+$ . We say that the probability measure  $\mathbf{P}$  on  $\Omega_+$  satisfies the *right-sided restriction property* if it is  $\mathcal{A}_+$  covariant and scale invariant. In other words,  $\mathbf{P}[K \cap (A \cdot A') = \emptyset] = \mathbf{P}[K \cap A = \emptyset] \mathbf{P}[K \cap A' = \emptyset]$  and  $\mathbf{P}[K \cap A = \emptyset] = \mathbf{P}[K \cap (\lambda A) = \emptyset]$  hold for all  $A, A' \in \mathcal{Q}_+$ ,  $\lambda > 0$ .

The proof of Proposition 3.3 shows that if  $\mathbf{P}$  satisfies the right-sided restriction property, then there exists a constant  $\alpha \geq 0$  such that for all  $A \in \mathcal{Q}_+$ ,

$$\mathbf{P}[K \cap A = \emptyset] = \Phi'_A(0)^\alpha.$$

Conversely, for all  $\alpha \geq 0$ , there exists at most one such probability measure  $\mathbf{P}$ . If it exists, we call it the *right-sided restriction measure with exponent  $\alpha$*  and denote it by  $\mathbf{P}_\alpha^+$ . For  $\alpha \geq 5/8$ , we may obtain  $\mathbf{P}_\alpha^+$  by applying  $\mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}}$  to a sample from the two-sided restriction measure  $\mathbf{P}_\alpha$ . (Recall the notation  $\mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}}$  from Section 2.)

In the following, we will see two other constructions of  $\mathbf{P}_\alpha^+$ , which are valid for all  $\alpha > 0$ , the first is based on reflected Brownian motion, while the second is an SLE type construction, where an appropriate drift is added to the driving process of  $\text{SLE}_{8/3}$ . We will also be able to conclude that  $\mathbf{P}_\alpha$  does not exist when  $\alpha < 5/8$ .

We generally ignore the uninteresting case  $\alpha = 0$ , where  $K = (-\infty, 0]$  a.s.

**8.2. Excursions of reflected Brownian motions.** We now construct  $\mathbf{P}_\alpha^+$  for all  $\alpha > 0$  using reflected Brownian motions, or, more precisely, Brownian excursions conditioned to avoid  $(0, \infty)$  and reflected at angle  $\theta$  off  $(-\infty, 0]$ . In order to define this, fix  $\theta \in (0, \pi)$  and let  $c = c_\theta = -\cot \theta$ . We first consider Brownian excursions in the wedge

$$W := W(\theta) = \{re^{i\varphi} : r > 0 \text{ and } \varphi \in (0, \pi - \theta)\}$$

reflected in the horizontal direction off the boundary line  $x = cy$ . Let  $(Y_t, t \geq 0)$  denote a three-dimensional Bessel process started from 0 (i.e., a one-dimensional Brownian motion conditioned to stay in  $(0, \infty)$ ). Let  $(X_t, t \geq 0)$  be a one-dimensional Brownian motion, acting independently of  $Y$  when inside  $W(\theta)$ , which is reflected off the “moving boundary”  $cY_t$ . More precisely, if  $\tilde{X}$  is standard Brownian motion started from 0, independent of  $Y$ , then  $X_t$  is the unique continuous function such that  $X_t \geq cY_t$  and  $X_t = \tilde{X}_t + \ell_t$ , where  $\ell$  is a nondecreasing continuous function with  $\int 1_{X_t > cY_t} d\ell_t = 0$ . (See, e.g., [41] for more on Skorokhod’s reflection lemma.)

Define

$$Z_t = X_t + iY_t.$$

Let  $V$  denote a random variable which has the same law as  $Z_{T_1}$  where  $T_R$  denotes for all  $R > 0$  the hitting time of  $R$  by  $Y$ . Then, for all  $r < R$ , the process  $X + iY$  on  $[T_r, T_R]$  is started with the same distribution as  $rV$  and then evolves like two-dimensional Brownian motion which is reflected horizontally off the line  $y = cx$  and conditioned to hit  $\mathbb{R} + iR$  before  $\mathbb{R}$  (this event is independent of  $X_{T_r}$  and has probability  $r/R$  for the unconditioned reflected Brownian motion).

Let  $b(z) = b_\theta(z) = z^{\pi/(\pi-\theta)}$ . If  $A \in \mathcal{Q}_+$ , let  $F = F_{A,\theta} = b^{-1} \circ \Phi_{b(A)} \circ b$ . Then  $F$  is a conformal transformation of  $W \setminus A$  onto  $W$  with  $F(0) = 0$  and  $|F(z) - z|$  bounded. It is straightforward to show that the image under  $F$  of a horizontally reflected Brownian motion in  $W$ , up to the first time it hits  $\mathbb{R} \cup (W \setminus A)$ , is a (time-changed) horizontally reflected Brownian motion in  $W$ : as long as it is away from the line  $y = cx$ , this is just conformal invariance of planar Brownian motion, and since  $F'$  is real on the line  $y = cx$ , it follows that  $F(W)$  also gets a horizontal push when it hits the line; that is,  $d\tilde{\ell}_t = F'(Z_t) d\ell_t$  defines the corresponding push for  $\text{Re}(F(Z_t))$ . It therefore follows just as in the case of the Brownian excursion in  $\mathbb{H}$  (which corresponds to the limiting case  $\theta = 0$ ) that for all small  $r > 0$ ,

$$\mathbf{P}[Z[T_r, \infty) \cap A = \emptyset] = \frac{\mathbf{E}[\text{Im}(F(Z_{T_r}))]}{r}.$$

Hence, letting  $r \rightarrow 0$ , we get by dominated convergence that

$$\mathbf{P}[Z \cap A = \emptyset] = \Phi'_{b(A)}(0)^{1-(\theta/\pi)}.$$

We now define the “reflected Brownian excursion” in  $\mathbb{H}$  (in short RBE) as  $B = \{b_\theta(Z_t) : t \geq 0\}$ . Then, the previous equation for  $\mathbf{P}[Z \cap A = \emptyset]$  may be rewritten

$$\mathbf{P}[B \cap A = \emptyset] = \mathbf{P}[Z \cap b_\theta^{-1}(A) = \emptyset] = \Phi'_A(0)^{1-(\theta/\pi)},$$

which shows that  $\mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}^+}(B)$  satisfies right-sided restriction with exponent  $\alpha = 1 - \theta/\pi$ .

Note that the limiting cases  $\theta = 0$  and  $\theta = \pi$  correspond, respectively, to the Brownian excursion ( $\alpha = 1$ ) and to the ray  $(-\infty, 0]$  that stays on the boundary ( $\alpha = 0$ ).

Reflected Brownian excursions therefore show that for all  $\alpha \in (0, 1]$ , the right-sided restriction measure with exponent  $\alpha$  exists. Taking unions of independent hulls which satisfy the right-sided restriction property, yields a realization of another right-sided restriction measure (and the exponents add up). We summarize this in a proposition.

**Proposition 8.1.** *The right-sided restriction measures  $P_\alpha^+$  exist for all  $\alpha > 0$ . If  $\alpha = a_1 + \dots + a_k$  where  $k$  is a positive integer and  $a_1, \dots, a_k \in (0, 1]$ , then  $\mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}^+}$  applied to the union of  $k$  independent RBEs with respective angles  $\theta_1 = \pi(1 - a_1), \dots, \theta_k = \pi(1 - a_k)$  has law  $P_\alpha^+$ .*

This, together with the observation that  $P_\alpha^+$  can be realized as the “left-filling” of samples from  $P_\alpha$  (when the latter exists) implies various rather surprising identities in the law between “right boundaries” of different processes:

**Corollary 8.2.**

- (1) *The right boundary of an RBE with angle  $3\pi/8$  has the same law as  $SLE_{8/3}$ . In particular, its law is symmetric with respect to the imaginary axis.*
- (2) *The right boundary of the union of  $n$  independent RBE’s with angles  $\pi - \theta_1, \dots, \pi - \theta_n$  has the same law as the right boundary of an RBE with angle  $\pi - (\theta_1 + \dots + \theta_n)$ , provided that  $\theta_1 + \dots + \theta_n < \pi$ .*
- (3) *The right boundary of the union of two independent RBE’s which are orthogonally reflected on the negative half-axis has the same law as the right boundary of a Brownian excursion.*

The first statement shows that the Brownian frontier (outer boundary) looks like a locally symmetric path. This, in spirit, answers a question raised by Chris Burdzy after Benoit Mandelbrot noted (based on simulations) the similarity between the dimension of self-avoiding walks and the Brownian frontier and proposed [34] the name “self-avoiding Brownian motion” for the Brownian frontier. Burdzy’s question was whether the Brownian frontier is [locally] symmetric. There are several different precise formulations of this question. See Section 10 for more about this issue.

Note that the last two statements (and their proofs) do not use SLE. The first statement yields an extremely fast algorithm to simulate chordal  $SLE_{8/3}$  and therefore also the scaling limit of self-avoiding half-plane walks (modulo the conjecture [30] that chordal  $SLE_{8/3}$  is the scaling limit of the half-plane self-avoiding walk) as the right boundary of a reflected excursion. See [20] for an algorithm to simulate directly such walks.

**8.3. The  $SLE(\kappa, \rho)$  process.** We will now describe the right boundaries of these sets in terms of SLE-type paths that are driven by Bessel-type processes.

Before introducing these processes, let us give a brief heuristic. Let  $\gamma$  denote the right boundary of a Brownian excursion in  $\mathbb{H}$ . Let us condition on a piece  $\gamma[0, t]$ . For the future of  $\gamma$  beyond time  $t$ , the right-hand boundary of  $\gamma[0, t]$  acts just like the positive real axis, and  $\gamma[t, \infty)$  is “conditioned” not to hit the right-hand side of  $\gamma[0, t]$ . If we believe in conformal invariance of the process, then we may ignore all the geometry of the domain  $\mathbb{H} \setminus \gamma[0, t]$  and map it onto the upper half-plane. However, we should keep track of the left image of 0 under the uniformizing map  $g_t$ . It is reasonable to believe that this is all that would be relevant to the distribution of  $g_t(\gamma[t, \infty))$ . (We will a posteriori see that this is the case.) Let  $W_t = g_t(\gamma(t))$ , let  $O_t$  be the left image of 0 under  $g_t$ , and take  $t$  to be the half-plane capacity



FIGURE 8.1. A sample of the beginning of a (long) reflected random walk with angle  $3\pi/8$  (its right boundary looks like  $\text{SLE}_{8/3}$ ).

parametrization for  $\gamma[0, t]$ . Then the pair  $(W_t, O_t)$  is a continuous Markov process, and the chordal version of Loewner's theorem gives  $dO_t/dt = 2/(O_t - W_t)$ . Scale and translation invariance show that it is enough to know what happens to  $W$  infinitesimally when  $O = 0$  and  $W = 1$ . The natural guess is that at that moment we have  $dW = \sqrt{\kappa} dB + \rho dt$ , for some constants  $\kappa > 0$  and  $\rho \in \mathbb{R}$ . Scaling this to other values of  $W$  gives  $\text{SLE}(\kappa, \rho)$ , as will be defined shortly.

Suppose that  $\kappa > 0$ ,  $\rho > -2$  and that  $B_t$  is a standard one-dimensional Brownian motion. Let  $(O_t, W_t)$  be the solution of

$$dO_t = \frac{2 dt}{O_t - W_t}, \quad dW_t = \frac{\rho dt}{W_t - O_t} + \sqrt{\kappa} dB_t$$

with  $O_0 = W_0 = 0$  and  $O_t \leq W_t$ . The meaning of this evolution is straightforward at times when  $W_t > O_t$ , but it is a bit more delicate when  $W_t = O_t$ . One way to construct  $(O_t, W_t)$  is to first define  $Z_t$  (later to become  $W_t - O_t$ ) as the solution to the Bessel equation

$$dZ_t = \frac{(\rho + 2) dt}{Z_t} + \sqrt{\kappa} dB_t$$

started from  $Z_0 = 0$ . More precisely,  $Z_t$  is  $\sqrt{\kappa}$  times a  $d$ -dimensional Bessel process where

$$(8.1) \quad d = 1 + \frac{2(\rho + 2)}{\kappa}.$$

It is well known (e.g., [41]) that this process is well defined (for all  $\rho > -2$  and all  $t \geq 0$ ). Note also that  $\int_0^t du/Z_u = (Z_t - \sqrt{\kappa}B_t)/(\rho + 2) < \infty$  for all  $t \geq 0$ . Then, set

$$\begin{aligned} O_t &= -2 \int_0^t \frac{du}{Z_u}, \\ W_t &= Z_t + O_t. \end{aligned}$$

If we then define the family of conformal maps  $g_t$  by  $\partial_t g_t(z) = 2(W_t - g_t(z))^{-1}$  and  $g_0(z) = z$  (for  $z \in \overline{\mathbb{H}}$ ), we get a Loewner chain that we call chordal  $\text{SLE}(\kappa, \rho)$ . Note



that when  $\rho = 0$ , we get the ordinary chordal  $SLE_\kappa$ . Intuitively, the definition of  $SLE(\kappa, \rho)$  can be understood as follows:  $O_t$  is the left-most point of  $g_t(\partial K_t)$  (when  $K_t$  is a simple path, this is simply the “left” image of the origin under  $g_t$ ); the  $W_t$  gets a push away from this point if  $\rho > 0$  (or towards this point if  $-2 < \rho < 0$ ), and this push is “constant” modulo scaling.

The next lemma lists a few basic properties of  $SLE(\kappa, \rho)$ , which are generalizations of known results for  $SLE$ , i.e., for the case  $\rho = 0$ .

**Lemma 8.3.** *Let  $\kappa > 0$ ,  $\rho > -2$  and set  $\rho_0 := -2 + \kappa/2$ . Let  $K_t$  denote the evolving hulls of  $SLE(\kappa, \rho)$  and  $K_\infty := \bigcup_{t>0} K_t$ .*

- (1) *The distribution of  $SLE(\kappa, \rho)$  is scale-invariant. More precisely, if  $\lambda > 0$ , then  $(K_t, t \geq 0)$  has the same distribution as  $(\lambda^{-1} K_{\lambda^2 t}, t \geq 0)$ .*
- (2) *If  $\kappa \leq 4$  and  $\rho \geq \rho_0$ , then a.s.  $K_\infty \cap \mathbb{R} = \{0\}$ .*
- (3) *If  $\kappa \leq 4$  and  $\rho < \rho_0$ , then a.s.  $K_\infty \cap \mathbb{R} = (-\infty, 0]$ .*
- (4)  *$K_\infty$  is a.s. unbounded.*

Recall that a.s. the  $d$ -dimensional Bessel process returns to zero if and only if  $d < 2$ . This will be essential in the proof of (2) and (3).

*Proof.* Clearly,  $(W_t, O_t)_{t \geq 0}$  has the same scaling property as Brownian motion, and (1) follows.

Now assume  $\kappa \leq 4$ . Let  $\tau_1 = \sup\{t \geq 0 : 1 \notin K_t\}$ . We want to show that  $\tau_1 = \infty$  a.s. Set  $x_t = g_t(1)$  for  $t < \tau_1$  and observe that  $x_t - O_t$  is monotone increasing. In particular,  $x_t - O_t \geq 1$ ,  $t < \tau_1$ . On the set of times  $t < \tau_1$  such that  $x_t - W_t < 1/2$ , we therefore have  $dW_t \leq 2|\rho| dt + \sqrt{\kappa} dB_t$ . Setting  $\tilde{x}_t = x_t - W_t$ , we get

$$d\tilde{x}_t \geq -2|\rho| dt - \sqrt{\kappa} dB_t + (2/\tilde{x}_t) dt,$$

on the set of times  $t$  such that  $\tilde{x}_t < 1/2$ . If  $\rho = 0$ , by comparing with the Bessel process we see that a.s.  $\tilde{x}_t$  never hits 0 and so  $\tau_1 = \infty$ . For  $\rho \neq 0$ , note that for any finite fixed  $t_0 > 0$  and any  $c \in \mathbb{R}$  the law of the process  $(B_t + ct, t \leq t_0)$  is equicontinuous with the law of  $(B_t, t \leq t_0)$ . (In fact, after conditioning on the position of the process at time  $t_0$ , their distributions are identical.) Therefore, also in this case  $\tilde{x}_t$  never hits 0 and  $\tau_1 = \infty$ . Hence, a.s.  $1 \notin K_t$  for all  $t \geq 0$ . This also implies that  $K_t \cap [1, \infty) = \emptyset$  a.s. for all  $t \geq 0$ , since  $K_t \cap \mathbb{R}$  is an interval. Scale invariance then implies  $K_\infty \cap (0, \infty) = \emptyset$  a.s.

Now suppose  $\rho \geq \rho_0$ . Then the Bessel process  $Z_t/\sqrt{\kappa}$  has dimension  $d \geq 2$ , as given by (8.1). Consequently, a.s.  $W_t - O_t = Z_t > 0$  for all  $t > 0$ . If  $x < 0$ , then  $g_t(x) \leq O_t$  for all  $t \geq 0$ . Hence,  $K_\infty \cap (-\infty, 0) = \emptyset$  a.s.

Now take  $\rho \in (-2, \rho_0)$ . Set  $y_t = g_t(-1)$  for  $t < \tau_{-1} := \sup\{t \geq 0 : -1 \notin K_t\}$ . Using  $W_t - y_t \geq W_t - O_t$  and  $\rho < 0$ , we get for  $t < \tau_{-1}$

$$\begin{aligned} W_t - y_t &= \sqrt{\kappa} B_t + \int_0^t ds \left( \frac{\rho}{W_s - O_s} + \frac{2}{W_s - y_s} \right) \\ &\leq \sqrt{\kappa} B_t + \int_0^t \frac{(\rho + 2) ds}{W_s - y_s}. \end{aligned}$$

So that  $W_t - y_t$  is smaller than a Bessel process that hits zero a.s. Hence, a.s.  $-1 \in K_\infty$ . This implies  $[-1, 0] \subset K_\infty$  a.s., and by scaling  $(-\infty, 0] \subset K_\infty$  a.s. This completes the proof of (2) and (3).

Statement (4) easily follows from (1), for example. One could also use the fact that the half-plane capacity of  $K_t$  is  $2t$ . □

The  $SLE(8/3, \rho)$ 's are related to the measures  $\mathbf{P}_\alpha^+$  via the following theorem that will be proved in the next subsection.

**Theorem 8.4.** *Let  $\rho > -2$ , and let  $K = \mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}^+}(\overline{K}_\infty)$ , where  $K_t$  is the hull of  $SLE(8/3, \rho)$  and  $K_\infty = \bigcup_{t \geq 0} K_t$ . Then  $K$  satisfies the right-sided restriction property with exponent*

$$\alpha = \frac{20 + 16\rho + 3\rho^2}{32} = \frac{(3\rho + 10)(2 + \rho)}{32}.$$

Note that when  $\rho$  spans  $(-2, \infty)$ ,  $\alpha$  spans  $(0, \infty)$ . This theorem has several nice corollaries, some of which we now briefly discuss.

**Corollary 8.5.** *If  $\alpha \geq 5/8$ , the right boundary of the two-sided restriction measure  $\mathbf{P}_\alpha$  has the same law as the  $SLE(8/3, \rho(\alpha))$  path, where*

$$\rho(\alpha) = \frac{-8 + 2\sqrt{24\alpha + 1}}{3}.$$

*In particular, the right boundary of a Brownian excursion has the law of (the path of)  $SLE(8/3, 2/3)$  and the right boundary of the union of two Brownian excursions has the law of  $SLE(8/3, 2)$ .*

**Corollary 8.6.** *For all  $\alpha < 5/8$ , the two-sided restriction probability measure  $\mathbf{P}_\alpha$  does not exist.*

*Proof.* Note that when  $\rho < 0$ ,  $W_t - \sqrt{\kappa}B_t$  is decreasing. It follows easily that the probability that  $i$  ends up eventually to “the right” of the right-hand boundary of  $SLE(8/3, \rho)$  (i.e.,  $i$  is separated from 1 by  $K_\infty \cup (-\infty, 0]$ ) is strictly larger than the corresponding quantity for  $SLE(8/3, 0)$ , which is  $1/2$  by symmetry. However, the same symmetry argument shows that for any  $\alpha > 0$ , if the two-sided probability measure with exponent  $\alpha > 0$  exists, then the  $\mathbf{P}_\alpha$  probability that  $i$  ends up to the “right” of  $K$  is at most  $1/2$  (it can be smaller if  $K$  is of positive Lebesgue measure). If  $K$  has law  $\mathbf{P}_\alpha$  with some  $\alpha < 5/8$ , then  $\mathcal{F}_{\mathbb{H}^+}^{\mathbb{R}^+}(K)$  has law  $\mathbf{P}_\alpha^+$ , which is described using  $SLE(8/3, \rho)$  for some  $\rho < 0$ . This contradicts the fact that the probability that it passes to the left of  $i$  is at least  $1/2$ .  $\square$

**Corollary 8.7.** *The boundary of the right-sided restriction measure intersects the negative half-line if and only if  $\alpha < 1/3$ . In particular, the reflected Brownian excursion with reflection angle  $\theta$  on the negative half-line has cut-points on the negative half-line if and only if  $\theta > 2\pi/3$ .*

*Proof.* This is just a combination of Lemma 8.3, Theorem 8.4 and Proposition 8.1.  $\square$

**Remark 8.8.** Note that nonexistence of cut-points on the negative half-line for the angle  $2\pi/3$  proves (via the correspondence between reflected Brownian motion and the  $SLE_6$  hull that is discussed in Section 9) nonexistence of cut-points for the  $SLE_6$  hull on the positive and negative half-line (and therefore also nonexistence of double points for  $SLE_6$  that are also local cut-points for the  $SLE_6$  path). In the discrete case (i.e., critical site percolation on the triangular grid), van den Berg and Jarai [7] have recently derived a stronger version of this result (with decay rates for probabilities).

8.4. **Proof of Theorem 8.4.** Fix  $\rho > -2$  and let

$$c = \frac{3\rho}{8} \text{ and } b = \frac{\rho(4 + 3\rho)}{32}.$$

Let  $(O_t, W_t)$  generate an SLE(8/3,  $\rho$ ) process so that

$$dW_t = \frac{\rho}{W_t - O_t} dt + \sqrt{8/3} dB_t, \quad dO_t = \frac{2}{O_t - W_t} dt.$$

Let  $A \in \mathcal{Q}_+$  be a given smooth hull, and let  $\Phi = \Phi_A$ ,  $T = T_A$ , and let  $h_t$  be as in Section 5 and define (for  $t < T$ ),

$$M_t := h'_t(W_t)^{5/8} h'_t(O_t)^b \left[ \frac{h_t(W_t) - h_t(O_t)}{W_t - O_t} \right]^c.$$

Of course, when  $W_t = O_t$ , we take  $M_t = h'_t(W_t)^{(5/8)+b+c}$ .

**Lemma 8.9.**  $(M_t, t < T)$  is a local martingale.

*Proof.* Plugging the explicit choice of  $(W_t, t \geq 0)$  into the results of Section 5 shows that for  $t < T$ ,

$$\begin{aligned} d[h_t(W_t)] &= \left( \frac{\rho h'_t(W_t)}{W_t - O_t} - (5/3) h''_t(W_t) \right) dt + \sqrt{8/3} h'_t(W_t) dB_t, \\ d[h'_t(W_t)] &= \left( \frac{\rho h''_t(W_t)}{W_t - O_t} + \frac{h''_t(W_t)^2}{2h'_t(W_t)} \right) dt + \sqrt{8/3} h''_t(W_t) dB_t, \\ d[h_t(O_t)] &= \frac{2h'_t(W_t)^2}{h_t(O_t) - h_t(W_t)} dt, \\ d[h'_t(O_t)] &= \left( \frac{2 h'_t(O_t)}{(O_t - W_t)^2} - \frac{2 h'_t(W_t)^2 h'_t(O_t)}{(h_t(O_t) - h_t(W_t))^2} \right) dt. \end{aligned}$$

Using these expressions in Itô's formula for  $dM_t$ , one can now compute the semimartingale decomposition of  $M_t$ . This is tedious but straightforward, so we omit the detailed calculation here. The drift term of  $dM_t$  turns out to be  $M_t$  times

$$\begin{aligned} &\left( \frac{5\rho}{8} - \frac{5c}{3} \right) \frac{h''_t(W_t)}{h'_t(W_t)(W_t - O_t)} \\ &+ (2b - c(\rho + 2) + \frac{4}{3}c(c + 1)) \frac{1}{(W_t - O_t)^2} \\ &+ (-2b + 2c + \frac{4}{3}c(c - 1)) \frac{h'_t(W_t)^2}{(h_t(W_t) - h_t(O_t))^2} \\ &+ \left( -\frac{8}{3}c^2 + \rho c \right) \frac{h'_t(W_t)}{(W_t - O_t)(h_t(W_t) - h_t(O_t))}. \end{aligned}$$

The terms in  $h''_t(W_t)^2/h'_t(W_t)^2$  and in  $h''_t(W_t)/(h_t(W_t) - h_t(O_t))$  happen to vanish because of the choice of the exponent 5/8 (and  $\kappa = 8/3$ ). The lemma follows as this drift term vanishes for the appropriate choice of  $b$  and  $c$ .  $\square$

**Lemma 8.10.** *There exists  $\epsilon > 0$  such that  $M_t \leq h'_t(W_t)^\epsilon$  for all  $t < T$ . In particular,  $M_t \leq 1$ .*

*Proof.* When  $\rho \geq 0$ , the statement is trivial since  $b, c \geq 0$  and  $h'_t(W_t)$ ,  $h'_t(O_t)$  and  $(h_t(W_t) - h_t(O_t))/(W_t - O_t)$  are all in  $[0, 1]$ . One has to be a little bit careful when  $\rho < 0$  as  $c < 0$  and  $b$  can be negative as well. Let

$$\alpha = \frac{5}{8} + b + c = \frac{(3\rho + 10)(2 + \rho)}{32}$$

and note that  $\alpha > 0$ .

We now want to show that

$$(8.2) \quad h'_t(W_t) \leq \frac{h_t(W_t) - h_t(O_t)}{W_t - O_t} \leq h'_t(O_t) \leq 1.$$

This will be established by showing that  $h'_t$  is decreasing in  $(-\infty, W_t]$ . Recall that  $h_t = g_{g_t(A)}$ . In fact, the following argument shows that  $g'_{A^*}$  is monotone decreasing on  $x < \inf(A^* \cap \mathbb{R})$  for every smooth hull  $A^*$ . Applying this with  $A^* = g_t(A)$  then yields (8.2). To prove this monotonicity result, we realize the map  $g_{A^*} = g_{g_t(A)} = h_t$  as a map in a Loewner chain, as follows. Let  $\beta : [0, S] \rightarrow \overline{\mathbb{H}}$  be the smooth path  $\overline{\partial A^*} \cap \overline{\mathbb{H}}$  starting from  $\beta(0) = \inf(A^* \cap \mathbb{R})$  and parametrized by half-plane capacity from  $\infty$ . Set  $\hat{g}_s = g_{\beta[0, s]}$  and  $x_s = \hat{g}_s(\beta(s))$ . By the chordal version of Loewner's theorem,  $\partial_s \hat{g}_s(z) = 2/(\hat{g}_s(z) - x_s)$ . Then  $\hat{g}_S = h_t$ , since both are equal to the normalized map from  $\mathbb{H} \setminus g_t(A)$  onto  $\mathbb{H}$ . Since  $\partial_s \hat{g}'_s(z) = -2\hat{g}'_s(z)/(\hat{g}_s(z) - x_s)^2$ , it follows that

$$(8.3) \quad \partial_s \log \hat{g}'_s(z) = \frac{-2}{(\hat{g}_s(z) - x_s)^2}.$$

Therefore,  $h'_t$  is decreasing on  $x < \beta(0)$ , which proves (8.2). This implies that in the case where  $\rho \in [-4/3, 0)$  (because then  $b \leq 0$  and  $c \leq 0$ ),

$$M_t \leq h'_t(W_t)^{5/8+b+c} = h'_t(W_t)^\alpha \leq 1.$$

Now suppose  $\rho \in (-2, -4/3)$ , which gives  $c < 0$  and  $b > 0$ . For this case, we use a similar argument involving the Loewner chain  $(\hat{g}_s)$ , but a little more care is necessary. Suppose that  $o < w < x_0$  and let  $w_s = \hat{g}_s(w)$ ,  $o_s = \hat{g}_s(o)$ . From the expression for  $\partial_s \hat{g}_s$  we have

$$\partial_s \log(w_s - o_s) = \frac{-2}{(x_s - w_s)(x_s - o_s)}.$$

Combining this with (8.3) shows that

$$(8.4) \quad M_t = \exp \int_0^S \left( \frac{-2(5/8)}{(x_s - w_s)^2} + \frac{-2c}{(x_s - w_s)(x_s - o_s)} + \frac{-2b}{(x_s - o_s)^2} \right) ds,$$

where  $w = W_t$  and  $o = O_t$ . But  $o_s < w_s < x_s$  for all  $s \leq S$ . Hence (using the explicit values of  $b$  and  $c$ , and  $\rho \in (-2, -4/3)$ ),

$$\frac{(5/8) - \epsilon}{(x_s - w_s)^2} + \frac{c}{(x_s - w_s)(x_s - o_s)} + \frac{b}{(x_s - o_s)^2} \geq 0$$

for some positive  $\epsilon = \epsilon(\rho)$ . The lemma finally follows. □

**End of the proof of Theorem 8.4.** It now remains to study the behavior of the bounded martingale  $(M_t, t < T)$  when  $t \rightarrow T-$ . Let  $K_t$  be the growing hull of the SLE(8/3,  $\rho$ ) process. Recall that  $A$  is a smooth hull in  $\mathcal{Q}_+$  and that  $K_\infty \cap (0, \infty) = \emptyset$ . Hence, if  $T < \infty$ , then  $K_T \cap A \neq \emptyset$ . Lemma 6.3 shows that

$$\lim_{t \rightarrow T-} h'_t(W_t) = 0,$$

and Lemma 8.10 implies that  $\lim_{t \rightarrow T-} M_t = 0$  if  $T < \infty$ . Let us now suppose that  $T = \infty$ . Lemma 6.2 shows that a.s. on the event  $T = \infty$ ,

$$\lim_{r \rightarrow \infty} h'_{T_r}(W_{T_r}) = 1.$$

By (8.2), it follows that  $\lim_{r \rightarrow \infty} M_{T_r} = 1$ . Hence, since  $M_t$  converges a.s. and in  $L^1$  when  $t \rightarrow T$ , it follows that the limit is  $1_{T=\infty}$  and

$$\mathbf{P}[K_\infty \cap A = \emptyset] = M_0 = \Phi'_A(0)^\alpha.$$

It remains to prove that a.s.  $K_\infty \cap A \neq \emptyset$  if and only if  $\overline{K}_\infty \cap A \neq \emptyset$ . As  $K_t$  is closed for each  $t$ , the proof of this fact is essentially identical to the argument showing that  $\bigcap_{s>0} \overline{\Xi}_s = \emptyset$  given at the end of the proof of Theorem 7.3.  $\square$

**8.5. Formal calculations.** In this subsection we discuss how one can guess the form of the martingales  $M_t$  giving the intersection probabilities. Since this is not part of the proof, we will not be rigorous; however, much of this discussion can be made rigorous and may be used to further study restriction measures.

Let  $\mathcal{Q}_1$  denote the set of  $A \in \mathcal{Q}$  such that  $A \cap \mathbb{R} \subset (1, \infty)$ . For  $A \in \mathcal{Q}_1$ , let  $\phi_A$  denote the unique conformal map  $\phi : \mathbb{H} \setminus A \rightarrow \mathbb{H}$  which fixes each of the three points  $0, 1, \infty$ . Suppose that  $X$  is a random set in  $\mathbb{H}$ , whose law is covariant with respect to the semigroup  $\Lambda = \{\phi_A : A \in \mathcal{Q}_1\}$ . An example of such a set should be given by an SLE(8/3,  $\rho$ ) started with  $O_0 = 0$  and  $W_0 = 1$ .

One can also associate to  $A$  the unique conformal map  $g_A : \mathbb{H} \setminus A \rightarrow \mathbb{H}$  that is normalized at infinity. Note that

$$\phi_A(z) = (g_A(z) - g_A(0)) / (g_A(1) - g_A(0)).$$

Define now

$$\tilde{H}(g_A) := H(\phi_A) := \mathbf{P}[X \subset \phi^{-1}(\mathbb{H})].$$

Our goal is to show that  $\tilde{H}(g_A)$  is of the form

$$g'_A(0)^a g'_A(1)^b (g_A(1) - g_A(0))^c.$$

It is more convenient to work first with  $H$  since  $\Lambda$  is a semigroup while the family  $\{g_A : A \in \mathcal{Q}_1\}$  is not. The function  $H$  is a semigroup homomorphism from  $\Lambda$  into the multiplicative semigroup  $[0, 1]$ . Consequently,  $dH$  is a Lie algebra homomorphism into  $\mathbb{R}$ . The “basic” vector fields generating  $\Lambda$  have the form

$$A(x) = \frac{z(1-z)}{z-x}, \quad x > 1.$$

(This vector field corresponds to an infinitesimal slit at  $x$ . Note that flowing along  $A(x)$  preserves  $0, 1, \infty$ .) This is a one real-parameter ( $x$ ) family of vector fields in the  $z$ -plane. The commutator of  $A(x)$  and  $A(y)$  turns out to be

$$[A(x), A(y)] = A(x)\partial_z A(y) - A(y)\partial_z A(x) = \frac{(x-y)(z-1)^2 z^2}{(x-z)^2 (y-z)^2}.$$

This is supposed to be annihilated by  $dH$ , since  $[0, 1]$  is commutative. Hence, if we divide by  $x - y$  and take a limit as  $y \rightarrow x$ , it will also be annihilated by  $dH$ . This is the vector field

$$(8.5) \quad \hat{A}(x) := \lim_{y \rightarrow x} (x - y)^{-1} [A(x), A(y)] = \frac{(1 - z)^2 z^2}{(x - z)^4}.$$

To understand  $H$ , we want to determine the function

$$h(x) = dH(A(x)).$$

So we want to extract from  $dH(\hat{A}(x)) = 0$  information about  $dH(A(x))$ . For this, we write  $\hat{A}(x)$  as a linear combination of the derivatives  $\partial_x^j A(x)$  with coefficients functions of  $x$ . Direct computation gives

$$\hat{A}(x) = -\partial_x A(x) + \frac{1}{2} (1 - 2x) \partial_x^2 A(x) + \frac{1}{6} (x - x^2) \partial_x^3 A(x).$$

Since  $dH$  is linear, it commutes with  $\partial_x$ , and we get

$$\begin{aligned} 0 &= dH(\hat{A}(x)) \\ &= -\partial_x dH(A(x)) + \frac{1}{2} (1 - 2x) \partial_x^2 dH(A(x)) + \frac{1}{6} (x - x^2) \partial_x^3 dH(A(x)) \\ &= -h'(x) + \frac{1}{2} (1 - 2x) h''(x) + \frac{1}{6} (x - x^2) h'''(x). \end{aligned}$$

The general solution of this equation turns out to be very simple. It is

$$h(x) = \frac{c_0 + c_1 x + c_2 x^2}{x(1 - x)} = a_0 \left( \frac{x - 1}{x} \right) + a_1 + a_2 \left( \frac{x}{x - 1} \right).$$

This, in fact, already determines the general form of  $H$ , since any  $\phi_A$  can be obtained in a Loewner-equation way from the infinitesimal fields  $A$ .

We now want to translate this information in terms of  $\tilde{H}$ , since this is the framework that we are working with (even though the present analysis shows that it is not the most natural one here, but we have some formulas worked out already, so it is more economical at this point). Suppose that  $g_A$  is obtained via a Loewner chain driven by a continuous function  $(x_s, s \leq S)$ . Then  $\partial_s \hat{g}_s = 2/(\hat{g}_s(z) - x_s)$  and  $\hat{g}_S = g_A$ . Associate to each  $\hat{g}_s$  the corresponding function  $\phi_s = (\hat{g}_s - \hat{g}_s(0))/(\hat{g}_s(1) - \hat{g}_s(0))$ , which is normalized at  $0, 1, \infty$ . Then,

$$\partial_s \phi_s = \frac{-2}{(\hat{g}_s(0) - x_s)(\hat{g}_s(1) - x_s)} A \left( \frac{x_s - \hat{g}_s(0)}{\hat{g}_s(1) - \hat{g}_s(0)} \right) \circ \phi_s.$$

Since  $H$  is multiplicative,  $\partial_s H(\phi_s) = H(\phi_s) dH((\partial_s \phi_s) \circ \phi_s^{-1})$ . It therefore follows readily that  $\tilde{H}(\hat{g}_S) = H(\phi_S) = \int_0^S \partial_s H(\phi_s) ds$  is equal to

$$\exp \int_0^S ds \left( \frac{a'}{(x_s - \hat{g}_s(1))^2} + \frac{c'}{(x_s - \hat{g}_s(0))(x_s - \hat{g}_s(1))} + \frac{b'}{(x_s - \hat{g}_s(0))^2} \right),$$

as we had in (8.4). The “good” values of  $a', b', c'$  can then be determined by inspection.

As was just pointed out, it can be quite useful to study the  $SLE(\kappa, \rho)$  in the context of conformal maps that fix  $0, 1$  and  $\infty$ . It is therefore natural to define

$$G_t(z) := \frac{g_t(z) - O_t}{W_t - O_t}$$

where  $(g_t, t \geq 0)$  is the Loewner chain associated with  $SLE(\kappa, \rho)$  and  $O_t$  is the “leftmost” image of 0 under  $g_t$ . The evolution equation for  $G_t(z)$  is

$$d_t G_t(z) = \frac{-\sqrt{\kappa}}{W_t - O_t} G_t(z) dB_t + \frac{1}{(W_t - O_t)^2} G_t(z) \left( \frac{2}{G_t(z) - 1} + \kappa - \rho - 2 \right) dt.$$

If one then defines a time-change

$$u(t) = \int_1^t \frac{dv}{(W_v - O_v)^2}$$

and  $\tilde{G}_u(z) := G_{t(u)}(z)$  for all real  $u$ , then

$$d\tilde{G}_u(z) = \tilde{G}_u(z) \sqrt{\kappa} d\tilde{B}_u + \tilde{G}_u(z) \left( \frac{2}{\tilde{G}_u(z) - 1} + \kappa - \rho - 2 \right) dt,$$

for a two-sided Brownian motion  $\tilde{B}$  satisfying  $d\tilde{B}_{u(t)} = -dB_t / (W_t - O_t)$ .

9. EQUIVALENCE OF THE FRONTIERS OF  $SLE_6$  AND BROWNIAN MOTION

Brownian motion and  $SLE_6$  are both conformally invariant and local. We shall now see that this implies a fundamental equivalence between the hulls that they generate. Some of the results presented in this section were announced in [50] and have been presented in seminars for some years now.

**9.1. Full-plane  $SLE_6$  and planar Brownian motion.** The simplest version of the equivalence between the boundary of  $SLE_6$  and planar Brownian motion involves full-plane  $SLE_6$ , whose definition we now recall. Let  $\xi : \mathbb{R} \rightarrow \partial\mathbb{U}$  be continuous. It is well known [39] that there is a unique one-parameter family of conformal maps  $f_t : \mathbb{U} \rightarrow \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  such that the inverses  $g_t = f_t^{-1}$  satisfy Loewner’s equation

$$(9.1) \quad \partial_t g_t(z) = -g_t(z) \frac{g_t(z) + \xi(t)}{g_t(z) - \xi(t)}$$

and the normalization (for all  $t \in \mathbb{R}$ )

$$(9.2) \quad \lim_{z \rightarrow \infty} z g_t(z) = e^t.$$

Set  $K_t := \hat{\mathbb{C}} \setminus f_t(\mathbb{U})$ . This is called the *hull* of the family  $(g_t)$ . Then  $K_t \supset K_s$  when  $t \geq s$ , i.e.,  $K_t$  is an increasing family of compact sets with  $\bigcap_t K_t = \{0\}$ . The relation (9.2) implies  $\text{cap}(K_t) = t$ , where  $\text{cap}$  denotes (logarithmic) capacity. (The capacity of a nonempty closed connected set  $K \subset \mathbb{C}$  can be defined as  $\lim_{z \rightarrow 0} \log |z g(z)|$ , where  $g : \mathbb{U} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{F}_{\mathbb{C}}(K)$  is any conformal map satisfying  $g(0) = \infty$ .)

We remark that in the presence of (9.1), if (9.2) holds for one  $t \in \mathbb{R}$ , it also holds for all  $t \in \mathbb{R}$ . The proof of uniqueness is based on the fact that if  $g_t$  and  $\hat{g}_t$  both satisfy (9.1) and (9.2) for  $t \in [-t_0, \infty)$ , then  $|g_t - \hat{g}_t|$  is necessarily small away from 0 if  $t_0$  is large. In other words, the far away past matters very little.

Now let  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  be two-sided real Brownian motion with  $\beta(0) = 0$ , and let  $b_0$  be random-uniform in  $\partial\mathbb{U}$  and independent from  $\beta$ . Set  $\xi(t) = b_0 \exp(i\sqrt{\kappa}\beta(t))$ . With this choice of  $\xi$ , the above  $(g_t, t \in \mathbb{R})$  (or  $(K_t, t \in \mathbb{R})$ ) is called *full-plane  $SLE_\kappa$* . As for other  $SLE_\kappa$  (see [42, 29]), there is a continuous path  $\gamma : [-\infty, \infty] \rightarrow \hat{\mathbb{C}}$  with  $\gamma(-\infty) = 0$  such that  $K_t = \mathcal{F}_{\mathbb{C}}(\gamma[-\infty, t])$ . Also, given  $\gamma[-\infty, s]$ , the evolution

of  $\gamma(t), t \geq s$ , is the same as the conformal image of the radial  $SLE_\kappa$  path. In particular, full-plane  $SLE_6$  satisfies the locality property.

For the remainder of this section, we will fix  $\kappa = 6$  and use  $K_t$  to refer to the hull of  $SLE_6$ . If  $X_t, 0 \leq t < \infty$ , is a complex Brownian motion starting at the origin, let  $\hat{X}_t = \mathcal{F}_\mathbb{C}(X[0, t])$  denote the *Brownian hull* at time  $t$ . The *frontier* or *outer boundary* at time  $t$  is  $\partial\hat{X}_t$  and the elements of  $\bigcup_{t \geq 0} \partial\hat{X}_t$  are called the *pioneer points*. If  $D$  is a simply connected domain containing the origin, let  $T_D = \inf\{t : K_t \in D\}$ ,  $\tau_D = \inf\{t : X_t \in D\}$ .

**Theorem 9.1.** *Let  $K_t$  denote the hull of full-plane  $SLE_6$  and  $\hat{X}_t$  a planar Brownian hull as above. Let  $D \subset \mathbb{C}$  be a simply connected domain containing the origin other than  $\mathbb{C}$ . Then  $K_{T_D}$  and  $\hat{X}_{\tau_D}$  have the same law.*

The proof of the theorem is based on the following lemma.

**Lemma 9.2** (Hitting measure for full-plane  $SLE_6$ ).  *$K_{T_D} \cap \partial D$  is a single point a.s. and the law of this point is the same as the law of  $X_{\tau_D}$ , i.e., harmonic measure from 0.*

*Proof of Theorem 9.1 (using Lemma 9.2).* Consider some closed set  $A \subset \hat{\mathbb{C}}$  such that  $D' = D \setminus A$  is simply connected. Then the law of  $K_{T_{D'}} \cap \partial D'$  is harmonic measure from 0 on  $\partial D'$ . Consequently, the probability that  $K_{T_{D'}} \cap A \neq \emptyset$  is the same as the probability that  $X_{\tau_{D'}} \cap \partial D' \neq \emptyset$ . The former is equal to  $\mathbf{P}[K_{T_D} \cap A \neq \emptyset]$  and the latter is equal to  $\mathbf{P}[\hat{X}_{\tau_D} \cap A \neq \emptyset]$ . Hence, the theorem follows by the corresponding analogue of Lemma 3.2.  $\square$

*Proof.* Proof of Lemma 9.2 Since radial  $SLE_6$  satisfies locality [24], it follows that  $\gamma(t)$  does too and  $K_{T_D}$  is covariant with respect to conformal maps  $\phi : D \rightarrow D^*$ . (One uses here the fact mentioned above of stability with respect to the far-away past.) Therefore, the law of  $K_{T_D} \cap \partial D$  is harmonic measure.  $\square$

*Remark.* Let  $K^s = K_{T_{s\mathbb{U}}}$ ,  $\hat{X}^s = X_{\tau_{s\mathbb{U}}}$ . Then it is easy to see that  $s^{-1}K^s, s^{-1}\hat{X}^s$  are continuous time Markov chains on the space of closed connected sets  $K$  contained in  $\overline{\mathbb{U}}$  with  $0 \in K$  such that  $K \cap \partial\mathbb{U}$  is a single point. This theorem can be interpreted as saying that the two chains have the same invariant distribution. (By choosing full-plane  $SLE_6$ , we have effectively started the chains in equilibrium.) However, it is not difficult (using the fact that the Brownian motion crosses itself but the  $SLE_6$  path does not) to show that the two chains are not the same.

**9.2. Chordal  $SLE_6$  and reflected Brownian motion.** Let  $A \subset \overline{\mathbb{H}} \setminus \{0\}$  be a closed subset of  $\mathbb{C}$  such that the component of 0 in  $\overline{\mathbb{H}} \setminus A$  is bounded. Consider chordal  $SLE_6$  in the upper-half plane and let  $T := \inf\{t : K_t \cap A \neq \emptyset\}$  denote the first time at which its hull  $K_t \subset \overline{\mathbb{H}}$  hits  $A$ .

On the other hand, define a reflected Brownian motion in  $\mathbb{H}$ ,  $(B_t, t \geq 0)$  that is started from 0 and reflected on  $\mathbb{H}$  with an angle  $2\pi/3$  pointing “away” from the origin. In other words,

$$B_t = W_t + \int_0^t (e^{\pi i/3} d\ell_s^+ + e^{2\pi i/3} d\ell_s^-),$$

where  $W$  is standard two-dimensional Brownian motion and  $\ell^+$  (resp.  $\ell^-$ ) is a continuous process that increases only when  $B \in (0, \infty)$  (resp.  $(-\infty, 0)$ ). It is well



known (see, e.g., [48]) that this process exists. Let  $\hat{B}_t := \mathcal{F}_{\mathbb{H}}(B[0, t])$ , and let  $\tau = \tau_A$  denote the hitting time of  $A$  by  $B$ .

**Theorem 9.3.**  *$K_T$  and  $\hat{B}_\tau$  have the same law.*

*Proof.* As in the proof of Theorem 9.1, it suffices to show that the law of  $K_T \cap A$  is the same as that of  $\hat{B}_\tau \cap A$ .

Let  $S$  be the triangle with corners  $0, e^{-2\pi i/3}, e^{-\pi i/3}$ . Let  $E_0$  denote the lower edge of  $S$ , and let  $E_1, E_2$  denote the other two edges. Let  $\phi : \mathbb{H} \setminus A \rightarrow S$  be the conformal map which takes  $0$  to  $0$  and maps  $(\mathbb{H} \setminus A) \cap A$  onto  $E_0$ . The Carthy-Carleson formula for  $SLE_6$  [23] says that  $\phi(K_T) \cap \partial S$  is a uniformly chosen point on  $E_0$ . We need to prove the same for  $\phi(\hat{B}_\tau) \cap \partial S$ . Reflected Brownian motion in smooth domains is conformally invariant up to a time change: the conformal invariance follows from uniqueness and the fact that the conformal map preserves angles up to the boundary by an application of Itô’s formula. (Itô’s formula is valid at the reflection times too.) Hence, it suffices to consider the hitting point on  $E_0$  by Brownian motion in  $S$  starting from  $0$ , reflected at angle  $2\pi/3$  away from  $0$  along the edges  $E_1$  and  $E_2$ .

Let  $n$  be large, and consider the triangular grid in  $\bar{S}$  of mesh  $1/n$ , where the edges of  $S$  are covered by edges of the grid. Consider the Markov chain  $Y$  on the vertices in  $\bar{S}$  starting from  $0$  with the following transition probabilities. When  $Y$  is at vertices interior to  $S$ , let  $Y$  move with equal probability to each of the neighbors. At vertices  $v \in (E_1 \cup E_2) \setminus (E_0 \cup \{0\})$  let  $Y$  stay in  $v$  with probability  $1/6$ , move to the neighbor below  $v$  on  $E_1 \cup E_2$  with probability  $1/3$ , and move to any of the other three neighbors in  $\bar{S}$  with probability  $1/6$  each. When  $Y$  is at  $0$ , let it stay in  $0$  with probability  $1/3$  and move to each of its two neighbors in  $\bar{S}$  with probability  $1/3$  each. Let  $Y$  stop when it hits  $E_0$ . Induction shows that for  $t = 0, 1, 2, \dots$  conditioned on  $\text{Im } Y(t) = h$  the distribution of  $Y(t)$  is uniform among vertices in  $\bar{S}$  satisfying  $\text{Im } v = h$ . In particular, the vertex where  $Y$  hits  $E_0$  is uniform among the vertices in  $E_0$ .

It is not hard to verify that as  $n$  tends to  $\infty$  the walk  $Y(n^2t)$  converges to the above reflected Brownian motion in  $S$ . □

Variations on this uniform hitting distribution property for reflected Brownian motions will be developed in [12].

A closed monotone class  $\mathcal{P}$  is a collection of nonempty closed subsets  $A \subset \hat{\mathbb{C}}$  that is closed in the Hausdorff topology and such that  $A \in \mathcal{P}$  and  $A' \supset A$  implies  $A' \in \mathcal{P}$  when  $A' \subset \hat{\mathbb{C}}$  is closed. For example,  $\mathcal{P}$  might be the collection of closed sets intersecting some fixed closed set  $Y$  or the collection of closed connected sets whose capacity is at least  $r$ . We now present a generalization of Theorems 9.1 and 9.3.

**Theorem 9.4.** *Let  $\mathcal{P}$  be a closed monotone class. Let  $T_{\mathcal{P}} := \inf\{t \in \mathbb{R} : K_t \in \mathcal{P}\}$ , where  $K_t$  is the hull of full-plane  $SLE_6$  starting from  $0$ . Let  $X_t$  be planar Brownian motion starting from  $X_0 = 0$  and  $\hat{X}_t := \mathcal{F}_{\mathbb{C}}(X_t)$ . Let  $\tau_{\mathcal{P}} := \inf\{t \geq 0 : \hat{X}_t \in \mathcal{P}\}$ . Then  $K_{T_{\mathcal{P}}}$  and  $\hat{X}_{\tau_{\mathcal{P}}}$  have the same law.*

*A corresponding generalization also holds for Theorem 9.3.*

In the above, we take  $K_{\infty} = \hat{X}_{\infty} = \hat{\mathbb{C}}$ . (This is relevant if  $T_{\mathcal{P}} = \infty$  or  $\tau_{\mathcal{P}} = \infty$  with positive probability.)

*Proof.* As the proof in the chordal case is the same, we will only treat the full-plane setting. If  $R > 0$ , let  $\mathcal{P}_R$  be the union of  $\mathcal{P}$  together with all closed sets

intersecting  $R\partial\mathbb{U}$ . Let  $D$  be a simply connected domain containing 0 other than  $\mathbb{C}$ . Let  $T_D = \sup\{t \in \mathbb{R} : K_t \subset D\}$  and  $\tau_D = \sup\{t \geq 0 : \hat{X}_t \subset D\}$ . By Theorem 9.1,

$$\mathbf{P}[K_{T_D} \in \mathcal{P}_R] = \mathbf{P}[\hat{X}_{\tau_D} \in \mathcal{P}_R].$$

Observe that

$$\mathbf{P}[K_{T_D} \in \mathcal{P}_R] = \mathbf{P}[T_D \geq T_{\mathcal{P}_R}] = \mathbf{P}[K_{T_{\mathcal{P}_R}} \subset D] + \mathbf{P}[T_D = T_{\mathcal{P}_R}],$$

and similarly

$$\mathbf{P}[\hat{X}_{\tau_D} \in \mathcal{P}_R] = \mathbf{P}[\hat{X}_{\tau_{\mathcal{P}_R}} \subset D] + \mathbf{P}[\tau_{\mathcal{P}_R} = \tau_D].$$

Shortly, we will prove

$$(9.3) \quad \mathbf{P}[T_D = T_{\mathcal{P}_R}] = \mathbf{P}[\tau_{\mathcal{P}_R} = \tau_D].$$

Together with the above equalities this implies

$$\mathbf{P}[K_{T_{\mathcal{P}_R}} \subset D] = \mathbf{P}[\hat{X}_{\tau_{\mathcal{P}_R}} \subset D].$$

The corresponding analogue of Lemma 3.2 then proves that the laws of  $K_{T_{\mathcal{P}_R}}$  and  $\hat{X}_{\tau_{\mathcal{P}_R}}$  are the same. The theorem follows by letting  $R \rightarrow \infty$ . It therefore remains to prove (9.3).

We claim that  $\lim_{t \nearrow T_D} K_t = K_{T_D}$  a.s. in the Hausdorff metric. By conformal invariance, it suffices to prove this for  $D = \mathbb{U}$ . The times of discontinuity of  $K_t$  (with respect to the Hausdorff metric) are times  $s$  where  $K_s \setminus \lim_{t \nearrow s} K_t$  contains a nonempty open set. If  $T_{\mathbb{U}}$  is a time of discontinuity with positive probability, then the same would be true for  $T_{r\mathbb{U}}$  for every  $r > 0$ . This would then contradict the fact that the (expected) area of  $K_{T_{\mathbb{U}}}$  is finite. Hence  $\lim_{t \nearrow T_D} K_t = K_{T_D}$  almost surely. By monotonicity of  $\mathcal{P}_R$ , it follows that  $\mathbf{P}[T_D = T_{\mathcal{P}_R}]$  is equal to the probability that  $K_{T_D} \in \mathcal{P}_R$  but every compact subset of  $K_{T_D} \cap D$  is not in  $\mathcal{P}_R$ . Now (9.3) follows, because the analogous argument applies to  $\mathbf{P}[\tau_{\mathcal{P}_R} = \tau_D]$  and the law of  $\hat{X}_{\tau_D}$  is the same as that of  $K_{T_D}$ .  $\square$

**9.3. Chordal SLE<sub>6</sub> as Brownian motion reflected on its past hull.** By iterating the above results we will obtain an “emulation” of chordal SLE<sub>6</sub> using reflected Brownian motion. Roughly, what we show is that the SLE<sub>6</sub> path is Brownian motion that is reflected off its past filling with angle  $2\pi/3$  towards infinity.

Let  $(B_t^n, t \geq 0)_{n \geq 1}$  be a sequence of independent samples of reflected planar Brownian motion in  $\mathbb{H}$  started from 0 that are reflected off the real axis with angle  $2\pi/3$  away from 0 (as before). Define  $\tilde{B}^1 = B^1$ ,  $\hat{K}^0 := \emptyset$  and define inductively:

- $\tau_n := \inf\{t \geq 0 : |\tilde{B}_t^n - \tilde{B}_0^n| \geq \epsilon\}$ ,
- $\hat{K}^n := \mathcal{F}_{\mathbb{H}}(\hat{K}^{n-1} \cup \tilde{B}^n[0, \tau_n])$ ,
- $\phi_n : \mathbb{H} \rightarrow \mathbb{H} \setminus \hat{K}^n$  is the conformal map normalized by  $\phi_n(0) = \tilde{B}_{\tau_n}^n$ ,  $\phi_n(\infty) = \infty$ ,  $\phi_n'(\infty) = 1$ ,
- $\tilde{B}_t^{n+1} := \phi_n(B_t^{n+1})$ .

**Corollary 9.5.** *Fix  $\epsilon > 0$ . Let  $\gamma$  denote the chordal SLE<sub>6</sub> path,  $K_t = \mathcal{F}_{\mathbb{H}}(\gamma[0, t])$  the SLE hull,  $T_0 := 0$  and inductively,  $T_{n+1} := \inf\{t \geq T_n : |\gamma(t) - \gamma(T_n)| \geq \epsilon\}$ . Then the sequence  $(\hat{K}^0, \hat{K}^1, \dots)$  defined above has the same law as the sequence  $(K_{T_0}, K_{T_1}, \dots)$ . Consequently, after reparameterization the path  $\hat{\gamma}$  obtained by concatenating the paths  $(\tilde{B}_t^n, t \in [0, \tau_n])$  stays within distance  $2\epsilon$  from the path  $\gamma$ ; that is,  $\sup_{t \geq 0} |\gamma(t) - \hat{\gamma}(t)| \leq 2\epsilon$ .*

*Proof.* This easily follows from Theorem 9.3 and induction. In the inductive step, we assume that  $(\hat{K}^0, \dots, \hat{K}^n)$  and  $(K_{T_0}, \dots, K_{T_n})$  have the same distribution. Note that (when  $n \geq 1$ ) this implies that  $(\hat{K}^0, \dots, \hat{K}^n, \tilde{B}_{\tau_n}^n)$  and  $(K_{T_0}, \dots, K_{T_n}, \gamma(T_n))$  have the same distribution, since  $\tilde{B}_{\tau_n}^n$  is a.s. the unique point in  $\hat{K}^n$  at distance  $\epsilon$  from  $\hat{K}^{n-1}$ . An application of Theorem 9.3 and conformal invariance now complete the induction step and the proof.  $\square$

One can also choose time sequences other than  $(T_n, \tau_n)$ . For instance, one can compare the sequence  $(K_{t_n}, n \geq 1)$  (where  $(t_n, n \geq 1)$  is a deterministic sequence) with  $(\hat{K}^1, \hat{K}^2, \dots)$  where the definition of  $\tau_n$  is replaced by

$$\tau_n := \inf\{t \geq 0 : a(\mathcal{F}_{\mathbb{H}}(\hat{K}^{n-1} \cup \tilde{B}^n[0, t])) \geq 2t_n\}.$$

This time, one can for instance identify the tip of the SLE curve (or of the stopped reflected Brownian motion) as the only accumulation point of cut-points of the hull. We leave the details to the interested reader.

**9.4. Nonequivalence of pioneer points and  $SLE_6$ .** Given these results, it is natural to try to better understand the differences between planar Brownian motion and  $SLE_6$ . How far does the equivalence go? Consider, for example, the setting of Theorem 9.1. Let  $\theta(t) := \inf\{s \geq 0 : \text{cap}(\hat{X}_s) \geq t\}$ . Theorem 9.4 implies that for all  $t \geq 0$ , the distributions of  $K_t$  and  $\hat{X}_{\theta(t)}$  are the same. However, the processes  $(K_t : t \geq 0)$  and  $(\hat{X}_{\theta(t)} : t \geq 0)$  are different. In fact, it is not hard to show that the joint distributions  $(K_s, K_t)$  and  $(\hat{X}_{\theta(s)}, \hat{X}_{\theta(t)})$  do not agree when  $s \neq t$ . It is also true that the random set  $X[0, \tau_U]$  does not have the same distribution as  $\gamma[-\infty, T_U]$ ; in fact, the first has Hausdorff dimension 2 and the latter dimension  $7/4$  [4].

The set  $Z = \bigcup_{t \leq \tau_U} \partial \hat{X}_t$  of pioneer points of  $X$  up to time  $\tau_U$  does have dimension  $7/4$  a.s. [26]. However,  $Z$  does not have the same law as  $\gamma[-\infty, T_U] = \bigcup_{t \leq T_U} \partial K_t$ . We now give the outline of one possible proof of this fact.

We say that  $(z_0, z_1, z_2, z_3)$  is a good configuration for  $Z$  if

- $z_0$  and  $z_3$  are cut-points of  $Z$ ,
- any subpath of  $Z$  from  $z_0$  to  $z_3$  goes through  $z_1$  or  $z_2$ , and
- there exist subpaths of  $Z$  from  $z_0$  to  $z_3$  that go through  $z_1$  (resp. through  $z_2$ ) and not through  $z_2$  (resp. through  $z_1$ ).

The set of good configurations of  $Z$  is comparable to the set of cut-points of  $Z$  in the sense that with positive probability, one can find four sets  $Z_0, Z_1, Z_2, Z_3$  of Hausdorff dimension  $3/4$  each (recall [24] that the Hausdorff dimension of the set of cut-points of the Brownian trace is  $3/4$ ) such that any  $(z_0, z_1, z_2, z_3) \in Z_0 \times Z_1 \times Z_2 \times Z_3$  is a good configuration. This is due to the fact that for a Brownian path as shown in Figure 9.1  $(z_0, z_1, z_2, z_3)$  is a good configuration.

On the other hand, a.s. the  $SLE_6$  does not have good configurations. For topological reasons, if  $(z_0, z_1, z_2, z_3)$  is a good configuration for  $\gamma[-\infty, T_U]$ , then at least one of the four points is a double point of  $\gamma$  (hint: consider the two path-connected components of  $\gamma[-\infty, T_U] \setminus \{z_1, z_2\}$  which contain  $z_0$  and  $z_3$ , respectively, and the order in which  $\gamma$  visits them). In particular this point is simultaneously a local cut-point and a double point. Such points do not exist for  $SLE_6$  (as explained in Remark 8.8). This argument can be made into a proof that the  $SLE_6$  path image is not the same as the set of pioneer points of planar Brownian motion.

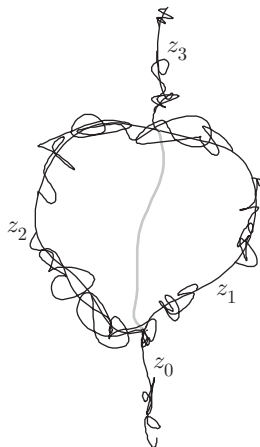


FIGURE 9.1. Good configurations for the set of pioneer points. The vertical path within the loop represents a segment of the Brownian path which does not contain any pioneer points, since it was traversed after the loop surrounding it.

**9.5. Conditioned  $SLE_6$ .** We have seen that the outer boundary of a planar Brownian path looks (locally) like an  $SLE_{8/3}$  path. More precisely, the right boundary (i.e., the right-hand side of the boundary of the filling) of a Brownian excursion in  $\mathbb{H}$  is the path of  $SLE(8/3, 2/3)$  and the right boundary of a reflected Brownian excursion with angle  $3\pi/8$  is  $SLE(8/3)$ . This gives some motivation to show that the outer boundaries of these Brownian excursions have the same law as that of some conditioned  $SLE_6$  processes, since this provides a description of the right boundary of conditioned  $SLE_6$  in terms of variants of the  $SLE_{8/3}$  paths.

In the spirit of the paper [27], it is not difficult to prove that if one considers reflected Brownian motion  $X$  in  $\mathbb{H}$  (with any given reflection angle) that is conditioned to hit (let  $T$  be this hitting time)  $(-\infty, -1/\epsilon) \cup (1/\epsilon, \infty)$  before  $(-1/\epsilon, -\epsilon) \cup (\epsilon, 1/\epsilon)$  and lets  $\epsilon \rightarrow 0$ , the limiting law of  $X[0, T]$  is exactly that of a Brownian excursion (that does not touch the real line except at the origin). In particular, this implies that the filling of chordal  $SLE_6$  conditioned not to intersect the real line (i.e., the limit when  $\epsilon \rightarrow 0$  of  $SLE_6$  conditioned not to intersect  $(-1/\epsilon, -\epsilon) \cup (\epsilon, 1/\epsilon)$ ) has law  $\mathbf{P}_1$ . In particular, its right boundary is  $SLE(8/3, 2/3)$ .

Similarly, the limit of the law of  $X[0, T]$  conditioned on  $X[0, T] \cap (\epsilon, 1/\epsilon) = \emptyset$  is simply the law of the reflected Brownian excursion. If the reflection angle is  $2\pi/3$  towards infinity, as before, then the law of the right boundary of this process is the  $SLE(8/3, -2/3)$  path. Hence, the right boundary of an  $SLE_6$  conditioned not to intersect the positive half-line is exactly  $SLE(8/3, -2/3)$ .

## 10. REMARKS

Let us briefly sum up some of the results that we have collected in the present paper concerning the description of the Brownian frontier.

- The filling of the union of five independent excursions has the same law as the filling of the union of eight independent chordal  $SLE_{8/3}$ .

- The right boundary of a Brownian excursion from 0 to infinity in the upper half-plane reflected on the negative half-line with reflection angle  $3\pi/8$  is  $SLE_{8/3}$ . This law is symmetric with respect to reflection in the imaginary axis.
- The right boundary of a Brownian excursion is  $SLE(8/3, 2/3)$ .
- The right boundary of the union of two independent Brownian excursions is  $SLE(8/3, 2)$ . As we shall mention shortly, the right and left boundaries can also be viewed as nonintersecting  $SLE_{8/3}$ 's.
- The Brownian bubble and the  $SLE_{8/3}$  bubbles are identical (up to scaling).

We conclude this paper by mentioning some closely related results that will be included in forthcoming papers:

- Analogous problems in the “radial case”, i.e., random subsets of the unit disk that contain one given boundary point and one given interior point, will be studied in [31]. A radial restriction property holds for  $SLE_{8/3}$ . In particular, if  $\gamma$  is a radial  $SLE_{8/3}$  path in  $\mathbb{U}$  from 1 to 0,  $A$  a compact set not containing 1, such that  $\mathbb{U} \setminus A$  is simply connected and contains 0, and  $\Psi_A$  is a conformal map from  $\mathbb{U} \setminus A$  onto  $\mathbb{U}$  with  $\Psi_A(0) = 0$ , then

$$\mathbf{P}[\gamma[0, \infty) \cap A = \emptyset] = |\Psi'_A(0)|^{5/48} |\Psi'_A(1)|^{5/8}.$$

- In the spirit of [32], the Brownian half-space intersection exponents computed in [23, 25] can be interpreted in terms of nonintersection of independent sets defined under different restriction measures. In particular, the measure  $\mathbf{P}_2$  can be viewed as the filling of two  $SLE_{8/3}$ 's that are conditioned not to intersect (of course, this event has probability 0, so this has to be taken as an appropriate limit). See, e.g., [51].
- In [33], a random countable set of loops  $L$  in the plane called the *Brownian loop soup* is constructed. Each  $\gamma \in L$  is a loop, that is, an equivalence class of periodic, continuous maps from  $\mathbb{R}$  to  $\mathbb{C}$ , where  $\gamma^1, \gamma^2$  are equivalent if for some  $r$ ,  $\gamma^1(t) = \gamma^2(t + r)$  holds for all  $t \in \mathbb{R}$ . Loosely speaking, each  $\gamma \in L$  is a *Brownian loop*. This loop soup is conformally invariant: for any conformal map  $\Phi : D \rightarrow D'$  the sets  $\{\Phi \circ \gamma : \gamma \in L, \gamma \subset D\}$  and  $\{\gamma : \gamma \in L, \gamma \subset D'\}$  have the same law, up to reparametrization of the loops.

It turns out that if one considers the set of loops in  $\mathbb{H}$ ,  $L(\mathbb{H}) = \{\gamma \in L : \gamma \subset \mathbb{H}\}$ , and any Loewner chain  $(K_t, t \in [0, T])$  generated by a continuous curve, then another (equivalent) way to add a Poisson cloud of Brownian bubbles to the Loewner chain (as in our construction of the general restriction  $\mathbf{P}_\alpha$  measures) is to add to the set  $K_T$  all the loops of  $L(\mathbb{H})$  that it intersects. Therefore,

$$\begin{aligned} & \mathbf{P}[(K_T \cup \{\text{cloud of bubbles}\}) \cap A = \emptyset] \\ &= \mathbf{P}[\text{No loop in } L(\mathbb{H}) \text{ intersects both } A \text{ and } K_T] \\ &= \mathbf{P}[K_T \cap (A \cup \{\text{loops that intersect } A\}) = \emptyset]. \end{aligned}$$

See [33] for more details. This Brownian loop soup is then used in [13, 51].

- Restriction formulas can also be derived for  $SLE(\kappa, \rho)$  processes, and, combined with the loop soup, they shed light on the relation between  $SLE_\kappa$  and the outer boundary of  $SLE_{16/\kappa}$  for  $\kappa < 4$ . See [13].

## ACKNOWLEDGMENTS

We thank Bálint Virág for fruitful conversations and for permitting us to include Proposition 4.1.

## REFERENCES

- [1] L.V. Ahlfors, *Conformal Invariants, Topics in Geometric Function Theory*, McGraw-Hill, New-York, 1973. MR **50**:10211
- [2] M. Bauer, D. Bernard (2002), SLE $_k$  growth processes and conformal field theories, *Phys. Lett. B* **543**, 135-138.
- [3] M. Bauer, D. Bernard (2002), *Conformal Field Theories of Stochastic Loewner Evolutions*, arXiv:hep-th/0210015.
- [4] V. Beffara (2002), Hausdorff dimensions for SLE $_6$ , arXiv:math.PR/0204208.
- [5] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov (1984), Infinite conformal symmetry of critical fluctuations in two dimensions, *J. Statist. Phys.* **34**, 763-774. MR **86e**:82019
- [6] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov (1984), Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B* **241**, 333-380. MR **86m**:81097
- [7] R. van den Berg, A. Jarai (2002), The lowest crossing in 2D critical percolation, arXiv:math.PR/0201030.
- [8] K. Burdzy (1987), *Multidimensional Brownian Excursions and Potential Theory*, Pitman Research Notes in Mathematics 164, John Wiley & Sons. MR **89d**:60146
- [9] J.L. Cardy (1984), Conformal invariance and surface critical behavior, *Nucl. Phys. B240 (FS12)*, 514-532.
- [10] J.L. Cardy (1992), Critical percolation in finite geometries, *J. Phys. A*, **25** L201-L206. MR **92m**:82048
- [11] L. Carleson, N. Makarov (2002), Laplacian path models, *J. Anal. Math.* **87**, 103-150.
- [12] J. Dubédat (2003), Reflected planar Brownian motions, intertwining relations and crossing probabilities, math.PR/0302250, preprint.
- [13] J. Dubédat (2003), *SLE*( $\kappa, \rho$ ) martingales and duality, math.PR/0303128, preprint.
- [14] B. Duplantier (1998), Random walks and quantum gravity in two dimensions, *Phys. Rev. Lett.* **81**, 5489-5492. MR **99j**:83034
- [15] B. Duplantier, K.-H. Kwon (1988), Conformal invariance and intersection of random walks, *Phys. Rev. Lett.* **61**, 2514-2517.
- [16] B. Duplantier, H. Saleur (1986), Exact surface and wedge exponents for polymers in two dimensions, *Phys. Rev. Lett.* **57**, 3179-3182. MR **88e**:82022
- [17] P.L. Duren, *Univalent functions*, Springer, 1983. MR **85j**:30034
- [18] R. Friedrich, W. Werner (2002), Conformal fields, restriction properties, degenerate representations and SLE, *C.R. Ac. Sci. Paris Ser. I Math* **335**, 947-952.
- [19] R. Friedrich, W. Werner (2003), Conformal restriction, highest-weight representations and SLE, math-ph/0301018, preprint.
- [20] T. Kennedy (2002), A faster implementation of the pivot algorithm for self-avoiding walks, *J. Stat. Phys.* **106**, 407-429.
- [21] T. Kennedy (2002), Monte Carlo tests of SLE predictions for the 2D self-avoiding walk, *Phys. Rev. Lett.* **88**, 130601.
- [22] T. Kennedy (2002), Conformal Invariance and Stochastic Loewner Evolution Predictions for the 2D Self-Avoiding Walk - Monte Carlo Tests, arXiv:math.PR/0207231.
- [23] G.F. Lawler, O. Schramm, W. Werner (2001), Values of Brownian intersection exponents I: Half-plane exponents. *Acta Mathematica* **187**, 237-273. MR **2002m**:60159a
- [24] G.F. Lawler, O. Schramm, W. Werner (2001), Values of Brownian intersection exponents II: Plane exponents. *Acta Mathematica* **187**, 275-308. MR **2002m**:60159b
- [25] G.F. Lawler, O. Schramm, W. Werner (2002), Values of Brownian intersection exponents III: Two-sided exponents. *Ann. Inst. Henri Poincaré* **38**, 109-123. MR **2003d**:60163
- [26] G.F. Lawler, O. Schramm, W. Werner (2002), Analyticity of planar Brownian intersection exponents. *Acta Mathematica* **189**, 179-201.
- [27] G.F. Lawler, O. Schramm, W. Werner (2001), Sharp estimates for Brownian non-intersection probabilities, in *In and Out of Equilibrium*, V. Sidoravicius, Ed., *Prog. Probab.*, Birkhauser, 113-131. MR **2003d**:60162

- [28] G.F. Lawler, O. Schramm, W. Werner (2002), One-arm exponent for critical 2D percolation, *Electronic J. Probab.* **7**, paper no. 2. MR **2002k**:60204
- [29] G.F. Lawler, O. Schramm, W. Werner (2001), Conformal invariance of planar loop-erased random walks and uniform spanning trees, arXiv:math.PR/0112234, *Ann. Prob.*, to appear.
- [30] G.F. Lawler, O. Schramm, W. Werner (2002), On the scaling limit of planar self-avoiding walks, math.PR/0204277, in *Fractal geometry and application, A jubilee of Benoit Mandelbrot*, AMS Proc. Symp. Pure Math., to appear.
- [31] G.F. Lawler, O. Schramm, W. Werner (2002), Conformal restriction: the radial case, in preparation.
- [32] G.F. Lawler, W. Werner (2000), Universality for conformally invariant intersection exponents, *J. Europ. Math. Soc.* **2**, 291-328. MR **2002g**:60123
- [33] G.F. Lawler, W. Werner (2003), The Brownian loop soup, preprint.
- [34] B.B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, 1982. MR **84h**:00021
- [35] B. Nienhuis, E.K. Riedel, M. Schick (1980), Magnetic exponents of the two-dimensional  $q$ -states Potts model in two dimensions, *J. Phys A* **13**, L. 189-192.
- [36] B. Nienhuis (1984), Critical behavior in two dimensions and charge symmetry of the Coulomb gas, *J. Stat. Phys.* **34**, 731-761.
- [37] M.P.M. den Nijs (1979), A relation between the temperature exponents of the eight-vertex and the  $q$ -state Potts model, *J. Phys. A* **12**, 1857-1868.
- [38] R.P. Pearson (1980), Conjecture for the extended Potts model magnetic eigenvalue, *Phys. Rev. B* **22**, 2579-2580.
- [39] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975. MR **58**:22526
- [40] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992. MR **95b**:30008
- [41] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 2nd Ed., 1994. MR **95h**:60072
- [42] S. Rohde, O. Schramm (2001), Basic properties of SLE, arXiv:math.PR/0106036, preprint.
- [43] H. Saleur, B. Duplantier (1987), Exact determination of the percolation hull exponent in two dimensions, *Phys. Rev. Lett.* **58**, 2325. MR **88d**:82073
- [44] O. Schramm (2000), Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* **118**, 221-288. MR **2001m**:60227
- [45] O. Schramm (2001), A percolation formula, *Electronic Comm. Probab.* **6**, 115-120. MR **2002h**:60227
- [46] S. Smirnov (2001), Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, *C. R. Acad. Sci. Paris Sér. I Math.* **333** no. 3, 239-244. MR **2002f**:60193
- [47] S. Smirnov, W. Werner (2001), Critical exponents for two-dimensional percolation, *Math. Res. Lett.* **8**, 729-744.
- [48] S.R.S. Varadhan, R.J. Williams (1985), Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38**, 405-443. MR **87c**:60066
- [49] B. Virág (2003), Brownian beads, in preparation.
- [50] W. Werner (2001), Critical exponents, conformal invariance and planar Brownian motion, in *Proceedings of the 4th ECM Barcelona 2000*, *Prog. Math.* **202**, Birkhäuser, 87-103. MR **2003f**:60181
- [51] W. Werner (2003), Girsanov's Theorem for SLE( $\kappa, \rho$ ) processes, intersection exponents and hiding exponents, math.PR/0302115, preprint.

DEPARTMENT OF MATHEMATICS, 310 MALOTT HALL, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853-4201

*E-mail address:* lawler@math.cornell.edu

MICROSOFT CORPORATION, ONE MICROSOFT WAY, REDMOND, WASHINGTON 98052

*E-mail address:* schramm@microsoft.com

DÉPARTEMENT DE MATHÉMATIQUES, BÂT. 425, UNIVERSITÉ PARIS-SUD, 91405 ORSAY CEDEX, FRANCE

*E-mail address:* wendelin.werner@math.u-psud.fr