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OORT'S CONJECTURE FOR $A_g \otimes \mathbb{C}$

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1. Introduction and results

Let A_g be the moduli space of principally polarized Abelian varieties and let $\mathbb{E}$ be the Hodge bundle.

1.1. Main Theorem. *Let X be a compact subvariety of $A_g \otimes \mathbb{C}$ such that $c_i(\mathbb{E})|_X = 0$ in $H^{2i}(X, \mathbb{R})$ for some $g \geq i \geq 0$. Then*

$$\dim(X) \leq \frac{i(i-1)}{2}, \tag{1}$$

with strict inequality if $i \geq 3$.

The Grothendieck-Riemann-Roch theorem implies that $c_g(\mathbb{E})$ is trivial on A_g (see [G99, 2.2]); thus we have the following corollary, conjectured by Oort [GO99, 3.5]:

1.2. Corollary. *There is no compact codimension g subvariety of $A_g \otimes \mathbb{C}$ for $g \geq 3$.*

Corollary 1.2 is striking in that it fails in positive characteristic: Oort has shown that the locus $Z \subset A_g \otimes \mathbb{F}_p$ of Abelian varieties of p-rank zero is complete and pure codimension g.

We also obtain

1.2.1. Corollary. *There is no compact codimension g subvariety of $M_g^c \otimes \mathbb{C}$ for $g \geq 3$.*

This is a formal consequence of (the $g = 3$ case of) Corollary 1.2 and the bounds of [Diaz87] on dimensions of compact subvarieties of $\overline{M}_g$; see §8.

Inequality (1) has been obtained previously, in all characteristics, by van der Geer [G99]. The improvement to strict inequality may seem rather humble, but in relation to Faber's conjectures, it is quite significant:

Faber has made the surprising conjecture that the subring $\mathcal{R}^*(M_g) \subset \mathrm{CH}^*(M_g)_{\mathbb{Q}} \otimes \mathbb{Q}$ generated by tautological classes *looks like* the rational cohomology of a smooth projective variety of dimension $g-2$; e.g., it satisfies Poincare duality and Grothendieck's standard conjectures. See [Faber99] for the precise statement and very compelling evidence. One naturally wonders if there is a projective $(g-2)$-dimensional variety with the right cohomology. The form of the conjectures, specifically his evaluation maps ([FP00, 0.4]), strongly suggests looking for a compact codimension

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$2g-1$ subvariety of M_g, whose cohomology class in $\overline{M}_g$ is a multiple of $c_g(\mathbb{E})c_{g-1}(\mathbb{E})$. There are analogous speculations for M_g^c (stable curves of compact type) ([FP00, 0.5]) that lead one to hope for a compact subvariety of codim g, with cohomology class a multiple of $c_g(\mathbb{E})$. Since $\mathbb{E}$ comes from A_g, it is natural to hope that the subvarieties do as well. More compellingly, there are natural analogs of Faber's conjectures for A_g and A_g° (the open subset of A_g whose complement corresponds to products of lower dimensional polarized varieties—note that the image of the Torelli map is a closed subset of A_g°) and corresponding evaluation maps which themselves raise the hope of compact subvarieties, of codimension g in A_g and of codimension $2g-1$ in A_g°. Indeed the tautological ring $\mathcal{R}^*(A_g)$ has been computed ([G99] and [EV02]) and the analog of Faber's conjecture holds, so if in particular one has a compact subvariety $Z \subset A_g$ with the desired cohomology class, then Poincare duality already gives that $\mathcal{R}^*(A_g) \to H^*(Z, \mathbb{Q})$ is injective.

Unfortunately, by Corollary 1.2 the hoped for subvariety of A_g does not exist in characteristic zero. Theorem 1.1 suggests a similar result may hold for A_g°: It is natural to wonder if $c_{g-1}(\mathbb{E})$ vanishes on A_g°. If it does, then by Theorem 1.1, with $i = g-1$, there is no compact subvariety of $A_g^\circ \otimes \mathbb{C}$ of codimension $2g-1$, for $g \geq 4$.

As noted, Oort's example $Z \subset A_g \otimes \mathbb{F}_p$ violates the analog of Corollary 1.2 for characteristic p. As Oort pointed out to us, since it is known that Z does not lift to char 0, uniqueness of Z would give a different, purely char p proof of Corollary 1.2. In this sense Corollary 1.2 is evidence for this uniqueness, which would obviously be attractive from the point of view of Faber's conjecture. In §7 we point out an intriguing parallel between our proof in char 0 and Oort's example; perhaps it can be exploited.

Finally, Theorem 1.1 is sharp for $i = 2$ and $i = 3$. There exists a compact surface $Z \subset A_3$ and a compact curve $C \subset A_2$. Taking products with fixed elliptic curves, one then obtains a compact surface $Z \subset A_g$, $g \geq 3$, and a compact curve $C \subset A_g$, $g \geq 2$, with $c_3(\mathbb{E})|_Z = 0$, $c_2(\mathbb{E})|_C = 0$.

After submitting this paper for publication, we learned from E. Colombo that the $i = g$ case of Corollary 2.6, the crucial technical step in the proof of Corollary 1.2, has been previously obtained by E. Izadi [Izadi98], using methods of Colombo and Pirola [CP90]. The main argument of Colombo-Pirola and Izadi is entirely different from ours (and quite ingenious). It is more elementary (no use is made of the vanishing of $c_g(\mathbb{E})$) and also more algebraic, so it might be useful for showing uniqueness of Oort's $Z \subset A_g \otimes \mathbb{F}_p$.

2. Proof of Main Theorem

In this section we give the main line of our argument for Theorem 1.1. The various constituent results will be proved in subsequent sections.

We use throughout the (orbifold) cover $q : \mathcal{H}_g \to A_g$ by Siegel's generalized upper half space. (We could as well replace A_g by a finite branched cover and q by an honest (unbranched) cover. Above and throughout the paper any statement about A_g should be interpreted in the orbifold sense.) $\mathcal{H}_g$ is by definition the space of symmetric complex $g \times g$ matrices with positive definite imaginary part, while S_g denotes the space of all symmetric $g \times g$ complex matrices.

2.1. Chern forms. $\mathbb{E}$ has a tautological Hermitian metric: A point of A_g is given by a pair $(V/L, \langle , \rangle)$, where $L \subset V$ is a lattice in a g-dim complex vector space and $\langle , \rangle$ is a positive definite Hermitian form (such that $\mathrm{Im}\langle , \rangle|_L$ is integral and unimodular). $\mathbb{E}$ is the Hermitian bundle with fibre V^* and metric dual to $\langle,\rangle$. (We abuse notation and denote this dual metric, and any other metric induced from $\langle,\rangle$, by $\langle,\rangle$). Griffiths showed that the curvature of $(\mathbb{E}, \langle,\rangle)$ is nonnegative [Griffiths84]. Our proof is based on the tension between this nonnegativity and the vanishing of $c_g(\mathbb{E})$ noted above: As $\mathbb{E} \geq 0$, it is natural to hope that $c_k(\mathbb{E})$ is represented by a nonnegative form $\tilde{c}_k$. (Recall a real (k,k) form is called nonnegative if its restriction to every complex k-plane in the tangent bundle is a nonnegative multiple of the volume form.) As we indicate in a moment, this is indeed the case. Thus if $Y \subset A_g$ is a compact k-fold and

$$0 = c_k(\mathbb{E}) \cap [Y] = \int_Y \tilde{c}_k, \tag{2}$$

then $\tilde{c}_k|_Y$ vanishes identically (on the smooth locus), and we get pointwise information on the tangent space of Y.

Here is how we obtain $\tilde{c}_k$. By Hodge theory, $\mathbb{E}$ sits in an exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^* \to 0, \tag{3}$$

where $\mathbb{H}$ is a flat bundle (with fibre $H^1(A, \mathbb{C})$ at the point $[A] \in A_g$ given by the Abelian variety A), whence the equality of cohomology classes

$$c(\mathbb{E}) \cdot c(\mathbb{E}^*) = 1. \tag{4}$$

Thus $c(\mathbb{E})$ is represented by the inverse Chern form of $(\mathbb{E}^*, \langle,\rangle)$. This is computed as follows:

2.2. Segre forms. Let $(F, \langle,\rangle)$ be a rank g Hermitian bundle, on a complex manifold B, with curvature Ω, let $G = \frac{1}{2\pi i}\Omega$, and let $p : \mathbb{P}(F) \to B$ be the projective bundle of lines in F. Let

$$A : T_{\mathbb{P}(F)} \to T_{\mathbb{P}(F)/B} \tag{5}$$

(where $T_{\mathbb{P}(F)/B}$ is the vertical tangent space, i.e., the kernel of dp) be the second fundamental form for the tautological sublinebundle $\mathcal{O}(-1) \subset p^*(F)$. Let $s(F, \langle,\rangle) = \sum s_k(F, \langle,\rangle)$ denote the Segre from, i.e.,

$$s_k(F) = p_*(c_1(\mathcal{O}(1), \langle,\rangle)^{g-k+1}). \tag{6}$$

2.3. Theorem. *The kernel of A defines a horizontal subspace, complementary to the vertical tangent space, which is identified with $p^*(T_B)$ by dp. Under this splitting of $T_{\mathbb{P}(F)}$ the first Chern form of $\mathcal{O}(1)$ at a line $L \subset F_b$, spanned by a unit vector v, is*

$$FS + \langle Gv, v\rangle, \tag{7}$$

where FS is the Fubini-Study form (on the vertical tangent space, normalized to represent the hyperplane class). The Segre form $s_k(F, \langle,\rangle)$ equals

$$\binom{g+k-1}{k} \int_{\mathbb{P}(F_b)} \left(\frac{\langle Gv, v\rangle}{\langle v, v\rangle}\right)^k d\mathrm{Vol}_{\mathbb{P}(F_b)}, \tag{8}$$

and the total Segre form $s(F, \langle,\rangle)$ equals $c(F, \langle,\rangle)^{-1}$ (pointwise!) in the algebra of even degree forms.

It is well known that the Segre class represents the inverse Chern class in cohomology; see, e.g., [BT82, p. 269] and [Fulton84, 3.1]. We do not know if this equality at the level of forms, or the rather canonical expressions above, has been previously observed.

The tangent space T_pA_g is canonically identified with $S(\mathbb{E}_p, \mathbb{E}_p^*)$, the space of symmetric linear maps from $\mathbb{E}_p$ to its dual. Theorem 2.3 applied to $(\mathbb{E}^*, \langle, \rangle)$, together with Griffiths' formula for the curvature, imply

2.4. Corollary. *The k^{th} Chern class of $\mathbb{E}$ is represented by $\tilde{c}_k := s_k(\mathbb{E}^*, \langle, \rangle)$. The form $\tilde{c}_k$ is nonnegative and vanishes on a (complex) i-plane Y in $T_pA_g = S(\mathbb{E}_p, \mathbb{E}_p^*)$ if and only if, for every $v \in \mathbb{E}_p$, the evaluation map $e_v : Y \to \mathbb{E}^*$ fails to be injective.*

For each line $L \subset \mathbb{E}_p$, choose $0 \neq w \in L$. Let $\langle, \rangle_L$ on $S(\mathbb{E}_p, \mathbb{E}_p^*)$ be the Hermitian form $e_w^*(\langle, \rangle_{\mathbb{E}_p^*})/\langle w, w\rangle$, i.e.,

$$\langle a, b\rangle_L = \frac{\langle e_w(a), e_w(b)\rangle}{\langle w, w\rangle}. \tag{9}$$

Up to a positive constant, the restriction of $\tilde{c}_k$ to p is the average of the k^{th} wedge powers of $-\operatorname{Im}(\langle, \rangle_L)$ over all lines $L \subset \mathbb{E}_p$.

We prove Theorem 2.3 and Corollary 2.4 in §3.

Given compact $X \subset A_g$ with $c_i(\mathbb{E})|_X = 0$, we may apply Corollary 2.4 to every i-plane in the tangent space at smooth points of X. (Since A_g is quasi-projective, each i-plane is realized as a tangent plane to a compact i-dimensional subvariety.)

We reformulate this using the following linear algebra result, proved in §4.

2.5. Theorem. *Let V be a complex vector space of dimension $g \geq 1$. Let i be an integer $g \geq i \geq 0$. Let $X \subset S(V, V^*)$ be a linear subspace of dimension at least $\max(i, \frac{i(i-1)}{2})$ such that for each $v \in V$ and every i-dimensional subspace $Y \subset X$ the evaluation map $e_v : Y \to V^*$ fails to be injective.*

Then $i \geq 3$, $X = W^\perp$ for some $(g - i + 1)$-dimensional linear subspace $W \subset V$, and X has dimension exactly $\frac{i(i-1)}{2}$.

Here $W^\perp \subset S(V, V^*)$ is the set of maps whose kernels contain W.

Combining this with Corollary 2.4, we find

2.6. Corollary. *Let $g \geq i \geq 3$. Let $X \subset A_g$ be a compact subvariety of dimension at least $\frac{i(i-1)}{2}$ such that $c_i(\mathbb{E})|_X = 0$. Then X has dimension exactly $\frac{i(i-1)}{2}$. Furthermore, at each smooth point $p \in X$, $T_pX = W^\perp \subset S(\mathbb{E}_p, \mathbb{E}_p^*)$ for some $(g - i + 1)$-dimensional subspace $W(p) \subset \mathbb{E}_p$.*

Let $X \subset A_g$ be as in Corollary 2.6, and let $\mathcal{X}$ be an irreducible component of $q^{-1}(X)$. By Corollary 2.6,

$$T_A X = W(A)^\perp \subset S_g \tag{10}$$

for each smooth $A \in X$.

In §5 we prove

2.7. Theorem. *The subspace $W(A)$ is constant (and henceforth denoted W). If $\mathcal{X}$ is an irreducible component of $q^{-1}(X)$, then for some fixed $\tau \in \mathcal{H}_g$, $\mathcal{X} \subset \mathcal{H}_g$ is the affine space*

$$X(W, \tau) := \{M \in \mathcal{H}_g | M|_W = \tau|_W\}. \tag{11}$$

Finally, in §6 we show

2.8. Theorem. *The image of $X(W,\tau)$ in A_g cannot be compact.*

3. Segre forms

Proof of Theorem 2.3. The restriction of A to the vertical tangent space is the identity, so A is indeed surjective. Now equation (7) follows by the definition of A; see [GH78, p. 78]. What remains is to show that the pointwise inverse of the total Chern form $c(F,\langle,\rangle)$ is given by the formula (8) and equals the Segre form.

If M is a positive Hermitian form on a g-dimensional complex vector space V with inner product $\langle,\rangle$, then

$$\pi^{-g}\det(M)\int_V e^{-\langle Mv,v\rangle}d\mathrm{Vol}_V = 1. \tag{12}$$

This identity remains true if M takes values in the commutative ring of even degree forms at a point, as long as the scalar part of M remains positive-definite.

Now let $V = F_p$, and let $M = I - G$. Then

$$\begin{aligned}
[c(F,\langle,\rangle)]^{-1} &= \pi^{-g}\int_{F_p} e^{-\langle (I-G)v,v\rangle}d\mathrm{Vol}_{F_p}\\
&= \pi^{-g}\int_{F_p}\sum_k \frac{\langle Gv,v\rangle^k}{k!}e^{-\langle v,v\rangle}d\mathrm{Vol}_{F_p}\\
&= \sum_k \binom{g+k-1}{k}\frac{1}{\mathrm{Vol}(S^{2g-1})}\int_{S^{2g-1}}\langle Gv,v\rangle^k d\mathrm{Vol}_{S^{2g-1}}\\
&= \sum_k \binom{g+k-1}{k}\frac{1}{\mathrm{Vol}(\mathbb{P}(F_p))}\int_{\mathbb{P}(F_p)}\left(\frac{\langle Gv,v\rangle}{\langle v,v\rangle}\right)^k d\mathrm{Vol}_{\mathbb{P}(F_p)}\\
&= \sum_k \int_{\mathbb{P}(F_p)}\left(FS + \frac{\langle Gv,v\rangle}{\langle v,v\rangle}\right)^{g+k-1}.
\end{aligned} \tag{13}$$

In going from the second line to the third, we integrate over radius, using the facts that $\int_0^\infty r^{2k}e^{-r^2}r^{2g-1}dr = (k+g-1)!/2$ and that the volume of S^{2g-1} is $2\pi^g/(g-1)!$. □

Proof of Corollary 2.4. The second paragraph implies the first, since $\langle,\rangle_L$ is semi-positive, and its restriction to an i-plane Y fails to be positive iff $e_{v^*}|_Y$ fails to be injective. Thus $\tilde{c}_k|_Y$ is nonnegative and vanishes iff e_{v^*} fails to be injective for all v.

What remains, then, is to compute the curvature of $\mathbb{E}^*$, from which we obtain the Segre form s_k by equation (8). This is given by Griffiths' formula [Griffiths84], but can also be computed from a direct calculation, which we include for the reader's convenience: The metric on $\mathbb{E}^*$ is given by the inner product $h = (\tau_I)^{-1}$, where τ_I denotes the imaginary part of τ; see [Kempf91, p. 59]. Thus the curvature is given by

$$\begin{aligned} \Omega =&\overline{\partial}(h^{-1}\partial h) \\ =&-\overline{\partial}[(\partial\tau_I)\tau_I^{-1}] \\ =&-(\partial\tau_I)\tau_I^{-1}(\overline{\partial\tau_I})\tau_I^{-1} \\ =&-\frac{1}{4}(\partial\tau)\tau_I^{-1}(\overline{\partial\tau})\tau_I^{-1}. \end{aligned} \tag{14}$$

If $v \in \mathbb{E}^*$, then $w := \tau_I^{-1}\overline{v} = \langle\cdot, v\rangle \in \mathbb{E}$, and

$$\begin{aligned} \langle Gv, v\rangle =&\frac{i}{8\pi}\overline{v^t}\tau_I^{-1}(\partial\tau)\tau_I^{-1}(\overline{\partial\tau})\tau_I^{-1}v \\ =&\frac{i}{8\pi}\langle e_w(\partial\tau), e_w(\partial\tau)\rangle \\ =&\frac{-1}{4\pi}\operatorname{Im}\left(e_w^*\langle,\rangle_{\mathbb{E}_p^*}\right). \end{aligned} \tag{15}$$

Note that $\langle v, v\rangle = \langle w, w\rangle$, and the unit sphere in $\mathbb{E}_p^*$ is naturally identified with the unit sphere in $\mathbb{E}_p$. As a result, averaging $\langle Gv, v\rangle^k/\langle v, v\rangle^k$ over all values of v is equivalent to averaging $(-\operatorname{Im}(\langle,\rangle_L)/4\pi)^k$ over all lines in $\mathbb{E}_p$. □

Remark. It is natural to wonder whether or not the Chern form $c_k(\mathbb{E}, \langle,\rangle)$ and the Segre form $s_k(\mathbb{E}^*, \langle,\rangle)$ are actually equal. We checked that this holds for $k = 1, 2$, and we suspect that it holds in general.

4. Linear algebra

As before, let $S(V, V^*)$ be the space of symmetric maps from the vector space V to its dual. For each $v \in V$, let $v^\perp \subset S(W, W^*)$ be the set of maps that vanish on v. Let $R_k \subset S(W, W^*)$ be the locus of maps whose image has dimension at most k. (This is a closed subset, not a vector subspace.)

We prove Theorem 2.5 in stages, beginning with the following standard fact:

4.1. Lemma. *Let $M \in R_k \setminus R_{k-1}$. R_k is smooth at M with tangent space*

$$T_M R_k = \{N \in S(V, V^*) | N(\ker(M)) \subset \operatorname{Im}(M)\}. \tag{16}$$

4.2. Corollary. *Let $Y \subset S(V, V^*)$ be a 2-plane spanned by rank-1 maps. Then $Y \subset R_2$, but $Y \not\subset R_1$.*

Proof. Let $M, N \in R_1$ be a basis of Y, and let $T \in Y$. Then $\ker(M) \cap \ker(N) \subset \ker(T)$, so $T \in R_2$. If $Y \subset R_1$, however, then $M \in T_N R_k$, so $M(\ker N) \subset \operatorname{Im}(N)$. Using that they are each rank one, this implies that M and N are dependent, a contradiction. □

4.3. Lemma. *Let $Y \subset S(V, V^*)$ be a linear subspace of R_2. Let $M \in Y$, $v \in V$ be such that $M(v)(v) \neq 0$. Then $v^\perp \cap Y \subset R_1$ and has dimension at most one.*

Proof. Suppose $N \in v^\perp \cap Y$ has rank 2. Since $M \in T_N R_2$, Lemma 4.1 implies that $M(v) = N(w)$ for some $w \in V$. But then $M(v)(v) = N(v)(w) = 0$, a contradiction. Thus $v^\perp \cap Y \subset R_1$. Now apply Corollary 4.2 to $v^\perp \cap Y$. □

It will be convenient to prove the cases $i = 1, 2$ of Theorem 2.5 separately

4.4. Proposition. *If $i < 3$, then the hypotheses of Theorem 2.5 are never met.*

Proof. We consider an i-dimensional subspace Y on which, for all nonzero v, e_v fails to be injective. That is, every vector in V is in the kernel of some nonzero element of Y, and we have

$$V = \bigcup_{Q \in \mathbb{P}(Y)} \ker(Q). \tag{17}$$

If $i = 1$, this means that all of V is in the kernel of a nonzero matrix, clearly impossible. If $i = 2$, this means that a g-dimensional vector space is the union of a 1-dimensional family of proper subspaces. Thus almost every element of Y has a $(g-1)$-dimensional kernel, i.e., is of rank 1. But this contradicts Corollary 4.2. $\square$

4.5. Proposition. *Let $i \geq 3$. If Theorem 2.5 holds for all spaces X of dimension equal to $i(i-1)/2$, then the hypotheses of the theorem cannot be satisfied by any X of dimension greater than $i(i-1)/2$.*

Proof. Let X have dimension greater than $i(i-1)/2$, satisfying the hypotheses of Theorem 2.5, and let X' be a subspace of dimension $i(i-1)/2$. The hypotheses of the theorem then apply to X', so $X' \subset W^{\perp}$ for some $(g-i+1)$-dimensional subspace $W \subset V$. Since their dimensions are equal, $X' = W^{\perp}$, so we can find $i-1$ rank-1 elements $x_1, \ldots x_{i-1}$ of X' whose ranges are linearly independent. Let $x_i \in X - X'$, and let Y be the span of $x_1, \ldots, x_n$. It is then straightforward to find a vector $v \in V$ for which e_v is injective on Y, which is a contradiction. $\square$

In light of Proposition 4.5, we will henceforth consider only spaces X of dimension equal to $i(i-1)/2$.

4.6. Proposition. *Theorem 2.5 holds for $i = 3$.*

Proof. Since $i = i(i-1)/2 = 3$, the only i-dimensional subspace Y of X is X itself. We first show that $Y \subset R_2$. This follows from equation (17) by counting dimensions. If $Y \not\subset R_2$, then there must exist a codimension-one locus $Z' \subset \mathbb{P}(Y)$ of rank-1 quadrics, and correspondingly a codimension-one closed cone $Z \subset Y \cap R_1$. But then, by Corollary 4.2, $Y \subset R_2$, which is a contradiction.

Having shown that $Y \subset R_2$, we use Lemma 4.3 to construct rank-1 elements of Y. Since Y is nonzero, we can find $M_0 \in Y$ and $v_0 \in V$ such that $M_0(v_0)(v_0) \neq 0$. Since e_{v_0} in not injective, $v_0^{\perp} \cap Y$ is nonempty and contains a rank-1 element M_1. Let $v_1 \in V$ be such that $M_1(v_1)(v_1) \neq 0$. By Lemma 4.3, we can then find a rank-1 element $M_2 \in v_1^{\perp}$.

Clearly M_1 and M_2 are independent. Let Z be the 2-plane they span, and let $W = \ker(M_1) \cap \ker(M_2)$. $W \subset V$ is codimension two and $Z \subset W^{\perp}$. Let H be another element of Y, independent of M_1 and M_2, so that Y is the span of M_1, M_2, and H.

By Proposition 4.4, there exists $t \in V$ such that

$$e_t : Z \to V^* \tag{18}$$

is injective. Injectivity is an open condition, so e_t is injective on Z for general t. However, by assumption, e_t is *never* injective on Y. Thus for each t there exists $M_3 \in Z$ such that $M_3(t) = H(t)$. However, that implies that, for every $w \in W$,

$$H(w)(t) = H(t)(w) = M_3(t)(w) = M_3(w)(t) = 0. \tag{19}$$

Since this is true for general t, $H(w)$ must be zero, so $H \in W^{\perp}$. But then $Y \subset W^{\perp}$.

Finally, the dimension of $W^{\perp}$ is exactly 3, so $X = Y = W^{\perp}$. $\square$

For $i > 3$, we will prove Theorem 2.5 by induction on i. The key to the inductive step is the following lemma:

4.7. Lemma. *Suppose that Theorem 2.5 applies for all values of $i \leq k$ and that X is a subspace of $\mathrm{Sym}(V, V^*)$ satisfying the hypotheses of Theorem 1.1 with $i = k+1$. Let $M \in X, v \in V$ be such that $M(v)(v) \neq 0$. There is a $(g-i+2)$-dimensional subspace $W \subset V$, containing v, such that*

$$W^{\perp} = v^{\perp} \cap X \subset S(V, V^*). \tag{20}$$

In particular $v^{\perp} \cap X$ contains nonzero elements of rank one.

Proof. If $W \subset V$ is a linear subspace and $W^{\perp} \subset S(V, V^*)$ is the subspace of maps f with $f|_W = 0$, then the natural map $W^{\perp} \to \mathrm{Hom}(V/W, (V/W)^*)$ identifies $W^{\perp}$ with $S(V/W, (V/W)^*)$.

Let $X_v := v^{\perp} \cap X$. By assumption, e_v is not injective on every i-dimensional subspace $Y \subset X$. This implies that the rank of $e_v : X \to V^*$ is at most $i-1$, so the kernel of e_v has codimension at most $i-1$. Thus

$$\begin{aligned} \dim(X_v) = \dim(X) - \mathrm{codim}(X_v, X) &\geq \frac{i(i-1)}{2} - (i-1) \\ &= \frac{(i-1)(i-2)}{2}. \end{aligned} \tag{21}$$

We claim that $X_v \subset S(V/v, (V/v)^*)$ satisfies the evaluation condition of Theorem 2.5. If not, then for some $w \neq 0$, the range of e_w on X_v has dimension $i-1$ and so equals the range of e_w on all of Y. Note that $e_w = e_{w+v}$ on $X_v \subset v^{\perp}$, so (after possibly replacing w by $w+v$) we can assume $M(w)(v) \neq 0$. But now $M(w) = N(w)$ for some $N \in v^{\perp}$; thus $M(w)(v) = N(w)(v) = N(v)(w) = 0$, a contradiction.

We now apply Theorem 2.5 to X_v. Viewed as an element of $S(V/v, (V/v)^*)$, X_v is the orthogonal complement of a $(g-i+1)$-dimensional subspace $W_0 \subset V/v$. This is tantamount to equation (20), where W is the preimage of W_0. $\square$

Proof of Theorem 2.5. In light of Propositions 4.4, 4.5 and 4.6, we need only prove Theorem 2.5 for $i > 3$ and for X of dimension exactly $i(i-1)/2$. We do this by induction on i, relying on the previously proven base case $i = 3$.

To establish the induction, we apply Lemma 4.7 to get a rank-1 element $M \in X$. Choose $v \in V$ with $M(v)(v) \neq 0$, and apply Lemma 4.7 again, so that

$$\overline{W}.^{\perp} = X_v \subset X \tag{22}$$

for some $(g-i+2)$-plane $\overline{W}. \subset V$ that contains v. Now let $W = \ker(M) \cap \overline{W}$. We claim that $X = W^{\perp}$.

To see this, let $Z \subset X$ be the span of $\overline{W}.^{\perp}$ and M, and let t be an element of V that is neither in $\overline{W}.$ nor in the kernel of M. (This is an open condition.) Note that $M(t)$ is a nonzero multiple of $M(v)$, so $M(t)(v) \neq 0$.

Since $t \notin W$, $e_t : W^{\perp} \to V^*$ has rank exactly $i-2$. Also $M(t) \notin e_t(\overline{W}.^{\perp})$, since that would imply $M(t)(v) = 0$. Thus $e_t(Z) \subset V^*$ has dimension $i-1$ and so is equal to $e_t(X)$. Now let x be an arbitrary element of X. Since $e_t(X) = e_t(Z)$, there exists an element $z \in Z \subset W^{\perp}$ such that $x(t) = z(t)$. But then, for any $w \in W$,

$$x(w)(t) = x(t)(w) = z(t)(w) = z(w)(t) = 0. \tag{23}$$

Since t was chosen arbitrarily from an open set, $x(w)$ must be zero. Since x and w were arbitrary, $X \subset W^{\perp}$. But X and $W^{\perp}$ have the same dimension, so $X = W^{\perp}$. $\square$

5. The tangent space is constant

5.1. Theorem. *Let $X \subset \mathcal{H}_g$ be a variety of dimension $\frac{i(i-1)}{2}$, with $i \geq 3$, such that at every smooth point $A \in X$ there is a $(g-i+1)$-plane $W(A) \subset \mathbb{C}^g$ such that*

$$T_A X = W(A)^\perp \subset T_A \mathcal{H}_g = S_g. \tag{24}$$

Then $W(A) = W$ is constant and $X \subset \mathcal{H}_g$ is the affine subspace

$$X = \{M \in \mathcal{H}_g | M|_W = \tau|_W\} \tag{25}$$

for some fixed $\tau \in \mathcal{H}_g$.

Proof. Once $W(A)$ is constant, the tangent space is constant and equation (25) follows. Let $I := \{1, \ldots, i-1\}$, $K = \{i, \ldots, g\}$. Pick a smooth point A_0. After changing basis, we can assume $W(A_0)$ is the span of $\{e_k\}, k \in K$, where $e_1, \ldots, e_g \in \mathbb{C}^g$ are the standard basis elements. Then on an open set U around A the coordinates A_{ab} of the upper left $(i-1) \times (i-1)$ block are analytic coordinates for X, and there exist holomorphic functions $z_{a,t}, a \in I, t \in K$, so that

$$v_t(A) := e_t + \sum_{s \in I} z_{s,t} e_s \tag{26}$$

form a basis of $W(A)$. We will show that $\partial_{ab} z_{s,t} = 0$ for all $a, b, s \in I, t \in K$ (where ∂_{ab} means differentiation with respect to the coordinate $A_{a,b}$). At $A \in U$ a basis for $T_A X \subset S_g$ is given by the matricies $\partial_{ab} A$. Note that the upper-left-hand entries of $\partial_{ab} A$ are particularly simple. If $a, b, c, d \in I$, then

$$(\partial_{ab} A)_{cd} = 1 \text{ if } (a = c \text{ and } b = d) \text{ or } (a = d \text{ and } b = c), \text{ 0 otherwise.} \tag{27}$$

By assumption $(\partial_{ab} A) \cdot v_t = 0$. Looking at the first i entries of this product gives

$$\begin{aligned} \partial_{ab} A_{c,t} &= 0 \quad a, b, c \in I, t \in K, c \notin \{a, b\}, \\ z_{a,t} + \partial_{ab} A_{b,t} &= 0 \quad a, b \in I, t \in K. \end{aligned} \tag{28}$$

If $i > 3$, or if $i = 3$ and $a = b$, then there exists $c \in I \setminus \{a, b\}$. Thus for $s \in I, t \in K$,

$$\partial_{ab} z_{s,t} = -\partial_{ab}(\partial_{sc} A_{c,t}) = -\partial_{sc}(\partial_{ab} A_{c,t}) = 0. \tag{29}$$

Finally, if $i = 3$ and $a, b \in I$, we have

$$\begin{aligned} \partial_{ab} z_{a,t} &= -\partial_{ab}(\partial_{aa} A_{a,t}) \\ &= -\partial_{aa}(\partial_{ba} A_{a,t}) \\ &= \partial_{aa} z_{b,t} = 0. \end{aligned} \tag{30}$$

□

6. Compactness

Here we argue that the image of $X(W, \tau)$ in A_g cannot be compact. The meaning of $X(W, \tau)$ is clearer in an alternative realization of $\mathcal{H}_g$:

$\mathcal{H}_g$, with its $\mathrm{Sp}(2g, \mathbb{R})$ action is canonically identified with the set of complex structures on $\mathbb{R}^{2g} = \mathbb{C}^g$ compatible with the standard symplectic form $(,)$, where $\mathrm{Sp}(2g, \mathbb{R})$ acts on complex structures by conjugation. For $M \in \mathcal{H}_g$ the corresponding complex structure on $\mathbb{R}^{2g}$ is given by identifying $\mathbb{R}^{2g}$ with $\mathbb{C}^g$ by sending a column vector $x \in \mathbb{C}^g$ to

$$f_M(x) = (\mathrm{Re}(x), -\mathrm{Re}(Mx)). \tag{31}$$

The above, and the next lemma, are all immediate from [Mumford83, 4.1].

6.1. Definition-Lemma. *For* $M \in X(W, \tau)$

$$V := f_M(W) = f_\tau(W) \tag{32}$$

is a nondegenerate subspace of $\mathbb{R}^{2g}$ *with a fixed complex structure compatible with the restriction of* $(,)$*. Under the above realization of* $\mathcal{H}_g$*,* $X(W, \tau)$ *is identified with the set of complex structures on* $\mathbb{R}^{2g}$*, compatible with* $(,)$*, extending the complex structure on* V*. This is naturally identified with compatible complex structures on* $(V^\perp, (,)|_{V^\perp})$*, and thus with* $\mathcal{H}_r$*, where* r *is the complex codimension of* W*.*

Let H (resp. $\Gamma \subset H$) be the subgroup of $\mathrm{Sp}(2g, \mathbb{R})$ (resp. $\mathrm{Sp}(2g, \mathbb{Z})$) that preserves $X(W, T)$—or equivalently by the above, the subgroups that preserve and act complex linearly on V. Restriction to $V, V^\perp$ identifies H with $\mathrm{U}(s) \times \mathrm{Sp}(2g, \mathbb{R})$, and under this identification, $\mathrm{U}(s) \times \{1\}$ is the stabilizer of τ. Recall that a subgroup of a semi-simple Lie group is called a **lattice** if it is discrete and the quotient has finite volume. It is called cocompact iff the quotient is compact. The following is clear:

6.2. Lemma. *Assume the image,* Z*, of* $X(W, \tau)$ *in* A_g *is a closed analytic subvariety. Then*

$$\Gamma \backslash (H / \mathrm{U}(s)) = \Gamma \backslash X(W, \tau) \tag{33}$$

is the normalization of Z*, and* $\Gamma \subset H$ *is a lattice and it is cocompact iff* Z *is compact.*

As $\mathrm{U}(s)$ is compact, $\Gamma \subset H$ is a lattice iff its image in $H/\mathrm{U}(s) = \mathrm{Sp}(2r, \mathbb{R})$ is a lattice. Furthermore, by the Borel Density Theorem ([Zimmer84, 3.2.5]), a lattice in a semi-simple Lie group without compact factors is Zariski dense.

Thus it suffices to prove the following, which was done jointly with (and mostly by) Scot Adams:

6.3. Theorem. *Let* $W \subset \mathbb{R}^{2g}$ *be a real subspace, of codimension* $2r$*, together with a complex structure compatible with* $(,)|_W$*. Let* $H \subset \mathrm{Sp}(2g, \mathbb{R})$ *be the subgroup of elements that preserves, and acts complex linearly on,* W*. Let* $\Gamma := H \cap \mathrm{Sp}(2g, \mathbb{Z})$*. The image of the restriction map*

$$\Gamma \to \mathrm{Sp}(W^\perp, \mathbb{R}) = \mathrm{Sp}(2r, \mathbb{R}) \tag{34}$$

is Zariski dense iff W *is spanned by integer vectors. In this case the image is a noncocompact lattice.*

Proof. Suppose first that W has a basis of integer vectors. Then the image of Γ in $H/\mathrm{U}(s) = \mathrm{Sp}(2r, \mathbb{R})$ is commensurable with (i.e., up to finite index subgroups equal to) $\mathrm{Sp}(2r, \mathbb{Z})$ a noncocompact lattice.

Now suppose that W is not spanned by integer vectors and that the image of Γ is dense. Let $G \subset H \subset \mathrm{Sp}(2g, \mathbb{R})$ be the Zariski closure of Γ. Restriction to $W^\perp$ gives a surjection

$$G(\mathbb{R}) \to \mathrm{Sp}(W^\perp, \mathbb{R}) = \mathrm{Sp}(2r, \mathbb{R}) \tag{35}$$

which together with restriction to W defines an embedding

$$G(\mathbb{R}) \subset \mathrm{U}(s) \times \mathrm{Sp}(2r, \mathbb{R}). \tag{36}$$

For any complex subspace $V \subset \mathbb{C}^{2g}$, nondegenerate with respect to the complex linear extension of $(,)$, let $G_V \subset \mathrm{Sp}(2g,\mathbb{C})$ be the subgroup of matricies that act trivially on V. Of course restriction to $V^\perp$ identifies this with $\mathrm{Sp}(V^\perp,\mathbb{C})$. We indicate its (complex) Lie algebra by

$$\mathrm{sp}_{V^\perp} \subset \mathrm{sp}(2g,\mathbb{C}). \tag{37}$$

We abuse notation and refer to $W \otimes_{\mathbb{R}} \mathbb{C}$ as W as well.

Note that if a Lie subalgebra $\mathfrak{h} \subset \mathrm{sp}(2g,\mathbb{C})$ is stable under $\mathrm{Gal}(\mathbb{C}/L)$ for a subfield $L \subset \mathbb{C}$, then $\mathfrak{h}$ is defined over L, i.e., is the extension of scalars of a Lie subalgebra of $\mathrm{sp}(2g, L)$, which we denote $\mathfrak{h}(L)$. The subalgebra $\mathfrak{h}(L) \subset \mathfrak{h}$ is simply the subset fixed by $\mathrm{Gal}(\mathbb{C}/L)$.

Since W is not spanned by rational vectors, there exists an element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ for which $\sigma(W) \neq W$. Let $V = \sigma(W)$. Let $\mathfrak{g} \subset \mathrm{sp}(2g,\mathbb{C})$ be the complexified Lie algebra of G.

By Levi's theorem, [Serre65, 4.1], (35) has a section at the level of real Lie algebras. Since there is no nontrivial homomorphism $\mathrm{sp}(2r,\mathbb{R}) \to \mathrm{u}(s)$, we conclude from (36) that $\mathrm{sp}_{W^\perp} \subset \mathfrak{g}$. Note that the image is an ideal, as it corresponds to the kernel of the restriction

$$G \to \mathrm{U}(s) \subset \mathrm{Sp}(W,\mathbb{R}). \tag{38}$$

If $V = \overline{V}$, let

$$r = \mathrm{sp}_{V^\perp} + \mathrm{sp}_{W^\perp} \subset \mathrm{sp}(2g,\mathbb{C}). \tag{39}$$

Since $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ preserves Γ, it preserves $\mathfrak{g}$, and thus

$$\mathrm{sp}_{V^\perp} = \sigma(\mathrm{sp}_{W^\perp}) \subset \mathfrak{g} \tag{40}$$

is another ideal. Since $\mathrm{sp}_{V^\perp}$ and $\mathrm{sp}_{W^\perp}$ are distinct simple ideals, their intersection is trivial, so r is the direct sum

$$r = \mathrm{sp}_{V^\perp} \oplus \mathrm{sp}_{W^\perp} \subset \mathfrak{g} \tag{41}$$

and is itself an ideal.

If $V \neq \overline{V}$, let

$$r = \mathrm{sp}_{V^\perp} + \mathrm{sp}_{\overline{V}^\perp} + \mathrm{sp}_{W^\perp} = \mathrm{sp}_{V^\perp} + \overline{\mathrm{sp}_{V^\perp}} + \mathrm{sp}_{W^\perp} \subset \mathrm{sp}(2g,\mathbb{C}). \tag{42}$$

By the same reasoning as before, each factor is a simple ideal, so r is the direct sum of its factors and is an ideal.

In either case, we have an ideal $r \subset \mathfrak{g}$. Since $r, \mathfrak{g}$ are preserved by complex conjugation, we have an induced real ideal $r(\mathbb{R}) \subset \mathfrak{g}(\mathbb{R})$. By construction, $r(\mathbb{R})$ is isomorphic to

$$\begin{aligned} &\mathrm{sp}(2r,\mathbb{R}) \oplus \mathrm{sp}(2r,\mathbb{R}) \quad \text{if } V = \overline{V}, \\ &\mathrm{sp}(2r,\mathbb{C}) \oplus \mathrm{sp}(2r,\mathbb{R}) \quad \text{if } V \neq \overline{V}. \end{aligned} \tag{43}$$

By (35) there is an induced surjection of real Lie algebras

$$r(\mathbb{R}) \to \mathrm{sp}(2r,\mathbb{R}) \tag{44}$$

whose kernel, by (36), is a subalgebra of $\mathrm{u}(s)$. As $r(\mathbb{R})$ is semi-simple, the kernel is a direct sum of simple factors of $r(\mathbb{R})$. Since no factor has a nontrivial homomorphism to $\mathrm{u}(s)$, we conclude r has only a single factor, which is a contradiction. □

7. Remarks on characteristic $p > 0$

In positive characteristic the only known codim g complete subvariety of A_g is the locus $Z \subset A_g \otimes \mathbb{F}_p$ of Abelian varieties of p-rank zero, discovered by Oort. Results of [Koblitz75] imply that the tangent space to Z is just as in Corollary 2.6 (with $i = g$):

7.1. Proposition (Koblitz). *At points $[A]$ in a Zariski dense subset of Z the following hold: The p-linear Frobenius*

$$H : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A) \tag{45}$$

has 1-dimensional cokernel. Let $L_{[A]} \subset H^0(A, \Omega^1_A) = \mathbb{E}_{[A]}$ be the dual line. Z is (orbifold) smooth at $[A]$ and its tangent space is given by

$$T_{[A]}Z = L^{\perp}_{[A]} \subset S(\mathbb{E}_{[A]}, \mathbb{E}^*_{[A]}) = T_{[A]}A_g. \tag{46}$$

Proof. This follows from [Koblitz75]. Here we give a few details for the readers convenience. The p-linear Frobenius gives a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(A, \Omega^1_A) & \longrightarrow & H^1_{\mathrm{DR}}(A) & \longrightarrow & H^1(A, \mathcal{O}_A) & \longrightarrow & 0 \\ & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle H}\big\downarrow & & \\ 0 & \longrightarrow & H^0(A, \Omega^1_A) & \longrightarrow & H^1_{\mathrm{DR}}(A) & \longrightarrow & H^1(A, \mathcal{O}_A) & \longrightarrow & 0 \end{array} \tag{47}$$

The snake lemma induces a canonical map $B : \ker(H) \to H^0(A, \Omega^1_A)$. This is the (restriction to $\ker(H)$ of the) map given by the matrix B in [Koblitz75]. The argument on pages 188-189 shows that the tangent space to Z is the perpendicular to the (1-dimensional) image of B. So it remains to show that the image of B is dual to the cokernel of H. This follows from the fact that the image of F is isotropic for the canonical pairing on H^1_{DR}. □

Given the parallel between Theorem 7.1 and our argument in characteristic zero, it is natural to wonder:

7.2. Question. *If $Z \subset A_g \otimes \mathbb{F}_p$ is complete and of codimension g, is the tangent space to Z as in Corollary 2.6? Is Oort's example the only possibility?*

8. M^c_g

Proof of Corollary 1.2.1. By [Diaz87] a compact subvariety of M^c_g or $M^c_{g,1}$ has codimension at least g, for any g, and a compact subvariety of M_g or $M_{g,1}$ has codimension at least $2g - 1$. Now assume $g \geq 3$ and let $Z \subset M^c_g$ be a compact subvariety of codimension at most g. We have a regular map

$$M^c_g \to A_g$$

with zero-dimensional fibres outside of ∂M^c_g (meaning the complement of $M_g \subset M^c_g$). Suppose first that $g = 3$. By Corollary 1.2, $Z \subset \partial M^c_3$ and so it projects to a complete surface in M^c_2, violating Diaz's bound. So we may assume $g \geq 4$. We proceed by induction.

By Diaz's bounds Z must meet the boundary. Let Z_i be the intersection of Z with δ_i.

If Z_i is nonempty, we have by Diaz's bounds

$$\begin{aligned} g &\geq \operatorname{codim}(Z, M_g^c) \geq \operatorname{codim}(Z_i, \delta_i) \\ &\geq \operatorname{codim}(\pi_i(Z_i), M_{i,1}^c) + \operatorname{codim}(\pi_{g-i}(Z_i), M_{g-i,1}^c) \end{aligned}$$

where π_k indicates the projection onto the factor $M_{k,1}^c$. Furthermore, by induction, we can replace the last term by $g+1$ if either i or $g-i$ is at least 3. Thus the only possibility is that $g = 4$ and Z meets only δ_2. But then its image in $M_{2,1}^c$ is a complete surface contained in $M_{2,1}$, contradicting Diaz's bounds. $\square$

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