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THE TAME AND THE WILD AUTOMORPHISMS OF POLYNOMIAL RINGS IN THREE VARIABLES

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1. Introduction

Let $C = F[x_1, x_2, \ldots, x_n]$ be the polynomial ring in the variables $x_1, x_2, \ldots, x_n$ over a field F, and let $Aut\,C$ be the group of automorphisms of C as an algebra over F. An automorphism $\tau \in Aut\,C$ is called *elementary* if it has a form

$$\tau : (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq \alpha \in F$, $f \in F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. The subgroup of $Aut\,C$ generated by all the elementary automorphisms is called the *tame subgroup*, and the elements from this subgroup are called *tame automorphisms* of C. Non-tame automorphisms of the algebra C are called *wild*.

It is well known [6], [9], [10], [11] that the automorphisms of polynomial rings and free associative algebras in two variables are tame. At present, a few new proofs of these results have been found (see [5], [8]). However, in the case of three or more variables the similar question was open and known as "The generation gap problem" [2], [3] or "Tame generators problem" [8]. The general belief was that the answer is negative, and there were several candidate counterexamples (see [5], [8], [12], [7], [19]). The best known of them is the following automorphism $\sigma \in Aut(F[x, y, z])$, constructed by Nagata in 1972 (see [12]):

$$\begin{aligned} \sigma(x) &= x + (x^2 - yz)z, \\ \sigma(y) &= y + 2(x^2 - yz)x + (x^2 - yz)^2 z, \\ \sigma(z) &= z. \end{aligned}$$

Observe that the Nagata automorphism is stably tame [17]; that is, it becomes tame after adding new variables.

The purpose of the present work is to give a negative answer to the above question. Our main result states that the tame automorphisms of the polynomial ring $A = F[x, y, z]$ over a field F of characteristic 0 are algorithmically recognizable. In particular, the Nagata automorphism σ is wild.

The approach we use is different from the traditional ones. The novelty consists of the imbedding of the polynomial ring A into the free Poisson algebra (or the

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algebra of universal Poisson brackets) on the same set of generators and of the systematical use of brackets as an additional tool.

The crucial role in the proof is played by the description of the structure of subalgebras generated by so-called $*$-reduced pairs of polynomials, given in [16]. More precisely, a lower estimate for degrees of elements of these subalgebras is essentially used in most of the proofs.

We follow the so-called "method of simple automorphisms", which was first developed in [1] for a characterization of tame automorphisms of two-generated free Leibniz algebras. Note that this method permits us also to establish directly the result of [4] concerning wild automorphisms of two-generated free matrix algebras, without using the results of [6], [9], [10], [11]. In fact, the first attempt to apply this method for a characterization of tame automorphisms of polynomial rings and free associative algebras in three variables was done by C. K. Gupta and U. U. Umirbaev in 1999. At that time, some results were obtained modulo a certain conjecture, which eventually proved not to be true for polynomial rings (see Example 1, Section 3). Really, the structure of tame automorphisms turns out to be much more complicated.

Observe that no analogues of the results of [16] are known for free associative algebras and for polynomial rings of positive characteristic, and the question on the existence of wild automorphisms is still open for these algebras.

The paper is organized as follows. In Section 2, some results are given, mainly from [16], which are necessary in the sequel. Some instruments for further calculations are also created here. In Section 3, elementary reductions and reductions of types I–IV are defined and characterized for automorphisms of the algebra A, and simple automorphisms of A are defined. The main part of the work, Section 4, is devoted to the proof of Theorem 1, which states that every tame automorphism of the algebra A is simple. The main results are formulated and proved in Section 5 as corollaries of Theorem 1.

2. Structure of two-generated subalgebras

Let F be an arbitrary field of characteristic 0, and let $A = F[x_1, x_2, x_3]$ be the ring of polynomials in the variables x_1, x_2, x_3 over F. Following [16], we will identify A with a certain subspace of the free Poisson algebra $P = PL\langle x_1, x_2, x_3\rangle$.

Recall that a vector space B over a field F, endowed with two bilinear operations $x \cdot y$ (a multiplication) and $[x, y]$ (a Poisson bracket), is called a *Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $[x, y]$, and B satisfies the *Leibniz identity*

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z]. \tag{1}$$

An important class of Poisson algebras is given by the following construction. Let L be a Lie algebra with a linear basis $l_1, l_2, \dots, l_k, \dots$. Denote by $P(L)$ the ring of polynomials on the variables $l_1, l_2, \dots, l_k, \dots$. The operation $[x, y]$ of the algebra L can be uniquely extended to a Poisson bracket $[x, y]$ on the algebra $P(L)$ by means of formula (1), and $P(L)$ becomes a Poisson algebra [15].

Now let L be a free Lie algebra with free generators $x_1, x_2, \dots, x_n$. Then $P(L)$ is a free Poisson algebra [15] with the free generators $x_1, x_2, \dots, x_n$. We will denote this algebra by $PL\langle x_1, x_2, \dots, x_n\rangle$. If we choose a homogeneous basis

$$x_1, x_2, \dots, x_n,\ [x_1, x_2], \dots, [x_1, x_n], \dots, [x_{n-1}, x_n],\ [[x_1, x_2], x_3] \dots$$

of the algebra L with nondecreasing degrees, then $PL\langle x_1, x_2, \dots, x_n\rangle$, as a vector space, coincides with the ring of polynomials on these elements. The space $PL\langle x_1, x_2, \dots, x_n\rangle$ is graded by degrees on x_i, and for every element $f \in PL\langle x_1, x_2, \dots, x_n\rangle$, the highest homogeneous part $\bar{f}$ and the degree $\deg f$ can be defined in an ordinary way. Note that

$$\overline{fg} = \bar{f}\bar{g}, \quad \deg(fg) = \deg f + \deg g, \quad \deg[f, g] \leq \deg f + \deg g.$$

In the sequel, we will identify the ring of polynomials $A = F[x_1, x_2, x_3]$ with the subspace of the algebra $PL\langle x_1, x_2, x_3\rangle$ generated by elements

$$x_1^{r_1} x_2^{r_2} x_3^{r_3}, \quad r_i \geq 0,\ 1 \leq i \leq 3.$$

Note that if $f, g \in A$, then

$$[f, g] = \gamma_{12}[x_1, x_2] + \gamma_{23}[x_2, x_3] + \gamma_{13}[x_1, x_3],$$
$$\gamma_{ij} = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}, \quad 1 \leq i < j \leq 3.$$

If $f_1, f_2, \dots, f_k \in A$, then by $\langle f_1, f_2, \dots, f_k\rangle$ we denote the subalgebra of the algebra A generated by these elements.

The following lemma is proved in [16].

Lemma 1. *Let $f, g, h \in A$. Then the following statements are true:*

1) *$[f, g] = 0$ iff f, g are algebraically dependent.*

2) *Suppose that $f, g, h \notin F$ and $m = \deg[f, g] + \deg h$, $n = \deg[g, h] + \deg f$, $k = \deg[h, f] + \deg g$. Then $m \leq \max(n, k)$. If $n \neq k$, then $m = \max(n, k)$.*

The next two simple statements are well known (see [5]):

F1) *If a, b are nonzero homogeneous algebraically dependent elements of A, then there exists an element $z \in A$ such that $a = \alpha z^n$, $b = \beta z^m$, $\alpha, \beta \in F$. In addition, the subalgebra $\langle a, b\rangle$ is one-generated iff $m|n$ or $n|m$.*

F2) *Let $f, g \in A$ and $\bar{f}, \bar{g}$ are algebraically independent. If $h \in \langle f, g\rangle$, then $\bar{h} \in \langle \bar{f}, \bar{g}\rangle$.*

Recall that a pair of elements f, g of the algebra A is called *reduced* (see [18]), if $\bar{f} \notin \langle \bar{g}\rangle$, $\bar{g} \notin \langle \bar{f}\rangle$. A reduced pair of algebraically independent elements $f, g \in A$ is called $*$-*reduced* (see [16]), if $\bar{f}, \bar{g}$ are algebraically dependent.

Let f, g be a $*$-reduced pair of elements of A and $n = \deg f < m = \deg g$. Put $p = \frac{n}{(n,m)}$, $s = \frac{m}{(n,m)}$,

$$N = N(f, g) = \frac{mn}{(m, n)} - m - n + \deg[f, g] = mp - m - n + \deg[f, g],$$

where (n, m) is the greatest common divisor of n, m. Note that $(p, s) = 1$, and by F1) there exists an element $a \in A$ such that $\bar{f} = \beta a^p$, $\bar{g} = \gamma a^s$. Sometimes, we will call a $*$-reduced pair of elements f, g also a p-*reduced* pair. Let $G(x, y) \in F[x, y]$. It was proved in [16] that if $\deg_y(G(x, y)) = pq + r$, $0 \leq r < p$, then

$$\deg(G(f, g)) \geq qN + mr, \tag{2}$$

and if $\deg_x(G(x, y)) = sq_1 + r_1$, $0 \leq r_1 < s$, then

$$\deg(G(f, g)) \geq q_1 N + nr_1. \tag{3}$$

It will be convenient for us to collect several evident properties of the $*$-reduced pair f, g in the following lemma.

Lemma 2. *Under the above notation,*
i) $p \geq 2$*;*
ii) $N = N(f,g) > \deg[f,g]$*;*
iii) if $p > 2$*, then* $N > m$*;*
iv) if $p = 2$*, then* $N > \frac{n}{2}$*.*

The properties $i) - iii)$ are evident. As for $iv)$, let $d = (n,m)$; then $n = 2d,\ m = sd$, and $N = sd - 2d + \deg[f,g] > (s-2)d \geq d = \frac{n}{2}$.
The statement of the following lemma is easily proved.

Lemma 3. *The elements of type* $f^i g^j$*, where* $j < p$*, have different degrees for different values of* i,j*.*

Inequality (2) and Lemma 3 imply

Corollary 1. *Let* $G(x,y) \in F[x,y],\ h = G(f,g)$*. Consider the following conditions:*
(i) $\deg h < N(f,g)$*;*
(ii) $\deg_y(G(x,y)) < p$*;*
(iii) $h = \sum_{i,j} \alpha_{ij} f^i g^j$*, where* $\alpha_{ij} \in F$ *and* $in + jm \leq \deg h$ *for all* i,j*;*
(iv) $\bar{h} \in \langle \bar{f}, \bar{g} \rangle$*.*
Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$*.*

Suppose that $p \geq 3$ or $\deg[f,g] > n$. Then obviously $N(f,g) > m$, and in the conditions of Corollary 1 we have $\bar{h} \in \langle \bar{f}, \bar{g} \rangle$ or $\deg h > m = \max(\deg f, \deg g)$. Note that the most complicated case in the investigation of tame automorphisms of A is represented by $*$-reduced pairs f,g with the condition $N(f,g) \leq m$, that is, by 2-reduced pairs f,g for which $\deg[f,g] \leq n$.

Lemma 4. *There exists a polynomial* $w(x,y) \in F[x,y]$ *of the type*

$$w(x,y) = y^p - \alpha x^s - \sum \alpha_{ij} x^i y^j,\ ni + mj < mp,$$

which satisfies the following conditions:
1) $\deg w(f,g) < pm$*;*
2) $\overline{w(f,g)} \notin \langle \bar{f}, \bar{g} \rangle$*.*

Proof. By F1), there exists a homogeneous element $a \in A$ such that $\bar{f} = \beta a^p$, $\bar{g} = \gamma a^s$. Then there exists $\alpha \in F$ such that $\bar{g}^p = \alpha \bar{f}^s$, and the elements of the type $\bar{f}^i \bar{g}^j$, $j < p$, form a basis of the subalgebra $\langle \bar{f}, \bar{g} \rangle$. Putting $h = g^p - \alpha f^s$, we have $\deg h < mp$. If $\bar{h} \in \langle \bar{f}, \bar{g} \rangle$, then $\bar{h} = \alpha_{ij} \bar{f}^i \bar{g}^j$, where $ni + mj < mp$. Change the element h to $h - \alpha_{ij} f^i g^j$. Then $\deg(h - \alpha_{ij} f^i g^j) < \deg h$. After several reductions of this type, we get an element

$$h = g^p - \alpha f^s - \sum \alpha_{ij} f^i g^j,\ ni + mj < mp,$$

for which $\bar{h} \notin \langle \bar{f}, \bar{g} \rangle$. Since f,g are algebraically independent, the equality $h = w(f,g)$ defines uniquely a polynomial $w(x,y)$ that satisfies the conditions of the lemma. □

A polynomial $w(x,y)$ satisfying the conditions of Lemma 4 we will call *a derivative polynomial* of the $*$-reduced pair f,g. Note that a derivative polynomial $w(x,y)$ is not uniquely defined in the general case. But the coefficient α in the conditions of Lemma 3 is uniquely defined by the equality $\bar{g}^p = \alpha \bar{f}^s$.

Lemma 5. *Let $w(x,y)$ be a derivative polynomial of a $*$-reduced pair f,g. Then the following statements are true:*

1) $\deg w(f,g)$ *is uniquely defined;*

2) *if* $\bar{f}$, $\overline{w(f,g)}$ *are algebraically dependent, then* $\overline{w(f,g)}$ *is uniquely defined;*

3) $\deg(w(f,g)) \geq N(f,g)$*;*

4) *if* $\deg(w(f,g)) < \deg g = m$*, then* $w(x,y)$ *is defined uniquely up to a summand* $q(x)$*, where* $n \cdot \deg(q(x)) \leq \deg(w(f,g))$*;*

5) *if* $\deg(w(f,g)) < \deg f$*, then* $w(x,y)$ *is defined uniquely up to a scalar summand from* F*.*

Proof. Let $w_1(x,y)$ be another derivative polynomial of the pair f,g. Since the coefficient α is uniquely defined in the conditions of Lemma 4, we have

$$h(x,y) = w(x,y) - w_1(x,y) = \sum \gamma_{ij} x^i y^j,$$

where $ni + mj < mp$. Now, if $\deg(w(f,g)) > \deg(w_1(f,g))$, then by Lemma 3 we get

$$\overline{h(f,g)} = \overline{w(f,g)} \in \langle \bar{f}, \bar{g} \rangle,$$

which contradicts the definition of $w(x,y)$.

Suppose that $\overline{w(f,g)} \neq \overline{w_1(f,g)}$. Then $\overline{h(f,g)} = \overline{w(f,g)} - \overline{w_1(f,g)}$. Since $\overline{h(f,g)} \in \langle \bar{f}, \bar{g} \rangle$, the elements $\bar{f}$, $\overline{h(f,g)}$ are algebraically dependent. Now, if $\bar{f}$, $\overline{w(f,g)}$ are algebraically dependent, then $\overline{w(f,g)}$, $\overline{h(f,g)}$ are algebraically dependent too. Furthermore, since $\deg(w(f,g)) = \deg(h(f,g))$, the elements $\overline{w(f,g)}$, $\overline{h(f,g)}$ are linearly dependent, and thus $\overline{w(f,g)} \in \langle \bar{f}, \bar{g} \rangle$. This again contradicts the definition of $w(x,y)$.

Note that $\deg(h(f,g)) \leq \deg(w(f,g))$. Since $\overline{w(f,g)} \notin \langle \bar{f}, \bar{g} \rangle$, Corollary 1 yields 3). If $\deg(w(f,g)) < \deg g$, then by Lemma 3 we get $h(x,y) = q(x)$, $\deg(q(f)) = \deg(h(f,g)) \leq \deg(w(f,g))$. This proves statements 4), 5) of the lemma. $\square$

Observe that in view of 3) and Lemma 2.iii), conditions 4), 5) of Lemma 5 may take place only for 2-reduced pairs.

Lemma 6. *Let $w(x,y)$ be a derivative polynomial of the pair f,g and $u = w(f,g)$. Then the highest homogeneous parts of the elements of the type*

$$f^i g^j u^t, \quad j < p, \quad 0 \leq t \leq 1,$$

are linearly independent.

Proof. Assuming the contrary, we get by Lemma 3 the equality of the form

$$\bar{f}^i \bar{g}^j \bar{u} = \beta \bar{f}^{i_1} \bar{g}^{j_1}, \quad j, j_1 < p.$$

If $i \leq i_1$, $j \leq j_1$, then this equality implies $\bar{u} \in \langle \bar{f}, \bar{g} \rangle$, which is impossible by the definition of $w(x,y)$. Assume that $j \leq j_1$, $i > i_1$. Then

$$\bar{f}^{i-i_1} \bar{u} = \beta \bar{g}^{j_1 - j}.$$

Since $\bar{u} \notin \langle \bar{f}, \bar{g} \rangle$, by Corollary 1 we have

$$\deg u \geq N(f,g) = pm - m - n + \deg[f,g].$$

Therefore,

$$\begin{aligned} \deg(\bar{g}^{j_1 - j}) = m(j_1 - j) &\geq n(i - i_1) + pm - m - n + \deg[f,g] \\ &\geq (p-1)m + \deg[f,g] > (p-1)m. \end{aligned}$$

This contradicts the inequality $j_1 - j \le p-1$. If $i \le i_1$, $j > j_1$, then

$$\bar{g}^{j-j_1}\bar{u} = \beta \bar{f}^{i_1-i}.$$

Since $j - j_1 \le p-1$ and $\bar{g}^p = \alpha \bar{f}^s$, we may assume that $i_1 - i < s$; otherwise $\bar{u} \in \langle \bar{f}, \bar{g}\rangle$. Thus,

$$\begin{aligned} \deg(\bar{f}^{i_1-i}) &= n(i_1-i) \ge m(j-j_1) + pm - m - n + \deg[f,g] \\ &\ge pm - n + \deg[f,g] > pm - n = ns - n = n(s-1), \end{aligned}$$

which is impossible. □

Lemma 7. *Let $w(x,y)$ be a derivative polynomial of the pair f, g, and $T(x,y) \in F[x,y]$, $\deg_y(T(x,y)) < 2p$. Then, the following statements are true:*

1) *$T(x,y)$ can be uniquely presented in the form*

$$T(x,y) = w(x,y)q(x,y) + s(x,y),$$

where $\deg_y(q(x,y))$, $\deg_y(s(x,y)) < p$;

2) *if $\deg(T(f,g)) \le t$, then*

$$\deg(w(f,g)) + \deg(q(f,g)) \le t, \ \deg(s(f,g)) \le t;$$

3) *if $\overline{T(f,g)} \notin \langle \bar{f}, \bar{g}\rangle$, then $q(x,y) \ne 0$ and*

$$\deg\left(\frac{\partial T}{\partial x}(f,g)\right) = \deg(q(f,g)) + n(s-1),$$

$$\deg\left(\frac{\partial T}{\partial y}(f,g)\right) = \deg(q(f,g)) + m(p-1).$$

4) *if $\overline{T(f,g)} \notin \langle \bar{f}, \bar{g}\rangle$ and $\deg(T(f,g)) < \min\{n+N, mp\}$, then $T(x,y) = \lambda w_1(f,g), 0 \ne \lambda \in F$, where $w_1(x,y)$ is also a derivative polynomial of the pair f, g.*

Proof. The first statement of the lemma follows from the division algorithm in the ring $(F[x])[y]$; one may divide $T(x,y)$ by $w(x,y)$ since the last polynomial is monic. Furthermore, the right part of the equality $T(f,g) = w(f,g)q(f,g) + s(f,g)$ is a linear combination of elements indicated in Lemma 6; therefore, by this lemma, only the elements of degree less than or equal to $\deg(T(f,g))$ may appear in this combination. This proves 2).

If $\overline{T(f,g)} \notin \langle \bar{f}, \bar{g}\rangle$, then by Corollary 1 we have $\deg_y T(x,y) \ge p$ and hence $q(x,y) \ne 0$. By Corollary 1 again, $\overline{s(f,g)} \in \langle \bar{f}, \bar{g}\rangle$; hence $\overline{T(f,g)} \ne \overline{s(f,g)}$. Consequently, by Lemma 6,

$$\deg(T(f,g)) = \deg(w(f,g)q(f,g)) \ge \deg(s(f,g)).$$

It follows from the definition of $w(x,y)$ in Lemma 4 that

$$\deg\left(\frac{\partial w}{\partial x}(f,g)\right) = n(s-1), \quad \deg\left(\frac{\partial w}{\partial y}(f,g)\right) = m(p-1).$$

Furthermore,

$$\frac{\partial T}{\partial x}(x,y) = \frac{\partial w}{\partial x}(x,y)q(x,y) + w(x,y)\frac{\partial q}{\partial x}(x,y) + \frac{\partial s}{\partial x}(x,y).$$

Easy calculations give

$$\begin{aligned}
\deg(w(f,g)\frac{\partial q}{\partial x}(f,g)) &\leq \deg(w(f,g)) + \deg(q(f,g)) - \deg f,\\
\deg(\frac{\partial s}{\partial x}(f,g)) &\leq \deg(T(f,g)) - \deg f\\
&= \deg(w(f,g)) + \deg(q(f,g)) - \deg f,\\
\deg(\frac{\partial w}{\partial x}(f,g)q(f,g)) &= \deg(q(f,g)) + n(s-1) = \deg(q(f,g)) + mp - n\\
&= \deg(w(f,g)) + \deg(q(f,g)) - n + (mp - \deg(w(f,g)))\\
&> \deg(w(f,g)) + \deg(q(f,g)) - n.
\end{aligned}$$

Therefore,

$$\deg\left(\frac{\partial T}{\partial x}(f,g)\right) = \deg(q(f,g)) + n(s-1).$$

Similar calculations give the value of $\deg(\frac{\partial T}{\partial y}(f,g))$.

To prove 4) we note first that by Lemma 5.3), $\deg(w(f,g)) \geq N$. Hence by statement 2) of this lemma, $\deg(q(f,g)) < n$ and $\deg(s(f,g)) < mp$. By Corollary 1, $\overline{q(f,g)} \in \langle \bar{f}, \bar{g}\rangle$. Hence $0 \neq q(x,y) = \lambda \in F$. Now it is easy to see that the polynomial $w_1(x,y) = \lambda^{-1}T(x,y) = w(x,y) + \lambda^{-1}s(x,y)$ is a derivative polynomial of the pair f, g. □

We give two corollaries that will be useful for references.

Corollary 2. *If $h \in \langle f, g\rangle \setminus F$ and $\deg h < n$, then $h = \lambda w(f,g)$, $0 \neq \lambda \in F$, where $w(x,y)$ is a derivative polynomial of the pair f, g.*

Proof. Put $h = T(f,g)$ and let $\deg_y T(x,y) = pq + r$, $r < p$. If $q = 0$, then Corollary 1 gives $\bar{h} \in \langle \bar{f}, \bar{g}\rangle$ and $\deg h \geq n$ or $h \in F$, a contradiction. Hence $q > 0$. Inequality (2) gives $\deg h \geq qN + mr$. Consequently, $r = 0$. If $q > 1$, then by Lemma 2, $\deg h \geq 2N > n$. Therefore, $q = 1$ and $\deg_y(T(x,y)) = p$. Then Lemma 7.4) proves the corollary. □

Corollary 3. *If $w(x,y)$ is a derivative polynomial of the pair f, g, then*

$$\deg\left(\frac{\partial w}{\partial x}(f,g)\right) = n(s-1),$$
$$\deg\left(\frac{\partial w}{\partial y}(f,g)\right) = m(p-1).$$

3. Reductions and simple automorphisms

A triple $\theta = (f_1, f_2, f_3)$ (or simply (f_1, f_2, f_3)) of elements of the algebra A below will always denote the automorphism θ of A such that $\theta(x_i) = f_i$, $1 \leq i \leq 3$. The number $\deg\theta = \deg f_1 + \deg f_2 + \deg f_3$ will be called a *degree* of the automorphism θ.

Recall that an *elementary transformation* of the triple (f_1, f_2, f_3) is, by definition, a transformation that changes only one element f_i to an element of the form $\alpha f_i + g$, where $0 \neq \alpha \in F$, $g \in \langle\{f_j | j \neq i\}\rangle$. The notation

$$(f_1, f_2, f_3) \to (g_1, g_2, g_3)$$

means that the triple (g_1, g_2, g_3) is obtained from (f_1, f_2, f_3) by a single elementary transformation. Observe that we do not assume that $\deg(g_1, g_2, g_3)$ should be smaller than $\deg\theta$. An automorphism (f_1, f_2, f_3) is called *tame* if there exists a sequence of elementary transformations of the form

$$(x_1, x_2, x_3) = (f_1^{(0)}, f_2^{(0)}, f_3^{(0)}) \to (f_1^{(1)}, f_2^{(1)}, f_3^{(1)}) \to \dots \\ \to (f_1^{(n)}, f_2^{(n)}, f_3^{(n)}) = (f_1, f_2, f_3).$$

The element f_1 of the automorphism $\theta = (f_1, f_2, f_3)$ is called *reducible*, if there exists $g \in \langle f_2, f_3\rangle$ such that $\bar{f}_1 = \bar{g}$; otherwise it is called *unreducible*. Put $f_1' = \alpha(f_1 - g)$, where $0 \neq \alpha \in F$; then $\deg f_1' < \deg f_1$ and $\deg(f_1', f_2, f_3) < \deg\theta$. In this case we will say also that f_1 *is reduced in* θ *by the automorphism* (f_1', f_2, f_3). If one of the elements f_1, f_2, f_3 of θ is reducible, then we will say that θ admits an *elementary reduction* or simply that θ is *elementary reducible.*

Lemma 8. *The elementary reducibility of automorphisms of the algebra A is algorithmically recognizable.*[1]

Proof. Let $\theta = (f_1, f_2, f_3)$ be an arbitrary automorphism of A. We will recognize the reducibility of f_3. If $\bar{f}_1, \bar{f}_2$ are algebraically independent, then f_3 is reducible if and only if $\bar{f}_3 \in \langle \bar{f}_1, \bar{f}_2\rangle$. Since $\bar{f}_1, \bar{f}_2$ are homogeneous, this question can be solved trivially, even without a reference to the solubility of the occurrence problem [13], [14]. If $\bar{f}_2 \in \langle \bar{f}_1\rangle$ and $\bar{f}_2 = \alpha \bar{f}_1^k$, then the element f_3 is reducible in θ if and only if it is reducible in the automorphism $(f_1, f_2 - \alpha f_1^k, f_3)$. Since $\deg(f_1, f_2 - \alpha f_1^k, f_3) < \deg\theta$, the statement of the lemma in this case can be proved by induction on $\deg\theta$.

Let now f_1, f_2 be a $*$-reduced pair and $\deg f_1 < \deg f_2$. Assume that there exists a polynomial $G(x, y) \in F[x, y]$ such that $\bar{f}_3 = \overline{G(f_1, f_2)}$. Inequalities (2), (3) gives a bound k for the numbers $\deg_x(G(x, y)), \deg_y(G(x, y))$. Then $G(f_1, f_2)$ is in the space generated by the elements $f_1^i f_2^j$, where $i, j \leq k$. The highest homogeneous parts of elements of this space can be described by triangulation. $\square$

Now we give an example of a tame automorphism, which does not admit an elementary reduction.

Example 1. Put

$$h_1 = x_1, \quad h_2 = x_2 + x_1^2, \quad h_3 = x_3 + 2x_1x_2 + x_1^3,$$
$$g_1 = 6h_1 + 6h_2h_3 + h_3^3, \quad g_2 = 4h_2 + h_3^2, \quad g_3 = h_3.$$

It is easy to show that (h_1, h_2, h_3) and (g_1, g_2, g_3) are tame automorphisms of the algebra A. Note that $\deg g_1 = 9$, $\deg g_2 = 6$, $\deg g_3 = 3$ and g_1, g_2 form a 2-reduced pair. A direct calculation shows that the element

$$f = g_1^2 - g_2^3$$

has degree 8. Hence $\deg f < \deg g_1$, and $\bar{f} \notin \langle \bar{g}_1, \bar{g}_2\rangle$.

Now we define a tame automorphism (f_1, f_2, f_3) by putting

$$f_1 = g_1 + g_3 + f, \quad f_2 = g_2, \quad f_3 = g_3 + f.$$

We have $\deg f_1 = 9$, $\deg f_2 = 6$, $\deg f_3 = 8$ and $\deg[f_i, f_j] > 9$, $1 \leq i < j \leq 3$. Then, using inequality (2), it is easy to check that the automorphism (f_1, f_2, f_3) does not admit an elementary reduction.

[1]In formulation of algorithmic results, we always assume that the ground field F is constructive.

Proposition 1. *Let $\theta = (f_1, f_2, f_3)$ be an automorphism of A such that $\deg f_1 = 2n$, $\deg f_2 = ns$, $s \geq 3$ is an odd number, $2n < \deg f_3 \leq ns$, $\bar{f}_3 \notin \langle \bar{f}_1, \bar{f}_2 \rangle$. Suppose that there exists $0 \neq \alpha \in F$ such that the elements $g_1 = f_1$, $g_2 = f_2 - \alpha f_3$ satisfy the conditions:*

i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = 2n$, $\deg g_2 = ns$;
ii) the element f_3 of the automorphism (g_1, g_2, f_3) is reduced by an automorphism (g_1, g_2, g_3) with the condition $\deg[g_1, g_3] < ns + \deg[g_1, g_2]$.

Then, the following statements are true:

1) $\overline{[f_1, f_2]} = \alpha\overline{[f_1, f_3]}$;
2) $\deg[f_i, f_j] > ns$, *where* $1 \leq i < j \leq 3$;
3) *if* $f \in \langle f_i, f_j \rangle$, *where* $1 \leq i < j \leq 3$, *then either* $\bar{f} \in \langle \bar{f}_i, \bar{f}_j \rangle$ *or* $\deg f > ns$;
4) $\deg f_1 + \deg f_3 > ns$.

Proof. We have

$$g_3 = \sigma f_3 + G(g_1, g_2), \quad \deg g_3 < \deg f_3, \tag{4}$$

where $0 \neq \sigma \in F$, $G(x, y) \in F[x, y]$. Hence

$$\overline{G(g_1, g_2)} = -\sigma \bar{f}_3. \tag{5}$$

If $\deg f_3 = ns$, then $\bar{f}_2, \bar{f}_3$ are linearly independent, since $\bar{f}_3 \notin \langle \bar{f}_1, \bar{f}_2 \rangle$. Therefore $\bar{f}_2, \bar{f}_3, \bar{g}_2$ are mutually linearly independent and $\bar{f}_3 \notin \langle \bar{f}_1, \bar{g}_2 \rangle = \langle \bar{g}_1, \bar{g}_2 \rangle$. If $\deg f_3 < ns$, then $\bar{f}_2 = \bar{g}_2$ and again we have $\bar{f}_3 \notin \langle \bar{g}_1, \bar{g}_2 \rangle$. Put $\deg_y(G(x, y)) = k = 2q + r$, $0 \leq r \leq 1$. The condition $\bar{f}_3 \notin \langle \bar{g}_1, \bar{g}_2 \rangle$ implies, by Corollary 1 and (5), that $q \geq 1$. Then inequality (2) gives $r = 0$ and

$$ns \geq \deg(G(g_1, g_2)) = \deg f_3 \geq q(ns - 2n + \deg[g_1, g_2]).$$

It is easy to deduce from here that if $s > 3$, then $q = 1$, $k = 2$, and if $s = 3$, then $q = 1, 2$, $k = 2, 4$. Besides, these inequalities imply $\deg[g_1, g_2] \leq 2n$ and statement 4) of the proposition.

Applying (1), we get from (4),

$$[g_1, g_3] = \sigma[g_1, f_3] + [g_1, g_2]\frac{\partial G}{\partial y}(g_1, g_2).$$

Since $\deg_y(\frac{\partial G}{\partial y}) = k - 1$ is an odd number, inequality (2) gives

$$\deg([g_1, g_2]\frac{\partial G}{\partial y}(g_1, g_2)) \geq \deg[g_1, g_2] + ns.$$

Consequently, by condition ii),

$$\deg[f_1, f_3] = \deg[g_1, f_3] \geq \deg[g_1, g_2] + ns.$$

Since $\deg[g_1, g_2] \leq 2n$ and $\alpha \neq 0$, the equality

$$[g_1, g_2] = [f_1, f_2] - \alpha[f_1, f_3]$$

gives statement 1) of the proposition and

$$\deg[f_1, f_2] = \deg[f_1, f_3] > ns.$$

If $\deg f_3 = ns$, then, as was remarked earlier, $\bar{f}_2, \bar{f}_3$ are algebraically independent, and so (see [16])

$$\deg[f_2, f_3] = \deg f_2 + \deg f_3 > ns.$$

If $\deg f_3 < ns$, then we have

$$\deg[f_1, f_2] + \deg f_3 < \deg[f_1, f_3] + \deg f_2.$$

By Lemma 1,

$$\deg[f_2, f_3] + \deg f_1 = \deg[f_1, f_3] + \deg f_2;$$

hence

$$\deg[f_2, f_3] = \deg[f_1, f_3] + n(s-2) > ns.$$

Thus statement 2) of the proposition is proved.

To prove 3), it suffices by F2) to consider only the case when $\bar{f}_i, \bar{f}_j$ are algebraically dependent. It is easily seen that f_1, f_2 and f_1, f_3 are $*$-reduced pairs. Suppose that $\bar{f}_2 \in \langle \bar{f}_3 \rangle$. If $\deg f_2 = \deg f_3$ then $\bar{f}_3 \in \langle \bar{f}_2 \rangle$, which contradicts the condition of the proposition. Otherwise $\deg f_2 \geq 2 \deg f_3 > \deg f_1 + \deg f_3$, which contradicts 4). Consequently, the pair f_i, f_j is $*$-reduced for every $i \neq j$ and Corollary 1 by Lemma 2.ii) implies 3). □

Definition 1. If an automorphism $\theta = (f_1, f_2, f_3)$ satisfies the conditions of proposition 1, then we will say that θ admits a reduction of type I, and the automorphism (g_1, g_2, g_3) will be called a reduction of type I of the automorphism θ, with an active element f_3.

The automorphism from Example 1 admits a reduction of type I.

Proposition 2. *Let $\theta = (f_1, f_2, f_3)$ be an automorphism of A such that $\deg f_1 = 2n$, $\deg f_2 = 3n$, $\frac{3n}{2} < \deg f_3 \leq 2n$, and $\bar{f}_1, \bar{f}_3$ are linearly independent. Suppose that there exist $\alpha, \beta \in F$, where $(\alpha, \beta) \neq (0,0)$, such that the elements $g_1 = f_1 - \alpha f_3$, $g_2 = f_2 - \beta f_3$ satisfy the conditions:*

- *i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = 2n$, $\deg g_2 = 3n$;*
- *ii) the element f_3 of the automorphism (g_1, g_2, f_3) is reduced by an automorphism (g_1, g_2, g_3) with the condition $\deg[g_1, g_3] < 3n + \deg[g_1, g_2]$.*

Then, the following statements are true:

1) *$\alpha \in F$ is the solution of the equation $\overline{[f_1, f_2]} = \alpha \overline{[f_3, f_2]}$, or $\alpha = 0$ if it has no solution;*
2) *$\beta \in F$ is the solution of the equation $\overline{[g_1, f_2]} = \beta \overline{[g_1, f_3]}$, or $\beta = 0$ if it has no solution;*
3) *$\deg[f_i, f_j] > 3n$, where $1 \leq i < j \leq 3$;*
4) *if $f \in \langle f_i, f_j \rangle$, where $1 \leq i < j \leq 3$, then either $\bar{f} \in \langle \bar{f}_i, \bar{f}_j \rangle$ or $\deg f > 3n$.*

Proof. Consider equalities (4), (5). If $\deg f_3 < 2n$, then obviously $\overline{G(g_1, g_2)} \notin \langle \bar{g}_1, \bar{g}_2 \rangle$. If $\deg f_3 = 2n$, then by the condition of the proposition, $\bar{f}_1, \bar{f}_3$ are linearly independent. Therefore, either $\alpha = 0$ and $\underline{g_1 = f_1}$, or $\alpha \neq 0$ and $\bar{g}_1, \bar{f}_1, \bar{f}_3$ are mutually linearly independent. In any case, $\overline{G(g_1, g_2)} \notin \langle \bar{g}_1, \bar{g}_2 \rangle$. Since $\deg f_3 \leq 2n$, then, as in the proof of Proposition 1, inequality (2) gives that $\deg_y(G(x,y)) = 2$, $\deg[g_1, g_2] \leq n$. Consequently, Lemma 7.4) gives that $G(x,y)$ is a derivative polynomial (up to a nonzero scalar factor) of the pair g_1, g_2, and by Corollary 3,

$$\deg \left(\frac{\partial G}{\partial y}(g_1, g_2) \right) = 3n.$$

From here, as in the proof of Proposition 1, we get

$$\deg[f_1, f_3] = \deg[g_1, f_3] = \deg[g_1, g_2] + 3n. \tag{6}$$

Consider the triple (g_1, g_2, f_3). By (6) and Lemma 1,

$$\deg[g_2, f_3] + \deg g_1 = \deg[g_1, f_3] + \deg g_2,$$

which yields

$$\deg[f_2, f_3] = \deg[g_2, f_3] = \deg[g_1, f_3] + n. \tag{7}$$

Furthermore,

$$[f_1, f_2] = [g_1 + \alpha f_3, g_2 + \beta f_3] = [g_1, g_2] + \beta[g_1, f_3] + \alpha[f_3, g_2].$$

Since $\deg[g_1, g_2] \le n$, this implies, by (6) and (7), that $\overline{[f_1, f_2]} = \alpha\overline{[f_3, g_2]} = \alpha\overline{[f_3, f_2]}$ if $\alpha \ne 0$, and $\overline{[f_1, f_2]} = \beta\overline{[g_1, f_3]}$ if $\alpha = 0$. Hence $\deg[f_1, f_2] > 3n$, and if $\alpha = 0$, then $[f_1, f_2]$, $[f_3, f_2]$ have different degrees. We have also

$$[g_1, f_2] = [g_1, g_2 + \beta f_3] = [g_1, g_2] + \beta[g_1, f_3].$$

Hence either $\beta \ne 0$ and $\overline{[g_1, f_2]} = \beta\overline{[g_1, f_3]}$, or the elements $[g_1, f_2] = [g_1, g_2]$ and $[g_1, f_3]$ have different degrees. This proves the statements 1), 2), 3) of the proposition. Finally, as in the proof of Proposition 1, Corollary 1 and Lemma 2.ii) give 4). $\square$

Definition 2. If an automorphism $\theta = (f_1, f_2, f_3)$ satisfies the conditions of proposition 2, then we will say that θ admits a reduction of type II, and the automorphism (g_1, g_2, g_3) will be called a reduction of type II of the automorphism θ, with an active element f_3.

Proposition 3. *Let $\theta = (f_1, f_2, f_3)$ be an automorphism of A such that $\deg f_1 = 2n$, and either $\deg f_2 = 3n$, $n < \deg f_3 \le \frac{3n}{2}$, or $\frac{5n}{2} < \deg f_2 \le 3n$, $\deg f_3 = \frac{3n}{2}$. Suppose that there exist $\alpha, \beta, \gamma \in F$ such that the elements $g_1 = f_1 - \beta f_3$, $g_2 = f_2 - \gamma f_3 - \alpha f_3^2$ satisfy the conditions:*

- *i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = 2n$, $\deg g_2 = 3n$;*
- *ii) there exists an element g_3 of the form*

$$g_3 = \sigma f_3 + g,$$

where $0 \ne \sigma \in F$, $g \in \langle g_1, g_2\rangle \setminus F$, such that $\deg g_3 \le \frac{3n}{2}$, $\deg[g_1, g_3] < 3n + \deg[g_1, g_2]$.

Then, the following statements are true:

1) *$\alpha \in F$ is the solution of the equation $\overline{[f_1, f_2]} = 2\alpha\overline{[f_1, f_3]}\bar{f}_3$, or $\alpha = 0$ if it has no solution;*
2) *$\beta \in F$ is the solution of the equation $\overline{[f_2 - \alpha f_3^2, f_1]} = \beta\overline{[f_2, f_3]}$, or $\beta = 0$ if it has no solution;*
3) *$\gamma \in F$ is the solution of the equation $\overline{[g_1, f_2 - \alpha f_3^2]} = \gamma\overline{[g_1, f_3]}$, or $\gamma = 0$ if it has no solution;*
4) *$\deg[f_1, f_3]$, $\deg[f_2, f_3] > 3n$;*
5) *if $(\alpha, \beta, \gamma) \ne (0, 0, 0)$, then $\deg[f_1, f_2] > 3n$; otherwise, $\deg[f_1, f_2] = \deg[g_1, g_2] \le \frac{n}{2}$;*
6) *if $\bar{g}_2 = -\alpha\bar{f}_3^2$, then $\frac{5n}{2} + \deg[g_1, g_2] \le \deg f_2 < 3n$; otherwise, $\deg f_2 = 3n$.*

Proof. Since $\deg f_3, \deg g_3 \le \frac{3n}{2}$, condition *ii)* yields that $\deg g \le \frac{3n}{2}$. Then, by Corollary 2, $g = \lambda w(g_1, g_2)$, where $w(x, y)$ is a derivative polynomial of the g_1, g_2.

Inequality (2) gives also $\deg[g_1, g_2] \leq \frac{n}{2}$. As in the proof of Propositions 1, 2, we obtain also (6), (7), which yields 4). Besides, we have

$$[f_1, f_2] = [g_1, g_2] + \gamma[g_1, f_3] + \beta[f_3, g_2] + 2\alpha[g_1, f_3]f_3.$$

Since $\deg f_3 > n$, this equality yields statements 1), 5) of the proposition. If $\bar{g}_2 = -\alpha\bar{f}_3^2$, then $\alpha \neq 0$, $\deg f_3 = \frac{3n}{2}$, $\deg f_2 < 3n$, and

$$\deg[f_1, f_2] = \deg[g_1, f_3] + \deg f_3 = \deg[g_1, g_2] + 3n + \frac{3n}{2}.$$

Consequently, $\deg f_2 \geq \deg[g_1, g_2] + \frac{5n}{2}$, which proves 6). We have also

$$\begin{aligned} [f_2 - \alpha f_3^2, f_1] &= [g_2, g_1] + \gamma[f_3, g_1] + \beta[g_2, f_3], \\ [g_1, f_2 - \alpha f_3^2] &= [g_1, g_2] + \gamma[g_1, f_3]. \end{aligned}$$

These equalities imply statements 2), 3) of the proposition. □

Definition 3. If an automorphism $\theta = (f_1, f_2, f_3)$ satisfies the conditions of Proposition 3, and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, $\deg g_3 < n + \deg[g_1, g_2]$, then we will say that θ admits a reduction of type III, and the automorphism (g_1, g_2, g_3) will be called a reduction of type III of the automorphism θ, with an active element f_3.

Corollary 4. *In the conditions of Proposition 3, if the automorphism* $\theta = (f_1, f_2, f_3)$ *admits a reduction of type III, then* $\deg(g_1, g_2, g_3) < \deg\theta$.

Proof. By Definition 3, we have

$$\deg(g_1, g_2, g_3) < 6n + \deg[g_1, g_2].$$

Hence it is sufficient to prove that

$$\deg\theta \geq 6n + \deg[g_1, g_2]. \tag{8}$$

It follows from (6) that $\deg f_3 \geq n + \deg[g_1, g_2]$. If $\bar{g}_2 \neq -\alpha\bar{f}_3^2$, then Proposition 3.6) gives $\deg f_2 = 3n$, which proves (8). If $\bar{g}_2 = -\alpha\bar{f}_3^2$, then, as shown above, $\deg f_3 = \frac{3n}{2}$ and by Proposition 3.6), $\deg f_2 \geq \frac{5n}{2} + \deg[g_1, g_2]$, which also gives (8). □

Definition 4. If an automorphism $\theta = (f_1, f_2, f_3)$ satisfies the conditions of Proposition 3, and there exists $0 \neq \mu \in F$ such that $\deg(g_2 - \mu g_3^2) \leq 2n$, then we will say that θ admits a reduction of type IV, and the automorphism $(g_1, g_2 - \mu g_3^2, g_3)$ will be called a reduction of type IV of the automorphism θ, with an active element f_3. In this case we will also call the automorphism (g_1, g_2, g_3) a *predreduction* of type IV of θ.

Corollary 5. *If an automorphism* $\theta = (f_1, f_2, f_3)$ *satisfies all the conditions of Proposition 3 and Definition 4, then* $\deg(g_1, g_2 - \mu g_3^2, g_3) < \deg\theta$.

Proof. Since $\deg(g_2 - \mu g_3^2) \leq 2n$, then $\bar{g}_2 = \mu\bar{g}_3{}^2$ and $\deg g_3 = \frac{3n}{2}$. Consequently, $\deg(g_1, g_2 - \mu g_3^2, g_3) \leq \frac{11n}{2}$. Inequality (8) completes the proof. □

Reductions of types I–IV, together with elementary reductions, permit us to introduce an auxiliary notion of a *simple automorphism*.

Definition 5. By induction on degree, we will define simple automorphisms of the algebra A as follows.

1) All the automorphisms of degree 3 are simple.

2) Suppose that the simple automorphisms of degree $< n$ are already defined.

3) An automorphism θ of degree $n > 3$ is called simple if there exists a simple automorphism of degree $< n$ that is either an elementary reduction or a reduction of type I–IV of θ.

Evidently, any simple automorphism is tame. Our principal goal is to prove the converse statement, that every tame automorphism is simple. We will do it in the next section.

Remark 1. If (f_1, f_2, f_3) is a simple automorphism, then the automorphisms (f_2, f_1, f_3) and $(\alpha f_1 + \beta, f_2, f_3)$, where $\alpha, \beta \in F$, $\alpha \neq 0$, are simple as well.

Really, if (g_1, g_2, g_3) is an elementary reduction or a reduction of type I–IV of (f_1, f_2, f_3), then (g_2, g_1, g_3) is a reduction of the same type for (f_2, f_1, f_3). It is also clear that we can always choose $\alpha_i, \beta_i \in F$ such that $(\alpha_1 g_1 + \beta_1, \alpha_2 g_2 + \beta_2, \alpha_3 g_3 + \beta_3)$ becomes a corresponding reduction for $(\alpha f_1 + \beta, f_2, f_3)$.

For convenience of terminology, we introduce also

Definition 6. An element f_i, $i = 1, 2, 3$, of the automorphism $\theta = (f_1, f_2, f_3)$ is called *simple reducible* if it is reduced by a simple automorphism.

4. A characterization of tame automorphisms

This section is devoted to the proof of our main result.

Theorem 1. *Every tame automorphism of the algebra A is simple.*

The plan of the proof. Assume that the statement of the theorem is not true. Then there exist tame automorphisms $\theta = (f_1, f_2, f_3)$, τ of A such that θ is simple, τ is not simple, and

$$\theta = (f_1, f_2, f_3) \to \tau.$$

In the set of all pairs of automorphisms with this property we choose and fix a pair θ, τ with the minimal $\deg \theta$.

In order to obtain a contradiction, it is enough to prove that τ is simple. The proof will consist of analysis of the cases, when θ admits an elementary reduction or a reduction of type I–IV to a simple automorphism of lower degree. If θ admits a reduction of type I–IV, then it will be convenient for us to fix the reduction of θ and consider one of the following variants for τ:

$$\tau = (f, f_2, f_3), \quad f = f_1 + a, \quad a \in \langle f_2, f_3 \rangle, \quad \deg a \leq \deg f_1, \tag{9}$$

$$\tau = (f_1, f, f_3), \quad f = f_2 + a, \quad a \in \langle f_1, f_3 \rangle, \quad \deg a \leq \deg f_2, \tag{10}$$

$$\tau = (f_1, f_2, f), \quad f = f_3 + a, \quad a \in \langle f_1, f_2 \rangle, \quad \deg a \leq \deg f_3. \tag{11}$$

Here, the restriction on $\deg a$ is imposed in order to exclude the trivial case when θ is an elementary reduction of τ. In the case when θ admits an elementary reduction, we will assume that τ has form (11). The proof of the theorem will be completed by Lemmas 9–17 and by Propositions 4, 5. □

The following evident statement is formulated for convenience of references.

Lemma 9. *Let ϕ be a simple automorphism of A such that $\deg \phi < \deg \theta$. If $\phi \to \psi$, then ψ is simple too.*

The proof follows immediately from the minimality condition for $\deg \theta$.

Corollary 6. *Suppose that there exists a sequence of automorphisms*

$$\phi_0 \to \phi_1 \to \ldots \to \phi_{k-1} \to \phi_k$$

of the algebra A such that $\deg \phi_i < \deg \theta$ for $0 \leq i \leq k-1$, and ϕ_0 is a simple automorphism. Then ϕ_k is also simple.

Lemma 10. *If θ admits a reduction of type I, then τ is simple.*

Proof. We adopt all the conditions and notation of Proposition 1 and by the definition of reduction we have that (g_1, g_2, g_3) is simple. Without loss of generality, we can also put

$$g_3 = f_3 + g, \quad g \in \langle g_1, g_2 \rangle, \ \deg g_3 < \deg f_3. \tag{12}$$

If τ is of form (9), then by Proposition 1.3) we have $\bar{a} \in \langle \bar{f}_2, \bar{f}_3 \rangle$. Since $\deg a \leq \deg f_1 < \min\{\deg f_2, \deg f_3\}$, this implies $a \in F$. Hence τ is simple by Remark 1.

Assume that τ has form (10). By statements 3) and 4) of Proposition 1, we have

$$\bar{a} = \beta \bar{f}_3 + \gamma (\bar{f}_1)^k, \ \beta, \gamma \in F, \ 2nk \leq \deg f_2.$$

Consider $a_1 = a - \beta f_3 - \gamma (f_1)^k$. Then again $a_1 \in \langle f_1, f_3 \rangle$ and $\deg a_1 < \deg a$; hence $\bar{a}_1 \in \langle \bar{f}_1 \rangle$, and it is easy to see that $a_1 \in \langle f_1 \rangle$. Thus we have

$$a = \beta f_3 + T(f_1), \quad \deg(T(f_1)) \leq \deg f_2, \quad f = f_2 + \beta f_3 + T(f_1).$$

Since $\deg f_1 = 2n$, $\deg f_2 = sn$, where s is odd, $\deg(T(f_1)) < \deg f_2$. Furthermore, $\deg f$ can be less than $\deg f_2$ only if $\bar{f}_2 = -\beta \bar{f}_3$. But $\bar{f}_3 \notin \langle \bar{f}_1, \bar{f}_2 \rangle$; hence $\deg f = \deg f_2$. Put

$$g_2' = f - (\alpha + \beta) f_3 = (f_2 - \alpha f_3) + T(f_1) = g_2 + T(g_1).$$

Then $\overline{g_2'} = \bar{g}_2$, and $g \in \langle g_1, g_2 \rangle = \langle g_1, g_2' \rangle$. If $\alpha + \beta \neq 0$, then (12) implies that (g_1, g_2', g_3) is a reduction of type I of τ. If $\alpha + \beta = 0$, then $\tau = (g_1, g_2', f_3)$ and the element f_3 is reduced in τ by (g_1, g_2', g_3). It remains to note that, by Lemma 9, (g_1, g_2', g_3) is a simple automorphism.

Now consider the case when τ has form (11). Proposition 1 gives, as before,

$$a = \beta f_2 + T(f_1), \quad \deg(T(f_1)) \leq \deg f_3, \quad f = f_3 + \beta f_2 + T(f_1),$$

and $\beta \neq 0$ is possible only if $\deg f_3 = \deg f_2$. Consider 3 cases:
1) $\beta = 0$, 2) $\beta(1 + \alpha\beta) \neq 0$, 3) $1 + \alpha\beta = 0$.

In case 1) we put

$$g_2' = f_2 - \alpha f = g_2 - \alpha T(g_1).$$

Since $\deg g_1 \not| \deg g_2$, the equality $\deg(T(g_1)) = \deg g_2$ is impossible, $\overline{g_2'} = \bar{g}_2$. By Lemma 9, the automorphism $\phi = (g_1, g_2', g_3)$ is simple again. Since

$$f = f_3 + T(f_1) = g_3 - g + T(g_1), \quad -g + T(g_1) \in \langle g_1, g_2 \rangle = \langle g_1, g_2' \rangle,$$

ϕ is a reduction of type I of τ.

In case 2) we put

$$g_2' = f_2 - \frac{\alpha}{1 + \alpha\beta} f = \frac{1}{1 + \alpha\beta} g_2 - \frac{\alpha}{1 + \alpha\beta} T(g_1).$$

A direct calculation gives

$$f = (1 + \alpha\beta) g_3 + g',$$

where

$$g' = -(1+\alpha\beta)g + \beta(1+\alpha\beta)g_2' + (1+\alpha\beta)T(g_1), \quad g' \in \langle g_1, g_2\rangle = \langle g_1, g_2'\rangle.$$

Since $\overline{g_2'} = \frac{1}{1+\alpha\beta}\,\bar{g}_2$, then (g_1, g_2', g_3) is a reduction of type I of τ.

In case 3) we have

$$\begin{aligned} f &= \beta(f_2 - \alpha f_3) + T(f_1) = \beta g_2 + T(g_1), \\ f_2 &= \alpha g_3 + (g_2 - \alpha g),\ g_2 - \alpha g \in \langle g_1, g_2\rangle = \langle g_1, f\rangle. \end{aligned}$$

Therefore, $\bar{f} = \beta\bar{g}_2$, and it is easy to check that f_2 is reduced in $\tau = (g_1, f_2, f)$ by (g_1, g_3, f). By Remark 1, (g_1, g_3, g_2) is simple. Thus, by Lemma 9, (g_1, g_3, f) is simple too. □

Lemma 11. *If θ admits a reduction of type II, then τ is simple.*

Proof. We adopt all the conditions and notation of Proposition 2, as well as equality (12). If τ has form (9), then Proposition 2.4) and the condition $\deg a \le \deg f_1$ give $a = \gamma f_3 + \lambda$. By Remark 1 we may assume that $\lambda = 0$. So

$$\begin{gathered} a = \gamma f_3, \quad f = f_1 + \gamma f_3, \\ f - (\gamma+\alpha)f_3 = g_1, \quad f_2 - \beta f_3 = g_2. \end{gathered}$$

If $(\gamma+\alpha, \beta) \ne (0,0)$, then it is easily checked that (g_1, g_2, g_3) is a reduction of type II of τ. Otherwise, the element f_3 is reduced in the automorphism $\tau = (g_1, g_2, f_3)$ by (g_1, g_2, g_3).

If τ has form (10), then by Proposition 2.4) (and Remark 1) we get

$$a = \gamma f_3 + \delta f_1, \quad f = f_2 + \gamma f_3 + \delta f_1.$$

Furthermore,

$$f - (\gamma + \beta + \delta\alpha)f_3 = g_2 + \delta g_1 = g_2', \quad f_1 - \alpha f_3 = g_1.$$

If $(\alpha, \gamma+\beta+\delta\alpha) \ne (0,0)$, then (g_1, g_2', g_3) is a reduction of type II of τ. Otherwise, (g_1, g_2', g_3) reduces the element f_3 of $\tau = (g_1, g_2', f_3)$.

Assume that τ has form (11). Proposition 2 and Remark 1 give

$$a = \gamma f_1, \quad f = f_3 + \gamma f_1,$$

and $\gamma \ne 0$ is possible only if $\deg f_1 = \deg f_3$. Since $\bar{f}_1, \bar{f}_3$ are linearly independent, $\deg f = \deg f_3$.

If $1 + \alpha\gamma \ne 0$, a direct calculation gives

$$\begin{aligned} f = f_3 + \gamma f_1 &= (1+\alpha\gamma)g_3 + (\gamma g_1 - (1+\alpha\gamma)g), \\ f_1 - \frac{\alpha}{1+\alpha\gamma}\, f &= \frac{1}{1+\alpha\gamma}\, g_1 = g_1', \\ f_2 - \frac{\beta}{1+\alpha\gamma}\, f &= g_2 - \frac{\beta\gamma}{1+\alpha\gamma}\, g_1 = g_2'. \end{aligned}$$

Since $\langle g_1, g_2\rangle = \langle g_1', g_2'\rangle$, it is easy to check that (g_1', g_2', g_3) is a reduction of type II of τ. By Corollary 6, the automorphism (g_1', g_2', g_3) is simple.

If $1 + \alpha\gamma = 0$, then

$$\begin{gathered} f = \gamma g_1, \ f_1 = \alpha g_3 + (g_1 - \alpha g), \\ f_2 - \frac{\beta}{\alpha} f_1 = g_2 - \frac{\beta}{\alpha} g_1 = g_2'. \end{gathered}$$

If $\beta = 0$, then the element f_1 is reduced in $\tau = (f_1, g_2', \gamma g_1)$ by $(g_3, g_2', \gamma g_1)$. Otherwise, it is easily checked that $(g_3, g_2', \gamma g_1)$ is a reduction of type II of τ with an active element f_1. □

Lemma 12. *If θ admits a reduction of type III or IV, then τ is simple.*

Proof. We adopt the conditions and notation of Proposition 3. Assume that τ has form (9). Let us show that in this case $a = \delta_1 f_3$.

Evidently, it suffices to prove that, for any $b \in \langle f_2, f_3\rangle$ with $\deg b \le \deg f_1 = 2n$, $\bar{b} \in \langle \bar{f_3}\rangle$ holds. Note that $\deg f_2 > \frac{5n}{2} > \deg b$. By F2) we may assume, without loss of generality, that $\bar{f}_2, \bar{f}_3$ are algebraically dependent. If $\bar{f}_2 \notin \langle \bar{f_3}\rangle$, then the pair f_2, f_3 is $*$-reduced, and the statement holds by Proposition 3.4), Corollary 1, and Lemma 2.ii). Otherwise $\deg f_3 = \frac{3n}{2}$, $\deg f_2 = 3n$, $\bar{f}_2 = \lambda(\bar{f}_3)^2$, where $0 \ne \lambda \in F$. Since $\deg g_2 = 3n$, then $\bar{f}_2 \ne \alpha \bar{f}_3^2$, i.e., $\lambda \ne \alpha$. Consider $f_2' = f_2 - \lambda f_3^2 = g_2 + \gamma f_3 + (\alpha - \lambda) f_3^2$. Then

$$[f_1, f_2'] = [g_1, g_2] + \gamma[g_1, f_3] + \beta[f_3, g_2] + 2(\alpha - \lambda)[g_1, f_3] f_3.$$

Since $(\alpha - \lambda) \ne 0$, we have, as in the proof of Proposition 3, that $\deg[f_1, f_2'] = \deg[g_1, g_2] + \frac{9n}{2}$, which yields $\deg f_2' > \frac{5n}{2}$. Note that $\langle f_2', f_3\rangle = \langle f_2, f_3\rangle$. If $\bar{f}_2', \bar{f}_3$ are algebraically independent, then by F2) we get $\bar{b} \in \langle \bar{f}_2', \bar{f}_3\rangle$. Otherwise, f_2', f_3 form a $*$-reduced pair, and since $\deg[f_2', f_3] = \deg[f_2, f_3] > 3n$, we have again $\bar{b} \in \langle \bar{f}_2', \bar{f}_3\rangle$ by Corollary 1 and Lemma 2.ii). But $\deg f_2' > \frac{5n}{2} > \deg b$, and so $\bar{b} \in \langle \bar{f}_3\rangle$.

Thus $a = \delta_1 f_3$, and so

$$f = f_1 + \delta_1 f_3 = g_1 + (\beta + \delta_1) f_3, \quad f_2 = g_2 + \gamma f_3 + \alpha f_3^2.$$

Since f_3 is preserved in the structure of $\tau = (f, f_2, f_3)$, it is easily checked that if θ admits a reduction of type IV, then (g_1, g_2, g_3) is a predreduction of type IV of τ. Suppose that θ admits a reduction of type III. If $(\alpha, \beta + \delta_1, \gamma) \ne (0, 0, 0)$, then (g_1, g_2, g_3) is a reduction of type III of τ. Otherwise, since $\deg g_3 < \deg f_3$ (see the proof of Corollary 4), the element f_3 is reduced in $\tau = (g_1, g_2, f_3)$ by (g_1, g_2, g_3).

Suppose that τ has form (10). Then, by Proposition 3.4) and Corollary 1 we have $\bar{a} \in \langle \bar{f}_1, \bar{f}_3\rangle$. Note that $\deg a \le \deg f_2 \le 3n$ and $\deg(f_1^2)$, $\deg(f_1 f_3)$, $\deg(f_3^3) > 3n$. So

$$a = \delta_1 f_3^2 + \sigma_1 f_3 + \mu_1 f_1,$$

and $\delta_1 \ne 0$ is possible only if $\deg f_2 \ge 2 \deg f_3$. Therefore,

$$f_1 = g_1 + \beta f_3, \quad f = g_2 + \mu_1 g_1 + (\alpha + \delta_1) f_3^2 + (\gamma + \sigma_1 + \mu_1 \beta) f_3.$$

Hence, if $\deg f \ne 3n$, then $\bar{g_2} + (\alpha + \delta_1)\bar{f_3}^2 = 0$, i.e., $\alpha + \delta_1 \ne 0$, $\deg f_3 = \frac{3n}{2}$. By Proposition 3,

$$\deg[f_3, g_1] = \deg[f_3, f_1] > 3n.$$

Since

$$[f, g_1] = [g_2, g_1] + 2(\alpha + \delta_1)[f_3, g_1] f_3 + (\gamma + \sigma_1 + \mu_1 \beta)[f_3, g_1],$$

then

$$\deg[f, g_1] > 3n + \frac{3n}{2} = \frac{9n}{2}.$$

Consequently, $\deg f > \frac{5n}{2}$. Now it is easy to show, that if θ admits a reduction of type IV, then $(g_1, g_2 + \mu_1 g_1, g_3)$ is a predreduction of type IV of τ. Suppose now that θ admits a reduction of type III. If $(\alpha + \delta_1, \beta, \gamma + \sigma_1 + \mu_1 \beta) \ne 0$, then

$(g_1, g_2 + \mu_1 g_1, g_3)$ is a reduction of type III of τ. Otherwise, the element f_3 is reduced in $\tau = (g_1, g_2 + \mu_1 g_1, f_3)$ by $(g_1, g_2 + \mu_1 g_1, g_3)$.

Assume that τ has form (11). If $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, then Proposition 3 yields $a \in F$. If $\alpha = \beta = \gamma = 0$, then θ admits a reduction of type IV. We have

$$f_1 = g_1, \quad f_2 = g_2, \quad f_3 = \frac{1}{\sigma} g_3 - \frac{1}{\sigma} g,$$

where $g \in \langle g_1, g_2 \rangle \setminus F$. Since $\deg a \leq \deg f_3 \leq \frac{3n}{2}$, we have

$$f = \frac{1}{\sigma} g_3 + c,$$

where $c = -\frac{1}{\sigma} g + a \in \langle g_1, g_2 \rangle$ and $\deg c \leq \frac{3n}{2}$. If $c \notin F$, then (g_1, g_2, g_3) is a predreduction of type IV of τ. Otherwise, by Remark 1 we can take $\tau = (g_1, g_2, \frac{1}{\sigma} g_3)$, and the element g_2 is reduced in τ by $(g_1, g_2 - \mu g_3^2, \frac{1}{\sigma} g_3)$. □

It remains now to consider the principal case, when θ admits an elementary reduction. It follows from Lemma 9 that if ϕ is simple and $\deg \phi < \deg \theta$, then every reducible element of ϕ is simple reducible. But one should carefully distinguish these notions if $\deg \phi \geq \deg \theta$.

Lemma 13. *Let $\phi = (g_1, g_2, g_3)$ be a simple automorphism and $\deg \phi \leq \deg \theta$. If g_1 is a simple reducible element of ϕ, then every elementary transformation ψ of ϕ changing only g_1 gives a simple automorphism.*

Proof. Assume that the element g_1 is reduced in ϕ by a simple automorphism $\phi' = (h_1, g_2, g_3)$. Then $\deg \phi' < \deg \theta$ and $h_1 = \alpha g_1 + g$, where $0 \neq \alpha \in F$, $g \in \langle g_2, g_3 \rangle$. Put $\psi = (\beta g_1 + T(g_2, g_3), g_2, g_3)$. Then

$$\beta g_1 + T(g_2, g_3) = \frac{\beta}{\alpha} h_1 + g', \quad g' \in \langle g_2, g_3 \rangle.$$

Hence $\phi' \to \psi$ and Lemma 9 completes the proof. □

In the sequel we will assume that τ has form (11) and θ admits an elementary reduction. Then Lemma 13 gives

Corollary 7. *If f_3 is a simple reducible element of θ, then τ is a simple automorphism.*

In the remainder of this section we will assume that f_3 is not a simple reducible element of θ. Then either f_1 or f_2 are simple reducible. For definiteness, we assume that f_2 is reduced in θ by a simple automorphism $\phi = (f_1, g_2, f_3)$, where

$$g_2 = f_2 + b, \quad b \in \langle f_1, f_3 \rangle, \quad \deg g_2 < \deg f_2. \tag{13}$$

Lemma 14. *The automorphism τ is simple if one of the following conditions is satisfied:*

1) $\bar{f}_2 \in \langle \bar{f}_1 \rangle$;
2) $\bar{f}_3 \in \langle \bar{f}_1 \rangle$;
3) *a does not depend on f_2;*
4) *$\bar{f}_1, \bar{f}_3$ are algebraically independent.*

Proof. If $\bar{f}_2 \in \langle \bar{f}_1 \rangle$, then by Lemma 13 we can choose an element b satisfying (13) such that $b \in \langle f_1 \rangle$. Then $\langle f_1, f_2 \rangle = \langle f_1, g_2 \rangle$. Consequently, there exists a sequence of elementary transformations of the form

$$\phi = (f_1, g_2, f_3) \to (f_1, g_2, f) \to (f_1, f_2, f) = \tau. \tag{14}$$

Since $\deg\phi$, $\deg(f_1, g_2, f) < \deg\theta$ (by (13)), it follows from Corollary 6 that the automorphism τ is simple.

If $\bar{f}_3 \in \langle \bar{f}_1 \rangle$ and $\bar{f}_3 = T(\bar{f}_1)$, then we put $g_3 = f_3 - T(f_1)$. There exists a sequence

$$\phi = (f_1, g_2, f_3) \to (f_1, g_2, g_3) \to (f_1, f_2, g_3) \to (f_1, f_2, f_3) = \theta.$$

By Corollary 6, the automorphism (f_1, f_2, g_3) is simple, and consequently, f_3 is a simple reducible element of θ, which is impossible.

If a does not depend on f_2, then sequence (14) proves the simplicity of τ.

Assume that $\bar{f}_1, \bar{f}_3$ are algebraically independent. Then by (13) we obtain that $\bar{f}_2 = -\bar{b} \in \langle \bar{f}_1, \bar{f}_3 \rangle$. By 1), we can assume that $\bar{f}_2 \notin \langle \bar{f}_1 \rangle$, i.e., $\bar{f}_2$ depends on $\bar{f}_3$. Then $\deg f_3 \leq \deg f_2$. Observe that $\bar{f}_1, \bar{f}_2$ are algebraically independent; otherwise, $\bar{f}_1, \bar{f}_3$ would be algebraically dependent. Therefore, $\bar{a} \in \langle \bar{f}_1, \bar{f}_2 \rangle$. By 3), we can also assume that a contains f_2. Then (11) gives $\deg f_2 \leq \deg a \leq \deg f_3$, and so $\deg f_2 = \deg f_3$. Thus

$$b = \alpha f_3 + T(f_1), \quad \alpha \neq 0, \quad \deg(T(f_1)) \leq \deg f_2.$$

We have the equalities

$$\begin{aligned} g_2 &= f_2 + \alpha f_3 + T(f_1), \\ f_2 &= g_2 - \alpha f_3 - T(f_1), \\ f_3 &= \frac{1}{\alpha} g_2 - \frac{1}{\alpha} f_2 - \frac{1}{\alpha} T(f_1), \end{aligned}$$

which induce the following sequence of elementary transformations:

$$(f_1, f_3, g_2) \to (f_1, f_2, g_2) \to (f_1, f_2, f_3) = \theta. \tag{15}$$

Since $\phi = (f_1, g_2, f_3)$ is simple, by Remark 1, (f_1, f_3, g_2) is simple too. By Lemma 9, sequence (15) gives simple reducibility of f_3 in θ, a contradiction. □

Thus, by Lemma 14, we can suppose that $\bar{f}_1, \bar{f}_3$ are algebraically dependent and $\bar{f}_3 \notin \langle \bar{f}_1 \rangle$. We will consider separately the 3 cases:

1) f_1, f_3 is a $*$-reduced pair, $\deg f_1 < \deg f_3$;
2) f_1, f_3 is a $*$-reduced pair, $\deg f_3 < \deg f_1$;
3) $\bar{f}_1 \in \langle \bar{f}_3 \rangle$, $\deg f_1 > \deg f_3$.

First, we will prove two propositions.

Proposition 4. *Let $\psi = (g_1, g_2, g_3)$ be a simple automorphism satisfying the following conditions:*

i) $\deg\psi \leq \deg\theta$*;*

ii) g_1, g_2 *is a 2-reduced pair and* $\deg g_1 = 2n$, $\deg g_2 = 3n$*;*

iii) $\bar{g}_1, \bar{g}_3$ *are linearly independent,* $\deg g_3 = m < 3n$*, and* g_3 *is not a simple reducible element of* ψ*.*

Then one of the following statements is satisfied:

1) $m < n + \deg[g_1, g_2]$*;*

2) ψ *admits a reduction of type IV with an active element* g_3*;*

3) $\deg[g_1, g_3] < 3n + \deg[g_1, g_2]$*, and there exists* $\alpha \in F$ *such that* $\deg(g_2 - \alpha g_3^2) \leq 2n$.

Proof. Assume that the proposition is not true, and let ψ be a counterexample of minimal degree. Then we have

$$n + \deg[g_1, g_2] \leq m < 3n. \tag{16}$$

Hence $\deg[g_1, g_2] < 2n$, and by Propositions 1, 2, 3, the automorphism ψ does not admit a reduction of types I–III. By its choice, neither admits ψ a reduction of type IV. So, by the definition of a simple automorphism, ψ admits an elementary reduction. By *iii*), either g_1 or g_2 is a simple reducible element of ψ. Since $\bar{g_1}, \bar{g_2}$ are algebraically dependent, this implies that $\bar{g_1}, \bar{g_2}, \bar{g_3}$ are mutually algebraically dependent. It follows from *iii*) and (16) that g_1, g_3 is a $*$-reduced pair.

Case 1. $2n < m < 3n$.

Assume that g_2 is a simple reducible element of ψ. The inequalities imposed on m make impossible the inclusion $\bar{g_2} \in \langle \bar{g_1}, \bar{g_3} \rangle$; hence Corollary 1 yields

$$\deg g_2 = 3n \geq N(g_1, g_3) = \frac{2n}{(2n, m)} m - m - 2n + \deg[g_1, g_3].$$

Thus $\frac{2n}{(2n,m)} \leq 3$. If $\frac{2n}{(2n,m)} = 2$, then $m = nt$, $t \geq 3$ is an odd number, which contradicts (16). Hence $\frac{2n}{(2n,m)} = 3$. Putting $(2n, m) = 2\rho$, we have $n = 3\rho$, $\deg g_1 = 6\rho$, $m = 2\rho t$, $t > 3$, $3 \nmid t$. Applying again Corollary 1, we get

$$9\rho > 6\rho t - 2\rho t - 6\rho,$$

i.e., $t < \frac{15}{4}$, which is impossible.

If g_1 is a simple reducible element of ψ, then the pair g_2, g_3 is $*$-reduced, and Corollary 1 yields

$$\deg g_1 = 2n \geq N(g_2, g_3) = \frac{m}{(3n, m)} 3n - 3n - m + \deg[g_2, g_3]. \tag{17}$$

Thus $\frac{m}{(3n,m)} = 2$ and $3n = (3n, m)t$, $t \geq 3$ is an odd number. Since $m > 2n$, we have $(3n, m) > n$ and $3n = (3n, m)t \geq (3n, m)3 > 3n$, a contradiction.

Case 2. $\frac{3n}{2} < m \leq 2n$.

Since $\bar{g_1}, \bar{g_3}$ are algebraically dependent, they would be linearly dependent if $m = 2n$, which contradicts *iii*). Therefore, $m < 2n$. If g_2 is a simple reducible element of ψ, then Corollary 1 gives

$$\deg g_2 = 3n \geq N(g_3, g_1) = \frac{m}{(2n, m)} 2n - 2n - m + \deg[g_1, g_3]. \tag{18}$$

Therefore, $\frac{m}{(2n,m)} \leq 3$. If $\frac{m}{(2n,m)} = 2$, then $2n = (2n, m)t$, $t \geq 3$ is an odd number. Hence $m = \frac{4n}{t}$, and the inequality $m > \frac{3n}{2}$ yields $t < \frac{8}{3}$, which is impossible. If $\frac{m}{(2n,m)} = 3$, then $2n = (2n, m)t$, $t > 3$, $3 \nmid t$. Thus $m = \frac{6n}{t} > \frac{3n}{2}$ and $t < 4$, which is also impossible.

If g_1 is a simple reducible element of ψ, then it follows from (17) that $\frac{m}{(3n,m)} = 2$, $3n = (3n, m)t$, $t \geq 3$ is an odd number. Since $\frac{3n}{2} < m = \frac{6n}{t} < 2n$, then $3 < t < 4$, which is impossible. So case 2 is done.

Now we can assume that $m \leq \frac{3n}{2}$. Then (16) gives $\deg[g_1, g_2] \leq \frac{n}{2}$, and consequently,

$$\deg[g_1, g_2] + \deg g_3 \leq 2n.$$

Then, by Lemma 1, we have

$$\deg[g_2, g_3] + \deg g_1 = \deg[g_1, g_3] + \deg g_2,$$

i.e.,

$$\deg[g_2, g_3] = \deg[g_1, g_3] + n. \tag{19}$$

Case 3. $m = \frac{3n}{2}$.

By putting $n = 2\rho$, we have $\deg g_1 = 4\rho$, $\deg g_2 = 6\rho$, $\deg g_3 = 3\rho$. Put also $\bar{g}_2 = \alpha \bar{g}_3{}^2$ (recall that $\bar{g}_2$ and $\bar{g}_3$ are algebraically dependent). We will first show that g_2 is a simple reducible element of ψ. Assume that g_1 is a simple reducible element of ψ which is reduced by a simple automorphism $\psi_1 = (h_1, g_2, g_3)$. Then $\deg \psi_1 < \deg \theta$, and by Lemma 9 the automorphism $\psi_2 = (h_1, g_2 - \alpha g_3^2, g_3)$ is also simple. The sequence

$$\psi_2 \to (g_1, g_2 - \alpha g_3^2, g_3) \to (g_1, g_2, g_3) = \psi$$

proves that g_2 is a simple reducible element of ψ.

Put

$$h_2 = g_2 - T(g_3, g_1),\ \deg h_2 < \deg g_2, \tag{20}$$

where h_2 is an unreducible element of $\xi = (g_1, h_2, g_3)$. Let $\deg_y(T(x,y)) = k = 3q + r$, $0 \le r < 3$. Since $\deg(T(g_3, g_1)) = \deg g_2 = 6\rho$, inequality (2) yields that either $q = 1$, $r = 0$, or $q = 0$, $r = 0, 1$.

Consider first the case $q = 1$. Note that g_3, g_1 is a 3-reduced pair and $\deg_y(w(x,y)) = 3$, where $w(x,y)$ is a derivative polynomial of the pair g_3, g_1. By Lemma 7, the polynomial $T(x,y)$ can be presented in the form

$$T(x,y) = w(x,y)q(x,y) + s(x,y),$$

where $\deg_y(q(x,y)) = 0$, $\deg_y(s(x,y)) < 3$. By Lemma 6.3),

$$\deg(w(g_3, g_1)) \ge N(g_3, g_1) = 3 \cdot 4\rho - 4\rho - 3\rho + \deg[g_1, g_3] > 5\rho.$$

Since $\deg(T(g_3, g_1)) = 6\rho$, Lemma 7.2) gives $\deg(q(g_3, g_1)) < \rho$, $\deg(s(g_3, g_1)) \le 6\rho$. The polynomials $q(x,y)$, $s(x,y)$ satisfy condition (ii) of Corollary 1. Hence they satisfy also (iii), and we get $q(x,y) = \lambda \ne 0$, $s(x,y) = \gamma x^2 + \delta x + \mu y$. Therefore,

$$T(x,y) = \lambda w(x,y) + \gamma x^2 + \delta x + \mu y.$$

Consequently, by Corollary 3,

$$\deg\left(\frac{\partial T}{\partial x}(g_3, g_1)\right) = 9\rho, \quad \deg\left(\frac{\partial T}{\partial y}(g_3, g_1)\right) = 8\rho.$$

By (1) and (20),

$$[g_1, h_2] = [g_1, g_2] - [g_1, g_3]\frac{\partial T}{\partial x}(g_3, g_1), \tag{21}$$

$$[h_2, g_3] = [g_2, g_3] - [g_1, g_3]\frac{\partial T}{\partial y}(g_3, g_1). \tag{22}$$

This yields, by (16) and (19),

$$\begin{aligned} \deg[g_1, h_2] &= \deg[g_1, g_3] + 9\rho, \\ \deg[h_2, g_3] &= \deg[g_1, g_3] + 8\rho. \end{aligned}$$

Therefore, $\deg h_2 \ge \deg[g_1, g_3] + 5\rho$. Since $\deg h_2 < \deg g_2 = 6\rho$, this yields $\deg[g_1, g_3] < \rho$. Now, applying F2) and Corollary 1, it is easy to show that g_1, g_3 are unreducible elements of $\xi = (g_1, h_2, g_3)$. Since $\deg[g_1, g_3] < \rho$, by Propositions 1, 2, 3, we conclude that ξ does not admit reductions of types I–IV. This contradicts the simplicity of ξ.

Hence $q = 0$, $r = 0, 1$. By Corollary 1 ((ii) implies (iii)), we have

$$T(x,y) = \alpha x^2 + \beta x + \gamma y.$$

Then $\deg(\frac{\partial T}{\partial y}(g_3, g_1)) \leq 0$ and by (19), (22), we conclude that

$$\deg[h_2, g_3] = \deg[g_2, g_3] = \deg[g_1, g_3] + 2\rho. \tag{23}$$

Now we will assume that $\deg h_2 > 2n$ and show that this case is impossible. If g_3 is a simple reducible element of ξ, then g_1, h_2 form a $*$-reduced pair, and it follows easily from (2) that this pair is 2-reduced. But it is impossible since $\deg h_2 < 3n$. Furthermore, if g_1 is a simple reducible element of ξ, then g_3, h_2 form a 2-reduced pair, that is, $\deg h_2 = \frac{9}{2}\rho$. Put

$$h_1 = g_1 + Q(g_3, h_2),\ \deg h_1 < \deg g_1,$$

where h_1 is an unreducible element of the simple automorphism $\xi_1 = (h_1, h_2, g_3)$.

Observe that ξ_1 satisfies all the conditions of the present proposition. By (2) and (23), we get $\deg_y(Q(x, y)) = 2$. Lemma 7.4) and Corollary 3 give

$$\deg\left(\frac{\partial Q}{\partial y}(g_3, h_2)\right) = \deg h_2 = \frac{9}{2}\rho.$$

Since

$$[h_1, g_3] = [g_1, g_3] + [h_2, g_3]\frac{\partial Q}{\partial y}(g_3, h_2),$$

by (23) we have

$$\deg[h_1, g_3] = \deg[h_2, g_3] + \frac{9}{2}\rho.$$

Therefore,

$$\deg h_1 \geq \deg[h_2, g_3] + \frac{3}{2}\rho = \frac{7}{2}\rho + \deg[g_1, g_3].$$

These inequalities show that ξ_1 does not satisfy the conclusions of the proposition. Since $\deg \xi_1 < \deg \psi$, this contradicts our choice of ψ.

We have thus shown that ξ does not admit elementary reductions. As for reductions of type I, the only possibility for ξ to have it is when g_1 is an active element of reduction and $\deg h_2 = \frac{9}{2}\rho$. Then there should exist $\alpha_1 \in F$ such that the element g_1 of the automorphism $(g_1, h_2 - \alpha_1 g_1, g_3)$ is reducible. Recall that $\overline{h_2} = \overline{g_2} - \alpha g_3^2 - \beta g_3 - \gamma g_1$. Set $h_2' = h_2 - \alpha_1 g_1 = g_2 - \alpha g_3^2 - \beta g_3 - (\gamma + \alpha_1)g_1$. Then $\overline{h_2'} = \overline{h_2}$ and we can replace h_2 by h_2' in our previous arguments. Then, as above, the condition for g_1 to be a reducible element of (g_1, h_2', g_3) gives a contradiction.

Furthermore, the comparison of the degrees of the components of ξ shows that ξ does not admit a reduction of type II. Thus ξ admits a reduction of type III or IV, with an active element g_3. Let (r_1, r_2, r_3) be a reduction of type III or a predreduction of type IV of ξ,

$$r_1 = g_1 - \beta_1 g_3, \quad r_2 = h_2 - \alpha_1 g_3^2 - \gamma_1 g_3.$$

Observe that, by definition, $\deg r_1 = 4\rho,\ \deg r_2 = 6\rho$. We have

$$\begin{aligned} r_2 &= g_2 - (\alpha + \alpha_1)g_3^2 - (\beta + \gamma_1)g_3 - \gamma g_1, \\ [r_1, r_2] &= [g_1, g_2] - 2(\alpha + \alpha_1)[g_1, g_3]g_3 \\ &\quad - (\beta + \gamma_1 + \gamma\beta_1)[g_1, g_3] + \beta_1[g_2, g_3]. \end{aligned}$$

By Proposition 3, applied to the triple $\xi = (g_1, h_2, g_3)$, $\deg[r_1, r_2] \le \rho$, $\deg[g_1, g_3] > 6\rho$. Furthermore, by (16) and (19), $\deg[g_1, g_2] \le \rho$, $\deg[g_1, g_3] > 8\rho$. Comparing the degrees of the left and right parts in the last equation gives

$$\alpha + \alpha_1 = 0, \quad \beta + \gamma_1 + \gamma\beta_1 = 0, \quad \beta_1 = 0.$$

Thus $r_1 = g_1$, $r_2 = g_2 - \gamma g_1$, $r_3 = \sigma g_3 + g$, where $g \in \langle r_1, r_2\rangle \setminus F$. Since g_3 is unreducible in ψ, $\deg r_3 = \deg g_3 = 3\rho$. But then

$$\deg r_1 + \deg r_2 + \deg r_3 = 13\rho > \deg g_1 + \deg h_2 + \deg g_3 = \deg \xi,$$

which contradicts Corollary 4.

Therefore, the automorphism (r_1, r_2, r_3) is not a reduction of type III of ξ. Assume that it is a predreduction of type IV of ξ; then it is easy to see that in this case ψ also admits a reduction of type IV with an active element g_3, which is impossible.

We have thus shown that the inequality $\deg h_2 > 2n$ is impossible. Hence $\deg h_2 \le 2n$. Since $h_2 = g_2 - \alpha g_3^2 - \beta g_3 - \gamma g_1$, this implies $\deg(g_2 - \alpha g_3^2) \le 2n$. Furthermore, it follows from (23) that

$$\deg[g_1, g_3] + 2\rho \le \deg h_2 + \deg g_3 \le 7\rho,$$

i.e., $\deg[g_1, g_3] \le 5\rho$. Thus the automorphism ψ satisfies statement 3) of the present proposition, which is impossible.

Case 4. $m < \frac{3n}{2}$.

If g_1 is a simple reducible element of ψ, then (17) gives that $\frac{m}{(3n,m)} = 2$. Applying once more (17) and (19), we get

$$2n \ge 3n - m + \deg[g_2, g_3] = 4n - m + \deg[g_1, g_3],$$

i.e., $m > 2n$, a contradiction.

Therefore, g_2 is a simple reducible element of ψ. By (18), $\frac{m}{(2n,m)} \le 3$.

Consider first the case $\frac{m}{(2n,m)} = 3$. Then $2n = (2n, m)t$, $t > 3$, $3 \nmid t$. Hence $m = \frac{6n}{t}$, and since $n < m < \frac{3n}{2}$, we have $t = 5$. By putting $n = 5\rho$, we have $\deg g_1 = 10\rho$, $\deg g_2 = 15\rho$, $\deg g_3 = 6\rho$. Then (18) gives $\deg[g_1, g_3] \le \rho$.

Consider equality (20) and put again $\deg_y T(x, y) = k = 3q + r$, $0 \le r < 3$. Since $\deg(T(g_3, g_1)) = \deg g_2 = 15\rho$, inequality (2) yields that either $q = 0$, $r = 1$ or $q = 1$, $r = 0$. In the first case, by Corollary 1, $\overline{T(g_3, g_1)} = \bar{g}_2 \in \langle \bar{g}_1, \bar{g}_3\rangle$, which is impossible. Thus $k = 3$ and by Lemma 7.4) $T(x, y)$ is a derivative polynomial (up to a nonzero scalar multiplier) of the pair g_3, g_1. Then Corollary 3 gives

$$\deg\left(\frac{\partial T}{\partial x}(g_3, g_1)\right) = 24\rho, \quad \deg\left(\frac{\partial T}{\partial y}(g_3, g_1)\right) = 20\rho.$$

From (21), (22), taking into account (16) and (19), we get

$$\deg[g_1, h_2] = \deg[g_1, g_3] + 24\rho,$$
$$\deg[h_2, g_3] = \deg[g_1, g_3] + 20\rho.$$

Hence $15\rho > \deg h_2 \ge 14\rho + \deg[g_1, g_3]$, and Corollary 1 yields that g_1, g_3 are unreducible elements of the automorphism $\xi = (g_1, h_2, g_3)$. Since $\deg[g_1, g_3] < \rho$, it is easy to check that ξ does not admit reductions of types I–IV, which contradicts the simplicity of ξ.

Now we consider the case $\frac{m}{(2n,m)} = 2$. By putting $(2n, m) = 2\rho$, we get $m = 4\rho$, $2n = 2\rho t$, $t \ge 3$ is an odd number. Then $m = \frac{4n}{t}$ and (16) implies $t < 4$. Hence $t = 3$ and $\deg g_1 = 6\rho$, $\deg g_2 = 9\rho$, $\deg g_3 = 4\rho$. Consider again equality (20) and put $\deg_y(T(x,y)) = k = 2q + r$, $0 \le r < 2$. Assume that $r = 1$. Then (2) yields $q = 1$, i.e., $k = 3$. By Lemma 7,

$$T(x,y) = w(x,y)q(x,y) + s(x,y),$$

where $w(x,y)$ is a derivative polynomial of the 2-reduced pair g_3, g_1, and $\deg_y(q(x,y)) = 1$, $\deg_y(s(x,y)) \le 1$. By Lemma 5.3),

$$\deg(w(g_3,g_1)) \ge N(g_3,g_1) = 12\rho - 6\rho - 4\rho + \deg[g_1,g_3] > 2\rho.$$

Now Lemma 7.2) gives $\deg(q(g_3,g_1)) < 7\rho$, and then Lemma 3 yields $\deg(q(g_3,g_1)) = 6\rho$. Thus, by Lemma 7.3), we get

$$\deg\left(\frac{\partial T}{\partial x}(g_3,g_1)\right) = 14\rho, \quad \deg\left(\frac{\partial T}{\partial y}(g_3,g_1)\right) = 12\rho.$$

Now (21) and (22) yield, by means of (16) and (19),

$$\begin{aligned} \deg[g_1,h_2] &= \deg[g_1,g_3] + 14\rho, \\ \deg[h_2,g_3] &= \deg[g_1,g_3] + 12\rho. \end{aligned}$$

Hence $\deg h_2 \ge 8\rho + \deg[g_1,g_3]$, and the elements g_1, g_3 of $\xi = (g_1,h_2,g_3)$ are unreducible. Since $\deg[g_1,g_3] < \rho$, it is easy to check that ξ does not admit reductions of type I–IV, a contradiction.

Thus $r = 0$, $k = 2q$. Inequality (2) gives $1 \le q \le 4$ and

$$\deg\left(\frac{\partial T}{\partial y}(g_3,g_1)\right) \ge (q-1)(2\rho + \deg[g_3,g_1]) + 6\rho.$$

Hence, by means of (19), (22),

$$\deg[h_2,g_3] \ge q(2\rho + \deg[g_3,g_1]) + 4\rho. \tag{24}$$

In particular,

$$\deg h_2 \ge q(2\rho + \deg[g_3,g_1]). \tag{25}$$

Assume that $\deg h_2 \ge 6\rho$. Since h_2 is an unreducible element of $\xi = (g_1,h_2,g_3)$, the elements $\bar{g_1}, \bar{h}_2$ are linearly independent, if $\deg h_2 = 6\rho$. Note that g_1, h_2 and g_3, h_2 do not compose 2-reduced pairs. Consequently, the elements g_1, g_3 of ξ are unreducible. In fact, assume that there exists $f \in \langle h_2, g_3\rangle$ such that $\bar{g}_1 = \bar{f}$. Since $\bar{g}_1 \notin \langle \bar{h}_2, \bar{g}_3\rangle$, the elements $\bar{h}_2, \bar{g}_3$ are algebraically dependent and the pair g_3, h_2 is $*$-reduced. It follows easily from (2) that this pair should be 2-reduced, a contradiction. Similarly, g_3 is unreducible. Furthermore, it follows from (25) that $\deg[g_3,g_1] < \deg h_2$. Hence, due to Definitions 1–4 and Propositions 1–3, ξ does not admit reductions of types I–IV. This contradicts the simplicity of ξ.

Therefore, $\deg h_2 < 6\rho$ and ξ satisfies all the conditions of the present proposition. Since $\deg\xi < \deg\psi$, then, by the choice of ψ, ξ should satisfy the conclusion of the proposition. It follows from (24), (25) that ξ does not satisfy statements 1), 3); hence ξ admits a reduction of type IV with an active element h_2. In this case, $\deg h_2 \le 3\rho$ and we have $q = 1$ in (25), which implies $\deg[g_1,g_3] \le \rho$. By statement 5) of Proposition 3, ξ has a predreduction of the form $(g_1, \tilde{h}_2, g_3)$, where

$$\tilde{h}_2 = \sigma h_2 + g,\ g \in \langle g_3, g_1\rangle \setminus F,\ \deg \tilde{h}_2 = 3\rho,\ \deg[g_3,\tilde{h}_2] < 6\rho + \deg[g_3,g_1].$$

By Corollary 2 we have $g = \lambda w(g_3,g_1)$, where $w(x,y)$ is a derivative polynomial of the pair g_3, g_1. Then $\deg_y(w(x,y)) = 2$. Now $\tilde{h}_2 = \sigma(g_2 - T(g_3,g_1)) + \lambda w(g_3,g_1)$ and $\sigma^{-1}\tilde{h}_2 = g_2 - \tilde{T}(g_3,g_1)$, where $\tilde{T}(x,y) = T(x,y) - \sigma^{-1}\lambda w(x,y)$. If we substitute

$\sigma^{-1}\tilde{h}_2$ instead of h_2 in (20) (with $\tilde{T}$ instead of T), we will have again $\deg_y(\tilde{T}(x,y)) = 2q$, $1 \le q \le 4$, since the equality $\deg_y(\tilde{T}(x,y)) = 3$ is impossible. Then inequalities (24) and (25) hold also for $\tilde{h}_2$. In particular,

$$\deg[g_3, \tilde{h}_2] \ge 6\rho + \deg[g_3, g_1],$$

which contradicts the previous inequality. □

Proposition 5. *Let $\psi = (g_1, g_2, g_3)$ be a simple automorphism satisfying the following conditions:*

i) $\deg \psi \le \deg \theta$*;*

ii) g_1, g_2 *is a* 2*-reduced pair and* $\deg g_1 = 2n$, $\deg g_2 = ns$*, where* $s \ge 5$ *is an odd number;*

iii) $\bar{g_3} \notin \langle \bar{g_1} \rangle$, $\deg g_3 = m < ns$*, and* g_3 *is not a simple reducible element of* ψ*.*

Then $m < n(s-2) + \deg[g_1, g_2]$*.*

Proof. Assuming the contrary, we have

$$n(s-2) + \deg[g_1, g_2] \le m < ns. \tag{26}$$

Hence $\deg[g_1, g_2] < 2n$. It is now easy to check that ψ does not admit reductions of type I–IV. Then ψ is elementary reducible, and either g_1 or g_2 is a simple reducible element of ψ. In particular, $\bar{g_1}, \bar{g_2}, \bar{g_3}$ are mutually algebraically dependent. It follows from (26) that $\bar{g_2} \notin \langle \bar{g_1}, \bar{g_3} \rangle$, $\bar{g_1} \notin \langle \bar{g_2}, \bar{g_3} \rangle$. Consequently, g_1, g_3 and g_2, g_3 are $*$-reduced pairs as well.

If g_2 is a simple reducible element of ψ, then by Corollary 1 we get

$$\deg g_2 = ns \ge N(g_1, g_3) = \frac{2n}{(2n,m)} m - m - 2n + \deg[g_1, g_3]. \tag{27}$$

Observe that by Lemma 2.i) we have $p = \frac{2n}{(2n,m)} \ge 2$. If $\frac{2n}{(2n,m)} \ge 4$, inequalities (26), (27) give $ns > 3m - 2n > 3n(s-2) - 2n$. Hence $s < 4$, which contradicts condition *ii*) of the proposition.

If $\frac{2n}{(2n,m)} = 2$, then $n = (2n, m)$, $m = nt$, where $t \ge 3$ is an odd number. From (26) we get $s - 2 < t < s$, which is impossible.

Therefore, $\frac{2n}{(2n,m)} = 3$. By putting $(2n, m) = 2\rho$, we have $n = 3\rho$, $\deg g_1 = 6\rho$, $m = 2\rho t$, where $t > 3$, $3 \not| t$. Inequalities (26) and (27) yield

$$3(s+2) > 4t, \quad 3(s-2) < 2t < 3s.$$

Hence $t = s = 5$, i.e., $\deg g_1 = 6\rho$, $\deg g_2 = 15\rho$, $\deg g_3 = 10\rho$. Applying (26), (27) once more, we get

$$\deg[g_1, g_2] \le \rho, \ \deg[g_1, g_3] \le \rho.$$

Let again $h_2 = g_2 - T(g_1, g_3)$, where $\deg h_2 < \deg g_2$ and h_2 is unreducible in $\xi = (g_1, h_2, g_3)$. Since $\deg(T(g_1, g_3)) = \deg g_2 = 15\rho$ and $\overline{T(g_1, g_3)} = \bar{g_2} \notin \langle \bar{g_1}, \bar{g_3} \rangle$, inequality (2) yields $\deg_y(T(x,y)) = 3$. By Lemma 7.4), up to a scalar multiplier, $T(x, y)$ is equal to a derivative polynomial of the pair g_1, g_3. Hence by Corollary 3,

$$\deg\left(\frac{\partial T}{\partial x}(g_1, g_3)\right) = 24\rho, \quad \deg\left(\frac{\partial T}{\partial y}(g_1, g_3)\right) = 20\rho.$$

Since

$$\begin{aligned} \deg[g_1, g_3] + \deg g_2 \quad &\le \quad 16\rho, \\ \deg[g_1, g_2] + \deg g_3 \quad &\le \quad 11\rho, \end{aligned}$$

Lemma 1 implies that

$$\deg[g_2, g_3] + \deg g_1 \leq 16\rho.$$

Therefore, $\deg[g_2, g_3] \leq 10\rho$. Now equalities (21), (22), with g_1 and g_3 permuted, give

$$\begin{aligned}\deg[g_1, h_2] &= \deg[g_1, g_3] + 20\rho,\\ \deg[h_2, g_3] &= \deg[g_1, g_3] + 24\rho.\end{aligned}$$

Hence $\deg h_2 \geq 14\rho + \deg[g_1, g_3]$, which implies easily by (2) that the elements g_1, g_3 are unreducible in $\xi = (g_1, h_2, g_3)$. Since $\deg[g_1, g_3] \leq \rho$, it is easily checked that ξ does not admit reductions of types I–IV. Thus, ξ is not a simple automorphism.

Now suppose that g_1 is a simple reducible element of ψ. Then Corollary 1 gives

$$\deg g_1 = 2n \geq N(g_3, g_2) = \frac{m}{(ns, m)} ns - ns - m + \deg[g_2, g_3].$$

Therefore, $\frac{m}{(ns,m)} = 2$ and $ns = (ns, m)t$, where $t \geq 3$ is an odd number. Then $m = \frac{2ns}{t}$ and (26) gives $s - 2 < \frac{2s}{t}$, i.e., $(t-2)(s-2) < 4$. Since $s \geq 5$, this yields $t = 3$, $s = 5$. By putting $(ns, m) = 5\rho$, we have again $\deg g_1 = 6\rho$, $\deg g_2 = 15\rho$, $\deg g_3 = 10\rho$. By (26), we have also $\deg[g_1, g_2] \leq \rho$. Suppose that

$$h_1 = g_1 + G(g_3, g_2), \quad \deg h_1 < \deg g_1,$$

where h_1 is an unreducible element of $\xi = (h_1, g_2, g_3)$. Since $\deg(G(g_3, g_2)) = \deg g_1 = 6\rho$, by Corollary 2 we have $G(x, y) = \lambda w(x, y)$, where $w(x, y)$ is a derivative polynomial of the pair g_3, g_2. Therefore, by Corollary 3,

$$\deg\left(\frac{\partial G}{\partial x}(g_3, g_2)\right) = 2 \cdot \deg g_3 = 20\rho.$$

Since

$$[h_1, g_2] = [g_1, g_2] + [g_3, g_2]\frac{\partial G}{\partial x}(g_3, g_2),$$

a comparison of degrees gives

$$\deg[h_1, g_2] = \deg[g_3, g_2] + 20\rho.$$

Thus,

$$\deg h_1 \geq 5\rho + \deg[g_3, g_2], \qquad \deg[g_3, g_2] < \rho.$$

It remains to note that $\xi = (h_1, g_2, g_3)$ satisfies all the conditions of Proposition 4 and does not satisfy conclusions 1), 3) of this proposition. If ξ had admitted a reduction of type IV with an active element h_1, then by the definition there would exist an element $g \in \langle g_3, g_2 \rangle \setminus F$ such that $\deg(h_1 + g) = \frac{15\rho}{2}$, i.e., $\deg g = \frac{15\rho}{2}$. Then by Corollary 2 we would have again $g = \alpha_1 w(g_3, g_2)$, which contradicts the equality $\deg(w(g_3, g_2)) = \deg(G(g_3, g_2)) = 6\rho$. $\square$

Lemma 15. *If f_1, f_3 is a $*$-reduced pair and $\deg f_1 < \deg f_3$, then τ is a simple automorphism.*

Proof. We consider first the case when $\deg f_2 > \deg f_3$. By (11), we have $a \in \langle f_1, f_2 \rangle$, $\deg a \leq \deg f_3$. By Lemma 14, we may assume that $\bar{f}_2 \notin \langle \bar{f}_1 \rangle$ and $a \notin \langle f_1 \rangle$. Since $\deg a < \deg f_2$, then by F2), $\bar{f}_1, \bar{f}_2$ are algebraically dependent. By Corollary

1, $\deg a \geq N = N(f_1, f_2)$. Hence $\deg f_2 > N$ and by Lemma 2.iii), the pair f_1, f_2 is 2-reduced, that is, $\deg f_1 = 2n$, $\deg f_2 = ns$, where $s \geq 3$ is an odd number, and

$$n(s-2) + \deg[f_1, f_2] \leq \deg a \leq \deg f_3.$$

Since f_3 is not a simple reducible element of θ (see Corollary 7), θ satisfies the conditions of one of Propositions 4, 5. Since $\deg f_3 > \deg f_1$, statements 2), 3) of Proposition 4 are not fulfilled for θ. Therefore,

$$\deg f_3 < n(s-2) + \deg[f_1, f_2],$$

which contradicts the previous inequality.

Suppose now that $\deg f_2 \leq \deg f_3$. If $\deg f_2 = \deg f_3$ and $\bar{f}_2, \bar{f}_3$ are linearly dependent, then the element b in (13) can be chosen as $b = \alpha f_3$. Then sequence (15) gives the simplicity of τ.

Thus we can assume that $\deg f_2 < \deg f_3$. Then using Lemma 14.i) we can furthermore assume that $\bar{b} = -\bar{f}_2 \notin \langle \bar{f}_1, \bar{f}_3 \rangle$. Since $\deg b \leq \deg f_3$, by Corollary 1 $N(f_1, f_3) \leq \deg f_3$, and by Lemma 2.iii) the elements f_1, f_3 form a 2-reduced pair. Put $\deg f_1 = 2n$, $\deg f_3 = ns$, where $s \geq 3$ is an odd number. The inequality $\deg f_3 = ns \geq N(f_1, f_3)$ gives also $\deg[f_1, f_3] \leq 2n$. We can also assume that g_2 is an unreducible element of $\phi = (f_1, g_2, f_3)$. Then ϕ satisfies the conditions of one of Propositions 4, 5.

Let $b = T(f_1, f_3)$, where $\deg_y(T(x, y)) = k$. Then (2) implies that $k = 2$ if $s > 3$, and $k = 2, 4$ if $s = 3$. Consequently, $\deg_y(\frac{\partial T}{\partial y}(x, y)) = k - 1 \in \{1, 3\}$, and (2) yields

$$\deg \left(\frac{\partial T}{\partial y}(f_1, f_3) \right) \geq ns.$$

By (13),

$$[f_1, g_2] = [f_1, f_2] + [f_1, f_3] \frac{\partial T}{\partial y}(f_1, f_3). \tag{28}$$

Assume first that $\deg g_2 < n(s-2) + \deg[f_1, f_3]$. Then (28) yields

$$\deg[f_1, f_2] \geq \deg[f_1, f_3] + ns. \tag{29}$$

Therefore, $\deg f_2 \geq n(s-2) + \deg[f_1, f_3] > n$ and $\bar{f}_1 \notin \langle \bar{f}_2 \rangle$. Consequently, either the elements $\bar{f}_1, \bar{f}_2$ are algebraically independent or f_1, f_2 form a $*$-reduced pair. Since $\deg[f_1, f_2] > ns \geq \deg a$, we have by Lemma 2.ii) and Corollary 1,

$$a = \alpha f_2^2 + \gamma f_2 + G(f_1), \quad \deg(G(f_1)) < \deg f_3,$$

where $\alpha \neq 0$ only if $\deg f_3 \geq 2 \deg f_2$. Since f_1, f_3 is a 2-reduced pair, then $f_1, f_3 + G(f_1)$ is also a 2-reduced pair. By Lemma 9, $(f_1, g_2, f_3 + G(f_1))$ is a simple automorphism. We have

$$(f_1, g_2, f_3 + G(f_1)) \to (f_1, f_2, f_3 + G(f_1)) \to (f_1, f_2, f_3 + \alpha f_2^2 + \gamma f_2 + G(f_1)) = \tau.$$

If $(\alpha, \gamma) \neq (0, 0)$, then it is easily checked that $(f_1, g_2, f_3 + G(f_1))$ is a reduction of τ of types I–III, with an active element f_2. Note that the type of the reduction depends on the degree of f_2:

1) if $2n < \deg f_2$, then $\deg(f_2^2) > n(s-2) + \deg[f_1, f_3] + 2n > ns = \deg f_3$, $\alpha = 0$, and τ admits a reduction of type I;

2) if $\frac{3n}{2} < \deg f_2 \leq 2n$, then $s = 3$ (otherwise f_2 is not reducible in θ), $\alpha = 0$ and τ admits a reduction of type II;

3) if $\deg f_2 \leq \frac{3n}{2}$, then $s = 3$ and τ admits a reduction of type III.

The case $\alpha = \gamma = 0$ follows from Lemma 14.

It remains to consider the cases when ϕ satisfies one of the conclusions 2), 3) of Proposition 4. If ϕ satisfies 3), then (28) again gives (29). Besides, in this case $s = 3$ and $\deg f_2 > \deg g_2 = \frac{3n}{2}$. Consequently, $a = \beta f_2 + \gamma f_1$. By Lemma 14 we can put $\beta \neq 0$. Then the automorphism $(f_1, g_2, f_3 + \gamma f_1)$ gives a reduction of type I or II of τ with an active element f_2.

Assume finally that ϕ admits a reduction of type IV with an active element g_2. Since $\deg[f_1, f_3] \leq 2n$, the scalars α, β, γ in Definition 4 are 0 by Proposition 3.5). Consequently, the reduction of type IV of ϕ has the form $(f_1, h_2, f_3 - \delta h_2^2)$, where

$$g_2 = h_2 + g,\ g \in \langle f_1, f_3\rangle \setminus F, \qquad \deg(f_3 - \delta h_2^2) \leq 2n.$$

Since $\deg g_2 \leq \deg h_2 = \frac{3n}{2}$, then $\deg g \leq \frac{3n}{2}$. Note that, by Definition 4, the automorphism (f_1, h_2, f_3) satisfies statement 3) of Proposition 4.

If $\deg f_2 > \frac{3n}{2}$, then we may take $\phi = (f_1, h_2, f_3)$, and this case may be reduced to the previous one.

Assume that $\deg f_2 \leq \frac{3n}{2}$. Then we write

$$f_2 = g_2 - b = h_2 + (g - b), \quad g - b \in \langle f_1, f_3\rangle.$$

If $g - b \notin F$, then θ admits the reduction of type IV $(f_1, h_2, f_3 - \delta h_2^2)$, with an active element f_2, $\alpha = \beta = \gamma = 0$, and the predreduction (f_1, h_2, f_3). This case was considered in Lemma 12. If $g - b = \alpha \in F$, then $f_2 = h_2 + \alpha$ and the automorphism $(f_1, f_2, f_3 - \delta h_2^2)$ is simple by Remark 1. Consequently, f_3 is a simple reducible element of θ, a contradiction. □

Lemma 16. *If f_1, f_3 is a $*$-reduced pair and $\deg f_3 < \deg f_1$, then τ is a simple automorphism.*

Proof. Consider first the case when $\deg f_1 < \deg f_2$. Since $\deg a \leq \deg f_3$ by (11), in this case $\bar{a} \notin \langle \bar{f}_1, \bar{f}_2\rangle$. Then the inclusion $a \in \langle f_1, f_2\rangle$ implies by F2) that the elements $\bar{f}_1, \bar{f}_2$ are algebraically dependent. By Lemma 14, we may assume that f_1, f_2 form a $*$-reduced pair. Since $\deg a < \deg f_1$, inequality (2) gives that $\deg f_1 = 2n$, $\deg f_2 = 3n$, $\deg[f_1, f_2] \leq 2n$. Besides, if $\deg f_3 < n + \deg[f_1, f_2] = N(f_1, f_2)$, then Corollary 1 implies $a \in F$. Thus, we may assume that

$$\deg f_3 \geq n + \deg[f_1, f_2].$$

Observe that θ satisfies all the conditions of Proposition 4; hence it should satisfy one of conlusions 2), 3) of this proposition. The case when θ admits a reduction of type IV was considered in Lemma 12. Therefore, we may assume that $\deg f_3 = \frac{3n}{2}$, and

$$\deg(f_2 - \alpha f_3^2) \leq 2n, \quad \deg[f_1, f_3] < 3n + \deg[f_1, f_2].$$

Observe that $\deg a$, $\deg f \leq \frac{3n}{2}$. By Lemma 13, the automorphism $(f_1, f_2 - \alpha f_3^2, f_3)$ is simple. If $a \notin F$, then $(f_1, f_2 - \alpha f_3^2, f_3)$ gives a reduction of type IV of τ.

Consider now the case when $\deg f_2 \leq \deg f_1$. If $\deg f_1 = \deg f_2$, then by Lemma 14 we may assume that $\bar{f}_1, \bar{f}_2$ are linearly independent. Suppose $\bar{f}_2 \in \langle \bar{f}_3\rangle$. If $\bar{f}_2, \bar{f}_3$ are linearly dependent, sequence (15) proves that τ is simple. Suppose that $\deg f_2 = t \cdot \deg f_3$, $t \geq 2$. Since $\deg f_3 < \min\{\deg f_1, \deg f_2\}$ and $\deg a \leq \deg f_3$, under the assumption that $a \notin F$, inequality (2) yields $\deg f_2 = 2n$, $\deg f_1 = 3n$, and

$$\deg f_3 \geq n + \deg[f_1, f_2] > \frac{2n}{t} = \deg f_3.$$

Therefore, $\bar{f}_2 \notin \langle \bar{f}_1, \bar{f}_3 \rangle$. Since f_2 is a simple reducible element of θ, it follows from Corollary 1 that $N(f_1, f_3) \leq \deg f_2 \leq \deg f_1$ and by Lemma 2.iii) the pair f_3, f_1 is 2-reduced. Without loss of generality, we can assume g_2 in (13) to be unreducible in $\phi = (f_1, g_2, f_3)$; then ϕ satisfies the conditions of one of Propositions 4, 5. The rest of the proof can now be fulfilled similarly to that of Lemma 15. □

Lemma 17. *If $\bar{f}_1 \in \langle \bar{f}_3 \rangle$ and $\deg f_1 > \deg f_3$, then τ is a simple automorphism.*

Proof. Put $n = \deg f_3$, $\bar{f}_1 = \alpha \bar{f}_3^{\,k}$, $k \geq 2$. Then $\deg f_1 = nk$. Consider the simple automorphism $\phi = (f_1, g_2, f_3)$ defined by (13). Since $\deg \phi < \deg \theta$, Lemma 9 implies that $\psi = (f_1 - \alpha f_3^k, g_2, f_3)$ is also a simple automorphism. Therefore, by Lemma 9 again, the sequence

$$\psi \to (f_1 - \alpha f_3^k, f_2, f_3) \to \theta$$

proves that f_1 is a simple reducible element of θ. Interchanging the elements f_1, f_2, by Lemmas 14, 15, 16, we can restrict ourselves to the case when $\bar{f}_2 \in \langle \bar{f}_3 \rangle$, $\deg f_2 > \deg f_3$. Thus, we can take $\deg f_2 = nr$, $r \geq 2$. By Lemma 14, we can assume that f_1, f_2 is a $*$-reduced pair, with $\deg f_1 = mp$, $\deg f_2 = ms$, $(p, s) = 1$, $m \geq n$. Then $N(f_1, f_2) > m(ps - p - s) > n \geq \deg a$. Hence by Corollary 1, $\bar{a} \in \langle \bar{f}_1, \bar{f}_2 \rangle$, which evidently implies that $a \in F$. □

This finishes the proof of Theorem 1.

5. The main results

Theorem 1 implies immediately

Theorem 2. *Let $\theta = (f_1, f_2, f_3)$ be a tame automorphism of the ring of polynomials $A = F[x_1, x_2, x_3]$ over a field F of characteristic 0. If $\deg \theta > 3$, then θ admits either an elementary reduction or a reduction of types I–IV.*

Corollary 8. *Under the conditions of Theorem 2, if $f_3 = x_3$, then θ admits an elementary reduction.*

Proof. In fact, it is easy to see that in this case θ does not admit reductions of types I–IV. □

Now we will consider the Nagata automorphism $\sigma = (f, g, h)$ (see [12]) of the polynomial ring $F[x, y, z]$, where

$$f = x + wz, \quad g = y + 2wx + w^2 z, \quad h = z, \quad w = x^2 - yz.$$

Corollary 9. *The Nagata automorphism of the polynomial ring $F[x, y, z]$ over a field F of characteristic 0 is wild.*

Proof. Note that $\bar{f} = wz$, $\bar{g} = w^2 z$, $\bar{h} = z$ are mutually algebraically independent, and none of the elements $\bar{f}, \bar{g}, \bar{h}$ is contained in the subalgebra generated by the other two elements. Consequently, the automorphism σ does not admit an elementary reduction. By Corollary 8, σ is wild. □

In [7], some examples of wild automorphisms of the algebra $F[z][x, y]$ were constructed. The next corollary shows that all those automorphisms are also wild as automorphisms of the algebra $F[x, y, z]$.

Corollary 10. *Let F be a field of characteristic 0. An automorphism (f, g) of the $F[z]$-algebra $F[z][x, y]$ is tame if and only if the automorphism (f, g, z) of the F-algebra $F[x, y, z]$ is tame.*

The proof follows easily from Corollary 8.

Theorem 3. *The tame and the wild automorphisms of the algebra $F[x_1, x_2, x_3]$ of polynomials in three variables over a constructive field F of characteristic 0 are algorithmically recognizable.*

Proof. By induction on degree, it suffices to recognize, for every automorphism $\theta = (f_1, f_2, f_3)$ with $\deg\theta > 3$, whether θ admits either an elementary reduction or a reduction of types I–IV. By Lemma 8, the elementary reducibility of θ is algorithmically recognizable.

Suppose that θ admits a reduction of types I–IV. If we assume that f_3 is an active element of the reduction and $\deg f_1 \leq \deg f_2$, then the roles of the elements of θ are defined uniquely. It follows from Propositions 1–3 that the coefficients α, β, γ and the elements g_1, g_2 satisfying the conditions of these propositions are uniquely defined and can be found effectively. Put $\phi = (g_1, g_2, f_3)$.

If the element f_3 of ϕ is unreducible, then it remains to check whether ϕ admits or not a reduction of type IV with an active element f_3. By Corollary 2, the element g_3 satisfying the condition of Proposition 3 will be defined effectively by the conditions

$$g_3 = f_3 + \delta w(g_1, g_2), \quad \delta \neq 0,$$
$$\deg[g_1, g_3] < \deg g_2 + \deg[g_1, g_2],$$

where $w(x, y)$ is a derivative polynomial of the pair g_1, g_2. Now it is easy to check the existence of $\mu \in F$ such that $\deg(g_2 - \mu g_3^2) \leq \deg g_1$.

If f_3 is a reducible element of ϕ, then by Lemma 8, we can effectively find an unreducible reduction g_3 of f_3. If

$$\deg g_3 < \deg g_2 - \deg g_1 + \deg[g_1, g_2],$$

then $\psi = (g_1, g_2, g_3)$ is a reduction of types I, II, III of θ. Otherwise it remains to check the validity of statements 2), 3) of Proposition 4 for ψ. It was already shown above that it can be done effectively. Then, as in the proof of Lemmas 15, 16, one can define which type of reduction θ admits. $\square$

Note that a reduction of type I consists of two elementary transformations, reductions of types II, III consist of three elementary transformations, and a reduction of type IV in general case consists of four elementary transformations. Then it follows from Theorem 2 that the degree of any tame nonlinear automorphism of A can be reduced by at most four elementary transformations. In this context, the following question seems very interesting.

Problem 1. Construct examples of tame automorphisms of the algebra A that admit reductions of types II–IV.

In other words, do there exist tame automorphisms of A whose degrees cannot be reduced by two (or even by three) elementary transformations? The automorphism (f_1, f_2, f_3) from Example 1 admits a reduction of type I and so its degree can be reduced by two (but not by one!) elementary transformations.

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