JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, Pages 193–215
S 0894-0347(04)00471-0
Article electronically published on November 3, 2004

THE ORBIFOLD CHOW RING OF TORIC DELIGNE-MUMFORD STACKS

LEV A. BORISOV, LINDA CHEN, AND GREGORY G. SMITH

1. Introduction

The orbifold Chow ring of a Deligne-Mumford stack, defined by Abramovich, Graber and Vistoli [2], is the algebraic version of the orbifold cohomology ring introduced by W. Chen and Ruan [7], [8]. By design, this ring incorporates numerical invariants, such as the orbifold Euler characteristic and the orbifold Hodge numbers, of the underlying variety. The product structure is induced by the degree zero part of the quantum product; in particular, it involves Gromov-Witten invariants. Inspired by string theory and results in Batyrev [3] and Yasuda [28], one expects that, in nice situations, the orbifold Chow ring coincides with the Chow ring of a resolution of singularities. Fantechi and Göttsche [14] and Uribe [25] verify this conjecture when the orbifold is $\mathrm{Sym}^n(S)$ where S is a smooth projective surface with $K_S = 0$ and the resolution is $\mathrm{Hilb}^n(S)$. The initial motivation for this project was to compare the orbifold Chow ring of a simplicial toric variety with the Chow ring of a crepant resolution.

To achieve this goal, we first develop the theory of toric Deligne-Mumford stacks. Modeled on simplicial toric varieties, a toric Deligne-Mumford stack corresponds to a combinatorial object called a *stacky fan*. As a first approximation, this object is a simplicial fan with a distinguished lattice point on each ray in the fan. More precisely, a stacky fan $\mathbf{\Sigma}$ is a triple consisting of a finitely generated abelian group N, a simplicial fan Σ in $\mathbb{Q} \otimes_{\mathbb{Z}} N$ with n rays, and a map $\beta \colon \mathbb{Z}^n \to N$ where the image of the standard basis in $\mathbb{Z}^n$ generates the rays in Σ. A rational simplicial fan Σ produces a canonical stacky fan $\mathbf{\Sigma} := (N, \Sigma, \beta)$ where N is the distinguished lattice and β is the map defined by the minimal lattice points on the rays. Hence, there is a natural toric Deligne-Mumford stack associated to every simplicial toric variety. A stacky fan $\mathbf{\Sigma}$ encodes a group action on a quasi-affine variety and the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is the quotient. If Σ corresponds to a smooth toric variety $X(\Sigma)$ and $\mathbf{\Sigma}$ is the canonical stacky fan associated to Σ, then we simply have $\mathcal{X}(\mathbf{\Sigma}) = X(\Sigma)$. We show that many of the basic concepts, such as open and closed toric substacks, line bundles, and maps between toric Deligne-Mumford stacks, correspond to combinatorial notions. We expect that many more results about toric varieties lift to the realm of stacks and hope that toric Deligne-Mumford stacks will serve as a useful testing ground for general theories.

Received by the editors September 30, 2003.
2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 14C15, 14M25.
Key words and phrases. Deligne-Mumford stack, Chow ring, toric variety, crepant resolution.
The first author was partially supported in part by NSF grant DMS-0140172.
The second author was partially supported in part by NSF VIGRE grant DMS-9810750.

Our description of the orbifold Chow ring of a toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ parallels the "Stanley-Reisner" presentation for the Chow ring of a simplicial toric variety. Specifically, the stacky fan $\mathbf{\Sigma}$ gives rise to the *deformed group ring* $\mathbb{Q}[N]^{\mathbf{\Sigma}}$. As a $\mathbb{Q}$-vector space, $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is the group algebra of N. Since N is abelian, we write $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{c \in N} \mathbb{Q}\, y^c$ where y is a formal variable. For $c \in N$, $\bar{c}$ denotes the image of c in $\mathbb{Q} \otimes_{\mathbb{Z}} N$. Multiplication in $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is defined by the equation

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is } \sigma \in \Sigma \text{ such that } \bar{c}_1 \in \sigma \text{ and } \bar{c}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let b_i be the image under the map $\beta\colon \mathbb{Z}^n \to N$ of the i-th standard basis vector. The map β endows $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ with a $\mathbb{Q}$-grading; if $\bar{c} = \sum_{\bar{b}_i \in \sigma} m_i \bar{b}_i$ where σ is the minimal cone in Σ containing $\bar{c}$ and m_i is a nonnegative rational number, then the $\mathbb{Q}$-grading is given by $\deg(y^c) := \sum_{\bar{b}_i \in \sigma} m_i$. Writing $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ for the orbifold Chow ring of $\mathcal{X}(\mathbf{\Sigma})$ with rational coefficient, our principal result is

Theorem 1.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a toric Deligne-Mumford stack with a projective coarse moduli space, then there is an isomorphism of $\mathbb{Q}$-graded rings*

$$A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong \frac{\mathbb{Q}[N]^{\mathbf{\Sigma}}}{\big\langle \sum_{i=1}^{n} \theta(b_i) y^{b_i} : \theta \in \operatorname{Hom}(N, \mathbb{Z}) \big\rangle} \,.$$

Using differential geometry, Jiang [16] establishes this result for the weighted projective space $\mathbb{P}(1,2,2,3,3,3)$.

Our proof of this theorem involves two steps. By definition, the orbifold Chow ring $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is isomorphic as an abelian group to the Chow ring of the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$. We first express $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ as a disjoint union of toric Deligne-Mumford stacks. This leads to a proof of Theorem 1.1 at the level of $\mathbb{Q}$-graded vector spaces. To compare the ring structures, we also express the moduli space $\mathcal{K}_{0,3}\big(\mathcal{X}(\mathbf{\Sigma}), 0\big)$ of 3-pointed twisted stable maps as a disjoint union of toric Deligne-Mumford stacks. This combinatorial description allows us to compute the virtual fundamental class of $\mathcal{K}_{0,3}\big(\mathcal{X}(\mathbf{\Sigma}), 0\big)$. We are then able to verify that multiplication in the deformed group ring coincides with the product in the orbifold Chow ring.

The paper is organized as follows. In Section 2, we extend Gale duality to maps of finitely generated abelian groups. This duality forms an essential link between stacky fans and toric Deligne-Mumford stacks. Nevertheless, this theory is entirely self-contained and may be of interest in other situations. The rudimentary theory of toric Deligne-Mumford stacks is developed in Sections 3 and 4. Specifically, we detail the correspondence between stacky fans and toric Deligne-Mumford stacks, we describe the open and closed toric substacks and we express the inertia stacks as disjoint unions of toric Deligne-Mumford stacks. The proof of Theorem 1.1 is given in Sections **??** and **??**. Finally in Section 7, we use our main result to compare the orbifold Chow rings of a simplicial toric variety and its crepant resolutions.

Conventions. Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers, we consider Chow rings and orbifold Chow rings with rational coefficients and we write $(-)^\star$ for the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$.

2. Gale duality with torsion

In this section, we generalize Gale duality. To orient the reader, we recall the basic form of Gale duality and its application to toric geometry. Given n vectors

$b_1, \ldots, b_n$ which span $\mathbb{Q}^d$, there is a dual configuration $[a_1 \cdots a_n] \in \mathbb{Q}^{(n-d)\times n}$ such that

$$0 \longrightarrow \mathbb{Q}^d \xrightarrow{[b_1 \cdots b_n]^{\mathsf{T}}} \mathbb{Q}^n \xrightarrow{[a_1 \cdots a_n]} \mathbb{Q}^{n-d} \longrightarrow 0 \tag{2.1}$$

is a short exact sequence; see Theorem 6.14 in [29]. The set of vectors $\{a_1, \ldots, a_n\}$ is uniquely determined up to a linear coordinate transformation in $\mathbb{Q}^{n-d}$. This duality plays a role in the study of smooth toric varieties. Specifically, let Σ be a fan with n rays such that the corresponding toric variety $X(\Sigma)$ is smooth. If $N \cong \mathbb{Z}^d$ is the lattice in Σ, then the minimal lattice points $b_1, \ldots, b_n$ generating the rays determine a map $\beta\colon \mathbb{Z}^n \to N$. By tensoring with $\mathbb{Q}$, we obtain a dual configuration $\{a_1, \ldots, a_n\}$. Since $X(\Sigma)$ is smooth, we have $a_i \in \mathbb{Z}^{n-d}$ and the set $\{a_1, \ldots, a_n\}$ is unique up to unimodular (determinant ± 1) coordinate transformations of $\mathbb{Z}^{n-d}$. Abbreviating $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ by $(-)^\star$, it follows that the list $\{a_1, \ldots, a_n\}$ defines a map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \mathbb{Z}^{n-d} \cong \operatorname{Pic}(X)$ and the short exact sequence (2.1) becomes $0 \to N^\star \xrightarrow{\beta^\star} (\mathbb{Z}^n)^\star \xrightarrow{\beta^\vee} \operatorname{Pic}(X) \to 0$; see Section 3.4 in [15]. Our goal is to extend this theory to a larger class of maps.

Let N be a finitely generated abelian group and consider a group homomorphism $\beta\colon \mathbb{Z}^n \to N$. The map β is determined by a finite list $\{b_1, \ldots, b_n\}$ of elements in N. The dual map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \operatorname{DG}(\beta)$ is defined as follows. Choose projective resolutions $\boldsymbol{E}$ and $\boldsymbol{F}$ of the $\mathbb{Z}$-modules $\mathbb{Z}^n$ and N, respectively. Theorem 2.2.6 in [27] shows that $\beta\colon \mathbb{Z}^n \to N$ lifts to a morphism $\boldsymbol{E} \to \boldsymbol{F}$ and Subsection 1.5.8 in [27] shows that the mapping cone $\operatorname{Cone}(\beta)$ fits into an exact sequence of cochain complexes $0 \to \boldsymbol{F} \to \operatorname{Cone}(\beta) \to \boldsymbol{E}[1] \to 0$. Since $\boldsymbol{E}$ is projective, we have the exact sequence of cochain complexes

$$0 \longrightarrow \boldsymbol{E}[1]^\star \longrightarrow \operatorname{Cone}(\beta)^\star \longrightarrow \boldsymbol{F}^\star \longrightarrow 0 \tag{2.2}$$

and the associated long exact sequence in cohomology contains the exact sequence

$$N^\star \xrightarrow{\beta^\star} (\mathbb{Z}^n)^\star \longrightarrow H^1\big(\operatorname{Cone}(\beta)^\star\big) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \longrightarrow 0\,. \tag{2.3}$$

Set $\operatorname{DG}(\beta) := H^1\big(\operatorname{Cone}(\beta)^\star\big)$ and define the dual map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \operatorname{DG}(\beta)$ to be the second map in (2.3). Since $\mathbb{Z}^n$ is projective, $\beta^\vee$ is in fact the only nontrivial connecting homomorphism in the long exact sequence associated to (2.2). This abstract definition guarantees the naturality of this construction. Indeed, mapping cones are natural in the following sense: for every commutative diagram of cochain complexes

$$\begin{tikzcd}
\boldsymbol{E} \arrow[r, "\beta"] \arrow[d, "\nu"] & \boldsymbol{F} \arrow[d, "\upsilon"] \\
\boldsymbol{E}' \arrow[r, "\beta'"] & \boldsymbol{F}'\,,
\end{tikzcd}$$

the map $\operatorname{Cone}(\beta) \to \operatorname{Cone}(\beta')$ given by $(b, c) \mapsto \big(\nu(b), \upsilon(c)\big)$ is a morphism. Moreover, this map is an isomorphism if both ν and υ are isomorphisms. It follows that both $\operatorname{DG}(\beta)$ and $\beta^\vee$ are well defined up to natural isomorphism.

On the other hand, there is an explicit description of the dual map $\beta^\vee$ and the dual group $\operatorname{DG}(\beta)$. If N has rank d, then the structure theorem of finitely generated abelian groups implies that there exists an integer matrix Q such that $0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to 0$ is a projective resolution of N. The map $\beta\colon \mathbb{Z}^n \to N$ lifts to a map $\mathbb{Z}^n \to \mathbb{Z}^{d+r}$ given by a matrix B. Since $\mathbb{Z}^n$ is projective, the cochain complex with $E^0 = \mathbb{Z}^n$ and $E^i = 0$ for all $i \neq 0$ is a projective resolution of $\mathbb{Z}^n$. With these choices, $\operatorname{Cone}(\beta)$ is the complex $0 \to \mathbb{Z}^{n+r} \xrightarrow{[B\,Q]} \mathbb{Z}^{d+r} \to 0$ and we obtain the sequence (2.3) by applying the Snake Lemma (Lemma 1.3.2 in [27]) to the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & (\mathbb{Z}^{d+r})^\star \arrow[r] \arrow[d, "{[B\,Q]^\star}"] & (\mathbb{Z}^{d+r})^\star \arrow[r] \arrow[d, "Q^\star"] & 0 \\
0 \arrow[r] & (\mathbb{Z}^n)^\star \arrow[r] & (\mathbb{Z}^{n+r})^\star \arrow[r] & (\mathbb{Z}^r)^\star \arrow[r] & 0.
\end{tikzcd}$$

Hence, $\operatorname{DG}(\beta) = (\mathbb{Z}^{n+r})^\star / \operatorname{Im}([B\,Q]^\star)$ and the map $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^\star \to (\mathbb{Z}^{n+r})^\star$ and the quotient map $(\mathbb{Z}^{n+r})^\star \to \operatorname{DG}(\beta)$.

Example 2.1. The list $\{(2,1), (-3,0)\} \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ yields a map $\beta \colon \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In this case, $Q = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 2 & -3 \\ 1 & 0 \end{smallmatrix}\right]$. Since the vector $[6\,4\,-3]^\star$ spans the integer kernel of matrix $\left[\begin{smallmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right]$, $\operatorname{DG}(\beta) \cong \mathbb{Z}^3 / \operatorname{Im}\big(\left[\begin{smallmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right]^\star\big) \cong \mathbb{Z}$ and $\beta^\vee \colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by $[6\,4]$.

We are especially interested in the map $\beta \colon \mathbb{Z}^n \to N$ when it has a finite cokernel.

Proposition 2.2. *Let $\beta \colon \mathbb{Z}^n \to N$ be a homomorphism of finitely generated abelian groups. The map β is naturally isomorphic to $\beta^{\vee\vee}$ if and only if the cokernel of β is finite. Moreover, if the cokernel of β is finite, then the kernel of $\beta^\vee$ is $N^\star$.*

Proof. Suppose that $\operatorname{Coker}(\beta)$ is not finite. The sequence (2.3) implies that the $\operatorname{Coker}(\beta^{\vee\vee})$ is $\operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{DG}(\beta), \mathbb{Z})$. Since $\operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{DG}(\beta), \mathbb{Z})$ is finite, we see that β cannot be isomorphic to $\beta^{\vee\vee}$.

Conversely, assume that the cokernel of β is finite. To compute $\beta^{\vee\vee}$, we first construct a projective resolution of $\operatorname{DG}(\beta) = (\mathbb{Z}^{n+r})^\star / \operatorname{Im}([B\,Q]^\star)$. Applying the Snake Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^r \arrow[r] \arrow[d, equal] & \mathbb{Z}^{n+r} \arrow[r] \arrow[d, "{[B\,Q]}"] & \mathbb{Z}^n \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "Q"] & \mathbb{Z}^{d+r} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

shows that $\operatorname{Coker}([B\,Q]) = \operatorname{Coker}(\beta)$ and $\operatorname{Ker}([B\,Q]) = \operatorname{Ker}(\beta)$. Hence, the complex $0 \to \operatorname{Ker}(\beta) \to \mathbb{Z}^{n+r} \xrightarrow{[B\,Q]} \mathbb{Z}^{d+r} \to 0$ is a projective resolution of $\operatorname{Coker}(\beta)$. Since $\operatorname{Ext}^i_{\mathbb{Z}}(\operatorname{Coker}(\beta), \mathbb{Z})$ can be computed from this resolution and $\operatorname{Coker}(\beta)^\star = 0$, we see that $[B\,Q]^\star$ is injective and $0 \to (\mathbb{Z}^{d+r})^\star \xrightarrow{[B\,Q]^\star} (\mathbb{Z}^{n+r})^\star \to 0$ is a projective resolution of $\operatorname{DG}(\beta)$.

Since the dual map $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^\star \to (\mathbb{Z}^{n+r})^\star$ and the quotient map $(\mathbb{Z}^{n+r})^\star \to \operatorname{DG}(\beta)$, it follows that the dual group $\operatorname{DG}(\beta^\vee)$ is $(\mathbb{Z}^{n+d+r})^{\star\star} / \operatorname{Im}\left(\left[\begin{smallmatrix} I_n & B^\star \\ 0 & Q^\star \end{smallmatrix}\right]^\star\right)$ and the map $\beta^{\vee\vee}$ is the composition of inclusion $(\mathbb{Z}^n)^{\star\star} \to (\mathbb{Z}^{n+d+r})^{\star\star}$ and the quotient map $(\mathbb{Z}^{n+d+r})^{\star\star} \to \operatorname{DG}(\beta^\vee)$. Because $\mathbb{Z}^m$ is naturally isomorphic to $(\mathbb{Z}^m)^{\star\star}$, it follows that $\operatorname{DG}(\beta^\vee)$ is naturally isomorphic to $(\mathbb{Z}^{d+r} / \operatorname{Im}(Q)) = N$ and $\beta^{\vee\vee}$ is naturally isomorphic to β. Lastly, our resolution of $\operatorname{DG}(\beta)$ also implies that $H^0\big(\operatorname{Cone}(\beta)^\star\big) = 0$ and thus the long exact sequence which gives (2.3) proves the second part of the proposition. $\square$

The operator $(-)^\vee$ is also well behaved in short exact sequences.

Lemma 2.3. *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_2} & \longrightarrow & \mathbb{Z}^{n_3} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_1} & & \Big\downarrow{\scriptstyle \beta_2} & & \Big\downarrow{\scriptstyle \beta_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \end{array} \tag{2.4}$$

in which the rows are exact and the columns have finite cokernels, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}^{n_3})^\star & \longrightarrow & (\mathbb{Z}^{n_2})^\star & \longrightarrow & (\mathbb{Z}^{n_1})^\star & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_3^\vee} & & \Big\downarrow{\scriptstyle \beta_2^\vee} & & \Big\downarrow{\scriptstyle \beta_1^\vee} & & \\ 0 & \longrightarrow & \mathrm{DG}(\beta_3) & \longrightarrow & \mathrm{DG}(\beta_2) & \longrightarrow & \mathrm{DG}(\beta_1) & \longrightarrow & 0. \end{array} \tag{2.5}$$

Proof. For $1 \leq i \leq 3$, choose $\boldsymbol{E}_i := (\mathbb{Z}^{n_i})[0]$ as a projective resolution of $\mathbb{Z}^{n_i}$. Using Lemma 2.2.8 in [27], the bottom row of (2.4) lifts to an exact sequence of projective resolutions $0 \to \boldsymbol{F}_1 \to \boldsymbol{F}_2 \to \boldsymbol{F}_3 \to 0$. Hence, the diagram (2.4) produces a commutative diagram of cochain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{E}_1 & \longrightarrow & \boldsymbol{E}_2 & \longrightarrow & \boldsymbol{E}_3 & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \boldsymbol{F}_1 & \longrightarrow & \boldsymbol{F}_2 & \longrightarrow & \boldsymbol{F}_3 & \longrightarrow & 0. \end{array}$$

The naturality of the mapping cone and the functor $(-)^\star$ yield a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \boldsymbol{E}_3[1]^\star & \longrightarrow & \boldsymbol{E}_2[1]^\star & \longrightarrow & \boldsymbol{E}_1[1]^\star & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Cone}(\beta_3)^\star & \longrightarrow & \mathrm{Cone}(\beta_2)^\star & \longrightarrow & \mathrm{Cone}(\beta_1)^\star & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \boldsymbol{F}_3^\star & \longrightarrow & \boldsymbol{F}_2^\star & \longrightarrow & \boldsymbol{F}_1^\star & \longrightarrow & 0. \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.6}$$

Since $\mathrm{Coker}(\beta_i)$ is finite and $\boldsymbol{E}_i = (\mathbb{Z}^{n_i})[0]$, we have $H^j\big(\mathrm{Cone}(\beta_i)^\star\big) = 0$ and $H^j(\boldsymbol{E}_i[1]^\star) = 0$ for all $j \neq 1$ and $1 \leq i \leq 3$. Hence, taking the cohomology of (2.6) yields (2.5). □

3. Toric Deligne-Mumford stacks

The purpose of this section is to associate a smooth Deligne-Mumford stack to certain combinatorial data. This construction is inspired by the quotient construction for toric varieties; for example see [6].

Let N be a finitely generated abelian group of rank d. We write $\overline{N}$ for the lattice generated by N in the d-dimensional $\mathbb{Q}$-vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural map $N \to \overline{N}$ is denoted by $b \mapsto \bar{b}$. Let Σ be a rational *simplicial* fan in $N_{\mathbb{Q}}$; every cone $\sigma \in \Sigma$ is generated by linearly independent vectors. Let $\rho_1, \dots, \rho_n$ be the rays (one-dimensional cones) in Σ. We assume that $\rho_1, \dots, \rho_n$ span $N_{\mathbb{Q}}$ and we fix an element $b_i \in N$ such that $\bar{b}_i$ generates the cone ρ_i for $1 \le i \le n$. The list $\{b_1, \dots, b_n\}$ defines a homomorphism $\beta\colon \mathbb{Z}^n \to N$ with finite cokernel. The triple $\mathbf{\Sigma} := \big(N, \Sigma, \beta\big)$ is called a *stacky fan*.

The stacky fan $\mathbf{\Sigma}$ encodes a group action on a quasi-affine variety Z. To describe this action, let $\mathbb{C}[z_1, \dots, z_n]$ be the coordinate ring of $\mathbb{A}^n$. The quasi-affine variety Z is the open subset defined by the reduced monomial ideal $J_\Sigma := \big\langle \prod_{\rho_i \not\subseteq \sigma} z_i : \sigma \in \Sigma \big\rangle$; in other words, $Z := \mathbb{A}^n - \mathbb{V}(J_\Sigma)$. The $\mathbb{C}$-valued points of Z are the $z \in \mathbb{C}^n$ such that the cone generated by the set $\{\rho_i : z_i = 0\}$ belongs to Σ. We equip Z with an action of the group $G := \operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$ as follows. By applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the dual map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \mathrm{DG}(\beta)$ (see Section 2), we obtain a homomorphism $\alpha\colon G \to (\mathbb{C}^*)^n$. The natural action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$ induces an action of G on $\mathbb{A}^n$. Since $\mathbb{V}(J_\Sigma)$ is a union of coordinate subspaces, Z is G-invariant.

The quotient stack $\mathcal{X}(\mathbf{\Sigma}) := [Z/G]$ is the Artin stack associated to the groupoid $s, t\colon Z \times G \rightrightarrows Z$ where s is the projection onto the first factor and t is given by the G-action on Z. If S is a scheme, then the objects in $[Z/G](S)$ are principal G-bundles $E \to S$ with a G-equivariant map $E \to Z$ and the morphisms are isomorphisms which preserve the map to Z. Since Z is smooth and separated, $\mathcal{X}(\mathbf{\Sigma})$ is a smooth separated algebraic stack; see Remark 10.13.2 in [19]. The next proposition shows that $\mathcal{X}(\mathbf{\Sigma})$ is in fact a Deligne-Mumford stack. We call $\mathcal{X}(\mathbf{\Sigma})$ the *toric Deligne-Mumford stack* associated to the stacky fan $\mathbf{\Sigma}$.

Lemma 3.1. *The map $Z \times G \to Z \times Z$ with $(z, g) \mapsto (z, z \cdot g)$ is a finite morphism.*

Proof. The morphism of schemes $\alpha\colon G \to (\mathbb{C}^*)^n$ corresponds to the map of rings $\mathbb{C}[(\mathbb{Z}^n)^\star] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{C}[\mathrm{DG}(\beta)]$. Since the cokernel of $\beta^\vee$ is finite, the ring $\mathbb{C}[\mathrm{DG}(\beta)]$ is integral over $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $G \to \operatorname{Im}(\alpha)$ is a finite morphism. Hence, it suffices to prove that $\xi\colon \operatorname{Im}(\alpha) \times Z \to Z \times Z$ is also a finite morphism. Because $\operatorname{Ker}(\beta^\vee) \cong N^\star$, $\operatorname{Im}(\alpha) = \operatorname{Spec}\big(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]/\langle \prod_{i=1}^n t_i^{\theta(b_i)} - 1 : \theta \in N^\star \rangle\big)$.

We next show that $\xi\colon \operatorname{Im}(\alpha) \times Z \to Z \times Z$ is an affine morphism. For each $\sigma \in \Sigma$, set $z_{\widehat{\sigma}} := \prod_{\rho_i \not\subseteq \sigma} z_i$ and let $U_\sigma := \mathbb{C}^n - \mathbb{V}(z_{\widehat{\sigma}})$. The coordinate ring of the open affine subset U_σ is $\mathbb{C}[z_1, \dots, z_n, z_{\widehat{\sigma}}^{-1}]$ and the collection $\{U_\sigma : \sigma \in \Sigma\}$ covers Z. Therefore, $\{U_\sigma \times U_{\sigma'} : \sigma, \sigma' \in \Sigma\}$ is an open affine cover of $Z \times Z$ and $U_\sigma \times U_{\sigma'} = \operatorname{Spec}(B_{\sigma,\sigma'})$ where $B_{\sigma,\sigma'} = \mathbb{C}[z_1, \dots, z_n, z_{\widehat{\sigma}}^{-1}, z_1', \dots, z_n', (z_{\widehat{\sigma}'}')^{-1}]$. Since coordinate subspaces are G-invariant, $\xi^{-1}(U_\sigma \times U_{\sigma'})$ is the affine set

$$\operatorname{Im}(\alpha) \times (U_\sigma \cap U_{\sigma'}) = \operatorname{Spec} A_{\sigma,\sigma'} = \operatorname{Spec}\left(\frac{\mathbb{C}\big[t_1^{\pm 1}, \dots, t_n^{\pm 1}, z_1, \dots, z_n, z_{\widehat{\sigma}}^{-1}, z_{\widehat{\sigma}'}^{-1}\big]}{\big\langle \prod_{i=1}^n t_i^{\theta(b_i)} - 1 : \theta \in N^\star \big\rangle}\right).$$

The restriction of ξ to this affine set corresponds to the map $\zeta\colon B_{\sigma,\sigma'} \to A_{\sigma,\sigma'}$ given by $z_i \mapsto z_i$ and $z_i' \mapsto t_i z_i$ for $1 \le i \le n$.

To prove that ξ is finite, we show that $A_{\sigma,\sigma'}$ is a finitely generated $B_{\sigma,\sigma'}$-module. Clearly, the $z_i \in A_{\sigma,\sigma'}$ and $(z_{\widehat{\sigma}})^{-1}$ are integral over $B_{\sigma,\sigma'}$. Since we have

$$t_i = \zeta\Big((z_{\widehat{\sigma}})^{-1} z_i' \prod_{\substack{\rho_j \not\subseteq \sigma' \\ j \ne i}} z_j\Big) \text{ and } t_i^{-1} = \zeta\Big((z_{\widehat{\sigma}'}')^{-1} z_i \prod_{\substack{\rho_j \not\subseteq \sigma' \\ j \ne i}} z_j'\Big),$$

both t_i for $\bar{b}_i \notin \sigma$ and t_i^{-1} for $\bar{b}_i \notin \sigma'$ are integral over $B_{\sigma,\sigma'}$. Thus, $t_i^{\pm 1}$ is integral when $\bar{b}_i \notin \sigma \cup \sigma'$. The Separation Lemma (see Section 1.2 in [15]) implies there is a $\theta \in N^\star$ such that $\theta(b_i) > 0$ if $\bar{b}_i \in \sigma$ and $\bar{b}_i \notin \sigma'$; $\theta(b_i) < 0$ if $\bar{b}_i \notin \sigma$ and $\bar{b}_i \in \sigma'$; and $\theta(b_i) = 0$ if $\bar{b}_i \in \sigma \cap \sigma'$. Hence, the relation $\prod_i t_i^{\theta(b_i)} = 1$ can be rewritten as $t_i^{\theta(b_i)} = \prod_{j \neq i} t_j^{-\theta(b_j)}$ and our assumptions on θ imply that the right-hand side is integral over $B_{\sigma,\sigma'}$. It follows that $t_i^{\pm 1}$ is integral over $B_{\sigma,\sigma'}$ when $\bar{b}_i \notin \sigma \cap \sigma'$. Because $\sigma \cap \sigma'$ is simplicial, $\bar{b}_i \in \sigma \cap \sigma'$ implies that the relations $\{\prod_i t_i^{\theta(b_i)} = 1 : \theta \in N^\star\}$ allow one to express a power of $t_i^{\pm 1}$ as a product of $t_j^{\pm 1}$ for $\bar{b}_j \notin \sigma \cap \sigma'$. This shows that $t_i^{\pm 1}$ for $1 \leq i \leq n$ is integral over $B_{\sigma,\sigma'}$. Lastly, we have $(z_{\widehat{\sigma}'})^{-1} = \zeta((z'_{\widehat{\sigma}'})^{-1}) \prod_{\rho_i \notin \sigma'} t_i$ which implies $A_{\sigma,\sigma'}$ is integral over $B_{\sigma,\sigma'}$ and completes the proof. □

Proposition 3.2. *The quotient $\mathcal{X}(\mathbf{\Sigma})$ is a separated Deligne-Mumford stack.*

Proof. By Corollary 2.2 in [10] (or Example 7.17 in [26]), it is enough to show that the stabilizers of the geometric points of Z are finite and reduced. Lemma 3.1 shows that the map $Z \times G \to Z \times Z$ defined by $(z, g) \mapsto (z, z \cdot g)$ is a finite morphism. It follows that each stabilizer is a finite group scheme. Since we are working in characteristic zero, all finite group schemes are reduced. □

Remark 3.3. In [18], a "toric stack" is defined to be the quotient of a toric variety by its torus. Since such a quotient is never a Deligne-Mumford stack, $\mathcal{X}(\mathbf{\Sigma})$ is *not* a "toric stack".

Remark 3.4. The definition of $\mathcal{X}(\mathbf{\Sigma})$ does not depend on the fan Σ being simplicial. However, $\mathcal{X}(\mathbf{\Sigma})$ is a Deligne-Mumford stack if and only if the fan Σ is simplicial.

As the next example indicates, our construction produces some classic Deligne-Mumford stacks.

Example 3.5. Let Σ be the complete fan in $\mathbb{Q}$ and consider the list $\{(2,1), (-3,0)\}$ of elements from $N := \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This data defines a stacky fan $\mathbf{\Sigma}$. From Example 2.1, we know $\beta^\vee \colon \mathbb{Z}^2 \to \mathrm{DG}(\beta) \cong \mathbb{Z}$ is given by the matrix $[6\,4]$. Furthermore, $Z := \mathbb{A}^2 - \{(0,0)\}$ and $\lambda \in G \cong \mathbb{C}^*$ acts by $(z_1, z_2) \mapsto (\lambda^6 z_1, \lambda^4 z_2)$. In this case, $\mathcal{X}(\mathbf{\Sigma})$ is precisely the moduli stack of elliptic curves $\overline{\mathcal{M}}_{1,1}$; see [9].

To illustrate that a toric Deligne-Mumford stack depends on the set $\{b_i\}$, we include the following:

Example 3.6. Let Σ be the complete fan in $\mathbb{Q}$, which implies $Z := \mathbb{A}^2 - \{(0,0)\}$, and let $N := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. If $\beta_1 \colon \mathbb{Z}^2 \to N$ corresponds to the list $\{(1,0), (-1,1)\}$ and $\mathbf{\Sigma}_1 = \big(N, \Sigma, \beta_1\big)$, then $\beta_1^\vee \colon \mathbb{Z}^2 \to \mathrm{DG}(\beta) \cong \mathbb{Z}$ is given by the matrix $[3\,3]$ and $\lambda \in G_1 \cong \mathbb{C}^*$ acts by $(z_1, z_2) \mapsto (\lambda^3 z_1, \lambda^3 z_2)$. On the other hand, if $\beta_2 \colon \mathbb{Z}^2 \to N$ corresponds to the list $\{(1,0), (-1,0)\}$, then $\beta_2^\vee \colon \mathbb{Z}^2 \to \mathrm{DG}(\beta) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is given by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $(\lambda_1, \lambda_2) \in G_2 \cong \mathbb{C}^* \times \boldsymbol{\mu}_3$ acts by $(z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_1 z_2)$. Therefore, for the stacky fan $\mathbf{\Sigma}_2 = (N, \Sigma, \beta_2)$, $\mathcal{X}(\mathbf{\Sigma}_2)$ is the quotient of $\mathbb{P}^1$ by a trivial action of $\mathbb{Z}/3\mathbb{Z}$ and $\mathcal{X}(\mathbf{\Sigma}_1) \ncong \mathcal{X}(\mathbf{\Sigma}_2)$.

The last result in this section makes the relationship between toric Deligne-Mumford stacks and toric varieties more explicit. Recall that a *coarse moduli space* of a Deligne-Mumford stack $\mathcal{X}$ is an algebraic space X with a morphism $\pi \colon \mathcal{X} \to X$

such that

- for all algebraically closed fields k, the map $\pi(k)\colon \mathcal{X}(k) \to X(k)$ is a bijection;
- given any algebraic space X' and any morphism $\pi'\colon \mathcal{X} \to X'$, there is a unique morphism $\chi\colon X \to X'$ such that $\pi' = \chi \circ \pi$.

Proposition 3.7. *The toric variety $X(\Sigma)$ is the coarse moduli space of $\mathcal{X}(\mathbf{\Sigma})$.*

Proof. By Proposition 4.2 in [10], it is enough to show that the toric variety $X(\Sigma)$ is the universal geometric quotient of Z by G. Under the additional assumptions that $N = \overline{N}$ and that the $b_i = \bar{b}_i$ are the unique minimal lattice points generating the rays in Σ, this is Theorem 2.1 in [6]. The reader can verify that the proof presented in [6] extends to our situation without any significant changes. □

Proposition 3.7 implies that $\mathcal{X}(\mathbf{\Sigma})$ has a projective coarse moduli space if and only if Σ is the normal fan of a polytope.

4. Closed and open substacks

This section explains how the stacky fan $\mathbf{\Sigma}$ encodes certain closed and open substacks of $\mathcal{X}(\mathbf{\Sigma})$. We also express the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ as a disjoint union of certain closed substacks.

To describe the connection between the combinatorics of the stacky fan $\mathbf{\Sigma}$ and the substacks of $\mathcal{X}(\mathbf{\Sigma})$, we use the theory of groupoids; see [20] for an introduction. Recall that a homomorphism of groupoids $\Theta\colon (R' \rightrightarrows U') \longrightarrow (R \rightrightarrows U)$ is called a *Morita equivalence* if

(1) the square

$$\begin{array}{ccc} R' & \xrightarrow{(s,t)} & U' \times U' \\ {\scriptstyle\Theta}\big\downarrow & & \big\downarrow{\scriptstyle\Theta\times\Theta} \\ R & \xrightarrow{(s,t)} & U \times U \end{array}$$

is Cartesian, and

(2) the morphism $t \circ \mathrm{pr}_1\colon R \times_{s,U,\Theta} U' \to U$ is locally surjective. In other words, U has an open covering $\{U_i \to U\}$ in the étale topology such that each $U_i \to U$ factors through $R \times_{s,U,\Theta} U'$.

The key observation is that two groupoids are Morita equivalent if and only if the associated stacks are isomorphic.

Fix a cone σ in the fan Σ. Let N_σ be the subgroup of N generated by the set $\{b_i : \rho_i \subseteq \sigma\}$ and let $N(\sigma)$ be the quotient group N/N_σ. By extending scalars, the quotient map $N \to N(\sigma)$ becomes the surjection $N_{\mathbb{Q}} \to N(\sigma)_{\mathbb{Q}}$. The quotient fan Σ/σ in $N(\sigma)_{\mathbb{Q}}$ is the set $\{\tilde{\tau} = \tau + (N_\sigma)_{\mathbb{Q}} : \sigma \subseteq \tau \text{ and } \tau \in \Sigma\}$ and the link of σ is the set $\mathrm{link}(\sigma) := \{\tau : \tau + \sigma \in \Sigma, \tau \cap \sigma = 0\}$. For each ray ρ_i in $\mathrm{link}(\sigma)$, we write $\tilde{\rho}_i$ for the ray in Σ/σ and $\tilde{b}_i$ for the image of b_i in $N(\sigma)$. To ensure that the quotient fan satisfies our hypothesis for constructing toric Deligne-Mumford stacks, we require the following:

Condition 4.1. The rays $\tilde{\rho}_i$ span $N(\sigma)_{\mathbb{Q}}$.

Note that if Σ is a complete fan, then every cone σ satisfies Condition 4.1.

Let ℓ be the number of rays in $\operatorname{link}(\sigma)$ and let $\beta(\sigma)\colon \mathbb{Z}^\ell \to N(\sigma)$ be the map determined by the list $\{\tilde{b}_i : \rho_i \in \operatorname{link}(\sigma)\}$. The quotient stacky fan $\mathbf{\Sigma/\sigma}$ is the triple $\big(N(\sigma), \Sigma/\sigma, \beta(\sigma)\big)$.

Proposition 4.2. *If σ is a cone in the stacky fan $\mathbf{\Sigma}$ which satisfies Condition 4.1, then $\mathcal{X}(\mathbf{\Sigma/\sigma})$ defines a closed substack of $\mathcal{X}(\mathbf{\Sigma})$.*

Proof. By definition, $\mathcal{X}(\mathbf{\Sigma})$ is $[Z/G]$. Let $W(\sigma)$ be the closed subvariety of Z defined by the ideal $J(\sigma) := \langle z_i : \rho_i \subseteq \sigma \rangle$ in $\mathbb{C}[z_1, \dots, z_n]$. The $\mathbb{C}$-valued points of $W(\sigma)$ are the $z \in \mathbb{C}^n$ such that the cone spanned by $\{\rho_i : z_i = 0\}$ contains σ and belongs to Σ. Hence, $\rho_i \not\subseteq \sigma \cup \operatorname{link}(\sigma)$ implies that $z_i \neq 0$. Since $J(\sigma)$ defines a coordinate subspace, $W(\sigma)$ is G-invariant and the groupoid $W(\sigma) \times G \rightrightarrows W(\sigma)$ defines a closed substack of $\mathcal{X}(\mathbf{\Sigma})$. It remains to show that $\mathcal{X}(\mathbf{\Sigma/\sigma})$ is the stack associated to $W(\sigma) \times G \rightrightarrows W(\sigma)$.

To begin, we construction a homomorphism from $W(\sigma) \times G \rightrightarrows W(\sigma)$ to the defining groupoid of $\mathcal{X}(\mathbf{\Sigma/\sigma})$. By renumbering the ρ_i, we may assume that $\tilde{\rho}_1, \dots, \tilde{\rho}_\ell$ are the rays in $\operatorname{link}(\sigma)$. If $\mathbb{C}[\tilde{z}_1, \dots, \tilde{z}_\ell]$ is the coordinate ring of $\mathbb{A}^\ell$, then

$$J_{\Sigma/\sigma} := \left\langle \textstyle\prod_{\rho_i \not\subseteq \tau} \tilde{z}_i : \sigma \subseteq \tau \text{ and } \tau \in \Sigma \right\rangle .$$

By definition, $\mathcal{X}(\mathbf{\Sigma/\sigma}) := [Z(\sigma)/G(\sigma)]$ where $Z(\sigma) := \mathbb{A}^\ell - \mathbb{V}(J_{\Sigma/\sigma})$ and $G(\sigma)$ is the group $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta(\sigma)), \mathbb{C}^*)$. Let $m := \dim \sigma$. The description of the $\mathbb{C}$-valued points of $W(\sigma)$ shows the projection $\mathbb{A}^n \to \mathbb{A}^\ell$ induces a surjection $\varphi_0\colon W(\sigma) \to Z(\sigma)$ with $\operatorname{Ker}(\varphi_0) = (\mathbb{C}^*)^{n-\ell-m}$. Applying Lemma 2.3 to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{n-\ell} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^\ell & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{\beta}} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta(\sigma)} & & \\
0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0
\end{array}$$

produces the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}^\ell)^\star & \longrightarrow & (\mathbb{Z}^n)^\star & \longrightarrow & (\mathbb{Z}^{n-\ell})^\star & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta(\sigma)^\vee} & & \Big\downarrow{\scriptstyle \beta^\vee} & & \Big\downarrow{\scriptstyle \tilde{\beta}^\vee} & & \\
0 & \longrightarrow & \mathrm{DG}(\beta(\sigma)) & \longrightarrow & \mathrm{DG}(\beta) & \longrightarrow & \mathrm{DG}(\tilde{\beta}) & \longrightarrow & 0.
\end{array} \tag{4.1}$$

Since the cone σ is simplicial, $N_\sigma \cong \mathbb{Z}^m$ and $\mathrm{DG}(\tilde{\beta}) \cong \mathbb{Z}^{n-\ell-m}$. Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to (4.1) gives the diagram with split exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{C}^*)^{n-\ell-m} & \longrightarrow & G & \xrightarrow{\varphi_1} & G(\sigma) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha(\sigma)} & & \\
0 & \longrightarrow & (\mathbb{C}^*)^{n-\ell} & \longrightarrow & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^\ell & \longrightarrow & 0.
\end{array}$$

Hence, $\Phi := (\varphi_0 \times \varphi_1, \varphi_0)$ is a homomorphism of groupoids from $W(\sigma) \times G \rightrightarrows W(\sigma)$ to $Z(\sigma) \times G(\sigma) \rightrightarrows Z(\sigma)$.

To prove that $\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma})$ is the stack associated to $W(\sigma)\times G \rightrightarrows W(\sigma)$, it suffices to show that Φ is a Morita equivalence. First, the commutative diagram

$$\begin{tikzcd} Z(\sigma)\times G(\sigma)\times(\mathbb{C}^*)^{2(n-\ell-m)} \arrow[d, "{(s,t,\mathrm{id})}"] & W(\sigma)\times G \arrow[l, "\cong"'] \arrow[r, "\varphi_0\times\varphi_1"] \arrow[d, "{(s,t)}"] & Z(\sigma)\times G(\sigma) \arrow[d, "{(s,t)}"] \\ Z(\sigma)\times Z(\sigma)\times(\mathbb{C}^*)^{2(n-\ell-m)} & W(\sigma)\times W(\sigma) \arrow[l, "\cong"'] \arrow[r, "\varphi_0\times\varphi_0"] & Z(\sigma)\times Z(\sigma) \end{tikzcd}$$

shows that $W(\sigma)\times G = \big(Z(\sigma)\times G(\sigma)\big)\times_{\varphi_0\times\varphi_0, Z(\sigma)\times Z(\sigma),(s,t)}\big(W(\sigma)\times W(\sigma)\big)$. Second, we have $\big(Z(\sigma)\times G(\sigma)\big)\times_{s,Z(\sigma),\varphi_0} W(\sigma)\cong Z(\sigma)\times G(\sigma)\times\mathbb{C}^{n-\ell-m}$ which implies that the map $t\circ \mathrm{pr}_1\colon \big(Z(\sigma)\times G(\sigma)\big)\times_{s,Z(\sigma),\varphi_0} W(\sigma)\to Z(\sigma)$ splits. Therefore, Φ is a Morita equivalence and $\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma})$ defines a closed substack of $\mathcal{X}(\mathbf{\Sigma})$. $\square$

Viewing $\sigma\in\Sigma$ as the fan consisting of the cone σ and all its faces, we can identify σ with an open substack of $\mathcal{X}(\mathbf{\Sigma})$. This substack has a particularly nice description when σ is of maximal dimension; $\dim\sigma = d = \operatorname{rank} N$. In this case, let $\beta_\sigma\colon \mathbb{Z}^d\to N$ be the map determined by the list $\{b_i : \rho_i\subseteq\sigma\}$. The induced stacky fan $\boldsymbol{\sigma}$ is the triple $\big(N,\sigma,\beta_\sigma\big)$.

Proposition 4.3. *If σ is a d-dimensional cone in the stacky fan $\mathbf{\Sigma}$, then $\mathcal{X}(\boldsymbol{\sigma})$ defines an open substack of $\mathcal{X}(\mathbf{\Sigma})$. Moreover, $\mathcal{X}(\boldsymbol{\sigma})$ is isomorphic to the quotient of $\mathbb{C}^d$ by the finite abelian group $N(\sigma)$.*

Proof. As in Lemma 3.1, let U_σ be the open subvariety of Z defined by the monomial $z_{\widehat{\sigma}} := \prod_{\rho_i\not\subseteq\sigma} z_i$. The $\mathbb{C}$-valued points of U_σ are the $z\in\mathbb{C}^n$ such that for each $z_i=0$ the ray ρ_i is contained in σ. Since $\mathbb{V}(z_{\widehat{\sigma}})$ is a union of coordinate subspaces, U_σ is G-invariant and the groupoid $U_\sigma\times G\rightrightarrows U_\sigma$ defines an open substack of $\mathcal{X}(\mathbf{\Sigma})$. It remains to show that $\mathcal{X}(\boldsymbol{\sigma})$ is the stack associated to $U_\sigma\times G\rightrightarrows U_\sigma$.

We construct a homomorphism from the defining groupoid of $\mathcal{X}(\boldsymbol{\sigma})$ to the groupoid $U_\sigma\times G\rightrightarrows U_\sigma$. Since σ is a d-dimensional simplicial cone, $J_\sigma=\langle 1\rangle$ and $Z_\sigma := \mathbb{A}^d$. By definition, $\mathcal{X}(\boldsymbol{\sigma}) := [Z_\sigma/G_\sigma]$ where $G_\sigma := \operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta_\sigma),\mathbb{C}^*)$. The description of the $\mathbb{C}$-valued points of U_σ yields a closed embedding $\psi_0\colon Z_\sigma\to U_\sigma$ where $\psi_0(Z_\sigma)=\mathbb{C}^d\times\mathbf{1}\subset\mathbb{C}^d\times(\mathbb{C}^*)^{n-d}\cong U_\sigma$. Applying Lemma 2.3 and the functor $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^d \arrow[r] \arrow[d, "\beta_\sigma"] & \mathbb{Z}^n \arrow[r] \arrow[d, "\beta"] & \mathbb{Z}^{n-d} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r, "\mathrm{id}"'] & N \arrow[r] & 0 & \end{tikzcd}$$

produces the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G_\sigma \arrow[r] \arrow[d, "\alpha_\sigma"] & G \arrow[r, "\psi_1"] \arrow[d, "\alpha"] & (\mathbb{C}^*)^{n-d} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & (\mathbb{C}^*)^d \arrow[r] & (\mathbb{C}^*)^n \arrow[r] & (\mathbb{C}^*)^{n-d} \arrow[r] & 0. \end{tikzcd} \tag{4.2}$$

Hence, $\Psi := (\psi_0\times\psi_1,\psi_0)$ is a homomorphism of groupoids from $Z_\sigma\times G_\sigma\rightrightarrows Z_\sigma$ to $U_\sigma\times G\rightrightarrows U_\sigma$ and an element $g\in G$ belongs to G_σ if and only if $(Z_\sigma\cdot g)\cap Z_\sigma\neq\emptyset$.

Next, we establish that $G_\sigma\cong N(\sigma)$. The definition of $N(\sigma)$ gives the exact sequence $0\longrightarrow\mathbb{Z}^{d+r}\xrightarrow{[B_\sigma\, Q]}\mathbb{Z}^{d+r}\longrightarrow N(\sigma)\longrightarrow 0$ where B_σ is the submatrix of B whose columns correspond to the $\rho_i \subseteq \sigma$. Since $N(\sigma)^\star = 0$, we obtain the exact sequence $0 \longrightarrow (\mathbb{Z}^{d+r})^\star \xrightarrow{[B_\sigma\, Q]^\star} (\mathbb{Z}^{d+r})^\star \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}\big(N(\sigma), \mathbb{Z}\big) \longrightarrow 0$ which implies that $\operatorname{DG}(\beta) = \operatorname{Ext}^1_{\mathbb{Z}}\big(N(\sigma), \mathbb{Z}\big) = \operatorname{Hom}_{\mathbb{Z}}\big(N(\sigma), \mathbb{Q}/\mathbb{Z}\big)$. Hence, the group G_σ is $\operatorname{Hom}_{\mathbb{Z}}\big(\operatorname{Hom}_{\mathbb{Z}}\big(N(\sigma), \mathbb{Q}/\mathbb{Z}\big), \mathbb{C}^*\big)$. We identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$ via the map $p \mapsto \exp(2\pi\sqrt{-1}p)$ to obtain a natural homomorphism from $N(\sigma)$ to G_σ. By expressing $N(\sigma)$ as a direct sum of cyclic groups, one verifies that this map is an isomorphism.

Finally, to prove that $\mathcal{X}(\boldsymbol{\sigma})$ is the stack associated to $U_\sigma \times G \rightrightarrows U_\sigma$, it suffices to show that Ψ is a Morita equivalence. First, because an element $g \in G$ belongs to G_σ if and only if $(Z_\sigma \cdot g) \cap Z_\sigma \neq \emptyset$, the commutative diagram

$$\begin{tikzcd} Z_\sigma \times G_\sigma \arrow[r, "\psi_0 \times \psi_1"] \arrow[d, "(s,t)"] & U_\sigma \times G \arrow[d, "(s,t)"] \\ Z_\sigma \times Z_\sigma \arrow[r, "\psi_0 \times \psi_0"] & U_\sigma \times U_\sigma \end{tikzcd}$$

establishes that $Z_\sigma \times G_\sigma = \big(Z_\sigma \times Z_\sigma\big) \times_{\psi_0 \times \psi_0, Z_\sigma \times Z_\sigma, (s,t)} \big(U_\sigma \times G\big)$. Secondly, we have $\big(U_\sigma \times G\big) \times_{s, U_\sigma, \psi_0} Z_\sigma \cong Z_\sigma \times G$ which implies that $\operatorname{pr}_1 \colon \big(U_\sigma \times G\big) \times_{s, U_\sigma, \psi_0} Z_\sigma \to U_\sigma \times G$ corresponds to the closed immersion $\psi_0 \times \operatorname{id} \colon Z_\sigma \times G \to U_\sigma \times G$. Lemma 3.1 implies that $t \colon U_\sigma \times G \to U_\sigma$ is finite. Since the action of $\operatorname{Coker}(\psi_1)$ on $\psi_0(Z_\sigma)$ surjects onto U_σ, we deduce that $t \circ \operatorname{pr}_1 \colon \big(U_\sigma \times G\big) \times_{s, U_\sigma, \psi_0} Z_\sigma \to U_\sigma$ is a finite surjective morphism of nonsingular varieties and hence is flat. Because the geometric fibers of $t \circ \operatorname{pr}_1$ correspond to G_σ, a finite set of reduced points, the map $t \circ \operatorname{pr}_1$ is also étale and therefore locally surjective. We conclude that Ψ is a Morita equivalence and $\mathcal{X}(\boldsymbol{\sigma})$ defines an open substack of $\mathcal{X}(\boldsymbol{\Sigma})$. □

Remark 4.4. Assuming that every cone in Σ is contained in a d-dimensional cone, Proposition 4.3 produces an étale atlas of $\mathcal{X}(\boldsymbol{\Sigma})$.

Remark 4.5. More generally, if $\boldsymbol{\Sigma}' := (N', \Sigma', \beta')$ and $\boldsymbol{\Sigma} := (N, \Sigma, \beta)$ are two stacky fans, then a *morphism of stacky fans* is a homomorphism $\phi \colon N' \to N$ satisfying

- for each cone $\sigma' \in \Sigma'$, there exists a $\sigma \in \Sigma$ such that $\phi_{\mathbb{Q}}(\sigma') \subseteq \sigma$ where $\phi_{\mathbb{Q}} \colon N' \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$;
- the element $\phi(b'_i)$ is an integer combination of the $b_j \in N$ where $\bar{b}'_i \in \sigma'$, $\bar{b}_j \in \sigma$ and $\sigma \in \Sigma$ is any cone that contains $\phi_{\mathbb{Q}}(\sigma')$.

For each morphism $\phi \colon \boldsymbol{\Sigma}' \to \boldsymbol{\Sigma}$, there is a morphism $\mathcal{X}(\boldsymbol{\Sigma}') \to \mathcal{X}(\boldsymbol{\Sigma})$. Since we do not make use of this construction, the proof is left to the reader.

For each d-dimensional cone σ in the stacky fan $\boldsymbol{\Sigma}$, we define $\operatorname{Box}(\sigma)$ to be the set of elements $v \in N$ such that $\bar{v} = \sum_{\rho_i \subseteq \sigma} q_i \bar{b}_i$ for some $0 \leq q_i < 1$. Hence, the set $\operatorname{Box}(\sigma)$ is in one-to-one correspondence with the elements in the finite group $N(\sigma)$. Let $\operatorname{Box}(\boldsymbol{\Sigma})$ be the union of $\operatorname{Box}(\sigma)$ for all d-dimensional cones $\sigma \in \Sigma$. For each $v \in N$, we write $\sigma(\bar{v})$ for the unique minimal cone containing $\bar{v}$.

Lemma 4.6. *If Σ is a complete fan, then the elements $v \in \operatorname{Box}(\boldsymbol{\Sigma})$ are in one-to-one correspondence with elements $g \in G$ which fix a point of Z. Moreover, we have $[Z^g/G] \cong \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$.*

Proof. By definition, an element $v \in \operatorname{Box}(\boldsymbol{\Sigma})$ corresponds to an element in $N(\tau)$ for some d-dimensional cone $\tau \in \Sigma$. In the proof of Proposition 4.3, we give an isomorphism between $N(\tau)$ and G_τ. Hence, there is a bijection sending v to an element g in the subgroup $G_\tau \subseteq G$. In addition, (4.2) implies that g acts trivially on points $z \in Z$ with $z_i = 0$ for all $\rho_i \subseteq \tau$ which shows that g fixes a point in Z.

Conversely, suppose $g \in G$ fixes a point $z \in Z$. Since the action of G on Z is defined via the map $\alpha\colon G \to (\mathbb{C}^*)^n$ where $g \mapsto \big(\alpha_1(g), \dots, \alpha_n(g)\big)$, we see that either $\alpha_i(g) = 1$ or $z_i = 0$ for all $1 \leq i \leq n$. The definition of Z guarantees that there exists a cone in Σ containing all the rays ρ_i for which $z_i = 0$. Let σ be the minimal cone with this property. Because Σ is a simplicial fan, the ray ρ_i is contained in σ if and only if $\alpha_i(g) \neq 1$. Thus, the closed subvariety $W(\sigma)$ defined in Proposition 4.2 is equal to the invariant subvariety Z^g. Moreover, our choice of σ implies that the element g stabilizes $\psi_0(Z_\tau)$ for every d-dimensional cone τ which contains σ. It follows that g corresponds to an element $v \in \operatorname{Box}(\mathbf{\Sigma})$. Finally, σ is clearly the intersection of all maximal cones τ for which v corresponds to an element in $N(\tau)$. Therefore, $\sigma = \sigma(\bar{v})$ and Proposition 4.2 establishes that $[Z^g/G] = [W(\sigma)/G] \cong \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$. $\square$

For a Deligne-Mumford stack $\mathcal{X}$, its *inertia stack* $\mathcal{I}(\mathcal{X})$ is defined to be the fibered product $\mathcal{X} \times_{\Delta, \mathcal{X}\times\mathcal{X}, \Delta} \mathcal{X}$ where Δ denotes the diagonal map. For a scheme S, an object in $\mathcal{I}\big(\mathcal{X}\big)(S)$ can be identified with the pair (x, ϕ) where x is an object in $\mathcal{X}(S)$ and ϕ is an automorphism of x. A morphism from $(x,\phi) \to (x', \phi')$ is a morphism $\gamma\colon x \to x'$ in $\mathcal{X}(S)$ such that $\gamma \circ \phi = \phi' \circ \gamma$. Since we are working over $\mathbb{C}$, the inertia stack $\mathcal{I}(\mathcal{X})$ is naturally isomorphic to the stack of representable morphisms from constant cyclotomic gerbes to $\mathcal{X}$; see Section 4.4 in [2].

Proposition 4.7. *If Σ is a complete fan, then $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big) = \coprod_{v \in \operatorname{Box}(\mathbf{\Sigma})} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$ where $\sigma(\bar{v})$ is the minimal cone in Σ containing $\bar{v}$.*

Proof. Let S be a connected scheme. An object x of $\mathcal{X}(\mathbf{\Sigma})(S)$ is a principal G-bundle $E \to S$ with a G-equivariant morphism $f\colon E \to Z$. An automorphism ϕ is an automorphism of the principal G-bundle $E \to S$ that is compatible with $E \to Z$. Since S is connected, ϕ corresponds to multiplication by an element $g \in G$. Moreover, because f is G-equivariant and $f = f \circ \phi$, the map f factors through Z^g. Hence, the principal G-bundle $E \to S$ with $E \to Z^g$ is an object in $[Z^g/G](S)$.

For an arbitrary scheme S and an object in $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)(S)$, we can assign an object in $\coprod_{g\in G}[Z^g/G](S)$ by considering the connected components of S. Finally, Lemma 4.6 shows that $Z^g \neq \emptyset$ if and only if g corresponds to an element $v \in \operatorname{Box}(\mathbf{\Sigma})$ and that $[Z^g/G] \cong \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$. $\square$

Remark 4.8. By combining Proposition 3.7 and Proposition 4.7, we see that the coarse moduli space of $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is isomorphic to the disjoint union of $X\big(\Sigma/\sigma(\bar{v})\big)$ for all $v \in \operatorname{Box}(\Sigma)$. In particular, we recover the description of the twisted sectors in Section 6 of [22].

5. Module structure on $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$

The goal of this section is to describe the orbifold Chow ring of a complete toric Deligne-Mumford stack as an abelian group. Throughout this section, we assume all fans are complete and simplicial and all Chow rings have rational coefficients.

We first introduce the *deformed group ring* $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma} = \big(N, \Sigma, \beta\big)$. As a vector space, $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is simply the group ring $\mathbb{Q}[N]$; in other words, $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{c \in N} \mathbb{Q} \cdot y^c$ where y is a formal variable. Multiplication in $\mathbb{Q}[N]^{\mathbf{\Sigma}}$

is defined as follows:

$$(5.1) \qquad y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there exists } \sigma \in \Sigma \text{ such that } \bar{c}_1 \in \sigma \text{ and } \bar{c}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

We endow $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ with a $\mathbb{Q}$-grading as follows: if $\bar{c} = \sum_{\rho_i \subseteq \sigma(\bar{c})} m_i \bar{b}_i$ where $\sigma(\bar{c})$ is the minimal cone in Σ containing $\bar{c}$, then $\deg(y^c) := \sum m_i \in \mathbb{Q}$.

Given a stacky fan $\mathbf{\Sigma}$, we denote by $S_{\mathbf{\Sigma}}$ the subring of $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ generated over $\mathbb{Q}$ by the monomials y^{b_i}. Since Σ is simplicial, the ring $S_{\mathbf{\Sigma}}$ is isomorphic to the quotient $\mathbb{Q}[x_1, \ldots, x_n]/I_\Sigma$ where the ideal I_Σ is generated by the square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ with $\rho_{i_1} + \cdots + \rho_{i_s} \notin \Sigma$. In particular, $S_{\mathbf{\Sigma}}$ is a $\mathbb{Z}$-graded ring and I_Σ is the Stanley-Reisner ideal associated to Σ.

To describe the Chow ring of $\mathcal{X}(\mathbf{\Sigma})$, we need certain line bundles corresponding to the rays $\rho_1, \ldots, \rho_n$. Since the category of coherent sheaves on $\mathcal{X}(\mathbf{\Sigma})$ is equivalent to the category of G-equivariant sheaves on Z (Example 7.21 in [26]), we can define L_i for $1 \leq i \leq n$ to be the line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponding to the trivial line bundle $\mathbb{C} \times Z$ on Z with the G-action on $\mathbb{C}$ given by the i-th component α_i of $\alpha \colon G \to (\mathbb{C}^*)^n$.

We first calculate the non-orbifold Chow ring of $\mathcal{X}(\mathbf{\Sigma})$.

Lemma 5.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then there is an isomorphism of $\mathbb{Z}$-graded rings*

$$\frac{S_{\mathbf{\Sigma}}}{\left\langle \sum_{i=1}^n \theta(\bar{b}_i) \cdot y^{b_i} : \theta \in N^\star \right\rangle} \longrightarrow A^*\big(\mathcal{X}(\mathbf{\Sigma})\big)$$

defined by $y^{b_i} \mapsto c_1(L_i)$.

Proof. For $1 \leq i \leq n$, let a_i denote the unique minimal lattice generator of ρ_i in Σ and let ℓ_i be the positive integer satisfying the relation $\bar{b}_i = \ell_i a_i$. The Jurkiewicz-Danilov Theorem (see page 134 in [21]) states that there is a surjective homomorphism of graded rings from $\mathbb{Q}[x_1, \ldots, x_n]$ to $A^*\big(X(\Sigma)\big)$ given by $x_i \mapsto D_i$ where D_i is the torus invariant Weil divisor on $X(\Sigma)$ associated with ρ_i. The kernel of this map is the ideal I_Σ plus the ideal generated by the linear relations $\sum_{i=1}^n \theta(a_i) \cdot x_i$ for all $\theta \in N^\star$. Example 6.7 in [26] establishes a natural isomorphism $A^*\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong A^*\big(X(\Sigma)\big)$ defined by $c_1(L_i) \mapsto \ell_i^{-1} \cdot D_i$. Since we have $\sum_{i=1}^n \theta(a_i) \cdot \ell_i \cdot x_i = \sum_{i=1}^n \theta(\bar{b}_i) \cdot x_i$ for all $\theta \in N^\star$, the composition of these two isomorphism establishes the claim. $\square$

This lemma allow us to establish Theorem 1.1 at the level of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces. More precisely, we prove the following result. If M is a $\mathbb{Q}$-graded module and c is a rational number, then we write $M[c]$ for the c-th shift of M; it is defined by the formula $M[c]_{c'} = M_{c'+c}$.

Proposition 5.2. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then there is an isomorphism of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces:*

$$\frac{\mathbb{Q}[N]^{\mathbf{\Sigma}}}{\langle \sum_{i=1}^n \theta(b_i) \cdot y^{b_i} : \theta \in N^\star \rangle} \cong \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big] \,.$$

Proof. The definition of $S_{\mathbf{\Sigma}}$ and $\mathrm{Box}(\mathbf{\Sigma})$ implies that $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} y^v \cdot S_{\mathbf{\Sigma}}$. We first analyze the individual summands. Fix an element $v \in \mathrm{Box}(\mathbf{\Sigma})$ and let $\tau := \sigma(\bar{v})$ be the minimal cone in Σ containing $\bar{v}$. It follows from the definition of multiplication in the deformed group ring that $y^v \cdot S_{\mathbf{\Sigma}}$ is isomorphic to the quotient of $S_{\mathbf{\Sigma}}$ by the ideal generated by the elements y^c where c lies outside the cones in Σ containing τ.

Let $S_{\mathbf{\Sigma}/\boldsymbol{\tau}}$ denote the subring of $\mathbb{Q}[N(\tau)]^{\mathbf{\Sigma}/\boldsymbol{\tau}}$ generated by $y^{\tilde{b}_i}$ for $\rho_i \in \operatorname{link}(\tau)$. By renumbering the rays in Σ, we may assume that $\tilde{\rho}_1, \dots, \tilde{\rho}_\ell$ are the rays in $\operatorname{link}(\tau)$. Recall that $\tilde{b}_i$ is the image of b_i in $N(\tau)$. For each ray $\rho_i \in \tau$, choose an element $\theta_i \in N^\star$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_j) = 0$ for all $\bar{b}_i \neq \bar{b}_j \in \tau$. Consider the map defined by

$$y^{b_i} \mapsto \begin{cases} y^{\tilde{b}_i} & \text{for } \rho_i \subseteq \operatorname{link}(\tau), \\ -\sum_{j=1}^{\ell} \theta_i(b_j) \cdot y^{\tilde{b}_j} & \text{for } \rho_i \subseteq \tau, \\ 0 & \text{for } \rho_i \not\subseteq \tau \cup \operatorname{link}(\tau). \end{cases}$$

Since this map is compatible with the multiplicative structures on $S_{\mathbf{\Sigma}}$ and $S_{\mathbf{\Sigma}/\boldsymbol{\tau}}$, it induces a surjective homomorphism from $S_{\mathbf{\Sigma}}$ to $S_{\mathbf{\Sigma}/\boldsymbol{\tau}}$. Clearly, the kernel contains the elements $\theta_i(b_i) \cdot y^{b_i} + \sum_{j=1}^{\ell} \theta_i(b_j) y^{b_j}$ for all $\rho_i \in \tau$ and the elements y^c where c lies outside the cones in Σ containing τ. Given any other element of the kernel, we can use these relations to obtain a linear combination of monomials y^w with $\bar{w} \in \operatorname{link}(\tau)$ which also belongs to the kernel. However, this is only possible if all the coefficients of y^w are zero, which implies that the given elements generate the kernel.

Since Lemma 5.1 establishes that

$$\frac{S_{\mathbf{\Sigma}/\boldsymbol{\tau}}}{\langle \sum_{i=1}^{\ell} \tilde{\theta}(\tilde{b}_i) \cdot y^{\tilde{b}_i} : \tilde{\theta} \in N(\tau)^\star \rangle} \cong A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\tau})\big) \,,$$

we have a surjective $\mathbb{Q}$-graded $\mathbb{Q}$-linear map from $y^v \cdot S_{\mathbf{\Sigma}}$ to $A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\tau})\big)[\deg(y^v)]$ whose kernel is generated by the elements $\theta_i(b_i) \cdot y^{b_i} + \sum_{j=1}^{\ell} \theta_i(b_j) y^{b_j}$ for all $\rho_i \in \tau$ and the pullbacks of the linear relations $\sum_{i=1}^{\ell} \tilde{\theta}(\tilde{b}_i) \cdot y^{\tilde{b}_i}$ where $\tilde{\theta} \in N(\tau)^\star$. Finally, taking the direct sum over all $v \in \operatorname{Box}(\mathbf{\Sigma})$ produces a surjective $\mathbb{Q}$-graded $\mathbb{Q}$-linear map from $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ to $\bigoplus_{v \in \operatorname{Box}(\mathbf{\Sigma})} A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big]$ whose kernel is generated by the elements $\sum_{i=1}^{n} \theta(b_i) \cdot y^{b_i}$ where $\theta \in N^\star$. $\square$

Remark 5.3. Although the elements θ_i in the proof of Proposition 5.2 are not uniquely determined, the possible choices differ by elements in $N(\tau)^\star$. It follows that the surjection from $y^v \cdot S_{\mathbf{\Sigma}}$ to $S_{\mathbf{\Sigma}/\boldsymbol{\tau}}[\deg(y^v)]$ is not canonically defined, but the surjection from $y^v \cdot S_{\mathbf{\Sigma}}$ to $A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\tau})\big)\big[\deg(y^v)\big]$ is.

Remark 5.4. The degree shift in Proposition 5.2 is also called the *age* of the component of the inertia stack; see Subsection 7.1 in [2].

6. The product structure on $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$

In this section, we study multiplication in $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$. Specifically, we complete the proof of Theorem 1.1 by showing that multiplication in the deformed group ring coincides with the orbifold product. To use the results on twisted stable curves in [1] [2], we assume that $\mathcal{X}(\mathbf{\Sigma})$ has a projective coarse moduli space. Proposition 3.7 implies that this is equivalent to saying Σ is the normal fan of a polytope.

To compare the two products, we first give a combinatorial description of the moduli space $\mathcal{K} := \mathcal{K}_{0,3}\big(\mathcal{X}(\mathbf{\Sigma}), 0\big)$ of 3-pointed twisted stable maps of genus zero and degree zero to $\mathcal{X}(\mathbf{\Sigma})$. The moduli space $\mathcal{K}$ is a smooth proper Deligne-Mumford stack with a projective coarse moduli space; see Theorem 3.6.2 in [2]. By identifying the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ with the stack of representable morphisms from a constant cyclotomic gerbe to $\mathcal{X}(\mathbf{\Sigma})$, Lemma 6.2.1 in [2] produces three evaluation maps denoted $\mathrm{ev}_i \colon \mathcal{K} \to \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ for $1 \leq i \leq 3$. The existence of ev_i stems from the following. Given a scheme S and a family of twisted stable maps in $\mathcal{K}(S)$, the coarse curve is just $\mathbb{P}^1 \times S$ and the markings can be identified as $\{0\} \times S$, $\{1\} \times S$ and $\{\infty\} \times S$. Constructing a lifting ev_i is equivalent to constructing a root of the normal bundle of the i-th marking, functorially in S. This is equivalent to constructing a root of the tangent space of $\mathbb{P}^1$ restricted to the i-th marking and a line bundle over a point obviously has the appropriate root.

Proposition 4.7 shows that $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big) = \coprod_{v \in \mathrm{Box}(\mathbf{\Sigma})} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$, so we can index the components of $\mathcal{K}$ by the images of the evaluation maps. Let $\mathcal{K}_{v_1,v_2,v_3}$ be the component of $\mathcal{K}$ such that ev_i maps to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ for $1 \leq i \leq 3$. For brevity, we write $v_1 + v_2 + v_3 \equiv 0$ to indicate that there exists a cone $\sigma \in \Sigma$ containing $\bar{v}_i$ for $1 \leq i \leq 3$ such that the sum $v_1 + v_2 + v_3$ belongs to the subgroup N_σ in N.

Proposition 6.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a toric Deligne-Mumford stack with a projective coarse moduli space, then*

$$\mathcal{K} = \coprod_{\substack{(v_1,v_2,v_3) \in \mathrm{Box}(\mathbf{\Sigma})^3 \\ v_1+v_2+v_3 \equiv 0}} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big)\,,$$

where $\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ is the minimal cone in Σ containing $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$.

Proof. We begin by examining the geometric points of $\mathcal{K}$. A $\mathbb{C}$-valued point of $\mathcal{K}$ is a representable morphism f from a twisted curve $\mathcal{C}$ to $\mathcal{X}(\mathbf{\Sigma})$ such that the induced map on coarse moduli spaces sends $\mathbb{P}^1$ to a point $x \in X(\Sigma)$. Hence, the map f factors through a closed substack $\mathcal{B}G'$ in $\mathcal{X}(\mathbf{\Sigma})$ where $G' \subseteq G$ is the isotropy group of $x \in \mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{B}G'$ is the classifying stack $[x/G']$. Corollary 1.6.2 in [19] shows that the morphism from $\mathcal{C}$ to $\mathcal{B}G'$ is also representable, which implies that the fibered product $\widehat{C} := \mathcal{C} \times_{\mathcal{B}G'} x$ is a scheme. Since $\mathcal{C}$ is smooth, we see that $\widehat{C}$ is a smooth curve, although it is typically disconnected. Let H be the subgroup of G' that acts trivially on the set of connected components of $\widehat{C}$. Since G' is abelian, the group H is the stabilizer of each connected component of $\widehat{C}$. By choosing a connected component C of $\widehat{C}$, we obtain $\mathcal{C} \cong [C/H]$. Assuming the points $\{0, 1, \infty\}$ in $\mathbb{P}^1$ correspond to the markings on $\mathcal{C}$, the properties of a twisted curve imply that the map $\mathcal{C} \to \mathbb{P}^1$ is an isomorphism over $\mathbb{P}^1 - \{0, 1, \infty\}$. It follows that C is a proper smooth Galois cover of $\mathbb{P}^1$ with Galois group H branched over 0, 1 and ∞. Specifically, if $\gamma_1, \gamma_2, \gamma_3$ are the generators of the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ corresponding to counterclockwise loops around 0, 1, ∞, respectively, then C is induced by a homomorphism $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to G$ sending γ_1 to g_i such that $g_1 \cdot g_2 \cdot g_3 = 1$ and g_1, g_2, g_3 generate H as a subgroup of G.

By definition, the map ev_i is induced by the representable morphism from the cyclotomic gerbe in $\mathcal{C}$ lying over the corresponding point in $\mathbb{P}^1$ to $\mathcal{X}(\mathbf{\Sigma})$; recall that over $\mathbb{C}$ the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is canonically isomorphic to the stack of representable morphisms from a constant cyclotomic gerbe to $\mathcal{X}(\mathbf{\Sigma})$. Hence, the evaluation map ev_i sends f to the geometric point (x, g_i) in the inertia stack. Because g_i belongs to the isotropy group of x, it fixes a point in Z. Thus, Lemma 4.6 shows that g_i corresponds to an element $v_i \in \mathrm{Box}(\mathbf{\Sigma})$ and ev_i maps to the component $[Z^{g_i}/G] = \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ of the inertia stack. Moreover, the condition that

$g_1 \cdot g_2 \cdot g_3 = 1$ means that there exists a cone $\sigma \in \Sigma$ containing $\bar{v}_1, \bar{v}_2, \bar{v}_3$ and the sum $v_1 + v_2 + v_3$ belongs to the subgroup N_σ in N. Therefore, the component $\mathcal{K}_{v_1,v_2,v_3}$ is nonempty if and only if $v_1 + v_2 + v_3 \equiv 0$.

The morphisms $\mathrm{ev}_i \colon \mathcal{K}_{v_1,v_2,v_3} \to \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ are compatible with the inclusion maps into $\mathcal{X}(\boldsymbol{\Sigma})$ for $1 \leq i \leq 2$, which yields a morphism

$$e \colon \mathcal{K}_{v_1,v_2,v_3} \to \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1)\big) \times_{\mathcal{X}(\boldsymbol{\Sigma})} \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_2)\big) = [Z^{g_1}/G] \times_{[Z/G]} [Z^{g_2}/G]\,.$$

Because H is the subgroup of G generated by g_1 and g_2 (note: $g_3 = g_1^{-1} g_2^{-1}$), we have $Z^{g_1} \times_Z Z^{g_2} = Z^{\langle g_1, g_2 \rangle} = Z^H$. It follows that $[Z^{g_1}/G] \times_{[Z/G]} [Z^{g_2}/G] = [Z^H/G]$. Our analysis of the geometric points of $\mathcal{K}$ shows that e induces a bijection between the $\mathbb{C}$-valued points of the coarse moduli spaces of $\mathcal{K}_{v_1,v_2,v_3}$ and $[Z^H/G]$. Since both $\mathcal{K}_{v_1,v_2,v_3}$ and $[Z^H/G]$ are smooth Deligne-Mumford stacks, their coarse moduli spaces have at worst quotient singularities. Applying Theorem VI.1.5 in [17], we deduce that, in fact, e produces an isomorphism between the coarse moduli spaces.

To prove that e is an isomorphism of stacks, it remains to show that e gives an isomorphism between the isotropy groups of $\mathbb{C}$-valued points. Indeed, since $\mathcal{K}$ is smooth (see page 18 in [2]) and e is representable, the isomorphism follows from a similar statement for the lifting of e to the atlases. Proposition 7.1.1 in [1] indicates that the automorphism group of a twisted stable curve is the direct product of the automorphism groups of the nodes, which implies that our curve $\mathcal{C}$ has only the trivial automorphism. Hence, an isotropy of the twisted stable map $f \colon \mathcal{C} \to \mathcal{B}G' \subseteq \mathcal{X}(\boldsymbol{\Sigma})$ corresponds to a diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & = & \mathcal{C} \end{array}$$

where ϕ is a G'-equivariant map of principal G'-bundles over $\mathcal{C}$. Since $\mathcal{C}$ is connected, the map ϕ is multiplication by an element of G'. Therefore, the isotropy group of the map f is precisely G', which completes the proof. $\square$

Proposition 6.1 also provides a presentation for the universal twisted stable curve over $\mathcal{K}$. To describe the universal curve, we focus on the component $\mathcal{K}_{v_1,v_2,v_3}$. As above, we write H for the subgroup of G corresponding to $\{v_1, v_2, v_3\}$ and $C \to \mathbb{P}^1$ for the associated Galois cover. Consider the quotient stack

$$\mathcal{U}_{v_1,v_2,v_3} := [(Z^H \times C)/(G \times H)] = [Z^H/G] \times [C/H]\,.$$

If S is a scheme, then the objects in $\mathcal{U}_{v_1,v_2,v_3}(S)$ are principal $(G \times H)$-bundles $E \to S$ with a $(G \times H)$-equivariant map $E \to Z^H \times C$. The *twisted projection map* π from $\mathcal{U}_{v_1,v_2,v_3}$ to $\mathcal{K}_{v_1,v_2,v_3} = [Z^H/G]$ is defined as follows: If H acts on E via the map $h \mapsto (h^{-1}, h) \in G \times H$, then E/H is a principal G-bundle over S. To obtain an object in $\mathcal{K}_{v_1,v_2,v_3}(S)$, observe that the $(G \times H)$-equivariant map $E \to Z^H \times C$ induces a G-equivariant map from E/H to Z^H. By verifying that π is compatible with morphisms in $\mathcal{U}_{v_1,v_2,v_3}(S)$ and $\mathcal{K}_{v_1,v_2,v_3}(S)$, we conclude that π is a morphism of stacks. With these definitions, we have

Corollary 6.2. *The universal twisted stable curve over $\mathcal{K}_{v_1,v_2,v_3} \cong [Z^H/G]$ is given by the twisted projection map $\pi \colon \mathcal{U}_{v_1,v_2,v_3} = [(Z^H \times C)/(G \times H)] \to [Z^H/G]$.*

Proof. Fix a map $S \to [Z^H/G]$ where S is a scheme and consider the fibered product $\mathcal{D} := \mathcal{U}_{v_1,v_2,v_3} \times_{[Z^H/G]} S$. Assuming that $S \to [Z^H/G]$ corresponds to the principal G-bundle $E \to S$ with a G-equivariant map $E \to Z^H$, it follows that $\mathcal{D}$ equals $[(E\times C)/(G\times H)]$ where the $(G\times H)$-action is given by $(e,c,g,h) \mapsto (e\cdot gh^{-1}, c\cdot h)$. The twisted projection map π induces a map $\mathcal{D} \to [E/G] = S$. Because the *anti-diagonal* action of H on $E \times C$ is free, the quotient $Y := (E \times C)/H$ is a scheme. Hence, we have $\mathcal{D} = [Y/G]$ where the G-action on Y is induced by the action on $E \times C$. Since H acts trivially on Z^H, the G-equivariant map $E \to Z^H$ induces a G-equivariant map $Y \to Z^H$ which shows that $\mathcal{D}$ maps to $[Z^H/G] \subseteq [Z/G] = \mathcal{X}(\mathbf{\Sigma})$. Moreover, if $R = R_1 + R_2 + R_3$ is the ramification divisor of the Galois cover $C \to \mathbb{P}^1$, then the image of the open set $E \times (C - R)$ gives an open substack of $[Y/G]$ which is isomorphic to $S\times\big(\mathbb{P}^1 - \{0,1,\infty\}\big)$. By definition, the evaluation map ev_i from $\mathcal{D}$ to the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ arises from the representable morphism from $[(E \times R_i)/(G \times H)]$ to $\mathcal{X}(\mathbf{\Sigma})$. In particular, ev_i is induced by the closed embedding $[Z^H/G] \to [Z^{g_i}/G] \cong \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$. We conclude that $\mathcal{U}_{v_1,v_2,v_3}$ is a family of twisted stable curves over $[Z^H/G]$ with a map $f\colon \mathcal{U}_{v_1,v_2,v_3} \to \mathcal{X}(\mathbf{\Sigma})$ and evaluation maps $\mathrm{ev}_i\colon \mathcal{U}_{v_1,v_2,v_3} \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big) \subseteq \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ for $1 \le i \le 3$.

Let $\mathcal{U}'$ denote the universal family of twisted stable curves over $\mathcal{K}_{v_1,v_2,v_3}$. By the universal mapping property of $\mathcal{U}'$, there is a map $\mu\colon [Z^H/G] \to \mathcal{K}_{v_1,v_2,v_3}$ such that

$$\begin{array}{ccc} \mu^*(\mathcal{U}') & \longrightarrow & \mathcal{U}' \\ \big\downarrow & & \big\downarrow \\ [Z^H/G] & \xrightarrow{\ \mu\ } & \mathcal{K}_{v_1,v_2,v_3} \end{array}$$

is a Cartesian diagram. Combining the definition of e with the first paragraph, we see that $e \circ \mu = \mathrm{id}$. Since Proposition 6.1 shows that e is an isomorphism, we conclude that μ is also an isomorphism and $\mathcal{U}_{v_1,v_2,v_3}$ is isomorphic to $\mathcal{U}'$. $\square$

Next, we describe the virtual fundamental class on $\mathcal{K}$. Recall that L_k denotes the line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponding to the line bundle $\mathbb{C} \times Z$ on Z where the G-action on $\mathbb{C}$ is given by the k-th component α_k of $\alpha\colon G \to (\mathbb{C}^*)^n$.

Proposition 6.3. *Let $\mathcal{K}_{v_1,v_2,v_3}$ be a component of the moduli space $\mathcal{K}$. If the integers $m_k \in \{1,2\}$ are defined by the relation $v_1 + v_2 + v_3 = \sum_{\rho_k \subseteq \sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)} m_k b_k$ in N, then the virtual fundamental class of the component $\mathcal{K}_{v_1,v_2,v_3}$ is*

$$\prod_{m_k=2} c_1(L_k)\big|_{\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1,\bar{v}_2,\bar{v}_3))}\,.$$

Proof. Let $f\colon \mathcal{U}_{v_1,v_2,v_3} \to \mathcal{X}(\mathbf{\Sigma})$ be the natural map and let $\pi\colon \mathcal{U}_{v_1,v_2,v_3} \to [Z^H/G]$ be the twisted projection map. Since $\mathcal{K}_{v_1,v_2,v_3}$ is smooth, the virtual fundamental class of $\mathcal{K}$ is given by the top Chern class of the bundle $R^1\pi_* f^*(T_{\mathcal{X}(\mathbf{\Sigma})})$; see Section 6.2 in [2]. To calculate this Chern class, observe that the pullback of the tangent bundle $f^*(T_{\mathcal{X}(\mathbf{\Sigma})})$ corresponds to a $(G \times H)$-equivariant bundle $\mathcal{V}$ on $Z^H \times C$; $\mathcal{V}$ is a trivial vector bundle of rank n where the $(G \times H)$-action is induced by the map $\alpha\colon G \to (\mathbb{C}^*)^n$ on its basis. Let $p\colon Z^H \times C \to Z^H$ be the projection map and let p_*^H be the invariant pushforward (pushing forward and taking invariant sections). Since the associated derived functor $R^1p_*^H$ sends $(G \times H)$-equivariant sheaves on $Z^H \times C$ to G-equivariant sheaves on Z^H, it suffices to compute $R^1p_*^H(\mathcal{V})$.

Let $\mathcal{W}_k$ be the trivial line bundle on $Z^H \times C$ with $(G \times H)$-action induced by the k-th component α_k of $\alpha\colon G \to (\mathbb{C}^*)^n$ and consider the following exact sequence of vector bundles on $Z^H \times C$: $0 \to p^*(T_{Z^H}) \to \mathcal{V} \to \bigoplus_{\rho_k \in \sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)} \mathcal{W}_k \to 0$. Since the H-invariant part of $R^1p_*p^*(T_{Z^H}) = R^1p_*(\mathcal{O}_{Z^H \times C}) \otimes T_{Z^H}$ is trivial, it suffices to calculate $R^1p_*^H(\mathcal{W}_k)$. Given a point $z \in Z^H$, the restriction of $\mathcal{W}_k$ to $z \times C$ is isomorphic to the trivial line bundle $\mathcal{L}_k$ on C with the H-action induced by α_k. Since the Leray spectral sequence degenerates, $H^1(C, \mathcal{L}_k) \cong H^1\big(\mathbb{P}^1, (p')_*^H(\mathcal{L}_k)\big)$ where $p'\colon C \to \mathbb{P}^1$ is the Galois cover. Because $v_j \in \mathrm{Box}(\mathbf{\Sigma})$ for $1 \leq j \leq 3$, there are $a_{j,k} \in \mathbb{Q}$ such that $0 \leq a_{j,k} < 1$ and $\bar{v}_j = \sum a_{j,k}\bar{b}_k$ where $\rho_k \in \sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. By hypothesis, we have $v_1 + v_2 + v_3 \equiv 0$ which means that $a_{1,k} + a_{2,k} + a_{3,k}$ is an integer between 0 and 2. Lemma 4.6 establishes that v_j corresponds to an element $g_j \in G$ and the proof of Proposition 4.3 shows that $\alpha_k(g_j) = \exp(2\pi\sqrt{-1}a_{j,k})$. From this explicit description of the H-action on $\mathcal{L}_k$, it follows that $(p')_*^H(\mathcal{L}_k) \cong p_*^H\big(\mathcal{W}_k\big|_{z \times C}\big)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a_{1,k} - a_{2,k} - a_{3,k})$. Since

$$\dim H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_{1,k} - a_{2,k} - a_{3,k})\big) = 1 \text{ when } a_{1,k} + a_{2,k} + a_{3,k} = 2,$$

we deduce that $R^1p_*^H(\mathcal{W}_k)$ is the line bundle $\mathbb{C} \times Z$ on Z where the G-action on $\mathbb{C}$ is given by the k-th component α_k. When $a_{1,k} + a_{2,k} + a_{3,k} \neq 2$, the cohomology group vanishes and $R^1p_*^H(\mathcal{W}_k)$ is zero. Therefore, we have

$$R^1\pi_*f^*(T_{\mathcal{X}(\mathbf{\Sigma})}) \cong \bigoplus_{m_k=2} L_k\big|_{[Z^H/G]}$$

and taking the top Chern class completes the proof. $\square$

Remark 6.4. The virtual classes in Proposition 6.3 are analogous to the classes $c(g,h)$ in [14]. However, we do not need the language of parabolic bundles because we give an explicit description for the H-action on the trivial line bundles $\mathcal{L}_k$.

We end this section with a proof of Theorem 1.1. Let $\iota\colon \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big) \to \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ denote the natural involution on the inertia stack defined by $(x,\phi) \mapsto (x, \phi^{-1})$ and let $\check{\mathrm{ev}}_3 := \iota \circ \mathrm{ev}_3$ be the twisted evaluation map; see Section 4.5 in [2]. If $\gamma_1, \gamma_2 \in A^*\big(\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)\big)$, then the orbifold product (Definition 6.2.2 in [2]) is

$$\gamma_1 * \gamma_2 := (\check{\mathrm{ev}}_3)_*\big(\mathrm{ev}_1^*(\gamma_1) \cap \mathrm{ev}_2^*(\gamma_2) \cap [\mathcal{K}]^{vir}\big)$$

where $[\mathcal{K}]^{vir}$ denotes the virtual fundamental class on $\mathcal{K}$. This definition agrees with the definition of the quantum product in degree zero.

Remark 6.5. Proposition 6.1 shows that the component $\mathcal{K}_{v_1,v_2,v_3}$ of the moduli stack is nonempty if and only if $v_1 + v_2 + v_3 \equiv 0$. Hence, if $\gamma_1 \in A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1))\big)$ and $\gamma_2 \in A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_2))\big)$, then the orbifold product $\gamma_1 * \gamma_2$ is nonzero only if there is a cone in Σ containing $\bar{v}_1$ and $\bar{v}_2$.

Proof of Theorem 1.1. By combining Proposition 4.7 and Proposition 5.2, we obtain the following isomorphism of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces:

$$A_{orb}^*\big(\mathcal{X}(\mathbf{\Sigma})\big) = \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big] \cong \frac{\mathbb{Q}[N]^{\mathbf{\Sigma}}}{\langle \sum_{i=1}^n \theta(b_i) \cdot y^{b_i} : \theta \in N^\star \rangle}\,.$$

It remains to show that the orbifold product agrees with the product structure on the deformed group ring. Since the elements of $\mathrm{Box}(\mathbf{\Sigma})$ generate $A_{orb}^*\big(\mathcal{X}(\mathbf{\Sigma})\big)$ as a module over the y^{b_i}, it suffices to show that $y^c * y^{b_i} = y^c \cdot y^{b_i}$ and $y^{v_1} * y^{v_2} = y^{v_1} \cdot y^{v_2}$ where $c \in N$ and $v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma})$.

We first consider the product $y^c * y^{b_i}$ where $c \in N$. By taking advantage of the linear relations $\sum_{i=1}^n \theta(b_i) \cdot y^{b_i}$ for $\theta \in N^\star$, we reduce to the case that b_i does not lie in the minimal cone $\sigma(\bar{c})$ containing $\bar{c}$. Let v be the representative of c in $\mathrm{Box}(\mathbf{\Sigma})$. By Remark 6.5, the only contribution to the product $y^c * y^{b_i}$ comes from the component $\mathcal{K}_{v,0,v'}$ where $v' \in \mathrm{Box}(\mathbf{\Sigma})$ is defined by the equation $v + v' = \sum_{\rho_i \subseteq \sigma(\bar{c})} b_i$. Hence, $\mathcal{K}_{v,0,v'}$ is isomorphic to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$, both $\mathrm{ev}_1, \check{\mathrm{ev}}_3 \colon \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big) \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$ are the identity map and $\mathrm{ev}_2 \colon \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big) \to \mathcal{X}(\mathbf{\Sigma})$ is the closed embedding. The restriction of y^{b_i} from $\mathcal{X}(\mathbf{\Sigma})$ to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$ is equal to $y^{\tilde{b}_i}$ if $\bar{b}_i$ and $\sigma(\bar{c})$ lie in a cone of Σ and it is equal to zero otherwise. Since Proposition 6.3 shows that the virtual fundamental class is 1, if $\mathrm{ev}_2^*(y^{b_i}) \neq 0$, then $y^c * y^{b_i}$ is simply multiplication in $A^*(\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big))$ and Proposition 5.2 shows that this agrees with multiplication in the deformed group ring. Moreover, when $\mathrm{ev}_2^*(y^{b_i}) = 0$, we have $y^c * y^{b_i} = 0 = y^c \cdot y^{b_i}$.

Next, consider the product $y^{v_1} * y^{v_2}$ where $v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma})$. If $\bar{v}_1$ and $\bar{v}_2$ are not contained in a cone, then Remark 6.5 implies that $y^{v_1} * y^{v_2} = 0$ and (5.1) implies that $y^{v_1} \cdot y^{v_2} = 0$. On the other hand, suppose the cone $\sigma \in \Sigma$ contains $\bar{v}_1$ and $\bar{v}_2$. Let $v_3 \in \mathrm{Box}(\mathbf{\Sigma})$ be the element such that $\bar{v}_3 \in \sigma(\bar{v}_1, \bar{v}_2)$ and $v_1 + v_2 + v_3 \equiv 0$; in other words, there exist integers m_i such that $v_1 + v_2 + v_3 = \sum_{\rho_i \subseteq \sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)} m_i b_i$ and $1 \leq m_i \leq 2$. Proposition 6.1 shows that the component $\mathcal{K}_{v_1,v_2,v_3}$ is isomorphic to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big)$ and the evaluation map ev_i corresponds to the closed embedding $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big) \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$. If I is the set of indices i such that $m_i = 2$, then Proposition 6.3 shows that the virtual fundamental class on $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big)$ is the product of the pullbacks of the divisor classes y^{b_i} where $i \in I$. Because of the degree shift, the class $y^{v_i} \in A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is identified with the class $1 \in A^*\big(\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)\big)$ and $y^{v_1} * y^{v_2}$ is the image of the virtual fundamental class under the twisted evaluation map $\check{\mathrm{ev}}_3$. In particular, if J denotes the set of indices i such that $\bar{b}_i \in \sigma(\bar{v}_1, \bar{v}_2)$ but $b_i \notin \sigma(\bar{v}_3)$, then unraveling the identification maps shows that $y^{v_1} * y^{v_2} = y^{\check{v}_3} \cdot \prod_{i \in I} y^{b_i} \cdot \prod_{j \in J} y^{b_j}$ where $\check{v}_3$ is the representation of $-v_3 \in \mathrm{Box}(\mathbf{\Sigma})$. The factor $y^{\check{v}_3}$ arises from the involution $\iota \colon \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big) \to \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$. Since $\check{v}_3 + \sum_{i \in I} b_i + \sum_{j \in J} b_j = v_1 + v_2$, we conclude that $y^{v_1} * y^{v_2} = y^{v_1} \cdot y^{v_2}$. $\square$

7. Applications to crepant resolutions

In this section, we relate the orbifold Chow ring to the Chow ring of a crepant resolution by showing that both rings are fibers of a flat family. This provides a new proof that the graded components of these Chow rings have the same dimension. On the other hand, we also establish that these Chow rings are not generally isomorphic.

A rational fan Σ with n rays produces a canonical stacky fan $\mathbf{\Sigma} := (N, \Sigma, \beta)$ where N is the distinguished lattice in the vector space containing Σ and $\beta \colon \mathbb{Z}^n \to N$ is the map defined by the minimal lattice points on the rays. Hence, there is a natural toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ associated to every toric variety $X(\Sigma)$. Proposition 3.7 shows that $X(\Sigma)$ is the coarse moduli space of $\mathcal{X}(\mathbf{\Sigma})$.

Theorem 7.1. *Let $X(\Sigma)$ be a projective simplicial toric variety and let $\mathcal{X}(\mathbf{\Sigma})$ be the associated toric Deligne-Mumford stack. If Σ' is a regular subdivision of Σ such that $X(\Sigma')$ is a crepant resolution of $X(\Sigma)$, then there is a flat family $T \to \mathbb{P}^1$ of schemes such that $T_0 \cong \mathrm{Spec}\, A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ and $T_\infty \cong \mathrm{Spec}\, A^*\big(X(\Sigma')\big)$.*

Remark 7.2. Any regular subdivision Σ' of Σ induces a morphism $X(\Sigma') \to X(\Sigma)$; see Section 1.4 in [15]. This morphism is a crepant resolution if and only if there is a Σ-linear support function $h' \colon \mathbb{Q}^d \to \mathbb{Q}$ such that $h'(0) = 0$ and $h'(b_i) = -1$ where $b_1, \dots, b_m$ are the minimal lattice points on the rays in Σ'; see Section 3.4 in [15].

Proof. We construct a family of algebras over $\mathbb{P}^1$ such that the fiber over zero is isomorphic to $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ and the fiber over ∞ is isomorphic to $A_\infty \cong A^*\big(\mathcal{X}(\mathbf{\Sigma'})\big)$. We also prove that this family is flat outside of a Zariski closed subset of $\mathbb{P}^1 - \{0, \infty\}$. The family $T \to \mathbb{P}^1$ is obtained by extending our family over this finite set.

We begin with some notation. Let $b_1, \dots, b_n$ be the minimal lattice points on the rays in Σ and let $b_{n+1}, \dots, b_m$ be the minimal lattice points on the additional rays in Σ'. Since $X(\Sigma')$ is smooth, the lattice points $b_1, \dots, b_m$ generate the group N. Hence, the ring $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is isomorphic to the quotient of the polynomial ring $S := \mathbb{Q}[y^{b_1}, \dots, y^{b_m}]$ by the binomial ideal I_2 which encodes the multiplication rules in (5.1). Fix a $\mathbb{Z}$-basis $\theta_1, \dots, \theta_d$ for $N^\star := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and let I_1 be the ideal in S generated by linear equations $\sum_{i=1}^m \theta_j(b_i)\, y^{b_i}$ for $1 \leq j \leq d$. The assumption that Σ' is a regular subdivision of Σ means that there is a Σ'-linear support function $h \colon N \to \mathbb{Z}$ such that $h(b_i) = 0$ for $1 \leq i \leq n$, $h(b_i) > 0$ for $n+1 \leq i \leq m$ and $h(c_1 + c_2) \geq h(c_1) + h(c_2)$ for all lattice points c_1, c_2 lying in the same cone of Σ. This inequality is strict unless c_1 and c_2 lie in the same cone of Σ'.

To describe our family over $\mathbb{P}^1 - \{\infty\}$, let $\breve{I}_1$ be the ideal in $S[t_1]$ generated by $\sum_{i=1}^m \theta_j(b_i)\, y^{b_i} t_1^{h(b_i)}$ for $1 \leq j \leq d$. Since $h(b_i) = 0$ if and only if $1 \leq i \leq n$, Theorem 1.1 implies that

$$\frac{S[t_1]}{\breve{I}_1 + I_2 + \langle t_1 \rangle} \cong \frac{S}{\langle \sum_{i=1}^n \theta_j(b_i) y^{b_i} : 1 \leq j \leq d \rangle + I_2} \cong A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)\,.$$

Moreover, it follows from the decomposition of I_2 in Proposition 4.8 in [13] that the sequence $\sum_{i=1}^n \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ forms a homogeneous system of parameters on S/I_2. Lemma 4.6 in [24] shows that S/I_2 is a Cohen-Macaulay ring, so we deduce that $\sum_{i=1}^n \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ is a regular sequence. Being a regular sequence is an open condition on the set of d-tuples of degree one elements in a finitely generated $\mathbb{Q}$-algebra. Therefore, the Hilbert function of the family $S[t_1]/(\breve{I}_1 + I_2)$ over $\mathbb{Q}[t_1]$ is constant outside a finite set in $\mathbb{Q}^*$.

For the family over $\mathbb{P}^1 - \{0\}$, let $\breve{I}_2$ be the binomial ideal in $S[t_2]$ which encodes the product rule

$$y^{c_1} \cdot y^{c_2} = \begin{cases} y^{c_1 + c_2}\, t_2^{h(c_1+c_2) - h(c_1) - h(c_2)} & \text{if there exists } \sigma \in \Sigma \text{ such that } c_1, c_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Because $h(c_1 + c_2) \geq h(c_1) + h(c_2)$ and equality holds if and only if c_1 and c_2 lie in the same cone of Σ', this product becomes

$$y^{c_1} \cdot y^{c_2} = \begin{cases} y^{c_1 + c_2} & \text{if there exists } \sigma' \in \Sigma' \text{ such that } c_1, c_2 \in \sigma', \\ 0 & \text{otherwise} \end{cases}$$

over $t_2 = 0$. Hence, $S[t_2]/(\breve{I}_2 + \langle t_2 \rangle) \cong S/I_{\Sigma'}$, where $I_{\Sigma'}$ is the Stanley-Reisner ideal associated to Σ', and Lemma 5.1 shows that $S[t_2]/(I_1 + \breve{I}_2 + \langle t_2 \rangle) \cong A^*\big(\mathcal{X}(\mathbf{\Sigma})\big)$. As $\mathbb{Q}$-vector spaces, both S/I_2 and $S/I_{\Sigma'}$ have a basis consisting of the monomials in S corresponding to lattice points in N. Proposition 4.8 in [13] implies that the sequence $\sum_{i=1}^n \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ forms a homogeneous system of parameters on S/I_2, and Theorem 5.1.16 in [5] shows that this sequence is also a homogeneous system of parameters on $S/I_{\Sigma'}$. Thus, $\sum_{i=1}^{m} \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ is a regular sequence on both S/I_2 and $S/I_{\Sigma'}$ and the Hilbert functions of $S/(I_1 + I_2)$ and $S/(I_1 + I_{\Sigma'})$ are equal.

We combine the two one-parameter families by using the automorphisms φ_k for $k = 1, 2$ of $S[t_k, t_k^{-1}]$ defined by $\varphi_k(y^{b_i}) = y^{b_i} t_k^{h(b_i)}$. Since φ_k takes $I_k \cdot S[t_k, t_k^{-1}]$ to $\breve{I}_k \cdot S[t_k, t_k^{-1}]$, these automorphisms induce the following isomorphisms:

$$\frac{S[t_1, t_1^{-1}]}{\breve{I}_1 + I_2} \cong \frac{S}{I_1 + I_2} \cong \frac{S[t_2, t_2^{-1}]}{I_1 + \breve{I}_2} .$$

Since a family of affine cones is a flat family if and only if the Hilbert function is constant (see Proposition III-56 in [12]), we conclude that our family is flat on a Zariski open subset of $\mathbb{P}^1$ which contains both 0 and ∞. □

Remark 7.3. In analogy with Theorem 15.17 in [11], the flat family constructed in the proof of Theorem 7.1 can be interpreted as a pair of Gröbner deformations with respect to the appropriate weight orders connecting the ideal $\breve{I}_1 + \breve{I}_2$ with its initial ideals $\langle \sum_{i=1}^{n} \theta_j(b_i) y^{b_i} : 1 \leq j \leq d \rangle + \breve{I}_2$ and $\breve{I}_1 + I_{\Sigma'}$.

Remark 7.4. According to the Cohomological Crepant Resolution Conjecture of Ruan [23], the orbifold cohomology of $\mathcal{X}(\mathbf{\Sigma})$ should be isomorphic to the (small) quantum cohomology of $X(\Sigma')$ for a very special choice of parameters of quantum cohomology. It is plausible that all fibers of the flat family of Theorem 7.1 appear as the quantum cohomology of $X(\Sigma')$ for a suitable choice of parameters. However, the Cohomological Crepant Resolution Conjecture itself is by no means easy to settle even in the toric case, since it is generally rather difficult to calculate the quantum cohomology of toric varieties.

We end with an example in which $A^*\big(X(\Sigma')\big)$ is not isomorphic to $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$.

Example 7.5. Let $N = \mathbb{Z}^2$ and let $\Sigma \subseteq \mathbb{R}^2$ be the complete fan in which the rays are generated by the lattice points $b_1 := (1, 0)$, $b_2 := (0, -1)$ and $b_3 := (-1, 2)$. Hence, the toric variety $X(\Sigma)$ is the weighted projective space $\mathbb{P}(1, 2, 1)$ and the associated toric Deligne-Mumford stack is the quotient $[(\mathbb{C}^3 - \{0\})/\mathbb{C}^*]$ where the action is given by $(z_1, z_2, z_3) \cdot \lambda = (\lambda z_1, \lambda^2 z_2, \lambda z_3)$. If we simply write x_i for the element $y^{b_i} \in \mathbb{Q}[N]$, then Theorem 1.1 implies that

$$A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{\big\langle x_1 x_3 - x_4^2, x_2 x_4, x_1 - x_3, -x_2 + 2x_3 \big\rangle} \cong \frac{\mathbb{Q}[x_3, x_4]}{\big\langle x_3^2 - x_4^2, x_3 x_4 \big\rangle} .$$

Let Σ' be the fan obtained from Σ by inserting the ray generated by $b_4 := (0, 1)$. It follows that $X(\Sigma')$ is the Hirzeburch surface $\mathbb{F}_2$, $X(\Sigma') \to X(\Sigma)$ is a crepant resolution (it blows down the (-2)-curve in $\mathbb{F}_2$), and Lemma 5.1 gives

$$\begin{aligned} A^*\big(X(\Sigma')\big) &\cong \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{\big\langle x_1 x_3, x_2 x_4, x_1 - x_3, -x_2 + 2x_3 + x_4 \big\rangle} \\ &\cong \frac{\mathbb{Q}[x_3, x_4]}{\big\langle x_3^2, 2x_3 x_4 + x_4^2 \big\rangle} = \frac{\mathbb{Q}[x_3, x_4]}{\big\langle x_3^2, (x_3 + x_4)^2 \big\rangle} . \end{aligned}$$

Since there is a degree one element $x \in A^*\big(X(\Sigma')\big)$ such that $x^2 = 0$ and there is no such element in $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$, we conclude that $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \ncong A^*\big(X(\Sigma')\big)$.

Acknowledgments

We thank Dan Abramovich, Bob Friedman, Bill Fulton, Tom Graber, Paul Horja, Andrew Kresch, Martin Olsson, Michael Thaddeus, Hsian-Hua Tseng, Howard Thompson and Ravi Vakil for useful discussions. We also thank the referees for many helpful suggestions.

References

[1] D. Abramovich, A. Corti, and A. Vistoli, *Twisted bundles and admissible covers*, Comm. Algebra **31** (2003), 3547–3618. MR2007376

[2] D. Abramovich, T. Graber, and A. Vistoli, "Algebraic orbifold quantum products" in *Orbifolds in mathematics and physics*, pp. 1–24, Contemp. Math. 310, Amer. Math. Soc., Providence, RI, 2002. MR1950940 (2004c:14104)

[3] V. V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math.Soc. **1** (1999), 5–33. MR1677693 (2001j:14018)

[4] L. A. Borisov, *String cohomology of a toroidal singularity*, J. Algebraic Geom. **9** (2000), 289–300. MR1735773 (2001b:14079)

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Math. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)

[6] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), 17–50. MR1299003 (95i:14046)

[7] W. Chen and Y. Ruan, *A New Cohomology Theory for Orbifold*, `arXiv:math.AG/0004129`.

[8] W. Chen and Y. Ruan, "Orbifold Gromov-Witten theory" in *Orbifolds in mathematics and physics*, pp. 25–85, Contemp. Math. 310, Amer. Math. Soc., Providence, RI, 2002. MR1950941 (2004k:53145)

[9] P. Deligne and M. Rapoport, "Les schémas de modules de courbes elliptiques" in *Modular functions of one variable, II*, pp. 143–316, Lecture Notes in Math. 349, Springer, Berlin, 1973. MR0337993 (49:2762)

[10] D. Edidin, "Notes on the construction of the moduli space of curves" in *Recent progress in intersection theory*, pp. 85–113, Trends Math., Birkhäuser Boston, Boston, MA, 2000. MR1849292 (2002f:14039)

[11] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Math. 150, Springer-Verlag, New York, 1995. MR1322960 (97a:13001)

[12] D. Eisenbud and J. Harris, *The geometry of schemes*, Graduate Texts in Math. 197, Springer-Verlag, New York, 2000. MR1730819 (2001d:14002)

[13] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), 1–45. MR1394747 (97d:13031)

[14] B. Fantechi and L. Göttsche, *Orbifold cohomology for global quotients*, Duke Math. J. **117** (2003), 197–227. MR1971293 (2004h:14062)

[15] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies 131, Princeton University Press, Princeton, NJ, 1993. MR1234037 (94g:14028)

[16] Y. Jiang, *The Chen-Ruan cohomology of weighted projective spaces*, `arXiv:math.AG/0304140`.

[17] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 32, Springer-Verlag, Berlin, 1996. MR1440180 (98c:14001)

[18] L. Lafforgue, *Pavages des simplexes, schémas de graphes recollés et compactification des* PGL_r^{n+1}/PGL_r, Invent. Math. **136** (1999), 233–271. MR1681089 (2000i:14071)

[19] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 39, Springer-Verlag, Berlin, 2000. MR1771927 (2001f:14006)

[20] I. Moerdijk, "Orbifolds as groupoids: an introduction" in *Orbifolds in mathematics and physics*, pp. 205–222, Contemp. Math. 310, Amer. Math. Soc., Providence, RI, 2002. MR1950948 (2004c:22003)

[21] T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer-Verlag, Berlin, 1988. MR0922894 (88m:14038)

[22] M. Poddar, "Orbifold cohomology group of toric varieties" in *Orbifolds in mathematics and physics*, pp. 223–231, Contemp. Math. 310, Amer. Math. Soc., Providence, RI, 2002. MR1950949 (2003j:14068)

[23] Y. Ruan,*Cohomology Ring of Crepant Resolutions of Orbifolds*, `arXiv:math.AG/0108195`.

[24] R. Stanley, "Generalized H-vectors, intersection cohomology of toric varieties, and related results" in *Commutative algebra and combinatorics*, pp. 187–213, Adv. Stud. Pure Math. 11, North-Holland, Amsterdam, 1987. MR0951205 (89f:52016)

[25] B. Uribe, *Orbifold cohomology of the symmetric product*, Comm. Anal. Geom. (to appear), arXiv:math.AT/0109125.

[26] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), 613–670. MR1005008 (90k:14004)

[27] C. A. Weibel, *An introduction to homological algebra*. Cambridge Studies in Advanced Math. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)

[28] T. Yasuda, *Twisted jets, motivic measure and orbifold cohomology*, Compos. Math. **140** (2004), 396–422. MR2027195

[29] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Math. 152, Springer-Verlag, New York, 1995. MR1311028 (96a:52011)

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: borisov@math.wisc.edu

Department of Mathematics, Columbia University, New York, New York 10027
Current address: Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, Ohio 43210
E-mail address: lchen@math.ohio-state.edu

Department of Mathematics, Barnard College, Columbia University, New York, New York 10027
Current address: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6 Canada
E-mail address: ggsmith@mast.queensu.ca