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WELL-POSEDNESS OF THE WATER-WAVES EQUATIONS

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1. Introduction

1.1. **Presentation of the problem.** The water-waves problem for an ideal liquid consists of describing the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity. In this paper, we restrict our attention to the case when the surface is a graph parameterized by a function $\zeta(t, X)$, where t denotes the time variable and $X = (X_1, \dots, X_d) \in \mathbb{R}^d$ the horizontal spatial variables. The method developed here works equally well for any integer $d \geq 1$, but the only physically relevant cases are of course $d = 1$ and $d = 2$. The layer of fluid is also delimited from below by a not necessarily flat bottom parameterized by a time-independent function $b(X)$. We denote by Ω_t the fluid domain at time t. The incompressibility of the fluids is expressed by

$$\operatorname{div} V = 0 \quad \text{in} \quad \Omega_t, \qquad t \geq 0, \tag{1.1}$$

where $V = (V_1, \dots, V_d, V_{d+1})$ denotes the velocity field ($V_1, \dots, V_d$ being the horizontal, and V_{d+1} the vertical components of the velocity). Irrotationality means that

$$\operatorname{curl} V = 0 \quad \text{in} \quad \Omega_t, \qquad t \geq 0. \tag{1.2}$$

The boundary conditions on the velocity at the surface and at the bottom are given by the usual assumption that they are both bounding surfaces, i.e. surfaces across which no fluid particles are transported. At the bottom, this is given by

$$V_n|_{\{y=b(X)\}} := \mathbf{n}_- \cdot V|_{\{y=b(X)\}} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{1.3}$$

where $\mathbf{n}_- := \frac{1}{\sqrt{1+|\nabla_X b|^2}}(\nabla_X b, -1)^T$ denotes the outward normal vector to the lower boundary of Ω_t. At the free surface, the boundary condition is kinematic and is given by

$$\partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2} V_n|_{\{y=\zeta(X)\}} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{1.4}$$

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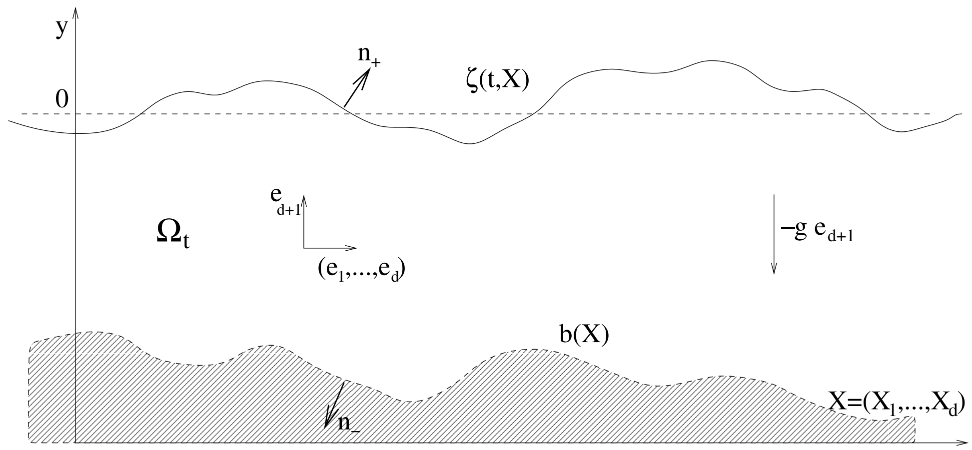

where $V_n|_{\{y=\zeta(X)\}} := \mathbf{n}_+ \cdot V|_{\{y=\zeta(X)\}}$, with $\mathbf{n}_+ := \frac{1}{\sqrt{1+|\nabla_X\zeta|^2}}(-\nabla_X\zeta, 1)^T$ denoting the outward normal vector to the free surface.

Neglecting the effects of surface tension yields that the pressure P is constant at the interface. Up to a renormalization, we can assume that

$$P|_{\{y=\zeta(X)\}} = 0 \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d. \tag{1.5}$$

Finally, the set of equations is closed with Euler's equation within the fluid,

$$\partial_t V + V \cdot \nabla_{X,y} V = -ge_{d+1} - \nabla_{X,y} P \quad \text{in} \quad \Omega_t, \quad t \geq 0, \tag{1.6}$$

where $-ge_{d+1}$ is the acceleration of gravity.

Early works on the well-posedness of Eqs. (1.1)-(1.6) within a Sobolev class go back to Nalimov [27], Yosihara [38] and Craig [10], as far as 1D-surface waves are concerned. All these authors work in a Lagrangian framework, which allows one to consider surface waves which are not graphs, and rely heavily on the fact that the fluid domain is two dimensional. In this case, complex coordinates are canonically associated to the $\mathbb{R}^2$-coordinates, and the incompressibility and irrotationality conditions (1.1) and (1.2) can be seen as the Cauchy-Riemann equations for the complex mapping $V_1 - iV_2$. There is therefore a singular integral operator on the top surface recovering boundary values of V_2 from boundary values of V_1. The water-waves equations (1.1)-(1.6) can then be reduced to a set of two nonlinear evolution equations, which can be "quasi-linearized" using a subtle cancellation property noticed by Nalimov. It seems that this cancellation property was the main reason why the Lagrangian framework was used. A major restriction of these works is that they only address the case of small perturbations of still water. The reasons for this restriction are quite technical, but the most fundamental is that this smallness assumption ensures that a generalized Taylor criterion is satisfied, thus preventing formation of Taylor instabilities (see [33, 4] and the introduction of [36]). Physically speaking, this criterion assumes that the surface is not accelerating into the fluid region more rapidly than the normal acceleration of gravity. From a mathematical viewpoint, this condition is crucial because the quasilinear system thus obtained is not strictly hyperbolic (zero is a multiple eigenvalue with a Jordan block) and requires a Lévy condition on the subprincipal symbol to be well-posed; one can see Taylor's criterion precisely as such a Lévy condition (see Section

4.1 below). In [3], Beale et al. proved that the linearization of the water-waves equations around a presumed solution is well-posed, provided this exact solution satisfies the generalized Taylor's sign condition (which is a weaker assumption than the smallness conditions of [27, 38, 10]). Wu's major breakthrough was to prove in [36] that Taylor's criterion always holds for solutions of the water-waves equations, as soon as the surface is nonself-intersect. Her energy estimates are also better than those of [3] and allow her to solve the full (nonlinear) water-waves equations, locally in time, and without restriction (other than smoothness) on the initial data, but in the case of a layer of fluid of infinite depth. The only existing theorems dealing with the case of finite depth require smallness conditions on the initial data when the bottom is flat [10], and an additional smallness condition on the variations of the bottom parameterization b when the bottom is uneven [38].

Very few papers deal with the well-posedness of the water-waves equations in Sobolev spaces in the three-dimensional setting (i.e. for a 2D surface). In [22], the generalization of the results of [3] to the three-dimensional setting is proved. More precisely, the authors show, in the case of a fluid layer of infinite depth, that the linearization of the water-waves equations around a presumed solution is well-posed, provided this exact solution satisfies the generalized Taylor's sign condition. As in [3], the energy estimates provided are not good enough to allow the resolution of the nonlinear water-waves equations by an iterative scheme. In [37], S. Wu (still in the case of a fluid layer of infinite depth) solved the nonlinear equations. Her proof relies heavily on Clifford analysis in order to extend to the 3D case (some of) the results provided by harmonic analysis in 2D. In the case of finite depth, no results exist.

1.2. **Presentation of the results.** In this paper, we deliberately chose to work in the Eulerian (rather than Lagrangian) setting, since it is the easiest to handle, especially when asymptotic properties of the solutions are concerned. Inspired by [29, 13] we use an alternate formulation of the water-waves equation (1.1)-(1.6). From the incompressibility and irrotationality assumptions (1.1) and (1.2), there exists a potential flow ϕ such that $V = \nabla_{X,y}\phi$ and

$$\Delta_{X,y}\phi = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0; \tag{1.7}$$

the boundary conditions (1.3) and (1.4) can also be expressed in terms of ϕ:

$$\partial_{\mathbf{n}_-}\phi|_{\{y=b(X)\}} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{1.8}$$

and

$$\partial_t\zeta - \sqrt{1+|\nabla_X\zeta|^2}\partial_{\mathbf{n}_+}\phi|_{\{y=\zeta(t,X)\}} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{1.9}$$

where we used the notation $\partial_{\mathbf{n}_-} := \mathbf{n}_- \cdot \nabla_{X,y}$ and $\partial_{\mathbf{n}_+} = \mathbf{n}_+ \cdot \nabla_{X,y}$. Finally, Euler's equation (1.6) can be put into Bernouilli's form

$$\partial_t\phi + \frac{1}{2}|\nabla_{X,y}\phi|^2 + gy = -P \quad \text{in} \quad \Omega_t, \quad t \geq 0. \tag{1.10}$$

As in [13], we reduce the system (1.7)-(1.10) to a system where all the functions are evaluated at the free surface only. For this purpose, we introduce the trace of the velocity potential ϕ at the surface

$$\psi(t,X) := \phi(t,X,\zeta(t,X)),$$

and the (rescaled) Dirichlet-Neumann operator $G(\zeta,b)$ (or simply $G(\zeta)$ when no confusion can be made on the dependence on the bottom parameterization b), which is a linear operator defined as

$$G(\zeta)\psi := \sqrt{1+|\nabla_X\zeta|^2}\partial_{\mathbf{n}_+}\phi_{|_{\{y=\zeta(t,X)\}}}.$$

Taking the trace of (1.10) on the free surface and using the chain rule shows that (1.7)-(1.10) are equivalent to the system

$$(1.11)\qquad \begin{cases} \partial_t\zeta - G(\zeta)\psi = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{1}{2(1+|\nabla_X\zeta|^2)}\left(G(\zeta)\psi + \nabla_X\zeta\cdot\nabla_X\psi\right)^2 = 0, \end{cases}$$

which is an evolution equation for the elevation of the free surface $\zeta(t,X)$ and the trace of the velocity potential on the free surface $\psi(t,X)$. Our results in this paper are given for this system.

The first part of this work consists in developing simple tools in order to make the proof of the well-posedness of the water-waves equations as simple as possible. It is quite obvious from the equations (1.11) that the Dirichlet-Neumann operator will play a central role in the proof; we give here a self-contained and quite elementary proof of the properties of the Dirichlet-Neumann operator that we shall need. A major difficulty lies in the dependence on ζ of the operator $G(\zeta)\cdot$. It is known that such operators depend analytically on the parameterization of the surface. Coifman and Meyer [9] considered small Lipschitz perturbations of a line or plane, and Craig *et al.* [12, 13] C^1 perturbations of hyperplanes in any dimension. Seen as an operator acting on Sobolev spaces, $G(\zeta)\cdot$ is of order one. In [13], an estimate of its operator norm is given in the form:

$$(1.12)\qquad |G(\zeta)\psi|_{H^k} \leq C\left(k,|\zeta|_{C^1}\right)\left(|\zeta|_{C^{k+1}}|\psi|_{H^1} + |\psi|_{H^{k+1}}\right),$$

for all integer $k\geq 0$ (estimates in L^q-based Sobolev spaces are also provided). In order to obtain this estimate, the authors give an expression of $G(\zeta)\cdot$ as a singular integral operator (inspired by the early works of Garabedian and Schiffer [17] and Coifman and Meyer [9] on Cauchy integrals) and use a multiple commutator estimate of Christ and Journé [6]. Estimate (1.12) has the interest of being "tame" (in the sense of Hamilton [21]; i.e., the control in the norms depending on the regularity index k is linear), but is only proved for flat bottoms and requires too much smoothness on ζ: a control of $|\zeta|_{C^{k+1}}$ is needed in (1.12), and hence of $|\zeta|_{H^s}$, with $s > d/2+k+1$, if one works in a Sobolev framework. A rapid look at equations (1.11) shows that one would like to allow only a control of ζ in $H^{k+1}(\mathbb{R}^d)$ (i.e., ζ and ψ should have the same regularity). Using an expression of $G(\zeta)\cdot$ involving tools of Clifford Algebras [18] and deep results of Coifman, McIntosh and Meyer [8] and Coifman, David and Meyer [7], S. Wu obtained in [37] another estimate with a sharp dependence on the smoothness of ζ:

$$(1.13)\qquad |G(\zeta)\psi|_{H^s} \leq C\left(s,|\zeta|_{H^{s+1}}\right)|\psi|_{H^{s+1}},$$

for all real numbers s large enough. If estimate (1.13) is obviously better than (1.12), it has two drawbacks. First, it is not tame, and hence not compatible for later use in a Nash-Moser convergence scheme. Second, its proof requires very deep results, which make its generalization to the present case of finite and uneven bottom highly nontrivial. In this paper, we prove in Theorem 3.6 the following estimate:

$$|G(\zeta)\psi|_{H^{k+1/2}} \leq C\left(k, |\zeta|_{H^{s_0}}\right)\left(|\zeta|_{H^{k+3/2}}|\nabla_X\psi|_{H^{s_0-1}} + |\nabla_X\psi|_{H^{k+1/2}}\right), \tag{1.14}$$

for all $k \in \mathbb{N}$, and where s_0 is a fixed positive real number. This estimate has the sharp dependence on ζ of (1.13) and is tame as (1.12). Moreover, it is sharper than the above estimates in the sense that only the gradient of ψ is involved; this will prove very useful here. Estimate (1.14) also holds for uneven bottoms and its proof uses only elementary tools of PDE: since the fluid layer is diffeomorphic to the flat strip $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$, we first transform the Laplace equation (1.7) with Dirichlet condition $\phi = \psi$ at the surface and homogeneous Neumann condition $\partial_{\mathbf{n}_-}\phi = 0$ at the bottom into an elliptic boundary value problem (BVP) with variable coefficients defined in the flat strip $\mathcal{S}$. The Dirichlet-Neumann operator $G(\zeta)\cdot$ can be expressed in terms of the solution to this new BVP (see Prop. 3.4). We give sharp tame estimates for a wide class of such elliptic problems in Theorem 2.9. Choosing the most simple diffeomorphism between the fluid domain and $\mathcal{S}$ as in [12, 2] and applying Theorem 2.9 to the elliptic problem thus obtained, we can obtain, via Prop. 3.4, a tame estimate on $G(\zeta)\cdot$. However, this estimate is not sharp since instead of $|\zeta|_{H^{k+3/2}}$ as in (1.14), one would need a control of $|\zeta|_{H^{k+2}}$. We must therefore gain half a derivative more to obtain (1.14). The trick consists in proving (see Prop. 2.13) that there exists a "regularizing" diffeomorphism between the fluid domain and the flat strip $\mathcal{S}$.

We also need further information on the Dirichlet-Neumann operator. In Theorem 3.10, we give the principal symbol of $G(\zeta)\cdot$: for all $f \in H^{1/2}(\mathbb{R}^d)$,

$$\left|\big(G(\zeta) - g_\zeta(X, D)\big)f\right|_{H^{j/2}} \leq \mathrm{Cst}\ |f|_{H^{j/2}}, \qquad j = -1, 0, 1,$$

where $g_\zeta(X,\xi) := \sqrt{|\xi|^2 + |\nabla_X\zeta|^2|\xi|^2 - (\nabla_X\zeta\cdot\xi)^2}$, and where the constant involves the L^∞-norm of a finite number of derivatives of ζ. Note in particular that for 1D surfaces, $g_\zeta(X, D) = |D|$, while for 2D surfaces it is a pseudo-differential operator (and not a simple Fourier multiplier). We then give tame estimates of the commutator of $G(\zeta)\cdot$ with spatial (in Prop. 3.15) and time (in Prop. 3.19) derivatives. Finally, we give in Theorem 3.20 an explicit expression of the shape derivative of $G(\zeta)\cdot$, i.e. the derivative of the mapping $\zeta \mapsto G(\zeta)\cdot$, and tame estimates of this and higher derivatives are provided in Prop. 3.25.

Note that all the above results are proved for a general constant coefficient elliptic operator $-\nabla_{X,y}\cdot P\nabla_{X,y}\phi = 0$ instead of $-\Delta_{X,y}$ in (1.7). This is useful if one wants to work with nondimensionalized equations. This first set of results consists therefore in preliminary tools for the study of the water-waves problem; we would like to stress the fact that they are sharp and only use the classical tools of PDE.

We then turn to investigate the water-waves equations (1.11). The first step consists of course in solving the linearization of (1.11) around some reference state $\underline{U} = (\underline{\zeta}, \underline{\psi})$, and in giving energy estimates on the solution. Using the explicit expression of the shape derivative of the Dirichlet-Neumann operator given in Theorem 3.20, we can give an explicit expression of the linearized operator $\underline{\mathcal{L}}$. Having the previous works on the water-waves equations in mind, it is not surprising to find that $\underline{\mathcal{L}}$ is hyperbolic, but that its principal symbol has an eigenvalue of multiplicity two (i.e., it is not strictly hyperbolic). In the works quoted in the previous section, this double eigenvalue is zero. Due to the fact that we work here in Eulerian, as opposed to Lagrangian, variables, this double eigenvalue is not zero anymore, but $i\underline{\mathbf{v}}\cdot\xi$, ξ being the dual variable of X, and $\underline{\mathbf{v}}$ being the horizontal component of the velocity at the surface of the reference state $\underline{U}$. It is natural to seek a linear change of unknowns which transforms the principal part of $\underline{\mathcal{L}}$ into its canonical expression consisting of an upper triangular 2×2 matrix with double eigenvalue $i\underline{\mathbf{v}}\cdot\xi$ and a Jordan block. Prop. 4.2 gives a striking result: this *a priori* pseudo-differential change of unknown is not even differential, and the commutator terms involving the Dirichlet-Neumann operator that should appear in the lower-order terms all vanish! This simplifies greatly the sequel.

Having transformed the linearized operator $\underline{\mathcal{L}}$ into an operator $\underline{\mathcal{M}}$ whose principal part exhibits the Jordan block structure inherent to the water-waves equations, we turn to study this operator $\underline{\mathcal{M}}$. The Lévy condition needed on the subprincipal symbol of $\underline{\mathcal{M}}$ in order for the associated Cauchy problem to be well-posed is quite natural, due to the peculiar structure of $\underline{M}$: a certain function $\underline{\mathfrak{a}}$ depending only on the reference state $\underline{U}$ must satisfy $\underline{\mathfrak{a}}\geq c_0>0$ for some positive constant c_0 (this is almost a necessary condition, since the linearized water-waves equations would be ill-posed if one had $\underline{\mathfrak{a}}<0$). It appears in Prop. 4.4 that this sign condition is exactly the generalized Taylor's sign condition of [3, 22, 36, 37]. Assuming for the moment that this condition holds, we use the tools developed in the first sections to show, in Prop. 4.5, that the Cauchy problem associated to $\underline{\mathcal{M}}$ is well-posed in Sobolev spaces, and to give energy estimates on the solution. There is a classical loss of information of half a derivative on this solution due to the Jordan block structure, but also a more dramatic loss of information with respect to the reference state $\underline{U}$, which makes a Picard iterative scheme inefficient for solving the nonlinear equation. Fortunately, the energy estimates given in Prop. 4.5 are tame, and Nash-Moser theory will provide a good iterative scheme. Inverting the change of unknown of Prop. 4.2, tame estimates are deduced in Prop. 4.14 for the solution of the Cauchy problem associated to the linearized operator $\underline{\mathcal{L}}$. The last step of the proof consists in solving the nonlinear equations (1.11) via a Nash-Moser iterative scheme. This requires proving that Taylor's sign condition $\underline{\mathfrak{a}}\geq c_0>0$ holds at each step of the scheme (and of course that the surface elevation $\zeta-b$ remains positive!). It is quite easy to see that it is sufficient for this condition to be satisfied that the first iterate satisfies it. Wu proved that this is always the case in infinite depth. We prove in Prop. 4.15 that this result remains true in the case of flat bottoms. For uneven bottoms, however, we must assume that the generalized Taylor's sign condition holds for the initial data. This can be ensured by smallness conditions on the initial data, but we also give a sufficient condition stating that Taylor's sign condition can be satisfied for initial data of arbitrary size provided that the bottom is "slowly variable" in the sense that

$$\mathrm{II}_b(V_{0\tau},V_{0\tau})\leq\frac{g}{\sqrt{1+|\nabla_X b|^2}},$$

where b is the bottom parameterization, II_b the second fundamental form associated to the surface $\{(X,y)\in\mathbb{R}^{d+1}, y=b(X)\}$, and $V_{0\tau}$ the tangential component of the initial velocity field V_0 evaluated at the bottom.

Our final result is then given in Theorem 5.3. For flat bottoms (i.e. $b(X) = b = \mathrm{Cst} < 0$), it can be stated as:

Theorem 1.1. *Let $\zeta_0 \in H^{s+1}(\mathbb{R}^d)$ and ψ_0 be such that $\nabla_X \psi_0 \in H^s(\mathbb{R}^d)^d$, with $s > M$ (M depending only on d). Assume moreover that*

$$\zeta_0 - b \geq 2h_0 \quad \text{on} \quad \mathbb{R}^d \quad \text{for some} \quad h_0 > 0.$$

Then there exists $T > 0$ and a unique solution (ζ, ψ) to the water-waves equations (1.11) *with initial conditions (ζ_0, ψ_0) and such that $(\zeta, \psi - \psi_0) \in C^1\big([0,T], H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)\big)$.*

Organization of the paper. Section 2 is devoted to the study of the Laplace equation (1.7) in the fluid domain, or more precisely to the equation $-\nabla_{X,y} \cdot P \nabla_{X,y} \phi = 0$, where P is a constant coefficient, symmetric and coercive $(d+1) \times (d+1)$ matrix. In Section 2.1, we show that this equation can be reduced to an elliptic boundary problem with variable coefficients on a flat strip, and sharp tame elliptic estimates for such problems are given in Section 2.2. We then show in Section 2.3 that among the various diffeomorphisms between the fluid domain and the flat strip, there are some that are particularly interesting, which we call "regularizing diffeomorphisms" and which allow the gain of half a derivative with respect to the regularity of the surface parameterization.

Section 3 is entirely devoted to the properties of the Dirichlet-Neumann operator. Basic properties (including the sharp estimate (1.14) mentioned above) are gathered in Section 3.1. In Section 3.2, we are concerned with the derivation of the principal part of the Dirichlet-Neumann operator, and in Section 3.3 with its commutator properties with space or time derivatives. Finally its shape derivatives are studied in Section 3.4.

The linearized water-waves equations are the object of Section 4. We first show in Section 4.1 that the linearized equations can be made trigonal and prove in Section 4.2 that the Cauchy problem associated to the trigonal operator is well-posed in Sobolev spaces, assuming that a Lévy condition on the subprincipal symbol holds. We also provide in this section tame estimates on the solution. The link with the solution of the original linearized water-waves equations is made in Section 4.3, and the Lévy condition is discussed in Section 4.4.

The fully nonlinear water-waves equations are solved in Section 5. A simple Nash-Moser implicit function theorem is first recalled in Section 5.1 and then used in Section 5.2 to obtain our final well-posedness result.

Finally, a technical proof needed in Section 2.1 has been postponed to Appendix A.

1.3. **Notation.** Here is a set of notation we shall use throughout this paper:
- Cst always denotes a numerical constant which may change from one line to another. If the constant depends on some parameters $\lambda_1, \lambda_2, \dots$, we denote it by $C(\lambda_1, \lambda_2, \dots)$.
- For any $\alpha = (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}^{d+1}$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_{d+1}$.
- For all $i = 1, \dots, d$, we write $\partial_i = \partial_{X_i}$; similarly, we write $\partial_{d+1} = \partial_y$, and, for all $\alpha \in \mathbb{N}^{d+1}$, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_{d+1}^{\alpha_{d+1}}$.
- We denote by $C_b^k(\mathbb{R}^d)$ the set of functions continuous and bounded on $\mathbb{R}^d$ together with their derivatives of order less than or equal to k, endowed with its canonical norm $|\cdot|_{k,\infty} = \sum_{|\alpha| \leq k} |\partial^\alpha \cdot|_{L^\infty}$. We denote also $C_b^\infty = \bigcap_k C_b^k$.

- We denote by $(\cdot,\cdot)$ the usual scalar product on $L^2(\mathbb{R}^d)$.
- We denote by $\Lambda = \Lambda(D)$, or $\langle D\rangle$, the Fourier multiplier with symbol $\Lambda(\xi) = \langle \xi\rangle = (1+|\xi|^2)^{1/2}$.
- For all $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ the space of distributions f such that $|f|_{H^s} := \big(\int \Lambda(\xi)^s |\widehat{f}(\xi)|^2\big)^{1/2} < \infty$, where $\widehat{f}$ denotes the Fourier transform of f. We also denote $H^\infty = \bigcap_s H^s$.
- If $f \in C([0,T], H^s(\mathbb{R}^d))$, we write $|f|_{H^s_T} = \sup_{t\in[0,T]} |f(t)|_{H^s}$.
- If B is a Banach space and if $f, g \in B$, then we write $|f,g|_B = |f|_B + |g|_B$. If $F = (f_1,\dots,f_n)^T \in B^n$, then $|F|_B := |f_1|_B + \cdots + |f_n|_B$.
- For all $s \in \mathbb{R}$, $\lceil s\rceil$ denotes the first integer strictly larger than s (so that $\lceil 1\rceil = 2$).

2. Elliptic boundary value problems on a strip

Throughout this section, we work on a domain Ω defined as

$$\Omega = \{(X,y) \in \mathbb{R}^{d+1}, b(X) < y < a(X)\},$$

where a and b satisfy the following condition:

$$\exists h_0 > 0, \qquad \min\{-b, a-b\} \geq h_0 > 0 \text{ on } \mathbb{R}^d \tag{2.1}$$

(this assumption means that we exclude beaches or islands for the fluid domain, either perturbed or at rest).

We also consider a constant coefficients elliptic operator $\mathbf{P} = -\nabla_{X,y}\cdot P\nabla_{X,y}$, where P is a symmetric matrix satisfying the following condition:

$$\exists p > 0 \quad \text{such that} \quad P\Theta\cdot\Theta \geq p|\Theta|^2, \quad \forall \Theta \in \mathbb{R}^{d+1}. \tag{2.2}$$

Finally, we consider boundary value problems of the form

$$\begin{cases} \mathbf{P}u = h & \text{on } \Omega, \\ u|_{\{y=a(X)\}} = f, & \partial_n^P u|_{\{y=b(X)\}} = g, \end{cases} \tag{2.3}$$

where h is a function defined on Ω and f, g are functions defined on $\mathbb{R}^d$. Moreover, $\partial_n^P u|_{\{y=b(X)\}}$ denotes the conormal derivative associated to $\mathbf{P}$ of u at the boundary $\{y = b(X)\}$,

$$\partial_n^P u|_{\{y=b(X)\}} = -\mathbf{n}_-\cdot P\nabla_{X,y} u|_{\{y=b(X)\}}, \tag{2.4}$$

where $\mathbf{n}_-$ denotes the outwards normal derivative at the bottom.

Notation 2.1. For all open sets $U \subset \mathbb{R}^{d+1}$, we denote by $\|\cdot : U\|_p$, $\|\cdot : U\|_{k,\infty}$ and $\|\cdot : U\|_{k,2}$ the canonical norms of $L^p(U)$, $W^{k,\infty}(U)$ and $H^k(U)$ respectively. When no confusion is possible on the domain U, we write simply $\|\cdot\|_p$, $\|\cdot\|_{k,\infty}$ and $\|\cdot\|_{k,2}$.

2.1. Reduction to an elliptic equation on a flat strip.

Throughout this section, we denote by R any diffeomorphism between Ω and the flat strip $\mathcal{S} = \mathbb{R}^d \times (0,1)$, which we assume to be of the form

$$R: \begin{array}{ccc} \Omega & \to & \mathcal{S} \\ (X,y) & \mapsto & (X, r(X,y)), \end{array} \tag{2.5}$$

and we denote its inverse R^{-1} by S,

$$S: \begin{array}{ccc} \mathcal{S} & \to & \Omega \\ (\widetilde{X},\widetilde{y}) & \mapsto & (\widetilde{X}, s(\widetilde{X},\widetilde{y})). \end{array} \tag{2.6}$$

We always assume the following on s:

Assumption 2.2. One has $s \in W^{1,\infty}(\mathcal{S})$ with $s|_{\widetilde{y}=0} = a$ and $s|_{\widetilde{y}=-1} = b$. Moreover, there exists $c_0 > 0$ such that $\partial_{\widetilde{y}} s \geq c_0$ on $\overline{\mathcal{S}}$.

Finally, we need the following definition:

Definition 2.3. Let $k \in \mathbb{N}$. The mapping s, given by (2.6), is called k-regular if it satisfies Assumption 2.2 and can moreover be decomposed into $s = s_1 + s_2$ with $s_1 \in C_b^k(\overline{\mathcal{S}})$ and $s_2 \in H^k(\mathcal{S})$, and if $\partial_{\widetilde{y}} s_1 \geq c_0$ on $\overline{\mathcal{S}}$.

Remark 2.4. The most simple diffeomorphism R between Ω and $\mathcal{S}$ is given by

$$r(X,y) = \frac{y - a(X)}{a(X) - b(X)},$$

and hence $s(\widetilde{X},\widetilde{y}) = (a(\widetilde{X}) - b(\widetilde{X}))\widetilde{y} + a(\widetilde{X})$. If $a \in H^k \cap W^{1,\infty}(\mathbb{R}^d)$ and $b \in C_b^k(\mathbb{R}^d)$, it is clear that s is k-regular, with $s_1(\widetilde{X},\widetilde{y}) := -b(\widetilde{X})\widetilde{y}$, $s_2(\widetilde{X},\widetilde{y}) := (1+\widetilde{y})a(\widetilde{X})$, and $c_0 = h_0$.

To any distribution u defined on Ω one can associate, using the diffeomorphism R and its inverse S given by (2.5)-(2.6), a distribution $\widetilde{u}$ defined on $\mathcal{S}$ as

$$\widetilde{u} = u \circ S, \tag{2.7}$$

and vice-versa,

$$u = \widetilde{u} \circ R. \tag{2.8}$$

The following lemma shows that the constant coefficients elliptic equation $\mathbf{P}u = 0$ on Ω can equivalently be formulated as a variable coefficients elliptic equation $\widetilde{\mathbf{P}}\widetilde{u} = 0$ on $\mathcal{S}$.

Lemma 2.5. *Suppose that the mapping s, given by* (2.6) *satisfies Assumption* 2.2. *Let $\mathbf{P} = -\nabla_{X,y} \cdot P\nabla_{X,y}$ with P satisfying* (2.2). *Then the equation $\mathbf{P}u = h$ holds in $\mathcal{D}'(\Omega)$ if and only if the equation $\widetilde{\mathbf{P}}\widetilde{u} = (\partial_{\widetilde{y}} s)\widetilde{h}$ holds in $\mathcal{D}'(\mathcal{S})$, where $\widetilde{u}$ and $\widetilde{h}$ are deduced from u and h via formula* (2.7), *and $\widetilde{\mathbf{P}} := -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}$, with*

$$\widetilde{P} = \frac{1}{\partial_{\widetilde{y}} s} \begin{pmatrix} \partial_{\widetilde{y}} s Id_{d\times d} & 0 \\ -\nabla_{\widetilde{X}} s^T & 1 \end{pmatrix} P \begin{pmatrix} \partial_{\widetilde{y}} s Id_{d\times d} & -\nabla_{\widetilde{X}} s \\ 0 & 1 \end{pmatrix}.$$

Moreover, one has, for all $\Theta \in \mathbb{R}^{d+1}$,

$$\widetilde{P}\Theta \cdot \Theta \geq \widetilde{p}|\Theta|^2, \qquad \textit{with} \qquad \widetilde{p} = \mathrm{Cst}\, p \frac{c_0^2}{\|\partial_{\widetilde{y}} s\|_\infty (1 + \|\nabla_{\widetilde{X},\widetilde{y}} s\|_\infty^2)}.$$

Proof. By definition, $\mathbf{P}u = h$ in $\mathcal{D}'(\Omega)$ if and only if

$$\int_\Omega \mathbf{P}u\varphi = \int_\Omega h\varphi, \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.9}$$

By definition of $\mathbf{P}$, one also has

$$\begin{aligned} \int_\Omega \mathbf{P}u\varphi &= \int_\Omega P\nabla_{X,y} u \cdot \nabla_{X,y}\varphi \\ &= \int_\Omega P \begin{pmatrix} (\nabla_{\widetilde{X}}\widetilde{u}) \circ R + \nabla_X r (\partial_{\widetilde{y}}\widetilde{u}) \circ R \\ \partial_y r (\partial_{\widetilde{y}}\widetilde{u}) \circ R \end{pmatrix} \cdot \begin{pmatrix} (\nabla_{\widetilde{X}}\widetilde{\varphi}) \circ R + \nabla_X r (\partial_{\widetilde{y}}\widetilde{\varphi}) \circ R \\ \partial_y r (\partial_{\widetilde{y}}\widetilde{\varphi}) \circ R \end{pmatrix} \\ &= \int_{\mathcal{S}} |\partial_{\widetilde{y}} s| P \begin{pmatrix} \nabla_{\widetilde{X}} + (\nabla_X r) \circ S \partial_{\widetilde{y}} \\ (\partial_y r) \circ S \partial_{\widetilde{y}} \end{pmatrix} \widetilde{u} \cdot \begin{pmatrix} \nabla_{\widetilde{X}} + (\nabla_X r) \circ S \partial_{\widetilde{y}} \\ (\partial_y r) \circ S \partial_{\widetilde{y}} \end{pmatrix} \widetilde{\varphi}. \end{aligned}$$

Integrating by parts yields therefore that $\int_\Omega \mathbf{P}u\varphi$ is equal to

$$-\int_{\mathcal{S}} \widetilde{\varphi} \begin{pmatrix} \nabla_{\widetilde{X}} \cdot + \partial_{\widetilde{y}}((\nabla_X r)\circ S \cdot) \\ \partial_{\widetilde{y}}((\partial_y r)\circ S\cdot) \end{pmatrix} \cdot |\partial_{\widetilde{y}} s| P \begin{pmatrix} \nabla_{\widetilde{X}} + (\nabla_X r)\circ S\partial_{\widetilde{y}} \\ (\partial_y r)\circ S\partial_{\widetilde{y}} \end{pmatrix} \widetilde{u}$$

and thus to

$$-\int_{\mathcal{S}} \widetilde{\varphi} \nabla_{\widetilde{X},\widetilde{y}} \cdot \begin{pmatrix} Id & 0 \\ ((\nabla_X r)\circ S)^T & (\partial_y r)\circ S \end{pmatrix} |\partial_{\widetilde{y}} s| P \begin{pmatrix} Id & (\nabla_X r)\circ S \\ 0 & (\partial_y r)\circ S \end{pmatrix} \nabla_{\widetilde{X},\widetilde{y}} \widetilde{u}.$$

By definition of r and s, one has $r(\widetilde{X}, s(\widetilde{X},\widetilde{y})) = \widetilde{y}$ for all $(\widetilde{X},\widetilde{y}) \in \mathcal{S}$. Differentiating this identity with respect to $\widetilde{X}$ and $\widetilde{y}$ respectively yields

$$(\nabla_X r)\circ S + (\partial_y r)\circ S \nabla_{\widetilde{X}} s = 0, \qquad \partial_{\widetilde{y}} s (\partial_y r)\circ S = 1.$$

Using these expressions in the above expressions gives the equality

$$\int_\Omega \mathbf{P}u\varphi = \int_{\mathcal{S}} \widetilde{\mathbf{P}}\widetilde{u}\widetilde{\varphi}, \tag{2.10}$$

where $\widetilde{\mathbf{P}}$ is as given in the statement of the lemma. Since one clearly has

$$\int_\Omega h\varphi = \int_{\mathcal{S}} \partial_y s \widetilde{h}\widetilde{\varphi},$$

the first claim of the lemma follows from (2.9) and (2.10).

We now prove the coercivity of $\widetilde{\mathbf{P}}$. One has, for all $\Theta \in \mathbb{R}^{d+1}$,

$$\widetilde{P}\Theta\cdot\Theta = \frac{1}{\partial_{\widetilde{y}} s} PA\Theta\cdot A\Theta, \qquad \text{with} \qquad A := \begin{pmatrix} \partial_{\widetilde{y}} s Id_{d\times d} & -\nabla_{\widetilde{X}} s \\ 0 & 1 \end{pmatrix},$$

and owing to (2.2) we have therefore

$$\widetilde{P}\Theta\cdot\Theta \geq \frac{p}{\partial_{\widetilde{y}} s} |A\Theta|^2. \tag{2.11}$$

The matrix A is invertible, and its inverse is given by

$$A^{-1} = \frac{1}{\partial_{\widetilde{y}} s} \begin{pmatrix} Id_{d\times d} & \nabla_{\widetilde{X}} s \\ 0 & \partial_{\widetilde{y}} s \end{pmatrix},$$

so that $\Theta = A^{-1}A\Theta$ can be bounded as

$$|\Theta| \leq \text{Cst } \frac{1}{c_0}\big(1 + \|\nabla_{\widetilde{X},\widetilde{y}} s\|_\infty\big)|A\Theta|.$$

Together with (2.11), this estimate yields the result of the lemma. □

The next lemma shows how the boundary conditions are transformed by the diffeomorphism R.

Lemma 2.6. *Suppose that the mapping s, given by* (2.6), *satisfies Assumption* 2.2. *For all $u \in C^1(\overline{\Omega})$, one has*

$$u|_{\{y=a\}} = \widetilde{u}|_{\{\widetilde{y}=0\}} \qquad \textit{and} \qquad \partial_n^P u|_{\{y=b\}} = \frac{1}{\sqrt{1+|\nabla_X b|^2}} \partial_n^{\widetilde{P}} \widetilde{u}|_{\{\widetilde{y}=-1\}}.$$

Proof. The first assertion of the lemma is straightforward. We now prove the second. By definition,

$$\begin{aligned} \partial_n^{\widetilde{P}} \widetilde{u}|_{\widetilde{y}=-1} &= -(-e_{d+1})\cdot \widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}|_{\widetilde{y}=-1} \\ &= -(-e_{d+1})\cdot \widetilde{P} \begin{pmatrix} \nabla_X u|_{y=b} + \nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\partial_y u|_{y=b} \\ \partial_{\widetilde{y}} s|_{\widetilde{y}=-1}\partial_y u|_{y=b} \end{pmatrix}. \end{aligned}$$

Replacing $\widetilde{P}$ by its expression given in Lemma 2.5, one obtains easily that

$$\begin{aligned}\partial_n^{\widetilde{P}}\widetilde{u}|_{\widetilde{y}=-1} &= -\begin{pmatrix}\nabla_X s|_{\widetilde{y}=-1}\\ -1\end{pmatrix}\cdot P\nabla_{X,y}u|_{y=b}\\ &= \sqrt{1+|\nabla_X b|^2}\partial_n^P u|_{y=b},\end{aligned}$$

which ends the proof of the lemma. □

Lemmas 2.5 and 2.6 show that the study of the boundary problems (2.3) can be deduced from the study of elliptic boundary value problems on a flat strip:

Proposition 2.7. *Suppose that the mapping s, given by* (2.6)*, satisfies Assumption* 2.2*. Then u is a (variational, classical) solution of* (2.3) *if and only if $\widetilde{u}$ given by* (2.7) *is a (variational, classical) solution of*

$$\begin{cases}\widetilde{\mathbf{P}}\widetilde{u} = (\partial_{\widetilde{y}}s)\widetilde{h} & \text{on } \mathcal{S},\\ \widetilde{u}|_{\widetilde{y}=0} = f, & \partial_n^{\widetilde{P}}\widetilde{u}|_{\widetilde{y}=-1} = \sqrt{1+|\nabla_X b|^2}g,\end{cases} \tag{2.12}$$

where $\widetilde{\mathbf{P}}$ is as given in Lemma 2.5*.*

The next section is therefore devoted to the study of the well-posedness of such variable coefficients elliptic boundary value problems on a flat strip. Before this, let us state a lemma dealing with the smoothness of the coefficients of $\widetilde{P}$. Its proof is given in Appendix A.

Lemma 2.8. *Let $k\in\mathbb{N}$ and assume that the mapping s, given by* (2.6)*, is* $(1+k)$*-regular. Then one can write $\widetilde{P}=\widetilde{P}_1+\widetilde{P}_2$ with $\widetilde{P}_1\in C_b^k(\overline{\mathcal{S}})^{(d+1)^2}$, $\widetilde{P}_2\in H^k(\mathcal{S})^{(d+1)^2}$ and*

$$\begin{aligned}\|\widetilde{P}_1\|_{k,\infty} &\leq C\big(\frac{1}{c_0},\|s_1\|_{k+1,\infty}\big),\\ \|\widetilde{P}_2\|_{k,2} &\leq C\big(\frac{1}{c_0},\|s_1\|_{1+k,\infty},\|s_2\|_{1,\infty}\big)\|s_2\|_{1+k,2}.\end{aligned}$$

2.2. **Variable coefficients elliptic equations on a flat strip.** We have seen in the previous section that the theory of elliptic equations on a general strip of type (2.3) can be deduced from the study of elliptic equations on a flat strip, but with variable coefficients. In this section, we study the following generic problem:

$$\begin{cases}\mathbf{Q}u := -\nabla_{X,y}\cdot Q\nabla_{X,y}u = h & \text{on } \mathcal{S},\\ u|_{y=0}=f, & \partial_n^Q u|_{y=-1}=g,\end{cases} \tag{2.13}$$

where we recall that ∂_n^Q denotes the conormal derivative associated to $\mathbf{Q}$,

$$\partial_n^Q u|_{y=0} = -e_{d+1}\cdot Q\nabla_{X,y}u|_{y=0},\qquad \partial_n^Q u|_{y=-1} = -(-e_{d+1})\cdot Q\nabla_{X,y}u|_{y=-1}. \tag{2.14}$$

We also assume that Q satisfies the following coercivity assumption:

$$\exists q>0 \text{ such that } Q(X,y)\Theta\cdot\Theta \geq q|\Theta|^2,\qquad \forall\Theta\in\mathbb{R}^{d+1},\quad \forall(X,y)\in\mathcal{S}. \tag{2.15}$$

The main result of this section is the following theorem.

Theorem 2.9. *Let $k\in\mathbb{N}$, $m_0=\lceil\frac{d+1}{2}\rceil$. Let $f\in H^{k+3/2}(\mathbb{R}^d)$, $g\in H^{k+1/2}(\mathbb{R}^d)$ and $h\in H^k(\mathcal{S})$.*

i. *If $Q\in W^{1+k}(\mathcal{S})^{(d+1)^2}$ satisfies* (2.15)*, then there exists a unique solution $u\in H^{k+2}(\mathcal{S})$ to* (2.13)*. Moreover,*

$$\|u\|_{k+2,2}\leq C\big(\frac{1}{q},\|Q\|_{1+k,\infty}\big)\big(\|h\|_{k,2}+|f|_{H^{k+3/2}}+|g|_{H^{k+1/2}}\big).$$

ii. *If* $Q_1 \in C_b^{k+1}(\overline{\mathcal{S}})^{(d+1)^2}$ *and* $Q_2 \in H^{1+k} \cap W^{m_0,\infty}(\mathcal{S})^{(d+1)^2}$ *are such that* $Q := Q_1 + Q_2$ *satisfies* (2.15), *then there exists a unique solution* $u \in H^{k+2}(\mathcal{S})$ *to* (2.13). *Moreover, when* $k \geq m_0$,

$$\begin{aligned}\|u\|_{k+2,2} \quad &\leq \quad C_k \times (\|h\|_{k,2} + |f|_{H^{k+3/2}} + |g|_{H^{k+1/2}}) \\ &+ \quad C_k \times (\|h\|_{m_0-1,2} + |f|_{H^{m_0+1/2}} + |g|_{H^{m_0-1/2}}) \|Q_2\|_{1+k,2},\end{aligned}$$

where $C_k = C\big(\frac{1}{q}, \|Q_1\|_{1+k,\infty}, \|Q_2\|_{m_0+1,\infty}\big)$.

Remark 2.10. **i.** The proof below shows that the quantity $\|\nabla_{X,y} u\|_{k+1,2}$ can be estimated more precisely than $\|u\|_{k+2,2}$. Namely, one can replace the quantities $|f|_{H^{k+3/2}}$ and $|f|_{H^{m_0+1/2}}$ in both estimates of the theorem by $|\nabla_X f|_{H^{k+1/2}}$ and $|\nabla_X f|_{H^{m_0-1/2}}$ respectively. This remark is very useful when giving estimates on the Dirichlet-Neumann operator.
ii. The second estimate of the theorem remains of course valid when $k < m_0$, but in that case, the first estimate of the theorem is more precise.

Proof. Even though $\mathcal{S}$ is unbounded, the proof follows the same lines as the usual proofs of existence and regularity estimates of solutions to elliptic equations on regular bounded domains ([26, 19]), but special care must be paid to use the specific Sobolev regularity of the coefficients of $\mathbf{Q}$. We only prove the second point of the theorem since the first one can be obtained by skipping the fourth step of the proof below.
Step 1. Construction of a variational solution to (2.13). We first introduce $f^\sharp(y,\cdot) := \chi(y|D|)f$, where χ is a smooth compactly supported function such that $\chi(0) = 1$. Classically, one has $f^\sharp|_{y=0} = f$ and

$$\text{(2.16)} \quad \|\nabla_{X,y} f^\sharp\|_{1,2} \leq \text{Cst } |\nabla_X f|_{H^{1/2}}, \quad |\partial_n^Q f^\sharp_{|_{y=-1}}|_{H^{1/2}} \leq \text{Cst } \|Q\|_{1,\infty} |\nabla_X f|_{H^{1/2}}.$$

It follows that u is a variational solution of (2.13) if and only if $u^\sharp := u - f^\sharp$ is a variational solution to

$$\text{(2.17)} \qquad \left\{ \begin{array}{l} \mathbf{Q} u^\sharp = h - \mathbf{Q}(f^\sharp) := h^\sharp, \\ u^\sharp|_{y=0} = 0, \qquad \partial_n^Q u^\sharp|_{y=-1} = \widetilde{g}, \end{array}\right.$$

where $\widetilde{g} := g - \partial_n^Q f^\sharp|_{y=-1}$.

Define the space V as $V := \overline{\mathcal{D}(\mathbb{R}^d \times [-1,0))}$, where the closure is taken relative to the $H^1(\mathcal{S})$-norm. It is a classical consequence of Lax-Milgram's theorem that there exists a unique $u^\sharp \in V$ such that

$$\text{(2.18)} \qquad \int_{\mathcal{S}} Q \nabla_{X,y} u^\sharp \cdot \nabla_{X,y} v = \int_{\mathcal{S}} h^\sharp v - \int_{y=-1} \widetilde{g} v, \qquad \forall v \in V.$$

Step 2. Regularity of the variational solution. We show that $u^\sharp \in H^2(\mathcal{S})$ using the classical method of Nirenberg's tangential differential quotients. For all $v \in V$ and $i = 1, \ldots, d$, one has $\rho_{i,h} v \in V$, where $\rho_{i,h} v$ is defined as

$$\rho_{i,h} v := \frac{\tau_{i,h} v - v}{h}, \qquad \text{with} \qquad \tau_{i,h}\varphi := \varphi(\cdot + h e_1), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d \times [-1,0));$$

we also recall that the adjoint operator of $\rho_{i,h}$ is $-\rho_{i,-h}$ and that one has the product rule $v_1 \rho_{i,h} v_2 = \rho_{i,h}(v_1 v_2) - \rho_{i,h} v_1 \tau_{i,h} v_2$. Using (2.18) with $\rho_{i,h} v$ instead of v, one gets therefore

$$\int_{\mathcal{S}} (\tau_{i,-h} Q) \nabla_{X,y} \rho_{i,-h} u^\sharp \cdot \nabla_{X,y} v = \int_{-1} \widetilde{g} \rho_{i,h} v - \int_{\mathcal{S}} \big(h^\sharp \rho_{i,h} v - \rho_{i,-h} Q \nabla_{X,y} u^\sharp \cdot \nabla_{X,y} v\big).$$

By the trace theorem and Poincaré's inequality, we get $|\rho_{i,h}v|_{H^{-1/2}} \leq \text{Cst } \|\nabla_{X,y}v\|_2$, so that the r.h.s of the above inequality can be bounded from above by

$$\text{Cst } \big(|\widetilde{g}|_{H^{1/2}} + \|h^\sharp\|_2 + \|Q\|_{1,\infty}\|\nabla_{X,y}u^\sharp\|_2\big)\,\|\nabla_{X,y}v\|_2. \tag{2.19}$$

Taking $v = \rho_{i,-h}u^\sharp$ as a test function in (2.19), using condition (2.15), and letting $h \to 0$, one gets therefore

$$\|\partial^2_{ij}u^\sharp\|_2 \leq \text{Cst } \frac{1}{q}\big(|\widetilde{g}|_{H^{1/2}} + \|h^\sharp\|_2 + \|Q\|_{1,\infty}\|\nabla_{X,y}u^\sharp\|_2\big) \tag{2.20}$$

for all $1 \leq i,j \leq d+1$ such that $i+j \leq 2d+1$. The missing term $\partial^2_{yy}u^\sharp$ is obtained as usual using the equation,

$$-\partial^2_{yy}u^\sharp = \frac{1}{q_{d+1,d+1}}\Big(\mathbf{Q}u^\sharp + \sum_{i+j\leq 2d+1}\partial_i\left(q_{i,j}\partial_j u^\sharp\right) + (\partial_y q_{d+1,d+1})\partial_y u^\sharp\Big), \tag{2.21}$$

where $Q = (q_{i,j})_{i,j}$, from which it follows easily that

$$\|\partial^2_{yy}u^\sharp\|_2 \leq \frac{\text{Cst}}{q}\Big(\|h^\sharp\|_2 + \|Q\|_{1,\infty}\big(\sum_{i+j\leq 2d+1}\|\partial^2_{ij}u^\sharp\|_2 + \|\nabla_{X,y}u^\sharp\|_2\big)\Big). \tag{2.22}$$

From (2.20) and (2.22), it follows that $u^\sharp \in H^2(\mathcal{S})$ and satisfies

$$\|\nabla_{X,y}u^\sharp\|_{1,2} \leq C\big(\frac{1}{q},\|Q\|_{1,\infty}\big)\Big(|\widetilde{g}|_{H^{1/2}} + \|h^\sharp\|_2 + \|\nabla_{X,y}u^\sharp\|_2\Big).$$

Replacing $u^\sharp, h^\sharp$ and $\widetilde{g}$ by their expressions in the above inequality, and using the estimates (2.16) yields

$$\|\nabla_{X,y}u\|_{1,2} \leq C\big(\frac{1}{q},\|Q\|_{1,\infty}\big)\Big(\|h\|_2 + \|\nabla_{X,y}u\|_2 + |\nabla_X f|_{H^{1/2}} + |g|_{H^{1/2}}\Big). \tag{2.23}$$

Step 3. Further regularity. We show by finite induction on k that for all $u \in H^{2+k}(\mathcal{S})$, $k = 0,\dots,m_0-1$, one has

$$\begin{aligned}\|\nabla_{X,y}u\|_{1+k,2} \leq{}& C\big(\frac{1}{q},\|Q\|_{1+k,\infty}\big)\\ &\times\ \Big(\|\mathbf{Q}u\|_{k,2} + \|\nabla_{X,y}u\|_{k,2} + |\nabla_X u_{|_{y=0}}|_{H^{1/2+k}} + |\partial^Q_n u_{|_{y=-1}}|_{H^{1/2+k}}\Big).\end{aligned} \tag{2.24}$$

By Step 2, this assertion is true when $k=0$. Let $m_0 > k \geq 1$ and assume it is also true for $0 \leq l \leq k-1$. For all $i = 1,\dots,d$, we apply $(2.24)_l$ with $l = k-1$ to the function $\rho_{i,h}u$:

$$\begin{aligned}\|\nabla_{X,y}\rho_{i,h}u\|_{k,2} \quad\leq\quad & C\big(\frac{1}{q},\|Q\|_{k,\infty}\big)\Big(\|\mathbf{Q}\rho_{i,h}u\|_{k-1,2} + \|\nabla_{X,y}\rho_{i,h}u\|_{k-1,2}\\ & +|\nabla_x\rho_{i,h}u_{|_{y=0}}|_{H^{k-1/2}} + |\partial^Q_n(\rho_{i,h}u)|_{y=-1}|_{H^{-1/2+k}}\Big).\end{aligned} \tag{2.25}$$

We now estimate the four terms which appear in the r.h.s. of (2.25). Since $\mathbf{Q}\rho_{i,h}u = \rho_{i,h}(\mathbf{Q}u) + \nabla_{X,y}\cdot(\rho_{i,h}Q)\nabla_{X,y}\tau_{i,h}u$, one has

$$\|\mathbf{Q}\rho_{i,h}u\|_{k-1,2} \leq \|\mathbf{Q}u\|_{k,2} + \text{Cst } \|Q\|_{k+1,\infty}\|\nabla_{X,y}u\|_{k,2}. \tag{2.26}$$

The second and third terms of (2.25) are very easily controlled. For the fourth one, we use the explicit expression of $\partial^Q_n(\rho_{i,h}u)$, use the trace theorem, and proceed as for the derivation of (2.26) to obtain

$$|\partial^Q_n(\rho_{i,h}u)|_{y=-1}|_{H^{k-1/2}} \leq |\partial^Q_n u_{|_{y=-1}}|_{H^{k+1/2}} + \text{Cst } \|Q\|_{k+1,\infty}\|\nabla_{X,y}u\|_{k,2}. \tag{2.27}$$

From (2.25), (2.26) and (2.27) (and letting $h \to 0$), it follows that $\|\partial_i u\|_{k+1,2}$ is bounded from above by the right-hand side of (2.24). In order to complete the proof, we still need an estimate of $\partial_y^2 u$ in $H^k(\mathcal{S})$. As in Step 2, such an estimate is obtained using (2.21).

Step 4. Further regularity. We show by induction on k that (2.24) can be generalized for $k \geq m_0$ as

$$\begin{aligned} \|\nabla_{X,y} u\|_{1+k,2} \quad &\leq \quad C_k \times \Big(\|\mathbf{Q}u\|_{k,2} + \|\nabla_{X,y}u\|_{k,2} + \|Q_2\|_{1+k,2}\|\nabla_{X,y}u\|_{m_0,2} \\ &\qquad + |\nabla_X u_{|_{y=0}}|_{H^{1/2+k}} + |\partial_n^Q u_{|_{y=-1}}|_{H^{1/2+k}}\Big), \end{aligned} \tag{2.28}$$

where $C_k = C\big(\frac{1}{q}, \|Q_1\|_{1+k,\infty}, \|Q_2\|_{m_0+1,\infty}\big)$.

The procedure is absolutely similar to Step 3. It is strictly unchanged until Eq. (2.26) where we now use Moser's tame estimates on products (e.g. [1]):

Lemma 2.11. *Let $l \in \mathbb{N}$ and $u, v \in H^l(\mathcal{S}) \cap L^\infty(\mathcal{S})$. Then one has*

$$\|uv\|_{l,2} \leq \text{Cst } \left(\|u\|_{l,2}\|v\|_\infty + \|v\|_{l,2}\|u\|_\infty\right).$$

This yields

$$\begin{aligned} &\|\mathbf{Q}\rho_{i,h} u\|_{k-1,2} \leq \|\mathbf{Q}u\|_{k,2} \\ &\quad + \quad \text{Cst } \big((\|Q_1\|_{k+1,\infty} + \|Q_2\|_{1,\infty})\|\nabla_{X,y}u\|_{k,2} + \|Q_2\|_{1+k,2}\|\nabla_{X,y}u\|_\infty\big). \end{aligned}$$

Estimate (2.27) is modified along the same lines and it follows from the Sobolev embedding $H^{m_0}(\mathcal{S}) \subset L^\infty(\mathcal{S})$ that $\|\partial_i u\|_{k+1,2}$ is bounded from above by the right-hand side of (2.28).

An estimate on $\partial_y^2 u$ in $H^k(\mathcal{S})$ is then provided as before using (2.21), which concludes the induction.

Step 5. Endgame. From the variational formulation of the problem, one easily gets the following lemma, whose proof we omit.

Lemma 2.12. *Let $h \in L^2(\mathcal{S})$, $f \in H^{1/2}(\mathbb{R}^d)$ and $g \in H^{-1/2}(\mathbb{R}^d)$. If $u \in H^2(\mathcal{S})$ solves the boundary value problem* (2.13), *then*

$$\|\nabla_{X,y}u\|_2 \leq C\big(\frac{1}{q}, \|Q\|_\infty\big)\left(\|h\|_2 + |\nabla_X f|_{H^{-1/2}} + |g|_{H^{-1/2}}\right)$$

and

$$\|u\|_{1,2} \leq C\big(\frac{1}{q}, \|Q\|_\infty\big)\left(\|h\|_2 + |f|_{H^{1/2}} + |g|_{H^{-1/2}}\right).$$

Iterating estimates (2.24) and (2.28) and using the lemma gives the theorem. $\square$

2.3. **Regularizing diffeomorphisms.** If u solves the boundary value problem (2.3), then one can give precise estimates on $\widetilde{u} = u \circ S$, owing to Prop. 2.7 and 2.8, and using Theorem 2.9. However, these estimates depend strongly on the diffeomorphism S chosen to straighten the fluid domain. The trivial diffeomorphism given in Remark 2.4 is not the best choice possible: in order to control the $H^k(\mathcal{S})$-norm of its Sobolev component, one needs to control the $H^k(\mathbb{R}^d)$-norm of the surface parameterization a. The next proposition shows that there exist "regularizing" diffeomorphisms for which a linear control of the $H^{k-1/2}$-norm suffices.

Proposition 2.13. *Let $k \in \mathbb{N}$, $k - \frac{1}{2} > 1 + \frac{d}{2}$, and let $b \in C_b^k(\mathbb{R}^d)$, $a \in H^{k-1/2}(\mathbb{R}^d)$. If there exists $h_0 > 0$ such that $a - b \geq h_0$ on $\mathbb{R}^d$, then there exists a diffeomorphism*

S of the form (2.6) *such that*

- *s is k-regular (with $c_0 = h_0/2$);*
- *one has $\partial_{\widetilde{y}} s|_{\widetilde{y}=0} = a - b$;*
- *one has $s_1 = -b(\widetilde{X})\widetilde{y}$ and $\|s_2\|_{k,2} \leq \mathrm{Cst}\ |a|_{H^{-1/2+k}}$.*

Remark 2.14. **i.** The diffeomorphism S provided by this lemma is a perturbation of the trivial diffeomorphism given in Remark 2.4. The C_b^k-component s_2 remains unchanged, and the behavior at the surface is exactly the same. However, the Sobolev component s_2 is half a derivative smoother here than for the trivial diffeomorphism (where it has the smoothness of a). This is why we say that the diffeomorphism is "regularizing".
ii. Note that if $a \in C([0,T], H^{-1/2+k}(\mathbb{R}^d))$ for some $T > 0$, and if the condition $a - b > h_0$ is satisfied uniformly in $t \in [0,T]$, then one has $\partial_t s = \partial_t s_2$ and $\|\partial_t s_2\|_{k,2} \leq C(|\partial_t a(t)|_{H_T^{-1/2+k}})$. This will be used in the proof of Prop. 3.19.
iii. If $\underline{a} \in H^{k+3/2}(\mathbb{R}^d)$ with $k > d/2$ is such that $a - b \geq h_0 > 0$ on $\mathbb{R}^d$, then one can find a neighbourhood $\mathcal{U}_{\underline{a}}$ of $\underline{a}$ in $H^{k+3/2}$ such that for all $a \in \mathcal{U}_{\underline{a}}$, $a - b \geq \frac{3}{4}h_0$. To each of these $a \in \mathcal{U}_{\underline{a}}$, one can associate a regularizing diffeomorphism $S_a(X,y) = (X, s_a(X,y))$ by Prop. 2.13. The proof shows that if $\mathcal{U}_{\underline{a}}$ is small enough, then the mapping $a \mapsto s_a$ is affine. This mapping is therefore smooth and (using the notation of the proof) one can check that for all $h \in H^{k+3/2}(\mathbb{R}^d)$, one has $d_{\underline{a}} s \cdot h = (\widetilde{y}+1)h_{\underline{\lambda}}$ for some $\underline{\lambda} > 0$. Hence,

$$d_{\underline{a}} s \cdot h|_{\widetilde{y}=0} = h, \qquad d_{\underline{a}} s \cdot h|_{\widetilde{y}=-1} = 0, \qquad \partial_{\widetilde{y}} d_{\underline{a}} s \cdot h|_{\widetilde{y}=0} = h.$$

Proof. Note that the Jacobian of the mapping $(\widetilde{X}, \widetilde{y}) \in \mathcal{S} \mapsto (\widetilde{X}, s(\widetilde{X}, \widetilde{y})) \in \Omega$ is equal to $|\partial_{\widetilde{y}} s|$. Therefore, if s satisfies the properties stated in the lemma, S is indeed a diffeomorphism between $\mathcal{S}$ and Ω.

Let $s_1 \in C_b^k(\overline{\mathcal{S}})$ be given by

$$s_1(\widetilde{X}, \widetilde{y}) = -b(\widetilde{X})\widetilde{y}, \qquad \forall (\widetilde{X}, \widetilde{y}) \in \mathcal{S};$$

we look for $s_2 \in H^k(\mathcal{S})$ such that $s := s_1 + s_2$ satisfies

$$\partial_{\widetilde{y}} s \geq \frac{h_0}{2} \quad \text{on} \quad \overline{\mathcal{S}}, \qquad s_2|_{\widetilde{y}=0} = a, \qquad \partial_{\widetilde{y}} s_2|_{\widetilde{y}=0} = a, \qquad s_2|_{\widetilde{y}=-1} = 0. \tag{2.29}$$

We construct such a mapping s_2 using a Poisson kernel extension of a. Let χ be a smooth, compactly supported, function defined on $\mathbb{R}$ and such that $\chi(0) = 1$ and $\chi'(0) = 0$. For any $\lambda > 0$, and $a \in H^{k-1/2}(\mathbb{R}^d)$, we define $a_\lambda \in H^k(\mathcal{S})$ as $a_\lambda(\cdot, \widetilde{y}) = \chi(\lambda \widetilde{y} \langle D \rangle)a$. From this definition it follows also that for all $(\widetilde{X}, \widetilde{y}) \in \overline{\mathcal{S}}$, one has

$$\begin{aligned} |\partial_{\widetilde{y}} a_\lambda(\widetilde{X}, \widetilde{y})| &= |\lambda \chi'(\lambda \widetilde{y} \langle D \rangle)\langle D \rangle a| \leq \lambda |\chi'|_\infty \int_{\mathbb{R}^d} \langle \xi \rangle |\widehat{a}(\xi)| d\xi \\ &\leq \mathrm{Cst}\ \lambda |\chi'|_\infty |a|_{H^{k-1/2}}, \end{aligned} \tag{2.30}$$

since $k - 1/2 > 1 + d/2$.

Define now $s_2 := (\widetilde{y}+1)a_\lambda$. It is obvious that s_2 satisfies the last three conditions of (2.29). For the first one, remark that

$$-b + \partial_{\widetilde{y}} s_2 = (-b + a) + (a_\lambda - a) + (1 + \widetilde{y})\partial_{\widetilde{y}} a_\lambda \tag{2.31}$$

and that

$$|a_\lambda(\widetilde{X},\widetilde{y})-a(\widetilde{X})| \quad = \quad \Big|\int_0^{\widetilde{y}} \partial_{\widetilde{y}} a_\lambda(\widetilde{X},\widetilde{y}')d\widetilde{y}'\Big| \leq \sup_{\overline{\mathcal{S}}} |\partial_{\widetilde{y}} a_\lambda|. \tag{2.32}$$

Taking λ small enough, one can complete the proof from (2.30), (2.31), (2.32) and the assumption $a-b\geq h_0$. □

3. The Dirichlet-Neumann operator

The aim of this section is to investigate the properties of the Dirichlet-Neumann operator associated to a class of boundary value problems included in the general framework studied in Section 2. It is known that such operators depend analytically on the parameterization of the surface. Coifman and Meyer [9] considered small Lipschitz perturbations of a line or plane, and Craig *et al.* [12, 13] C^1 perturbations of hyperplanes in any dimension. These studies rely on subtle estimates of singular integral operators. More recently, Nicholls and Reitich [28] addressed the analyticity of the Dirichlet-Neumann operator using a simple method based on a change of variables (see also [2]). Here, we are also interested in the dependence of the Dirichlet-Neumann operator on the fluid domain, but from a Sobolev rather than an analytical viewpoint. The sharp elliptic estimate of the previous section allows us to give "tame" estimates on the action of the Dirichlet-Neumann operator on Sobolev spaces. In this section, we also compute the principal symbol of the DN operator, give tame estimates of its commutators with spatial or time derivatives, and also study carefully its shape derivatives.

3.1. **Definition and basic properties.** As in Section 2, we consider a fluid domain Ω of the form

$$\Omega=\{(X,y)\in\mathbb{R}^{d+1}, a(X)<y<b(X)\},$$

where a and b satisfy

$$\exists h_0>0, \qquad \min\{-b,a-b\}\geq h_0>0 \text{ on } \mathbb{R}^d. \tag{3.1}$$

We also consider a constant coefficients elliptic operator $\mathbf{P}=-\nabla_{X,y}\cdot P\nabla_{X,y}$, satisfying the coercivity condition (2.2). The boundary value problems we consider in this section are a particular case of the boundary value problems (2.3) since we only consider the case of a homogeneous source term and the Neumann boundary condition at the bottom. More precisely, let u solve

$$\begin{cases} \mathbf{P}u=0 & \text{on } \Omega, \\ u|_{\{y=a(X)\}}=f, & \partial_n^P u|_{\{y=b(X)\}}=0, \end{cases} \tag{3.2}$$

where we recall that, as defined in (2.4), ∂_n^P denotes the conormal derivative associated to P.

For all $k\in\mathbb{N}$ and $f\in H^{k+3/2}(\mathbb{R}^d)$, and provided that a and b are smooth enough, we know by Theorem 2.9 that $u\in H^{k+2}(\Omega)$ exists and is unique. Therefore, the following definition makes sense.

Definition 3.1. Let $k\in\mathbb{N}$, and assume that $a,b\in W^{2,\infty}(\mathbb{R}^d)$ satisfy condition (3.1). We define the Dirichlet-Neumann operator to be the operator $G(a,b)$ given by

$$G(a,b): \begin{array}{lcl} H^{k+3/2}(\mathbb{R}^d) & \to & H^{k+1/2}(\mathbb{R}^d) \\ f & \mapsto & -\sqrt{1+|\nabla_X a|^2}\partial_n^P u_{|\{y=a(X)\}}, \end{array}$$

where u denotes the solution of (3.2).

Remark 3.2. **i.** Thus defined, $G(a,b)$ is not exactly the Dirichlet-Neumann operator because of the scaling factor $\sqrt{1+|\nabla_X a|^2}$; yet, we use this terminology for the sake of simplicity.
ii. Thanks to the minus sign in the definition, $G(a,b)$ maps the Dirichlet data to the (rescaled) outward normal derivative when $P = Id$.

As in Section 2, we can associate to (3.2) an elliptic boundary value problem on the flat strip $\mathcal{S} = \mathbb{R}^d \times (0,1)$: denoting by R the "regularizing" diffeomorphism between $\mathcal{S}$ and Ω (and by S its inverse) given in Prop. 2.13, and $\widetilde{u} = u \circ S$, one has

$$\begin{cases} \widetilde{\mathbf{P}}\widetilde{u} = 0 \qquad \text{on } \mathcal{S}, \\ \widetilde{u}_{|\{\widetilde{y}=0\}} = f, \qquad \partial_n^{\widetilde{P}}\widetilde{u}_{|\{\widetilde{y}=-1\}} = 0, \end{cases} \tag{3.3}$$

where $\widetilde{\mathbf{P}} = -\nabla_{\widetilde{X},\widetilde{y}}\widetilde{P}\cdot\nabla_{\widetilde{X},\widetilde{y}}$ is as given in Lemma 2.5.

Notation 3.3. We denote by $f^\flat$ the solution of the b.v.p. (3.3).

Proceeding as in the proof of Lemma 2.6, one can define the Dirichlet-Neumann operator in terms of $f^\flat$.

Proposition 3.4. *Under the same assumptions as in Def.* 3.1*, one has*

$$G(a,b)f = -\partial_n^{\widetilde{P}} f^\flat_{|\widetilde{y}=0}, \qquad \forall f \in H^{3/2}(\mathbb{R}^d),$$

where $f^\flat$ is as defined in Notation 3.3.

Before stating our main estimates on the DN operator, let us state some notation.

Notation 3.5. **i.** When a bottom parameterization $b \in W^{k,\infty}(\mathbb{R}^d)$ $(k \in \mathbb{N}\cup\{\infty\})$ is given, we generically write $B = |b|_{W^{k,\infty}}$.
ii. For all $r,s \in \mathbb{R}$, we denote generically by $M(s)$ (resp. $M_r(s)$) constants which depend on B and $|a|_{H^s}$ (resp. r, B and $|a|_{H^s}$).

The next theorem shows that the DN operator is of order one and gives precise estimates on its operator norm.

Theorem 3.6. *Let $m_0 = \lceil \frac{d+1}{2} \rceil$ and a,b be two continuous functions satisfying* (3.1)*. Then:*
i. *For all $k \in \mathbb{N}$, if $a,b \in W^{k+2,\infty}(\mathbb{R}^d)$, then for all f such that $\nabla_X f \in H^{k+1/2}(\mathbb{R}^d)^2$, one has*

$$|G(a,b)f|_{H^{k+1/2}} \leq C\big(|a|_{W^{k+2,\infty}}, |b|_{W^{k+2,\infty}}\big)|\nabla_X f|_{H^{k+1/2}}.$$

ii. *For all $k \in \mathbb{N}$, if $a \in H^{2m_0+1/2} \cap H^{k+3/2}(\mathbb{R}^d)$ and if $b \in W^{k+2}(\mathbb{R}^d)$, then*

$$|G(a,b)f|_{H^{k+1/2}} \leq M_s(2m_0+1/2)\left(|\nabla_X f|_{H^{k+1/2}} + |a|_{H^{k+3/2}}|\nabla_X f|_{H^{m_0+1/2}}\right),$$

for all f such that $\nabla_X f \in H^{k+1/2} \cap H^{m_0+1/2}$, and where we used Notation 3.5.

Remark 3.7. Note that the DN operator is defined for functions f whose gradient is in some Sobolev space, but which are not necessarily in a Sobolev space themselves.

Proof. We just prove the second part of the theorem; the proof of the first part is very similar. Owing to Prop. 3.4, we have

$$|G(a,b)f|_{H^{k+1/2}} = \big|\partial_n^{\widetilde{P}}\widetilde{u}|_0\big|_{H^{k+1/2}} = \big|\widetilde{P}|_{\widetilde{y}=0}e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}|_{\widetilde{y}=0}\big|_{H^{k+1/2}}.$$

By the trace theorem, this yields

$$|G(a,b)f|_{H^{k+1/2}} \leq \mathrm{Cst}\ \big\|\widetilde{P}e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\big\|_{k+1,2},$$

where the notation $\|\cdot\|$ is as in Notation 2.1.

Using the decomposition $\widetilde{P} = \widetilde{P}_1 + \widetilde{P}_2$ of Lemma 2.8 and the tame product estimate of Lemma 2.11, one obtains

$$\begin{aligned} |G(a,b)f|_{H^{k+1/2}} &\leq \mathrm{Cst}\ \|\widetilde{P}_1\|_{k+1,\infty}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} \\ &+ \mathrm{Cst}\ \|\widetilde{P}_2\|_{\infty}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} + \mathrm{Cst}\ \|\widetilde{P}_2\|_{k+1,2}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{m_0,2}. \end{aligned} \tag{3.4}$$

Now, remark that when the diffeomorphism S between the flat strip $\mathcal{S}$ and the fluid domain Ω is the regularizing diffeomorphism of Prop. 2.13, the estimates of Lemma 2.8, together with the Sobolev embedding $H^{m_0}(\mathcal{S}) \subset L^\infty(\mathcal{S})$, give

$$\begin{aligned} \|\widetilde{P}_1\|_{k+1,2} &\leq C\,(B)\,, \\ \|\widetilde{P}_2\|_{k+1,2} &\leq C\,(B, |a|_{H^{m_0+1/2}})\,|a|_{H^{k+3/2}}. \end{aligned} \tag{3.5}$$

Similarly, the constant C_k which appears in Theorem 2.9 when one takes $Q = \widetilde{P}$ can be bounded from above by $C\,(B, |a|_{H^{2m_0+1/2}})$ and the result follows therefore from (3.4), Theorem 2.9 and Remark 2.10. $\square$

Some important properties of the DN operator are listed in the next proposition.

Proposition 3.8. *Let $a, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (3.1). Then:*
i. *The operator $G(a,b)$ is self-adjoint:*

$$(G(a,b)f, g) = (f, G(a,b)g), \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

ii. *The operator $G(a,b)$ is positive:*

$$(G(a,b)f, f) \geq 0, \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

iii. *We also have the estimates*

$$\big|\big(G(a,b)f, g\big)\big| \leq M(m_0 + 1/2)|f|_{H^{1/2}}|g|_{H^{1/2}}, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d),$$

and for all $\mu \geq \frac{\widetilde{p}}{3}$, where $\widetilde{p}$ is given in Lemma 2.5, one has

$$\big|\big([G(a,b) + \mu]f, f\big)\big| \geq \mathrm{Cst}\ \widetilde{p}|f|_{H^{1/2}}, \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Remark 3.9. Using the self-adjointness of $G(a,b)$, one could extend this operator to all Sobolev spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$.

Proof. **i.** According to Prop. 3.4, and using Notation 3.3, one has $(G(a,b)f, g) = (\partial_n^{\widetilde{P}} f^\flat|_0, g)$, and Green's identity yields $(\partial_n^{\widetilde{P}} f^\flat|_0, g) = (f, \partial_n^{\widetilde{P}} g^\flat|_0)$. Using Prop. 3.4 once again yields the result.
ii. Writing $(G(a,b)f, f) = (\partial_n^{\widetilde{P}} f^\flat|_0, f)$ and integrating by parts, one obtains

$$\begin{aligned} (G(a,b)f, f) &= \int_{\mathcal{S}} \nabla_{X,y} f^\flat \cdot \widetilde{P}\nabla_{X,y} f^\flat \\ &\geq \widetilde{p}\|\nabla_{X,y} f^\flat\|_2^2, \end{aligned} \tag{3.6}$$

where the last inequality uses the coercivity of $\widetilde{\mathbf{P}}$ proved in Lemma 2.5.
iii. Proceeding as in **ii**, one has

$$\begin{aligned}(G(a,b)f,g) &= \int_{\mathcal{S}} \nabla_{X,y} f^\flat \cdot \widetilde{P} \nabla_{X,y} g^\flat \\ &\leq \|\widetilde{P}\|_\infty \|\nabla_{X,y} f^\flat\|_2 \|\nabla_{X,y} g^\flat\|_2,\end{aligned}$$

and the first estimate follows from Lemma 2.12.

To prove the second estimate, remark first that by Poincaré's inequality, we have $\|f^\flat\|_2 \leq 2\|\partial_y f^\flat\|_2 + |f|_2$, and therefore $\|\nabla_{X,y} f^\flat\|_2 \geq \frac{1}{2}(\|f^\flat\|_2 - |f|_2)$. As a consequence, we obtain $\|\nabla_{X,y} f^\flat\|_2 \geq \frac{1}{3}\|f^\flat\|_{1,2} - \frac{1}{3}|f|_2$. Using (3.6) and the estimate $|f|_{H^{1/2}} \leq \mathrm{Cst}\ \|f^\flat\|_{1,2}$, we deduce $(G(a,b)f,f) \geq C|f|_{H^{1/2}} - \frac{\widetilde{p}}{3}|f|_2$. The end of the proof is then straightforward. □

3.2. **Symbol of the Dirichlet-Neumann operator.** In order to compute the commutator of $G(a,b)$ with differential operators, which is a crucial step to obtaining energy estimates for the water-waves equations, we need to know its principal symbol. Since this result is interesting in itself, we state it as a theorem (we use the classical notation $\sigma(x,D)$ to denote the pseudo-differential operator associated to the symbol $\sigma(x,\xi)$).

Theorem 3.10. *There exists an integer q_0, depending only on d, such that if $a \in H^{q_0+1/2}(\mathbb{R}^d)$ and $b \in C_b^\infty(\mathbb{R}^d)$ satisfy* (3.1)*, then, for $j = -1, 0, 1$, one has*

$$\forall f \in H^{j/2}(\mathbb{R}^d), \qquad \big|\big(G(a,b) - g_a(X,D)\big) f\big|_{H^{j/2}} \leq M(q_0 + 1/2)|f|_{H^{j/2}},$$

where $M(\cdot)$ is as defined in Notation 3.5 *and the symbol $g_a(X,\xi)$ is given by*

$$g_a(X,\xi) = \sqrt{(PN\cdot N)\begin{pmatrix}\xi\\0\end{pmatrix}\cdot P\begin{pmatrix}\xi\\0\end{pmatrix} - \left[N\cdot\begin{pmatrix}\xi\\0\end{pmatrix}\right]^2},$$

with $N := (-\nabla_X a, 1)^T$.

Remark 3.11. **i.** The estimate of the theorem can be extended to higher-order Sobolev spaces, but we do not need such a result here.
ii. The parameterization b of the bottom does not appear in the principal symbol of $G(a,b)$. This is not surprising since the contribution to the surface of the bottom is "smoothed" by the elliptic equation.
iii. For the water-waves equations, one has $P = Id_{(d+1)\times(d+1)}$ and g_a takes the simple form

$$g_a(X,D) = \sqrt{|\xi|^2 + |\nabla_X a|^2|\xi|^2 - (\nabla_X a\cdot\xi)^2}.$$

There is therefore an interesting phenomenological difference between the 1D and the 2D cases. In the latter, the principal symbol of the Dirichlet-Neumann operator is a pseudo-differential operator, while in the former, it is simply a Fourier multiplier: $g_a(D) = |D|$, which does not depend on the fluid domain.

Proof. The proof of the theorem relies strongly on the factorization procedure of elliptic operators, as set forth in [35] (see also [30]). Recall that, according to Prop. 3.4, one has $G(a,b)f = -\partial_n^{\widetilde{P}} f^\flat|_0$, where $f^\flat$ denotes the solution of (3.3). The idea is to deduce an approximation of $G(a,b)f$ from an approximation $f^\flat_{app}$ of $f^\flat$ for which the conormal derivative at the surface can be explicitly computed. In order to find such an approximation, we first approximate the elliptic operator $\widetilde{\mathbf{P}}$; this is where we need the factorization procedure mentioned above.

Writing $\widetilde{P} = \begin{pmatrix} \widetilde{P}_1 & \widetilde{\mathbf{p}} \\ \widetilde{\mathbf{p}}^T & \widetilde{p}_{d+1} \end{pmatrix}$, one can check that the operator $\widetilde{\mathbf{P}}$ can be written in the form

$$\begin{aligned} \widetilde{\mathbf{P}} \quad = \quad & -\widetilde{p}_{d+1}\partial_{\widetilde{y}}^2 - \big(2\widetilde{\mathbf{p}}\cdot\nabla_{\widetilde{X}} + (\partial_{\widetilde{y}}\widetilde{p}_{d+1} + \nabla_{\widetilde{X}}\cdot\widetilde{\mathbf{p}})\big)\partial_{\widetilde{y}} \\ & + P_1\Delta_{\widetilde{X}} + \big((\nabla_{\widetilde{X}}\cdot P_1) + \partial_{\widetilde{y}}\widetilde{\mathbf{p}}\big)\cdot\nabla_{\widetilde{X}}. \end{aligned} \tag{3.7}$$

We now look for an approximation of $\widetilde{\mathbf{P}}$ of the form

$$\widetilde{\mathbf{P}}_{\mathbf{app}} := -\widetilde{p}_{d+1}(\partial_y - \eta_-(\widetilde{X},\widetilde{y},D))(\partial_y - \eta_+(\widetilde{X},\widetilde{y},D)), \tag{3.8}$$

where, for all $\widetilde{y}\in[-1,0]$, $\eta_\pm(\widetilde{X},\widetilde{y},D)$ denotes the pseudo-differential operator of the symbol $\eta_\pm(\widetilde{X},\widetilde{y},\xi)$. Obviously, if one wants the highest-order terms of $\widetilde{\mathbf{P}}_{\mathbf{app}}$ to match those of $\widetilde{\mathbf{P}}$, $\eta_\pm(\widetilde{X},\widetilde{y},\xi)$ must be the roots of the second-order polynomial $\Sigma(\widetilde{\mathbf{P}}) = -\widetilde{p}_{d+1}\eta^2 - 2i\widetilde{\mathbf{p}}\cdot\xi\eta + \xi\cdot\widetilde{P}_1\xi$; namely,

$$\eta_\pm(X,y,\xi) = \frac{1}{\widetilde{p}_{d+1}}\Big(- i\widetilde{\mathbf{p}}\cdot\xi \pm \sqrt{\widetilde{p}_{d+1}\xi\cdot\widetilde{P_1}\xi - (\widetilde{\mathbf{p}}\cdot\xi)^2}\Big). \tag{3.9}$$

We now take the function $f^\flat_{app}$ we are looking for as an approximate solution of the equation $\widetilde{\mathbf{P}}_{\mathbf{app}}\phi = 0$; from (3.8), it suffices to take an approximate solution of the *backward* evolution equation

$$(\partial_{\widetilde{y}} - \eta_+(\widetilde{X},\widetilde{y},D))u = 0, \quad u|_{\widetilde{y}=0} = f, \quad \text{for } \widetilde{y}\in[-1,0]; \tag{3.10}$$

we therefore take

$$f^\flat_{app}(\widetilde{X},\widetilde{y}) := \sigma_{app}(\widetilde{X},\widetilde{y},D)f \tag{3.11}$$

where $\sigma_{app}(\widetilde{X},\widetilde{y},\xi) := \exp\big(-\int_{\widetilde{y}}^0 \eta_+(\widetilde{X},y',\xi)dy'\big)$.

Since the real part of η_+ is always positive, $\sigma_{app}(\widetilde{X},\widetilde{y},D)$ is smoothing for all $y\in[-1,0)$. As a consequence, one has: □

Lemma 3.12. *Let* $m_0 = \lceil\frac{d+1}{2}\rceil$. *Let* $a\in H^{2m_0+1/2}(\mathbb{R}^d)$ *and* $b\in W^{d+1,\infty}(\mathbb{R}^d)$ *satisfy condition* (3.1). *Then*

$$\|f^\flat_{app}\|_2 \le M(2m_0+1/2)|f|_{H^{-1/2}} \quad \text{and} \quad \|f^\flat_{app}\|_{1,2} \le M(2m_0+1/2)|f|_{H^{1/2}},$$

where $M(\cdot)$ *is as defined in Notation* 3.5.

Proof. From the explicit expression of η_+ given in (3.9), one deduces

$$\frac{\|\widetilde{P}\|_\infty}{\widetilde{p}}|\xi| \ge \Re\big(\eta_+(\widetilde{X},\widetilde{y},\xi)\big) \ge C_+|\xi|,$$

where C_+ is a positive constant which depends on h_0, p, $|b|_{1,\infty}$ and $|a|_{H^{m_0+1/2}}$. Let us define $\widetilde{\sigma}_{app}$ as

$$\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},\xi) := \sigma_{app}(\widetilde{X},\widetilde{y},\xi)\exp\big(-\frac{C_+}{2}\widetilde{y}|\xi|\big);$$

it is clear that $\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},\xi)$ is a symbol of order zero (uniformly in $\widetilde{y}\in[-1,0]$). The operator $\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},D)$ acts therefore continuously on $L^2(\mathbb{R}^d)$. Moreover, its operator norm can be bounded in terms of a finite number of L^∞-norms of space-frequency derivatives of the symbol $\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},\xi)$ (d-derivatives with respect to $\widetilde{X}$

and d derivatives with respect to ξ are enough; see [23] and also [25]). Using the Sobolev embedding $H^s(\mathbb{R}^d)\subset L^\infty(\mathbb{R}^d)$ for $s>d/2$, it follows that

$$|\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},D)\phi|_2 \le M(2m_0+1/2)|\phi|_2, \qquad \forall \phi\in L^2(\mathbb{R}^d).$$

Thus,

$$\begin{aligned}\|f^\flat_{app}\|_2 &= \Big(\int_{-1}^0 \big|\widetilde{\sigma}_{app}(\widetilde{X},y,D)\big(\exp(\frac{C_+}{2}y|D|)f\big)\big|_2^2 dy\Big)^{1/2}\\ &\le M(2m_0+1/2)\Big(\int_{-1}^0 \big|\exp(\frac{C_+}{2}y|D|)f\big|_2^2 dy\Big)^{1/2}.\end{aligned}$$

The gain of half a derivative claimed in the first estimate of the lemma is deduced from this expression by a classical computation (see e.g. Prop. 12.4 of [34]).

One can estimate the first-order derivatives of $f^\flat$ in the same way, which yields the second estimate of the lemma. □

We now prove that $f^\flat_{app}$ is indeed an approximate solution of (3.10) and hence of the equation $\widetilde{\mathbf{P}}_{\mathbf{app}}f^\flat_{app}=0$.

Lemma 3.13. *There exists an integer q_0, depending only on d, such that if $a\in H^{q_0+1/2}(\mathbb{R}^d)$ and $b\in C_b^\infty(\mathbb{R}^d)$ satisfy condition* (3.1), *then*

$$\|\widetilde{\mathbf{P}}_{\mathbf{app}}f^\flat_{app}\|_2 \le M(q_0+1/2)|f|_{H^{1/2}}.$$

Proof. Simple computations yield

$$\begin{aligned}&\widetilde{\mathbf{P}}_{\mathbf{app}}f^\flat_{app} = -\widetilde{p}_{d+1}(\partial_{\widetilde{y}}-\eta_-(\widetilde{X},\widetilde{y},D))\\ &\times\big((\eta_+\widetilde{\sigma}_{app})(\widetilde{X},\widetilde{y},D)-\eta_+(\widetilde{X},\widetilde{y},D)\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},D)\big)\big(\exp(\frac{C_+}{2}\widetilde{y}|D|)f\big),\end{aligned}\tag{3.12}$$

where $\widetilde{\sigma}_{app}$ and C_+ are as in the proof of Lemma 3.12.

It is easy to check that $\eta_-(\widetilde{X},\widetilde{y},D)$ is of order one, so that it acts continuously on $H^1(\mathbb{R}^d)$ with values in $L^2(\mathbb{R}^d)$. As in the proof of Lemma 3.12 above, we can bound its norm in terms of a finite number of derivatives of the symbol. For q_0 large enough, we therefore have

$$\begin{aligned}&\|\widetilde{\mathbf{P}}_{\mathbf{app}}f^\flat_{app}\|_2 \le M(q_0+1/2)\\ &\times\ \Big\|\Big((\eta_+\widetilde{\sigma}_{app})(\widetilde{X},\widetilde{y},D)-\eta_+(\widetilde{X},\widetilde{y},D)\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},D)\Big)\big(\exp(\frac{C_+}{2}\widetilde{y}|D|)f\big)\Big\|_{1,2},\end{aligned}$$

where $M(\cdot)$ is as defined in Notation 3.5.

Similarly, the operator $(\eta_+\widetilde{\sigma}_{app})(\widetilde{X},\widetilde{y},D)-\eta_+(\widetilde{X},\widetilde{y},D)\widetilde{\sigma}_{app}(\widetilde{X},\widetilde{y},D)$ is of order 0, so that (taking a larger q_0 if necessary), one has

$$\|\widetilde{\mathbf{P}}_{\mathbf{app}}f^\flat_{app}\|_2 \le M(q_0+1/2)\big\|\exp(\frac{C_+}{2}y|D|)f\big\|_{1,2},$$

and one can conclude the proof as for Lemma 3.12. □

We now proceed to estimate the difference $f^\flat - f^\flat_{app}$:

Lemma 3.14. *There exists an integer q_0, depending only on d, such that if $a\in H^{q_0+1/2}(\mathbb{R}^d)$ and $b\in C_b^\infty(\mathbb{R}^d)$ satisfy* (3.1), *then*

$$\|f^\flat - f^\flat_{app}\|_{2,2} \le M(q_0+1/2)|f|_{H^{1/2}}.$$

Proof. Since by definition $\widetilde{\mathbf{P}} f^\flat = 0$, one gets

$$\begin{aligned}\widetilde{\mathbf{P}}(f^\flat - f^\flat_{app}) &= -(\widetilde{\mathbf{P}} - \widetilde{\mathbf{P}}_{\mathbf{app}}) f^\flat_{app} - \widetilde{\mathbf{P}}_{\mathbf{app}} f^\flat_{app} \\ &:= h^1_{app} + h^2_{app}, \end{aligned} \tag{3.13}$$

together with the boundary conditions $(f^\flat - f^\flat_{app})|_0 = 0$ and $\partial_n^{\widetilde{P}}(f^\flat - f^\flat_{app})|_{-1} = -\partial_n^{\widetilde{P}} f^\flat_{app}|_{-1}$. Using Theorem 2.9, we therefore find

$$\|f^\flat - f^\flat_{app}\|_{2,2} \leq M(2m_0 + 1/2)\big(\|h^1_{app}\|_2 + \|h^2_{app}\|_2 + |\partial_n^{\widetilde{P}} f^\flat_{app}|_{-1}|_{H^{1/2}}\big). \tag{3.14}$$

Using (3.7) and the definition of $\widetilde{\mathbf{P}}_{\mathbf{app}}$, one checks easily that $\widetilde{\mathbf{P}} - \widetilde{\mathbf{P}}_{\mathbf{app}}$ is a first-order operator, so that with the help of Lemma 3.12, one gets the bound $\|h^1_{app}\|_2 \leq M(q_0 + 1/2)|f|_{H^{1/2}}$. Owing to Lemma 3.13, the same bound also holds on $\|h^2_{app}\|_2$. Finally, since $f \mapsto \partial_n^{\widetilde{P}} f^\flat_{app}|_{-1}$ is a smoothing operator, such an estimate also holds for $|\partial_n^{\widetilde{P}} f^\flat_{app}|_{-1}|_{H^{1/2}}$, and the proof of the lemma is complete. □

We are now ready to finish the proof of the theorem. First remark that $G(a,b)f + \partial_n^{\widetilde{P}} f^\flat_{app}|_0 = -\partial_n^{\widetilde{P}}(f^\flat - f^\flat_{app})|_{\widetilde{y}=0}$ so that

$$\big|G(a,b)f + \partial_n^{\widetilde{P}} f^\flat_{app}|_0\big|_{H^{1/2}} \leq M(m_0 + 1/2)\|f^\flat - f^\flat_{app}\|_{2,2};$$

by Lemma 3.14, we therefore have, for some $q_0 \in \mathbb{N}$,

$$\big|G(a,b)f + \partial_n^{\widetilde{P}} f^\flat_{app}|_0\big|_{H^{1/2}} \leq M(q_0 + 1/2)|f|_{H^{1/2}}. \tag{3.15}$$

To prove that (3.15) coincides with the estimate of the theorem in the case $j = 1/2$, we must show that $\partial_n^{\widetilde{P}} f^\flat_{app}|_{\widetilde{y}=0} = -g_a(X,D)f$, which we do now.

Thanks to Lemmas 2.5 and 2.13, we know the explicit expression of $\widetilde{P}|_{y=0}$; from the definition of the conormal derivative, one can then compute easily

$$\partial_n^{\widetilde{P}} f^\flat_{app}|_{\widetilde{y}=0} = -N \cdot P \begin{pmatrix} \nabla_X f \\ 0 \end{pmatrix} - \frac{N \cdot PN}{a-b} \partial_{\widetilde{y}} f^\flat_{app}|_{\widetilde{y}=0}, \tag{3.16}$$

where $N := (-\nabla_X a, 1)^T$.

The explicit expression of η_+ given in (3.9) yields also

$$\begin{aligned}\partial_y f^\flat_{app}|_{y=0} = &\frac{a-b}{PN \cdot N} \\ &\times \left(-N \cdot P \begin{pmatrix} \nabla_X f \\ 0 \end{pmatrix} + \sqrt{(PN \cdot N) \begin{pmatrix} D \\ 0 \end{pmatrix} \cdot P \begin{pmatrix} D \\ 0 \end{pmatrix} - \left[N \cdot \begin{pmatrix} D \\ 0 \end{pmatrix} \right]^2} f \right).\end{aligned}$$

Plugging this expression into (3.16) yields $\partial_n^{\widetilde{P}} f^\flat_{app}|_{\widetilde{y}=0} = -g_a(X,D)f$, which concludes the $H^{1/2}$-estimate of the theorem. Recalling that the DN operator is self-adjoint (see Prop. 3.8), one deduces the $H^{-1/2}$-estimate by a standard duality argument; finally, the L^2-estimate is obtained by interpolation.

3.3. **Commutator estimates.** This section is devoted to the proof of tame estimates of the commutator of the Dirichlet-Neumann operator with spatial derivatives and time derivative. The next proposition deals with the case of spatial derivatives.

Proposition 3.15. *There exists an integer* q_0*, depending only on* d*, such that if* $a \in H^{q_0+1/2}(\mathbb{R}^d)$ *and* $b \in C_b^\infty(\mathbb{R}^d)$ *satisfy* (3.1)*, then for all* $k \in \mathbb{N}$ *and* $\alpha \in \mathbb{N}^d$*,* $|\alpha| \le k$*, one has*

$$\big|[\Lambda^{1/2}\partial^\alpha, G(a,b)]f\big|_2 \le M_k(q_0+1/2)\left(|f|_{H^{k+1/2}} + |a|_{H^{k+3/2}}|f|_{H^{m_0+3/2}}\right),$$

where $M_k(\cdot)$ *is as defined in Notation* 3.5.

Remark 3.16. The interest of this commutator estimate is that it is "tame": even though we have a loss of derivative (in the sense that one needs to control the $H^{k+3/2}$-norm of a and not only its $H^{k+1/2}$-norm), this loss is linear, and the multiplicative constant which appears in front of it involves only Sobolev norms of f *independent* of k. This point is crucial to obtaining tame energy estimates later.

Proof. First remark that the following identity holds for all $\alpha \in \mathbb{N}^d$, $|\alpha| \le k$:

$$[\Lambda^{1/2}\partial^\alpha, G(a,b)] = [\Lambda^{1/2}, G(a,b)]\partial^\alpha + \Lambda^{1/2}[\partial^\alpha, G(a,b)],$$

so that

$$\big|[\Lambda^{1/2}\partial^\alpha, G(a,b)]f\big|_2 \le \big|[\Lambda^{1/2}, G(a,b)]\partial^\alpha f\big|_2 + \big|[\partial^\alpha, G(a,b)]f\big|_{H^{1/2}}. \tag{3.17}$$

Estimate of $\big|[\mathbf{\Lambda^{1/2}}, \mathbf{G(a,b)}]\partial^\alpha \mathbf{f}\big|_\mathbf{2}$**.** The idea is to replace $G(a,b)$ by its principal symbol $g_a(X,D)$ computed in the previous section. One has, denoting $G_0 := G(a,b) - g_a(X,D)$,

$$\begin{aligned} [\Lambda^{1/2}, G(a,b)]\partial^\alpha f &= [\Lambda^{1/2}, g_a(X,D)]\partial^\alpha f + \Lambda^{1/2}\big(G_0\partial^\alpha f\big) - G_0\big(\Lambda^{1/2}\partial^\alpha f\big) \\ &:= A_1 + A_2 + A_3. \end{aligned} \tag{3.18}$$

The operator $[\Lambda^{1/2}, g_a(X,D)]$ is of order $1/2$, and one can bound its operator norm $\|[\Lambda^{1/2}, g_a(X,D)]\|_{H^{1/2}\to L^2}$, as in the proofs of Lemmas 3.12 and 3.13, in terms of the derivatives of the symbol $g_a(X,\xi)$ given in Theorem 3.10. Thus, for some $q_0 \in \mathbb{N}$,

$$|A_1|_2 \le C\left(|a|_{H^{q_0+1/2}}\right)|f|_{H^{k+1/2}}. \tag{3.19}$$

Both A_2 and A_3 can be bounded using Theorem 3.10:

$$|A_j|_2 \le M(q_0+1/2)|f|_{H^{k+1/2}}, \qquad j = 2,3. \tag{3.20}$$

From (3.18)-(3.20), we deduce

$$\big|[\Lambda^{1/2}, G(a,b)]\partial^\alpha f\big|_2 \le M(q_0+1/2)|f|_{H^{k+1/2}}. \tag{3.21}$$

Estimate of $|[\partial^\alpha, \mathbf{G(a,b)}]\mathbf{f}|_{\mathbf{H^{1/2}}}$**.** Using Notation 3.3, it is easy to check that

$$[\partial^\alpha, G(a,b)]f = [\partial^\alpha, \widetilde{P}|_0 e_{d+1}]\cdot\nabla_{\widetilde{X},\widetilde{y}} f^\flat|_0 - \partial_n^{\widetilde{P}} v|_0, \tag{3.22}$$

with $v := \partial^\alpha f^\flat - (\partial^\alpha f)^\flat$.

The first term of the r.h.s. of (3.22) is estimated as follows:

$$\big|[\partial^\alpha, \widetilde{P}|_0 e_{d+1}]\cdot\nabla_{\widetilde{X},\widetilde{y}} f^\flat|_0\big|_{H^{1/2}} \le \mathrm{Cst}\ \big\|[\partial^\alpha, \widetilde{P}]\nabla_{\widetilde{X},\widetilde{y}} f^\flat\big\|_{1,2}. \tag{3.23}$$

In order to estimate the second term of (3.22), first remark that v solves the b.v.p.

$$\begin{cases} \widetilde{\mathbf{P}} v = \nabla_{\widetilde{X},\widetilde{y}}\cdot[\partial^\alpha, \widetilde{P}]\nabla_{\widetilde{X},\widetilde{y}} f^\flat \\ v|_0 = 0, \qquad \partial_n^{\widetilde{P}} v|_{-1} = -[\partial^\alpha, \widetilde{P}|_{-1} e_{d+1}]\cdot\nabla_{\widetilde{X},\widetilde{y}} f^\flat|_{-1}; \end{cases} \tag{3.24}$$

by the trace theorem and Theorem 2.9, one therefore gets

$$\big|\partial_n^{\widetilde{P}} v_{|_0}\big|_{H^{1/2}} \le M(m_0+3/2)\big\|[\partial^\alpha, \widetilde{P}]\nabla_{\widetilde{X},\widetilde{y}} f^\flat\big\|_{1,2}. \tag{3.25}$$

The term $\big\|[\partial^\alpha,\widetilde{P}]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}$ which appears in both (3.23) and (3.25) is estimated in the following lemma:

Lemma 3.17. *Under the same assumptions as in the proposition, one has*

$$\big\|[\partial^\alpha,\widetilde{P}]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}\leq M_k(2m_0+1/2)\left(|f|_{H^{k+1/2}}+|a|_{H^{k+3/2}}|f|_{H^{m_0+3/2}}\right).$$

Proof. By Lemma 2.8, we can decompose $\widetilde{P}$ into $\widetilde{P}=\widetilde{P}_1+\widetilde{P}_2$, so that $[\partial^\alpha,\widetilde{P}]=[\partial^\alpha,\widetilde{P}_1]+[\partial^\alpha,\widetilde{P}_2]$.

One has $\big\|[\partial^\alpha,\widetilde{P}_1]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}\leq\|\widetilde{P}_1\|_{k+1,\infty}\|f^\flat\|_{k+1,2}$, which is itself smaller than the r.h.s. of the estimate of the lemma, thanks to the estimates (3.5) and Theorem 2.9.

In order to bound $\big\|[\partial^\alpha,\widetilde{P}_2]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}$ from above, remark that

$$\big\|[\partial^\alpha,\widetilde{P}_2]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}\sim\big\|[\partial^\alpha,\widetilde{P}_2]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_2+\sum_{j=1}^{d+1}\big\|\partial_j\left([\partial^\alpha,\widetilde{P}_2]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\right)\big\|_2$$

and that for all $j=1,\dots,d+1$, one has $\partial_j[\partial^\alpha,\widetilde{P}_2]=[\partial^\alpha,\widetilde{P}_2]\partial_j+[\partial^\alpha,(\partial_j\widetilde{P}_2)]$. It is then easy to obtain, using Lemma 2.11 and the estimates (3.5), that

$$\big\|[\partial^\alpha,\widetilde{P}_2]\nabla_{\widetilde{X},\widetilde{y}}f^\flat\big\|_{1,2}\leq M_k(2m_0+1/2)\big(\|f^\flat\|_{k+1,2}+|a|_{H^{k+3/2}}\|f^\flat\|_{m_0+2,2}\big).$$

Owing to Theorem 2.9, the r.h.s. of this latter estimate is smaller than the r.h.s. of the estimate given in the lemma, so that the proof is complete. ☐

From (3.22), (3.23), (3.25) and the lemma one obtains

$$|[\partial^\alpha,G(a,b)]f|_{H^{1/2}}\leq M_k(2m_0+1/2)\left(|f|_{H^{k+1/2}}+|a|_{H^{k+3/2}}|f|_{H^{m_0+3/2}}\right).\tag{3.26}$$

The proposition is therefore a consequence of (3.17), (3.21) and (3.26). ☐

We end this section with two propositions concerning the commutator properties of the Dirichlet-Neumann operator with a general scalar-valued differential operator of order one, and with the time derivative (when the surface depends on time).

Proposition 3.18. *Let $m_0=\lceil\frac{d+1}{2}\rceil$ and suppose that $a\in H^{m_0+3/2}(\mathbb{R}^d)$ and $b\in C_b^\infty(\mathbb{R}^d)$ satisfy (3.1). Let $O(\partial_X)$ be a first-order differential operator on $\mathbb{R}^d$ with coefficients in $W^{1,\infty}(\mathbb{R}^d)$. Then, for all $f\in H^{1/2}(\mathbb{R}^d)$, one has*

$$\big|\big([O(\partial_X),G(a,b)]f,f\big)\big|\leq M(m_0+3/2)C(|O|_{1,\infty})|f|_{H^{1/2}}^2,$$

where $M(\cdot)$ is as in Notation 3.5, and $|O|_{1,\infty}$ denotes the sum of the $W^{1,\infty}$-norm of all the coefficients of $O(\partial_X)$.

Proof. With the same techniques as in the proof of the previous proposition, one can show that

$$[O(\partial_X),G(a,b)]f=e_{d+1}\cdot\big([O(\partial_{\widetilde{X}}),\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}]\big)f^\flat|_{\widetilde{y}=0}-\partial_n^{\widetilde{P}}v|_{\widetilde{y}=0},\tag{3.27}$$

where v is the solution of the boundary value problem

$$\left\{\begin{array}{l}\widetilde{\mathbf{P}}v=\nabla_{\widetilde{X},\widetilde{y}}\cdot[O(\partial_{\widetilde{X}}),\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}]f^\flat-(\nabla_{\widetilde{X},\widetilde{y}}O(\partial_{\widetilde{X}}))\cdot\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}f^\flat\\ v|_{\widetilde{y}=0}=0,\qquad\partial_n^{\widetilde{P}}v|_{\widetilde{y}=-1}=-e_{d+1}\cdot\big([O(\partial_{\widetilde{X}}),\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}]f^\flat\big)|_{\widetilde{y}=-1}.\end{array}\right.\tag{3.28}$$

Green's identity asserts that

$$(\partial_n^{\widetilde{P}}v|_{\widetilde{y}=0},f)+(\partial_n^{\widetilde{P}}v|_{\widetilde{y}=-1},f^\flat|_{\widetilde{y}=-1})=\int_{\mathcal{S}}\widetilde{\mathbf{P}}vf^\flat-\int_{\mathcal{S}}\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}v\cdot\nabla_{\widetilde{X},\widetilde{y}}f^\flat.\tag{3.29}$$

We also know that

$$\int_{\mathcal S}\widetilde{\mathbf P}vf^\flat=\int_{\mathcal S}\big(\nabla_{\widetilde X,\widetilde y}\cdot[O(\partial_{\widetilde X}),\widetilde P\nabla_{\widetilde X,\widetilde y}]f^\flat-(\nabla_{\widetilde X,\widetilde y}O(\partial_{\widetilde X}))\cdot\widetilde P\nabla_{\widetilde X,\widetilde y}f^\flat\big)f^\flat.$$

Integrating by parts the first term of the r.h.s., one finds (recall that $\partial_n^{\widetilde P}v|_{\widetilde y=-1}=-e_{d+1}\cdot([O(\partial_{\widetilde X}),\widetilde P\nabla_{\widetilde X,\widetilde y}]f^\flat)|_{\widetilde y=-1})$,

$$\begin{aligned}\int_{\mathcal S}\widetilde{\mathbf P}vf^\flat\quad&=\quad-\int_{\mathcal S}[O(\partial_{\widetilde X}),\widetilde P\nabla_{\widetilde X,\widetilde y}]f^\flat\cdot\nabla_{\widetilde X,\widetilde y}f^\flat\\&\quad-\int_{\mathcal S}\big((\nabla_{\widetilde X,\widetilde y}O(\partial_{\widetilde X}))\cdot\widetilde P\nabla_{\widetilde X,\widetilde y}f^\flat\big)f^\flat\\&\quad+\big(e_{d+1}\cdot\big([O(\partial_{\widetilde X}),\widetilde P\nabla_{\widetilde X,\widetilde y}]f^\flat\big)|_{\widetilde y=0},f\big)+(\partial_n^{\widetilde P}v|_{\widetilde y=-1},f^\flat|_{\widetilde y=-1}).\end{aligned}\tag{3.30}$$

From (3.27)-(3.30) we deduce

$$\begin{aligned}\big([O(\partial_{\widetilde X}),G(a,b)]f,f\big)&=\int_{\mathcal S}[O(\partial_{\widetilde X}),\widetilde P\nabla_{\widetilde X,\widetilde y}]f^\flat\cdot\nabla_{\widetilde X,\widetilde y}f^\flat\\&+\int_{\mathcal S}\widetilde P\nabla_{\widetilde X,\widetilde y}v\cdot\nabla_{\widetilde X,\widetilde y}f^\flat+\int_{\mathcal S}\big((\nabla_{\widetilde X,\widetilde y}O(\partial_{\widetilde X}))\cdot\widetilde P\nabla_{\widetilde X,\widetilde y}f^\flat\big)f^\flat,\end{aligned}$$

from which one obtains easily

$$\big([O(\partial_X),G(a,b)]f,f\big)\le C\big(\|\widetilde P\|_{1,\infty},|O|_{1,\infty}\big)\|f^\flat\|_{1,2}^2+\|\widetilde P\|_\infty\|v\|_{1,2}\|f^\flat\|_{1,2}.\tag{3.31}$$

Multiplying (3.28) by v, integrating by parts and using Poincaré's inequality, one obtains

$$\|v\|_{1,2}\le M(m_0+3/2)C(|O|_{1,\infty})\|f^\flat\|_{1,2},$$

and (3.31) yields therefore

$$\big([O(\partial_X),G(a,b)]f,f\big)\le M(m_0+3/2)C(|O|_{1,\infty})\|f^\flat\|_{1,2}^2,$$

and one concludes the proof with the help of Lemma 2.12. □

With only minor modifications, and using Remark 2.14, the same proof gives:

Proposition 3.19. *Let $m_0=\lceil\frac{d+1}{2}\rceil$ and $T>0$. Let $a\in C^1([0,T],H^{m_0+3/2}(\mathbb R^d))$ and $b\in C_b^\infty(\mathbb R^d)$ satisfy* (3.1) *uniformly for $t\in[0,T]$. Then, for all $f\in H^{1/2}(\mathbb R^d)$, one has*

$$|([\partial_t,G(a,b)]f,f)|\le M(m_0+3/2)C\left(|\partial_t a|_{L^\infty_T}\right)|f|^2_{H^{1/2}},$$

where $M(\cdot)$ is as in Notation 3.5.

3.4. **Shape derivative of the Dirichlet-Neumann operator.** The Dirichlet-Neumann operator is linear but depends nonlinearly on the parameterization a of the surface. It is known that this dependence is smooth, and even analytical [9, 13, 28]. The next theorem gives an explicit expression of its shape derivative, that is, of its derivative with respect to the surface parameterization. In Prop. 3.25 below we give tame estimates on the first and second shape derivatives.

Theorem 3.20. *Let $m_0=\lceil\frac{d+1}{2}\rceil$ and $k\in\mathbb N$, $k\ge m_0$. Suppose that $\underline a\in H^{k+3/2}(\mathbb R^d)$ and $b\in C_b^\infty(\mathbb R^d)$ satisfy* (3.1). *Then there exists a neighborhood $\mathcal U_{\underline a}$ of $\underline a$ in $H^{k+3/2}(\mathbb R^d)$ such that for all given $f\in H^{k+3/2}(\mathbb R^d)$, the mapping*

$$a\in\mathcal U_{\underline a}\subset H^{k+3/2}(\mathbb R^d)\mapsto G(a,b)f\in H^{k+1/2}(\mathbb R^d)$$

is well defined and differentiable. Moreover, for all $h \in H^{k+3/2}(\mathbb{R}^d)$*, one has*

$$d_{\underline{a}} G(\cdot, b) f \cdot h = -G(\underline{a}, b)\,(h\underline{Z}) - \nabla_X \cdot (h\underline{\mathbf{v}}),$$

where

$$\underline{Z} := \frac{1}{N \cdot PN}\Big(G(\underline{a}, b) f - PN \cdot \begin{pmatrix} \nabla_X f \\ 0 \end{pmatrix}\Big),$$

with

$$N := (-\nabla_X \underline{a}, 1)^T \quad \textit{and} \quad \underline{\mathbf{v}} := \nabla_X f - \underline{Z} \nabla_X \underline{a}.$$

Remark 3.21. For the water-waves equations, one has $P = Id_{(d+1)\times(d+1)}$, and $\underline{Z}$ is simply given by $\underline{Z} = \dfrac{1}{1 + |\nabla_X \underline{a}|^2}\,(G(\underline{a}, b) f + \nabla_X \underline{a} \cdot \nabla_X f)$.

Proof. We can choose a neighborhood $\mathcal{U}_{\underline{a}} \subset H^{k+3/2}$ of $\underline{a}$ such that for all $a \in \mathcal{U}_{\underline{a}}$, condition (3.1) is satisfied (taking h_0 smaller if necessary). To each $a \in \mathcal{U}_{\underline{a}}$ it is therefore possible to associate a regularizing diffeomorphism $S_a(X, y) = (X, s_a(X, y))$ as in Prop. 2.13. Taking $\mathcal{U}_{\underline{a}}$ smaller if necessary, and using Remark 2.14, we can assume that the mapping $a \mapsto s_a$ is affine. We denote by $d_{\underline{a}} s$ its derivative at $\underline{a}$. Since the matrix $\widetilde{P}_a$, given by Lemma 2.5 with $s = s_a$, has coefficients in $H^{k+1}(\mathcal{S})$, it follows that the mapping

$$a \in \mathcal{U}_{\underline{a}} \subset H^{k+3/2}(\mathbb{R}^d) \mapsto \widetilde{P}_a \in H^{k+1}(\mathcal{S})^{(d+1)^2}$$

is smooth. We denote by $d_{\underline{a}} \widetilde{P}$ its derivative at $\underline{a}$. Let us also denote by $f_a^\flat$ the solution of the boundary value problem

$$\left\{\begin{array}{ll} -\nabla_{\widetilde{X}, \widetilde{y}} \cdot \widetilde{P}_a \nabla_{\widetilde{X}, \widetilde{y}} f_a^\flat = 0 & \text{in } \mathcal{S}, \\ f_a^\flat|_{\widetilde{y}=0} = f, \qquad \partial_n^{\widetilde{P}_a} f_a^\flat|_{\widetilde{y}=-1} = 0. & \end{array}\right. \tag{3.32}$$

By Theorem 2.9, we know that $f_a^\flat \in H^{k+2}(\mathcal{S})$. It is quite easy to prove that the mapping $\mathcal{B}$ defined as

$$\mathcal{B}:\ a \in \mathcal{U}_{\underline{a}} \subset H^{k+3/2}(\mathbb{R}^d) \mapsto f_a^\flat \in H^{k+2}(\mathcal{S})$$

is continuous. Differentiating (3.32) with respect to a, it is easy to show that $\mathcal{B}$ is differentiable at $\underline{a}$ and that for all $h \in H^{k+3/2}(\mathbb{R}^d)$, $\widetilde{v}_{\underline{a},h} := d_{\underline{a}} \mathcal{B} \cdot h$ solves

$$\left\{\begin{array}{ll} -\nabla_{\widetilde{X}, \widetilde{y}} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X}, \widetilde{y}} \widetilde{v}_{\underline{a},h} = \nabla_{\widetilde{X}, \widetilde{y}} \cdot d_{\underline{a}} \widetilde{P} \cdot h \nabla_{\widetilde{X}, \widetilde{y}} f_a^\flat & \text{in } \mathcal{S}, \\ \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} = 0, \qquad \partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=-1} = -e_{d+1} \cdot \Big(d_{\underline{a}} \widetilde{P} \cdot h \nabla_{\widetilde{X}, \widetilde{y}} f_a^\flat \Big)|_{\widetilde{y}=-1}. & \end{array}\right. \tag{3.33}$$

The following is a key lemma. It gives an explicit function solving (3.33) except for the Dirichlet condition at the surface.

Lemma 3.22. *For all* $h \in H^{k+3/2}(\mathbb{R}^d)$*, the function* $\widetilde{v}^\flat_{\underline{a},h} := \frac{d_{\underline{a}} s \cdot h}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_y f^\flat_{\underline{a}}$ *solves*

$$\left\{\begin{array}{l} -\nabla_{\widetilde{X}, \widetilde{y}} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X}, \widetilde{y}} \widetilde{v}^\flat_{\underline{a},h} = \nabla_{\widetilde{X}, \widetilde{y}} \cdot d_{\underline{a}} \widetilde{P} \cdot h \nabla_{\widetilde{X}, \widetilde{y}} f^\flat_{\underline{a}}, \\ \widetilde{v}^\flat_{\underline{a},h}|_{\widetilde{y}=0} = \frac{h}{\underline{a} - b} \partial_{\widetilde{y}} f^\flat_{\underline{a}}|_{\widetilde{y}=0}, \qquad \partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^\flat_{\underline{a},h} = -e_{d+1} \cdot \Big(d_{\underline{a}} \widetilde{P} \cdot h \nabla_{\widetilde{X}, \widetilde{y}} f^\flat_{\underline{a}} \Big)|_{\widetilde{y}=-1}. \end{array}\right.$$

Remark 3.23. The expression of $\widetilde{v}^\flat_{\underline{a},h}$ given in the above lemma might not seem obvious. We sketch here a way to find it in the case where $P = Id$ and for 1D surfaces. Denote by u_a the solution of the Laplace equation (1.7) in $\Omega_{a,b}$ with Dirichlet condition f at the surface and homogeneous Neumann condition at the bottom. First write in variational form that u_a solves this boundary value problem and then differentiate this variational equality with respect to a using the classical work of Hadamard on shape functionals [20] (see also Lemma 5.1 of [15]). This yields an expression of the derivative of the mapping $a \mapsto u_a$. Pulling this expression back by the regularizing diffeomorphism S yields an expression of the derivative of $\mathcal{B}$ and hence of $\widetilde{v}^\flat_{\underline{a},h}$. The expression given in Lemma 3.22 is just a generalization of this expression found formally in the case of multi-dimensional surface waves.

Proof. Let us compute (writing (X, y) instead of $(\widetilde{X}, \widetilde{y})$),

$$\nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \nabla_{X,y} \widetilde{v}^\flat_{\underline{a},h} = \frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \partial_y \nabla_{X,y} f^\flat_{\underline{a}}$$
$$+ \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \cdot \widetilde{P}_{\underline{a}} \partial_y \nabla_{X,y} f^\flat_{\underline{a}} + \nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} (\partial_y f^\flat_{\underline{a}}) \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big).$$

Using the fact that $\nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}} = 0$, we obtain

$$\nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \nabla_{X,y} \widetilde{v}^\flat_{\underline{a},h} = -\nabla_{X,y} \cdot \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \partial_y \widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}} \big)$$
$$+ \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \cdot \partial_y (\widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}}) + \nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} (\partial_y f^\flat_{\underline{a}}) \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big).$$

Still using the identity $\nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}} = 0$, one can remark that

$$\begin{aligned} \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \cdot \partial_y (\widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}}) &= \partial_y \big(\widetilde{P}_{\underline{a}} \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \cdot \nabla_{X,y} f^\flat_{\underline{a}} \big) \\ &- \nabla_{X,y} \cdot \big(\partial_y \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \widetilde{P}_{\underline{a}} \nabla_{X,y} f^\flat_{\underline{a}} \big), \end{aligned}$$

and therefore, one can write

$$\nabla_{X,y} \cdot \widetilde{P}_{\underline{a}} \nabla_{X,y} \widetilde{v}^\flat_{\underline{a},h} = \nabla_{X,y} \cdot \widetilde{Q}_a \nabla_{X,y} f^\flat_{\underline{a}}, \tag{3.34}$$

where the symmetric matrix $\widetilde{Q}_{\underline{a}}$ is equal to

$$-\partial_y \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \widetilde{P}_{\underline{a}} \big) + \begin{pmatrix} 0_{d+1 \times d} & \widetilde{P}_{\underline{a}} \nabla_{X,y} \left(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \right) \end{pmatrix} + \begin{pmatrix} 0_{d \times d+1} \\ \big(\widetilde{P}_{\underline{a}} \nabla_{X,y} \big(\frac{d_{\underline{a}} s \cdot h}{\partial_y s_{\underline{a}}} \big) \big)^T \end{pmatrix}.$$

We now prove that $\widetilde{Q}_{\underline{a}} = -d_{\underline{a}} \widetilde{P} \cdot h$. In order to do so, let us write the matrix P in the form $P = \begin{pmatrix} P_1 & \mathbf{p} \\ \mathbf{p}^T & p_{d+1} \end{pmatrix}$, where P_1 is a $d \times d$ symmetric matrix, $\mathbf{p} \in \mathbb{R}^d$ and $p_{d+1} \in \mathbb{R}$. The matrix $\widetilde{P}_{\underline{a}}$ given by Lemma 2.5 can therefore be written

$$\widetilde{P}_{\underline{a}} = \begin{pmatrix} \partial_y s_{\underline{a}} P_1 & -P_1 \nabla_X s_{\underline{a}} + \mathbf{p} \\ (-P_1 \nabla_X s_{\underline{a}} + \mathbf{p})^T & \frac{1}{\partial_y s_{\underline{a}}} (\nabla_X s_{\underline{a}} \cdot P_1 \nabla_X s_{\underline{a}} + p_{d+1} - 2\mathbf{p} \cdot \nabla_X s_{\underline{a}}) \end{pmatrix},$$

and it follows that for any $h \in H^{k+3/2}(\mathbb{R}^d)$, the matrix $d_{\underline{a}} \widetilde{P} \cdot h$ is given by

$$\begin{pmatrix} \partial_y (d_{\underline{a}} s \cdot h) P_1 & -P_1 \nabla_X (d_{\underline{a}} s \cdot h) \\ (-P_1 \nabla_X (d_{\underline{a}} s \cdot h))^T & \frac{1}{\partial_y s_{\underline{a}}} \big(2\nabla_X (d_{\underline{a}} s \cdot h) \cdot P_1 \nabla_X s_{\underline{a}} - 2\mathbf{p} \cdot \nabla_X (d_{\underline{a}} s \cdot h) \big) \\ & - \frac{\partial_y (d_{\underline{a}} s \cdot h)}{\partial_y s_{\underline{a}}} \frac{\nabla_X s_{\underline{a}} \cdot P_1 \nabla_X s_{\underline{a}} + p_{d+1} - 2\mathbf{p} \cdot \nabla_X s_{\underline{a}}}{\partial_y s_{\underline{a}}} \end{pmatrix}.$$

It is then easy, though tedious, to check that $\widetilde{Q}_{\underline{a}} = -d_{\underline{a}}\widetilde{P}\cdot h$. From (3.34) we obtain therefore $-\nabla_{X,y}\cdot\widetilde{P}_{\underline{a}}\nabla_{X,y}\widetilde{v}^\flat_{\underline{a},h} = \nabla_{X,y}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f^\flat_{\underline{a}}$, and it remains only to check that $\widetilde{v}^\flat_{\underline{a},h}$ satisfies the boundary conditions to conclude the proof of the lemma.

From Prop. 2.13 and Remark 2.14, one has $d_{\underline{a}}s\cdot h|_{y=0} = h$ and $\partial_y s_{\underline{a}}|_{y=0} = \underline{a}-b$ so that $\widetilde{v}^\flat_{\underline{a},h}$ satisfies the Dirichlet boundary condition stated in the lemma on the upper boundary of the strip $\mathcal{S}$. To check that the Neumann condition of the lower boundary is also satisfied, recall that by definition

$$\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_{y=-1} = e_{d+1}\cdot\big(\widetilde{P}_{\underline{a}}\nabla_{X,y}\big(\frac{d_{\underline{a}}s\cdot h}{\partial_y s_{\underline{a}}}\partial_y f^\flat_{\underline{a}}\big)\big)\big|_{y=-1}.$$

Now, recall that owing to Remark 2.14, one has $d_{\underline{a}}s\cdot h|_{y=-1}=0$, so that

$$\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_{y=-1} = \big(\frac{\partial_y(d_{\underline{a}}s\cdot h)}{(\partial_y s_{\underline{a}})^2}\nabla_X s_{\underline{a}}\cdot P_1\nabla_X s_{\underline{a}} + p_{d+1} - 2\mathbf{p}\cdot\nabla_X s_{\underline{a}}\big)\partial_y f^\flat_{\underline{a}}\big|_{y=-1}.$$

One can check that this latter expression equals $-e_{d+1}\cdot\big(d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f^\flat_{\underline{a}}\big)|_{y=-1}$, which concludes the proof. $\square$

From (3.33) and Lemma 3.22, $\widetilde{v}_{\underline{a},h} - \widetilde{v}^\flat_{\underline{a},h}$ solves

$$\begin{cases} -\nabla_{X,y}\cdot\widetilde{P}_{\underline{a}}\nabla_{X,y}(\widetilde{v}_{\underline{a},h} - \widetilde{v}^\flat_{\underline{a},h}) = 0 \\ (\widetilde{v}_{\underline{a},h} - \widetilde{v}^\flat_{\underline{a},h})|_{y=0} = -\frac{h}{a-b}\partial_y f^\flat_{\underline{a}}|_{y=0}, \qquad \partial_n^{\widetilde{P}_{\underline{a}}}(\widetilde{v}_{\underline{a},h} - \widetilde{v}^\flat_{\underline{a},h})|_{y=-1} = 0; \end{cases}$$

by definition of the DN operator $G(\underline{a},b)$, it follows that $-\partial_n^{\widetilde{P}_{\underline{a}}}(\widetilde{v}_{\underline{a},h} - \widetilde{v}^\flat_{\underline{a},h})|_{y=0} = G(\underline{a},b)\big(-\frac{h}{\underline{a}-b}\partial_y f^\flat_{\underline{a}}|_{y=0}\big)$, or equivalently

$$-\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{y=0} = -\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_{y=0} - G(\underline{a},b)\big(\frac{h}{\underline{a}-b}\partial_y f^\flat_{\underline{a}}|_{y=0}\big). \tag{3.35}$$

To finish the proof, we write $d_{\underline{a}}G(\cdot,b)f\cdot h$ in terms of $-\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{y=0}$.

One has $G(\underline{a},b)f = e_{d+1}\cdot\widetilde{P}_{\underline{a}}\nabla_{X,y}f^\flat_{\underline{a}}|_{y=0}$; hence, using the fact that $\widetilde{v}_{\underline{a},h}$ denotes the derivative of the mapping $a\mapsto f^\flat_a$ at $\underline{a}$ applied to $h\in H^{k+3/2}(\mathbb{R}^d)$,

$$d_{\underline{a}}G(\cdot,b)f\cdot h = e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f^\flat_{\underline{a}}|_{y=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{y=0}. \tag{3.36}$$

Together with (3.35), and using the identity $\partial_y f^\flat_{\underline{a}}|_{y=0} = (a-b)\underline{Z}$, with $\underline{Z}$ as defined in the statement of the theorem, this yields

$$d_{\underline{a}}G(\cdot,b)f\cdot h = e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f^\flat_{\underline{a}}|_{y=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_{y=0} - G(\underline{a},b)\,(h\underline{Z})\,. \tag{3.37}$$

Lemma 3.24. *Under the assumptions and with the notation of the theorem, one has*

$$e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f^\flat_{\underline{a}}|_{y=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_{y=0} = -\begin{pmatrix}\nabla_X\\0\end{pmatrix}\cdot\Big[h\widetilde{P}_{\underline{a}}\begin{pmatrix}\underline{\mathbf{v}}\\ \underline{Z}\end{pmatrix}\Big].$$

Proof. Recall that owing to Prop. 2.13 and Remark 2.14, one has $s_{\underline{a}}|_0 = \underline{a}$, $\partial_y s_{\underline{a}}|_0 = a-b$, $d_{\underline{a}}s\cdot h|_0 = h$ and $\partial_y d_{\underline{a}}s\cdot h|_0 = h$. Using the same notation as in the proof of Lemma 3.22, one obtains

$$\begin{aligned} -\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^\flat_{\underline{a},h}|_0 \quad = \quad & -(P_1\nabla_X\underline{a} - \mathbf{p})\nabla_x h\underline{Z} \\ & + \frac{h}{\underline{a}-b}\left(\nabla_X\underline{a}\cdot P_1\nabla_X\underline{a} + p_{d+1} - 2\mathbf{p}\cdot\nabla_X\underline{a}\right)\underline{Z} \\ & -h\nabla_X\cdot P_1\nabla_X f - h\nabla_X\cdot\left[(-P_1\nabla_X\underline{a} + \mathbf{p})\underline{Z}\right]. \end{aligned}$$

Using the expression of $d_{\underline{a}}\widetilde{P}\cdot h$ given in the proof of Lemma 3.22, we also compute

$$\begin{aligned} e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f_{\underline{a}}^{\flat}|_{y=0} &= -P_1\nabla_X h\cdot\nabla_X f+2(P_1\nabla_X\underline{a}-\mathbf{p})\cdot\nabla_X h\underline{Z} \\ &\quad -\frac{h}{\underline{a}-b}(\nabla_X\underline{a}\cdot P_1\nabla_X\underline{a}+p_{d+1}-2\mathbf{p}\cdot\nabla_X\underline{a})\underline{Z}, \end{aligned}$$

and the lemma follows. ☐

The theorem is then a simple consequence of (3.37) and Lemma 3.24. ☐

Theorem 3.20 is crucial in the symbolic analysis of the linearized water-wave equations. However, one can notice that the explicit expression it gives is not very useful at the time of giving estimates of the shape derivatives. Indeed, both terms of this expression are in $H^{k-1/2}(\mathbb{R}^d)$, while the derivative of the DN operator belongs to $H^{k+1/2}(\mathbb{R}^d)$. This means that there is a cancellation of the most singular components of both terms. Estimates of the shape derivatives have therefore to be done at an upper level.

Proposition 3.25. *Let $m_0=\lceil\frac{d+1}{2}\rceil$ and $k\in\mathbb{N}$, $k\geq m_0$. Suppose that $\underline{a}\in H^{k+3/2}\cap H^{2m_0+1/2}(\mathbb{R}^d)$, $\nabla_X f\in H^{k+1/2}(\mathbb{R}^d)^d$ and $b\in C_b^\infty(\mathbb{R}^d)$ satisfy (3.1). Then the mapping*

$$a\in\mathcal{U}_{\underline{a}}\subset H^{k+3/2}(\mathbb{R}^d)\mapsto G(a,b)f\in H^{k+1/2}(\mathbb{R}^d)$$

is C^∞ and the successive derivatives are "tame":

i. *For all $h\in H^{k+3/2}(\mathbb{R}^d)$, one has*

$$\begin{aligned} |d_{\underline{a}}G(\cdot,b)f\cdot h|_{H^{k+1/2}} &\leq C\left(k,B,|\underline{a}|_{H^{2m_0+1/2}},|\nabla_X f|_{H^{m_0-1/2}}\right) \\ &\quad\times\left(|h|_{H^{k+3/2}}+|h|_{H^{m_0+1/2}}(|\underline{a}|_{H^{k+3/2}}+|\nabla_X f|_{H^{k+1/2}})\right); \end{aligned}$$

ii. *For all $h_1,h_2\in H^{k+3/2}(\mathbb{R}^d)$,*

$$\begin{aligned} \left|d_{\underline{a}}^2G(\cdot,b)f\cdot(h_1,h_2)\right|_{H^{k+1/2}} &\leq C\left(k,B,|\underline{a}|_{H^{2m_0+1/2}},|\nabla_X f|_{H^{m_0-1/2}}\right) \\ &\quad\times\Big(|h_1|_{H^{k+3/2}}|h_2|_{H^{m_0+1/2}}+|h_2|_{H^{k+3/2}}|h_1|_{H^{m_0+1/2}} \\ &\quad+|h_1|_{H^{m_0+1/2}}|h_2|_{H^{m_0+1/2}}(|\underline{a}|_{H^{k+3/2}}+|\nabla_X f|_{H^{k+1/2}})\Big); \end{aligned}$$

iii. *Similar estimates hold for $d_{\underline{a}}^jG(\cdot,b)f$, $j\geq 3$.*

Proof. Recall that if the diffeomorphism s_a is the regularizing diffeomorphism constructed in Prop. 2.13, one has $d_{\underline{a}}s\cdot h=(y+1)\chi(\underline{\lambda}y|D|)h$ for some $\underline{\lambda}>0$ and where χ is the same compactly supported function as in the proof of Lemma 2.13. Therefore, for all $k\geq m_0$, $\|d_{\underline{a}}s\cdot h\|_{k+1,2}\leq \mathrm{Cst}\ |h|_{H^{k+1/2}}$. From the explicit expression of $\widetilde{P}_{\underline{a}}$ given in Lemma 2.5, and with the same computations as for Lemma 2.8, one obtains therefore

$$\|d_{\underline{a}}\widetilde{P}\cdot h\|_{k,2}\leq C\left(B,|\underline{a}|_{H^{m_0+1/2}}\right)\left(|h|_{H^{k+1/2}}+|h|_{H^{m_0+1/2}}|\underline{a}|_{H^{k+1/2}}\right); \tag{3.38}$$

recall also that owing to Theorem 2.9 and Remark 2.10 (with $Q=\widetilde{P}_{\underline{a}}$), the solution $f_{\underline{a}}^\flat$ to (3.32) satisfies for all $k\geq 0$ the tame estimate

(3.39)

$$\|\nabla_{X,y}f_{\underline{a}}^\flat\|_{k+1,2}\leq C(k,B,|\underline{a}|_{H^{2m_0+1/2}})\Big(|\nabla_X f|_{H^{k+1/2}}+|\nabla_X f|_{H^{m_0-1/2}}|\underline{a}|_{H^{k+1/2}}\Big).$$

Now, recall that we saw in (3.36) that

$$d_{\underline{a}}G(\cdot,b)f\cdot h=e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{X,y}f_{\underline{a}}^\flat|_{y=0}-\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{y=0},$$

where $\widetilde{v}_{\underline{a},h}$ solves (3.33). From (3.38) and (3.39), together with Lemma 2.11, it is easy to see that the first term of the r.h.s. satisfies the estimate of the proposition. The estimate on the second term of the r.h.s. is deduced from Theorem 2.9 applied to the boundary value problem (3.33).

Since the methods for obtaining the estimates on higher derivatives of $G(\cdot, b)f$ are absolutely similar, we omit the proof. □

4. The linearized water-waves equations

4.1. **Trigonalization of the linearized system.**

As seen in the introduction, the water-waves equations are

$$\begin{cases} \partial_t\zeta - G(\zeta)\psi = 0 \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{1}{2(1+|\nabla_X\zeta|^2)}\left(G(\zeta)\psi + \nabla_X\zeta\cdot\nabla_X\psi\right)^2 = 0, \end{cases} \tag{4.1}$$

where, for the sake of simplicity, we wrote $G(\zeta)$ instead of $G(\zeta, b)$, b being the parameterization of the bottom.

We can write this system in condensed form as

$$\partial_t U + \mathcal{F}(U) = 0, \tag{4.2}$$

with $U = (\zeta, \psi)^T$ and

$$\mathcal{F}(U) = \Big(- G(\zeta)\psi, g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{\left(G(\zeta)\psi + \nabla_X\zeta\cdot\nabla_X\psi\right)^2}{2(1+|\nabla_X\zeta|^2)}\Big)^T. \tag{4.3}$$

This section is devoted to the study of the linearized water-waves equations around an *admissible* reference state, in the following sense:

Definition 4.1. Let $T > 0$. We say that $\underline{U} = (\underline{\zeta}, \underline{\psi})^T$ is an admissible reference state if $(\underline{\zeta}, \underline{\psi} - \underline{\psi}_{|t=0})^T \in C([0,T]; H^\infty(\mathbb{R}^d)^2)$ and $\nabla_X\underline{\psi}_{|t=0} \in H^\infty(\mathbb{R}^d)^d$, and if moreover

$$\exists h_0 > 0 \quad \text{such that} \quad \min\{-b, \underline{\zeta} - b\} \geq h_0 \text{ on } [0,T]\times\mathbb{R}^d,$$

where we recall that $y = b(X)$ is a parameterization of the bottom.

By definition, the linearized operator $\underline{\mathcal{L}}$ associated to (4.2) is given by $\underline{\mathcal{L}} := \partial_t + d_{\underline{U}}\mathcal{F}$; from the explicit expression of $\mathcal{F}$ given above, one computes

$$d_{\underline{U}}\mathcal{F} = \begin{pmatrix} -d_{\underline{\zeta}}G(\cdot)\underline{\psi} & -G(\underline{\zeta}) \\ -\underline{Z}d_{\underline{\zeta}}G(\cdot)\underline{\psi} - \underline{Z}\underline{\mathbf{v}}\cdot\nabla_X + g & \underline{\mathbf{v}}\cdot\nabla_X - \underline{Z}G(\underline{\zeta}) \end{pmatrix}, \tag{4.4}$$

with $\underline{Z} = Z(\underline{U})$, $\underline{\mathbf{v}} := \mathbf{v}(\underline{U})$ and, for all $U = (\zeta, \psi)^T$ smooth enough,

$$Z(U) := \frac{1}{1+|\nabla_X\zeta|^2}\left(G(\zeta)\psi + \nabla_X\zeta\cdot\nabla_X\psi\right) \tag{4.5}$$

and

$$\mathbf{v}(U) := \nabla_X\psi - Z(U)\nabla_X\zeta. \tag{4.6}$$

According to Theorem 3.20, we have, for all $\zeta \in H^\infty(\mathbb{R}^d)$,

$$d_{\underline{\zeta}}G(\cdot)\underline{\psi}\cdot\zeta = -G(\underline{\zeta})(\underline{Z}\zeta) - \nabla_X\cdot(\zeta\underline{\mathbf{v}}),$$

so that $\underline{\mathcal{L}}$ becomes

$$\underline{\mathcal{L}} = \partial_t + \begin{pmatrix} G(\underline{\zeta})(\underline{Z}\cdot) + \nabla_X\cdot(\cdot\,\underline{\mathbf{v}}) & -G(\underline{\zeta})\cdot \\ \underline{Z}G(\underline{\zeta})(\underline{Z}\cdot) + (g + \underline{Z}\nabla_X\cdot\underline{\mathbf{v}}) & \underline{\mathbf{v}}\cdot\nabla_X\cdot - \underline{Z}G(\underline{\zeta})\cdot \end{pmatrix}.$$

One can check that the principal part of the above operator admits $i\underline{\mathbf{v}}\cdot\xi$ as an eigenvalue of multiplicity two and a nontrivial Jordan block. Taking $V := (\zeta, \psi - \underline{Z}\zeta)^T$ as a new unknown makes this Jordan block appear under its canonical form. Unexpectedly enough, this change of unknowns not only makes trigonal the principal symbol of $d_{\underline{U}}\mathcal{F}$ but also gives an explicit and extremely simple expression of the lower-order terms:

Proposition 4.2. *Let $T > 0$, $\underline{U}$ be an admissible reference state, and $G = (G_1, G_2)^T \in C^2([0,T], H^\infty(\mathbb{R}^d)^2)$.*
The following two assertions are equivalent:
i. *the pair $U = (\zeta, \psi)^T$ solves $\underline{\mathcal{L}}U = G$ on $[0,T]\times\mathbb{R}^d$;*
ii. *the pair $V := (\zeta, \psi - \underline{Z}\zeta)^T$ solves $\underline{\mathcal{M}}V = H$ on $[0,T]\times\mathbb{R}^d$, with*

$$H := \begin{pmatrix} G_1 \\ G_2 - \underline{Z}G_1 \end{pmatrix} \quad \text{and} \quad \underline{\mathcal{M}} := \partial_t + \begin{pmatrix} \nabla_X\cdot(\cdot\underline{\mathbf{v}}) & -G(\underline{\zeta})\cdot \\ \underline{\mathfrak{a}} & \underline{\mathbf{v}}\cdot\nabla_X \end{pmatrix},$$

where $\underline{\mathfrak{a}} := g + \partial_t\underline{Z} + \underline{\mathbf{v}}\cdot\nabla_X\underline{Z}$.

Notation 4.3. For all U smooth enough, we write

$$\mathfrak{a}(U) := g + \partial_t Z(U) + \mathbf{v}(U)\cdot\nabla_X Z(U), \tag{4.7}$$

where $Z(U)$ and $\mathbf{v}(U)$ are as defined in (4.5)-(4.6), so that $\underline{\mathfrak{a}} = \mathfrak{a}(\underline{U})$.

The coefficient $\underline{\mathfrak{a}}$ appearing in the trigonal operator $\underline{\mathcal{M}}$ obviously plays an important role. It is therefore interesting to give it a physical meaning. The pair $(\underline{\zeta}, \underline{\psi})$ being given as in Prop. 4.2, we can define a velocity potential $\underline{\phi}$ by solving the Laplace equation (1.7) in the fluid domain with Dirichlet condition $\underline{\phi} = \underline{\psi}$ at the surface and homogeneous Neumann condition at the bottom. In accordance with (1.10), we introduce the pressure $\underline{P}$ as

$$-\underline{P} = \partial_t\underline{\phi} + \frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2 + gy. \tag{4.8}$$

The following proposition shows that if $(\underline{\zeta}, \underline{\psi})$ solves the water-waves equations (4.1) at some time t_0, then the pressure $\underline{P}$ defined in (4.8) vanishes at the surface and the normal derivative of the pressure at the surface coincides with $-\underline{\mathfrak{a}}$. The condition $\underline{\mathfrak{a}} \geq c_0 > 0$ we shall impose later (see (4.10)) coincides therefore with the traditional Taylor criterion [33, 3, 22, 37] that the interface is not accelerating into the fluid region more rapidly than the normal component of the gravity.

Proposition 4.4. *Let $T > 0$ and $\underline{U}$ be an admissible reference state. If for some $t_0 \in [0,T]$, $\underline{U}$ solves the water-waves equations* (4.1)*, then $\underline{P}$, defined in* (4.8)*, satisfies*

$$P|_{\{y=\zeta(t_0,X)\}} = 0 \quad \text{and} \quad -\partial_{\mathbf{n}_+}\underline{P}|_{\{y=\zeta(t_0,X)\}} = \underline{\mathfrak{a}}(t_0,\cdot).$$

Proof. Let us remark that

$$\begin{aligned} \partial_t\underline{\phi}|_{\{y=\zeta(t,X)\}} &= \partial_t\underline{\psi} - \partial_y\underline{\phi}|_{\{y=\zeta(t,X)\}}\partial_t\underline{\zeta}, \\ \nabla_X\underline{\phi}|_{\{y=\zeta(t,X)\}} &= \nabla_X\underline{\psi} - \partial_y\underline{\phi}|_{\{y=\zeta(t,X)\}}\nabla_X\underline{\zeta}, \\ \partial_y\underline{\phi}|_{\{y=\zeta(t,X)\}} &= \underline{Z}, \end{aligned}$$

where $\underline{Z} = Z(\underline{U})$ is defined in (4.5). It follows therefore from (4.8) that

$$-\underline{P}|_{\{y=\zeta(t,X)\}} = \partial_t\underline{\psi} + g\underline{\zeta} + \frac{1}{2}|\nabla_X\underline{\psi}|^2 \ - \frac{1}{2}\underline{Z}\left(2\partial_t\underline{\zeta} - G(\underline{\zeta})\underline{\psi} + \nabla_X\underline{\psi}\cdot\nabla_X\underline{\zeta}\right).$$

From this expression, one deduces easily that $\underline{P}|_{\{y=\zeta(t_0,X)\}}=0$ if $\underline{U}$ solves (4.1) at time $t=t_0$.

We now prove the second statement of the proposition. One has by definition

$$-\partial_{\mathbf{n}_+}\underline{P}|_{\{y=\zeta(t,X)\}}=-\frac{1}{1+|\nabla_X\underline{\zeta}|^2}\left(-\nabla_X\underline{\zeta}\cdot\nabla_X\underline{P}+\partial_y\underline{P}\right)\big|_{\{y=\zeta(t,X)\}}.$$

At time $t=t_0$, we just saw that $P|_{\{y=\zeta(t_0,X)\}}=0$, from which one deduces easily that $-\partial_{\mathbf{n}_+}\underline{P}|_{\{y=\zeta(t_0,X)\}}=-\partial_y\underline{P}|_{\{y=\zeta(t_0,X)\}}$. Now, from the definition (4.8) of $\underline{P}$, one computes

$$-\partial_y\underline{P}=\partial_t\partial_y\underline{\phi}+\nabla_X\underline{\phi}\cdot\nabla_X\partial_y\underline{\phi}+\partial_y\underline{\phi}\partial_y^2\underline{\phi}+g.$$

Remarking that

$$\begin{array}{rcl}(\partial_t\partial_y\underline{\phi})|_{\{y=\zeta(t,X)\}} &=& \partial_t\underline{Z}-\partial_y^2\underline{\phi}|_{\{y=\zeta(t,X)\}}\partial_t\underline{\zeta},\\ \nabla_X\partial_y\underline{\phi}|_{\{y=\zeta(t,X)\}} &=& \nabla_X\underline{Z}-\partial_y^2\underline{\phi}|_{\{y=\zeta(t,X)\}}\nabla_X\underline{\zeta},\end{array}$$

one obtains finally $-\partial_y\underline{P}|_{\{y=\zeta(t_0,X)\}}=\underline{\mathfrak{a}}(t_0,\cdot)$, which concludes the proof. □

4.2. **Study of the trigonal operator $\underline{\mathcal{M}}$.** Because the principal part of $\underline{\mathcal{M}}$ has a Jordan block, the Cauchy problem

$$\left\{\begin{array}{l}\underline{\mathcal{M}}V=H,\\ V|_{t=0}=V_0\end{array}\right. \tag{4.9}$$

could be either ill- or well-posed. Such situations have been extensively studied for differential systems (see [16] and the references therein for the study of general non-strictly-hyperbolic problems, and [11] for a more related situation), and seem inherent to the water-waves problem [10, 36, 37]: in order to be well-posed, a Lévy condition is needed on the sub-principal symbol of $\underline{\mathcal{M}}$. Since the operator $G(\underline{\zeta})$ is positive, the Lévy condition on $\underline{\mathcal{M}}$ becomes

$$\exists c_0>0\quad\text{such that}\quad \underline{\mathfrak{a}}(t,X)\geq c_0,\qquad\forall(t,X)\in[0,T]\times\mathbb{R}^d, \tag{4.10}$$

where $\underline{\mathfrak{a}}$ is defined in terms of $\underline{U}$ as in Prop. 4.2. The next proposition shows that under this condition, the Cauchy problem associated to the trigonal operator $\underline{\mathcal{M}}$ is well-posed, and that one gets tame estimates on the solution.

Proposition 4.5. *Let $m_0=\lceil\frac{d+1}{2}\rceil$, $T>0$ and let $\underline{U}$ be an admissible reference state. Also let $H\in C([0,T]\times H(\mathbb{R}^d)^2)$ and $V_0\in H^\infty(\mathbb{R}^d)^2$. Then there is a unique solution $V\in C^1([0,T],H^\infty(\mathbb{R}^d)^2)$ to (4.9) and for all $k\in\mathbb{N}$, there exist $\underline{\kappa}_k$, $\underline{\nu}_k$ such that*

(4.11)

$$\begin{array}{l}|V(t)|_{H^{k+1/2}\times H^{k+1}}\leq\underline{\kappa}_k e^{\underline{\nu}_k t}|V_0|_{H^{k+1/2}\times H^{k+1}}\\ \quad+\quad \underline{\kappa}_k\displaystyle\int_0^t e^{\underline{\nu}_k(t-t')}|H(t')|_{H^{k+1/2}\times H^{k+1}}dt'\\ \quad+\quad \underline{\kappa}_k\displaystyle\int_0^t e^{\underline{\nu}_k(t-t')}\left(|\underline{\zeta}|_{H^{k+3/2}}+|\underline{\mathbf{v}}|_{H^{k+2}}+|\underline{\mathfrak{a}}-g|_{H^{k+1}}\right)|V|_{H^{m_0+3/2}\times H^{m_0+2}}dt'.\end{array}$$

The constants $\underline{\kappa}_k$, $\underline{\nu}_k$ depend on b and $\underline{U}$ through

$$\begin{array}{rcl}\underline{\kappa}_k &=& C\big(k,B,|\underline{\mathfrak{a}}-g|_{H_T^{m_0-1/2}},|\underline{\zeta}|_{H_T^{m_0+1/2}}\big),\\ \underline{\nu}_k &=& C\big(k,B,|\underline{\mathfrak{a}}-g|_{H_T^{1/2+m_0}},|\underline{\mathbf{v}}|_{H_T^{m_0+3/2}},|\underline{\zeta}|_{H_T^{q_0+1/2}},|\partial_t\underline{\mathfrak{a}}|_{H_T^{m_0-1/2}},|\partial_t\underline{\zeta}|_{H_T^{m_0-1/2}}\big),\end{array}$$

where q_0 is an integer depending only on d, and B is as in Notation 3.5.

4.2.1. *Proof of Prop.* 4.5. As is often the case for equations similar to (4.9) (see e.g. [36, 37]), we first consider a parabolic regularization of (4.9):

$$\left\{\begin{array}{l}\underline{\mathcal{M}}_\varepsilon V = H\\ V|_{t=0} = V_0,\end{array}\right. \qquad \text{with} \quad \underline{\mathcal{M}}_\varepsilon := \underline{\mathcal{M}} + \begin{pmatrix} 0 & -\varepsilon^2\Lambda^2\\ 0 & 0\end{pmatrix}. \tag{4.12}$$

Even for (4.12), well-posedness is not straightforward. As in [36, 37], we choose to use an iterative scheme to prove it. Let us first introduce the notation

$$\nabla_{\underline{\mathbf{v}}} f := \frac{1}{2}(\nabla_X\cdot(f\underline{\mathbf{v}}) + \underline{\mathbf{v}}\cdot\nabla_X f), \qquad \forall f\in\mathcal{S}(\mathbb{R}^d), \tag{4.13}$$

and

$$A_\varepsilon := \begin{pmatrix} \nabla_{\underline{\mathbf{v}}} & -\varepsilon^2\Lambda\\ \underline{\mathfrak{a}} & \nabla_{\underline{\mathbf{v}}}\end{pmatrix}, \qquad A := \begin{pmatrix} \frac{1}{2}\nabla_X\cdot\underline{\mathbf{v}} & -G(\underline{\zeta})\\ 0 & -\frac{1}{2}\nabla_X\cdot\underline{\mathbf{v}}\end{pmatrix},$$

so that $\underline{\mathcal{M}}_\varepsilon = \partial_t + A_\varepsilon + A$.

We seek a solution of (4.12) as a limit of the sequence $(V^n)_n$ defined for all $n\in\mathbb{N}$ as

$$\left\{\begin{array}{l}\partial_t + A_\varepsilon)V^{n+1} = H - AV^n,\\ V^{n+1}|_{t=0} = V_0,\end{array}\right. \qquad \text{and} \quad V^0 = V_0. \tag{4.14}$$

Well-posedness of Cauchy problems of type (4.14) is ensured by the next lemma.

Lemma 4.6. *Let $T>0$, $\underline{U}$ be an admissible reference state, and also let $H = (H_1,H_2)^T\in C([0,T],H^\infty(\mathbb{R}^d)^2)$ and $V_0\in H^\infty(\mathbb{R}^d)^2$. For all $\varepsilon\in(0,1)$, the Cauchy problem*

$$\left\{\begin{array}{l}(\partial_t + A_\varepsilon)V = H,\\ V|_{t=0} = V_0\end{array}\right.$$

admits a unique solution $V\in C^1([0,T],H^\infty(\mathbb{R}^d)^2)$. Moreover, for all $s\in\mathbb{R}$ there exist $\lambda_s = \lambda_s(\varepsilon,\underline{U})$ and $C_s = C(s,\varepsilon,\underline{U})$ such that

$$|V|_{H^s\times H^{s+1}} \leq C_s\left(e^{\lambda_s t}|V_0|_{H^s\times H^{s+1}} + \int_0^t e^{\lambda_s(t-t')}|H(t')|_{H^s\times H^{s+1}}dt'\right).$$

Proof. In order to perform energy estimates on the equation, we seek a change of unknowns which symmetrizes the operator A_ε. Let $S_\varepsilon = \begin{pmatrix}\sqrt{\underline{\mathfrak{a}}} & 0\\ 0 & \varepsilon\Lambda\end{pmatrix}$; one has $S_\varepsilon^{-1} = \begin{pmatrix}\frac{1}{\sqrt{\underline{\mathfrak{a}}}} & 0\\ 0 & \frac{1}{\varepsilon}\Lambda^{-1}\end{pmatrix}$ (note the importance here of the Lévy condition (4.10)). The operator S_ε is a symmetrizer of A_ε in the sense that $S_\varepsilon A_\varepsilon S_\varepsilon^{-1} = A_\varepsilon^1 + A_\varepsilon^0$, with

$$A_\varepsilon^1 = \begin{pmatrix}\nabla_{\underline{\mathbf{v}}} & -\varepsilon\sqrt{\underline{\mathfrak{a}}}\Lambda\\ \varepsilon\Lambda(\sqrt{\underline{\mathfrak{a}}}\cdot) & \nabla_{\underline{\mathbf{v}}}\end{pmatrix} \qquad \text{and} \qquad A_\varepsilon^0 = \begin{pmatrix}\sqrt{\underline{\mathfrak{a}}}[\nabla_{\underline{\mathbf{v}}},\frac{1}{\sqrt{\underline{\mathfrak{a}}}}] & 0\\ 0 & \Lambda[\nabla_{\underline{\mathbf{v}}},\Lambda^{-1}]\end{pmatrix};$$

that is, the principal part A_ε^1 of $S_\varepsilon A_\varepsilon S_\varepsilon^{-1}$ is an anti-adjoint operator of order one. The natural energy $E_{s,\varepsilon}$ associated to the equation is therefore defined as $E_{s,\varepsilon}(V) := \big|S_\varepsilon\Lambda^s V\big|_2^2 = \big(\Lambda^s V, S_\varepsilon^2\Lambda^s V\big)$. As usual, one computes

$$\begin{aligned}\frac{d}{dt}e^{-2\lambda t}E_{s,\varepsilon}(V) &= -2\lambda e^{-2\lambda t}E_{s,\varepsilon}(V) + 2e^{-2\lambda t}\big(\Lambda^s(\partial_t + A_\varepsilon)V, S_\varepsilon^2\Lambda^s V\big)\\ &- 2e^{-2\lambda t}\big(\Lambda^s A_\varepsilon V, S_\varepsilon^2\Lambda^s V\big) + e^{-2\lambda t}\big(\Lambda^s V,[\partial_t,S_\varepsilon^2]\Lambda^s V\big).\end{aligned} \tag{4.15}$$

Estimate of $\big(\mathbf{\Lambda^s(\partial_t + A_\varepsilon)V, S_\varepsilon^2\Lambda^s V}\big)$. By Cauchy-Schwartz and then Hölder's inequality, one obtains easily

$$\big|\big(\Lambda^s(\partial_t + A_\varepsilon)V, S_\varepsilon^2\Lambda^s V\big)\big| \leq \frac{1}{2}E_{s,\varepsilon}(V) + \frac{1}{2}E_{s,\varepsilon}((\partial_t + A_\varepsilon)V). \tag{4.16}$$

Estimate of $\big(\mathbf{\Lambda^s A_\varepsilon V, S_\varepsilon^2\Lambda^s V}\big)$. One has

$$\big(\Lambda^s A_\varepsilon V, S_\varepsilon^2\Lambda^s V\big) = (S_\varepsilon\Lambda^s A_\varepsilon\Lambda^{-s}S_\varepsilon^{-1}S_\varepsilon\Lambda^s V, S_\varepsilon\Lambda^s V),$$

and since the principal symbols of $S_\varepsilon\Lambda^s A_\varepsilon\Lambda^{-s}S_\varepsilon^{-1}$ and $S_\varepsilon A_\varepsilon S_\varepsilon^{-1}$ are the same, we deduce from the decomposition $S_\varepsilon A_\varepsilon S_\varepsilon^{-1} = A_\varepsilon^1 + A_\varepsilon^0$ above that the operator $S_\varepsilon\Lambda^s A_\varepsilon\Lambda^{-s}S_\varepsilon^{-1}$ is of order one with skew-symmetric principal symbol. Classical results of pseudo-differential calculus yield therefore

$$\begin{aligned}\big|\big(\Lambda^s A_\varepsilon V, S_\varepsilon^2\Lambda^s V\big)\big| &\leq C\,(\varepsilon, s, \underline{U})\,|S_\varepsilon\Lambda^s V|_2^2\\ &= C\,(\varepsilon, s, \underline{U})\,E_{s,\varepsilon}(V).\end{aligned} \tag{4.17}$$

Estimate of $\big(\mathbf{\Lambda^s V, [\partial_t, S_\varepsilon^2]\Lambda^s V}\big)$. Since $[\partial_t, S_\varepsilon^2] = \begin{pmatrix}\partial_t\underline{\mathfrak{a}} & 0\\ 0 & 0\end{pmatrix}$, one obtains easily $\big(\Lambda^s V, [\partial_t, S_\varepsilon^2]\Lambda^s V\big) = (\Lambda^s V_1, \partial_t\underline{\mathfrak{a}}\Lambda^s V_1)$ and thus

$$\big|\,\big(\Lambda^s V, [\partial_t, S_\varepsilon^2]\Lambda^s V\big)\,\big| \leq C\big(\underline{U}\big)E_{s,\varepsilon}(V). \tag{4.18}$$

Endgame. Using (4.15), (4.16), (4.17) and (4.18) one obtains

$$\begin{aligned}\frac{d}{dt}e^{-2\lambda t}E_{s,\varepsilon}(V) &\leq e^{-2\lambda t}\,(1 + C(s, \varepsilon, \underline{U}) - 2\lambda)\,E_{s,\varepsilon}(V)\\ &+ e^{-2\lambda t}E_{s,\varepsilon}((\partial_t + A_\varepsilon)V).\end{aligned}$$

For λ large enough (in order for the prefactor of $E_{s,\varepsilon}$ to be negative in the r.h.s. of the inequality above),we have therefore

$$E_{s,\varepsilon}(V(t)) \leq e^{2\lambda t}E_{s,\varepsilon}(V_0) + \int_0^t e^{2\lambda(t-t')}E_{s,\varepsilon}\big((\partial_t + A_\varepsilon)V(t')\big)dt'. \tag{4.19}$$

Now, remark that $\frac{1}{\kappa}|V|^2_{H^s\times H^{s+1}} \leq E_{s,\varepsilon}(V) \leq \kappa|V|^2_{H^s\times H^{s+1}}$, for some constant κ depending on ε, s, $\frac{1}{c_0}$ and $\underline{U}$. Equation (4.19) gives therefore the desired energy estimate in the $H^s\times H^{s+1}$-norm, and it is routine to conclude the proof by classical duality arguments. ☐

Owing to this lemma, we have the following estimate for (4.14):

$$\big|V^{n+1}\big|_{H^s\times H^{s+1}} \leq C_s \times \big(e^{\lambda_s t}|V_0|_{H^s\times H^{s+1}} + \int_0^t e^{\lambda_s(t-t')}|H - AV^n|_{H^s\times H^{s+1}}dt'\big).$$

From the definition of A, one obtains easily, for all $s > d/2$,

$$|AV^n|_{H^s\times H^{s+1}} \leq C\,(B, \underline{U})\,|V^n|_{H^s\times H^{s+1}},$$

so that one has finally, for all $s > d/2$,

$$\begin{aligned}&\big|V^{n+1}\big|_{H^s\times H^{s+1}} \leq C\,(s, \varepsilon, B, \underline{U})\\ &\times\big(e^{\lambda_s t}|V_0|_{H^s\times H^{s+1}} + \int_0^t e^{\lambda_s(t-t')}\,(|H(t')|_{H^s\times H^{s+1}} + |V^n(t')|_{H^s\times H^{s+1}})\,dt'\big).\end{aligned}$$

Proving the convergence of the iterative scheme (4.14) is then classical. We have therefore:

Lemma 4.7. *Let $T>0$ and $\underline{U}$ be an admissible reference state satisfying* (4.10)*. Also let $H=(H_1,H_2)^T\in C([0,T],H^\infty(\mathbb{R}^d)^2)$ and $V_0\in H^\infty(\mathbb{R}^d)^2$. Then, for all $\varepsilon\in(0,1)$, there exists a unique solution $V\in C^1([0,T],H^\infty(\mathbb{R}^d)^2)$ to* (4.12).

We now turn to give precise energy estimates on the solution V to (4.12) given by Lemma 4.7.

Let us denote by $\underline{M}_\varepsilon$ the spatial part of the operator $\underline{\mathcal{M}}_\varepsilon$, so that $\underline{\mathcal{M}}_\varepsilon=\partial_t+\underline{M}_\varepsilon$, and decompose it as $\underline{M}_\varepsilon=\underline{M}_{1,\varepsilon,\mu}+\underline{M}_{0,\mu}$ with

$$\underline{M}_{1,\varepsilon,\mu}=\begin{pmatrix}\nabla_{\underline{\mathbf{v}}} & -\underline{G}_{\varepsilon,\mu}\\ \underline{\mathfrak{a}} & \nabla_{\underline{\mathbf{v}}}\end{pmatrix}\quad\text{and}\quad \underline{M}_{0,\mu}=\begin{pmatrix}\frac12\nabla_X\cdot\underline{\mathbf{v}} & \mu\\ 0 & -\frac12\nabla_X\cdot\underline{\mathbf{v}}\end{pmatrix},$$

where $\underline{G}_{\varepsilon,\mu}:=G(\underline{\zeta})+\varepsilon^2\Lambda^2+\mu$ and μ is some real positive constant (which we add here because we will need the operator $G(\underline{\zeta})+\mu$ to control the $H^{1/2}$-norm as in Prop. 3.8).

As in the proof of Lemma 4.6, the strategy consists of symmetrizing the principal part of the operator, namely, $\underline{M}_{1,\varepsilon,\mu}$. The operator $S_{\varepsilon,\mu}$ which symmetrizes $\underline{M}_{1,\varepsilon,\mu}$ is given here by

$$S_{\varepsilon,\mu}=\begin{pmatrix}\sqrt{\underline{\mathfrak{a}}} & 0\\ 0 & \underline{G}^{1/2}_{\varepsilon,\mu}\end{pmatrix},$$

where $\underline{G}^{1/2}_{\varepsilon,\mu}$ denotes the square root of the operator $\underline{G}_{\varepsilon,\mu}$. The natural energy to consider here is therefore

$$\mathcal{E}_{s,\varepsilon,\mu}(V)=(\Lambda^sV,S^2_{\varepsilon,\mu}\Lambda^sV)\quad\text{with}\quad S^2_{\varepsilon,\mu}=\begin{pmatrix}\underline{\mathfrak{a}} & 0\\ 0 & \underline{G}_{\varepsilon,\mu}\end{pmatrix}.\tag{4.20}$$

In fact, we do not work directly with all $s\in\mathbb{R}$: the estimates of Theorem 2.9 show that it is convenient to work with Sobolev spaces $H^{k+1/2}(\mathbb{R}^d)$, $k\in\mathbb{N}$. Instead of taking $s=k+1/2$ in the definition above, we change it slightly as

$$\mathcal{E}_{k+1/2,\varepsilon,\mu}(V)=\sum_{\alpha\in\mathbb{N}^d,|\alpha|\le k}\big(\Lambda^{1/2}\partial^\alpha V,S^2_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V\big);\tag{4.21}$$

when $\varepsilon=0$, we write simply $\mathcal{E}_{k+1/2,\mu}$ instead of $\mathcal{E}_{k+1/2,0,\mu}$. The link between spaces of finite energy for (4.21) and Sobolev spaces is made in the next lemma.

Lemma 4.8. *Let $T>0$ and $\underline{U}$ be a reference state satisfying* (4.10)*. Then there exists $\underline{\mu}>0$ such that for all $V\in H^\infty(\mathbb{R}^d)^2$ and $k\in\mathbb{N}$,*

$$\frac{1}{\underline{\kappa}_k}|V|^2_{H^{k+1/2}\times H^{k+1}}\le\mathcal{E}_{k+1/2,\varepsilon,\underline{\mu}}(V)-\varepsilon^2|V_2|^2_{H^{k+3/2}}\le\underline{\kappa}_k|V|^2_{H^{k+1/2}\times H^{k+1}},$$

where $\underline{\kappa}_k$ is as in the statement of Prop. 4.5.

Notation 4.9. From now on, we always take $\mu=\underline{\mu}$ and write simply $\mathcal{E}_{k+1/2,\varepsilon}$ instead of $\mathcal{E}_{k+1/2,\varepsilon,\underline{\mu}}$.

Proof. For all $\alpha\in\mathbb{N}^d$, $|\alpha|\le k$, write $\big(\Lambda^{1/2}\partial^\alpha V,S^2_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V\big)=A_1+A_2$ with

$$A_{1,\alpha}=\big(\Lambda^{1/2}\partial^\alpha V_1,\underline{\mathfrak{a}}\Lambda^{1/2}\partial^\alpha V_1\big),\qquad A_{2,\alpha}=\big(\Lambda^{1/2}\partial^\alpha V_2,\underline{G}_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V_2\big).\tag{4.22}$$

Upper and lower bounds for A_1 are easy to find:

$$c_0|V_1|_{H^{k+1/2}}^2 \leq \sum_{\alpha\in\mathbb{N}^d,|\alpha|\leq k} A_{1,\alpha} \leq (g+|\underline{\mathfrak{a}}-g|_\infty)|V_1|_{H^{k+1/2}}^2. \tag{4.23}$$

Remark now that $A_{2,\alpha} = \big(\Lambda^{1/2}\partial^\alpha V_2, (G(\underline{\zeta})+\mu)\Lambda^{1/2}\partial^\alpha V_2\big) + \varepsilon^2|\Lambda^{3/2}\partial^\alpha V_2|_2^2$, so that using Prop. 3.8 (and assuming that μ is large enough), one obtains

$$\frac{1}{\underline{C}}|V_2|_{H^{k+1}}^2 \leq \sum_{\alpha\in\mathbb{N}^d,|\alpha|\leq k} A_{2,\alpha} - \varepsilon^2|V_2|_{H^{k+3/2}}^2 \leq \underline{C}|V_2|_{H^{k+1}}^2, \tag{4.24}$$

where $\underline{C} = C\left(B,\mu,|\underline{\zeta}|_{H^{m_0+1/2}}\right)$, and B is as in Notation 3.5.

The lemma follows therefore from (4.21) and (4.22)-(4.24). $\square$

Before addressing the heart of the proof, let us recall some useful nonlinear estimates.

Lemma 4.10. *Let $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^d$ such that $|\alpha| = k$. Let $a\in H^\infty(\mathbb{R}^d)$ and $\mathbf{v}\in H^\infty(\mathbb{R}^d)^d$ and define $\nabla_{\mathbf{v}}$ as in* (4.13)*. Then:*
i. *For all $s\geq 0$ and $u\in H^\infty(\mathbb{R}^d)$, one has*

$$\big|[\Lambda^{1/2}\partial^\alpha, a]u\big|_{H^s} \leq \mathrm{Cst}\ \left(|a|_{1,\infty}|u|_{H^{k-1/2+s}} + |a|_{H^{k+1/2+s}}|u|_\infty\right).$$

ii. *For all $s\geq 0$, and $u\in H^\infty(\mathbb{R}^d)$, one has*

$$\big|[\Lambda^{1/2}\partial^\alpha, \nabla_{\mathbf{v}}]u\big|_{H^s} \leq \mathrm{Cst}\ \left(|\mathbf{v}|_{2,\infty}|u|_{H^{k+1/2+s}} + |\mathbf{v}|_{H^{k+3/2+s}}|u|_{1,\infty}\right).$$

Proof. The first point of the lemma is the classical Kato-Ponce estimate [24]. The second one is a consequence of this estimate since one has

$$[\Lambda^{1/2}\partial^\alpha, \nabla_{\mathbf{v}}]u = [\Lambda^{1/2}\partial^\alpha, \mathbf{v}]\cdot\nabla_X u + \frac{1}{2}[\Lambda^{1/2}\partial^\alpha, \nabla_X\cdot\mathbf{v}]u. \qquad\square$$

Lemma 4.11. *Let $T>0$ and $\underline{U}$ be an admissible reference state satisfying* (4.10)*. Then, for all $k\in\mathbb{N}$, the solution V to* (4.12) *satisfies*

$$\begin{aligned}
\mathcal{E}_{k+1/2,\varepsilon}(V(t)) &\leq e^{2\underline{\nu}_k t}\mathcal{E}_{k+1/2,\varepsilon}(V_0) + \int_0^t e^{2\underline{\nu}_k(t-t')}\mathcal{E}_{k+1/2,\varepsilon}(H(t'))dt'\\
&+ \int_0^t e^{2\underline{\nu}_k(t-t')}\left(|\underline{\zeta}(t')|_{H^{k+3/2}}^2 + |\underline{\mathbf{v}}(t')|_{H^{k+2}}^2 + |\underline{\mathfrak{a}}(t')-g|_{H^{k+1}}^2\right)\mathcal{E}_{m_0+3/2}(V(t'))dt'\\
&+ \varepsilon^2\int_0^t e^{2\underline{\nu}_k(t-t')}\left(|\underline{\mathfrak{a}}(t')-g|_{H^{k+3/2}}^2 + |\underline{\mathbf{v}}(t')|_{H^{k+5/2}}^2\right)\mathcal{E}_{m_0+3/2}(V(t'))dt',
\end{aligned}$$

where the constant ν_k is as in the statement of Prop. 4.5.

Proof. Throughout this proof, we write $s_0 := m_0 - 1/2 > d/2$. We proceed as in the proof of Lemma 4.6. One computes

$$\begin{aligned}\frac{d}{dt}e^{-2\nu t}\mathcal{E}_{k+1/2,\varepsilon}(V) &= -2\nu e^{-2\nu t}\mathcal{E}_{k+1/2,\varepsilon}(V)\\ &+ 2e^{-2\nu t}\sum_\alpha \big(\Lambda^{1/2}\partial^\alpha(\partial_t+\underline{M}_\varepsilon)V, S^2_{\varepsilon,\underline{\mu}}\Lambda^{1/2}\partial^\alpha V\big)\\ &- 2e^{-2\nu t}\sum_\alpha \big(\Lambda^{1/2}\partial^\alpha \underline{M}_{1,\varepsilon,\underline{\mu}}V, S^2_{\varepsilon,\underline{\mu}}\Lambda^{1/2}\partial^\alpha V\big)\\ &- 2e^{-2\nu t}\sum_\alpha \big(\Lambda^{1/2}\partial^\alpha \underline{M}_{0,\underline{\mu}}V, S^2_{\varepsilon,\underline{\mu}}\Lambda^{1/2}\partial^\alpha V\big)\\ &+ e^{-2\nu t}\sum_\alpha \big(\Lambda^{1/2}\partial^\alpha V, [\partial_t, S^2_{\varepsilon,\underline{\mu}}]\Lambda^{1/2}\partial^\alpha V\big),\end{aligned}\tag{4.25}$$

where the sums are taken over all $\alpha\in\mathbb{N}^d$, $|\alpha|\leq k$.

Estimate of $\big(\mathbf{\Lambda^{1/2}}\partial^\alpha\underline{\mathbf{M}}_{\mathbf{1},\varepsilon,\underline{\mu}}\mathbf{V},\mathbf{S}^{\mathbf{2}}_{\varepsilon,\underline{\mu}}\mathbf{\Lambda^{1/2}}\partial^\alpha\mathbf{V}\big)$. From the definitions of $\underline{M}_{1,\varepsilon,\underline{\mu}}$ and $S_{\varepsilon,\underline{\mu}}$, one computes easily

$$\begin{aligned}\big(\Lambda^{1/2}\partial^\alpha\underline{M}_{1,\varepsilon,\underline{\mu}}V, S^2_{\varepsilon,\underline{\mu}}\Lambda^{1/2}\partial^\alpha V\big) &= \big(\Lambda^{1/2}\partial^\alpha(\nabla_{\underline{\mathbf{v}}}V_1), \underline{\mathfrak{a}}\Lambda^{1/2}\partial^\alpha V_1\big)\\ &+ \big(\Lambda^{1/2}\partial^\alpha(\nabla_{\underline{\mathbf{v}}}V_2), \underline{G}_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V_2\big)\\ &+ \big[\big(\Lambda^{1/2}\partial^\alpha(\underline{\mathfrak{a}}V_1), \underline{G}_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V_2\big) - \big(\Lambda^{1/2}\partial^\alpha\underline{G}_{\varepsilon,\mu}V_2, \underline{\mathfrak{a}}\Lambda^{1/2}\partial^\alpha V_1\big)\big]\\ &:= I_1+I_2+I_3.\end{aligned}\tag{4.26}$$

• Estimate of I_1. Using the fact that the operator $\nabla_{\underline{\mathbf{v}}}$ is anti-adjoint, one finds

$$I_1 = -\big(\underline{\mathfrak{a}}\Lambda^{1/2}\partial^\alpha V_1, [\nabla_{\underline{\mathbf{v}}}, \Lambda^{1/2}\partial^\alpha]V_1\big) - \frac{1}{2}\big(\Lambda^{1/2}\partial^\alpha V_1, [\nabla_{\underline{\mathbf{v}}}, \underline{\mathfrak{a}}]\Lambda^{1/2}\partial^\alpha V_1\big).$$

Using Lemma 4.10 one can control the first term of the r.h.s. and remarking that $[\nabla_{\underline{\mathbf{v}}}, \underline{\mathfrak{a}}] = \underline{\mathbf{v}}\cdot\nabla_X\underline{\mathfrak{a}}$, one can control also the second one:

$$\begin{aligned}|I_1| &\leq \text{Cst } |\underline{\mathfrak{a}}\Lambda^{1/2}\partial^\alpha V_1|_2\left(|\underline{\mathbf{v}}|_{H^{s_0+2}}|V_1|_{H^{k+1/2}} + |\underline{\mathbf{v}}|_{H^{k+3/2}}|V_1|_{H^{1+s_0}}\right)\\ &+ \text{Cst } |\underline{\mathbf{v}}|_{H^{s_0}}|\underline{\mathfrak{a}}-g|_{H^{s_0+1}}|V_1|^2_{H^{k+1/2}}.\end{aligned}$$

Using Hölder's inequality and Lemma 4.8, one obtains therefore

$$|I_1| \leq \underline{D}_k\mathcal{E}_{k+1/2,\varepsilon}(V) + \mathcal{E}_{m_0+1/2}(V)|\underline{\mathbf{v}}|^2_{H^{k+3/2}},\tag{4.27}$$

where, throughout this proof, $\underline{D}_k$ is a positive constant which depends on the same parameters as $\underline{\nu}_k$ in the statement of Prop. 4.5.

• Estimate of I_2. Using the fact that the operators $\nabla_{\underline{\mathbf{v}}}$ and $\underline{G}_{\varepsilon,\mu}$ are respectively anti- and self-adjoint, one computes

$$\begin{aligned}I_2 &= \big(\underline{G}_{\varepsilon,\mu}\Lambda^{1/2}\partial^\alpha V_2, [\Lambda^{1/2}\partial^\alpha, \nabla_{\underline{\mathbf{v}}}]V_2\big) - \frac{1}{2}\big(\Lambda^{1/2}\partial^\alpha V_2, [\nabla_{\underline{\mathbf{v}}}, \underline{G}_{\varepsilon,\mu}]\Lambda^{1/2}\partial^\alpha V_2\big)\\ &:= I_{21}+I_{22}.\end{aligned}\tag{4.28}$$

By Prop. 3.8 we have

$$\begin{aligned}|I_{21}| &\leq C\big(B,\underline{\mu},|\underline{\zeta}|_{H^{m_0+1/2}}\big)|V_2|_{H^{k+1}}\big|[\Lambda^{1/2}\partial^\alpha,\nabla_{\underline{\mathbf{v}}}]V_2\big|_{H^{1/2}}\\ &+ \varepsilon^2|V_2|_{H^{k+3/2}}\big|[\Lambda^{1/2}\partial^\alpha,\nabla_{\underline{\mathbf{v}}}]V_2\big|_{H^1};\end{aligned}$$

we then use Lemmas 4.8 and 4.10, as well as Hölder's inequality to find

$$|I_{21}| \leq \underline{D}_k\mathcal{E}_{k+1/2,\varepsilon}(V) + \mathcal{E}_{m_0+1/2}(V)\left(|\underline{\mathbf{v}}|^2_{H^{k+2}} + \varepsilon^2|\underline{\mathbf{v}}|^2_{H^{k+5/2}}\right).\tag{4.29}$$

To control I_{22}, one uses successively Prop. 3.18 and Lemma 4.8 to find

$$|I_{22}| \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V). \tag{4.30}$$

From (4.28), (4.29) and (4.30), we obtain finally

$$|I_2| \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) + \mathcal{E}_{m_0+1/2}(V) \left(|\underline{\mathbf{v}}|^2_{H^{k+2}} + \varepsilon^2 |\underline{\mathbf{v}}|^2_{H^{k+5/2}} \right). \tag{4.31}$$

• Estimate of I_3. One has

$$\begin{aligned} I_3 &= \left(\underline{\mathfrak{a}} \Lambda^{1/2} \partial^\alpha V_1, [\underline{G}_{\varepsilon,\mu}, \Lambda^{1/2}\partial^\alpha] V_2 \right) + \left([\Lambda^{1/2}\partial^\alpha, \underline{\mathfrak{a}}] V_1, \underline{G}_{\varepsilon,\mu} \Lambda^{1/2} \partial^\alpha V_2 \right) \\ &:= I_{31} + I_{32}. \end{aligned} \tag{4.32}$$

Using the Cauchy-Schwartz inequality and Prop. 3.15, we obtain

$$|I_{31}| \leq |\underline{\mathfrak{a}} \Lambda^{1/2} \partial^\alpha V_1|_2 M(q_0 + 1/2) \left(|V_2|_{H^{k+1/2}} + |\underline{\zeta}|_{H^{k+3/2}} |V_2|_{H^{m_0+3/2}} \right),$$

where q_0 is the same as in Prop. 3.15.

It is then easy to deduce that

$$|I_{31}| \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) + |\underline{\zeta}|^2_{H^{k+3/2}} \mathcal{E}_{m_0+3/2}(V). \tag{4.33}$$

For I_{32}, we proceed as for I_{21} and find

$$|I_{32}| \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) + \left(|\underline{\mathfrak{a}} - g|^2_{H^{k+1}} + \varepsilon^2 |\underline{\mathfrak{a}} - g|^2_{H^{k+3/2}} \right) \mathcal{E}_{m_0+1/2}(V). \tag{4.34}$$

From (4.32), (4.33) and (4.34), we have therefore

$$|I_3| \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) + \left(|\underline{\zeta}|^2_{H^{k+3/2}} + |\underline{\mathfrak{a}} - g|^2_{H^{k+1}} + \varepsilon^2 |\underline{\mathfrak{a}} - g|^2_{H^{k+3/2}} \right) \mathcal{E}_{m_0+3/2}(V). \tag{4.35}$$

Finally, from (4.26), (4.27), (4.31) and (4.35) one obtains the estimate:

$$\begin{aligned} \sum_{|\alpha| \leq k} \left(\Lambda^{1/2} \partial^\alpha \underline{M}_{1,\varepsilon,\underline{\mu}} V, S^2_{\varepsilon,\underline{\mu}} \Lambda^{1/2} \partial^\alpha V \right) &\leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) \\ &+ \left(|\underline{\zeta}|^2_{H^{k+3/2}} + |\underline{\mathbf{v}}|^2_{H^{k+2}} + |\underline{\mathfrak{a}} - g|^2_{H^{k+1}} \right) \mathcal{E}_{m_0+3/2}(V) \\ &+ \varepsilon^2 \left(|\underline{\mathfrak{a}} - g|^2_{H^{k+3/2}} + |\underline{\mathbf{v}}|^2_{H^{k+5/2}} \right) \mathcal{E}_{m_0+3/2}(V). \end{aligned} \tag{4.36}$$

Estimate of $\left(\mathbf{\Lambda^{1/2}}\partial^\alpha \underline{\mathbf{M}}_{\mathbf{0},\underline{\mu}} \mathbf{V}, \mathbf{S}^2_{\varepsilon,\underline{\mu}} \mathbf{\Lambda^{1/2}} \partial^\alpha \mathbf{V}\right)$. Without any particular difficulty, this term is bounded from above by

$$\underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V) + \left(|\underline{\mathbf{v}}|^2_{H^{k+3/2}} + \varepsilon^2 |\underline{\mathbf{v}}|^2_{H^{k+5/2}} \right) \mathcal{E}_{m_0+3/2}(V). \tag{4.37}$$

Estimate of $\left(\mathbf{\Lambda^{1/2}}\partial^\alpha \mathbf{V}, [\partial_\mathbf{t}, \mathbf{S}^2_{\varepsilon,\underline{\mu}}] \mathbf{\Lambda^{1/2}} \partial^\alpha \mathbf{V}\right)$. Remark that this term can be decomposed into $\left(\Lambda^{1/2}\partial^\alpha V_1, \partial_t \underline{\mathfrak{a}} \Lambda^{1/2} \partial^\alpha V_1\right) + \left(\Lambda^{1/2} \partial^\alpha V_2, [\partial_t, G(\underline{\zeta})] \Lambda^{1/2} \partial^\alpha V_2\right)$; the first term of this decomposition is easy to bound; for the second, we use Prop. 3.19, so that finally

$$\sum_{|\alpha| \leq k} \left(\Lambda^{1/2} \partial^\alpha V, [\partial_t, S^2_{\varepsilon,\underline{\mu}}] \Lambda^{1/2} \partial^\alpha V \right) \leq \underline{D}_k \mathcal{E}_{k+1/2,\varepsilon}(V). \tag{4.38}$$

End of the proof. From (4.25), (4.36), (4.38) and (4.38), we obtain, as in the proof of Lemma 4.6,

$$\begin{aligned} \frac{d}{dt} e^{-2\nu t} \mathcal{E}_{k+1/2,\varepsilon}(V) &\leq (1 + \underline{D}_k - 2\nu) \mathcal{E}_{k+1/2,\varepsilon}(V) + e^{-2\nu t} \mathcal{E}_{k+1/2,\varepsilon}((\partial_t + \underline{M}_\varepsilon)) \\ &+ e^{-2\nu t} \left(|\underline{\zeta}|^2_{H^{k+3/2}} + |\underline{\mathbf{v}}|^2_{H^{k+2}} + |\underline{\mathfrak{a}} - g|^2_{H^{k+1}} \right) \mathcal{E}_{m_0+3/2}(V) \\ &+ e^{-2\nu t} \left(\varepsilon^2 \left(|\underline{\mathfrak{a}} - g|^2_{H^{k+3/2}} + |\underline{\mathbf{v}}|^2_{H^{k+5/2}} \right) \right) \mathcal{E}_{m_0+3/2}(V). \end{aligned}$$

When $(1 + \underline{D}_k - 2\nu)$ is negative, the estimate of the lemma follows easily from this expression. □

We can now prove the well-posedness of (4.9). In order to do this, we show that the sequence $(V^\varepsilon)_{\varepsilon\in(0,1)}$, where V^ε denotes the solution to (4.12), converges to a solution of (4.9) when $\varepsilon \to 0$.

Let us first prove that $(V^\varepsilon)_\varepsilon$ is a Cauchy sequence. Let $0 < \varepsilon_2 < \varepsilon_1 < 1$ and write $W = V^{\varepsilon_1} - V^{\varepsilon_2}$. One has

$$\begin{cases} \underline{\mathcal{M}}_{\varepsilon_1} W = H_{\varepsilon_1,\varepsilon_2} \\ W|_{t=0} = 0 \end{cases} \qquad \text{with} \qquad H_{\varepsilon_1,\varepsilon_2} := \begin{pmatrix} -(\varepsilon_1^2 - \varepsilon_2^2)\Lambda^2 V_2^{\varepsilon_2} \\ 0 \end{pmatrix}.$$

Remark now that, as a first consequence of Lemma 4.11, for all $k \in \mathbb{N}$, there exists $M_k > 0$ such that $|V_2^\varepsilon|_{H_T^{k+1}} \le M_k$, for all $\varepsilon \in (0,1)$. Applying Lemma 4.11 to W yields therefore

$$\mathcal{E}_{k+1/2,\varepsilon_1}(W(t)) \le (\varepsilon_1^2 - \varepsilon_2^2) C(\underline{U}, T) M_{k+1} + \int_0^T e^{2\nu(t-t')} C(\underline{U}) \mathcal{E}_{m_0+1/2}(W(t'))dt'.$$

From a Gronwall-type argument, we deduce

$$\sup_{t\in[0,T]} \mathcal{E}_{k+1/2,\varepsilon_1}(W(t)) \to 0 \qquad \text{as} \qquad \varepsilon_1 \to 0,$$

and it follows therefore from Lemma 4.8 that $(V^\varepsilon)_\varepsilon$ is a Cauchy sequence in $C([0,T], H^{k+1/2}(\mathbb{R}^d) \times H^{k+1}(\mathbb{R}^d))$. The sequence is therefore convergent in this space, and the limit solves (4.9). The estimate given in the proposition is simply obtained by taking $\varepsilon = 0$ in Lemmas 4.8 and 4.11.

4.3. **Tame estimates for the water-waves equations.** In this section, we give our main result concerning the linearized water-waves equations: the Cauchy problem

$$\begin{cases} \underline{\mathcal{L}} U = G, \\ U|_{t=0} = U_0 \end{cases} \tag{4.39}$$

is well-posed, and the solution U satisfies tame estimates. We first need to introduce two scales of Banach spaces, namely E_a and F_a, in which the estimates can be written simply, and in which a Nash-Moser scheme can be constructed.

Definition 4.12. Let $T > 0$ and $a \in \mathbb{R}$. Define the Banach spaces E_a and F_a as

$$E_a := \bigcap_{j=0}^{2} C^j\left([0,T], H^{a+2-j}(\mathbb{R}^d)^2\right),$$

$$F_a := \Big(\bigcap_{j=0}^{1} C^j\left([0,T], H^{a+1-j}(\mathbb{R}^d)^2\right)\Big) \times H^{a+2}(\mathbb{R}^d)^2,$$

and endow them with the norms

$$|f|_{E_a} := \sum_{j=0}^{2} |\partial_t^j f|_{H_T^{a+2-j}}, \qquad |(g,h)|_{F_a} := \sum_{j=0}^{1} |\partial_t^j g|_{H_T^{a+1-j}} + |h|_{H^{a+2}}.$$

Notation 4.13. An admissible reference state $\underline{U} = (\underline{\zeta}, \underline{\psi})^T$ does not necessarily belong to the Banach scale E_a because $\underline{\psi}_{|t=0}$ is not necessarily in a Sobolev space (though its gradient is). However, we abusively use the notation $|\underline{U}|_{E^a}$ to denote the quantity

$$|\underline{U}|_{E_a} = |\underline{U} - \underline{U}_{|t=0}|_{E_a} + |\nabla_x \underline{U}_{|t=0}|_{H^{a+1}}.$$

Proposition 4.14. *Let $m_0=\lceil\frac{d+1}{2}\rceil$, $T>0$ and $\underline{U}$ be an admissible reference state satisfying (4.10). Also let $G\in C^1([0,T]\times H^\infty(\mathbb{R}^d)^2)$ and $U_0\in H^\infty(\mathbb{R}^d)^2$. Then there is a unique solution $U\in C^2([0,T],H^\infty(\mathbb{R}^d)^2)$ to (4.39). Moreover, for all $a\in\mathbb{R}$, $a\geq m_0+1$, the following estimate holds:*

$$|U|_{E_a}\leq C\big(k,B,|\underline{U}|_{E_{q_0+1/2}},T\big)\left[|(G,U_0)|_{F_{a+3/2}}+|(G,U_0)|_{F_{m_0+1}}|\underline{U}|_{E_{a+5/2}}\right],$$

for some $q_0\in\mathbb{N}$ depending only on d.

Proof. Denote $U_0=(U_{01},U_{02})^T$ and let $V_0:=(U_{01},U_{02}-\underline{Z}|_{t=0}U_{01})^T$ and $H:=(G_1,G_2-\underline{Z}G_1)^T$. Prop. 4.5 asserts that there exists a unique solution $V\in C^1([0,T],H^\infty(\mathbb{R}^d)^2)$ to the Cauchy problem (4.9). Owing to Prop. 4.2, we know that $U:=(V_1,V_2+\underline{Z}V_1)^T$ solves the Cauchy problem (4.39). We now proceed to derive tame estimates on U from the energy estimate (4.12).

Taking $k=m_0+1$ in (4.12), one obtains by a simple Gronwall argument that

$$|V|_{H_T^{m_0+3/2}\times H_T^{m_0+2}}\leq C\big(B,|\underline{U}|_{E_{q_0+1/2}},T\big)\times\big(|U_0|_{H^{m_0+2}}+|G|_{H_T^{m_0+2}}\big),\tag{4.40}$$

for some q_0 depending only on d. Plugging this expression into $(4.12)_{k+2}$, and estimating the quantities $|H(t)|_{H^{k+5/2}\times H^{k+3}}$ and $\big(|\underline{\zeta}|_{H^{k+7/2}}+|\underline{\mathbf{v}}|_{H^{k+4}}+|\underline{\mathfrak{a}}-g|_{H^{k/3}}\big)$ which appear in (4.12) in terms of $\underline{U}$, U_0 and G by standard tame estimates, one obtains (taking a larger q_0 if necessary),

$$\begin{aligned}|V|_{H_T^{k+5/2}\times H_T^{k+3}}\ \leq\ & C\big(k,B,|\underline{U}|_{E_{q_0+1/2}},T\big)\\ &\times\left[|(G,U_0)|_{F_{k+2}}+|(G,U_0)|_{F_{m_0+1}}|\underline{U}|_{E_{k+3}}\right],\end{aligned}$$

from which it is easy to deduce (using the formula $U=(V_1,V_2+\underline{Z}V_1)^T$),

$$|U|_{H_T^{k+5/2}}\leq C\big(k,B,|\underline{U}|_{E_{q_0+1/2}},T\big)\left[|(G,U_0)|_{F_{k+2}}+|(G,U_0)|_{F_{m_0+1}}|\underline{U}|_{E_{k+3}}\right].\tag{4.41}$$

In order to obtain a control of U in $E_{k+1/2}$ we still need to control $\partial_t U$ and $\partial_t^2 U$ in $H^{k+3/2}$ and $H^{k+1/2}$ respectively.

Since $\partial_t U=-d_{\underline{U}}\mathcal{F}\cdot U+G$ one has $|\partial_t U|_{H_T^{k+3/2}}\leq|d_{\underline{U}}\mathcal{F}\cdot U|_{H_T^{k+3/2}}+|G|_{H_T^{k+3/2}}$; from the expression of $d_{\underline{U}}\mathcal{F}$ given in (4.4) and the tame estimates of Prop. 3.25, one deduces

$$|\partial_t U|_{H_T^{k+3/2}}\leq C\big(k,B,|\underline{U}|_{E_{q_0+1/2}}\big)\big(|U|_{H_T^{k+5/2}}+|U|_{H_T^{m_0+1/2}}|\underline{U}|_{H_T^{k+5/2}}\big)+|G|_{H_T^{k+3/2}},$$

which, together with (4.41), yields

$$|\partial_t U|_{H_T^{k+3/2}}\leq C\big(k,B,|\underline{U}|_{E_{q_0+1/2}},T\big)\left[|(G,U_0)|_{F_{k+2}}+|(G,U_0)|_{F_{m_0+1}}|\underline{U}|_{E_{k+3}}\right].\tag{4.42}$$

Finally, one has $\partial_t^2 U=-d^2_{\underline{U}}\mathcal{F}\cdot(\partial_t\underline{U},U)-d_{\underline{U}}\mathcal{F}\cdot\partial_t U+\partial_t G$. One can compute $d^2_{\underline{U}}\mathcal{F}$ from the expression of $d_{\underline{U}}\mathcal{F}$ given in (4.4) and prove that it is a tame bilinear mapping using Prop. 3.25. Using (4.41) and (4.42) we can then obtain a tame estimate on $\partial_t^2 U$ (we do not detail the proof since it does not raise any particular difficulty). Namely,

$$|\partial_t^2 U|_{H_T^{k+1/2}}\leq C\big(k,B,|\underline{U}|_{E_{q_0+1/2}},T\big)\left[|(G,U_0)|_{F_{k+2}}+|(G,U_0)|_{F_{m_0+1}}|\underline{U}|_{E_{k+3}}\right].\tag{4.43}$$

The proposition is then a consequence of (4.41), (4.42) and (4.43) for all $a=k+1/2$, $k\in\mathbb{N}$, $k\geq m_0+1$. By interpolation, we deduce it for all $a\in\mathbb{R}$, $a\geq m_0+1$. $\square$

4.4. **On the Lévy condition $\underline{\mathfrak{a}} \geq \mathbf{c_0} > \mathbf{0}$.** As seen in Prop. 4.4, the Lévy condition (4.10), namely $\underline{\mathfrak{a}} \geq c_0 > 0$, is equivalent to the traditional Taylor criterion. Early works [27, 10, 38] assume smallness conditions on $\underline{U}$, which implies that this criterion holds. One of Wu's key results [36, 37] is that, both for 1D or 2D surface waves, one has indeed $\underline{\mathfrak{a}} = \mathfrak{a}(\underline{U}) \geq c_0 > 0$ as soon as the reference state $\underline{U}$ solves the water-wave equations (4.1). We investigate in this section if this result extends to the present case of finite depth. We first set some notation.

Let $\Gamma_b := \{(X, b(X)), X \in \mathbb{R}^d\}$ be the lower boundary of the fluid domain. One can define the mapping $\mathbf{n}$ on Γ_b as

$$\mathbf{n}: \begin{array}{ccc} \Gamma_b & \to & S^d \\ \sigma & \mapsto & -\mathbf{n}_-(\sigma), \end{array}$$

so that $\mathbf{n}(\sigma)$ is the inward unit normal vector to Γ_b at $\sigma \in \Gamma_b$. This mapping is regular and its derivative $d_\sigma n$ at σ is a linear map from $T_\sigma\Gamma_b$ into $T_{\mathbf{n}(\sigma)}S^d$. Since $T_{\mathbf{n}(\sigma)}S^d = T_\sigma\Gamma_b$ by construction, $d_\sigma\mathbf{n}$ is an endomorphism of $T_\sigma\Gamma_b$. By definition, the second fundamental form of Γ_b is defined as

$$\mathrm{II}_b(\sigma)(p,q) = (d_\sigma\mathbf{n}p, q)_{\mathbb{R}^{d+1}}, \qquad \forall p,q \in T_\sigma\Gamma_b, \tag{4.44}$$

where $(\cdot,\cdot)_{\mathbb{R}^{d+1}}$ denotes the usual scalar product of $\mathbb{R}^{d+1}$.

In the next proposition, we show that the Lévy condition (4.10) is satisfied provided that a certain smallness condition holds on the second fundamental form evaluated at the bottom values of the velocity field.

Proposition 4.15. *Let $T > 0$ and $\underline{U} = (\underline{\zeta}, \underline{\psi})^T$ be an admissible reference state, and denote by $\underline{\phi}$ the velocity potential associated to $\underline{\psi}$. Assume that for some $t_0 \in [0,T]$, $\underline{U}$ solves the water-waves equations* (4.1) *and that*

$$\mathrm{II}_b(\nabla_{X,y}\underline{\phi}_{|\Gamma_b}, \nabla_{X,y}\underline{\phi}_{|\Gamma_b}) \leq \frac{g}{\sqrt{1+|\nabla_X b|^2}}. \tag{4.45}$$

Then there exists $c_0 > 0$ such that $\underline{\mathfrak{a}}(t_0,\cdot) \geq c_0$ on $\mathbb{R}^d$.

Remark 4.16. **i.** The velocity potential $\underline{\phi}$ associated to $\underline{\psi}$ is found by solving the Laplace equation (1.7) in the fluid domain, with Dirichlet condition $\underline{\psi}$ at the surface and homogeneous Neumann boundary condition at the bottom. This latter condition ensures that for all $\sigma \in \Gamma_b$, $\nabla_{X,y}\underline{\phi}(\sigma)$ lives in $T_\sigma\Gamma_b$, so that the expression $\mathrm{II}_b(\nabla_{X,y}\underline{\phi}_{|\Gamma_b}, \nabla_{X,y}\underline{\phi}_{|\Gamma_b})$ makes sense.
ii. If the bottom is flat, then $\mathrm{II}_b = 0$ everywhere, and criterion (4.45) is always satisfied. Thus, in the case of flat bottoms, Wu's result remains true: the generalized Taylor's sign condition $-\partial_{\mathbf{n}_+}\underline{P}|(\cdot,\underline{\zeta}(t_0,\cdot)) \geq c_0 > 0$ holds provided that the reference state $\underline{U}$ solves the water-waves equations (1.11) at time t_0.
iii. By continuity arguments, Wu's result can also be extended to "nearly flat" bottoms: no smallness condition on the reference state $\underline{U}$ is required for the generalized Taylor's sign condition to hold, provided that the bottom parameterization b is flat enough (how flat depending on $\underline{U}$).
iv. In 1D, the criterion given in the proposition reads simply

$$b''(\partial_x\underline{\phi})^2 \leq g,$$

and is therefore always satisfied in the regions where the bottom surface is concave.
v. As we will see later, Taylor's sign criterion $\underline{\mathfrak{a}}(0,\cdot)$ is a sufficient condition for

the well-posedness of the water-waves equations for small times. This condition is almost necessary, but the criterion given in Lemma 4.15 gives only a sufficient condition for Taylor's sign condition to be satisfied. Its interest lies in its simple geometric form. It is for instance obvious that this sufficient condition is fulfilled for flat or nearly flat bottoms, which is far from transparent if one works directly with Taylor's sign condition.

Proof. Recall that $\underline{\mathfrak{a}}(t_0,\cdot) = g + (\partial_t \underline{Z})(t_0,\cdot) + (\underline{\mathbf{v}}\cdot\nabla_X \underline{Z})(t_0,\cdot)$, where $\underline{Z} = Z(\underline{U})$ and $\underline{\mathbf{v}} = \mathbf{v}(\underline{U})$ are given by (4.5) and (4.6). Since $\underline{U}$ (and its derivatives involved in $\underline{Z}$ and $\underline{\mathbf{v}}$) vanishes at infinity, so do $(\partial_t\underline{Z})(t_0,\cdot)$ and $(\underline{\mathbf{v}}\cdot\nabla_X\underline{Z})(t_0,\cdot)$; the acceleration of gravity g being strictly positive, one deduces that there exist $c_1>0$ and $R>0$ such that $\underline{\mathfrak{a}}(t_0,X)\geq c_1$ whenever $|X|>R$, which is precisely the property we want to prove. The remainder of the proof consists therefore in showing that there exists $c_2>0$ such that $\underline{\mathfrak{a}}(t_0,X)\geq c_2$ on the ball $|X|\leq R$.

We know by Prop. 4.4 that $\underline{\mathfrak{a}}(t_0,\cdot) = -\partial_{\mathbf{n}_+}\underline{P}|_{\{y=\zeta(t_0,X)\}}$, where $-\underline{P} = \partial_t\underline{\phi} + \frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2 + gy$. Prop. 4.4 also asserts that $\underline{P}=0$ on the surface; it follows that $\underline{P}$ solves the boundary value problem

$$\begin{cases} -\Delta\underline{P} = \Delta\left(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2\right), \\ \underline{P}|_{\{y=\underline{\zeta}(t_0,X)\}} = 0, \qquad \partial_{\mathbf{n}_-}\underline{P}|_{\Gamma_b} = -\partial_{\mathbf{n}_-}\left(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2\right)|_{\Gamma_b} - \partial_{\mathbf{n}_-}(gy). \end{cases}$$

The next lemma makes the link between the Neumann condition at the bottom and the second fundamental form II_b (recall that by assumption, $\nabla_{X,y}\underline{\phi}|_{\Gamma_b}(\sigma)$ belongs to $T_\sigma\Gamma_b$).

Lemma 4.17. *The velocity potential $\underline{\phi}$ being defined as above, one has*

$$-\partial_{\mathbf{n}_-}\big(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2\big)\Big|_{\Gamma_b} = -\mathrm{II}_b(\nabla_{X,y}\underline{\phi}|_{\Gamma_b}, \nabla_{X,y}\underline{\phi}|_{\Gamma_b}).$$

Proof. **Step 1.** Geometric tools. The first step consists in reparameterizing the fluid domain Ω in the neighborhood of Γ_b. For $\eta>0$ small enough, one can define the mapping

$$\mathbf{\Psi}: \begin{array}{ccc} \Gamma_b\times(0,\eta) & \to & \omega\subset\Omega \\ (\sigma,z) & \mapsto & \sigma + z\mathbf{n}(\sigma); \end{array}$$

if η is small enough, Ψ is a C^∞-parameterization of its range ω. We now want to define the gradient in these new coordinates. Let us denote by ∇_{Γ_b} the gradient on the submanifold Γ_b and introduce $\nabla_{\Gamma_b(z)}$ defined as

$$\nabla_{\Gamma_b(z)} := (Id + zd_\sigma\mathbf{n})^{-1}\nabla_{\Gamma_b}. \tag{4.46}$$

One can prove ([5], see also [15] for the 1D case) that for any function w defined on ω one has

$$\nabla_{X,y}w(X,y) = \frac{\partial\widetilde{w}}{\partial z}(P(X,y),\varphi(X,y))\mathbf{n}(P(X,y)) + (\nabla_{\Gamma_b(\varphi(X,y))})(P(X,y)), \tag{4.47}$$

where $\widetilde{w} := w\circ\Psi$, $P(X,y)$ denotes the orthogonal projection of (X,y) on Γ_b (which is unique if η is small enough) and $\varphi(X,y) := |(X,y)-P(X,y)|$. From (4.47), it follows in particular that

$$\partial_{\mathbf{n}_-}w|_{\Gamma_b} = -\partial_{\mathbf{n}}w|_{\Gamma_b} = -\partial_z\widetilde{w}|_{z=0}, \tag{4.48}$$

and that the tangential component of $\nabla_{X,y}w|_{\Gamma_b}$ is exactly $\nabla_{\Gamma_b}w$.

Step 2. We now use the tools introduced above to prove the result. According to (4.48) and with the same notation as in the first step, one finds $\partial_{\mathbf{n}_-}\nabla_{X,y}\underline{\phi}|_{\Gamma_b} =$

$-\partial_z\widetilde{\nabla_{X,y}\underline{\phi}}|_{z=0}$. By definition, one also has $\widetilde{\nabla_{X,y}\underline{\phi}} = \nabla_{X,y}\underline{\phi}\circ\Psi$, so that using (4.47), one obtains

$$\frac{\partial}{\partial z}\big(\widetilde{\nabla_{X,y}\underline{\phi}}\big)|_{z=0}(\sigma) = \frac{\partial}{\partial z}\big(\frac{\partial\widetilde{\underline{\phi}}}{\partial z}(\sigma,z)\big)|_{z=0}\mathbf{n}(\sigma) + \frac{\partial}{\partial z}(\nabla_{\Gamma_b(z)}\widetilde{\underline{\phi}})|_{z=0}(\sigma).$$

Using (4.46), this yields

$$\frac{\partial}{\partial z}\big(\widetilde{\nabla_{X,y}\underline{\phi}}\big)|_{z=0}(\sigma) = \frac{\partial}{\partial z}\big(\frac{\partial\widetilde{\underline{\phi}}}{\partial z}(\sigma,z)\big)|_{z=0}\mathbf{n}(\sigma) - d_\sigma\mathbf{n}\nabla_{\Gamma_b}\widetilde{\underline{\phi}}. \tag{4.49}$$

Since by (4.48) we have $-\partial_{\mathbf{n}_-}\left(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2\right)|_{\Gamma_b} = \nabla_{X,y}\underline{\phi}|_{\Gamma_b}\cdot\frac{\partial}{\partial z}\left(\widetilde{\nabla_{X,y}\underline{\phi}}\right)\Big|_{z=0}$ and because by assumption $\nabla_{X,y}\underline{\phi}|_{\Gamma_b}$ is tangent to Γ_b, it follows from (4.49) that $-\partial_{\mathbf{n}_-}\left(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2\right)|_{\Gamma_b} = -\nabla_{X,y}\underline{\phi}|_{\Gamma_b}\cdot d_\sigma\mathbf{n}\nabla_{X,y}\underline{\phi}|_{\Gamma_b}$, which is the result claimed in the lemma. □

Remarking that $-\partial_{\mathbf{n}_-}(gy) = \dfrac{g}{\sqrt{1+|\nabla_x b|^2}}$, the assumption made in the statement of the proposition ensures that $\partial_{\mathbf{n}_-}\underline{P}|_{\Gamma_b}\geq 0$. Now, remark that $\underline{P}$ is subharmonic because $\Delta(\frac{1}{2}|\nabla_{X,y}\underline{\phi}|^2) = \sum_{j=1}^{d+1}|\nabla_{X,y}\partial_j\underline{\phi}|^2\geq 0$; whenever $\underline{P}$ reaches its minimum, it is therefore necessarily on the boundary of the fluid domain Ω and at such a point the outward normal derivative is strictly negative. From the observation made above, $\underline{P}$ cannot reach its minimum on Γ_b. Its minimum is therefore reached on the surface, where $\underline{P}$ vanishes identically. Hence, $\underline{P}$ is positive in the fluid domain. Moreover, any point of the surface being a minimum for the subharmonic function $\underline{P}$, one has $\partial_{\mathbf{n}_+}\underline{P} < 0$ everywhere on the surface.

As said above, one has $\underline{\mathfrak{a}}(t_0,X) = -\partial_{\mathbf{n}_+}\underline{P}(X,\zeta(t_0,X))$. It follows that one has $\underline{\mathfrak{a}}(t_0,X) > 0$ everywhere on $\mathbb{R}^d$. By a continuity argument, there exists $c_2>0$ such that $\underline{\mathfrak{a}}(t_0,X)\geq c_2$ for all X in the ball $|X|\leq R$. Taking $c_0=\min\{c_1,c_2\}$ concludes the proof of the proposition. □

5. The nonlinear equations

In this section, we construct a solution to the water-waves equations. The crucial step is the tame estimate on the linearized equation proved in the previous section. The iterative scheme we use here is of Nash-Moser type. We first state a Nash-Moser implicit function theorem in Section 5.1 and then use it to solve the water-waves equations in 5.2.

5.1. **A simple Nash-Moser implicit function theorem.** For the sake of simplicity, we do not use an optimal form of the Nash-Moser theorem. A very simple version of this result can be found in [31]; for the sake of completeness, we reproduce here this result.

Let E_a and F_a, $a\geq 0$ be two scales of Banach spaces and denote $E_\infty = \bigcap_{a\geq 0}E_a$, $F_\infty=\bigcap_{a\geq 0}F_a$. Assume also that there exist some smoothing operators $(S_\theta)_{\theta>1}$: $E_\infty\to E_\infty$ satisfying for every $V\in E_\infty$, $\theta>1$ and s and $t\geq 0$,

$$\begin{cases} |S_\theta V|_{E_s}\leq C_{s,t}\theta^{s-t}|V|_{E_t} & \text{if}\quad s\geq t;\\ |V-S_\theta V|_{E_s}\leq C_{s,t}\theta^{s-t}|V|_{E_t} & \text{if}\quad s\leq t.\end{cases} \tag{5.50}$$

We also assume that $|V|_{E_s}\leq |V|_{E_t}$ whenever $s\leq t$.

Theorem 5.1. *Let $\Phi : E_\infty \to F_\infty$ and assume that there exist $\overline{U} \in E_\infty$, an integer $m > 0$, a real number δ and constants C_1, C_2 and $(C_a)_{a \geq m}$ such that for any $U, V, W \in E_\infty$,*

$$|U - \overline{U}|_{E_{3m}} < \delta \Rightarrow \left\{ \begin{array}{l} \forall a \geq m, \quad |\Phi(U)|_{F_a} \leq C_a(1 + |U|_{E_{a+m}}) \\ |d_U\Phi \cdot V|_{F_{2m}} \leq C_1 |V|_{E_{3m}} \\ |d_U^2 \Phi \cdot (V, W)|_{F_{2m}} \leq C_2 |V|_{E_{3m}} |W|_{E_{3m}}. \end{array} \right. \tag{5.51}$$

Moreover, one assumes that for every $U \in E_\infty$ such that $|U - \overline{U}|_{3m} < \delta$, there exists an operator $\Psi(U) : F_\infty \to E_\infty$ satisfying for any $\varphi \in F_\infty$, $d_U\Phi \cdot \Psi(U)\varphi = \varphi$ and

$$\forall a \geq m, \quad |\Psi(U)\varphi|_{E_a} \leq C_a \left(|\varphi|_{F_{a+m}} + |U|_{E_{a+m}} |\varphi|_{F_{2m}} \right). \tag{5.52}$$

Then if $|\Phi(\overline{U})|_{F_{2m}}$ is sufficiently small (with respect to some upper bound of $1/\delta$, $|\overline{U}|_M$ and $(C_a)_{a \leq M}$ where M depends only on m), there exists a function $U \in E_\infty$ such that $\Phi(U) = 0$.

Remark 5.2. The proof of [31] shows in fact that $M \geq 3m$ and that for all $a \geq M$, assuming that $\overline{U} \in E_a$ instead of $\overline{U} \in E_\infty$ ensures the existence of a solution $U \in E_a$ instead of E_∞.

5.2. **Resolution of the water-waves equations.** We are now ready to state the main theorem of this paper (recall that II_b denotes the second fundamental form of the bottom, as defined in (4.44)):

Theorem 5.3. *Let $b \in C_b^\infty(\mathbb{R}^d)$, $\zeta_0 \in H^{s+1}(\mathbb{R}^d)$ and ψ_0 be such that $\nabla_X \psi_0 \in H^s(\mathbb{R}^d)^d$, with $s > M$ (M depending only on d). Assume moreover that*

$$\min\{\zeta_0 - b, -b\} \geq 2h_0 \quad \text{on} \quad \mathbb{R}^d \quad \text{for some} \quad h_0 > 0$$

and

$$\mathrm{II}_b(V_{0|_{\{y=b(X)\}}}, V_{0|_{\{y=b(X)\}}}) \leq \frac{g}{\sqrt{1 + |\nabla_X b|^2}},$$

where V_0 is the velocity field associated to ψ_0. Then there exists $T > 0$ and a unique solution (ζ, ψ) to the water-waves equations (1.11) *with initial conditions (ζ_0, ψ_0) and such that $(\zeta, \psi - \psi_0) \in C^1\big([0,T], H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)\big)$.*

Remark 5.4. **i.** The initial velocity field V_0 associated to ψ_0 is given by the expression $V_0 = \nabla_{X,y}\phi_0$, where ϕ_0 is the velocity potential found by solving the Laplace equation (1.7) in the fluid domain $\{(X, y) \in \mathbb{R}^{d+1}, b(X) < y < \zeta_0(X)\}$ with Dirichlet condition ψ_0 at the surface and homogeneous boundary condition at the bottom.

ii. In the case of flat bottoms, $\mathrm{II}_b = 0$ everywhere and the assumption on II_b made in the theorem is always satisfied. For uneven bottoms, the smallness assumption made on II_b is weaker than the smallness assumptions made, in the case of 1D surface waves, by Yosihara [38].

iii. One can replace the assumption on II_b by the (sharper) assumption that $\mathfrak{a}(\overline{U})|_{t=0} \geq c_0 > 0$ on $\mathbb{R}^d$, where $\mathfrak{a}$ is defined in (4.7) and $\overline{U} = U_0 - t\mathcal{F}(U_0)$, with $U_0 = (\zeta_0, \psi_0)^T$ and $\mathcal{F}$ defined as in (4.3).

iv. It is physically reasonable to assume that the velocity decays at infinity, but it would be too restrictive to suppose that the velocity potential also does. This is why we take ψ_0 such that $\nabla_x \psi_0 \in H^s(\mathbb{R}^d)^d$, and not simply $\psi_0 \in H^{s+1}(\mathbb{R}^d)$.

Proof. The result is obtained as a consequence of the Nash-Moser Theorem 5.1. We work here with the scale of Banach spaces $(E_a)_a$ and $(F_a)_a$ given in Def. 4.12. It is classical that E_∞ is equipped with a family of smoothing operators $(S_\theta)_{\theta>0}$ satisfying (5.50). Direct use of Nash-Moser's theorem would restrict us to the case of small initial data U_0. To avoid this, we proceed as in [21] (p. 195), exploiting the fact that the water-waves equations are solvable at $t = 0$. Given any initial condition $U_0 = (\zeta_0, \psi_0)$ such that $(\zeta_0, \nabla_X\psi_0) \in H^\infty(\mathbb{R}^d)^{1+d}$, one can find $\overline{U} \in C^3([0,T], H^\infty(\mathbb{R}^d)^2) \subset E_\infty$ such that

$$\overline{U}|_{t=0} = 0, \quad \left[\partial_t\overline{U} + \mathcal{F}(\overline{U} + U_0)\right]|_{t=0} = 0, \quad \left[\partial_t^2\overline{U} + \partial_t\left(\mathcal{F}(\overline{U} + U_0)\right)\right]|_{t=0} = 0.$$

We then define $\overline{G}$ as $\overline{G} = \partial_t\overline{U} + \mathcal{F}(\overline{U} + U_0)$ and introduce the mapping Φ:

$$\Phi : \begin{array}{ccc} E_\infty & \to & F_\infty \\ U & \mapsto & (\partial_t U + \mathcal{F}(U + U_0), U|_{t=0}), \end{array}$$

so that $\Phi(\overline{U}) = (\overline{G}, 0)$. Clearly, if $\Phi(U) = 0$, then $U + U_0$ furnishes a solution to the Cauchy problem (1.11) with initial condition (ζ_0, ψ_0).

Let us check that the assumptions of Theorem 5.1 are satisfied. One has, for all $a \geq 0$,

$$\begin{aligned} |\Phi(U)|_{F_a} &= |\partial_t U + \mathcal{F}(U + U_0)|_{H_T^{a+1}} + \left|\partial_t^2 U + d_{U+U_0}\mathcal{F}\cdot\partial_t U\right|_{H_T^a} + |U|_{t=0}|_{H^{a+2}} \\ &\leq |U|_{E_a} + |\mathcal{F}(U + U_0)|_{H_T^{a+1}} + |d_{U+U_0}\mathcal{F}\cdot\partial_t U|_{H_T^a}. \end{aligned}$$

From the explicit expression of $\mathcal{F}$ given by (4.3) and the tame estimates on the Dirichlet-Neumann operator and its derivatives given in Theorem 3.6 and Prop. 3.25, it is easy to deduce that for all $a \geq m_0 + 1/2$,

$$|\Phi(U)|_{F_a} \leq C\big(a, B, |\zeta_0|_{H^{a+2}}, |\nabla_X\psi_0|_{H^{a+1}}, |U|_{E_{2m_0+1/2}},\big)\big(1 + |U|_{E_a}\big) \tag{5.53}$$

(note the above estimate only involves the gradient of ψ_0, which is made possible by Theorem 3.6; see Remark 3.7).

Taking $m \geq m_0$ and some $\delta > 0$, the condition $|U - \overline{U}|_{E_{3m}} \leq \delta$ implies that $|U|_{E_{3m}}$ and hence $|U|_{E_{2m_0+1/2}}$ remains bounded. Defining C_a as the supremum of all the constants which appear in (5.53) when U remains in the ball $|U - \overline{U}|_{E_{3m}} \leq \delta$ gives therefore the first condition of (5.51).

For all $H, H_1, H_2 \in E_\infty$, one has

$$d_U\Phi\cdot H = (\partial_t H + d_{U+U_0}\mathcal{F}\cdot H, H|_{t=0}) \tag{5.54}$$

and

$$d_U^2\Phi\cdot(H_1, H_2) = \big(d_{U+U_0}^2\mathcal{F}\cdot(H_1, H_2), 0\big);$$

checking that the last two conditions of (5.51) are satisfied is thus obtained in the same way as for the first one, using Prop. 3.25.

We now turn to check condition (5.52). From the expression of $d_U\Phi$ given in (5.54), it is obvious that the right inverse $\Psi(U)$ must be defined as

$$\forall (G, V_0) \in F_\infty, \quad \Psi(U)(G, V_0) = V, \quad \text{where} \quad \left\{\begin{array}{l} \partial_t V + d_{U+U_0}\mathcal{F}\cdot V = G \\ V|_{t=0} = V_0. \end{array}\right.$$

In order to deduce the estimate (5.52) from Prop. 4.14, we must show that for all $U \in E_\infty$ in the ball $|U - \overline{U}|_{E_{3m}} < \delta$, $U + U_0$ is an admissible reference state satisfying (4.10) uniformly, i.e. that there exists $h_0 > 0$ and $c_0 > 0$ such that

$$\forall U \in E_\infty, \quad |U - \overline{U}|_{E_{3m}} < \delta, \quad U_1 + \zeta_0 - b \geq h_0 \quad \text{on} \quad [0,T]\times\mathbb{R}^d, \tag{5.55}$$

and

$$\text{(5.56)} \qquad \forall U \in E_\infty, \quad |U-\overline{U}|_{E_{3m}} < \delta, \quad \mathfrak{a}(U+U_0) \geq c_0 \quad \text{on} \quad [0,T]\times\mathbb{R}^d,$$

where $\mathfrak{a}(u)$ is as defined in (4.7).

Lemma 5.5. *Under the assumptions of the theorem, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then (5.55) and (5.56) are satisfied (for a possibly smaller $T > 0$).*

Proof. To prove (5.55), write $U_1(t)+\zeta_0-b = \int_0^t \partial_t U_1(t')dt' + U_1|_{t=0}+\zeta_0-b$, so that using the assumption made on the initial data, $U_1(t)+\zeta_0-b \geq 2h_0 - T|\partial_t U_1|_{L^\infty_T} - |(U_1-\overline{U}_1)|_{t=0}|$, where we used the fact that $\overline{U}|_{t=0}=0$. Sobolev embeddings then yield $U_1(t)-b \geq 2h_0 - \text{Cst } T|U|_{E^{3m}} - \text{Cst } \delta$, from which the conclusion is easy.

To prove (5.56), remark that $\mathfrak{a}(U(t)+U_0) = \mathfrak{a}(U(t)+U_0) - \mathfrak{a}(\overline{U}(t)+U_0) + \int_0^t \partial_t \big(\mathfrak{a}(\overline{U}(t')+U_0)\big)\,dt' + \mathfrak{a}(U_0+\overline{U}|_{t=0})$. It follows that $\mathfrak{a}(U(t)+U_0) \geq \mathfrak{a}(\overline{U}|_{t=0}+U_0) - C(|U_0|_{E_{3m}}, |\overline{U}|_{E_{3m}})(T+\delta)$. Since by construction, $\overline{U}+U_0$ solves the water-waves equations (1.11) at time $t=0$, we deduce from Props. 4.4 and 4.15 that there exists $c_0 > 0$ such that $\mathfrak{a}(\overline{U}|_{t=0}+U_0) \geq 2c_0$. The end of the proof is then straightforward. □

This lemma shows that the estimate (5.52) assumed in Theorem 5.1 is a consequence of Prop. 4.14 (taking a larger m if necessary). We can therefore use Theorem 5.1, which asserts that one can solve the equation $\Phi(U)=0$ provided that $|\Phi(\overline{U})|_{F_{2m}} \leq M_0$ for some $M_0>0$. Now, recall that $\Phi(\overline{U}) = (\overline{G},0)$ and that, by construction, $\overline{G}|_{t=0} = \partial_t\overline{G}|_{t=0} = 0$. One has therefore

$$\big|\Phi(\overline{U})\big|_{F_{2m}} = \big|\overline{G}\big|_{H_T^{2m+1}} + \big|\partial_t\overline{G}\big|_{H_T^{2m}} \leq T\big(\,\big|\partial_t\overline{G}\big|_{H_T^{2m+1}} + \big|\partial_t^2\overline{G}\big|_{H_T^{2m}}\,\big),$$

which, taking a smaller T if necessary, is smaller than M_0.

We have therefore proved the existence of a solution $U\in E_\infty$ to $\Phi(U)=0$, i.e. a solution to the water-waves equations (1.11); the case of finitely regular solutions is handled as in Remark 5.2.

We now turn to prove uniqueness. Let U_1 and U_2 be two solutions in E_{a+m}, for some $a \geq m_0+1/2$, m being as above. The difference $U=U_2-U_1$ solves therefore

$$\begin{cases} \partial_t U + d_{U_1+U_0}\mathcal{F}\cdot U = G, \\ U|_{t=0}=0, \end{cases} \qquad \text{with} \quad G := -\int_0^1 (1-t)d^2_{U_0+U_1+t(U_2-U_1)}\mathcal{F}\cdot(U,U)dt.$$

Using Prop. 3.25, it is easy to obtain that for all $s \geq 2m_0+1/2$, one has $|G|_{H^s} \leq C_s|U|_{H^{m_0+1/2}}$, where the constant C_s depends on the norm of U_1 and U_2 in E_s. Proceeding as in the proof of Prop. 4.14, one obtains the estimate

$$|U|_{H^a_T} \leq C_{a+m}C(T)\int_0^t |U(t)|_{H^{m_0+1/2}}dt,$$

for some integer $m>0$. Bounding $|U(t)|_{H^{m_0+1/2}}$ from above by $|U(t)|_{H^a}$ and using a classical Gronwall argument yields $U=0$, whence the uniqueness. □

Appendix A. Proof of Lemma 2.8

Owing to Lemma 2.5, the nonconstant coefficients of $\widetilde{P}$ are of the form (up to a multiplicative constant)

$$A = \partial_i s, \quad i = 1, \dots, d+1, \quad B = \frac{1}{\partial_{\widetilde{y}} s} \text{ or } C = \frac{\partial_i s \partial_j s}{\partial_{\widetilde{y}} s}, \text{ with } 1 \le i, j \le d.$$

It is clear that one can write $A = A_1 + A_2$, with $A_1 = \partial_i s_1$ and $A_2 = \partial_i s_2$, so that

$$\|A_1\|_{k,\infty} \le \|s_1\|_{k+1,\infty}, \qquad \|A_2\|_{k,2} \le \|s_2\|_{k+1,2}.$$

Similarly, one can write $B = B_1 + B_2$ and $C = C_1 + C_2$ with

$$B_1 = \frac{1}{\partial_{\widetilde{y}} s_1}, \qquad B_2 = \frac{-\partial_{\widetilde{y}} s_2}{\partial_{\widetilde{y}} s_1 \partial_{\widetilde{y}} s},$$

and

$$C_1 = \frac{\partial_i s_1 \partial_j s_1}{\partial_{\widetilde{y}} s_1}, \quad C_2 = \frac{(\partial_i s_2 \partial_j s_2 + \partial_i s_1 \partial_j s_2 + \partial_i s_2 \partial_j s_1)\partial_{\widetilde{y}} s_1 - \partial_{\widetilde{y}} s_2 \partial_i s_1 \partial_j s_1}{\partial_{\widetilde{y}} s_1 \partial_{\widetilde{y}} s}.$$

It follows easily that

$$\|B_1\|_{k,\infty} \le C\big(\frac{1}{c_0}, \|s_1\|_{k+1,\infty}\big), \qquad \|C_1\|_{k,\infty} \le C\big(\frac{1}{c_0}, \|s_1\|_{k+1,\infty}\big),$$

which achieves the proof of the first estimate of the lemma.
We now turn to estimate the Sobolev norms of B_2 and C_2. Remark that they are both of the form $\frac{f_2}{g_1+g_2}$, with $f_2, g_2 \in H^k(\mathcal{S})$, $g_1 \in C_b^k(\overline{\mathcal{S}})$ and

$$\|f_2, g_2\|_{k,2} \le C(\|s_1\|_{k+1,\infty}, \|s_2\|_{1,\infty})\|s_2\|_{k+1,2}, \quad \|g_1\|_{k,\infty} \le C(\|s_1\|_{k+1,\infty}). \tag{A.1}$$

Let us denote $g := g_1 + g_2$. For all $\alpha \in \mathbb{N}^{d+1}$, $|\alpha| = k$, one can show by induction that $\partial^\alpha \left(\frac{f_2}{g}\right)$ is a sum of terms of the form

$$I = \frac{1}{g^{1+|\alpha|}} \partial^\beta f_2 \prod_{n=0}^{|\alpha|-|\beta|} \prod_{J_n \in \mathbb{N}^{d+1}, |J_n| = n} (\partial^{J_n} g)^{r_{J_n}}, \tag{A.2}$$

where $\beta \in \mathbb{N}^{d+1}$, $r_{J_n} \in \mathbb{N}$ satisfy the relation

$$|\beta| + \sum_{n=0}^{|\alpha|-|\beta|} n \sum_{J_n \in \mathbb{N}^{d+1}, |J_n| = n} r_{J_n} = k. \tag{A.3}$$

Decomposing g into $g = g_1 + g_2$, one obtains the following estimate:

$$\|I\|_2 \le C\big(\frac{1}{c_0}, \|g_1\|_{k,\infty}, \|g_2\|_\infty\big) \Big\| \partial^\beta f_2 \prod_{n=1}^{|\alpha|-|\beta|} \prod_{J_n \in \mathbb{N}^{d+1}, |J_n| = n} (\partial^{J_n} g_2)^{r'_{J_n}} \Big\|_2, \tag{A.4}$$

where the r'_{J_n} are such that $0 \le r'_{J_n} \le r_{J_n}$.

Let l be defined as

$$l := |\beta| + \sum_{n=1}^{|\alpha|-|\beta|} n \sum_{J_n\in\mathbb{N}^{d+1},|J_n|=n} r'_{J_n}, \tag{A.5}$$

so that by (A.3), one has $0 \le l \le k$.

• If $l = 0$, then necessarily

$$\partial^\beta f_2 \prod_{n=1}^{|\alpha|-|\beta|} \prod_{J_n\in\mathbb{N}^{d+1},|J_n|=n} (\partial^{J_n} g_2)^{r'_{J_n}} = f_2,$$

and therefore

$$\|I\|_2 \le C\big(\frac{1}{c_0}, \|g_1\|_{k,\infty}, \|g_2\|_\infty\big)\|f_2\|_2. \tag{A.6}$$

• If $l \ge 1$, then remark that

$$\frac{1}{2l/|\beta|} + \sum_{n=1}^{|\alpha|-|\beta|} \sum_{J_n\in\mathbb{N}^{d+1},|J_n|=n} \frac{r'_{J_n}}{2l/n} = \frac{1}{2}.$$

Denoting by J the L^2-norm which appears in (A.4) and using Young's inequality, one has therefore

$$J \le \|\partial^\beta f_2\|_{2l/|\beta|} \prod_{n=1}^{|\alpha|-|\beta|} \prod_{J_n\in\mathbb{N}^{d+1},|J_n|=n} \|\partial^{J_n} g_2\|_{2l/n}^{r'_{J_n}}.$$

Recalling that for all $\phi \in \mathcal{D}(\mathcal{S})$, one has

$$\|\partial^\gamma \phi\|_{2l/|\gamma|} \le \mathrm{Cst}\ \|\phi\|_\infty^{1-|\gamma|/l}\|\phi\|_{l,2}^{|\gamma|/l}, \qquad \gamma \in \mathbb{N}^{d+1}, \qquad 0 \le |\gamma| \le l,$$

and using (A.5), it follows that

$$J \le C\left(\|g_2\|_\infty\right)\|f_2\|_\infty^{1-|\beta|/l}\|g_2\|_{l,2}^{1-|\beta|/l}\|f_2\|_{l,2}^{|\beta|/l}.$$

Plugging the estimates (A.1) into this inequality and using (A.4) and (A.6), one obtains the second estimate of the lemma.

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