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CUSPS AND $\mathcal{D}$-MODULES

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1. Introduction

1.1. **$\mathcal{D}$-modules on singular varieties.** Let Y denote a variety over a field and let $\mathcal{D}_Y$ denote the full sheaf of differential operators on Y (in characteristic zero $\mathcal{D}_Y$ is the familiar sheaf of differential operators, and in characteristic p the sheaf $\mathcal{D}_Y$ will contain all divided powers of operators ∂). It is well known that on a *general* singular variety Y, the ring of differential operators is badly behaved: in particular, it need not be Noetherian nor, if Y is affine, a simple ring (see [BGG]).

Following Grothendieck, there is an alternative notion of a sheaf with "infinitesimal parallel transport" called a *(co)stratification* (see [Be2], [Be3], [BD1] or Section 2), which agrees with the notion of left (respectively, right) $\mathcal{D}_Y$-module when Y is smooth but will not agree, in general, when Y is singular. Our first result presents a simple vanishing condition that guarantees that the notions of $\mathcal{D}$-module and (co)stratification on a singular variety coincide.

Theorem 1.1. *Let Y be a good Cohen-Macaulay variety (see Definition 2.9). Then the categories of left $\mathcal{D}_Y$-modules and stratifications on Y are naturally equivalent, as are the categories of right $\mathcal{D}_Y$-modules and costratifications on Y.*

The class of good Cohen-Macaulay varieties, characterized by the vanishing of higher local cohomology sheaves along the diagonal, includes the smooth varieties and those for which the diagonal is a set-theoretic local complete intersection, in particular varieties with cusp singularities. Our technique is based on an extension to this setting (Theorem 2.15) of the Grothendieck-Sato formula

$$\mathcal{D}_X = \underline{H}^d_\Delta(X \times X, \mathcal{O}_X \boxtimes \omega_X)$$

describing the sheaf of differential operators on a smooth variety X of dimension d as a local cohomology sheaf along the diagonal. This formula is a generalization of the description of differential operators on a smooth curve as kernel functions with poles along the diagonal, through the Cauchy integral formula (see e.g. [BS]). The theorem then follows from the observation that $\mathcal{D}_Y$ is *flat* over $\mathcal{O}_Y$ whenever the variety Y is a good Cohen-Macaulay variety in the sense of Definition 2.9.

1.2. **Cusp Morita equivalence.** We will refer to a map $X \to Y$ as a *cuspidal quotient* morphism when Y is the quotient of X by an infinitesimal equivalence relation (so that f is a homeomorphism on underlying ringed spaces—we give a precise definition in Section 2.1). Examples include the normalization map of a

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curve with cusp singularities, the normalization map $\mathfrak{h} \to X_m$ of the space of quasi-invariants for a Coxeter group (see below), and the Frobenius homeomorphism in characteristic p. A good cuspidal quotient is one that satisfies a relative version of the good Cohen-Macaulay condition (vanishing of local cohomology along the graph), and hence benefits from a relative version of the Grothendieck-Sato formula. By exploiting the resulting identification of $\mathcal{D}$-modules and stratifications, we obtain the following:

Theorem 1.2. *Let $f : X \to Y$ be a good cuspidal quotient morphism between good Cohen-Macaulay varieties (in particular, any cuspidal quotient morphism from a smooth variety X to a CM variety Y). Then $\mathcal{D}_X$ and $\mathcal{D}_Y$ are canonically Morita equivalent.*

Corollary 1.3. *If $X \to Y$ is a universal homeomorphism from a smooth variety X to a CM variety Y, then $\mathcal{D}_Y$ is (left and right) Noetherian. If Y is affine, then $\mathcal{D}(Y)$ is a simple ring.*

More precisely, we show that the usual description (in the smooth setting) of $\mathcal{D}$-module pushforward and pullback by bimodules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ may be adapted to this singular setting, and that the functors between left and right $\mathcal{D}$-modules given by these bimodules are equivalences.

$\mathcal{D}$-modules on singular curves over $\mathbf{C}$ (or an algebraically closed field of characteristic zero) have been extensively studied [Sm], [Mu], [DE], [BW2]; in particular, Smith and Stafford [SS] proved, using techniques of noncommutative algebra, that the category of $\mathcal{D}$-modules on a cuspidal curve is Morita equivalent to the category of $\mathcal{D}$-modules on its (smooth) normalization. This result was extended to a class of singular surfaces by Hart and Smith [HS], who conjectured an extension to arbitrary Cohen-Macaulay varieties with a smooth bijective normalization; further refinements also appeared in [CS], [J]. Higher-dimensional examples of cuspidal quotients are given by the normalization map of the variety of m-quasi-invariants X_m for a Coxeter group W acting by reflections on a vector space $\mathfrak{h}$. The variety X_m sits in a diagram $\mathfrak{h} \to X_m \to \mathfrak{h}/W$ with $\mathfrak{h} \to X_m$ bijective, and arises in the study of the quantum Calogero-Moser dynamical system; it was proven in [BEG] that $\mathcal{D}(X_m)$ is simple. Theorem 1.2 generalizes and clarifies these results of [SS], [HS], [BEG] (and resolves the conjecture of [HS]). In the case of the inspiring but atypical *flat* example of a cusp quotient, the Frobenius homeomorphism, our result becomes the Cartier descent for stratifications due to Berthelot [Be3].

1.3. **Crystals on cusps.** The notion of right $\mathcal{D}$-module or costratification may be abstracted further into the definition of a !-*crystal* (on the infinitesimal site—see [BD1], 7.10), which is a sheaf endowed with compatible extensions ("parallel transport") to arbitrary nilpotent thickenings. Crystals have many good properties, and, in particular, are characterized by the "Kashiwara theorem": for any closed embedding of Y in a smooth variety Z, !-crystals on Y are just right $\mathcal{D}_Z$-modules supported on Y. Generally, $\mathcal{D}_Y$-modules and costratifications also diverge from crystals on Y. However, we explain in Proposition 3.14 that when X is smooth, it is a formal consequence of the Kashiwara theorem that the categories of !-crystals on X and its cuspidal quotients are equivalent. When combined with the above descent for $\mathcal{D}$-modules, this implies that right $\mathcal{D}_Y$-modules are equivalent to !-crystals on Y and therefore satisfy the Kashiwara theorem.

Corollary 1.4 (Cusp Kashiwara Theorem)**.** *Suppose $X \to Y$ is a universal homeomorphism from a smooth variety X to a CM variety Y. Right $\mathcal{D}_Y$-modules (or costratifications) are equivalent to !-crystals on Y and thus are equivalent to right $\mathcal{D}_Z$-modules supported on Y for any closed embedding $Y \hookrightarrow Z$ of Y in a smooth variety Z.*

This generalizes the result of [DE] for curves, and resolves Conjecture 9.9 of [BEG] for arbitrary cusps (we note that this conjecture in the original case of Cherednik algebras follows from the Morita equivalence of [BEG] and the standard descent for crystals of Proposition 3.14).

The intuition behind all these results comes from the description of stratifications as sheaves equipped with the structure of infinitesimal parallel transport, that is, equivariance for the deRham groupoid on X given by the formal neighborhood of the diagonal in $X \times X$. This means that the structure of stratification (or $\mathcal{D}$-module in the smooth case) may be interpreted as a kind of descent datum, indicating how to descend a sheaf on X to the quotient of X by the deRham groupoid (known as the deRham space of X). On the other hand, cuspidal quotients $X \to Y$ are precisely the quotients of X by *subgroupoids* of the deRham groupoid (that is, by an equivalence relation living along the diagonal in $X \times X$). In other words, a stratification on X already comes equipped with descent data for any cuspidal quotient Y. Furthermore, one may hope that the descended sheaf retains enough of the infinitesimal structure from X to be itself a stratification on Y. The cusp structures, which are certain slight shrinkings of the sheaf of functions on X, may be imagined to arise by letting the smooth variety X "drip" or "pinch" a little; one thereby obtains a system of "dripping varieties" all of which are dripping down toward the deRham space of X, hence have the same collection of $\mathcal{D}$-modules (which are, morally speaking, sheaves pulled back from this deRham space).

Unfortunately, the naive $\mathcal{O}$-module descent is not the correct adjoint to the pullback functor in the category of deRham-equivariant sheaves—in fact, effective descent fails for $\mathcal{O}$-modules in the cusp setting. However, perhaps surprisingly this descent picture (made precise in a suitable way) *does* still apply for the category of deRham-equivariant sheaves, giving the equivalence of categories of $\mathcal{D}$-modules of Theorem 1.2.

1.4. **Cusp induction.** The Morita equivalence for cusps has applications to the study of $\mathcal{D}$-modules on a smooth variety X as well. There is an important subcategory of the category of (left or right) $\mathcal{D}_X$-modules consisting of *induced* $\mathcal{D}$-modules (see [Sa], [BD2]): an induced right $\mathcal{D}$-module is one of the form $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$. This category generates the derived category of $\mathcal{D}_X$-modules and is convenient to work with in many ways, but it is rather small and seems to have no known intrinsic characterization as a subcategory of the category of $\mathcal{D}_X$-modules. However, we can use the Morita equivalence of Theorem 1.2 to "collect" the categories of induced $\mathcal{D}$-modules from all cuspidal quotients of X. It is tempting to think of this construction as a substitute for Cartier descent in characteristic p, defining integrable connections on X by pullback of quasicoherent sheaves under powers of Frobenius: in the absence of this canonical cofinal collection of cusp quotient maps in characteristic zero, we use the collection of all cusps. By collecting modules we obtain a larger category sharing all the good properties of induced $\mathcal{D}$-modules and, in the case of curves, admitting a simple intrinsic characterization. We call a $\mathcal{D}_X$-module *cusp-induced* if it lies in the essential image of the category of induced $\mathcal{D}_Y$-modules under the equivalence of Theorem 1.2 for some Cohen-Macaulay cuspidal quotient $X \to Y$. We thereby obtain an equivalence of categories:

Theorem 1.5. *There is an equivalence of categories between the direct limit of $\mathcal{O}_Y$-modules with differential operators as morphisms and cusp-induced $\mathcal{D}$-modules:*

$$(\varinjlim_{X\to Y} \mathrm{qcoh}(\mathcal{O}_Y), \mathrm{Diff}) \to \text{cusp-ind}(\mathcal{D}_X).$$

A quasi-inverse functor is given by the deRham functor.

This theorem may be considered as a "cuspidal Riemann-Hilbert correspondence", describing the deRham functor on the full subcategory of cusp-induced $\mathcal{D}$-modules. In the case when X is a smooth curve, the category of cusp-induced $\mathcal{D}$-modules is precisely identified with the category of $\mathcal{D}_X$-modules that are generically torsion-free (see Proposition 5.14). In particular, this applies to (locally) *projective* $\mathcal{D}$-modules, called *$\mathcal{D}$-bundles* in [BD2]. As we explain in Sections 5.2, 5.3, and 5.4, the theorems above reproduce and generalize results of Cannings and Holland [CH1], [CH2] describing $\mathcal{D}$-bundles on a nonsingular curve. More precisely, $\mathcal{D}$-bundles are classified by their deRham data, which are torsion-free sheaves on deep enough cusps (a rigified form of the deRham datum, known as a "fat sheaf", is used in [BGK2] to study quiver varieties). The deRham data are parametrized by an infinite-dimensional Grassmannian—the *adelic Grassmannian* $\mathbf{Gr}^{ad}(X)$ of [W]—which may be described as the direct limit of the compactified Picard varieties (moduli of rank 1 torsion-free sheaves) of the dripping curves Y. Note that the projective rank 1 *induced* $\mathcal{D}$-modules are classified simply by $\mathrm{Pic}(X)$; more generally, a submodule of a locally free, or even cusp-induced, $\mathcal{D}$-module with a finitely supported cokernel is cusp-induced in any dimension.

The adelic Grassmannian and related moduli spaces of projective $\mathcal{D}$-modules ("$\mathcal{D}$-bundles", [BD2]) have appeared recently in several contexts [BGK1], [BGK2], [BW1], [BW2]. In particular, the adelic Grassmannian was introduced by Wilson ([W] for $X = \mathbb{A}^1$) to collect the data for the algebraic (Krichever) solutions to the KP hierarchy coming from all cusp quotients Y of X, which we see are naturally described by $\mathcal{D}$-line bundles on the smooth curve X. In [BN1], we give a completely different (morally "Fourier dual") construction of solutions of the KP hierarchy from $\mathcal{D}$-bundles, in particular explaining the mysterious link between solitons and many-body (Calogero-Moser) systems discovered in [AMM], [Kr1], [Kr2] and deepened in [W]. In [BN2], we use $\mathcal{D}$-bundles (specifically, a factorization structure on the adelic Grassmannian) to give a geometric construction of the $\mathcal{W}_{1+\infty}$-vertex algebra (associated to the central extension of the Lie algebra of differential operators on the circle) and localization for its representations.

The reader may also wish to see [BW3] for a recent review of a problem closely related to the subject of the present paper, the problem of classifying affine varieties up to differential isomorphism, where cuspidal quotients of affine curves play an important role.

1.5. **Overview.** In Section 2 we prove the Grothendieck-Sato description of differential operators in the generality we require. In Section 3 we use the Grothendieck-Sato formula to compare the categories of $\mathcal{D}$-modules and stratifications. Section

4 contains the proof of the Morita equivalence for $\mathcal{D}$-modules under cuspidal quotients. Finally, in Section 5 we introduce cusp-induced $\mathcal{D}$-modules and describe their main properties.

2. Grothendieck-Sato formula for differential operators

Convention 2.1. *Fix a ground field k of any characteristic. For us, all schemes and morphisms are defined over k. By a* variety *we will mean an integral, separated scheme of finite type over k.*

2.1. **Cuspidal quotients and jets.** Recall that a morphism of schemes $f : X \to Y$ is a *universal homeomorphism* if for every morphism $Y' \to Y$ the pullback $f_{(Y')} : Y' \times_Y X \to Y'$ is a homeomorphism. It follows from Prop. IV.2.4.5 of [EGA] that a morphism $f : X \to Y$ of k-varieties is a universal homeomorphism if and only if f satisfies:

(1) f is a finite morphism;
(2) f is surjective;
(3) for every algebraically closed field K, the map $X(K) \xrightarrow{f(K)} Y(K)$ is injective.

Example 2.2. If Y is a k-variety, then the normalization map $\widetilde{Y} \to Y$ is a universal homeomorphism if and only if Y is geometrically unibranch (see Section IV.6.15 of [EGA])—this follows from Lemme 0.23.2.2 of [EGA].

Note that a variety is geometrically unibranch if and only if it is étale-locally irreducible (this is a consequence of [EGA, IV.18.8.15]).

Definition 2.3. A *cuspidal quotient morphism* $X \to Y$ is a universal homeomorphism between Cohen-Macaulay varieties X and Y over k.

Example 2.4. Suppose X is a nonsingular variety defined over a field k of characteristic p. Then the Frobenius morphism $X \to X'$ is a cuspidal quotient morphism (see Exp. XV, Section 1, Prop. 2 of [SGA5]).

Definition 2.5 (Jets)**.** For a variety X over k, write $\mathcal{J}_X = \mathcal{O}_{\widehat{X \times X}} = \varprojlim \mathcal{O}_{X\times X}/I_\Delta^k$, considered as a pro-coherent $\mathcal{O}_X$-bimodule. The bimodule $\mathcal{J}_X$, which is actually a Hopf algebroid, is called the *jet algebroid* of X. The associated formal groupoid $\widehat{\Delta} = \mathrm{FSpec}(\mathcal{J}_X)$ is the *jet groupoid* of X. More generally, for any finite morphism $f : X \to Y$ we write $\mathcal{J}_{Y\leftarrow X} = \varprojlim \mathcal{O}_{Y\times X}/I^k_{\Gamma_f}$ for the ring of functions on the formal completion of $Y \times X$ along the graph Γ_f of f, and $\mathcal{J}_{X\to Y}$ for the completion of the graph in $X \times Y$.

Remark 2.6 (Characteristic p). As we mentioned in the introduction, in characteristic p we work always with the full formal neighborhood of the diagonal as our jet groupoid, with the result that the ring $\mathcal{D}$ of differential operators that we consider is the full ring of differential operators (and, in particular, includes all divided powers). This has the advantage that it allows us to use Kashiwara's theorem in Sections 4 and 5 (which would not hold if we used a smaller subgroupoid).

Remark 2.7 (Pro-coherent sheaves). The jet algebroid is not quasicoherent, but rather *pro-coherent*, that is, it is a limit of a filtered system of coherent sheaves; in working with pro-coherent objects, one remembers the inverse system from which the limit arose, up to a relaxed notion of isomorphism (see the appendix of [AM] or

Deligne's appendix to [Ha1] for details). In working with these algebroids, all operations (tensor product, $\underline{\mathrm{Hom}}$, etc.) are taken in the category of pro-quasi-coherent sheaves, where the quasi-coherent sheaves are taken as constant inverse systems (in topological terms, with the discrete topology). So, for example, if $\{\mathcal{J}^n\}_{n\geq 0}$ is an inverse system of coherent $\mathcal{O}_X$-modules, the tensor product of $\{\mathcal{J}^n\}_{n\geq 0}$ with the quasicoherent $\mathcal{O}_X$-module M corresponds to the inverse system $\{M \otimes_{\mathcal{O}_X} \mathcal{J}^n\}$.

One may define appropriate notions of Hopf algebroid as well as quasicoherent comodule and (as we shall define later) cocomodule for a Hopf algebroid in the pro-quasicoherent category: here the required modification is that the morphisms for the coaction, counit, etc. are morphisms in the pro-category.

2.2. **$\mathcal{D}$-modules and local cohomology.** The Grothendieck-Sato formula describes the relationship between $\mathcal{D}_Y$ and the formal completion of the diagonal in $Y \times Y$, as we will explain.

Proposition 2.8. *Suppose X is a Cohen-Macaulay variety over k and $f : X \to Y$ is a finite morphism. Let $d = \dim(Y) = \dim(X)$. For any quasicoherent $\mathcal{O}_Y$-module M, one has*

$$\underline{\mathrm{Ext}}^i_Y(\mathcal{J}_{Y\leftarrow X}, M) = \underline{H}^{i+d}_{\Gamma_f}(Y \times X, M \boxtimes \omega_X)$$

for all $i \in \mathbf{Z}$. In particular, the local cohomology sheaf $\underline{H}^{i+d}_{\Gamma_f}(Y \times X, M \boxtimes \omega_X)$ vanishes for all $i < 0$.

Proof. The formula follows from Grothendieck duality. By assumption, X is of finite type; hence by Nagata's Theorem there is an embedding $X \overset{i}{\hookrightarrow} \overline{X}$ in a proper k-scheme $\overline{X}$. Let $\Gamma_f^{(k)}$ denote the kth-order neighborhood of the graph of f in $Y \times \overline{X}$. Let $\pi : Y \times \overline{X} \to Y$ denote projection on the first factor. Then for any quasicoherent $\mathcal{O}_Y$-module M, Grothendieck duality implies that

$$\mathbf{R}\pi_*\left(\mathbf{R}\underline{\mathrm{Hom}}_{Y\times\overline{X}}\left(\mathcal{O}_{\Gamma_f^{(k)}}, \pi^! M\right)\right) \simeq \mathbf{R}\underline{\mathrm{Hom}}_Y\left(\mathbf{R}\pi_*\mathcal{O}_{\Gamma_f^{(k)}}, M\right).$$

Because $\mathcal{O}_{\Gamma_f^{(k)}}$ is supported on the subscheme $\Gamma_f^{(k)} \subset Y \times \overline{X}$ that is finite over Y and on the scheme $Y \times X$, which is Cohen-Macaulay over Y, this reduces (see [Co]) to

$$\pi_*\underline{\mathrm{Ext}}^{i+d}_{Y\times X}(\mathcal{O}_{\Gamma_f^{(k)}}, M \boxtimes \omega_X) \simeq \underline{\mathrm{Ext}}^i_Y(\pi_*\mathcal{O}_{\Gamma_f^{(k)}}, M).$$

Taking colimits over k, the left-hand side becomes local cohomology, giving the desired formula and vanishing. □

We next wish to determine when the local cohomology groups of Proposition 2.8 vanish for $i > 0$; unfortunately, this is a slightly more complicated question.

Definition 2.9 (Good morphism, good CM variety)**.** A *good* cuspidal quotient morphism $f : X \to Y$ between Cohen-Macaulay varieties of dimension d is a cuspidal quotient morphism with the property that for any quasicoherent $\mathcal{O}_Y$-module M, the local cohomology sheaf $\underline{H}^{d+1}_{\Gamma_f}(Y \times X, M \boxtimes \omega_X)$ vanishes. The cuspidal quotient morphism f is *very good* if for any quasicoherent $\mathcal{O}_{Y\times X}$-module M, the local cohomology sheaf $\underline{H}^{d+i}_{\Gamma_f}(Y \times X, M)$ vanishes for all $i > 0$. A CM variety Y of dimension d is *good* (*very good*) if the identity morphism $\mathrm{id}_Y : Y \to Y$ is a good (respectively, very good) morphism.

Of course, a very good cuspidal quotient morphism is good.

Definition 2.10. We will call an integral closed subscheme $Z \subset W$ of an integral scheme W a *set-theoretic local complete intersection* if, for each point $z \in Z \subset W$, there is a regular sequence $x_1, \ldots, x_k$ in $\mathcal{O}_{W,z}$ such that $\operatorname{Spec} \mathcal{O}_{W,z}/(x_1, \ldots, x_k)$ is a nilpotent thickening of its closed subscheme $\operatorname{Spec} \mathcal{O}_{Z,z}$.

We next give some conditions that imply that a variety or morphism is very good.

Proposition 2.11.

(1) *If X is a smooth k-variety of dimension d, then X is very good.*
(2) *Suppose that $f : X \to Y$ is a cuspidal quotient morphism. Then the following are equivalent:*
   (a) *X is very good;*
   (b) *f is very good;*
   (c) *Y is very good.*
(3) *Suppose that $f : X \to Y$ is a cuspidal quotient morphism and that $\Gamma_f \subset Y \times X$ is a set-theoretic local complete intersection in $Y \times X$. Then f is very good.*

Proof. Part (1) is well known: it follows from the fact that Δ_X is a local complete intersection in $X \times X$ when X is smooth over k. For part (2), because we wish to compute local cohomology sheaves along Δ_X, Δ_Y and the graph of f, we may assume that X and Y are affine schemes. Let $U_Y \subset Y \times Y$ (respectively U_X, respectively U_f) denote the complement of Δ_Y in $Y \times Y$ (respectively of Δ_X in $X \times X$, respectively of Γ_f in $Y \times X$). It follows from the assumption that $X \to Y$ is a cuspidal quotient morphism that the natural maps $X \times X \xrightarrow{f \times 1} Y \times X \xrightarrow{1 \times f} Y \times Y$ are homeomorphisms and that the maps $\Delta_X \to X \times_Y X \to \Delta_Y$ are bijective. So $U_X = (f \times 1)^{-1}(U_f) = (f \times f)^{-1}(U_Y)$ and $(1 \times f)^{-1}(U_Y) = U_f$; moreover, the maps $U_X \to U_f \to U_Y$ are surjective and finite. By Proposition 1.1 of [Ha3], this implies that the cohomological dimensions of U_X, U_f and U_Y are the same. By Proposition 2.2 of [Ha2], one has $H^i(U, M) = H^{i+1}_\Gamma(M)$ for every quasicoherent M and every $i \geq 1$ (here Γ stands for one of Δ_X, Δ_Y or Γ_f and U stands for the corresponding open complement). Since we have reduced to the case of an affine ambient variety, the local cohomology sheaf is the sheaf associated to the module $H^{i+1}_\Gamma(M)$, and consequently the maximal degrees for nonzero local cohomology sheaves along Δ_X, Δ_Y and Γ_f all coincide.

For part (3), the question is local on Y. Thus, we may assume that there is a regular sequence $x_1, \ldots, x_d$ in $\mathcal{O}_{Y \times X}$ such that $\operatorname{Spec} \mathcal{O}_{Y \times X}/(x_1, \ldots, x_d)$ is a nilpotent thickening of Γ_f. But then the local cohomology along Γ_f coincides with local cohomology with respect to $(x_1, \ldots, x_d)$, which is computed by the direct limit of Koszul cohomology groups by Theorem 2.3 of [Ha2], hence vanishes above degree d. $\square$

Recall (following Grothendieck) that $\mathcal{D}_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y, \mathcal{O}_Y)$, and more generally $\mathcal{D}_Y(M, N) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y \otimes_{\mathcal{O}_Y} M, N)$ for any quasicoherent $\mathcal{O}_Y$-modules M and N; we will require also the relative version for a map $X \to Y$, replacing $\mathcal{J}$ by its analogs from Definition 2.5:

Definition 2.12.

(1) Let $\mathcal{D}_{X \to Y} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_{X \to Y}, \mathcal{O}_X)$.

(2) Let $\mathcal{D}_{Y\leftarrow X} = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, \mathcal{O}_Y)$.

Lemma 2.13. *Let $f : X \to Y$ be an affine morphism between k-varieties.*

(1) *The pro-coherent sheaf $\mathcal{J}_{Y\leftarrow X}$ (respectively $\mathcal{J}_{X\to Y}$) associated to the completion of $Y \times X$ ($X \times Y$) along the graph of f is $(1 \times f)^*\mathcal{J}_Y$ (respectively $(f \times 1)^*\mathcal{J}_Y$).*

(2) *If $X \xrightarrow{f} Y$ is a universal homeomorphism, then the jet algebroid of X is the pullback of the jet algebroid of Y; that is, $\mathcal{J}_X = (f \times f)^*\mathcal{J}_Y$ as pro-coherent Hopf algebroids.*

Proof. In (1), the statements for $\mathcal{J}_{Y\leftarrow X}$ and $\mathcal{J}_{X\to Y}$ hold because the ideal of Γ_f is generated by elements $a \otimes 1 - 1 \otimes a$ where a is a local section of $\mathcal{O}_Y$.

For (2), we have $\mathcal{J}_X = \varprojlim \mathcal{O}_{X\times X}/I^k_\Delta$ and $\mathcal{J}_Y = \varprojlim \mathcal{O}_{Y\times Y}/I^k_\Delta$. Locally on either X or Y, I_Δ is generated by elements of the form $a \otimes 1 - 1 \otimes a$ for local sections $a \in \mathcal{O}$. It follows that one has an exact sequence

$$(f \times f)^* I_{\Delta_Y} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{X\times_Y X} \to 0$$

on $X\times X$. In particular, we find that the image of $(f\times f)^*I_{\Delta_Y}$ in $\mathcal{O}_{X\times X}$ is contained in I_{Δ_X}. Moreover, since f is a universal homeomorphism, $\mathcal{O}_{X\times_Y X}$ is supported in a finite-order neighborhood of the diagonal in $X\times X$. It thus follows that the image of $(f \times f)^*I_{\Delta_Y}$ contains $I^n_{\Delta_X}$ for some $n \geq 1$. Therefore, the two limits coincide. Since the two inverse systems are cofinal, the pro-structures also agree. □

Corollary 2.14. *Suppose that $f : X \to Y$ is a cuspidal quotient morphism. Then $\mathcal{D}_{X\to Y}$ is canonically identified with the sheaf $\mathcal{D}_Y(\mathcal{O}_Y, \mathcal{O}_X)$ of $\mathcal{O}_Y$-differential operators from $\mathcal{O}_Y$ to $\mathcal{O}_X$, and similarly $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_Y(\mathcal{O}_X, \mathcal{O}_Y)$.*

Proof. By part (1) of Lemma 2.13 we have $\mathcal{J}_{X\to Y} = \mathcal{O}_X \otimes \mathcal{J}_Y$. Hence, we get $\mathcal{D}_{X\to Y} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X \otimes \mathcal{J}_Y, \mathcal{O}_X)$. The right-hand side equals $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y, \mathcal{O}_X)$ by the usual adjoint associativity, and this is $\mathcal{D}_Y(\mathcal{O}_Y, \mathcal{O}_X)$ by Grothendieck's definition of differential operators.

Part (1) of Lemma 2.13 also gives $\mathcal{J}_{Y\leftarrow X} = \mathcal{J}_Y \otimes \mathcal{O}_X$. So the equation $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_Y(\mathcal{O}_X, \mathcal{O}_Y)$ follows from Grothendieck's definition. □

See Lemma 4.2 and Remark 4.5 for an alternative definition of $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ and their relation with the standard bimodules of $\mathcal{D}$-module theory.

Theorem 2.15 (Grothendieck-Sato Formula)**.** *Suppose $f : X \to Y$ is a good cuspidal quotient morphism of k-varieties. Let $d = \dim(Y) = \dim(X)$. Then for any quasicoherent $\mathcal{O}_Y$-module M,*

$$\underline{H}^d_{\Gamma_f}(Y \times X, M \boxtimes \omega_X) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, M) = M \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y\leftarrow X}. \tag{2.1}$$

Proof. The first equality of (2.1) is immediate from Proposition 2.8. To prove the second equality of (2.1), observe that the vanishing of $\underline{\mathrm{Ext}}^1_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, M) = \underline{H}^{d+1}_{\Gamma_f}(Y \times X, M \boxtimes \omega_X)$ for every quasicoherent M implies that $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, \cdot)$ is exact, and therefore this exactness follows from f being good. Given a finite presentation $\mathcal{O}^A_Y \xrightarrow{\alpha} \mathcal{O}^B_Y \xrightarrow{\beta} M \to 0$ of a coherent $\mathcal{O}_Y$-module M, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y^A \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \arrow[r] \arrow[d, "\cong"] & \mathcal{O}_Y^B \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \arrow[r] \arrow[d, "\cong"] & M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \arrow[r] \arrow[d] & 0 \\
\underline{\mathrm{Hom}}_Y(\mathcal{J}_{Y\leftarrow X}, \mathcal{O}_Y^A) \arrow[r] & \underline{\mathrm{Hom}}_Y(\mathcal{J}_{Y\leftarrow X}, \mathcal{O}_Y^B) \arrow[r] & \underline{\mathrm{Hom}}_Y(\mathcal{J}_{Y\leftarrow X}, M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, which proves (2.1) for coherent M. Finally, local cohomology [Ha2, Proposition 1.12] and tensor product both commute with colimits, implying (2.1) for all quasicoherent M. □

Corollary 2.16. *For $f : X \to Y$ as in Theorem 2.15, the sheaf $\mathcal{D}_{Y\leftarrow X}$ is flat over $\mathcal{O}_Y$. In particular, if Y is a good Cohen-Macaulay variety, then $\mathcal{D}_Y$ is flat as a left $\mathcal{O}_Y$-module.*

Applying the theorem to the identity morphism of a good Cohen–Macaulay variety, we obtain the corollary:

Corollary 2.17. *The Grothendieck–Sato formula*

$$\mathcal{D}_Y = \underline{H}^d_\Delta(Y \times Y, \mathcal{O}_Y \boxtimes \omega_Y)$$

holds for good Cohen–Macaulay varieties Y.

3. Differential operators, jets and stratifications

In this section, after reviewing some of the formalism of descent, we compare left and right $\mathcal{D}$-modules with stratifications and costratifications, or equivalently jet comodules and jet cocomodules, on a good Cohen-Macaulay variety Y. These comparisons follow from the extension of the Grothendieck-Sato formula, Theorem 2.15 (applied to the identity map of Y). We also explain the descent for crystals under cuspidal quotients. An excellent reference for stratifications and costratifications in the smooth setting is provided by [Be2], [Be3].

3.1. **$*$-Equivariance.** For background on the relationship between groupoids and Hopf algebroids, see, for example, [Ho].

Definition 3.1. Suppose $\mathcal{G} \rightrightarrows X$ is a groupoid over X (we suppress the unit in this notation). We say that an $\mathcal{O}_X$-module M is *equivariant with respect to the groupoid* $\mathcal{G} \rightrightarrows X$ if M is equipped with an isomorphism $p_1^*M \to p_0^*M$ that is compatible with composition over $\mathcal{G} \times_X \mathcal{G}$ in the usual sense (see Section 1.6 of [De]).

We work with groupoids that are affine or pro-affine over X, and write $\mathcal{H}$ for the Hopf algebroid associated to $\mathcal{G}$. By the adjunction of pushforward and pullback, we have

$$\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{G}}(p_1^*M, p_0^*M) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(M, p_{1*}p_0^*M).$$

We identify $p_{1*}p_0^*M = M \underset{\mathcal{O}_X}{\otimes} \mathcal{H}$, considered as an $\mathcal{O}_X$-module via the right $\mathcal{O}_X$-structure on $\mathcal{H}$ and where the tensor product is taken over the left structure.

Lemma 3.2. *The category of $\mathcal{G}$-equivariant modules is equivalent to the category of right $\mathcal{H}$-comodules that are counital in the sense that*

$$M \to M \underset{\mathcal{O}_X}{\otimes} \mathcal{H} \xrightarrow{\text{counit}} M \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X = M$$

is the identity.

Remark 3.3. The counital condition on $\mathcal{H}$-comodules is equivalent to the requirement that the corresponding map $p_0^*M \to p_1^*M$ is an isomorphism.

Proposition 3.4. *Suppose $f : X \to Y$ is affine. Let $\mathcal{H}_Y$ be a Hopf algebroid on Y, and $\mathcal{H}_X = (f \times f)^*\mathcal{H}_Y$. Then $\mathcal{H}_X$ is a Hopf algebroid on X and f^* induces a functor from* $\mathrm{comod}(\mathcal{H}_Y)$ *to* $\mathrm{comod}(\mathcal{H}_X)$.

Proof. The Hopf algebroid structure on $\mathcal{H}_X$ is defined as follows: the comultiplication on $\mathcal{H}_X = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{H}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_X$ is given by means of the comultiplication on $\mathcal{H}_Y$ and the map $\mathcal{H}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{H}_Y \to \mathcal{H}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{H}_Y$ induced from the structure morphism $\mathcal{O}_Y \to \mathcal{O}_X$. The other structures of Hopf algebroid are immediate, as is the functor on comodules. $\square$

3.2. **!-Equivariance.** We now replace the adjoint pair of functors (f^*, f_*) by the adjoint pair $(f_*, f^!)$.

Definition 3.5 (See Section 7.10 of [BD1])**.** For a finite morphism of schemes f, or more generally for an ind-finite morphism of formal schemes, the pushforward $f_* : \mathcal{O}_X\text{-}\mathrm{mod} \to \mathcal{O}_Y\text{-}\mathrm{mod}$ has a right adjoint $f^!$, defined as follows: for an $\mathcal{O}_Y$-module N, $f^!N$ is the $\mathcal{O}_X$-module corresponding to the $f_*\mathcal{O}_X$-module $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, N)$ on Y.

In our situation, f is affine; consequently, we will omit the notation f_* for the pushforward unless we wish to emphasize that we are forgetting the $\mathcal{O}_X$-module structure down to $\mathcal{O}_Y$.

Let N be an $\mathcal{O}_X$-module that is !-equivariant with respect to the groupoid $\mathcal{G}$ over X: in other words, we are given an isomorphism $p_0^!N \to p_1^!N$ compatible with compositions. By the adjunction of !-pullback and pushforward, we have

$$\underline{\mathrm{Hom}}_{\mathcal{G}}(p_0^!N, p_1^!N) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(p_{1*}p_0^!N, N).$$

It follows that the analog of Lemma 3.2 identifies the notion of !-equivariant sheaf with that of *cocomodule*:

Definition 3.6 (Cocomodules)**.** Let $\mathcal{H}$ be a Hopf algebroid on X. An $\mathcal{H}$-cocomodule N is an $\mathcal{O}_X$-module N, equipped with a morphism $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, N) \to N$, so that the two compositions $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H} \otimes \mathcal{H}, N) \to N$ given by the coproduct

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H} \otimes \mathcal{H}, N) \xrightarrow{\Delta^*} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, N) \to N$$

and the induced map

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H} \otimes \mathcal{H}, N) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, N)) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, N) \to N$$

agree. We further require N to be unital, so that the composition

$$N = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, N) \xrightarrow{\text{counit}} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, N) \to N$$

is the identity.

Remark 3.7. If $\mathcal{H}$ is an $\mathcal{O}_X$-coalgebra that is *projective* over $\mathcal{O}_X$, with dual algebra $\mathcal{H}^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{O}_X)$, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, M) = M \otimes \mathcal{H}^*$ and the structure of $\mathcal{H}$-cocomodule is equivalent to that of right $\mathcal{H}^*$-module.

Proposition 3.8. *Suppose $f : X \to Y$ is finite. Let $\mathcal{H}_Y$ be a Hopf algebroid on Y and $\mathcal{H}_X$ the corresponding algebroid on X, $\mathcal{H}_X = (f \times f)^*\mathcal{H}_Y$. Then $f^!$ induces a functor from $\mathcal{H}_Y$-cocomodules to $\mathcal{H}_X$-cocomodules.*

Proof. Let $(p_{0,Y}, p_{1,Y}) : \mathcal{G}_Y \to Y \times Y$ be the groupoid corresponding to $\mathcal{H}_Y$ and $(p_{0,X}, p_{1,X}) : \mathcal{G}_X \to X \times X$ be that corresponding to $\mathcal{H}_X$. Thus $f \circ p_{i,X} = p_{i,Y} \circ (f \times f)$. If N is an $\mathcal{H}_Y$-cocomodule, that is, an $\mathcal{O}_Y$-module with an isomorphism $p_{0,Y}^! N \to p_{1,Y}^! N$ satisfying a composition rule, we obtain an isomorphism

$$p_{0,X}^! f^! N = (f \times f)^! p_{0,Y}^! N \to (f \times f)^! p_{1,Y}^! N = p_{1,X}^! f^! N,$$

which defines the desired $\mathcal{H}_X$- cocomodule structure on $f^! N$. □

3.3. **Stratifications and costratifications coincide with $\mathcal{D}$-modules.**

Definition 3.9.

(1) A stratification M on X is a quasicoherent sheaf with a right comodule structure for the jet algebroid on X. Equivalently, M is a sheaf equivariant with respect to the deRham groupoid $\widehat{X \times X}$, or a sheaf equipped with an isomorphism $p_1^* M \to p_0^* M$ on $\widehat{X \times X}$ compatible with composition on the triple product.

(2) A costratification M on X is a quasicoherent sheaf with a cocomodule structure for the jet algebroid on X. Equivalently, M is a sheaf !-equivariant with respect to the deRham groupoid $\widehat{X \times X}$, or a sheaf equipped with an isomorphism $p_0^! M \to p_1^! M$ on $\widehat{X \times X}$ compatible with composition on the triple product.

Remark 3.10 (Left and Right). It is clear from the description in terms of equivariance for a groupoid that the datum of a counital right comodule is equivalent to the datum of a counital left comodule. When one dualizes a comodule structure, however, to obtain a module over the algebra dual to the Hopf algebroid, one makes a choice—in our case, the algebra $\mathcal{D}$ is the *left* dual of jets—which breaks the symmetry. This explains our insistence on the use of right comodules.

Theorem 3.11. *Suppose Y is a good Cohen-Macaulay variety.*

(1) *The categories of quasicoherent $\mathcal{J}_Y$-cocomodules (or costratifications on Y) and right $\mathcal{D}_Y$-modules are equivalent.*

(2) *The categories of quasicoherent right $\mathcal{J}_Y$-comodules (or stratifications on Y) and left $\mathcal{D}_Y$-modules are equivalent.*

Recall that quasicoherence for $\mathcal{J}_Y$-cocomodules and $\mathcal{D}_Y$-modules means quasicoherence over $\mathcal{O}_Y$.

Proof. The first assertion, the equivalence of $\mathcal{J}_Y$-cocomodules and right $\mathcal{D}_Y$-modules, is immediate from the formula $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y, M) = M \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ of (2.1) using standard methods (see [Be3, Prop. 1.1.4] for the smooth case).

The proof of the second assertion also follows the standard outline (see [Be1, II, 4.1–4.2]), using a general algebraic fact that we now explain. To begin, let L and R be commutative rings, and let $Q = \varprojlim Q_n$ denote a pro-object in the category of (L, R)-bimodules that are finitely generated over L. Let $P = \varinjlim \mathrm{Hom}_L(Q_n, L)$ denote the L-dual of Q. For simplicity, we assume that the maps $Q_{n+1} \to Q_n$ are surjections.

Lemma 3.12. *If the natural map $N \otimes_L P \to \mathrm{Hom}_L(Q, N)$ is an isomorphism for all L-modules N, then the natural map*

$$\mathbf{F} : \mathrm{Hom}_L(P \otimes_R M, N) \to \mathrm{Hom}_R(M, N \otimes_L Q) \tag{3.1}$$

is an isomorphism of (L, R)*-bimodules for all* L*-modules* N *and* R*-modules* M*.*

The method of proof is standard (with the usual modifications for the category of pro-objects):

Proof of Lemma. The hypotheses of the lemma give a canonical element $1_Q \in \mathrm{Hom}_L(Q, Q) = Q \otimes_L P$. Using the action of R on P, this element determines a map $\delta : R \to Q \otimes_L P$. We also have the natural contraction maps $\mathrm{tr}_n : P_n \otimes Q_n \to L$ for all n; by abuse of notation, we write this system of homomorphisms as $\mathrm{tr} : P \otimes Q \to L$. The hypotheses of the lemma give identities

$$(1_Q \otimes \mathrm{tr}) \circ (\delta \otimes 1_Q) = 1_Q \ \text{ and } \ (\mathrm{tr} \otimes 1_P) \circ (1_P \otimes \delta) = 1_P \tag{3.2}$$

(where the first is an identity in the pro-category).

Recall that the map $\mathbf{F}$ of (3.1) is given by taking $\phi : P \otimes M \to N$ to the composite $M \xrightarrow{M \otimes \delta} M \otimes P \otimes Q \xrightarrow{\phi \otimes Q} N \otimes Q$. Similarly, one may define a map $\mathbf{F}^{-1}$ by taking $\psi : M \to N \otimes Q$ to the composite $M \otimes P \xrightarrow{\psi \otimes P} N \otimes Q \otimes P \xrightarrow{N \otimes \mathrm{tr}} N$. It is then straightforward to check that $\mathbf{F}$ and $\mathbf{F}^{-1}$ are mutually inverse using (3.2). □

Part (2) of Theorem 3.11 now follows by taking $L = R = \mathcal{O}_Y$, $Q = \mathcal{J}_Y$, $P = \mathcal{D}_Y$, and $M = N$: the hypothesis of Lemma 3.12 follows from Equation (2.1), and so by the lemma we obtain from (3.1) an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} M, M) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(M, M \otimes_{\mathcal{O}_Y} \mathcal{J}_Y).$$

The compatibilities required for module and comodule structures are immediate from the constructions. □

Corollary 3.13. *Suppose* $f : X \to Y$ *is a cuspidal quotient morphism between good CM varieties. Then the naive functors* f^* *and* $f^!$ *on* $\mathcal{O}_Y$*-modules determine pullback functors from the categories of left and right* $\mathcal{D}_Y$*-modules to the categories of left and right (respectively)* $\mathcal{D}_X$*-modules.*

This follows from the existence of the corresponding pullbacks for jet comodules and cocomodules.

3.4. **Descent for crystals.** The theory of !-crystals is developed in [BD1], section 7.10, whose definitions we follow. It is important to note that even in characteristic p we allow *all* nilpotent thickenings (not just those with fixed divided power structure) in the definition of a !-crystal, and thus our notion does not coincide with the more usual crystalline terminology in characteristic p.

Let $f : X \to Y$ denote a cuspidal quotient of a smooth variety X, $i : Y \to Z$ a closed embedding into a smooth variety Z, and $g = i \circ f : X \to Z$. In this section we describe descent for !-crystals, namely the statement that the functor $f^!$ induces an equivalence of categories between !-crystals on X and !-crystals on Y. In fact, !-crystals on X are equivalent to right $\mathcal{D}_X$-modules while, thanks to the Kashiwara theorem for !-crystals, !-crystals on Y are equivalent to right $\mathcal{D}_Z$-modules supported on Y. Thus it suffices to prove that the standard $\mathcal{D}$-module functors define an equivalence of these two categories. The result seems to be well known, but since we could not find a reference we sketch a proof for the benefit of the reader. We are grateful to Dennis Gaitsgory for very valuable discussions of !-crystals and their properties.

Proposition 3.14. *The functors g_* and $g^!$ define quasi-inverse equivalences of the categories* $\mathrm{mod}-\mathcal{D}_X$ *of right $\mathcal{D}_X$-modules and* $(\mathrm{mod}-\mathcal{D}_Z)_Y$ *of right $\mathcal{D}_Z$-modules supported on Y.*

Proof. We use the adjunction between $g^!$ and g^* (note that g is proper). We first prove that the adjunction $Id \to g^! g_*$ is an isomorphism. Let $p_1, p_2 : X \times_Z X$ denote the two projections from the descent groupoid of X over Z to X. Then we have a natural base change equivalence $g^! g_* = (p_2)_*(p_1)^!$. Now by the Kashiwara equivalence for right $\mathcal{D}_{X\times X}$-modules supported on the diagonal $\Delta : X \to X \times X$ with right $\mathcal{D}_X$-modules, we have an isomorphism $(p_1)^! = \Delta_*$. It follows that $g^! g_* = (p_2)_*(p_1)^! = (p_2)_*\Delta_* = Id$ as desired.

For the converse, the isomorphism property of the adjunction $g_* g^! \to Id$, we use a flattening stratification of $f : X \to Y$. Note that for a *flat* cuspidal quotient, the equivalence of categories is a consequence of flat $*$- and !-descent for coherent sheaves—the proof is as in Section 2.4 of [Be3]. The conclusion now follows by induction: assuming (by way of the inductive hypothesis) the isomorphism for the subcategory of modules supported on a closed subvariety V which is a union of strata, we wish to add a stratum C to obtain a closed subvariety $V \cup C$ and check the isomorphism there. But the canonical (Cousin) decompositions of $g_* g^! M$ and M agree both on V and (by the stratum-by-stratum equivalence) on C; consequently they agree on all of $V \cup C$. It follows that the adjunction morphism is an isomorphism as claimed. □

4. Cusp Morita equivalence

In this section, we will prove that tensor products with the standard bimodules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ of $\mathcal{D}$-module theory give Morita equivalences of categories of $\mathcal{D}$-modules under cuspidal quotient morphisms. The functors from $\mathcal{D}_Y$-modules to $\mathcal{D}_X$-modules are given by $*$ and !-pullback, and one should think of the inverse functors as descent functors in the categories of jet (co)comodules.

Throughout this section, we fix a cuspidal quotient morphism $X \xrightarrow{f} Y$.

4.1. Morita equivalence.

Lemma 4.1. *Suppose $f : X \to Y$ is a finite, dominant map, and that M is a nonzero quasicoherent $\mathcal{O}_Y$-module. Then $f^! M$ is nonzero.*

Proof. Suppose $M \neq 0$. By left exactness of $\underline{\mathrm{Hom}}$, we may assume that $M = \mathcal{O}_Y/P$ for some prime P of $\mathcal{O}_Y$, and may further localize $\mathcal{O}_Y$ at P (which we suppress in our notation). Then $f^! M$ becomes $\underline{\mathrm{Hom}}_Y(\mathcal{O}_X, \mathcal{O}_Y/P) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y/P}(\mathcal{O}_X \otimes \mathcal{O}_Y/P, \mathcal{O}_Y/P)$, the dual of a vector space over the field $\mathcal{O}_Y/P$. Since $\mathcal{O}_X$ is supported at every point of Y, $\mathcal{O}_X \otimes \mathcal{O}_Y/P$ is nonzero, completing the proof. □

Lemma 4.2. *Suppose that $f : X \to Y$ is a cuspidal quotient morphism. Then:*

(1) $(1 \times f)^! \mathcal{D}_Y = \mathcal{D}_{Y\leftarrow X}$;
(2) *if Y is good, then* $(f \times 1)^* \mathcal{D}_Y = \mathcal{D}_{X\to Y}$;
(3) *if f is good, then* $(f \times 1)^* \mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X$;
(4) $(1 \times f)^! \mathcal{D}_{X\to Y} = \mathcal{D}_X$.

Furthermore, if Y and f are good, then for any quasicoherent $\mathcal{O}_Y$-module M we have

$$(1 \times f)^!(M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y) = M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}. \tag{4.1}$$

Proof. For (1), we have

$$(1\times f)^!\mathcal{D}_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{D}_Y) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y,\mathcal{O}_Y))$$
$$= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_X,\mathcal{O}_Y) = \mathcal{D}_{Y\leftarrow X}$$

using the definitions, adjunction, and part (1) of Lemma 2.13.

Parts (2) and (3) follow from the identifications

$$(f\times 1)^*\mathcal{D}_Y = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y,\mathcal{O}_X) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_{X\to Y},\mathcal{O}_X),$$

$$(f\times 1)^*\mathcal{D}_{Y\leftarrow X} = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X},\mathcal{O}_X) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_X,\mathcal{O}_X)$$

using the Grothendieck-Sato formula, adjunction and Lemma 2.13.

For part (4), we have

$$(1\times f)^!\mathcal{D}_{X\to Y} = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_{X\to Y},\mathcal{O}_X)) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_X,\mathcal{O}_X) = \mathcal{D}_X$$

using Lemma 2.13.

Finally, (4.1) follows by computing

$$(1\times f)^!(M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y,M)) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_X, M)$$
$$= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X},M) = M\otimes_{\mathcal{O}_Y}\mathcal{D}_{Y\leftarrow X}$$

using the Grothendieck-Sato formula. □

Theorem 4.3 (Cusp Morita equivalence)**.** *Suppose $f: X\to Y$ is a good cuspidal quotient morphism of good CM varieties. Then the bimodules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ induce Morita equivalences of the categories of (left or right) $\mathcal{D}_X$-modules and $\mathcal{D}_Y$-modules.*

Proof. We will show that $\mathcal{D}_{X\to Y}\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X$ and $\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y} = \mathcal{D}_Y$; the equivalences are then given by tensoring with one or the other of these bimodules.

It follows immediately from the Grothendieck-Sato formula and Lemma 4.2 that $\mathcal{D}_{X\to Y}\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X$: indeed, we have

$$\mathcal{D}_{X\to Y}\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X.$$

Observe that this morphism is just the one obtained by viewing $\mathcal{D}_{Y\leftarrow X}$ and $\mathcal{D}_{X\to Y}$ as sheaves of differential operators (as in Lemma 2.14) and taking $\phi\otimes\psi$ to the composite differential operator $\phi\circ\psi$ from $\mathcal{O}_X$ to $\mathcal{O}_X$.

Consider the natural map $\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\xrightarrow{\Phi}\mathcal{D}_Y$, which again comes from viewing $\mathcal{D}_{Y\leftarrow X}$ and $\mathcal{D}_{X\to Y}$ as sheaves of differential operators and taking composites. Tensoring on the right by $\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X}$ gives

$$\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_{Y\leftarrow X}\to\mathcal{D}_{Y\leftarrow X},$$

which is the identity map; letting $M = \mathrm{coker}(\Phi)$, we thus have $M\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} = 0$.

There is a coequalizer diagram

$$M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \rightrightarrows M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \to M\otimes_{\mathcal{D}_Y}\mathcal{D}_Y = M \tag{4.2}$$

defining the $\mathcal{D}$-tensor product for any quasicoherent right $\mathcal{D}_Y$-module M. The map $M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \to M$ is split surjective as a map of right $\mathcal{O}_Y$-modules (thanks to the unit map $\mathcal{O}_Y \to \mathcal{D}_Y$). Taking $f^!$ of (4.2), then, gives a surjective map

$$f^!(M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y) = M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \to f^!(M); \tag{4.3}$$

here the equality on the left follows from (4.1). The composites of (4.3) with the two arrows

$$f^!(M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y) = M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \rightrightarrows M \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \tag{4.4}$$

agree by construction. This implies that the coequalizer of (4.4), which is $M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}$, must map surjectively to $f^!(M)$. In particular, by Lemma 4.1, $M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X} = 0$ implies $M = 0$.

The conclusion of the last paragraph applied to $M = \operatorname{coker}(\Phi)$ thus implies that Φ is surjective. It then follows that Φ is injective as well; for, letting $K = \ker(\Phi)$, applying f^* to the exact sequence

$$0 \to K \to \mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \xrightarrow{\Phi} \mathcal{D}_Y \to 0$$

gives a sequence

$$f^*K \to \mathcal{D}_{X\to Y} \to \mathcal{D}_{X\to Y} \to 0,$$

in which the map from f^*K must be injective because $\mathcal{D}_Y$ is flat over $\mathcal{O}_Y$ (Corollary 2.16). But then $f^*K = 0$, which implies that the surjective map

$$\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} f^*K = \mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} K \to K$$

must have zero image, i.e., $K = 0$. This completes the proof of the theorem. □

Combining this result with Proposition 3.14 we obtain the following:

Corollary 4.4. *Suppose $f : X \to Y$ is a cuspidal quotient morphism from a smooth k-variety X. Then:*

(1) *The bimodules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ induce Morita equivalences of the categories of (left and right) $\mathcal{D}_X$-modules and $\mathcal{D}_Y$-modules.*
(2) *The category of right $\mathcal{D}_Y$-modules is equivalent to the category of $!$-crystals on Y and thus is equivalent to the category of right $\mathcal{D}_Z$-modules supported on Y for any closed embedding $Y \hookrightarrow Z$ of Y in a smooth variety Z.*

Remark 4.5. The bimodules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ (Definition 2.12) are adaptations to the singular context of the standard bimodules used for pushforward and pullback of $\mathcal{D}$-modules. The standard definition of the bimodule $\mathcal{D}_{X\to Y}$ (used for pullback of left $\mathcal{D}$-modules) is $\mathcal{D}_{X\to Y} = (f \times 1)^*\mathcal{D}_Y$, which agrees with our definition thanks to Lemma 4.2. Thus for any left $\mathcal{D}_Y$-module, we have a canonical isomorphism of $\mathcal{D}_X$-modules $f^*M = \mathcal{D}_{X\to Y} \underset{\mathcal{D}_Y}{\otimes} M$. Note, in particular, that under our equivalence, the left $\mathcal{D}_Y$-module $\mathcal{O}_Y$ corresponds to the left $\mathcal{D}_X$-module $\mathcal{O}_X$.

The definition given for $\mathcal{D}_{Y\leftarrow X}$ is likewise the replacement, for the not necessarily Gorenstein variety Y, of the standard definition of the bimodule $\mathcal{D}_{Y\leftarrow X}$ using the canonical line bundle. Recall that for a morphism of *smooth* varieties $f : X \to Y$, one defines an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\mathcal{D}_{Y\leftarrow X}$ as $\omega_X \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} \omega_Y^{-1})$, where we first turn the right $\mathcal{D}_Y$-structure of $\mathcal{D}_Y$ into a left structure, pull back to a left $\mathcal{D}_X$-module, and then reconvert to a right $\mathcal{D}_X$-module, using the canonical sheaves of Y and X. Thus, for a smooth morphism f we may write $\mathcal{D}_{Y\leftarrow X} = f^*\mathcal{D}_Y \underset{\mathcal{O}_X}{\otimes} \omega_{X/Y}$, the tensor product with the relative canonical bundle (here pullback is along the right structure—otherwise one utilizes the transposition isomorphism [Be3]). However, for a smooth affine morphism we also have the identification of $f^!M$ with the $\mathcal{O}_X$-module corresponding to the $f_*\mathcal{O}_X$-module $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, M) = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)\otimes M$, namely $f^!M = f^*M\otimes\omega_{X/Y}$. Thus, in this case, $\mathcal{D}_{Y\leftarrow X} = (1\times f)^!\mathcal{D}_Y$ and our definitions agree (again by Lemma 4.2).

Note also that for any quasicoherent right $\mathcal{D}_Y$-module M we have $f^!M = M\otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}$. Indeed, the proof of Theorem 4.3 shows that the map $M\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\leftarrow X} \to f^!M$ is surjective. It then follows by a diagram chase that if $M \to N$ is a surjective map of quasicoherent right $\mathcal{D}_Y$-modules then $f^!M \to f^!N$ is surjective. Thus, $f^!$ is an exact functor on quasicoherent right $\mathcal{D}_Y$-modules. Since $f^!\mathcal{D}_Y^I = \mathcal{D}_Y^I \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}$, taking a presentation of M we find that $f^!M = M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}$ as well.

The definition using jets or equivalently !-pullback has obvious functorial advantages over the definition using the canonical sheaf and, as we have seen, has the correct role in the cuspidal setting. Thus the Morita equivalences of Theorem 4.3 are given by the suitable adaptations of the pullback and pushforward functors for $\mathcal{D}$-modules.

4.2. **Dripping varieties.**

Definition 4.6. The *deRham space* Y_{dR} of a variety Y is the quotient, in the category of spaces (that is, *fppf* sheaves of sets), of Y by the formal groupoid $\widehat{Y\times Y}$.

As a functor, Y_{dR} assigns to a scheme S the set $Y_{dR}(S) = Y(S^{red})$, the set of Y-points of the reduced scheme of S. As a k-space, Y_{dR} is far from being algebraic.

Definition 4.7. A ($*$-)coherent sheaf on Y_{dR} is an equivariant sheaf for the formal groupoid of the diagonal under $*$-pullback, in other words a comodule for $\mathcal{J}_Y$. We can also define a !-coherent sheaf on Y_{dR} as an equivariant sheaf under !-pullback, namely a $\mathcal{J}_Y$-cocomodule.

For $f : X \to Y$ a cuspidal quotient, the deRham spaces X_{dR} and Y_{dR} are very similar: the groupoid $X\times_Y X$ defining the quotient map $X \to Y$ is a subgroupoid of the deRham groupoid of X. So we may expect the map $X \to X_{dR}$ to factor through $X \to Y$ and to identify the deRham spaces of X and Y. The directed system of all cusp quotients of X can be considered a finitary approximation of the deRham quotient. Thus we successively quotient out by increasing finite infinitesimal equivalence relations in order to approach the quotient of X by the full infinitesimal nearness relation $\widehat{X\times X}$: the smooth variety X "drips" down towards X_{dR} through the cuspidal varieties Y.

Corollary 4.8. *Suppose that $X \to Y$ is a good cuspidal quotient morphism between good CM varieties X and Y. Then the deRham spaces X_{dR} and Y_{dR} have equivalent categories of $*$-coherent sheaves and !-coherent sheaves (respectively).*

Thus the dripping varieties picture is accurate on the level of coherent sheaves: the ($*$ or !) pullback of $\mathcal{O}_Y$-modules from deepening cusps to X gives rise to sheaves with an increasingly large piece of a (left or right) $\mathcal{D}_X$-module structure, while the pullback of $\mathcal{D}$-modules from any cusp to X gives rise to all $\mathcal{D}_X$-modules.

5. Cusp induction

Throughout this section, $X \xrightarrow{f} Y$ will always denote a good cuspidal quotient morphism between good Cohen-Macaulay varieties over the field k, and all $\mathcal{D}$-modules and $\mathcal{O}$-modules are assumed to be quasicoherent.

Notation 5.1. *Let $\pi_{X\times X}, \pi_{Y\times X}$, and $\pi_{Y\times Y}$ be the projections onto the first factor from the formal completions of $X \times X$ along the diagonal, $Y \times X$ along the graph of f, and $Y \times Y$ along the diagonal, respectively.*

We recall the notion of induced $\mathcal{D}$-module from [Sa] (see also [BD2]). There is an exact faithful functor $\mathrm{Ind}_X : \mathcal{O}_X{-}\mathrm{mod} \to \mathrm{mod}{-}\mathcal{D}_X$ that sends an $\mathcal{O}_X$-module M to the induced $\mathcal{D}_X$-module

$$\mathrm{Ind}_X(M) = M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X = \underline{\mathrm{Hom}}_{\text{pro-mod}(\mathcal{O}_X)}(\mathcal{J}_X, M) = \pi^!_{X\times X} M.$$

The functor naturally lands in $(\mathcal{O}_X, \mathcal{D}_X)$-bimodules, and we then forget the commuting $\mathcal{O}_X$-structure.

One may define a similar induction functor from $\mathcal{O}_Y$-modules to $\mathcal{J}_Y$-cocomodules for any scheme Y. We wish to construct a large category of $\mathcal{D}_X$-modules on X, sharing some of the good properties of induced $\mathcal{D}_X$-modules, by collecting induced modules from all cuspidal quotients of X.

5.1. Exactness of cusp induction.

As before, let $\mathcal{J}_{Y\leftarrow X}$ denote the pro-coherent $(\mathcal{O}_Y, \mathcal{O}_X)$-bimodule (and commutative algebra) of functions on the formal completion $\widehat{Y \times X}$ of $Y \times X$ along the graph of $f : X \to Y$.

Proposition 5.2.

1. *The functor*
$$M_Y \mapsto \pi^!_{Y\times X} M_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, M_Y)$$
on $\mathcal{O}_Y{-}\mathrm{mod}$ is exact and equivalent to $M_Y \mapsto M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}$.
2. *The functor $\pi^!_{Y\times X}$ may be refined to an exact functor from $\mathcal{O}_Y{-}\mathrm{mod}$ to $\mathrm{mod}{-}\mathcal{D}_X$, the functor of* cusp induction *$M_Y \mapsto \mathrm{Ind}_{Y\leftarrow X} M_Y$.*

Proof. Part (1) is proven in Section 2.2. For part (2), the functor $M_Y \mapsto M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}$ clearly lands in the category of right $\mathcal{D}_X$-modules, and since the underlying functor to $\mathcal{O}_X$-modules is exact, it follows that the refined functor is exact as well. $\square$

5.2. Cusp-induced modules.

Let M_Y denote an $\mathcal{O}_Y$-module, and $M_X = f^* M_Y$ the corresponding $\mathcal{O}_X$-module. Let $\mathrm{Ind}_X M_X = M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ and $\mathrm{Ind}_Y M_Y = M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y$ be the corresponding induced right $\mathcal{D}_X$-module and right $\mathcal{D}_Y$-module. The cusp-induced $\mathcal{D}_X$-module $\mathrm{Ind}_{Y\leftarrow X} M_Y = M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}$ is an intermediate object.

Lemma 5.3. *We have canonical isomorphisms*

1. $(f\times 1)^* \mathrm{Ind}_{Y\leftarrow X} M_Y = \mathrm{Ind}_X M_X$ *with respect to the left $\mathcal{O}_Y$-structure on $\mathrm{Ind}_{Y\leftarrow X} M_Y$, and*
2. $(1\times f)^! \mathrm{Ind}_Y M_Y = \mathrm{Ind}_{Y\leftarrow X} M_Y$ *with respect to the right $\mathcal{O}_Y$-structure on $\mathrm{Ind}_Y M_Y$.*

This follows immediately from Lemma 4.2.

Recall from Section 2.1 the definition of $\mathcal{D}_X(N, M)$.

Notation 5.4. *If M_Y is an $\mathcal{O}_Y$-module equipped with an $\mathcal{O}_Y$-embedding $M_Y \subseteq M$ in an $\mathcal{O}_X$-module M and N is an $\mathcal{O}_X$-module, we let*

$$\mathcal{D}(N, M_Y) \subseteq \mathcal{D}_X(N, M)$$

denote the subsheaf that consists of those operators the image of which lies in M.

Proposition 5.5. *Suppose that X is a smooth k-variety. With the above notation, we have*

$$\mathcal{D}_Y(\mathcal{O}_X, M_Y) = \mathcal{D}(\mathcal{O}_X, M_Y).$$

Proof. Because Y is a cuspidal quotient of a smooth k-variety, it is very good Cohen-Macaulay; hence the functor $M_Y \mapsto \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, M_Y)$ is exact. In particular, the embedding $M_Y \to M$ induces an embedding

$$\mathcal{D}_Y(\mathcal{O}_X, M_Y) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, M) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_X, M) = \mathcal{D}_X(\mathcal{O}_X, M).$$

On the other hand, $\mathcal{D}_X(\mathcal{O}_X, M)$ consists of those k-linear maps $\theta : \mathcal{O}_X \to M$ for which $I^n_{\Delta_X} \cdot \theta = 0$ for some $n \geq 0$. So if $\theta \in \mathcal{D}_X(\mathcal{O}_X, M)$ takes $\mathcal{O}_X$ into M_Y, we have, in particular, $I^n_{\Delta_Y} \cdot \theta = 0$ and so θ lies in $\mathcal{D}_Y(\mathcal{O}_X, M_Y)$ as well, completing the proof. □

Corollary 5.6 (Agreement with the Cannings-Holland construction, [CH1])**.** *If M_Y is a rank 1 torsion-free $\mathcal{O}_Y$-module on a birational cuspidal quotient $X \to Y$ of a nonsingular curve X and $M_Y \subset K_X$ is an embedding of M_Y as an $\mathcal{O}_Y$-module, then the cusp-induced $\mathcal{D}_X$-module $\widetilde{M}$ equals the subsheaf $\mathcal{D}_X(\mathcal{O}_X, M_Y) \subset \mathcal{D}_X(\mathcal{O}_X, K_X)$ of operators with image in M_Y.*

5.3. **Riemann-Hilbert correspondence for cusp-induced modules.** Recall that an f-*differential* quasicoherent $(\mathcal{O}_Y, \mathcal{O}_X)$-bimodule M is a quasicoherent sheaf on $Y \times X$ that is torsion with respect to the ideal $I_{Y\leftarrow X}$ of the graph of f in $Y \times X$ (i.e., it is the union of its sections supported on finite-order neighborhoods of the graph of f, and hence underlies a $\mathcal{J}_{Y\leftarrow X}$-module). We consider M as a sheaf of $(\mathcal{O}_Y, \mathcal{O}_X)$-bimodules on Y. Conversely [BB, Section 1.1.3], if M is a sheaf of $(\mathcal{O}_Y, \mathcal{O}_X)$-bimodules on Y that is quasicoherent as an $\mathcal{O}_Y$-module and is torsion with respect to the ideal $I_{Y\leftarrow X} \subset \mathcal{O}_Y \otimes_k \mathcal{O}_X$ of the graph of f, then M is f-differential.

An $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodule is said to be f-differential if the underlying $(\mathcal{O}_Y, \mathcal{O}_X)$-bimodule is. The category of f-differential $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodules is denoted $\mathrm{mod}\text{–}(\mathcal{O}_Y, \mathcal{D}_X)_f$.

The deRham cohomology of right $\mathcal{D}_X$-modules $h_X : M \to M \otimes_{\mathcal{D}_X} \mathcal{O}_X$ defines a functor from differential bimodules to $\mathcal{O}_Y$-modules,

$$h_X : \mathrm{mod}\text{–}(\mathcal{O}_Y, \mathcal{D}_X)_f \longrightarrow \mathcal{O}_Y\text{–}\mathrm{mod}\,.$$

Note that we do not need to sheafify h_X on the category of differential bimodules, since $M \otimes_{\mathcal{D}_X} \mathcal{O}_X$ is automatically a quasicoherent $\mathcal{O}_Y$-module.

Theorem 5.7. *Cusp induction defines a fully faithful functor*

$$\mathrm{Ind}_{Y\leftarrow X} : \mathcal{O}_Y\text{–}\mathrm{mod} \longrightarrow \mathrm{mod}\text{–}(\mathcal{O}_Y, \mathcal{D}_X)_f,$$

which is right adjoint and right inverse to the deRham functor h_X.

Proof. A cusp-induced $\mathcal{D}_X$-module $\mathrm{Ind}_{Y\leftarrow X}(M_Y) = M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}$ carries an f-local $\mathcal{O}_Y$-structure commuting with its right $\mathcal{D}_X$-structure, since it is the tensor product of a coherent sheaf with the f-differential module $\mathcal{D}_{Y\leftarrow X}$; thus we may refine cusp induction to a functor to $(\mathcal{O}_Y, \mathcal{D}_X)_f$-bimodules. Furthermore, the de-Rham functor takes $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodules to $\mathcal{O}_Y$-modules.

By Theorem 4.3, $\mathcal{D}_{Y\leftarrow X}$ is flat as a right $\mathcal{D}_X$-module, and $\mathcal{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X = \mathcal{O}_Y$. It follows that for any $\mathcal{O}_Y$-module M,

$$\mathrm{Ind}_{Y\leftarrow X} M_Y \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X = M_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X = M_Y,$$

so that cusp-induced $\mathcal{D}$-modules are deRham exact, and the deRham functor applied to an induced $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodule $\mathrm{Ind}_{Y\leftarrow X} M_Y$ recovers M_Y as an $\mathcal{O}_Y$-module.

If $\widetilde{M}$ is an $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodule that is isomorphic to an induced bimodule $\widetilde{N} = N_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y\leftarrow X}$, then we have $h(\widetilde{M}) \cong h(\widetilde{N}) = N_Y$. So $\widetilde{M}$ is isomorphic to the induction of its deRham module. It follows that the deRham and induction functors define quasi-inverse equivalences of categories between the essential image of induction in $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodules and $\mathcal{O}_Y$-modules, provided we can show that the induction functor is full, which is a special case of the adjunction.

Suppose M is an f-differential bimodule and N is a quasicoherent $\mathcal{O}_Y$-module. We would like to show that $\mathcal{O}_Y$-morphisms $h_X(M) \to N$ are in bijection with $(\mathcal{O}_Y, \mathcal{D}_X)$-morphisms $M \to \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, N)$. Consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_X, M) \arrow[r, shift left, "u_r^*"] \arrow[r, shift right, "a"'] \arrow[d, "(1)"] & M \arrow[r, "p_M"] \arrow[d, "(2)"] & h_X(M) \arrow[d, dashed, "(3)"] \\
\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{J}_X, \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, N)) \arrow[r, shift left, "u_r^*"] \arrow[r, shift right, "\Delta^*"'] & \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{J}_{Y\leftarrow X}, N) \arrow[r, "u_l^*"] & N
\end{tikzcd}$$

where a and Δ^* are the action maps, and the rows are colimit diagrams. If we are given an $\mathcal{O}_Y$-module map (3), the composite $(3) \circ p_M$ induces, by adjunction, a $\mathcal{J}$-module map (2) such that $u_l^* \circ (2) = (3) \circ p_M$: the map is defined by $(2)(m)(j_1 \otimes j_2) = (3) \circ p_M(j_1 \otimes j_2 \cdot m)$ for $j_1 \otimes j_2$ a section of $\mathcal{J}$ and m a section of M. If (1) is the map induced by (2) under !-pullback, then $(2) \circ u_r^* = u_r^* \circ (1)$, and consequently $(2) \circ a = \Delta^* \circ (1)$: we have

$$(3) \circ p_M \circ a = (3) \circ p_M \circ u_r^* = u_l^* \circ (2) \circ u_r^* = u_l^* \circ u_r^* \circ (1) = u_l^* \circ \Delta^* \circ (1),$$

and so for a section s of $\underline{\mathrm{Hom}}_Y(\mathcal{J}, M)$ we find that

$$\begin{aligned}
(2)(a(s))(j_1 \otimes j_2) &= (3) \circ p_M(j_1 \otimes j_2 \cdot a(s)) = (3) \circ p_M \circ a(j_1 \otimes 1 \otimes j_2 \cdot s) \\
&= u_l^* \Delta^*(1)(j_1 \otimes 1 \otimes j_2 \cdot s) = u_l^*(j_1 \otimes j_2 \cdot \Delta^*(1)(s)) = \Delta^* \circ (1)(s)(j_1 \otimes j_2).
\end{aligned}$$

Thus (2) is an $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodule map. Conversely, starting from an $(\mathcal{O}_Y, \mathcal{D}_X)$-bimodule map (2), one has $(2) \circ a = \Delta^* \circ (1)$ and $(2) \circ u_r^* = u_r^* \circ (1)$. So the composite $u_l^* \circ (2)$ factors through $h(M)$ as $(3) \circ p_M$ for some $\mathcal{O}_Y$-module map (3). Under adjunction, moreover, $(3) \circ p_M$ corresponds to (2), as desired. This establishes the adjunction of h_X and $\mathrm{Ind}_{Y\leftarrow X}$, completing the proof. □

Lemma 5.8. *Let $g : Y \to Y'$ be a morphism of cusp quotients of X. Then the functors* Ind_Y *and* $\mathrm{Ind}_{Y'} \circ g_*$ *from $\mathcal{O}_Y$-modules to $\mathcal{D}_X$-modules are naturally isomorphic.*

Proof. The lemma follows from the identification $(g \times 1)^* \mathcal{J}_{Y' \leftarrow X} = \mathcal{J}_{Y \leftarrow X}$ (under the morphism $g \times 1 : Y \times X \to Y' \times X$) and the ensuing identification

$$\begin{aligned} \mathrm{Ind}_{Y' \leftarrow X}\, g_* M_Y &= \underline{\mathrm{Hom}}_{\mathcal{O}_{Y'}}(\mathcal{J}_{Y' \leftarrow X}, g_* M_Y) \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(g^* \mathcal{J}_{Y' \leftarrow X}, M_Y) = \mathrm{Ind}_{Y \leftarrow X}\, M_Y, \end{aligned}$$

which respects cocomodule structures. □

Definition 5.9. For $\mathcal{O}_Y$-modules M_Y and N_Y, the *differential morphisms* are the $\mathcal{D}_X$-module homomorphisms between cusp-induced modules,

$$\mathrm{Diff}(M_Y, N_Y) = \mathrm{Hom}_{\mathrm{mod}-\mathcal{D}_X}(\mathrm{Ind}_Y\, M_Y, \mathrm{Ind}_Y\, N_Y).$$

Definition 5.10. The category cusp-ind($\mathcal{D}_X$) of *cusp-induced $\mathcal{D}$-modules* on X is the full subcategory of mod-$\mathcal{D}_X$ consisting of $\mathcal{D}_X$-modules isomorphic to $\mathrm{Ind}_{Y \leftarrow X} M_Y$ for some cusp quotient $X \xrightarrow{f} Y$ and some $\mathcal{O}_Y$-module M_Y.

Corollary 5.11 (Cuspidal Riemann-Hilbert correspondence). *The functors* $\mathrm{Ind}_{Y \leftarrow X}$ *define an equivalence of categories*

$$\varinjlim_{X \to Y} (\mathrm{qcoh}(\mathcal{O}_Y), \mathrm{Diff}) \longrightarrow \text{cusp-ind}(\mathcal{D}_X).$$

A quasi-inverse functor is given by the deRham functor.

Proof. The functor $\mathrm{Ind}_{Y \leftarrow X}$ from $\mathcal{O}_Y-$mod to mod$-\mathcal{D}_X$ is faithful, since the $(\mathcal{O}_Y, \mathcal{D}_X)$-morphisms between cusp-induced modules inject into the $\mathcal{D}_X$-morphisms. It follows that by allowing differential morphisms we make this faithful functor full. By Lemma 5.8, it is compatible with deepening the cusps and so descends to the inductive limit. Since cusp-induced $\mathcal{D}_X$-modules are by definition the essential image of this functor, it follows that we have an equivalence of categories. □

The corollary is an extension of the equivalence between induced $\mathcal{D}_X$-modules and $\mathcal{O}_X$-modules with differential morphisms [Sa].

5.4. **Cusp-induced $\mathcal{D}$-modules on a curve.** Let X denote a smooth curve over a field k of characteristic zero; the category of cusp-induced $\mathcal{D}$-modules on X is particularly easy to describe in this case. The cusp-induction functor itself also becomes very concrete: as we have seen (Corollary 5.6), in the case of curves, the cusp-induction of an $\mathcal{O}_Y$-module $M \subset K_Y = K_X$ agrees with the Cannings-Holland construction, the sheaf $\mathcal{D}_X(\mathcal{O}_X, M)$ of differential operators with values in M.

Definition 5.12. For a right $\mathcal{D}_X$-module M, we let $T_{\mathcal{O}}(M)$ denote the $\mathcal{O}_X$-torsion submodule of M, and $T_{\mathcal{D}}(M)$ denote the $\mathcal{D}_X$-torsion submodule of M. We say that M is *generically torsion-free* if there is a nonempty open set U of X such that $M|_U$ is a torsion-free $\mathcal{D}_U$-module, in other words if $T_{\mathcal{D}}(M)$ is supported on a proper closed subvariety.

Lemma 5.13. *$T_{\mathcal{O}}(M)$ and $T_{\mathcal{D}}(M)$ are $\mathcal{D}_X$-submodules of M, which agree if M is generically torsion-free.*

Proposition 5.14. *A finitely-generated $\mathcal{D}_X$-module M is cusp-induced if and only if M is generically torsion-free.*

Proof. Suppose $M = \overline{M} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y \leftarrow X}$ for a cuspidal quotient $X \to Y$. $\overline{M}$ is an extension

$$0 \to T_{\mathcal{O}_Y}(\overline{M}) \to \overline{M} \to \overline{M}/T_{\mathcal{O}_Y}(\overline{M}) \to 0$$

where $T_{\mathcal{O}_Y}(\overline{M})$ is $\mathcal{O}_Y$-torsion and $\overline{M}/T_{\mathcal{O}_Y}(\overline{M})$ is $\mathcal{O}_Y$-torsion-free. Since $\mathcal{D}_{Y \leftarrow X}$ is flat over $\mathcal{O}_Y$, we obtain an exact sequence

$$0 \to T_{\mathcal{O}_Y}(\overline{M}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y \leftarrow X} \to M \to (\overline{M}/T_{\mathcal{O}_Y}(\overline{M})) \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y \leftarrow X} \to 0,$$

where $T_{\mathcal{O}_Y}(\overline{M}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y \leftarrow X}$ is an $\mathcal{O}_X$-torsion submodule of M. Moreover, because $\overline{M}/T_{\mathcal{O}_Y}(\overline{M})$ is $\mathcal{O}_Y$-torsion-free, it embeds locally in $\mathcal{O}_Y^n$ for some n; hence $(\overline{M}/T_{\mathcal{O}_Y}(\overline{M})) \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{Y \leftarrow X}$ embeds locally in $\mathcal{D}_X^n$ and so is $\mathcal{D}_X$-torsion-free.

Conversely, suppose M is a $\mathcal{D}_X$-module that is generically torsion-free. The $\mathcal{O}_X$-torsion submodule $T_{\mathcal{O}}(M)$ is supported on a finite collection of closed points; hence by Kashiwara's Theorem it is isomorphic to an induced $\mathcal{D}_X$-module. Moreover, $M/T_{\mathcal{O}}(M)$ is $\mathcal{D}_X$-torsion-free, hence is locally projective on X; therefore, the group $\mathrm{Ext}^1_{\mathcal{D}_X}(M/T_{\mathcal{O}}(M), T_{\mathcal{O}}(M))$ vanishes and so $M \cong T_{\mathcal{O}}(M) \oplus M/T_{\mathcal{O}}(M)$. Now $M/T_{\mathcal{O}}(M)$ embeds in a locally free (and therefore induced) $\mathcal{D}_X$-module with cosupport a finite set. It thus suffices to prove the following easy consequence of Kashiwara's theorem.

Lemma 5.15. *Suppose*

$$\mathbf{L}: \quad 0 \to \ker(\beta) \to M \xrightarrow{\beta} Q \to 0$$

is a short exact sequence of finitely generated $\mathcal{D}_X$-modules such that M is cusp-induced and Q is supported on a finite subset of X. Then there is a cuspidal quotient $X \to Y$ and an exact sequence of coherent $\mathcal{O}_Y$-modules

$$\mathbf{L}': \quad 0 \to K \to M' \to Q' \to 0$$

such that $\mathrm{Ind}_{Y \leftarrow X}(\mathbf{L}') \cong \mathbf{L}$ *as sequences of right $\mathcal{D}_X$-modules.*

This completes the proof of Proposition 5.14. □

The sheaf of algebras $\mathcal{D}_X$ locally has homological dimension one (see [SS, Section 1.4(e)]), from which it follows that any torsion-free $\mathcal{D}$-module is locally projective. Moreover, $\mathcal{D}_X$ possess a skew field of fractions (see [SS, Section 2.3]), from which it follows that any finitely generated locally projective $\mathcal{D}$-module has a well-defined rank.

Definition 5.16. [BD2] A $\mathcal{D}$-vector bundle (or $\mathcal{D}$-bundle for short) M on X is a locally projective (equivalently torsion-free) right $\mathcal{D}_X$-module of finite rank.

Corollary 5.17. *A $\mathcal{D}$-module M on X is a $\mathcal{D}$-bundle if and only if it is isomorphic to a $\mathcal{D}$-module cusp-induced from a torsion-free $\mathcal{O}_Y$-module for some cuspidal quotient Y of X.*

An important class of $\mathcal{D}$-bundles (of rank one) is provided by the right ideals in $\mathcal{D}_X$ (in fact any rank one $\mathcal{D}$-bundle may locally be embedded as a right ideal in $\mathcal{D}_X$).

Example 5.18. Consider the right ideal $M \subset \mathcal{D}_{\mathbf{A}^1} = \mathbf{C}\langle z, \partial\rangle/\{\partial z - z\partial - 1\}$ generated by z^2 and $1 - z\partial$. Then $M = \mathcal{D}(\mathcal{O}_{\mathbf{A}^1}, \mathcal{O}_Y)$ where $\mathcal{O}_Y = k[z^2, z^3]$ is the coordinate ring of the cuspidal cubic curve $y^2 = x^3$, i.e., M is cusp-induced from the structure sheaf of Y.

As the example illustrates, $\mathcal{D}$-bundles are not locally trivial—that is, they are not locally (on the base curve) isomorphic to a direct sum of copies of $\mathcal{D}$. It follows from the cusp-induced description, however, that $\mathcal{D}$-bundles are *generically* trivial. It is convenient to parametrize $\mathcal{D}$-bundles by picking such a generic trivialization. This leads to the description by Cannings and Holland [CH1], [CH2] of ideals in $\mathcal{D}$ by means of the *adelic Grassmannian* [W].

Definition 5.19. The adelic Grassmannian $\mathbf{Gr}^{ad}(X)$ is the set of isomorphism classes of $\mathcal{D}$-bundles M of rank one equipped with a generic trivialization, $M \otimes K_X \cong \mathcal{D}_X \otimes K_X$.

Remark 5.20. This set-theoretic definition may be refined to a moduli problem, giving the adelic Grassmannian an algebraic structure, which we study in [BN2]. More precisely, for a finite set I, there is an ind–scheme of ind–finite type $\mathbf{Gr}^{ad}_I(X)$ over X^I, parametrizing $\mathcal{D}$–bundles with trivializations outside of I–tuples of points of X. As we allow these prescribed positions of singularities to collide, we obtain a “nonlinear vertex algebra” structure on $\mathbf{Gr}^{ad}(X)$: the spaces $\{\mathbf{Gr}^{ad}_I(X)\}$, considered over the inductive system of spaces X^I with respect to partial diagonal maps (i.e., surjections of finite sets), form a factorization ind–scheme, as defined in [BD2]. We also identify the vertex (or chiral) algebras obtained by linearization of this factorization space with the $\mathcal{W}_{1+\infty}$-vertex algebra and its variants.

Corollary 5.21. $\mathbf{Gr}^{ad}(X)$ *is isomorphic to the direct limit over cusp quotients* $X \to Y$ *of the set of isomorphism classes of rank* 1 *torsion-free* $\mathcal{O}_Y$*-modules equipped with a generic trivialization.*

Proof. Observe that if $X \to Y$ is a cuspidal quotient morphism (over k of characteristic zero), then it is birational. It follows that $K_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y\leftarrow X} = K_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ canonically. So by Corollary 5.11 we get an embedding of the set of isomorphism classes of rank 1 torsion-free $\mathcal{O}_Y$-modules equipped with generic trivialization in $\mathbf{Gr}^{ad}(X)$. (Note that since we have rigidified our modules using the generic trivialization, the distinction between differential morphisms and $\mathcal{O}_Y$–morphisms goes away – this injectivity is certainly false without the rigidification.) It suffices, then, to prove that this embedding is surjective. So suppose M is a rank one $\mathcal{D}$-bundle equipped with an isomorphism $\phi : M \otimes K_X \to \mathcal{D} \otimes K_X$. Then there is an effective divisor D on X such that ϕ is of the form $\psi \otimes K_X$ for a $\mathcal{D}$-module homomorphism $\psi : M \to \mathcal{O}(D) \otimes_{\mathcal{O}_X} \mathcal{D}_X$ that is an isomorphism on a nonempty open set of X. Now Lemma 5.15 implies that there is a cuspidal quotient $X \to Y$ and a homomorphism $\psi' : M' \to \mathcal{O}(D)$ of $\mathcal{O}_Y$-modules such that $\mathrm{Ind}_{Y\leftarrow X}(\psi') = \psi$. The canonical map $\mathcal{O}(D) \to K_X$ now gives the desired generic trivialization, completing the proof. $\square$

As we let the cusps Y get deeper, the rings $\mathcal{O}_Y$ evaporate so that the adelic Grassmannian parametrizes simply certain linear algebra data. This is most succinctly explained in [BD2], Section 2.1: $\mathcal{D}$-submodules of any $\mathcal{D}$-module M that are cosupported at a point $x \in X$ are in canonical bijection (via the deRham functor h) with subspaces of the stalk of the deRham cohomology $h(M)_x$ at x that are open in a natural topology. In the above case, we have $\mathcal{D}$-submodules of $\mathcal{D}(K_X)$

cosupported at some finite collection of points, which correspond to collections of open subspaces of Laurent series $\mathcal{K}_{x_i}$ at x_i with respect to the usual topology.

Remark 5.22. The adelic Grassmannian (for $X = \mathbf{A}^1$) was used by G. Wilson to parametrize solutions of the KP hierarchy that are rational (and decay at infinity) in the first time variable. More precisely, these solutions correspond to Krichever data, which are rank one torsion-free sheaves on a cuspidal quotient of the affine line. (More generally, $\mathbf{Gr}^{ad}(X)$ for a curve X parametrizes Krichever data for X and all of its cuspidal quotients together.) Wilson then shows that under the action of the KP flows on the adelic Grassmannian, the underlying collections of points on $\mathbf{A}^1$ move according to the Calogero-Moser particle system. This extends the results of Airault-McKean-Moser [AMM], Krichever [Kr1] and Shiota [Sh] to allow collisions of the Calogero-Moser particles. In [BN1] we develop a $\mathcal{D}$-bundle point of view on the KP hierarchy, and obtain geometric proofs of this result and extend it to the rational, trigonometric and elliptic solutions of (multicomponent) KP hierarchies.

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