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INDEPENDENCE OF ℓ OF MONODROMY GROUPS

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Dedicated to Nicholas M. Katz on his 60th birthday

1. Introduction

Let X be a smooth curve over a finite field of characteristic p. Let E be a number field, and consider an E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$ of lisse sheaves on X. This means that for every place λ of E not lying over p, we are given a lisse E_λ-sheaf $\mathcal{L}_\lambda$ on X, and these lisse sheaves are E-compatible with one another in the sense that, for every closed point x of X, the polynomial

$$\det(1 - T\,\mathrm{Frob}_x, \mathcal{L}_\lambda)$$

has coefficients in E and is independent of the place λ.

Let $\bar{\eta} \to X$ be a geometric point of X. For every place λ of E not lying over p, the lisse E_λ-sheaf $\mathcal{L}_\lambda$, which forms the λ-component of the system $\mathbf{L}$, has its corresponding monodromy E_λ-representation

$$[\mathcal{L}_\lambda] : \pi_1(X, \bar{\eta}) \to \mathrm{GL}(\mathcal{L}_{\lambda\bar{\eta}}),$$

and its corresponding *arithmetic monodromy group* $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$, defined as the Zariski closure of the image of $\pi_1(X, \bar{\eta})$ under $[\mathcal{L}_\lambda]$. The group $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$ is an algebraic group over the local ℓ-adic field E_λ, and it is given with a faithful tautological E_λ-rational representation

$$\sigma_\lambda : \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \hookrightarrow \mathrm{GL}(\mathcal{L}_{\lambda\bar{\eta}}).$$

The philosophy of motives in algebraic geometry leads one to suspect that if our E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$ is "motivic", then the collection $\{\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}), \sigma_\lambda\}$ of ℓ-adic monodromy groups and their tautological representations should be "independent of ℓ" in a suitable sense. This philosophy can be made more precise in the form of the following conjecture, which we are going to address in this paper.

Conjecture 1.1. Let X be a smooth curve over a finite field of characteristic p. Let E be a number field, and let $\mathbf{L} = \{\mathcal{L}_\lambda\}$ be an E-compatible system of lisse sheaves on the curve X. Assume that the E-compatible system $\mathbf{L}$ is semisimple and pure of weight w for some integer w.

(i) There exist a finite extension F of E and an algebraic group G over the number field F such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, the F_λ-groups

$$G \otimes_F F_\lambda \qquad \text{and} \qquad \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \otimes_{E_\lambda} F_\lambda \qquad \text{are isomorphic.}$$

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(ii) Assume that (i) holds. After replacing F by a further finite extension, there exists an F-rational representation σ of G such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, and identifying the F_λ-groups $G \otimes_F F_\lambda$ and $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \otimes_{E_\lambda} F_\lambda$ via an isomorphism given in (i), the F_λ-rational representations

$$\sigma \otimes_F F_\lambda \qquad \text{and} \qquad \sigma_\lambda \otimes_{E_\lambda} F_\lambda \qquad \text{are isomorphic.}$$

In the situation of (i), one could be more optimistic and ask for an algebraic group G over the number field E itself, i.e., without first passing to a finite extension F. I do not know if that would constitute "asking for too much", because the philosophy of motives does not predict that the group G can be defined over the given field E. However, in the situation of (ii), even if the group G is defined over E, one would in general need to pass to a finite extension in order to find the representation σ.

A conjecture similar to (1.1) for compatible systems of cohomology sheaves has been studied previously by M. Larsen and R. Pink; see [LP95], Conj. 5.1. They have also addressed the related situation in the "abstract" setting; see [LP92], especially §9.

In the situation of (1.1), for each place $\lambda \in |E|_{\neq p}$ of E not lying over p, the group $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$ sits in a short exact sequence

$$1 \to \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0 \to \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \to \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \ / \ \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0 \to 1.$$

The restriction of σ_λ to $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0$ will be denoted by the same symbol. By a result of J.-P. Serre, one knows that the finite groups $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \ / \ \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0$ are independent of λ. More precisely:

Theorem 1.2 (J.-P. Serre)**.** (See [Ser00], Th. on p. 15.) *With the notation and hypotheses of* (1.1)*, the kernel of the surjective homomorphism*

$$\pi_1(X, \bar{\eta}) \xrightarrow{[\mathcal{L}_\lambda]} \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \twoheadrightarrow \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \ / \ \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0$$

is the same open subgroup of $\pi_1(X, \bar{\eta})$ for every place $\lambda \in |E|_{\neq p}$ of E not lying over p.

There are two proofs of this in [Ser00], pp. 15–20; a third proof can be found in [LP92], Prop. 6.14. As a consequence:

Corollary 1.3. *With the notation and hypotheses of* (1.1)*, there exists a finite group Γ such that for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, the finite groups*

$$\Gamma \qquad \textit{and} \qquad \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta}) \ / \ \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0 \qquad \textit{are isomorphic.}$$

A simplified version of our main result asserts the "independence of λ" of the identity component $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0$ of the monodromy groups $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$, and of their tautological representations σ_λ:

Theorem 1.4. *Assume the notation and hypotheses of* (1.1).

(i) *There exist a finite extension F of E and a connected split reductive algebraic group G_0 over the number field F such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, the connected F_λ-algebraic groups*

$$G_0 \otimes_F F_\lambda \qquad \textit{and} \qquad \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0 \otimes_{E_\lambda} F_\lambda \qquad \textit{are isomorphic.}$$

(ii) *There exists an F-rational representation σ_0 of G_0 such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, and identifying the F_λ-groups $G_0 \otimes_F F_\lambda$ and $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0 \otimes_{E_\lambda} F_\lambda$ via an isomorphism given in (i), the F_λ-rational representations*

$$\sigma_0 \otimes_F F_\lambda \qquad \textit{and} \qquad \sigma_\lambda \otimes_{E_\lambda} F_\lambda \qquad \textit{are isomorphic.}$$

Note that the isomorphism in (i) between the two groups is not unique, nor is there a canonical one, but the hypotheses of the theorem do allow us to rigidify the situation to some extent; we refer to (6.9) for the precise statement.

Suppose the curve X is geometrically connected over the base field k. Let $\overline{k}$ be the algebraic closure of k in $\kappa(\bar{\eta})$, and regard $\bar{\eta}$ also as a geometric point of $X \otimes_k \overline{k}$. Then for every place λ of E not lying over p, we may also consider the *geometric monodromy group* $\mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})$ of the lisse E_λ-sheaf $\mathcal{L}_\lambda$, i.e., the Zariski closure of the image of $\pi_1(X \otimes_k \overline{k}, \bar{\eta})$ under $[\mathcal{L}_\lambda]$; we write σ'_λ for its faithful tautological E_λ-rational representation. Thanks to Larsen and Pink, one has the analogue of (1.2):

Theorem 1.5 (M. Larsen and R. Pink)**.** (See [LP95], Prop. 2.2.) *With the notation and hypotheses of* (1.1)*, the kernel of the surjective homomorphism*

$$\pi_1(X \otimes_k \overline{k}, \bar{\eta}) \xrightarrow{[\mathcal{L}_\lambda]} \mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta}) \to \mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta}) \ / \ \mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})^0$$

is the same open subgroup of $\pi_1(X \otimes_k \overline{k}, \bar{\eta})$ for every place $\lambda \in |E|_{\neq p}$ of E not lying over p.

From a result of P. Deligne (cf. [Del80], Cor. 1.3.9), one infers that the identity component $\mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})^0$ of the geometric monodromy group is equal to the derived subgroup $\mathrm{Der}(\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})^0)$ of the identity component of the arithmetic monodromy group. Consequently, our main theorem (1.4) implies the following:

Theorem 1.6. *Assume the notation and hypotheses of* (1.1)*.*

(i) *There exist a finite extension F of E and a connected split semisimple algebraic group G'_0 over the number field F such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, the connected F_λ-algebraic groups*

$$G'_0 \otimes_F F_\lambda \qquad \textit{and} \qquad \mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})^0 \otimes_{E_\lambda} F_\lambda \qquad \textit{are isomorphic.}$$

More precisely, one can take G'_0 to be the derived subgroup of the group G_0 in (1.4)*(i).*

(ii) *There exists an F-rational representation σ'_0 of G'_0 such that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, and identifying the F_λ-groups $G'_0 \otimes_F F_\lambda$ and $\mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})^0 \otimes_{E_\lambda} F_\lambda$ via an isomorphism given in (i), the F_λ-rational representations*

$$\sigma'_0 \otimes_F F_\lambda \qquad \textit{and} \qquad \sigma'_\lambda \otimes_{E_\lambda} F_\lambda \qquad \textit{are isomorphic.}$$

More precisely, one can take σ'_0 to be the restriction to G'_0 of the representation σ_0 in (1.4)*(ii).*

A weaker form of (1.6), obtained earlier by Larsen and Pink (see [LP95], Th. 2.4), asserts that after scalar extensions from the various ℓ-adic local fields to a common algebraically closed field (such as $\mathbb{C}$), the various $\mathrm{G}_{\mathrm{geom}}(\mathcal{L}_\lambda, \bar{\eta})^0$ become isomorphic to one another, and that the same goes for the representations σ'_λ under the additional hypothesis that the compatible system is geometrically absolutely irreducible.

Their results were established using their theorem (cf. [LP90], Th. 1 and Th. 2) to the effect that a connected semisimple algebraic group over an algebraically closed field of characteristic 0 is determined up to (non-canonical) isomorphism by the *dimension data* of the group. As we shall explain in a moment, we adopt a completely different approach in this paper by making full use of the motivic origin of compatible systems on a curve.

The results of (1.4) and (1.6) (and for that matter, those of (1.2) and (1.5) as well) also hold when X is any irreducible normal variety of finite type over a finite field of characteristic p. This is seen by reducing to the case of curves — for instance, by using space-filling curves (cf. [Katz99], Th. 8 and Lemma 6).

The proof of (1.4) is given in §6; we outline the strategy here as a guide to the organization of this paper. For each $\lambda \in |E|_{\neq p}$, write G_{E_λ} for $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$. To fix ideas, let us assume (only for simplicity) that G_{E_λ} is connected for *one* $\lambda \in |E|_{\neq p}$; by (1.2), the same is then true for *every* $\lambda \in |E|_{\neq p}$. We want to show that, allowing E to be replaced by a finite extension, the G_{E_λ} for various λ are all obtained by scalar extensions from a common group G_0 over E.

The key result which lies at the core of our whole argument is an extension (4.6) of the fundamental theorem (4.1) of L. Lafforgue; it allows us to exploit the hypothesis of compatibility (cf. (6.12) and (6.8)) to establish that the Grothendieck rings of the various G_{E_λ} are all isomorphic in a way which identifies the irreducible representations and which respects the character of each irreducible representation. Roughly speaking, this means that the connected reductive groups G_{E_λ} all "have the same representation theory". Making use of the tight connection (cf. (6.7)) between the representation theory and the structure theory of connected reductive groups, we can then conclude that the various G_{E_λ} do come from a common source G_0, *provided that* they contain maximal tori coming from a common *split* torus T_0 over E. Granting this, the fact that the representations σ_λ of G_{E_λ} are all obtained by scalar extensions from a common representation σ_0 of G_0 is then an easy consequence (cf. (6.13)) of our constructions.

To get the torus T_0, we appeal to Serre's theory of Frobenius tori (cf. (5.7), (5.8)); accordingly, we can find a Frobenius element which generates an E-torus whose ℓ-adic scalar extensions yield maximal tori in the various G_{E_λ}; hence, if we make an extension of E to split this torus, we obtain the T_0 we want. However, in order to apply Serre's results, we have to verify that the lisse sheaves $\mathcal{L}_\lambda$ satisfy certain hypotheses. All but one of these hypotheses are provided by another theorem (3.1) of Lafforgue; the "missing" hypothesis (concerning boundedness in denominator) is a result (3.2) which we establish in §3.

In the course of this work, I have benefited tremendously from discussions with Pierre Deligne, Johan de Jong, Nicholas Katz and Laurent Lafforgue. It is a pleasure to acknowledge my intellectual debts to all of them. I am also grateful to the anonymous referees, whose valuable suggestions have helped improve and clarify the exposition here.

2. Definitions and notation

2.1. Let E be a number field. If p is a prime number, let $|E|_p$ denote the finite set of valuations of E lying over p, and let

$$|E|_{\neq p} := \bigcup_{\ell \text{ prime} \neq p} |E|_\ell$$

denote the set of valuations of E not lying over p. Let $|E|_\infty$ denote the finite set of archimedean absolute values of E. We regard each $\nu \in |E|_p$ as a homomorphism

$$\nu : E^\times \to \mathbb{Q}, \qquad \text{normalized so that } \nu(p) = 1,$$

and we regard each $|\cdot| \in |E|_\infty$ as a homomorphism

$$|\cdot| : E^\times \to \mathbb{R}^\times_{>0}, \qquad \text{normalized so that } |n| = n \text{ for every } n \in \mathbb{Z}_{>0}.$$

With these normalizations, the valuations and absolute values are compatible with passing to finite extensions of E.

Let $\alpha \in \overline{\mathbb{Q}}^\times$ be a nonzero algebraic number. Let p be a prime number, and let q be a positive power of p.

(1) Let $w \in \mathbb{Z}$ be an integer; we say that α is *pure of weight w with respect to q* if for every archimedean absolute value $|\cdot| \in |\mathbb{Q}(\alpha)|_\infty$ of $\mathbb{Q}(\alpha)$, one has

$$|\alpha| = q^{w/2}.$$

(2) We say that α is *plain of characteristic p* if α is an ℓ-adic unit for every prime $\ell \neq p$; in other words, for every non-archimedean valuation $\lambda \in |\mathbb{Q}(\alpha)|_{\neq p}$ of $\mathbb{Q}(\alpha)$ not lying over p, one has

$$\lambda(\alpha) = 0.$$

(3) Let $C \geq 0$ be a real number; we say that α is *C-bounded in valuation with respect to q* if for every non-archimedean valuation $\nu \in |\mathbb{Q}(\alpha)|_p$ of $\mathbb{Q}(\alpha)$ lying over p, one has

$$\left| \frac{\nu(\alpha)}{\nu(q)} \right| \leq C.$$

(4) Let $D > 0$ be an integer; we say that α is *D-bounded in denominator with respect to q* if for every non-archimedean valuation $\nu \in |\mathbb{Q}(\alpha)|_p$ of $\mathbb{Q}(\alpha)$ lying over p, one has

$$\frac{\nu(\alpha)}{\nu(q)} \in \tfrac{1}{D}\,\mathbb{Z}.$$

2.2. Let ℓ be a prime number, and let Λ be an ℓ-adic field (i.e., Λ is an algebraic extension of $\mathbb{Q}_\ell$). Let X be a connected normal scheme of finite type over $\mathrm{Spec}(\mathbb{Z}[1/\ell])$, and let $\bar\eta \to X$ be a geometric point of X. If $\mathcal{L}$ is a lisse Λ-sheaf on X, we let

$$[\mathcal{L}] : \pi_1(X, \bar\eta) \to \mathrm{GL}(\mathcal{L}_{\bar\eta})$$

denote the corresponding continuous monodromy Λ-representation of the arithmetic fundamental group $\pi_1(X, \bar\eta)$ of X. The *arithmetic monodromy group* $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar\eta)$ of $\mathcal{L}$ is the Zariski closure of the image of $\pi_1(X, \bar\eta)$ in $\mathrm{GL}(\mathcal{L}_{\bar\eta})$ under the monodromy representation $[\mathcal{L}]$; it is a linear algebraic group over Λ.

2.3. Let $|X|$ denote the set of closed points of X. For each $x \in |X|$, choose an algebraic geometric point $\bar x \to x \in X$ of X localized at x. The absolute Galois group $\mathrm{Gal}(\kappa(\bar x)/\kappa(x))$ is a free profinite group generated by the *geometric Frobenius at x*, $\mathrm{Frob}_x := \mathrm{Frob}_{\kappa(x)} \in \mathrm{Gal}(\kappa(\bar x)/\kappa(x))$. If $\mathcal{L}$ is a lisse Λ-sheaf on X, we denote its restriction to x by $\mathcal{L}(x)$, and we write

$$[\mathcal{L}(x)] : \mathrm{Gal}(\kappa(\bar x)/\kappa(x)) \to \mathrm{GL}(\mathcal{L}_{\bar x})$$

for the monodromy Λ-representation of $\mathcal{L}(x)$. Thus, $[\mathcal{L}(x)](\mathrm{Frob}_x)$ is an element of $\mathrm{GL}(\mathcal{L}_{\bar x})$.

Let $\overline{\Lambda}$ be an algebraically closed extension of Λ. A lisse Λ-sheaf $\mathcal{L}$ on X is

- *algebraic*,
- resp. *pure of weight* w (for an integer w),
- resp. *plain of characteristic* p (for a prime number p),
- resp. C*-bounded in valuation* (for a real number $C \geq 0$),
- resp. D*-bounded in denominator* (for an integer $D > 0$)

if for every closed point $x \in |X|$, and for every eigenvalue $\alpha \in \overline{\Lambda}^{\times}$ of $[\mathcal{L}(x)](\mathrm{Frob}_x)$,

- α is an algebraic number,
- resp. α is pure of weight w with respect to $\#\kappa(x)$,
- resp. α is plain of characteristic p,
- resp. α is C-bounded in valuation with respect to $\#\kappa(x)$,
- resp. α is D-bounded in denominator with respect to $\#\kappa(x)$.

These properties of lisse Λ-sheaves are stable under passage to subquotients and formation of extensions. Let $a : X' \to X$ be a morphism of finite type; if $\mathcal{L}$ on X has one of these properties, then so does $a^*\mathcal{L}$ on X', and the converse holds when a is surjective.

2.4. Let X be a scheme of finite type over a finite field of characteristic p. Let E be a number field. An *E-compatible system* $\mathbf{L}$ *on* X is a collection $\mathbf{L} = \{\mathcal{L}_\lambda\}$ of lisse sheaves on X indexed by the set $|E|_{\neq p}$ of places of E not lying over p, where, for every such place $\lambda \in |E|_{\neq p}$ of E, $\mathcal{L}_\lambda$ is a lisse E_λ-sheaf on X, and this collection of lisse sheaves satisfies the condition that they are E-compatible with one another: i.e., for every closed point $x \in |X|$ of X, and for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, the polynomial

$$\det(1 - T\,\mathrm{Frob}_x, \mathcal{L}_\lambda)$$

has coefficients in E and is independent of the place λ.

We say that the E-compatible system $\mathbf{L}$ is *pure of weight* w (for an integer w) if for any/every $\lambda \in |E|_{\neq p}$, the lisse E_λ-sheaf $\mathcal{L}_\lambda$ is pure of weight w. We say that $\mathbf{L}$ is *absolutely irreducible*, resp. *semisimple*, if for each $\lambda \in |E|_{\neq p}$, the lisse E_λ-sheaf $\mathcal{L}_\lambda$ has the corresponding property (i.e., is irreducible over an algebraic closure of E_λ, resp. is semisimple over E_λ).

Note that if $\mathbf{L} = \{\mathcal{L}_\lambda\}$ is an E-compatible system, then for every $\lambda \in |E|_{\neq p}$, the lisse E_λ-sheaf $\mathcal{L}_\lambda$ is necessarily plain of characteristic p. We will see later (cf. (4.6)) that this necessary condition is also sufficient for a lisse ℓ-adic sheaf to extend to an E-compatible system for some number field E.

3. Boundedness of denominator

Let us first recall the following fundamental result, first conjectured by P. Deligne, and now established by L. Lafforgue.

Theorem 3.1 (L. Lafforgue)**.** (See [Laf02], Théorème VII.6 (i–iv).) *Let X be a smooth curve over a finite field of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathcal{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X which is irreducible, of rank r, and whose determinant is of finite order. Then there exists a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that the lisse sheaf $\mathcal{L}$ on X is*

(1) *E-rational (hence algebraic);*
(2a) *pure of weight 0;*

(2b) *plain of characteristic p; and*
(2c) *C-bounded in valuation, where C may be taken to be $(r-1)^2/r$.*

In this section, we prove the following assertion (2d), which complements (3.1) above.

Theorem 3.2. *With the notation and hypotheses of* (3.1)*, there exists an integer $D > 0$ such that the lisse sheaf $\mathcal{L}$ on X is*

(2d) *D-bounded in denominator.*

Proof of (3.2)*, prelude.* We begin our argument as Lafforgue does for his proof of assertion (2c) in (3.1); see [Laf02], Th. VII.6, Démo., pp. 198–200, especially parts (v) and (iv). Accordingly, we consider the lisse sheaf

$$q_1{}^*\mathcal{L} \otimes q_2{}^*\mathcal{L}^\vee(1-r) \qquad \text{on the surface } X \times X,$$

where q_1 and q_2 denote the two projections from $X \times X$ onto X. The fundamental fact we need to know about this lisse sheaf is that it "appears" in the ℓ-adic cohomology of a certain stack over the surface. More precisely, there exists a stack $\mathfrak{X}$ over $X \times X$ — namely, $\overline{\mathrm{Cht}_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}}$ in the notation of [Laf02] — such that if

$$f : \mathfrak{X} \to X \times X$$

denotes the structural morphism, the semisimple[1] lisse sheaf $q_1{}^*\mathcal{L} \otimes q_2{}^*\mathcal{L}^\vee(1-r)$ on $X \times X$ occurs as a direct summand in the semisimplified lisse[2] cohomology sheaf $(\mathrm{R}^{2r-2}f_!(\overline{\mathbb{Q}}_\ell))^{\mathrm{ss}}$ of $\mathfrak{X}$ over the surface $X \times X$; for the justification of this key assertion, see [Laf02], Chap. VI, §3, especially the statements of Lemme VI.26 and Th. VI.27. For our present purpose of proving (3.2), we need not be concerned with the precise definition and moduli interpretation of the stack $\mathfrak{X}$; what will be important for us is the following assertion obtained from [Laf02], Th. V.2 and Lemme A.3: there is a finite extension $\mathbb{F}_{q'}$ of $\mathbb{F}_q$, and a scheme Z — namely, $(\overline{\mathrm{Cht}_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}})^{\mathrm{gr}} \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$ in the notation of [Laf02] — which is proper over the surface $V := (X \times X) \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$, such that if $p : Z \to V$ denotes the structural morphism, the cohomology sheaf $\mathrm{R}^{2r-2}p_!(\overline{\mathbb{Q}}_\ell)$ is equal to the pull-back of $\mathrm{R}^{2r-2}f_!(\overline{\mathbb{Q}}_\ell)$ to V. The point to note is that if $\mathcal{G}$ denotes the pull-back of the lisse sheaf $q_1{}^*\mathcal{L} \otimes q_2{}^*\mathcal{L}^\vee(1-r)$ to V, then every irreducible constituent of $\mathcal{G}$ is also an irreducible constituent of the lisse cohomology sheaf $\mathrm{R}^{2r-2}p_!(\overline{\mathbb{Q}}_\ell)$ of a proper *scheme* Z over V. We use this observation to deduce the following key lemma.

Lemma 3.3. *Let $\mathcal{F}$ be an irreducible constituent of the lisse sheaf $\mathcal{G}$ on V.*

(1) *There exist*
   (a) *a finite universal homeomorphism $\widetilde{V} \to V$,*
   (b) *an open dense subscheme $\widetilde{U}$ of $\widetilde{V}$, and*
   (c) *a scheme $Z_{\mathcal{F}}$ over $\widetilde{U}$ whose structural morphism $\widetilde{p}_{\mathcal{F}} : Z_{\mathcal{F}} \to \widetilde{U}$ is projective and smooth,*

   such that on the scheme $\widetilde{U}$, the lisse sheaf $\mathcal{F}|_{\widetilde{U}}$ is an irreducible constituent of the cohomology sheaf $\mathrm{R}^{2r-2}(\widetilde{p}_{\mathcal{F}})_!(\overline{\mathbb{Q}}_\ell)$ (the latter is a lisse sheaf on $\widetilde{U}$ since $\widetilde{p}_{\mathcal{F}}$ is proper and smooth).

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[1] Cf. [Laf02], remark after Déf. VI.14.
[2] Cf. [Laf02], Chap. VI, §2a, especially p. 165, third paragraph.

(2) *Let U denote the image of $\widetilde{U}$ in V. There exists an integer $D > 0$ such that the algebraic lisse sheaf $\mathcal{F}|_U$ on U is D-bounded in denominator.*

Proof of (3.3, (1)). Let η be the generic point of V. The irreducible lisse sheaf $\mathcal{F}$ on V restricts to an irreducible lisse sheaf $\mathcal{F}_\eta$ on η. We have the proper morphism $p_\eta : Z_\eta \to \eta$, whose cohomology sheaf $\mathrm{R}^{2r-2}(p_\eta)_!(\overline{\mathbb{Q}}_\ell)$ admits $\mathcal{F}_\eta$ as an irreducible subquotient.

We now construct a proper hypercovering $a : Z'_\bullet \to Z_\eta$ of Z_η (cf. [Del74b], (6.2.5)), but using the alteration theorem of de Jong (cf. [dJ96], Th. 4.1) instead of resolution. Accordingly, each Z'_i is a projective scheme over η, whose structural morphism $p_i : Z'_i \to \eta$ factors through some η_i in such a way that $p'_i : Z'_i \to \eta_i$ is smooth, and $\kappa(\eta_i) \supseteq \kappa(\eta)$ is a finite and purely inseparable extension (this factorization property of the structural morphisms can be derived from [dJ96], Remark 4.3). Since a proper hypercovering is of cohomological descent, the adjunction morphism $\overline{\mathbb{Q}}_\ell \to \mathrm{R}a_!\overline{\mathbb{Q}}_\ell$ is an isomorphism in the "derived" category $\mathrm{D}^b_c(Z_\eta, \overline{\mathbb{Q}}_\ell)$ of $\overline{\mathbb{Q}}_\ell$-sheaves on Z_η. We can then see from the resulting spectral sequence for $\mathrm{R}(p_\eta)_!$ that $\mathcal{F}_\eta$ is an irreducible subquotient of the cohomology sheaf $\mathrm{R}^j(p_i)_!(\overline{\mathbb{Q}}_\ell)$ of some Z'_i, for some cohomological degree j.

Let $\widetilde{V}$ be the normalization of V in η_i over η. Since V is itself normal and $\eta_i \to \eta$ is finite and purely inseparable, the morphism $\widetilde{V} \to V$ is a finite universal homeomorphism. The projective and smooth morphism $p'_i : Z'_i \to \eta_i$ "spreads out" to a projective and smooth morphism $\widetilde{p}_{\mathcal{F}} : Z_{\mathcal{F}} \to \widetilde{U}$ for some open dense subscheme $\widetilde{U}$ of $\widetilde{V}$ (i.e., the latter morphism exists, having the former morphism as its generic fiber). From the previous paragraph, it follows that $\mathcal{F}|\widetilde{U}$ is an irreducible subquotient of $\mathrm{R}^j(\widetilde{p}_{\mathcal{F}})_!(\overline{\mathbb{Q}}_\ell)$. Using Deligne's theory of weights (cf. [Del80]) to compare the weights of the lisse sheaves involved, we see that the cohomological degree j above is in fact equal to $2r-2$. □

Proof of (3.3, (2)). From part (1), we have the morphism $\widetilde{p}_{\mathcal{F}} : Z_{\mathcal{F}} \to \widetilde{U}$, which is projective and smooth. Let d be the rank of the lisse sheaf $\mathcal{H} := \mathrm{R}^{2r-2}(\widetilde{p}_{\mathcal{F}})_!(\overline{\mathbb{Q}}_\ell)$ on $\widetilde{U}$, and let $D := d!$. By part (1), $\mathcal{H}$ admits $\mathcal{F}|_{\widetilde{U}}$ as an irreducible subquotient. We will show that the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{H}$ on $\widetilde{U}$ is $\mathbb{Q}$-rational and is D-bounded in denominator; this will imply that $\mathcal{F}|_{\widetilde{U}}$ is also D-bounded in denominator, and thus the conclusion of part (2) will follow.

Let $u \in |\widetilde{U}|$ be a closed point, $k = \kappa(u)$ be its residue field, and $\overline{k}$ be an algebraic closure of k. We have to show that the $\overline{\mathbb{Q}}_\ell$-representation $[\mathcal{H}(u)]$ of $\mathrm{Gal}(\overline{k}/k)$ is $\mathbb{Q}$-rational and is $d!$-bounded in denominator with respect to $\#k$. Let Y be the fiber of $Z_{\mathcal{F}}$ over u; thus Y is a projective smooth scheme over the finite field k. By the proper base change theorem, $[\mathcal{H}(u)]$ is isomorphic to the representation of $\mathrm{Gal}(\overline{k}/k)$ on the ℓ-adic cohomology $\mathrm{H}^{2r-2}(Y \otimes_k \overline{k}, \overline{\mathbb{Q}}_\ell)$ of $Y \otimes_k \overline{k}$. The fact that $[\mathcal{H}(u)]$ is $\mathbb{Q}$-rational then follows from the Weil Conjectures as proved by Deligne (cf. [Del74a] or [Del80]); in fact, one knows that the "inverse" characteristic polynomial $\det(1 - T\,[\mathcal{H}(u)](\mathrm{Frob}_u))$ has coefficients in $\mathbb{Z}$. Note that d is the degree of this polynomial.

To see that $[\mathcal{H}(u)]$ is $d!$-bounded in denominator with respect to $\#k$, we appeal to the crystalline cohomology $H := \mathrm{H}^{2r-2}_{\mathrm{cris}}(Y/W(k))$ of Y with respect to the ring $W(k)$ of Witt vectors over k; for a concise summary of the assertions we need, see

[Ber75], §§1–2. The Frobenius endomorphism of Y relative to k,

$$\mathrm{Fr}_{Y/k} : Y \to Y, \qquad \text{defined by } y \mapsto y^{\#k},$$

induces an endomorphism $(\mathrm{Fr}_{Y/k})^* : H \to H$ which gives H the structure of an $(\mathrm{Fr}_{Y/k})^*$-crystal. Let K be the fraction field of $W(k)$, and let ν_K be the discrete valuation of K, which we regard as a homomorphism

$$\nu_K : K^\times \to \mathbb{Q}, \qquad \text{normalized so that } \nu_K(\#k) = 1.$$

One has the "inverse" characteristic polynomial

$$\det(1 - T\,(\mathrm{Fr}_{Y/k})^*, H \otimes_{W(k)} K) \in K[T]$$

of the endomorphism of $H \otimes_{W(k)} K$ induced by $(\mathrm{Fr}_{Y/k})^*$. By the compatibility theorem of Katz-Messing (cf. [KM74], Th. 1 and Cor. 1), one knows that the coefficients of this polynomial lie in $\mathbb{Z}$, and that one has

$$\det(1 - T\,(\mathrm{Fr}_{Y/k})^*, H \otimes_{W(k)} K) = \det(1 - T\,[\mathcal{H}(u)](\mathrm{Frob}_u)) \qquad (\text{equality in } \mathbb{Z}[T]),$$

and hence the $(\mathrm{Fr}_{Y/k})^*$-isocrystal $H \otimes_{W(k)} K$ has K-dimension d. The desired assertion that $[\mathcal{H}(u)]$ is $d!$-bounded in denominator with respect to $\#k$ is therefore equivalent to the assertion that the Newton polygon of the polynomial $\det(1 - T\,(\mathrm{Fr}_{Y/k})^*, H \otimes_{W(k)} K)$ with respect to the valuation ν_K has all its slopes lying in $\frac{1}{d!}\,\mathbb{Z}$. By a theorem of Yu. I. Manin (cf. [Ber75], Th. 1.3(ii)), the slopes of this Newton polygon are the slopes of the $(\mathrm{Fr}_{Y/k})^*$-isocrystal $H \otimes_{W(k)} K$, and so by the classification theorem of J. Dieudonné (cf. [Ber75], Th. 1.3(i)), these slopes are of the form r/s where $r \in \mathbb{Z}$, $s \in \{1, \ldots, d\}$, and $(r, s) = 1$; in particular, we see that these slopes all lie in $\frac{1}{d!}\,\mathbb{Z}$, which is what we want. $\square$

Proof of (3.2), *coda.* Applying (3.3) to every irreducible constituent of the lisse sheaf $\mathcal{G}$ on V and consolidating the conclusions thus obtained, we infer that there exist an open dense subscheme U' of V and an integer $D_0 > 0$ such that the algebraic lisse sheaf $\mathcal{G}|_{U'}$ on U' is D_0-bounded in denominator. Let U denote the image of U' under the finite étale map $V \to X \times X$. Then we see that the lisse sheaf $({q_1}^*\mathcal{L} \otimes {q_2}^*\mathcal{L}^\vee(1-r))|_U$ on U is also D_0-bounded in denominator. The complement of U in $X \times X$ is a finite union of divisors and closed points; hence

$$S := \{\ x \in |X| \ : \ {q_1}^{-1}(x) \cap U = \emptyset \quad \text{in } X \times X\ \} \qquad \text{is a finite set.}$$

Let $D = r\,D_0$. We claim that the algebraic lisse sheaf $\mathcal{L}|_{X-S}$ on $X - S$ is D-bounded in denominator. To see this, let $x \in |X| - S$ be a closed point, let $\alpha \in \overline{\mathbb{Q}}_\ell^{\,\times}$ be an eigenvalue of Frob_x acting on $\mathcal{L}|_{X-S}$, and let $\nu \in |\mathbb{Q}(\alpha)|_p$ be a non-archimedean valuation of $\mathbb{Q}(\alpha)$ lying over p. By the definition of S, we can find a closed point u of U with $q_1(u) = x$. Set $y := q_2(u) \in |X|$, and let $e = [\kappa(u) : \kappa(x)]$, $f = [\kappa(u) : \kappa(y)]$. If $\beta \in \overline{\mathbb{Q}}_\ell^{\,\times}$ is an eigenvalue of Frob_y acting on $\mathcal{L}$, then it follows that $\alpha^e\,\beta^{-f}\,(\#\kappa(u))^{2r-2} \in \overline{\mathbb{Q}}_\ell^{\,\times}$ is an eigenvalue of Frob_u acting on $({q_1}^*\mathcal{L} \otimes {q_2}^*\mathcal{L}^\vee(1-r))|_U$; so we have

$$\frac{\nu(\alpha^e\,\beta^{-f}\,(\#\kappa(u))^{2r-2})}{\nu(\#\kappa(u))} \in \tfrac{1}{D_0}\,\mathbb{Z},$$

and hence

$$\frac{e\,\nu(\alpha) - f\,\nu(\beta)}{\nu(\#\kappa(u))} \in \tfrac{1}{D_0}\,\mathbb{Z}.$$

By hypothesis, the determinant of $\mathcal{L}$ is of finite order, which implies that the product of the r many eigenvalues β of Frob_y acting on $\mathcal{L}$ is a root of unity; it follows that

$$\frac{re\,\nu(\alpha)}{\nu(\#\kappa(u))} = \frac{r\,\nu(\alpha)}{\nu(\#\kappa(x))} \in \tfrac{1}{D_0}\,\mathbb{Z},$$

which proves our claim.

Since S is a finite set, we may now replace D by a multiple to ensure that the algebraic lisse sheaf $\mathcal{L}$ on the whole of X is D-bounded in denominator. This completes the proof of (3.2). $\square$

Remark 3.4. Lafforgue has also shown (see [Laf02], Prop. VII.7) that, as predicted by Deligne's conjecture, assertions (2a), (2b) and (2c) of (3.1) generalize to the case when X is an arbitrary normal variety of finite type over a finite field of characteristic p. The proof is by reduction to the case of curves, using the fact that the assertions in (3.1) are *uniform* for any lisse sheaf of a given rank r on any curve. Unfortunately, our proof of (3.2) does not produce an expression for the integer $D > 0$ in terms of just the rank r of the lisse sheaf, and so we do not get the desired generalization of assertion (2d).

4. Compatible systems of Lisse sheaves

According to Deligne's conjecture (cf. [Del80], Conj. (1.2.10)(v)), on any normal connected scheme X over a finite field, an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$ whose determinant $\det \mathcal{L}$ is of finite order should "extend" to an absolutely irreducible E-compatible system $\{\mathcal{L}_\lambda\}$ of lisse sheaves for some number field E. Thanks to Lafforgue's proof of the Langlands Correspondence for GL_r over function fields, this conjecture is now known to hold when X is a curve:

Theorem 4.1. (cf. [Laf02], Théorème VII.6 (v) and [Chin03b]) *Let X be a smooth curve over a finite field of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathcal{L}$ be an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X whose determinant is of finite order. Then there exists a number field $E \subset \overline{\mathbb{Q}}_\ell$ and an absolutely irreducible E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$ on X which is E-compatible with the given lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$; i.e., for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, there exists an absolutely irreducible lisse E_λ-sheaf $\mathcal{L}_\lambda$ on X which is E-compatible with the given lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$.*

The purpose of this section is to give a mild generalization (4.6) of the above theorem, in which one replaces the assumption "determinant is of finite order" by a weaker hypothesis of plainness. We first recall (cf. (4.2)) the construction of rank 1 twisting lisse sheaves, and use these to give equivalent characterizations (cf. (4.3)) of algebraicity, purity and plainness for an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf.

4.2. Let $\mathbb{F}$ be a finite field of characteristic p, let $\ell \neq p$ be a prime number, and let Λ be an ℓ-adic field. For any ℓ-adic unit $\alpha \in \Lambda^\times$, one has a well-defined continuous homomorphism

$$\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \to \mathrm{GL}_1(\Lambda) = \Lambda^\times, \qquad \text{mapping } \mathrm{Frob}_{\mathbb{F}} \text{ to } \alpha,$$

which corresponds to a lisse Λ-sheaf of rank 1 on the scheme $\mathrm{Spec}(\mathbb{F})$; we let the symbol $\alpha^{\deg_{\mathbb{F}}}$ denote the pullback of this lisse Λ-sheaf to any scheme X of finite type over $\mathbb{F}$. If $x \in |X|$ is a closed point of X, then one has

$$\det(1 - T\,\mathrm{Frob}_x, \alpha^{\deg_{\mathbb{F}}}) = 1 - T\,\alpha^{\deg_{\mathbb{F}}(\kappa(x))} \qquad (\text{equality in } \Lambda[T]).$$

If $\alpha = 1/(\#\mathbb{F})$, the resulting lisse sheaf is the *Tate twist*. An important special case: if E is a number field and $\alpha \in E^\times$ is *plain of characteristic* p, then the rank 1 lisse E_λ-sheaf $\alpha^{\deg_\mathbb{F}}$ is defined for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, and these lisse sheaves together form an E-compatible system.

Proposition 4.3. *Let X be a smooth curve over a finite field $\mathbb{F}$ of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathcal{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X which is irreducible, of rank r. Let $w \in \mathbb{Z}$ be an integer. The following are equivalent:*

(1) *$\mathcal{L}$ is algebraic (resp. pure of weight w, resp. plain of characteristic p);*
(2) *for some closed point $x \in |X|$ of X, the $\overline{\mathbb{Q}}_\ell$-representation $[\mathcal{L}(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$ is algebraic (resp. pure of weight w with respect to $\#\kappa(x)$, resp. plain of characteristic p);*
(3) *$\det\mathcal{L}$ is algebraic (resp. pure of weight $r\,w$, resp. plain of characteristic p);*
(4) *for some closed point $x \in |X|$ of X, the $\overline{\mathbb{Q}}_\ell$-representation $[\det\mathcal{L}(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$ is algebraic (resp. pure of weight $r\,w$, resp. plain of characteristic p);*
(5) *there exist an integer $n > 0$ and an element $\beta \in \overline{\mathbb{Q}}_\ell^{\,\times}$ which is algebraic (resp. pure of weight $n\,r\,w$ with respect to q, resp. plain of characteristic p), such that $(\det\mathcal{L})^{\otimes n}$ is isomorphic to $\beta^{\deg_\mathbb{F}}$;*
(6) *there exists an element $\alpha \in \overline{\mathbb{Q}}_\ell^{\,\times}$ which is algebraic (resp. pure of weight w with respect to q, resp. plain of characteristic p), such that $\mathcal{L} \otimes \alpha^{\deg_\mathbb{F}}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant is of finite order.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and that (1) $\Rightarrow$ (3) $\Rightarrow$ (4). By a result of Deligne (cf. [Del80], Prop. (1.3.4)(i)), one knows that for some $n \in \mathbb{Z}_{>0}$, the n-th tensor power of $\det\mathcal{L}$ is geometrically constant, which means that its monodromy representation $[(\det\mathcal{L})^{\otimes n}]$ factors as

$$\pi_1(X, \bar{\eta}) \to \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \to \overline{\mathbb{Q}}_\ell^{\,\times};$$

hence $(\det\mathcal{L})^{\otimes n}$ is isomorphic to $\beta^{\deg_\mathbb{F}}$ where $\beta \in \overline{\mathbb{Q}}_\ell^{\,\times}$ is the image of the geometric Frobenius $\mathrm{Frob}_\mathbb{F} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$; thus (4) $\Rightarrow$ (5). Now let $\alpha := 1/\sqrt[nr]{\beta} \in \overline{\mathbb{Q}}^\times$ be an nr-th root of $1/\beta$ in $\overline{\mathbb{Q}}$; the determinant of the lisse sheaf $\mathcal{L} \otimes \alpha^{\deg_\mathbb{F}}$ is then of finite order dividing n, whence (5) $\Rightarrow$ (6). Finally, (6) $\Rightarrow$ (1) follows directly from parts (1), (2a) and (2b) of Lafforgue's theorem (3.1). $\square$

Remark 4.4. The equivalent characterizations in (4.3) above hold more generally when X is an arbitrary normal variety of finite type over a finite field of characteristic p. The proof is the same, using [Laf02], Prop. VII.7 (see (3.4)) instead of (3.1) in the final step.

Proposition 4.5. *Let X be a smooth curve over a finite field $\mathbb{F}$ of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathcal{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Suppose $\mathcal{L}$ is algebraic. Then $\mathcal{L}$ is:*

– *E-rational, for some number field $E \subset \overline{\mathbb{Q}}_\ell$,*
– *C-bounded in valuation, for some real number $C > 0$, and*
– *D-bounded in denominator, for some integer $D > 0$.*

Proof. It suffices to treat the case when $\mathcal{L}$ is irreducible, and in that case, the proposition follows from the equivalence (1) $\Leftrightarrow$ (6) in (4.3), together with Lafforgue's theorem (3.1) (assertions (1) and (2c)) and (3.2) (assertion (2d)). $\square$

Theorem 4.6. *Let X be a smooth curve over a finite field $\mathbb{F}$ of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathcal{L}$ be an irreducible (resp. semisimple) lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Suppose $\mathcal{L}$ is plain of characteristic p. Then there exists a number field $E \subset \overline{\mathbb{Q}}_\ell$ and an absolutely irreducible (resp. semisimple) E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$ on X which is E-compatible with the given lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$. If $\mathcal{L}$ is pure of weight w for some $w \in \mathbb{Z}$, then so is the system $\mathbf{L}$.*

Proof. It is clear that we only have to treat the case when $\mathcal{L}$ is irreducible. By the equivalence (1) $\Leftrightarrow$ (6) in (4.3), $\mathcal{L}$ is isomorphic to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf of the form $\mathcal{F} \otimes (1/\alpha)^{\deg_{\mathbb{F}}}$ where $\mathcal{F}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant is of finite order, and $\alpha \in \overline{\mathbb{Q}}_\ell^{\times}$ is plain of characteristic p. Applying (4.1) to $\mathcal{F}$, we obtain a number field E with $\alpha \in E$, and an E-compatible system $\mathbf{F} = \{\mathcal{F}_\lambda\}$ which is E-compatible with $\mathcal{F}$. The E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$, with $\mathcal{L}_\lambda := \mathcal{F}_\lambda \otimes (1/\alpha)^{\deg_{\mathbb{F}}}$ for each $\lambda \in |E|_{\neq p}$, satisfies the first assertion of the theorem. The second assertion is clear. $\square$

Note that by the equivalence (1) $\Leftrightarrow$ (2) in (4.3), in order to verify the hypotheses that $\mathcal{L}$ is plain of characteristic p (resp. is pure of weight w) in (4.6), it suffices to pick one closed point $x \in |X|$ of X and check that the $\overline{\mathbb{Q}}_\ell$-representation $[\mathcal{L}(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$ has the corresponding property. Likewise, in (4.5), it suffices to check that $[\mathcal{L}(x)]$ is algebraic for one closed point x in order to conclude that $\mathcal{L}$ is algebraic.

5. Frobenius tori

In this section, we review the theory of Frobenius tori due to J.-P. Serre, with the aim of establishing the two key results we need — namely, (5.7) and (5.8). The main ideas are all in [Ser00], §§4–5; we only have to make them applicable to general lisse ℓ-adic sheaves satisfying appropriate hypotheses.

5.1. Let Λ be a field of characteristic 0, and let $E \subset \Lambda$ be a subfield of Λ. Consider an r-dimensional E-rational Λ-representation σ of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, where $\mathbb{F}$ denotes an arbitrary finite field. The image $\sigma(\mathrm{Frob}) \in \mathrm{GL}_r(\Lambda)$ of the geometric Frobenius has a semisimple part

$$\sigma(\mathrm{Frob})^{\mathrm{ss}} \in \mathrm{GL}_r(\Lambda).$$

Since σ is E-rational, the E-algebra generated by the semisimple element $\sigma(\mathrm{Frob})^{\mathrm{ss}}$,

$$E[\sigma(\mathrm{Frob})^{\mathrm{ss}}] \;:=\; E[T] \;/\; (\text{ minimal } E\text{-polynomial of } \sigma(\mathrm{Frob})^{\mathrm{ss}} \text{ in the variable } T\,),$$

is a finite étale algebra over E. Therefore, the multiplicative group of this E-algebra defines an E-torus $\mathrm{Mult}(E[\sigma(\mathrm{Frob})^{\mathrm{ss}}])$. The group of E-points of $\mathrm{Mult}(E[\sigma(\mathrm{Frob})^{\mathrm{ss}}])$ is isomorphic to $E[\sigma(\mathrm{Frob})^{\mathrm{ss}}]^{\times}$, and so it contains the element $\sigma(\mathrm{Frob})^{\mathrm{ss}}$. We define the *Frobenius group $H(\sigma)$ of σ* as the Zariski closure of the subgroup of $\mathrm{Mult}(E[\sigma(\mathrm{Frob})^{\mathrm{ss}}])$ generated by $\sigma(\mathrm{Frob})^{\mathrm{ss}}$. It is a diagonalizable E-group, whose group of characters $\mathrm{X}(H(\sigma))$ as a $\mathrm{Gal}(\overline{E}/E)$-module is canonically isomorphic — via the "evaluation at Frob" map — to the subgroup of $\overline{E}^{\times}$ generated by the eigenvalues of $\sigma(\mathrm{Frob})$.

One can also consider the Zariski closure of the subgroup of $\mathrm{GL}_r(\Lambda)$ generated by the element $\sigma(\mathrm{Frob})^{\mathrm{ss}}$:

$$\overline{(\sigma(\mathrm{Frob})^{\mathrm{ss}})^{\mathbb{Z}}} \quad \subseteq \mathrm{GL}_r(\Lambda);$$

it is canonically isomorphic to the Λ-scalar extension $H(\sigma)_{/\Lambda}$ of the E-group $H(\sigma)$.

The *Frobenius torus* $A(\sigma)$ *of* σ is by definition the identity component $H(\sigma)^0$ of $H(\sigma)$; it is an E-torus, whose group of characters $\mathrm{X}(A(\sigma))$ as a $\mathrm{Gal}(\overline{E}/E)$-module is canonically isomorphic to the torsion-free quotient of $\mathrm{X}(H(\sigma))$.

Note that our definition differs from that in [Ser00] in that we first pass to the semisimplification of the element $\sigma(\mathrm{Frob})$; this allows our discussion to proceed without having to assume that σ is a semisimple representation.

5.2. The Frobenius group $H(\sigma)$ and the Frobenius torus $A(\sigma)$ of the E-rational representation σ are determined by the minimal E-polynomial of the semisimple element $\sigma(\mathrm{Frob})^{\mathrm{ss}}$, and hence also by the "inverse" characteristic polynomial

$$\det(1 - T\,\sigma(\mathrm{Frob})) \in E[T]$$

of $\sigma(\mathrm{Frob})$. Consequently, if Λ' is another field containing E and σ' is a Λ'-representation of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ which is E-compatible with σ, then the Frobenius groups $H(\sigma)$ and $H(\sigma')$ of σ and σ' respectively are in fact the same E-group; likewise the Frobenius tori $A(\sigma)$ and $A(\sigma')$ are the same E-tori.

5.3. The order of the finite group $H(\sigma)/A(\sigma)$ is an integer which divides the order of some root of unity belonging to the field E' obtained from E by adjoining all eigenvalues of $\sigma(\mathrm{Frob})$, and E' is of degree $\leq r!$ over E. Suppose E is a number field; then there are only finitely many roots of unity of degree $\leq r!$ over E. Letting N be the least common multiple of the orders of these roots of unity, we see that the order of $H(\sigma)/A(\sigma)$ divides N. Note that N depends only on the number field E and the integer $r > 0$, but is independent of σ.

5.4. In order to state the finiteness result of Serre, let us introduce the following notation. Let $r \geq 1$ be an integer, and let Λ be a field of characteristic 0. Let $E \subset \Lambda$ be a number field contained in Λ. Let $C \geq 0$ be a real number, and let $D > 0$ be an integer. Define the set

$$\Sigma^r_{E\subset\Lambda}(C,D)_p := \left\{ \begin{array}{l} r\text{-dimensional } \Lambda\text{-representations } \sigma \text{ of } \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \text{ such that} \\ \text{for some positive power } q \text{ of } p, \text{ and some integer } w,\ \sigma \text{ is:} \\ \qquad \text{(1) } E\text{-rational,} \\ \qquad \text{(2a) pure of weight } w \text{ with respect to } q, \\ \qquad \text{(2b) plain of characteristic } p, \\ \qquad \text{(2c) } C\text{-bounded in valuation with respect to } q, \text{ and} \\ \qquad \text{(2d) } D\text{-bounded in denominator with respect to } q \end{array} \right\}$$

and let

$$\Sigma^r_{E\subset\Lambda}(C,D) := \bigcup_{p \text{ prime}} \Sigma^r_{E\subset\Lambda}(C,D)_p.$$

Theorem 5.5 (J.-P. Serre). *With the above notation, as σ runs over the set $\Sigma^r_{E\subset\Lambda}(C,D)$, there are only finitely many possibilities for the $\mathrm{GL}_{r/\overline{E}}$-conjugacy class of the Frobenius torus $A(\sigma)$ of σ, where $\overline{E}$ denotes an algebraic closure of E.*

We leave to the reader the pleasant exercise of generalizing the argument found in [Ser00], Th. on p. 8 to the situation of (5.5); the case of weight $w = 0$ requires a little more argument. We note in passing that the assumption that C and D are *uniform bounds* for the valuation and denominator of the σ in question plays a key role in showing the finiteness assertion here.

5.6. Now let X be an irreducible normal scheme of finite type over a finite field of characteristic p, and let $\bar{\eta} \to X$ be a geometric point of X. Assume that Λ is an ℓ-adic field for some prime number $\ell \neq p$. Let $\mathcal{L}$ be a *semisimple* lisse Λ-sheaf on X; its arithmetic monodromy group $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$ of $\mathcal{L}$ is then a (possibly non-connected) reductive group over Λ. For every closed point $x \in |X|$, the Λ-representation $[\mathcal{L}(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$ factors through $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$; the semisimple element $[\mathcal{L}(x)](\mathrm{Frob}_x)^{\mathrm{ss}}$ therefore lies in $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$. Hence, for each $x \in |X|$, we have the diagonalizable Λ-group

$$H(x) := \overline{([\mathcal{L}(x)](\mathrm{Frob}_x)^{\mathrm{ss}})^{\mathbb{Z}}} \quad \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$$

and the Λ-torus

$$A(x) := H(x)^0 \quad \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0.$$

These are canonically isomorphic, over Λ, to the Frobenius group $H([\mathcal{L}(x)])_{/\Lambda}$ and the Frobenius torus $A([\mathcal{L}(x)])_{/\Lambda}$ associated to the representation $[\mathcal{L}(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$.

Theorem 5.7. (Cf. [Ser00], Th. on p. 12.) *Assume the notation and hypotheses of* (5.6), *and let $r \geq 1$ be the rank of the semisimple lisse Λ-sheaf $\mathcal{L}$ on X. Suppose there exist a number field $E \subset \Lambda$ contained in the ℓ-adic field Λ, an integer w, a real number $C \geq 0$, and an integer $D > 0$, such that the lisse Λ-sheaf $\mathcal{L}$ on X is*

(1) *E-rational,*
(2a) *pure of weight w,*
(2b) *plain of characteristic p,*
(2c) *C-bounded in valuation, and*
(2d) *D-bounded in denominator.*

Then there exists an open dense subset $U \subset \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$ of the identity component of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$, such that U is stable under conjugation by $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$, and such that for any closed point $x \in |X|$ of X with $[\mathcal{L}(x)](\mathrm{Frob}_x) \in U$, the Zariski closure

$$\overline{([\mathcal{L}(x)](\mathrm{Frob}_x))^{\mathbb{Z}}} \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$$

of the subgroup it generates is a maximal Λ-torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$.

Proof. We may assume that Λ is algebraically closed. By the list of hypotheses on the lisse Λ-sheaf $\mathcal{L}$, we may apply (5.5) to the collection of representations $[\mathcal{L}(x)]$ ($x \in |X|$), and it follows that as x runs over $|X|$, there are only finitely many possibilities for the GL_r-conjugacy class of the Frobenius torus $A([\mathcal{L}(x)])$ associated to $[\mathcal{L}(x)]$.

Let $T_0 \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$ be a maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$. We claim that each GL_r-conjugacy class of Frobenius tori contains only finitely many subtori $A \subseteq T_0$ of T_0. To see this, we just have to show that if two subtori A_1, A_2 of T_0 are conjugate to each other in GL_r, then they are already conjugate to each other under the action of the (finite) Weyl group of T_0 in GL_r. But if $g \in \mathrm{GL}_r$ conjugates $A_1 \subseteq T_0$ to $A_2 \subseteq T_0$, then both T_0 and the g-conjugate of T_0 are maximal tori contained in the centralizer of A_2 in GL_r; this centralizer is a connected reductive subgroup of GL_r, so one can adjust g by an element of this subgroup to assume that g normalizes

T_0, and this shows what we want. As a consequence of our claim, we see that if we define the set

$$\Phi' := \left\{ \begin{array}{l} \text{proper subtori } A \subsetneq T_0 \text{ of } T_0 \text{ such that, for some } x \in |X|, \\ A \text{ belongs to the } \mathrm{GL}_r\text{-conjugacy class of the Frobenius} \\ \text{torus } A([\mathcal{L}(x)]) \text{ associated to } [\mathcal{L}(x)] \end{array} \right\},$$

then Φ' is a finite set.

We are given the number field E and the integer $r > 0$. Let N be the least common multiple of the orders of the finitely many roots of unity of degree $\leq r!$ over E. Define the set

$$\Phi := \left\{ \begin{array}{l} \text{subgroups } H \subseteq T_0 \text{ of } T_0 \text{ such that} \\ \text{the identity component } H^0 \text{ of } H \text{ lies in } \Phi' \\ \text{and the order of } H/H^0 \text{ divides } N \end{array} \right\}.$$

From the finiteness of the set Φ', we see that Φ is also a finite set.

If $H \subseteq T_0$ belongs to the finite set Φ, then its identity component H^0 is a proper subgroup of T_0, and so the same is true for H. Therefore, if F_H denotes the Zariski closure of the union of all $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$-conjugates of H, then F_H is of dimension strictly less than that of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$. It follows that

$$U' := \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0 - \bigcup_{H \in \Phi} F_H$$

is an open dense subset of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$. We define U to be the intersection of U' with the open dense ("characteristic") subset of regular semisimple elements in the connected reductive group $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$. From its construction we see that U is stable under conjugation by $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$.

Let $x \in |X|$ be a closed point with $[\mathcal{L}(x)](\mathrm{Frob}_x) \in U$. Since $[\mathcal{L}(x)](\mathrm{Frob}_x)$ is semisimple, the Zariski closure

$$\overline{([\mathcal{L}(x)](\mathrm{Frob}_x))^{\mathbb{Z}}} \quad \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$$

of the subgroup generated by $[\mathcal{L}(x)](\mathrm{Frob}_x)$ is just the diagonalizable Λ-group $H(x)$ introduced in (5.6), and it is contained in some maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$. The identity component of $H(x)$ is the torus $A(x) = H(x)^0$, and by construction, it belongs to the GL_r-conjugacy class of the Frobenius torus $A([\mathcal{L}(x)])$ associated to $[\mathcal{L}(x)]$. From the assumption that the polynomial

$$\det(1 - T\,[\mathcal{L}(x)](\mathrm{Frob}_x))$$

has coefficients in the number field E and has degree r, we see (cf. (5.3)) that the order of $H(x)/A(x)$ divides the integer N above. If $A(x)$ is not a maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$, then $H(x)$ is $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$-conjugate to some subgroup H belonging to the set Φ, but this contradicts the fact that $[\mathcal{L}(x)](\mathrm{Frob}_x)$ does not lie in F_H. Thus $A(x)$ must be a maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$, and hence the same is true for $H(x)$. □

Corollary 5.8. *Assume the notation and hypotheses of* (5.6) *and* (5.7)*. Let* $x_0 \in |X|$ *be a closed point of* X*. The subgroup* $A(x_0) \subseteq \mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})$ *is a maximal torus of* $\mathrm{G}_{\mathrm{arith}}(\mathcal{L}, \bar{\eta})^0$ *if and only if the Frobenius torus* $A([\mathcal{L}(x_0)])$ *associated to* $[\mathcal{L}(x_0)]$ *has the maximum dimension (over* E*) among all Frobenius tori* $A([\mathcal{L}(x)])$ *as* x *runs over the set of closed points* $|X|$ *of* X*.*

Proof. By (5.7) and Čebotarev's density theorem (cf. [Ser65], Th. 7), there exists some closed point $y \in |X|$ such that $A(y)$ is a maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L},\bar{\eta})^0$; so the dimension of each $A(x)$ (over Λ) is at most that of $A(y)$. Therefore, $A(x_0)$ is a maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathcal{L},\bar{\eta})^0$ if and only if it has the maximum dimension (over Λ) among the tori $A(x)$ as x runs over $|X|$. Since one has the isomorphism $A(x) \cong A([\mathcal{L}(x)])_{/\Lambda}$ for each $x \in |X|$, the corollary follows. □

Remark 5.9. The results of (5.7) and (5.8) also hold when X is any irreducible normal scheme of finite type over $\mathbb{Z}[1/\ell]$, except that one would have to replace the assumption (2b) on the lisse sheaf $\mathcal{L}$ by an appropriately generalized notion of plainness (one in which the characteristic p is allowed to vary). The proof is the same, thanks to the very general nature of (5.5).

6. Independence of ℓ

In this section, we prove our main "independence of ℓ" theorem (1.4).

6.1. **Hypotheses.** Let X be a smooth curve over a finite field of characteristic p, let E be a number field, and let $\mathbf{L} = \{\mathcal{L}_\lambda\}$ be an E-compatible system of lisse sheaves on the curve X. We assume that the E-compatible system $\mathbf{L}$ is semisimple and pure of weight w for some integer w. For each $\lambda \in |E|_{\neq p}$, write

$$G_{E_\lambda} := \mathrm{G}_{\mathrm{arith}}(\mathcal{L}_\lambda, \bar{\eta})$$

for the arithmetic monodromy group of $\mathcal{L}_\lambda$, and write σ_λ for its tautological faithful representation. The group G_{E_λ} is a (possibly non-connected) reductive group over E_λ, and the representation σ_λ is an E_λ-rational representation of G_{E_λ}.

Our goal is to show that the identity component $G^0_{E_\lambda}$ of G_{E_λ} and its tautological representation σ_λ are "independent of λ" in the sense of (1.4). In (6.5), we shall construct a finite extension F of E, a connected split reductive group G_0 over F, and an F-rational representation σ_0 of G_0. We will show in (6.9) that for every place $\lambda \in |F|_{\neq p}$ of F not lying over p, writing λ also for its restriction to E, one has an isomorphism of connected F_λ-groups:

$$f_\lambda : G^0_{E_\lambda} \otimes_{E_\lambda} F_\lambda \xrightarrow{\cong} G_0 \otimes_F F_\lambda,$$

and we will show in (6.13) that with this isomorphism of groups, one also has an isomorphism of representations:

$$\sigma_\lambda \otimes_{E_\lambda} F_\lambda \cong \sigma_0 \otimes_F F_\lambda.$$

First, some preliminary reductions.

Lemma 6.2. *To prove* (1.4)*, we may assume that for each $\lambda \in |E|_{\neq p}$, the E_λ-monodromy group G_{E_λ} is* connected*.*

Proof. Indeed, under the hypotheses of (6.1), we may apply Serre's theorem (1.2) to see that as λ runs over $|E|_{\neq p}$, the kernel of the surjective homomorphism

$$\pi_1(X,\bar{\eta}) \xrightarrow{[\mathcal{L}_\lambda]} G_{E_\lambda} \to G_{E_\lambda}/G^0_{E_\lambda}$$

is the same open subgroup of $\pi_1(X,\bar{\eta})$, independent of $\lambda \in |E|_{\neq p}$. This open subgroup of $\pi_1(X,\bar{\eta})$ corresponds to a finite étale cover $a : X' \to X$ of X by a connected smooth curve X' pointed by the same geometric point $\bar{\eta}$. The inverse image $a^*\mathbf{L} := \{a^*\mathcal{L}_\lambda\}$ on X' of the E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$ is still a semisimple E-compatible system which is pure of integer weight. Moreover, for each $\lambda \in |E|_{\neq p}$, the arithmetic monodromy group of $a^*\mathcal{L}_\lambda$ is the connected reductive group $G^0_{E_\lambda}$, given with the same tautological representation σ_λ. Hence, to prove (1.4), we may replace X by X' and $\mathbf{L}$ by its inverse image. This proves the lemma. $\square$

6.3. **Notation.** Henceforth, we adopt both the hypotheses (6.1) and the assumption that for each $\lambda \in |E|_{\neq p}$, we have $G_{E_\lambda} = G^0_{E_\lambda}$.

Let $x \in |X|$ be a closed point of X. By the E-compatibility hypothesis on $\mathbf{L}$, as λ runs over $|E|_{\neq p}$, the Frobenius group $H([\mathcal{L}_\lambda(x)])$ (cf. (5.1)) associated to the E_λ-representation $[\mathcal{L}_\lambda(x)]$ of $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$ is a diagonalizable group over E which is independent of the place $\lambda \in |E|_{\neq p}$ (cf. (5.2)); we denote it by $H(\mathbf{L}(x))$. Similarly, the Frobenius torus $A([\mathcal{L}_\lambda(x)])$ is an E-torus which is independent of the place $\lambda \in |E|_{\neq p}$ and we denote it by $A(\mathbf{L}(x))$; it is also the identity component of $H(\mathbf{L}(x))$.

For each $\lambda \in |E|_{\neq p}$, we write

$$H(x)_{E_\lambda} := \overline{([\mathcal{L}_\lambda(x)](\mathrm{Frob}_x)^{\mathrm{ss}})^{\mathbb{Z}}} \quad \subseteq G_{E_\lambda}$$

for the Zariski closure of the subgroup of G_{E_λ} generated by $[\mathcal{L}(x)](\mathrm{Frob}_x)^{\mathrm{ss}}$. It is a diagonalizable subgroup of G_{E_λ} over E_λ, and we write

$$A(x)_{E_\lambda} := H(x)^0_{E_\lambda} \quad \subseteq G_{E_\lambda}$$

for its identity component, which is an E_λ-torus.

Lemma 6.4. *Assume* (6.1) *and* (6.3) *above. There exist infinitely many closed points $x \in |X|$ of X with the following property: for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, one has $A(x)_{E_\lambda} = H(x)_{E_\lambda}$, and it is a maximal E_λ-torus of G_{E_λ}.*

Proof. By our assumptions on the E-compatible system $\mathbf{L} = \{\mathcal{L}_\lambda\}$, for each place $\lambda \in |E|_{\neq p}$, the lisse E_λ-sheaf $\mathcal{L}_\lambda$ is E-rational (hence algebraic), pure of some integer weight, and plain of characteristic p; by (4.5), it is also C-bounded in valuation and D-bounded in denominator. Hence the hypotheses of (5.7) and (5.8) are all verified for these lisse sheaves.

Pick any place $\mu \in |E|_{\neq p}$ of E not lying over p. From (5.7), we obtain an open dense subset $U \subset G_{E_\mu}$, stable under conjugation by G_{E_μ}, such that for any $x \in |X|$ with $[\mathcal{L}_\mu(x)](\mathrm{Frob}_x) \in U$, the subgroup $H(x)_{E_\mu} \subseteq G_{E_\mu}$ is a maximal E_μ-torus of G_{E_μ} (recall from (6.2) that we have put ourselves in the situation where the monodromy group G_{E_λ} is connected for every $\lambda \in |E|_{\neq p}$). By Čebotarev's density theorem, there exist infinitely many closed points $x \in |X|$ of X with $[\mathcal{L}_\mu(x)](\mathrm{Frob}_x) \in U$. We claim that these closed points satisfy the conclusion of the lemma.

To see that, let x_0 be such a closed point. By (5.8) and the assumption on x_0, we infer that $A([\mathbf{L}(x_0)]) = H([\mathbf{L}(x_0)])$ and that the Frobenius torus $A([\mathbf{L}(x_0)])$ has the maximum dimension (over E) among all Frobenius tori $A([\mathbf{L}(x)])$ as x runs over $|X|$. Thus, for any place $\lambda \in |E|_{\neq p}$ of E not lying over p, it follows from (5.8) again that the E_λ-subgroup $A(x_0)_{E_\lambda} = H(x_0)_{E_\lambda} \subseteq G_{E_\lambda}$ is a maximal torus of G_{E_λ}. $\square$

6.5. **Constructions.** Assume the hypotheses in (6.1) and the notation in (6.3). We shall now make a series of constructions to be used for the rest of this section.

6.5.1. Choose once and for all a closed point $x \in |X|$ of X satisfying the conclusion of (6.4); thus for every place $\lambda \in |E|_{\neq p}$ of E not lying over p, $A(x)_{E_\lambda} = H(x)_{E_\lambda}$ is a maximal torus of G_{E_λ}.

6.5.2. Let F be the splitting field of the E-torus $A([\mathbf{L}(x)]) = H([\mathbf{L}(x)])$; it is the same as the splitting field of the polynomial $\det(1 - T\,\mathrm{Frob}_x, \mathcal{L}_\lambda) \in E[T]$, for any/every $\lambda \in |E|_{\neq p}$.

6.5.3. Let T_0 be the split F-torus defined by the Frobenius E-torus $A([\mathbf{L}(x)])$ via scalar extension:

$$T_0 := A([\mathbf{L}(x)]) \otimes_E F = H([\mathbf{L}(x)]) \otimes_E F.$$

6.5.4. For each $\lambda \in |F|_{\neq p}$, writing λ also for its restriction to E, we obtain the lisse F_λ-sheaf $\mathcal{F}_\lambda := \mathcal{L}_\lambda \otimes_{E_\lambda} F_\lambda$ by scalar extension. This gives us the collection of lisse sheaves $\mathbf{F} := \{\mathcal{F}_\lambda\}$ indexed by the places $|F|_{\neq p}$ of F not lying over p, and it is clear that $\mathbf{F}$ is an F-compatible system.

6.5.5. For each $\lambda \in |F|_{\neq p}$, the arithmetic monodromy group $\mathrm{G_{arith}}(\mathcal{F}_\lambda, \bar{\eta})$ of the lisse F_λ-sheaf $\mathcal{F}_\lambda$ is identified with $G_{E_\lambda} \otimes_{E_\lambda} F_\lambda$; we denote it by G_{F_λ}. Thus:

$$G_{F_\lambda} := G_{E_\lambda} \otimes_{E_\lambda} F_\lambda = \mathrm{G_{arith}}(\mathcal{F}_\lambda, \bar{\eta}).$$

The tautological representation of G_{F_λ} is identified with $\sigma_\lambda \otimes_{E_\lambda} F_\lambda$; we denote this by ρ_λ. Thus:

$$\rho_\lambda := \sigma_\lambda \otimes_{E_\lambda} F_\lambda.$$

6.5.6. For each $\lambda \in |F|_{\neq p}$, writing λ also for its restriction to E, we have the F_λ-torus

$$T_{F_\lambda} := A(x)_{E_\lambda} \otimes_{E_\lambda} F_\lambda = H(x)_{E_\lambda} \otimes_{E_\lambda} F_\lambda \quad \subseteq G_{F_\lambda}.$$

By (6.5.1) and (6.5.2), T_{F_λ} is in fact a *split* maximal torus of G_{F_λ}, whence G_{F_λ} is a connected *split* reductive group over F_λ.

6.5.7. For each $\lambda \in |F|_{\neq p}$, one has the canonical isomorphism (cf. (5.1))

$$f_\lambda : T_{F_\lambda} \xrightarrow{\cong} T_{0/F_\lambda} \qquad \text{of split } F_\lambda\text{-tori,}$$

where the right-hand side T_{0/F_λ} is the F_λ-scalar extension of the F-torus T_0.

6.5.8. Choose once and for all a place $\mu \in |F|_{\neq p}$ of F not lying over p.

6.5.9. Define the F-group G_0 as follows. First consider the F_μ-group G_{F_μ}; it is a connected split reductive group over F_μ, and so it is defined over any subfield of F_μ. We let G_0 be "the" connected split reductive group over F, whose isomorphism type over F_μ is that of G_{F_μ}, and which contains the split F-torus T_0 as its maximal torus. Thus, G_0 and T_0 fit into the commutative diagram

$$\begin{array}{ccc} T_{F_\mu} & \to & T_0 \\ \downarrow\cap & & \downarrow\cap \\ G_{F_\mu} & \to & G_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F_\mu) & \to & \mathrm{Spec}(F) \end{array}$$

in which the squares are cartesian.

6.5.10. Likewise, define the F-rational representation σ_0 of the F-group G_0 as follows. The F_μ-group G_{F_μ} is given with a tautological representation ρ_μ, and this is defined over any subfield of F_μ because G_{F_μ} is split. We let σ_0 be "the" F-rational representation of the F-group G_0 whose scalar-extension to F_μ is the representation ρ_μ. Thus:

$$\rho_\mu := \sigma_\mu \otimes_{E_\mu} F_\mu \cong \sigma_0 \otimes_F F_\mu.$$

Our goal now is to show that for each $\lambda \in |F|_{\neq p}$, the canonical isomorphism f_λ in (6.5.7) of split F_λ-tori *extends* to an isomorphism

$$f_\lambda : G_{F_\lambda} \xrightarrow{\cong} G_{0/F_\lambda} \qquad \text{of connected split reductive groups over } F_\lambda$$

(where the right-hand side G_{0/F_λ} is the F_λ-scalar extension of the F-group G_0), and that if we identify the two groups using this isomorphism, then we have an isomorphism of representations

$$\rho_\lambda := \sigma_\lambda \otimes_{E_\lambda} F_\lambda \cong \sigma_0 \otimes_F F_\lambda.$$

These will be achieved in (6.9) and in (6.13) respectively.

6.6. **Notation.** If G is a split reductive algebraic group over a field k, and $T \subseteq G$ is a maximal torus of G, then the group of characters of T over k is denoted by $\mathrm{X}_k(T)$, and we set

$$\mathrm{Irr}_k(G) := \{\text{isomorphism classes of irreducible } k\text{-rational representations of } G\}.$$

Note that since G is split, every object of $\mathrm{Irr}_k(G)$ is in fact absolutely irreducible. We write $\mathrm{K}(T)$ and $\mathrm{K}(G)$ for the Grothendieck ring of the category of k-rational representations of T and G respectively; these rings are given with the canonical basis of $\mathrm{X}_k(T)$ and $\mathrm{Irr}_k(G)$ respectively. Restriction of representations from G to T gives rise to the "character homomorphism" $\mathrm{ch}_G : \mathrm{K}(G) \to \mathrm{K}(T)$ of Grothendieck rings; this homomorphism is injective if G is connected.

The main representation-theoretic input we need is the fact that over a field of characteristic 0, the isomorphism type of a connected split reductive group can be determined from the character data of its irreducible representations; more precisely:

Theorem 6.7. *Let k be a field of characteristic* 0. *Let G and G' be connected split reductive algebraic groups over k, and let $T \subseteq G$ and $T' \subseteq G'$ be maximal tori, satisfying the hypotheses (i)–(iii) below.*

(i) *Suppose $\phi : \mathrm{Irr}_k(G') \xrightarrow{\cong} \mathrm{Irr}_k(G)$ is a bijection of sets; it induces an isomorphism of abelian groups $\phi : \mathrm{K}(G') \xrightarrow{\cong} \mathrm{K}(G)$ which makes the diagram*

$$\begin{array}{ccc} \mathrm{K}(G') & \xrightarrow[\cong]{\phi} & \mathrm{K}(G) \\ \uparrow & & \uparrow \\ \mathrm{Irr}_k(G') & \xrightarrow[\cong]{\phi} & \mathrm{Irr}_k(G) \end{array} \qquad \textit{commute.}$$

(ii) *Suppose $f_T : T \xrightarrow{\cong} T'$ is an isomorphism of tori; it induces an isomorphism of the group of characters $\mathrm{X}(f_T) : \mathrm{X}_k(T') \xrightarrow{\cong} \mathrm{X}_k(T)$ and an isomorphism* *of rings* $\mathrm{K}(f_T) : \mathrm{K}(T') \xrightarrow{\cong} \mathrm{K}(T)$ *which make the diagram*

$$\begin{array}{ccc} \mathrm{K}(T') & \xrightarrow[\cong]{\mathrm{K}(f_T)} & \mathrm{K}(T) \\ \big\uparrow & & \big\uparrow \\ \mathrm{X}_k(T') & \xrightarrow[\cong]{\mathrm{X}(f_T)} & \mathrm{X}_k(T) \end{array} \qquad \textit{commute.}$$

(iii) *Finally, suppose that the diagram*

$$\begin{array}{ccc} \mathrm{K}(T') & \xrightarrow[\cong]{\mathrm{K}(f_T)} & \mathrm{K}(T) \\ {\scriptstyle \mathrm{ch}_{G'}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ch}_G} \\ \mathrm{K}(G') & \xrightarrow[\cong]{\phi} & \mathrm{K}(G) \end{array} \qquad \textit{commutes.}$$

Then the following conclusions hold:

(1) *There exists an isomorphism of algebraic groups* $f : G \xrightarrow{\cong} G'$ *such that:*
   (a) f *extends the isomorphism* f_T *of maximal tori given in (ii), i.e., the diagram*

$$\begin{array}{ccc} T & \xrightarrow[\cong]{f_T} & T' \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow[\cong]{f} & G' \end{array} \qquad \textit{commutes;} \quad \textit{and}$$

   (b) *the bijection of sets*

$$\mathrm{Irr}_k(f) : \mathrm{Irr}_k(G') \xrightarrow{\cong} \mathrm{Irr}_k(G) \qquad \textit{given by } \rho' \mapsto \rho' \circ f$$

   is equal to the bijection of sets

$$\phi : \mathrm{Irr}_k(G') \xrightarrow{\cong} \mathrm{Irr}_k(G) \qquad \textit{given in (i).}$$

(2) *If* $f' : G \xrightarrow{\cong} G'$ *is another isomorphism of algebraic groups having the same properties as* f *in* (1a) *and* (1b) *above, then there exists a* k*-rational point* $t \in T(k)$ *of* T *such that*

$$f' = f \circ \textit{(conjugation by } t\textit{).}$$

This result is intuitively clear, despite its lengthy statement. It seems to be also well known to the experts in algebraic groups, although I have not been able to locate a satisfactory reference in the published literature. The prudent reader can refer to [Chin03a], Th. 1.4 for a detailed proof.

To proceed, let us assume for the moment the validity of the following:

Theorem 6.8. *Given the constructions* (6.5) *above, for each* $\lambda \in |F|_{\neq p}$*, there exists a unique bijection*

$$\phi : \mathrm{Irr}_{F_\lambda}(G_{0/F_\lambda}) \xrightarrow{\cong} \mathrm{Irr}_{F_\lambda}(G_{F_\lambda}) \qquad \textit{of sets,} \tag{6.8.1}$$

such that the diagram

$$\begin{array}{ccc} \mathrm{K}(T_{0/F_\lambda}) & \xrightarrow[\cong]{\mathrm{K}(f_\lambda)} & \mathrm{K}(T_{F_\lambda}) \\ {\scriptstyle \mathrm{ch}_{G_{0/F_\lambda}}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ch}_{G_{F_\lambda}}} \\ \mathrm{K}(G_{0/F_\lambda}) & \xrightarrow[\cong]{\phi} & \mathrm{K}(G_{F_\lambda}) \end{array} \qquad \textit{commutes.} \tag{6.8.2}$$

Now if we combine (6.7) with (6.8), we obtain the following strengthened version of our main result (1.4)(i):

Corollary 6.9. *Given the constructions* (6.5) *above, for each* $\lambda \in |F|_{\neq p}$, *there exists an isomorphism*

$$f_\lambda : G_{F_\lambda} \xrightarrow{\cong} G_{0/F_\lambda} \qquad \text{of connected split reductive groups over } F_\lambda,$$

extending the canonical isomorphism (cf. (6.5.7)*)*

$$f_\lambda : T_{F_\lambda} \xrightarrow{\cong} T_{0/F_\lambda} \qquad \text{of split } F_\lambda\text{-tori},$$

and inducing the bijection (6.8.1) *given by* (6.8)*. Moreover, if*

$$f'_\lambda : G_{F_\lambda} \xrightarrow{\cong} G_{0/F_\lambda} \qquad \text{is another isomorphism}$$

with the same properties as f_λ *above, then there exists* $t \in T_{F_\lambda}(F_\lambda)$ *such that*

$$f'_\lambda = f_\lambda \circ (\text{conjugation by } t).$$

We postpone the discussion of (1.4)(ii) to (6.13); for now, let us focus on the proof of (6.8).

6.10. Let $\lambda \in |F|_{\neq p}$ be a given place of F not lying over p. A bijection (6.8.1) which makes (6.8.2) commute is necessarily unique, as one sees from the fact that the two sides of (6.8.1) are the canonical basis sets for the bottom two terms of (6.8.2). Hence it suffices for us to show the existence of (6.8.1) making (6.8.2) commute.

From the constructions in (6.5) (especially (6.5.9)), we obtain the following commutative diagram:

$$\begin{tikzcd}
T_{F_\mu} \arrow[r] \arrow[d, hook] & T_0 \arrow[d, hook] & T_{0/F_\lambda} \arrow[l] \arrow[d, hook] & T_{F_\lambda} \arrow[l, "f_\lambda", "\cong"'] \arrow[d, hook] \\
G_{F_\mu} \arrow[r] \arrow[d] & G_0 \arrow[d] & G_{0/F_\lambda} \arrow[l] \arrow[d] & G_{F_\lambda} \arrow[d] \\
\mathrm{Spec}(F_\mu) \arrow[r] & \mathrm{Spec}(F) & \mathrm{Spec}(F_\lambda) \arrow[l] \arrow[r, equal] & \mathrm{Spec}(F_\lambda)
\end{tikzcd}$$

in which the left four squares are cartesian. Thanks to the fact that the tori and the connected reductive groups appearing above are all split, the functors of scalar extensions (from F to F_λ and from F to F_μ respectively) yield isomorphisms of character groups:

$$\mathrm{X}_{F_\mu}(T_{F_\mu}) \xleftarrow{\cong} \mathrm{X}_F(T_0) \xrightarrow{\cong} \mathrm{X}_{F_\lambda}(T_{0/F_\lambda})$$

and bijections of sets of irreducible representations:

$$\mathrm{Irr}_{F_\mu}(G_{F_\mu}) \xleftarrow{\cong} \mathrm{Irr}_F(G_0) \xrightarrow{\cong} \mathrm{Irr}_{F_\lambda}(G_{0/F_\lambda}),$$

and these induce the following commutative diagram of Grothendieck rings:

$$\begin{tikzcd}
\mathrm{K}(T_{F_\mu}) & \mathrm{K}(T_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \mathrm{K}(T_{0/F_\lambda}) & \mathrm{K}(T_{F_\lambda}) \arrow[l, "\mathrm{K}(f_\lambda)", "\cong"'] \\
\mathrm{K}(G_{F_\mu}) \arrow[u, hook, "\mathrm{ch}_{G_{F_\mu}}"] & \mathrm{K}(G_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, hook, "\mathrm{ch}_{G_0}"] & \mathrm{K}(G_{0/F_\lambda}) \arrow[u, hook, "\mathrm{ch}_{G_{0/F_\lambda}}"] & \mathrm{K}(G_{F_\lambda}) \arrow[u, hook, "\mathrm{ch}_{G_{F_\lambda}}"']
\end{tikzcd}$$

We use the top row of isomorphisms to identify $\mathrm{K}(T_{F_\mu})$ with $\mathrm{K}(T_{F_\lambda})$. It follows that to prove (6.8), it suffices for us to construct a map

(6.10.1) $$\Theta : \mathrm{Irr}_{F_\mu}(G_{F_\mu}) \to \mathrm{Irr}_{F_\lambda}(G_{F_\lambda})$$

which is bijective, and which makes the diagram

$$\begin{array}{ccc} \mathrm{K}(T_{F_\mu}) & = & \mathrm{K}(T_{F_\lambda}) \\ {\scriptstyle \mathrm{ch}_{G_{F_\mu}}} \big\uparrow & & \big\uparrow {\scriptstyle \mathrm{ch}_{G_{F_\lambda}}} \\ \mathrm{K}(G_{F_\mu}) & \xrightarrow{\Theta} & \mathrm{K}(G_{F_\lambda}) \end{array} \qquad \text{commute.} \tag{6.10.2}$$

Reformulating (6.8) in this way, we see that the two places λ, μ of the number field F play symmetric roles. In particular, we see that it suffices to construct *some* map Θ in (6.10.1) which makes the diagram (6.10.2) commute; it will automatically be bijective, for if we reverse the roles of λ and μ, we would obtain a map Θ' in the direction opposite to Θ in (6.10.1), and the commutativity of the diagram (6.10.2) (and the analogous one for Θ') would show that Θ and Θ' are inverses of each other.

6.11. **Constructions.** Recall from (6.5.4) and (6.5.5) that G_{F_λ} is the arithmetic monodromy group of the lisse F_λ-sheaf $\mathcal{F}_\lambda = \mathcal{L}_\lambda \otimes_{E_\lambda} F_\lambda$.

6.11.1. The monodromy representation $[\mathcal{F}_\lambda]$ of $\mathcal{F}_\lambda$ factorizes as a composite homomorphism

$$[\mathcal{F}_\lambda] : \pi_1(X, \bar{\eta}) \xrightarrow{\omega_\lambda} G_{F_\lambda} \xhookrightarrow{\rho_\lambda} \mathrm{GL}(\mathcal{F}_{\lambda\bar{\eta}}),$$

where we let ω_λ denote the continuous homomorphism from $\pi_1(X, \bar{\eta})$ into G_{F_λ} with a Zariski dense image, and ρ_λ is the faithful tautological F_λ-rational representation of G_{F_λ} (cf. (6.5.5)).

6.11.2. From (6.5.6), we see that the maximal torus T_{F_λ} of G_{F_λ} is the Zariski closure of the subgroup of G_{F_λ} generated by $\omega_\lambda(\mathrm{Frob}_x)^{\mathrm{ss}}$, where $x \in |X|$ is the point chosen in (6.5.1). The group of characters $\mathrm{X}(T_{F_\lambda})$ of T_{F_λ} is canonically isomorphic to the subgroup $\Psi \subset F^\times$ generated by the eigenvalues of Frob_x acting on $\mathcal{F}_{\lambda\bar{\eta}}$ (cf. (5.1)). The Grothendieck ring $\mathrm{K}(T_{F_\lambda})$ of T_{F_λ} is therefore canonically isomorphic to the group ring of Ψ over $\mathbb{Z}$.

6.11.3. Let $\tau : G_{F_\lambda} \to \mathrm{GL}_N$ be an F_λ-rational representation of G_{F_λ}. We may pull it back via ω_λ to obtain a continuous F_λ-representation of $\pi_1(X, \bar{\eta})$, and hence a lisse F_λ-sheaf on X, which we will denote by $\mathcal{F}_\lambda(\tau)$.

The restriction of τ to the maximal torus T_{F_λ} decomposes as a direct sum of characters of T_{F_λ}, and this decomposition is encoded by the element $\mathrm{ch}_{G_{F_\lambda}}(\tau)$ of $\mathrm{K}(T_{F_\lambda})$. Identifying $\mathrm{X}(T_{F_\lambda})$ with the group Ψ as in (6.11.2), we see that the characters of T_{F_λ} appearing in this decomposition are given by the eigenvalues of Frob_x acting on the lisse sheaf $\mathcal{F}_\lambda(\tau)$; in other words, the element $\mathrm{ch}_{G_{F_\lambda}}(\tau) \in \mathrm{K}(T_{F_\lambda})$ is determined by the "inverse" characteristic polynomial $\det(1 - T\,\mathrm{Frob}_x, \mathcal{F}_\lambda(\tau))$ of Frob_x acting on $\mathcal{F}_\lambda(\tau)$.

6.11.4. If τ is absolutely irreducible, then so is the lisse F_λ-sheaf $\mathcal{F}_\lambda(\tau)$; this follows from the fact that the monodromy representation of $\mathcal{F}(\tau)$ is the composite of the continuous homomorphism ω_λ (which has a Zariski dense image), followed by the representation τ of G_{F_λ}.

6.11.5. The above discussion holds just as well when λ is replaced by μ, and we shall employ the analogous notation for the objects constructed. Note that by our hypotheses, $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ are F-compatible with each other; likewise their duals ${\mathcal{F}_\lambda}^\vee$ and ${\mathcal{F}_\mu}^\vee$ are also F-compatible with each other.

The existence of the map Θ in (6.10.1) which makes the diagram (6.10.2) commute is a consequence of the following:

Theorem 6.12. *Given the constructions* (6.5) *and* (6.11) *above, for every irreducible F_μ-rational representation $\theta_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ of G_{F_μ}, there exists an irreducible F_λ-rational representation $\theta_\lambda \in \mathrm{Irr}_{F_\lambda}(G_{F_\lambda})$ of G_{F_λ} such that the F_μ-sheaf $\mathcal{F}_\mu(\theta_\mu)$ and the F_λ-sheaf $\mathcal{F}_\lambda(\theta_\lambda)$ are F-compatible. Moreover, one has*

$$\mathrm{ch}_{G_{F_\mu}}(\theta_\mu) = \mathrm{ch}_{G_{F_\lambda}}(\theta_\lambda) \qquad \textit{equality in } \mathrm{K}(T_{F_\mu}) = \mathrm{K}(T_{F_\lambda}).$$

Proof. We first work in the "μ-world". Recall that the group G_{F_μ} is given with the faithful tautological F_μ-rational representation ρ_μ. Let $\theta_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ be an irreducible F_μ-rational representation of G_{F_μ}. By a general result in representation theory (cf. [DG70], Chap. II, §2, Prop. 2.9), there exist non-negative integers $a, b \in \mathbb{Z}_{\geq 0}$ such that θ_μ occurs with positive multiplicity in the tensor power representation ${\rho_\mu}^{\otimes a} \otimes {\rho_\mu}^{\vee \otimes b}$. Consider the isotypic decomposition

$${\rho_\mu}^{\otimes a} \otimes {\rho_\mu}^{\vee \otimes b} \cong \bigoplus_{\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})} {\tau_\mu}^{\oplus n(\tau_\mu)},$$

where $n(\tau_\mu)$ is the multiplicity (almost all of which are zero) of τ_μ in ${\rho_\mu}^{\otimes a} \otimes {\rho_\mu}^{\vee \otimes b}$. The isotypic decomposition of the lisse F_μ-sheaf ${\mathcal{F}_\mu}^{\otimes a} \otimes {\mathcal{F}_\mu}^{\vee \otimes b}$ is then given by

$$(6.12.1) \qquad {\mathcal{F}_\mu}^{\otimes a} \otimes {\mathcal{F}_\mu}^{\vee \otimes b} \cong \bigoplus_{\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})} \mathcal{F}_\mu(\tau_\mu)^{\oplus n(\tau_\mu)}.$$

Since both $\mathcal{F}_\mu$ and ${\mathcal{F}_\mu}^\vee$ are plain of characteristic p, so is ${\mathcal{F}_\mu}^{\otimes a} \otimes {\mathcal{F}_\mu}^{\vee \otimes b}$. Hence, for each of the finitely many $\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ with $n(\tau_\mu) > 0$, the lisse F_μ-sheaf $\mathcal{F}_\mu(\tau_\mu)$ is also plain of characteristic p. Moreover, because τ_μ is absolutely irreducible, so is $\mathcal{F}_\mu(\tau_\mu)$. Therefore each of these $\mathcal{F}_\mu(\tau_\mu)$ satisfies the hypotheses of (4.6).

We now "pass from μ to λ". Applying (4.6), we see that there is a finite extension F' of F and a place λ of F' lying over the given place λ of F such that for each $\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ with $n(\tau_\mu) > 0$, there is an absolutely irreducible lisse F'_λ-sheaf $\mathcal{H}'(\tau_\mu)$ on X which is F-compatible with the lisse F_μ-sheaf $\mathcal{F}_\mu(\tau_\mu)$. We can thus form the lisse F'_λ-sheaf

$$(6.12.2) \qquad \mathcal{H}' := \bigoplus_{\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})} \mathcal{H}'(\tau_\mu)^{\oplus n(\tau_\mu)}$$

(note that the indexing set is still $\mathrm{Irr}_{F_\mu}(G_{F_\mu})$). Comparing this with (6.12.1), we see that $\mathcal{H}'$ is F-compatible with the lisse F_μ-sheaf ${\mathcal{F}_\mu}^{\otimes a} \otimes {\mathcal{F}_\mu}^{\vee \otimes b}$. As $\mathcal{F}_\mu$ (resp. ${\mathcal{F}_\mu}^\vee$) is F-compatible with $\mathcal{F}_\lambda$ (resp. ${\mathcal{F}_\lambda}^\vee$), the lisse F'_λ-sheaf $\mathcal{H}'$ is also F-compatible with the lisse F_λ-sheaf ${\mathcal{F}_\lambda}^{\otimes a} \otimes {\mathcal{F}_\lambda}^{\vee \otimes b}$. By Čebotarev's density theorem and the trace comparison theorem of Bourbaki (cf. [Bou58], §12, no. 1, Prop. 3), it follows that $\mathcal{H}'$ is isomorphic to the F'_λ-scalar extension of the lisse F_λ-sheaf ${\mathcal{F}_\lambda}^{\otimes a} \otimes {\mathcal{F}_\lambda}^{\vee \otimes b}$.

From this observation, we infer that the monodromy representation of the lisse F'_λ-sheaf $\mathcal{H}'$ factors through the group $(G_{F_\lambda})_{/F'_\lambda}$ obtained from the group G_{F_λ} by scalar extension to F'_λ. Consequently, each of the absolutely irreducible direct summands $\mathcal{H}'(\tau_\mu)$ of $\mathcal{H}'$ appearing in (6.12.2) is a lisse F'_λ-sheaf whose monodromy representation also factors through the group $(G_{F_\lambda})_{/F'_\lambda}$. In other words, for each $\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ with $n(\tau_\mu) > 0$, there is an F'_λ-rational representation τ'_λ of $(G_{F_\lambda})_{/F'_\lambda}$ such that the monodromy representation of $\mathcal{H}'(\tau_\mu)$ factorizes as

$$[\mathcal{H}'(\tau_\mu)] : \pi_1(X, \bar{\eta}) \xrightarrow{(\omega_\lambda)_{/F'_\lambda}} (G_{F_\lambda})_{/F'_\lambda} \xrightarrow{\tau'_\lambda} \mathrm{GL}(\mathcal{H}'(\tau_\mu)_{\bar{\eta}}).$$

The representation τ'_λ of $(G_{F_\lambda})_{/F'_\lambda}$ is necessarily irreducible, since this is so for $[\mathcal{H}'(\tau_\mu)]$.

Now it remains for us to descend from F'_λ to F_λ. Since the group G_{F_λ} is a split group over F_λ, the representation τ'_λ is defined over F_λ: there exists an irreducible F_λ-rational representation τ_λ of G_{F_λ} whose scalar extension to F'_λ is τ'_λ. Precomposing τ_λ with ω_λ yields the continuous F_λ-representation $\tau_\lambda \circ \omega_\lambda$ of $\pi_1(X, \bar{\eta})$, which is none other than the monodromy representation of the lisse F_λ-sheaf $\mathcal{F}_\lambda(\tau_\lambda)$, as constructed in (6.11.3). Since the F'_λ-scalar extension of $\mathcal{F}_\lambda(\tau_\lambda)$ is $\mathcal{H}'(\tau_\mu)$, we conclude that $\mathcal{F}_\lambda(\tau_\lambda)$ is F-compatible with $\mathcal{F}_\mu(\tau_\mu)$.

In summary, to each $\tau_\mu \in \mathrm{Irr}_{F_\mu}(G_{F_\mu})$ with $n(\tau_\mu) > 0$, we have constructed a corresponding $\tau_\lambda \in \mathrm{Irr}_{F_\lambda}(G_{F_\lambda})$ such that $\mathcal{F}_\lambda(\tau_\lambda)$ is F-compatible with $\mathcal{F}_\mu(\tau_\mu)$. Since a and b are chosen so that $n(\theta_\mu) > 0$, we may specialize to the case of $\tau_\mu = \theta_\mu$, and that proves the first assertion of the theorem. The second assertion follows from the first and the remarks in (6.11.3) and (6.11.4). This completes the proof of (6.12), and the proof of theorem (6.8) as well. □

6.13. **Proof of (1.4)(ii).** Fix $\lambda \in |F|_{\neq p}$ and use the isomorphism f_λ given in (6.9) to identify the groups G_{F_λ} and G_{0/F_λ}. We want to show that under this identification, the tautological representation ρ_λ of G_{F_λ} is isomorphic to $\sigma_0 \otimes_F F_\lambda$, where σ_0 is the F-rational representation of G_0 constructed in (6.5.10). It suffices to show that the characters $\mathrm{ch}_{G_{F_\lambda}}(\rho_\lambda)$ and $\mathrm{ch}_{G_{0/F_\lambda}}(\sigma_0 \otimes_F F_\lambda)$ are mapped to each other under the isomorphism

$$\mathrm{K}(f_\lambda) : \mathrm{K}(T_{F_\lambda}) \xrightarrow{\cong} \mathrm{K}(T_{0/F_\lambda}).$$

From the construction of σ_0 in (6.5.10) and the argument in (6.10), we see that this is the same as showing the equality of $\mathrm{ch}_{G_{F_\lambda}}(\rho_\lambda)$ and $\mathrm{ch}_{G_{F_\mu}}(\rho_\mu)$ as elements of $\mathrm{K}(T_{F_\lambda}) = \mathrm{K}(T_{F_\mu})$. In view of (6.11.3), this amounts to showing the equality

$$\det(1 - T\,\mathrm{Frob}_x, \mathcal{F}_\lambda) = \det(1 - T\,\mathrm{Frob}_x, \mathcal{F}_\mu),$$

but that follows from the fact that $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ are F-compatible. □

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