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POISSON BRACKETS AND TWO-GENERATED SUBALGEBRAS OF RINGS OF POLYNOMIALS

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1. Introduction

Let $A = F[x_1, x_2, \dots, x_n]$ be a ring of polynomials over a field F on the variables $x_1, x_2, \dots, x_n$. It is well known (see, for example, [11]) that the study of automorphisms of the algebra A is closely related with the description of its subalgebras. By the theorem of P. M. Cohn [4], a subalgebra of the algebra $F[x]$ is free if and only if it is integrally closed. The theorem of A. Zaks [13] says that the Dedekind subalgebras of the algebra A are rings of polynomials in a single variable. A. Nowicki and M. Nagata [8] proved that the kernel of any nontrivial derivation of the algebra $F[x, y]$, $char(F) = 0$, is also a ring of polynomials in a single generator. An original solution of the occurrence problem for the algebra A, using the Groebner basis, was given by D. Shannon and M. Sweedler [9]. However, the method of the Groebner basis does not give any information about the structure of concrete subalgebras. Recall that the solubility of the occurrence problem for rings of polynomials over fields of characteristic 0 was proved earlier by G. Noskov [7].

The present paper is devoted to the investigation of the structure of two-generated subalgebras of A. In the sequel, we always assume that F is an arbitrary field of characteristic 0. Let us denote by $\bar{f}$ the highest homogeneous part of an element $f \in A$, and by $\langle f_1, f_2, \dots, f_k\rangle$ the subalgebra of A generated by the elements $f_1, f_2, \dots, f_k \in A$.

Definition 1. A pair of polynomials $f_1, f_2 \in A$ is called $*$-reduced if they satisfy the following conditions:
1) $\bar{f}_1, \bar{f}_2$ are algebraically dependent;
2) f_1, f_2 are algebraically independent;
3) $\bar{f}_1 \notin \langle \bar{f}_2\rangle, \bar{f}_2 \notin \langle \bar{f}_1\rangle$.

Recall that a pair f_1, f_2 with condition 3) is usually called *reduced*. Condition 1) means that we exclude the trivial case when $\bar{f}_1, \bar{f}_2$ are algebraically independent. We do not consider the case when f_1, f_2 are algebraically dependent. Concerning this case, recall the well-known theorem of S. S. Abhyankar and T. -T. Moh [1], which says that if $f, g \in F[x]$ and $\langle f, g\rangle = F[x]$, then $\bar{f} \in \langle \bar{g}\rangle$ or $\bar{g} \in \langle \bar{f}\rangle$.

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The main result of this paper, formulated in Theorem 3, gives a lower bound for degrees of the elements of the subalgebra $\langle f_1, f_2\rangle$, where f_1, f_2 is a $*$-reduced pair. In particular, this estimate yields a new proof of the theorem of H. Jung [5] on automorphisms of the rings of polynomials in two variables. The estimate involves a certain invariant of the pair f_1, f_2 which depends on the degrees of f_1, f_2 and of their Poisson bracket.

The paper is structured as follows. In Section 2, we imbed the algebra A into the free Poisson algebra on the same set of variables. In this way, we introduce a Poisson bracket on A as a restriction of the bracket in the free Poisson algebra. We investigate elementary properties of the Poisson bracket in A and prove for it an analogue of G. M. Bergman's theorem [3] on centralizers (Theorem 1). In Section 3, the structure of subalgebras of A generated by a $*$-reduced pair f_1, f_2 with the condition $\deg[f_1, f_2] > \min(\deg f_1, \deg f_2)$ is investigated in detail, and a lower bound for the degrees of the elements of such subalgebras is obtained (Theorem 2). The main result is deduced from Theorem 2 in Section 4, where some relevant examples of two-generated subalgebras are also given.

2. Poisson brackets

First, we recall the definition of Poisson algebras (see [10]).

Definition 2. A vector space B over a field F endowed with two bilinear operations $x \cdot y$ (a multiplication) and $[x, y]$ (a Poisson bracket) is called a Poisson algebra, if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $[x, y]$, and B satisfies the following identity (the Leibniz identity):

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z]. \tag{1}$$

An important class of Poisson algebras is given by the following construction. Let L be a Lie algebra with a linear basis $l_1, l_2, \ldots, l_k, \ldots$. Denote by $P(L)$ the ring of polynomials on the variables $l_1, l_2, \ldots, l_k, \ldots$. The operation $[x, y]$ of the algebra L can be uniquely extended to a Poisson bracket $[x, y]$ on the algebra $P(L)$ by means of formula (1), and $P(L)$ becomes a Poisson algebra [10].

Now let L be a free Lie algebra with free generators $x_1, x_2, \ldots, x_n$. Then $P(L)$ is a free Poisson algebra on the same set of generators [10]. We will denote this algebra by $PL\langle x_1, x_2, \ldots, x_n\rangle$. If we choose a homogeneous basis

$$x_1, x_2, \ldots, x_n,\ [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n],\ [[x_1, x_2], x_3], \ldots \tag{2}$$

of the algebra L with nondecreasing degrees, then $PL\langle x_1, x_2, \ldots, x_n\rangle$, as a vector space, coincides with the algebra of polynomials on these elements. Evidently, the vector space $PL\langle x_1, x_2, \ldots, x_n\rangle$ is graded by degrees on x_i, and for every element $f \in PL\langle x_1, x_2, \ldots, x_n\rangle$, the highest homogeneous part $\bar{f}$ and the degree function $\deg f$ can be defined in an ordinary way. Note that

$$\overline{fg} = \bar{f}\bar{g}, \quad \deg(fg) = \deg f + \deg g, \quad \deg[f, g] \leq \deg f + \deg g.$$

It is natural to identify the ring of polynomials $A = F[x_1, x_2, \ldots, x_n]$ with the subspace of the algebra $PL\langle x_1, x_2, \ldots, x_n\rangle$ generated by the elements

$$x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}, \quad r_i \geq 0,\ 1 \leq i \leq n.$$

Put also

$$C = \bigoplus_{1 \leq i < j \leq n} [x_i, x_j]A \subseteq PL\langle x_1, x_2, \ldots, x_n\rangle.$$

The next lemma follows immediately from identity (1).

Lemma 1. *If $f, g \in A$, then*
$$[f,g] = \sum_{1\le i<j\le n} [x_i, x_j]\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right) \in C.$$

Thus, the Poisson bracket on $PL\langle x_1, x_2, \dots, x_n\rangle$ defines a mapping
$$[\cdot,\cdot] : A \times A \longrightarrow C, \quad (x,y) \mapsto [x,y].$$
It follows immediately from (1) that, for any $f \in A$, the mapping
$$ad(f) : A \longrightarrow C, \quad x \mapsto [x,f],$$
is a derivation of the algebra A with coefficients in the free A-module C (see [11]).

For any $f \in A$ we put also
$$\partial f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)^t,$$
where t is the transposition. The statement of the following lemma is well known for $k = n$ (see [6], [12]).

Lemma 2. *Elements $f_1, f_2, \dots, f_k$ of the algebra A are algebraically dependent if and only if the columns $\partial(f_1), \partial(f_2), \dots, \partial(f_k)$ are linearly dependent over A.*

Proof. Let $T(y_1, y_2, \dots, y_k) \in F[y_1, y_2, \dots, y_k]$ be a nonzero polynomial of minimal degree such that $T(f_1, f_2, \dots, f_k) = 0$. Applying the derivation ∂ to this equality, we get
$$\partial(f_1)\frac{\partial T}{\partial y_1}(f_1, f_2, \dots, f_k) + \dots + \partial(f_k)\frac{\partial T}{\partial y_k}(f_1, f_2, \dots, f_k) = 0.$$
This gives a nontrivial linear dependence of the elements $\partial(f_1), \partial(f_2), \dots, \partial(f_k)$ over A.

Now let elements $f_1, f_2, \dots, f_k \in A$ be algebraically independent. Complete them to an algebraically independent system $f_1, \dots, f_k, f_{k+1}, \dots, f_n$ of elements in the quotient field $Q(A) = F(x_1, x_2, \dots, x_n)$. Then, by results of [6], [12], we have
$$\det(\partial(f_1), \partial(f_2), \dots, \partial(f_k), \dots, \partial(f_n)) \neq 0.$$
Therefore, the columns $\partial(f_1), \partial(f_2), \dots, \partial(f_k)$ are linearly independent over A. $\square$

Corollary 1. *Elements $f_1, f_2, \dots, f_k \in A$ are algebraically dependent if and only if all the minors of order k of the matrix $(\partial(f_1), \partial(f_2), \dots, \partial(f_k))$ are equal to 0.*

Corollary 2. *Elements $f, g \in A$ are algebraically dependent if and only if*
$$\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} = 0, \ 1 \le i < j \le n.$$

Corollary 3. *Elements $f, g \in A$ are algebraically dependent if and only if $[f,g] = 0$.*

Of course, the Poisson bracket $[f,g]$ can be defined without using free Poisson algebras, just as the vector
$$(\gamma_{1,2}, \gamma_{1,3}, \dots, \gamma_{1,n}, \gamma_{2,3}, \dots, \gamma_{n-1,n}),$$

where

$$\gamma_{i,j} = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}, \quad 1 \leq i < j \leq n.$$

But our definition of Poisson bracket via free Poisson algebras has certain advantages for working with degrees and highest homogeneous parts. In particular,

$$\deg[f, g] = \deg f + \deg g$$

iff $\bar{f}, \bar{g}$ are algebraically independent.

The statement of the following lemma is well known (see [4]).

Lemma 3. *If homogeneous polynomials $f, g \in A$ are algebraically dependent, then there exists an element $a \in A$ such that $f = \alpha a^k, g = \beta a^r$, where $\alpha, \beta \in F, \ \ k, r \geq 0$.*

Definition 3. For every $f \in A$, the set of elements

$$\mathcal{C}(f) = \{g \in A \,|\, [f, g] = 0\}$$

is called a centralizer of f in A.

It follows immediately from (1) that $\mathcal{C}(f)$ is a subalgebra of A.

The next theorem presents an analogue of the theorem of G. Bergman [3] on centralizers in free associative algebras.

Theorem 1. *For every $f \in A \setminus F$, the centralizer $\mathcal{C}(f)$ is a ring of polynomials on a single variable.*

Proof. Let $\bar{f} = \alpha a^k$, where $\alpha \in F, \ k \geq 1$, and a is an element that is not a proper power. If $[\bar{f}, \bar{g}] \neq 0$, then $\overline{[f, g]} = [\bar{f}, \bar{g}] \neq 0$. Consequently, if $g \in \mathcal{C}(f)$, then $[\bar{f}, \bar{g}] = 0$. Therefore, by Corollary 3 and Lemma 3, we get $\bar{g} = \beta a^r$. Repeating the arguments of Bergman's proof in [3], one can easily show that the algebra $\mathcal{C}(f)$ is finitely generated. Furthermore, by Corollary 3, $\mathcal{C}(f)$ has transcendent degree one over F. Therefore, the Krull dimension of $\mathcal{C}(f)$ is equal to 1.

Suppose that $g \in Q(\mathcal{C}(f))$ is integral over $\mathcal{C}(f)$. Since $\mathcal{C}(f) \subseteq A$ and A is integrally closed in $Q(A)$, then $g \in A$. Let

$$g^k + g_1 g^{k-1} + \cdots + g_{k-1} g + g_k = 0, \ g_i \in \mathcal{C}(f), \ 1 \leq i \leq k,$$

be an integral equation of minimal degree for g over $\mathcal{C}(f)$. Applying the derivation $ad(f)$ to this equation, we get

$$[g, f](kg^{k-1} + (k-1)g_1 g^{k-2} + \cdots + g_{k-1}) = 0.$$

Hence $[f, g] = 0$, i.e., $g \in \mathcal{C}(f)$ and $\mathcal{C}(f)$ is integrally closed.

Thus, we have proved that $\mathcal{C}(f)$ is a Dedekind domain. Since $F \subset \mathcal{C}(f) \subseteq A$, by a theorem of A. Zaks [13], $\mathcal{C}(f)$ is a ring of polynomials on a single variable. $\square$

Problem 1. Is an analogue of Bergman's theorem true for centralizers of free Poisson algebras?

Now we will give two lemmas which will be useful for the calculation of Poisson brackets. Note that calculation of Poisson brackets is strongly related to the Jacobian conjecture (see [2]).

Lemma 4. *Let $a \in A \setminus F, \ c \in C \setminus \{0\}$. Then $[a, c] \neq 0$.*

Proof. It is sufficient to prove the statement of the lemma for homogeneous elements a, c. Without loss of generality, we may also assume that the field F is algebraically closed. We will use induction on the number of variables on which a depends. If $a \in F[x_1]$, then the equality $[a, c] = 0$ is equivalent to $[x_1, c] = 0$. Let us write

$$c = \sum_{i<j} c_{ij}[x_i, x_j],\ c_{ij} \in A.$$

Then we have the equality

$$[c, x_1] = \sum_{i<j} \sum_{k>1} \frac{\partial c_{ij}}{\partial x_k}[x_k, x_1][x_i, x_j] + \sum_{i<j} c_{ij}[[x_i, x_j], x_1] = 0.$$

The vector space $PL\langle x_1, x_2, \dots, x_n\rangle$ coincides with the space of polynomials on the variables (2). Therefore, the last equality implies

$$\sum_{i<j} c_{ij}[[x_i, x_j], x_1] = 0.$$

Moreover, since $[[x_i, x_j], x_1],\ 1 \le i < j \le n$, are linearly independent elements of the free Lie algebra L, we get $c_{ij} = 0,\ 1 \le i < j \le n$.

Now let $a = f(x_1, x_2, \dots, x_n),\ n > 1$, and $\deg a = k$. Consider the linear automorphism φ of A such that $\varphi(x_1) = x_1,\ \varphi(x_i) = x_i + \alpha_i x_1,\ 2 \le i \le n$. Then $\varphi(a) = x_1^k f(1, \alpha_2, \dots, \alpha_n) + g$, where $\deg_{x_1}(g) < k$. Choose $\alpha_2, \dots, \alpha_n$ such that $f(1, \alpha_2, \dots, \alpha_n) = 0$. If this is impossible, then $a = \alpha x_1^k \in F[x_1]$, and in this case the lemma has already been proved. So, we can assume that $\deg_{x_1}(a) = s < k$ and

$$a = a_0 + a_1 x_1 + \cdots + a_s x_1^s,$$

where $a_i \in F[x_2, \dots, x_n],\ \deg(a_i) = k - i,\ 0 \le i \le s$.

Analogously, the element c, as a polynomial in the variables (2), can be represented in the form

$$c = c_0 + c_1 x_1 + \cdots + c_r x_1^r,$$

where $c_i \in C \cap F[x_2, \dots, x_n, [x_1, x_2], \dots],\ 0 \le i \le r$. Then

$$[a, c] = [a_s, c_r]x_1^{s+r} + h,$$

where h has degree $< s + r$ on x_1, as a polynomial on variables (2). Since $a_s \in F[x_2, \dots, x_n]$ and $\deg(a_s) = k-s > 0$, by the induction assumption we get $[a_s, c_r] \neq 0$. Consequently, $[a, c] \neq 0$. □

Corollary 4. *Let $f, g \in A,\ h \in A \setminus F$. Then*

$$\deg([[f, g], h]) = \deg[f, g] + \deg h, \quad \overline{[[f, g], h]} = [\overline{[f, g]}, \bar{h}].$$

Lemma 5. *Let $f, g, h \in A \setminus F$. Put*

$$m = \deg[f, g] + \deg h,\ n = \deg[g, h] + \deg f,\ k = \deg[h, f] + \deg g.$$

Then $m \le \max(n, k)$. If $n \neq k$, then $m = \max(n, k)$.

Proof. Since $PL\langle x_1, x_2, \dots, x_n\rangle$ is a Lie algebra under the Poisson bracket $[x, y]$, we have

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0.$$

Now, the simple comparison of the highest parts of the elements $[[f, g], h],\ [[g, h], f]$, and $[[h, f], g]$, by Corollary 4, gives the statement of the lemma. □

3. Subalgebras with a condition on Poisson brackets

We fix a $*$-reduced pair of polynomials $f_1, f_2 \in A$ with the condition $\deg[f_1, f_2] > \min(\deg f_1, \deg f_2)$. For definiteness, we put $m_1 = \deg f_1 < \deg f_2 = m_2$, then $\deg([f_1, f_2]) = m_1 + m_0$, $m_0 > 0$. By Definition 1, the elements $\bar{f}_1, \bar{f}_2$ are algebraically dependent, and $\bar{f}_2 \notin \langle \bar{f}_1 \rangle$. Therefore, $m_1 \not| \, m_2$.

This section is devoted to the description of the highest homogeneous parts of elements of the subalgebra $\langle f_1, f_2 \rangle$. The following algorithm forms the main part of this description.

Algorithm 1. The initial step of algorithm 1 is numbered by 3, for simplicity of notation.

Step 3. Consider the equation

$$m_1 s_1 = m_2 s_2, \ s_1, s_2 \in Z.$$

We fix a minimal natural number s_2 for which this equation has an integer solution, and fix this solution s_1, s_2. Observe that in this solution we have

$$s_1 = \frac{m_2}{(m_1, m_2)}, \ s_2 = \frac{m_1}{(m_1, m_2)}. \tag{3}$$

Here and in the sequel, $(a_1, a_2, \dots, a_n)$ denotes the greatest common divisor of the elements $a_1, a_2, \dots, a_n$. Since $m_1 \not| \, m_2$, we have $s_2 > 1$.

Lemma 6. *The elements of the type*

$$f_1^{i_1} f_2^{i_2}, \ i_2 < s_2,$$

have different degrees for different values of i_1, i_2.

Proof. Keeping in mind the equality

$$\deg(f_1^{i_1} f_2^{i_2}) = m_1 i_1 + m_2 i_2,$$

suppose that

$$m_1 i_1 + m_2 i_2 = m_1 j_1 + m_2 j_2, \ i_2, j_2 < s_2.$$

Without loss of generality, we can assume that $i_2 \geq j_2$. Then

$$m_1(j_1 - i_1) = m_2(i_2 - j_2), \ 0 \leq i_2 - j_2 < s_2.$$

By the condition on choosing s_2, we have $i_2 = j_2$, and, consequently, $i_1 = j_1$. $\square$

Since $\deg(f_1^{s_1}) = m_1 s_1 = m_2 s_2 = \deg(f_2^{s_2})$ and $\bar{f}_1, \bar{f}_2$ are algebraically dependent, by Lemma 3 we may assume that $\bar{f}_1^{\,s_1} = \bar{f}_2^{\,s_2}$. Hence the element $f = f_2^{s_2} - f_1^{s_1}$ has degree less than $m_2 s_2$.

Assume that $\bar{f} \in \langle \bar{f}_1, \bar{f}_2 \rangle$. Since $\bar{f}_1, \bar{f}_2$ are algebraically dependent, by Lemma 3 there exists an element $a \in A$ such that $\bar{f}_1 = \alpha a^k, \bar{f}_2 = \beta a^s$ for some $\alpha, \beta \in F$. Therefore, the space of elements of type $\bar{f}_1^{\,i_1} \bar{f}_2^{\,i_2}$ of fixed degree is one dimensional. Hence $\bar{f} = \alpha_{i_1,i_2} \bar{f}_1^{\,i_1} \bar{f}_2^{\,i_2}$. Since $m_1 i_1 + m_2 i_2 = \deg f < m_2 s_2$, the element ${f_1}^{i_1} {f_2}^{i_2}$ is of the type given in Lemma 6. We replace f by $f - \alpha_{i_1,i_2} f_1^{i_1} f_2^{i_2}$ and note that $\deg(f - \alpha_{i_1,i_2} f_1^{i_1} f_2^{i_2}) < \deg f$. After several such reductions, we get an element

$$f_3 = f_2^{s_2} - f_1^{s_1} - \sum_{i_1,i_2} \alpha_{i_1,i_2} f_1^{i_1} f_2^{i_2},$$

where $m_1 i_1 + m_2 i_2 < m_2 s_2$ and $\bar{f}_3 \notin \langle \bar{f}_1, \bar{f}_2 \rangle$. Observe that $f_3 \neq 0$ since f_1, f_2 are algebraically independent. We put $m_3 = \deg f_3$.

Lemma 7. $\overline{[f_1, f_3]} = s_2\overline{[f_1, f_2]}\bar{f_2}^{s_2-1}$.

Proof. A straightforward calculation gives

$$[f_1, f_3] = [f_1, f_2](s_2 f_2^{s_2-1} - \sum_{i_1,i_2} i_2\alpha_{i_1,i_2} f_1^{i_1} f_2^{i_2-1}).$$

Since $m_1 i_1 + m_2 i_2 < m_2 s_2$, the inequality

$$\deg(f_1^{i_1} f_2^{i_2-1}) = m_1 i_1 + m_2(i_2 - 1) < m_2 s_2 - m_2 = \deg(f_2^{s_2-1})$$

completes the proof. □

Corollary 5. $m_0 + m_2(s_2 - 1) \leq m_3 < m_2 s_2$.

Proof. By Lemma 7 we get

$$m_1 + m_3 \geq \deg[f_1, f_3] = m_1 + m_0 + m_2(s_2 - 1);$$

hence $m_3 \geq m_0 + m_2(s_2 - 1)$. The inequality $m_3 < m_2 s_2$ follows from the definition of f_3. □

Step 3 of algorithm 1 completes its work by testing whether the elements $\bar{f}_1, \bar{f}_3$ are algebraically dependent. If they are algebraically independent, algorithm 1 finishes its work too.

Suppose now that, after $t \geq 3$ steps, algorithm 1 produces a set of polynomials

$$f_1, f_2, \dots, f_t,\ \deg f_i = m_i,\ 1 \leq i \leq t, \tag{4}$$

which satisfy the following conditions:

C1) Let $s_i,\ 2 \leq i \leq t-1$, be a minimal natural number for which the equation

$$m_1 r_1 + m_2 r_2 + \cdots + m_{i-1} r_{i-1} = m_i s_i$$

has an integer solution $r_1, r_2, \dots, r_{i-1}$. Then this equation has a solution satisfying the inequalities $0 < r_1,\ 0 \leq r_2 < s_2,\ \dots,\ 0 \leq r_{i-1} < s_{i-1}$.

C2) The elements $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_{t-1}$ are mutually algebraically dependent, and, for every j, $2 \leq j \leq t$, $\bar{f}_j \notin \langle \bar{f}_1, \bar{f}_2, \dots, \bar{f}_{j-1}\rangle$ holds.

C3) $\overline{[f_1, f_i]} = s_2 \dots s_{i-1}\overline{[f_1, f_2]}\bar{f_2}^{s_2-1} \dots \bar{f}_{i-1}^{s_{i-1}-1}$, for every i, $3 \leq i \leq t$.

C4) $m_0 + m_2(s_2 - 1) + \cdots + m_{i-1}(s_{i-1} - 1) \leq m_i < m_{i-1}s_{i-1}$, for all i, $3 \leq i \leq t$, and $(m_1, m_2) > (m_1, m_2, m_3) > \cdots > (m_1, m_2, \dots, m_{t-1})$.

If $\bar{f}_1, \bar{f}_t$ are algebraically independent, then algorithm 1 finishes its work. Otherwise, the next step starts.

Step $t + 1$ ($t \geq 3$). Denote by s_t a minimal natural number for which the equation

$$m_1 r_1 + m_2 r_2 + \cdots + m_{t-1} r_{t-1} = m_t s_t \tag{5}$$

has an integer solution $r_1, r_2, \dots, r_{t-1}$.

Lemma 8. *Equation* (5) *has an integer solution* $r_1, r_2, \dots, r_{t-1}$, *satisfying the inequalities* $0 < r_1,\ 0 \leq r_2 < s_2,\ \dots,\ 0 \leq r_{t-1} < s_{t-1}$.

Proof. Applying successively condition C1), we can substitute r_i in equation (5) by their remainders modulo s_i, for every $i = t-1, \dots, 2$. In this way, we find a solution of (5) satisfying the inequalities $0 \leq r_i < s_i,\ 2 \leq i \leq t-1$. By C4), we have

$$m_t \geq m_0 + m_2(s_2 - 1) + \cdots + m_{t-1}(s_{t-1} - 1).$$

Therefore,

$$m_1r_1 + m_2r_2 + \cdots + m_{t-1}r_{t-1} = m_ts_t \geq m_t \geq m_0 + m_2(s_2-1) + \cdots + m_{t-1}(s_{t-1}-1).$$

Since

$$m_2r_2 + \cdots + m_{t-1}r_{t-1} \leq m_2(s_2-1) + \cdots + m_{t-1}(s_{t-1}-1),$$

we get $m_1r_1 \geq m_0 > 0$, i.e., $r_1 > 0$. □

By Lemma 3 and Lemma 8, $\bar{f}_t \in \langle \bar{f}_1, \ldots, \bar{f}_{t-1}\rangle$ if and only if $s_t = 1$. Now condition C2) gives $s_t > 1$. Therefore, $(m_1, m_2, \ldots, m_{t-1}) > (m_1, m_2, \ldots, m_t)$, since otherwise $(m_1, m_2, \ldots, m_{t-1}) \,|\, m_t$ and equation (5) would admit a solution with $s_t = 1$.

Lemma 9. *The elements of the type*

$$f_1^{i_1} f_2^{i_2} \ldots f_t^{i_t},\ i_2 < s_2, \ldots, i_t < s_t,$$

have different degrees for different values of $i_1, i_2, \ldots, i_t$.

Proof. Consider the equation

$$m_1i_1 + m_2i_2 + \cdots + m_ti_t = m_1j_1 + m_2j_2 + \cdots + m_tj_t,$$

where $i_r, j_r < s_r,\ 2 \leq r \leq t$. By definition of s_t, we get $i_t = j_t$, which yields

$$m_1i_1 + m_2i_2 + \cdots + m_{t-1}i_{t-1} = m_1j_1 + m_2j_2 + \cdots + m_{t-1}j_{t-1}.$$

Applying successively condition C1), we get $i_{t-1} = j_{t-1},\ \ldots,\ i_1 = j_1$. □

In the sequel, we fix a solution $r_1, r_2, \ldots, r_{t-1}$ of equation (5) satisfying the conditions of Lemma 8. Then, by Lemma 3,

$$\bar{f}_t^{\,s_t} = \alpha_{r_1,\ldots,r_{t-1}} \bar{f}_1^{\,r_1} \bar{f}_2^{\,r_2} \ldots \bar{f}_{t-1}^{r_{t-1}}.$$

Consider the element

$$f = f_t^{s_t} - \alpha_{r_1,\ldots,r_{t-1}} f_1^{r_1} f_2^{r_2} \ldots f_{t-1}^{r_{t-1}}.$$

We have $\deg f < m_ts_t$. Assume that $\bar{f} \in \langle \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_t\rangle$. Since the elements $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_t$ are mutually algebraically dependent, the space spanned by the elements of the type $\bar{f}_1^{\,i_1} \bar{f}_2^{\,i_2} \ldots \bar{f}_t^{\,i_t}$ with a fixed degree is one dimensional. Therefore,

$$\bar{f} = \alpha_{i_1,i_2,\ldots,i_t} \bar{f}_1^{\,i_1} \bar{f}_2^{\,i_2} \ldots \bar{f}_t^{\,i_t},$$

for some $i_1, i_2, \ldots, i_t$ such that $m_1i_1 + m_2i_2 + \cdots + m_ti_t = \deg f$. Arguing as in the proof of Lemma 8, by Lemma 8 and condition C1), we may assume that $i_2 < s_2, \ldots, i_t < s_t$. Thus the element $f_1^{i_1} f_2^{i_2} \ldots f_t^{i_t}$ has a form given in Lemma 9, and we replace f by the element $f - \alpha_{i_1,i_2,\ldots,i_t} f_1^{i_1} f_2^{i_2} \ldots f_t^{i_t}$. Repeating, if necessary, such reductions, eventually we get an element

$$f_{t+1} = f_t^{s_t} - \alpha_{r_1,\ldots,r_{t-1}} f_1^{r_1} f_2^{r_2} \ldots f_{t-1}^{r_{t-1}} - \sum_{i_1,i_2,\ldots,i_t} \alpha_{i_1,i_2,\ldots,i_t} f_1^{i_1} f_2^{i_2} \ldots f_t^{i_t}, \tag{6}$$

where $m_1r_1 + \cdots + m_{t-1}r_{t-1} = m_ts_t$, $m_1i_1 + \cdots + m_ti_t < m_ts_t$, and $\bar{f}_{t+1} \notin \langle \bar{f}_1, \ldots, \bar{f}_t\rangle$ or $f_{t+1} = 0$. The next lemma shows, in particular, that $f_{t+1} \neq 0$. We put $m_{t+1} = \deg f_{t+1}$.

Lemma 10. $\overline{[f_1, f_{t+1}]} = s_2 \ldots s_t \overline{[f_1, f_2]} \bar{f}_2^{\,s_2-1} \ldots \bar{f}_t^{\,s_t-1}$.

Proof. By (1) and (6),

$$\begin{aligned}[f_1, f_{t+1}] &= s_t[f_1, f_t]f_t^{s_t-1} - \alpha_{r_1,\dots,r_{t-1}}(\sum_{j\geq 2} r_j[f_1,f_j]f_1^{r_1}\dots f_j^{r_j-1}\dots f_{t-1}^{r_{t-1}}) \\ &\quad - \sum_{i_1,i_2,\dots,i_t} \alpha_{i_1,i_2,\dots,i_t}(\sum_{j\geq 2} i_j[f_1,f_j]f_1^{i_1}\dots f_j^{i_j-1}\dots f_t^{i_t}).\end{aligned}$$

By condition C3),

$$\deg([f_1, f_t]f_t^{s_t-1}) = m_1 + m_0 + m_2(s_2-1) + \dots + m_t(s_t-1) = d.$$

Furthermore, equality (5) gives

$$\begin{aligned}&\deg([f_1,f_j]f_1^{r_1}\dots f_j^{r_j-1}\dots f_{t-1}^{r_{t-1}}) = m_1 + m_0 + m_2(s_2-1) + \dots + m_{j-1}(s_{j-1}-1) \\ &\quad + m_1r_1 + \dots + m_j(r_j-1) + \dots + m_{t-1}r_{t-1} \\ &= m_1 + m_0 + m_2(s_2-1) + \dots + m_{j-1}(s_{j-1}-1) + m_ts_t - m_j \\ &= d + m_t - m_j - m_j(s_j-1) - \dots - m_{t-1}(s_{t-1}-1).\end{aligned}$$

By C4), for every $j < t$,

$$\begin{aligned}&m_j + m_j(s_j-1) + \dots + m_{t-1}(s_{t-1}-1) = m_js_j + \dots + m_{t-1}(s_{t-1}-1) \\ &\quad > m_{j+1} + m_{j+1}(s_{j+1}-1) + \dots + m_{t-1}(s_{t-1}-1) \\ &\quad > \cdots \\ &\quad > m_{t-1} + m_{t-1}(s_{t-1}-1) = m_{t-1}s_{t-1} > m_t.\end{aligned}$$

Hence

$$\deg([f_1,f_j]f_1^{r_1}\dots f_j^{r_j-1}\dots f_{t-1}^{r_{t-1}}) < d.$$

Analogously, the condition $m_1i_1 + \dots + m_ti_t < m_ts_t$ gives

$$\deg([f_1,f_j]f_1^{i_1}\dots f_j^{i_j-1}\dots f_t^{i_t}) < d.$$

Now, by condition C3),

$$\overline{[f_1, f_{t+1}]} = s_t\overline{[f_1,f_t]}\bar{f}_t^{\,s_t-1} = s_2\dots s_t\overline{[f_1,f_2]}\bar{f}_2^{\,s_2-1}\dots\bar{f}_t^{\,s_t-1},$$

which proves the lemma. □

Corollary 6. $m_0 + m_2(s_2-1) + \dots + m_t(s_t-1) \leq m_{t+1} < m_ts_t$.

Thus, algorithm 1 is described and justified. It will necessarily stop after a finite number of steps because of the strict decrease of the sequence of greatest common divisors in C4).

In the sequel, we suppose that algorithm 1 has finished its work and produced a set of elements (4) which satisfies conditions C1)–C4) and the condition

C5) $\bar{f}_1, \bar{f}_t$ are algebraically independent.

Lemma 11. *The highest homogeneous parts of the elements of the type*

$$f_1^{i_1}f_2^{i_2}\dots f_t^{i_t},\ i_2 < s_2,\dots,i_{t-1} < s_{t-1}, \tag{7}$$

are linearly independent.

Proof. As in Lemma 9, it is easy to see that the elements of the type

$$f_1^{i_1} f_2^{i_2} \dots f_{t-1}^{i_{t-1}}, \ i_2 < s_2, \dots, i_{t-1} < s_{t-1},$$

have different degrees for different values of $i_1, i_2, \dots, i_{t-1}$.

Since the elements $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_{t-1}$ are mutually algebraically dependent, by Lemma 3 there exists an element $a \in A$ such that $\bar{f}_i = \alpha_i a^{k_i}$, $1 \le i \le t-1$. The linear dependence of the highest parts of the elements of type (7) would give the nontrivial equality of the form

$$F_0(a) + F_1(a)\bar{f}_t + \dots + F_s(a)\bar{f}_t^{\,s} = 0.$$

This contradicts condition C5). □

Consider now the set of commutative words

$$f_1^{i_1} f_2^{i_2} \dots f_t^{i_t} \tag{8}$$

on alphabet (4), without the restrictions imposed in (7). Define a degree function d on these words, by setting

$$d(f_1^{i_1} f_2^{i_2} \dots f_t^{i_t}) = \deg(f_2^{i_2} \dots f_t^{i_t}) = m_2 i_2 + \dots + m_t i_t.$$

By their construction in the process of algorithm 1 (see (6)), the elements f_{k+1} satisfy the following relations:

$$f_k^{s_k} = f_{k+1} + \alpha_{r_1,\dots,r_{k-1}} f_1^{r_1} f_2^{r_2} \dots f_{k-1}^{r_{k-1}} + \sum_{i_1, i_2, \dots, i_k} \alpha_{i_1, i_2, \dots, i_k} f_1^{i_1} f_2^{i_2} \dots f_k^{i_k}, \tag{9}$$

where $m_1 r_1 + \dots + m_{k-1} r_{k-1} = m_k s_k$, $m_1 i_1 + \dots + m_k i_k < m_k s_k$, $r_j, i_j < s_j$ for $2 \le j \le k-1$, and $i_k < s_k$, $2 \le k \le t-1$.

Lemma 12. *The set of words of type* (7) *forms a basis of the algebra* $\langle f_1, f_2 \rangle$.

Proof. We prove first that the words (8) can be expressed as linear combinations of words of type (7). The process of reduction consists of substituting the elements $f_k^{s_k}$, $2 \le k \le t-1$, by the right parts of the equalities (9). It suffices to prove that the d-degree of the right part of (9) is less than $d(f_k^{s_k})$; then our statement will easily follow by induction on the d-degree. By C4), we have

$$\begin{aligned} d(f_k^{s_k}) &= m_k s_k > m_{k+1} = d(f_{k+1}), \\ d(f_{k+1}) &\ge m_0 + m_2(s_2 - 1) + \dots + m_k(s_k - 1). \end{aligned}$$

Since $r_j, i_j < s_j$ for $2 \le j \le k-1$, and $i_k < s_k$, we get

$$\begin{aligned} d(f_1^{r_1} f_2^{r_2} \dots f_{k-1}^{r_{k-1}}) &= m_2 r_2 + \dots + m_{k-1} r_{k-1} < d(f_{k+1}), \\ d(f_1^{i_1} f_2^{i_2} \dots f_k^{i_k}) &= m_2 i_2 + \dots + m_k i_k < d(f_{k+1}). \end{aligned}$$

Thus, the linear subspace spanned by the words of type (7) forms a subalgebra which obviously coincides with $\langle f_1, f_2 \rangle$. Now, Lemma 11 completes the proof. □

Define a linear order $\le$ on the set of words of type (7), which corresponds to the inverse lexicographic order on the set of t-tuples $(i_1, i_2, \dots, i_t)$. For an element $f \in \langle f_1, f_2 \rangle$, we will denote by $\{f\}$ the leading term of f with respect to $\le$ (with coefficient 1). For a word $u = f_1^{i_1} f_2^{i_2} \dots f_t^{i_t}$, we put also $[u] = f_2^{i_2} \dots f_t^{i_t}$.

Lemma 13. *Let* u, v *be words of type* (7) *and* $u = f_1^{i_1}[u]$, $v = f_1^{j_1}[v]$. *Then* $u \le v$ *iff* $d(u) < d(v)$ *or* $d(u) = d(v)$, $i_1 \le j_1$. *Moreover,* $d(u) = d(v)$ *iff* $[u] = [v]$.

Proof. By definition, we have $u \le v$ iff $[u] < [v]$ or $[u] = [v]$, $i_1 \le j_1$. Therefore, it is sufficient to prove that if $[u] < [v]$, then $d(u) < d(v)$.

Take $[u] = f_2^{i_2} \dots f_t^{i_t}$, $[v] = f_2^{j_2} \dots f_t^{j_t}$, and assume that $i_t = j_t, \dots, i_{k+1} = j_{k+1}, i_k < j_k$. Then we obtain, by C4),

$$\begin{aligned} d(u) &= m_2 i_2 + \dots + m_k i_k + m_{k+1} i_{k+1} + \dots + m_t i_t \\ &\le m_2(s_2 - 1) + \dots + m_{k-1}(s_{k-1} - 1) + m_k i_k + m_{k+1} i_{k+1} + \dots + m_t i_t \\ &< m_k + m_k i_k + m_{k+1} i_{k+1} + \dots + m_t i_t \\ &\le m_k j_k + m_{k+1} j_{k+1} + \dots + m_t j_t \le d(v). \end{aligned}$$

□

This lemma and the proof of Lemma 12 imply

Corollary 7. *The leading term of the right part of equality* (*9*) *is equal to* f_{k+1}.

Lemma 14. *Let* u, v, w *be arbitrary words of type* (*7*). *If* $u < v$, *then* $\{uw\} < \{vw\}$.

Proof. We will prove the lemma by induction on $d(v) + d(w)$. A base for the induction is given by the evident case $d(v) + d(w) = 0$, when u, v, w depend only on f_1. Assume now that for arbitrary words u_1, v_1, w_1 of type (7), with the condition $u_1 < v_1$, $d(v_1) + d(w_1) < d(v) + d(w)$, the inequality $\{u_1 w_1\} < \{v_1 w_1\}$ is true. With this assumption, it is easy to see that for arbitrary elements $f, g \in \langle f_1, f_2 \rangle$, such that $d(f) + d(g) < d(v) + d(w)$, the equality

$$\{fg\} = \{\{f\}\{g\}\} \tag{10}$$

holds.

First, we consider the case $w = f_r$, $1 \le r \le t$. Note that the statement of the lemma is trivial for $w = f_1$. Moreover, without loss of generality, we can take $u = [u]$, $v = [v]$. Now we proceed with the reverse induction on r. If $w = f_t$, then the statement of the lemma is trivial too. Suppose that it is true for $w = f_l$, $l > r$.

Denote by k the maximal natural number such that v depends on f_k. Since $u < v$, then u also does not depend on $f_{k+1}, \dots, f_t$. Consequently, if $r > k$, we have

$$\{uf_r\} = uf_r < vf_r = \{vf_r\}.$$

If $r = k$, then we have $u = u_1 f_r^{i_r}$, $v = v_1 f_r^{j_r}$, where u_1, v_1 do not depend on f_r and $i_r \le j_r$. If $j_r < s_r - 1$, then

$$\{uf_r\} = u_1 f_r^{i_r+1} < v_1 f_r^{j_r+1} = \{vf_r\}.$$

If $j_r = s_r - 1$, then the subword $f_r^{s_r}$ of the word $vf_r = v_1 f_r^{s_r}$ is replaced by the right part of equality (9), with $k = r$. By Corollary 7 and equality (10), we obtain

$$\{vf_r\} = v_1 f_{r+1}.$$

In case $i_r = s_r - 1$ we have $u_1 < v_1$, and analogously

$$\{uf_r\} = u_1 f_{r+1} < v_1 f_{r+1}.$$

If $i_r < s_r - 1$, then by condition C4) we get

$$d(uf_r) \le m_2(s_2 - 1) + \dots + m_r(s_r - 1) < m_{r+1} \le d(v_1 f_{r+1}).$$

Now consider the case $r < k$. If u also depends on f_k, then we have $u = u_1 f_k$, $v = v_1 f_k$, $u_1 < v_1$. A composition of two inductions gives

$$\{uf_r\} = \{(u_1 f_r) f_k\} = \{\{u_1 f_r\} f_k\} < \{\{v_1 f_r\} f_k\} = \{(v_1 f_r) f_k\} = \{vf_r\}.$$

If u does not depend on f_k, then $v = v_1 f_k$ and by condition C4) we have $d(u) < m_k$. Since $f_k \leq v$, the previous case gives $\{f_k f_r\} \leq \{v f_r\}$, i.e., $f_k f_r \leq \{v f_r\}$. Note that

$$d(u f_r) < m_k + m_r = d(f_r f_k).$$

Then

$$\{u f_r\} < f_r f_k \leq \{v f_r\}.$$

We now turn to the case $w = w_1 w_2$, where w_1 and w_2 are nontrivial words of type (7). The induction on the length of w relatively to the variables (4) gives

$$\{uw\} = \{(uw_1)w_2\} = \{\{uw_1\}w_2\} < \{\{vw_1\}w_2\} = \{vw\}.$$

$\square$

Corollary 8. *Let $f, g \in \langle f_1, f_2 \rangle$ be arbitrary elements. Then*

$$\{fg\} = \{\{f\}\{g\}\}.$$

Lemma 15. $m_i \geq (m_0 + m_2(s_2 - 1))s_3 \dots s_{i-1},\ 3 \leq i \leq t.$

Proof. We prove the statement of the lemma by induction on i. If $i = 3$, it follows from condition C4). Assume that it is true for every $r < i$. By C4),

$$m_i \geq (m_0 + m_2(s_2 - 1)) + m_3(s_3 - 1) + \dots + m_{i-1}(s_{i-1} - 1).$$

Therefore, by the induction assumption,

$$\begin{aligned} m_i &\geq (m_0 + m_2(s_2 - 1))(1 + (s_3 - 1) + s_3(s_4 - 1) + \dots + s_3 \dots s_{i-2}(s_{i-1} - 1)) \\ &= (m_0 + m_2(s_2 - 1))s_3 \dots s_{i-1}. \end{aligned}$$

$\square$

For every natural number k, we define the numbers $r_2, r_3, \dots, r_{t-1}, q_t$, by means of the following equalities:

$$\begin{aligned} k &= s_2 q_3 + r_2,\ 0 \leq r_2 < s_2, \\ q_3 &= s_3 q_4 + r_3,\ 0 \leq r_3 < s_3, \\ &\dots \\ q_{t-1} &= s_{t-1} q_t + r_{t-1},\ 0 \leq r_{t-1} < s_{t-1}. \end{aligned} \tag{11}$$

Lemma 16. $\{f_2^k\} = f_2^{r_2} f_3^{r_3} \dots f_{t-1}^{r_{t-1}} f_t^{q_t}.$

Proof. We will show, by reverse induction on i, that

$$\{f_i^{q_i}\} = f_i^{r_i} \dots f_{t-1}^{r_{t-1}} f_t^{q_t},$$

for every $i = 2, \dots, t$, with $q_2 = k$. The base of the induction, for $i = t$, is evident. By Corollary 8, we have

$$\{f_{i-1}^{q_{i-1}}\} = \{(f_{i-1}^{s_{i-1}})^{q_i} f_{i-1}^{r_{i-1}}\} = \{\{\{f_{i-1}^{s_{i-1}}\}^{q_i}\} f_{i-1}^{r_{i-1}}\}.$$

Observe that, by Corollary 7, $\{f_{i-1}^{s_{i-1}}\} = f_i$. Hence, by the induction assumption,

$$\{f_{i-1}^{q_{i-1}}\} = \{\{f_i^{q_i}\} f_{i-1}^{r_{i-1}}\} = \{f_{i-1}^{r_{i-1}} f_i^{r_i} \dots f_{t-1}^{r_{t-1}} f_t^{q_t}\} = f_{i-1}^{r_{i-1}} f_i^{r_i} \dots f_{t-1}^{r_{t-1}} f_t^{q_t},$$

which proves the lemma. $\square$

Corollary 9. $\{f_1^r f_2^k\} = \{f_1^{r_1} f_2^{k_1}\}$ *iff* $r_1 = r,\ k_1 = k$.

Theorem 2. *Let $G(x,y) \in F[x,y]$ and $\deg_y(G(x,y)) = k$. Then*

$$\deg(G(f_1,f_2)) \geq m_2 k + (m_0 - m_2)\left[\frac{k\cdot(m_1,m_2)}{m_1}\right],$$

where $[\alpha]$ is the integer part of α.

Proof. By Corollary 9 and Lemma 11, we have

$$\deg(G(f_1,f_2)) \geq \deg(\{f_2^k\}).$$

Return to equalities (11). By Lemma 16,

$$\deg(\{f_2^k\}) = m_2 r_2 + \cdots + m_{t-1}r_{t-1} + m_t q_t.$$

Now, Lemma 15 gives

$$\deg(G(f_1,f_2)) \geq m_2 r_2 + (m_0 + m_2(s_2-1))(r_3 + s_3 r_4 + s_3 s_4 r_5 + \cdots + s_3 \ldots s_{t-1} q_t).$$

By (11),

$$r_3 + s_3(r_4 + s_4(r_5 + \cdots + s_{t-2}(r_{t-1} + s_{t-1}q_t)\ldots)) = q_3.$$

Consequently,

$$\begin{aligned}\deg(G(f_1,f_2)) &\geq m_2 r_2 + (m_0 + m_2(s_2-1))q_3 \\ &= m_2(r_2 + s_2 q_3) + (m_0 - m_2)q_3 = m_2 k + (m_0 - m_2)q_3.\end{aligned}$$

Since $q_3 = [\frac{k}{s_2}]$, by equality (3) we have $q_3 = \left[\frac{k\cdot(m_1,m_2)}{m_1}\right]$. This proves the theorem. □

4. The estimation of degrees in the general case

The main result of the paper is the following theorem.

Theorem 3. *Let g_1, g_2 be an arbitrary $*$-reduced pair of polynomials over a field F of characteristic 0, and let $n = \deg g_1 < m = \deg g_2$, $p = \frac{n}{(n,m)}$, $s = \frac{m}{(n,m)}$, $N = N(g_1,g_2) = \frac{mn}{(n,m)} - m - n + \deg[g_1,g_2]$. Suppose that $G(x,y) \in F[x,y]$. If $\deg_y(G(x,y)) = pq + r$, where $0 \leq r < p$, then*

$$\deg(G(g_1,g_2)) \geq qN + mr.$$

If $\deg_x(G(x,y)) = sq_1 + r_1$, where $0 \leq r_1 < s$, then

$$\deg(G(g_1,g_2)) \geq q_1 N + n r_1.$$

Proof. Put $f_1 = g_1, f_2 = g_1 g_2$. Since $\deg_y(G(x,y)) = pq + r = k$, then $x^k G(x,y) = H(x,xy)$, where $\deg_z(H(x,z)) = k$. Then

$$g_1^k G(g_1,g_2) = H(g_1, g_1 g_2) = H(f_1,f_2),$$

and consequently,

$$\deg(G(g_1,g_2)) = \deg(H(f_1,f_2)) - \deg(g_1^k).$$

Now we show that the elements f_1, f_2 satisfy all the conditions of Theorem 2. Since $\bar{g_1}, \bar{g_2}$ are algebraically dependent and $\bar{g_2} \notin \langle \bar{g_1} \rangle$, the elements $\bar{f_1} = \bar{g_1}$, $\bar{f_2} = \bar{g_1}\bar{g_2}$ are algebraically dependent as well, and $\bar{f_2} \notin \langle \bar{f_1} \rangle$. Put $m_1 = n = \deg f_1$, $m_2 = n + m = \deg f_2$. Since $\deg[g_1,g_2] \geq 2$, we have $\deg[f_1,f_2] = \deg(g_1[g_1,g_2]) = m_1 + \deg[g_1,g_2] > m_1$. Thus, all the conditions of Theorem 2 are fulfilled. Note that $(m_1,m_2) = (n, n+m) = (n,m)$, and in the notation of Theorem 2 we have

$m_0 = \deg[g_1, g_2]$. Applying Theorem 2, we obtain from the formula for degree $G(g_1, g_2)$ above

$$\begin{aligned}
\deg(G(g_1,g_2)) &\geq (n+m)k + (\deg[g_1,g_2] - n - m)\left[\frac{k}{p}\right] - nk \\
&= mk - (m+n-\deg[g_1,g_2])q \\
&= mpq + mr - mq - nq + q\deg[g_1,g_2] \\
&= q(pm - m - n + \deg[g_1,g_2]) + mr = qN + mr.
\end{aligned}$$

Similarly, putting $f_1 = g_2$, $f_2 = g_2 g_1$, we get the second part of the theorem. □

Evidently, $\frac{mn}{(m,n)} \geq m + n$ and so $N(g_1, g_2) \geq \deg[g_1, g_2] \geq 2$. Therefore, by [4, 6.9], Theorem 3 immediately implies the theorem of H. Jung [5] on tameness of automorphisms of polynomials in two variables.

Corollary 10 ([5]). *Every automorphism of a ring of polynomials in two variables over a field of characteristic* 0 *is tame.*

Corollary 11. *In the conditions of Theorem* 3*, for every $h \in \langle g_1, g_2\rangle$, either $\bar{h} \in \langle \bar{g_1}, \bar{g_2}\rangle$ or $\deg h \geq N(g_1, g_2)$.*

Proof. If $h = G(g_1, g_2)$ and $\deg_y(G(x, y)) < p$, then h is a linear combination of the elements of type $g_1^i g_2^j$, $j < p$. By the proof of Lemma 6, these elements have different degrees for different values of i, j. Consequently, $\bar{h} = \alpha \bar{g_1}^i \bar{g_2}^j$ and $\bar{h} \in \langle \bar{g_1}, \bar{g_2}\rangle$. If $\deg_y(G(x, y)) \geq p$, then in the conditions of Theorem 3 we have $q \geq 1$ and $\deg h \geq N(g_1, g_2)$. □

Note that if $p > 2$ or $\deg[g_1, g_2] > n$, then $N(g_1, g_2) > m$. We now give some examples of $*$-reduced pairs with the condition $p = 2$, $\deg[g_1, g_2] \leq n$.

Example 1. Let $a = x + y^2$, $b = y$, and

$$\begin{aligned}
g_1 &= b + a^2 = y + x^2 + 2xy^2 + y^4, \\
g_2 &= 3ab + 2a^3 = 3xy + 2x^3 + 3y^3 + 6x^2y^2 + 6xy^4 + 2y^6.
\end{aligned}$$

Then $[g_1, g_2] = 3b[b, a] = -3y[x, y]$, $\deg g_1 = n = 4$, $\deg g_2 = m = 6$, $\deg[g_1, g_2] = 3$, and in the notation of Theorem 3 we have $p = 2$. The element

$$f = 2g_2^2 - 8g_1^3 + 3g_2 = 6xy^4 + 12x^2y^2 + 6x^3 + y^3 + 9xy$$

has degree $5 = N(g_1, g_2) < m$, and the lower bound of the estimation of Theorem 3 is accessible.

Example 2. Let $a = x + y^2 + y^3$, $b = y$, and

$$\begin{aligned}
g_1 &= b + a^2 = y + x^2 + 2xy^2 + 2xy^3 + y^4 + 2y^5 + y^6, \\
g_2 &= 3ab + 2a^3 = 3xy + 2x^3 + 3y^3 + 6x^2y^2 + 3y^4 + 6x^2y^3 + 6xy^4 \\
&\quad + 12xy^5 + 2y^6 + 6xy^6 + 6y^7 + 6y^8 + 2y^9.
\end{aligned}$$

Then $[g_1, g_2] = 3b[b, a] = -3y[x, y]$, $\deg g_1 = n = 6$, $\deg g_2 = m = 9$, $\deg[g_1, g_2] = 3$, and in the notation of Theorem 3 we have $p = 2$. Consider the element

$$f = 4g_1^3 - g_2^2 = 3y^8 + 6y^7 + 3y^6 + 6xy^5 + 6xy^4 + 3x^2y^2 + 4y^3.$$

We have $\deg f = 8 > N(g_1, g_2) = 6$, and so the lower bound of the estimation of Theorem 3 is not accessible.

Problem 2. Does there exist an estimation, as in Theorem 3, for an algebraically dependent reduced pair of elements? In particular, will the estimation of Theorem 3 be true for an algebraically dependent reduced pair of elements if we change $\deg[g_1, g_2]$ to 1?

Certainly, a positive solution of this question would extend the theorem [1].

Example 3. If we set $x = 0$ in Example 1, then we get the algebraically dependent reduced pair of elements

$$g_1 = y + y^4,\ \ g_2 = 3y^3 + 2y^6.$$

The subalgebra $\langle g_1, g_2 \rangle$ contains the element y^3 of degree 3.

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