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SIMPLE HIRONAKA RESOLUTION IN CHARACTERISTIC ZERO

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0. Introduction

In the present paper we give a short proof of the Hironaka theorem on resolution of singularities. Recall that in the classical approach to the problem of embedded resolution originated by Hironaka [25] and later developed and simplified by Bierstone-Milman [8] and Villamayor [34] an invariant which plays the role of a measure of singularities is constructed. The invariant is upper semicontinuous and defines a stratification of the ambient space. This invariant drops after the blow-up of the maximal stratum. It determines the centers of the resolution and allows one to patch up local desingularizations to a global one. Such an invariant carries rich information about singularities and the resolution process. The definition of the invariant is quite involved. What adds to the complexity is that the invariant is defined within some rich inductive scheme encoding the desingularization and assuring its canonicity (Bierstone-Milman's towers of local blow-ups with *admissible centers* and Villamayor's *general basic objects*) (see also Encinas-Hauser [17]).

The idea of forming the invariant is based upon the observation due to Hironaka that the resolution process controlled by the order or the Hilbert-Samuel function can be reduced to the resolution process on some smooth hypersurface, called a *hypersurface of maximal contact* (see [25]). The reduction to a hypersurface of maximal contact is not canonical and for two different hypersurfaces of maximal contact we get two different objects loosely related but having the same

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invariant. To make all the processes canonical and relate the objects induced by restrictions we either interpret the invariant in a canonical though quite technical way (so called "Hironaka trick") or build a relevant canonical resolution datum ([9],[10],[11],[17],[23],[25],[34], [35],[36]).

The approach we propose in this paper is based upon the above-mentioned reduction procedure and two simple observations.

(1) The resolution process defined as a sequence of suitable blow-ups of ambient spaces can be applied simultaneously not only to the given singularities but rather to a class of equivalent singularities obtained by simple arithmetical modifications. This means that we can "tune" singularities before resolving them.
(2) In the equivalence class we can choose a convenient representative given by the *homogenized ideals* introduced in the paper. The restrictions of homogenized ideals to different hypersurfaces of maximal contact define locally analytically isomorphic singularities. Moreover the local isomorphism of hypersurfaces of maximal contact is defined by a local analytic automorphism of the ambient space preserving all the relevant resolutions.

This approach puts much less emphasis on the invariant which is defined only for the considered algorithm (mainly to mark the progress towards the desingularization). The inductive structure is reduced to the existence of a canonical functorial resolution in lower dimensions. "Canonicity" means here that étale isomorphic objects undergo the same resolution process. In the proofs by Bierstone-Milman and Villamayor the invariant is defined in a more general situation: not only for the constructed algorithm but for the larger class of test blow-ups. The idea goes back to Hironaka and is often referred to as Hironaka's trick. Using this more general language Bierstone and Milman prove that the objects have the same resolutions if they have the same sequences of test blow-ups (see [10],[11]).

The strategy of the proof we formulate here is essentially the same as the one found by Hironaka and simplified by Bierstone-Milman and Villamayor. Our algorithm and use of the invariant are very similar to that of Bierstone-Milman ([8], [9], [10]), though the preliminary setup and organization of the proof is very close to that of Villamayor ([34], [35], [36]). In particular we apply here one of Villamayor's key simplifications, eliminating the use of the Hilbert-Samuel function and the notion of normal flatness (see [13]). As a result, the resolution algorithm shown here is simpler though it is slightly weaker than the original Hironaka theorem. At the present moment it is not clear whether the technique of homogenization can be applied to the stronger version of the canonical embedded resolution.

The presented proof is elementary, constructive and self-contained.

The paper is organized as follows. In section 1 we formulate three main theorems: the theorem of canonical principalization (Hironaka's "Desingularization II"), the theorem of canonical embedded resolution (a slightly weaker version of Hironaka's "Desingularization I") and the theorem of canonical resolution. In section 2 we formulate the theorem of canonical resolution of marked ideals and show how it implies three main theorems (Hironaka's resolution principle). Section 3 gives important technical ingredients. In particular we introduce here the notion of homogenized ideals. In section 4 we formulate the resolution algorithm and prove the theorem of canonical resolution of marked ideals.

The methods used in this paper can be applied to desingularization of analytic spaces; we deal with the analytic case in a separate paper.

1. Formulation of the main theorems

All algebraic varieties in this paper are defined over a ground field of characteristic zero. The assumption of characteristic zero is needed for the local existence of a hypersurface of maximal contact (Lemma 3.3.4).

We give a proof of the following Hironaka Theorems (see [25]):

(1) **Principalization of sheaves of ideals**

Theorem 1.0.1. *Let $\mathcal{I}$ be a sheaf of ideals on a smooth algebraic variety X. There exists a principalization of $\mathcal{I}$, that is, a sequence*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ with smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*

(b) *The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor $\widetilde{E}$ which is a natural combination of the irreducible components of the divisor E_r.*

The morphism $(\widetilde{X}, \widetilde{\mathcal{I}}) \to (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms, embeddings of ambient varieties and (separable) ground field extensions.

(2) **Weak-Strong Hironaka Embedded Desingularization**

Theorem 1.0.2. *Let Y be a subvariety of a smooth variety X over a field of characteristic zero. There exists a sequence*

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \longleftarrow X_i$ with smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*

(b) *Let $Y_i \subset X_i$ be the strict transform of Y. All centers C_i are disjoint from the set $\mathrm{Reg}(Y) \subset Y_i$ of points where Y (not Y_i) is smooth (and are not necessarily contained in Y_i).*

(c) *The strict transform $\widetilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the exceptional divisor E_r.*

(d) *The morphism $(X, Y) \leftarrow (\widetilde{X}, \widetilde{Y})$ defined by the embedded desingularization commutes with smooth morphisms, embeddings of ambient varieties and (separable) ground field extensions.*

(e) *(Strengthening of Bravo-Villamayor [13])*

$$\sigma^*(\mathcal{I}_Y) = \mathcal{I}_{\widetilde{Y}} \mathcal{I}_{\widetilde{E}},$$

where $\mathcal{I}_{\widetilde{Y}}$ is the sheaf of ideals of the subvariety $\widetilde{Y} \subset \widetilde{X}$ and $\mathcal{I}_{\widetilde{E}}$ is the sheaf of ideals of a simple normal crossing divisor $\widetilde{E}$ which is a natural combination of the irreducible components of the divisor E_r.

(3) **Canonical Resolution of Singularities**

Theorem 1.0.3. *Let Y be an algebraic variety over a field of characteristic zero.*

There exists a canonical desingularization of Y that is a smooth variety $\widetilde{Y}$ together with a proper birational morphism $\mathrm{res}_Y : \widetilde{Y} \to Y$ which is functorial with respect to smooth morphisms. For any smooth morphism $\phi : Y' \to Y$ there is a natural lifting $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ which is a smooth morphism.

In particular $\mathrm{res}_Y : \widetilde{Y} \to Y$ is an isomorphism over the nonsingular part of Y.

Moreover res_Y commutes with (separable) ground field extensions.

Remarks. (1) By the exceptional divisor of the blow-up $\sigma : X' \to X$ with a smooth center C we mean the inverse image $E := \sigma^{-1}(C)$ of the center C. By the exceptional divisor of the composite of blow-ups σ_i with smooth centers C_{i-1} we mean the union of the strict transforms of the exceptional divisors of σ_i. This definition coincides with the standard definition of the exceptional set of points of the birational morphism in the case when $\mathrm{codim}(C_i) \geq 2$ (as in Theorem 1.0.2). If $\mathrm{codim}(C_{i-1}) = 1$ the blow-up of C_{i-1} is an identical isomorphism and defines a formal operation of converting a subvariety $C_{i-1} \subset X_{i-1}$ into a component of the exceptional divisor E_i on X_i. This formalism is convenient for the proofs. In particular it indicates that C_{i-1} identified via σ_i with a component of E_i has simple normal crossings with other components of E_i.

(2) In Theorem 1.0.2 we blow up centers of codimension ≥ 2 and both definitions coincide.

(3) Given a closed embedding of smooth varieties $i : X \hookrightarrow X'$, the coherent sheaf of ideals $\mathcal{I}$ on X defines a coherent subsheaf $i_*(\mathcal{I}) \subset i_*(\mathcal{O}_X)$ of the $\mathcal{O}_{X'}$-module $i_*(\mathcal{O}_X)$. Let $i^\sharp : O_{X'} \to i_*(\mathcal{O}_X)$ be the natural surjection of $\mathcal{O}_{X'}$-modules. The inverse image $\mathcal{I}' = (i^\sharp)^{-1}(i_*(\mathcal{I}))$ defines a coherent sheaf of ideals on X'. By abuse of notation $\mathcal{I}'$ will be denoted as $i_*(\mathcal{I}) \cdot O_{X'}$.

(4) Theorem 1.0.1 says that the canonical principalization of $\mathcal{I}$ on X commutes with closed embeddings of ambient varieties $X \subset X'$. This means that the canonical principalization of $\mathcal{I}' = i_*(\mathcal{I}) \cdot O_{X'}$ restricts to the canonical principalization of $\mathcal{I}$ on X. In fact we show even more: The canonical principalization of $\mathcal{I}$ with centers C_i defines the canonical principalization of $\mathcal{I}'$ on X' with the centers $i(C_i)$.

2. Preliminaries

To simplify our considerations we shall assume that the ground field is algebraically closed of characteristic zero. At the end of the paper we deduce the theorem for an arbitrary ground field of characteristic zero.

2.1. **Resolution of marked ideals.** For any sheaf of ideals $\mathcal{I}$ on a smooth variety X and any point $x \in X$ we denote by

$$\mathrm{ord}_x(\mathcal{I}) := \max\{i \mid \mathcal{I}_x \subset m_x^i\}$$

the *order* of $\mathcal{I}$ at x. (Here m_x denotes the maximal ideal of x.)

Definition 2.1.1 (Hironaka [25], [27], Bierstone-Milman [8], Villamayor [34])**.** A *marked ideal* (originally a *basic object* of Villamayor) is a collection $(X, \mathcal{I}, E, \mu)$, where X is a smooth variety, $\mathcal{I}$ is a sheaf of ideals on X, μ is a nonnegative integer and E is a totally ordered collection of divisors whose irreducible components are pairwise disjoint and all have multiplicity one. Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

Definition 2.1.2. (Hironaka [25], [27], Bierstone-Milman [8], Villamayor [34]) By the *support* (originally *singular locus*) of $(X, \mathcal{I}, E, \mu)$ we mean

$$\operatorname{supp}(X, \mathcal{I}, E, \mu) := \{x \in X \mid \operatorname{ord}_x(\mathcal{I}) \geq \mu\}.$$

Remarks. (1) The ideals with assigned orders or functions with assigned multiplicities and their supports are key objects in the proofs of Hironaka, Villamayor and Bierstone-Milman (see [25]). Hironaka introduced the notion of *idealistic exponent*. Then various modifications of this definition were considered in the papers of Bierstone-Milman (*presentation of invariant*) and Villamayor (*basic objects*). In our proof we stick to Villamayor's presentation of his basic objects (and their resolutions). Our marked ideals are essentially the same notion as basic objects. However because of some technical differences and in order to introduce more suggestive terminology we shall call them marked ideals.

(2) Sometimes for simplicity we shall represent marked ideals $(X, \mathcal{I}, E, \mu)$ as couples $(\mathcal{I}, \mu)$ or even ideals $\mathcal{I}$.

(3) For any sheaf of ideals $\mathcal{I}$ on X we have $\operatorname{supp}(\mathcal{I}, 1) = V(\mathcal{I})$.

(4) For any marked ideals $(\mathcal{I}, \mu)$ on X, $\operatorname{supp}(\mathcal{I}, \mu)$ is a closed subset of X (Lemma 3.2.2).

Definition 2.1.3 (Hironaka [25], [27], Bierstone-Milman [8], Villamayor [34])**.** By a *resolution* of $(X, \mathcal{I}, E, \mu)$ we mean a sequence of blow-ups $\sigma_i : X_i \to X_{i-1}$ of disjoint unions of smooth centers $C_{i-1} \subset X_{i-1}$,

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} \ldots X_i \longleftarrow \ldots \xleftarrow{\sigma_r} X_r,$$

which defines a sequence of marked ideals $(X_i, \mathcal{I}_i, E_i, \mu)$ where

1. $C_i \subset \operatorname{supp}(X_i, \mathcal{I}_i, E_i, \mu)$.
2. C_i has simple normal crossings with E_i.
3. $\mathcal{I}_i = \mathcal{I}(D_i)^{-\mu} \sigma_i^*(\mathcal{I}_{i-1})$, where $\mathcal{I}(D_i)$ is the ideal of the exceptional divisor D_i of σ_i.
4. $E_i = \sigma_i^{\mathrm{c}}(E_{i-1}) \cup \{D_i\}$, where $\sigma_i^{\mathrm{c}}(E_{i-1})$ is the set of strict transforms of divisors in E_{i-1}.
5. The order on $\sigma_i^{\mathrm{c}}(E_{i-1})$ is defined by the order on E_{i-1} while D_i is the maximal element of E_i.
6. $\operatorname{supp}(X_r, \mathcal{I}_r, E_r, \mu) = \emptyset$.

Definition 2.1.4. The sequence of morphisms which are either isomorphisms or blow-ups satisfying conditions (1)-(5) is called a *multiple test blow-up*. The number of morphisms in a multiple test blow-up will be called its *length*.

Definition 2.1.5. An *extension* of a sequence of blow-ups $(X_i)_{0 \leq i \leq m}$ is a sequence $(X'_j)_{0 \leq j \leq m'}$ of blow-ups and isomorphisms $X'_0 = X'_{j_0} = \ldots = X'_{j_1 - 1} \leftarrow X'_{j_1} = \ldots = X'_{j_2 - 1} \leftarrow \ldots X'_{j_m} = \ldots = X'_{m'}$, where $X'_{j_i} = X_i$.

In particular we shall consider *extensions of multiple test blow-ups.*

Remarks. (1) The definition of extension arises naturally when we pass to open subsets of the considered ambient variety X.

(2) The notion of a *multiple test blow-up* is analogous to the notions of *test* or *admissible* blow-ups considered by Hironaka, Bierstone-Milman and Villamayor.

2.2. **Transforms of marked ideals and controlled transforms of functions.** In the setting of the above definition we shall call

$$(\mathcal{I}_i, \mu) := \sigma_i^{\mathrm{c}}(\mathcal{I}_{i-1}, \mu)$$

a *transform of the marked ideal* or *controlled transform* of $(\mathcal{I}, \mu)$. It makes sense for a single blow-up in a multiple test blow-up as well as for a multiple test blow-up. Let $\sigma^i := \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ be a composition of consecutive morphisms of a multiple test blow-up. Then in the above setting

$$(\mathcal{I}_i, \mu) = (\sigma^i)^{\mathrm{c}}(\mathcal{I}, \mu).$$

We shall also denote the controlled transform $(\sigma^i)^{\mathrm{c}}(\mathcal{I}, \mu)$ by $(\mathcal{I}, \mu)_i$ or $[\mathcal{I}, \mu]_i$.

The controlled transform can also be defined for local sections $f \in \mathcal{I}(U)$. Let $\sigma : X \leftarrow X'$ be a blow-up with a smooth center $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ defining a transformation of marked ideals $\sigma^{\mathrm{c}}(\mathcal{I}, \mu) = (\mathcal{I}', \mu)$. Let $f \in \mathcal{I}(U)$ be a section of a sheaf of ideals. Let $U' \subseteq \sigma^{-1}(U)$ be an open subset for which the sheaf of ideals of the exceptional divisor is generated by a function y. The function

$$g = y^{-\mu}(f \circ \sigma) \in \mathcal{I}(U')$$

is a *controlled transform* of f on U' (defined up to an invertible function). As before we extend it to any multiple test blow-up.

The following lemma shows that the notion of controlled transform is well-defined.

Lemma 2.2.1. *Let $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ be a smooth center of the blow-up $\sigma : X \leftarrow X'$ and let D denote the exceptional divisor. Let $\mathcal{I}_C$ denote the sheaf of ideals defined by C. Then*

(1) $\mathcal{I} \subset \mathcal{I}_C^{\mu}$.
(2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^{\mu}$.

Proof. (1) We can assume that the ambient variety X is affine. Let $u_1, \ldots, u_k$ be parameters generating $\mathcal{I}_C$. Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^{\mu}$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$ for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x) u^\alpha$. Thus $\operatorname{ord}_x(\mathcal{I}) < \mu$. This contradicts the assumption $C \subset \operatorname{supp}(\mathcal{I}, \mu)$.

(2) $\sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^{\mu} = (\mathcal{I}_D)^{\mu}$. □

2.3. **Functorial properties of multiple test blow-ups.**

Lemma 2.3.1. *Let $\phi : X' \to X$ be a smooth morphism. Then*

(1) *For any sheaf of ideals $\mathcal{I}$ on X and any $x' \in X'$, $x = \phi(x') \in X$ we have* $\operatorname{ord}_{x'}(\phi^*(\mathcal{I})) = \operatorname{ord}_x(\mathcal{I})$.
(2) *Let E be a set of divisors with smooth disjoint components such that all components of all divisors have simultaneously simple normal crossings. Then the inverse images $\phi^{-1}(D)$ of divisors $D \in E$ have disjoint components and all the components of the divisors $\phi^{-1}(D)$ have simultaneously simple normal crossings.*

Proof. The assertions can be verified locally. Assume ϕ is of relative dimension r. Then it factors (locally) as $\phi = \pi\psi$ where $\psi : U_{x'} \to X \times \mathbf{A}^r$ is étale and $\pi : X \times \mathbf{A}^r \to X$ is the natural projection. Let $x'' := \psi(x')$. Since ψ defines a formal analytic isomorphism $\psi^* : \widehat{\mathcal{O}}_{x'',X\times\mathbf{A}^r} \simeq \widehat{\mathcal{O}}_{x',X}$ the assertions of the lemma are satisfied for ψ. Moreover they are satisfied for the natural projection π and for the composition $\phi = \pi\psi$. $\square$

Proposition 2.3.2. *Let X_i be a multiple test blow-up of a marked ideal $(X, \mathcal{I}, E, \mu)$ defining a sequence of marked ideals $(X_i, \mathcal{I}_i, E_i, \mu)$. Given a smooth morphism $\phi : X' \to X$, the induced sequence $X'_i := X' \times_X X_i$ is a multiple test blow-up of $(X', \mathcal{I}', E', \mu)$ such that*

1. *ϕ lifts to smooth morphisms $\phi_i : X'_i \to X_i$.*
2. *(X'_i) defines a sequence of marked ideals $(X'_i, \mathcal{I}'_i, E'_i, \mu)$ where $\mathcal{I}'_i = \phi_i^*(\mathcal{I}_i)$, the divisors in E'_i are the inverse images of the divisors in E_i and the order on E'_i is defined by the order on E_i.*
3. *If (X_i) is a resolution of $(X, \mathcal{I}, E, \mu)$ then (X'_i) is an extension of a resolution of $(X', \mathcal{I}', E', \mu)$.*

Proof. Induction on i. The pull-back $\sigma'_{i+1} : X'_i \leftarrow X'_{i+1}$ of the blow-up $\sigma_{i+1} : X_i \leftarrow X_{i+1}$ by the smooth morphism $\phi_i : X'_i \to X_i$ is either the blow-up with the smooth center $C'_i = \sigma_i^{-1}(C_i)$ or an isomorphism if $\sigma_i^{-1}(C) = \emptyset$. Since $\phi_i : X'_i \to X_i$ is smooth for any $x' \in X'_i$ and $x = \phi_i(x)$, $\operatorname{ord}_{x'}(\mathcal{I}'_i) = \operatorname{ord}_{x'}(\phi_i^*(\mathcal{I}_i)) = \operatorname{ord}_x(\mathcal{I}_i)$. Thus $C'_i \subset \operatorname{supp}(\mathcal{I}'_i, \mu)$. Moreover C'_i has simple normal crossings with E'_i. It is left to show that the transformation rules for the sheaves of ideals $\mathcal{I}'_i$ and sets of divisors E'_i are preserved by the induced smooth morphisms ϕ_i. Note that the inverse image of the exceptional divisor D_{i+1} of σ_{i+1} is the exceptional divisor $D'_{i+1} = \phi_i^{-1}(D_i)$ of σ'_{i+1}. Thus we have

$$\begin{aligned}
E'_{i+1} &= (\sigma'_i)^c(E'_i) \cup \{D'_{i+1}\} = (\sigma'_i)^c(\phi_i^{-1}(E_i)) \cup \phi_{i+1}^{-1}\{D_{i+1}\} \\
&= \phi_{i+1}^{-1}((\sigma_i)^c(E_i) \cup \{D_{i+1}\}) = \phi_{i+1}^{-1}(E_{i+1}), \\
\mathcal{I}'_{i+1} &= (\sigma'_{i+1})^c(\mathcal{I}'_i) = (\sigma'_{i+1})^*(\mathcal{I}'_i)\mathcal{I}(D'_{i+1})^{-\mu} \\
&= (\sigma'_{i+1})^*(\phi_i^*(\mathcal{I}_i)) \cdot \phi_{i+1}^*((\mathcal{I}(D_{i+1}))^{-\mu}) \\
&= \phi_{i+1}^*(\sigma_{i+1}^*(\mathcal{I}_i) \cdot (\mathcal{I}(D_{i+1}))^{-\mu}) = \phi_{i+1}^*(\sigma_{i+1}^c(\mathcal{I}_i) = \phi_{i+1}^*(\mathcal{I}_{i+1}). \quad \square
\end{aligned}$$

Definition 2.3.3. We say that the above multiple test blow-up (X'_i) is *induced* by ϕ_i and X. We shall denote (X'_i) and the corresponding marked ideals $(X', \mathcal{I}', E', \mu)$ by

$$\phi^*(X_i) := X'_i, \quad \phi^*(X_i, \mathcal{I}_i, E_i, \mu) := (X'_i, \mathcal{I}'_i, E'_i, \mu).$$

The above proposition and definition generalize to any sequence of blow-ups with smooth centers.

Proposition 2.3.4. *Let X_i be a sequence of blow-ups with smooth centers having simple normal crossings with exceptional divisors. Given a smooth morphism $\phi : X' \to X$, the induced sequence $X'_i := X' \times_X X_i$ is an extension of the sequence of blow-ups with smooth centers having simple normal crossings with exceptional divisors.* $\square$

2.4. **Canonicity revisited.** In this paper we prove the following theorem:

1. **Canonical resolution of marked ideals**

Theorem 2.4.1. *Let K be a field of characteristic zero. With any marked ideal $(X, \mathcal{I}, E, \mu)$ over K there is associated a resolution (X_i) called canonical such that*

(1) *For any smooth morphism $\phi : X' \to X$ the induced resolution $\phi^*(X_i)$ is an extension of the canonical resolution of $\phi^*(X, \mathcal{I}, E, \mu)$.*
(2) *If $E = \emptyset$, then (X_i) commutes with closed embeddings of the ambient varieties $X \hookrightarrow X'$; that is, the canonical resolution (X_i) of $(X, \mathcal{I}, \emptyset, \mu)$ with centers C_i defines the canonical resolution (X'_i) of $(X', \mathcal{I}', \emptyset, \mu)$, where $\mathcal{I}' = i_*(\mathcal{I}) \cdot \mathcal{O}_{X'}$, with the centers $i(C_i)$.*
(3) *(X_i) commutes with (separable) ground field extensions.*

Remark. The theorem will be proved first for algebraically closed fields (see Theorem 4.0.1). Then the general case will be deduced in section 5.7.

This theorem (as we show in the section below) implies slightly stronger theorems on canonical principalization and canonical embedded desingularization.

2. **Canonical principalization**

Theorem 2.4.2. *There exists a canonical principalization of a sheaf of ideals $\mathcal{I}$ on a smooth variety X, that is, a sequence of blow-ups (X_i) in the sense of Theorem* 1.0.1 *such that for any smooth morphism $\phi : X' \to X$ the induced sequence $\phi^*(X_i)$ of blow-ups is an extension of the canonical principalization of $\phi^*(\mathcal{I})$ on X'.*

3. **Canonical embedded desingularization**

Theorem 2.4.3. *There exists a canonical embedded desingularization of a subvariety Y of a smooth variety X, that is, a sequence of blow-ups (X_i) in the sense of Theorem* 1.0.2 *such that for any smooth morphism $\phi : X' \to X$ the induced sequence $\phi^*(X_i)$ of blow-ups is an extension of the canonical embedded desingularization of $Y' := \phi^{-1}(Y)$ on X'.*

2.5. **Hironaka resolution principle.** Our proof is based upon the following principle which can be traced back to Hironaka and was used by Villamayor in his simplification of Hironaka's algorithm:

Proposition 2.5.1. *Assume the ground field is of characteristic zero and is not necessarily algebraically closed. The following implications hold true:*

(1) ***Canonical resolution of marked ideals*** $(X, \mathcal{I}, E, \mu)$

$$\Downarrow$$

(2) ***Canonical principalization of the sheaves $\mathcal{I}$ on smooth ambient varieties*** X

$$\Downarrow$$

(3) ***Canonical weak embedded desingularization of subvarieties*** $Y \subset X$

$$\Downarrow$$

(4) ***Canonical desingularization of varieties over the fixed ground field*** K.

Proof. (1)⇒(2) **Canonical principalization**

Let $\sigma : X \leftarrow \widetilde{X}$ denote the morphism defined by the canonical resolution $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_k = \widetilde{X}$ of $(X, \mathcal{I}, \emptyset, 1)$. The controlled transform $(\widetilde{\mathcal{I}}, 1) = (\mathcal{I}_k, 1) = \sigma^{\mathrm{c}}(\mathcal{I}, 1)$ has empty support. Consequently, $V(\widetilde{\mathcal{I}}) = V(\mathcal{I}_k) = \emptyset$, which implies $\widetilde{\mathcal{I}} = \mathcal{I}_k = \mathcal{O}_{\widetilde{X}}$. By definition for $i = 1, \ldots, k$, we have $(\mathcal{I}_i, 1) = \sigma_i^c(\mathcal{I}_{i-1}) = \mathcal{I}(D_i)^{-1}\sigma^*(\mathcal{I}_{i-1})$, and thus

$$\sigma_i^*(\mathcal{I}_{i-1}) = \mathcal{I}_i \cdot \mathcal{I}(D_i).$$

Note that if $\mathcal{I}(D) = \mathcal{O}(-D)$ is the sheaf of ideals of a simple normal crossing divisor D on a smooth X and $\sigma : X' \to X$ is the blow-up with a smooth center C which has only simple normal crossings with D, then $\sigma^*(\mathcal{I}(D)) = \mathcal{I}(\sigma^*(D))$ is the sheaf of ideals of the divisor with simple normal crossings. The components of the induced Cartier divisors $\sigma^*(D)$ are either the strict transforms of the components of D or the components of the exceptional divisors. (The local equation $y_1^{a_1} \cdot \ldots \cdot y_l^{a_k}$ of D is transformed by the blow-up $(y_1, \ldots, y_n) \to (y_1, y_1y_2, y_1y_3, \ldots, y_1y_l, y_{l+1}, \ldots, y_n)$ into the equation $y_1^{a_1+\ldots a_l} y_2^{a_2} \ldots y_n^{a_n}$.) This implies by induction on i that

$$\sigma_i^* \sigma_{i-1}^* \ldots \sigma_2^* \sigma_1^*(\mathcal{I}_0) = \mathcal{I}_i \cdot \mathcal{I}(E_i)$$

where E_i is an exceptional divisor with simple normal crossings constructed inductively as

$$\mathcal{I}(E_i) = \sigma^*(\mathcal{I}(E_{i-1}))\mathcal{I}(D_i).$$

Finally the full transform $\sigma_k^*(\mathcal{I}) = \mathcal{I}_k \cdot \mathcal{I}(E_k) = \mathcal{O}_{\widetilde{X}} \cdot \mathcal{I}(E_k) = \mathcal{I}(E_k)$ is principal and generated by the sheaf of ideals of a divisor whose components are the exceptional divisors. The canonicity conditions for principalization follow from the canonicity of resolution of marked ideals.

The actual process of principalization is controlled by some invariants and is often achieved before $(X, \mathcal{I}, \emptyset, 1)$ has been resolved (Theorem 4.0.1 and section 5.5)

(2)⇒(3) **Canonical embedded desingularization**

Lemma 2.5.2. *The canonical principalization of $\mathcal{I}$ defines an isomorphism over $X \setminus V(\mathcal{I})$.*

Proof. The canonical principalization of $(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n})$ is an isomorphism over generic points and is equivariant with respect to translations; thus it is an isomorphism. The restriction of the canonical principalization $(\widetilde{X}, \widetilde{\mathcal{I}})$ of $(X, \mathcal{I})$ to an open subset U is the canonical principalization of $(U, \mathcal{I}_{|U})$. Let $\widetilde{X} \to X$ be the canonical principalization of $(X, \mathcal{O}_X)$ and $x \in X \setminus V(\mathcal{I})$. Locally we find an étale morphism from an open subset $U \subset X \setminus V(\mathcal{I})$ to $\mathbf{A}^n$, and the canonical principalization of $(U, \mathcal{I}_U) = (U, \mathcal{O}_U)$ is induced by the canonical principalization of $(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n})$ and thus it is an isomorphism. □

Let $Y \subset X$ be an irreducible subvariety. Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_k = \widetilde{X}$ be the canonical principalization of sheaves of ideals $\mathcal{I}_Y$ subject to conditions (a) and (b) from Theorem 1.0.1. Suppose all centers C_{i-1} of the blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ are disjoint from the generic points of strict transforms $Y_{i-1} \subset Y$ of Y. Then $\widetilde{\sigma}$ is an isomorphism over the generic points y of Y and $\widetilde{\sigma}^*(\mathcal{I})_y = \sigma^*(\mathcal{I})_y$. Moreover no exceptional divisor passes through y. This contradicts the condition $\widetilde{\sigma}^*(\mathcal{I}) = \mathcal{I}_{\widetilde{E}}$. Thus there is a smallest i_{res} with the property that $C_{i_{\mathrm{res}}}$ contains the strict transform $Y_{i_{\mathrm{res}}}$ and all centers C_j for $j < i_{\mathrm{res}}$ are disjoint from the generic points of strict transforms Y_j. Let $y \in Y_{i_{\mathrm{res}}}$ be a generic point for which $X_{i_{\mathrm{res}}} \to X$

is an isomorphism. Find an open set $U \subset X$ intersecting Y such that $X_{i_{\rm res}} \to X$ is an isomorphism over U. Then $Y_{i_{\rm res}} \cap U = Y \cap U$ and $C_{i_{\rm res}} \cap U \supseteq Y_{i_{\rm res}} \cap U$ by the definition of $Y_{i_{\rm res}}$. On the other hand, by the previous lemma $C_{i_{\rm res}} \cap U \subseteq Y_{i_{\rm res}} \cap U$, which gives $C_{i_{\rm res}} \cap U = Y_{i_{\rm res}} \cap U$. Finally, $Y_{i_{\rm res}}$ is an irreducible component of a smooth (possibly reducible) center C_i. This implies that $Y_{i_{\rm res}}$ is smooth and has simple normal crossings with the exceptional divisors. We define the canonical embedded resolution of (X, Y) to be

$$(X,Y) = (X_0, Y_0) \leftarrow (X_1, Y_1) \leftarrow (X_2, Y_2) \leftarrow \ldots \leftarrow (X_{i_{\rm res}}, Y_{i_{\rm res}}).$$

If $(X', Y') \to (X, Y)$ is a smooth morphism, then the induced sequence of blow-ups $(X'_i)_{0\le i\le k} = (X' \times_X X_i)_{0\le i\le k}$ is an extension of the canonical principalization $(X'_j)_{0\le j\le k'}$ of $(X', \mathcal{I}_{Y'})$. Moreover $X'_{j_{\rm res}} = X'_{i_{\rm res}}$ and $(X'_i)_{0\le i\le i_{\rm res}}$ is an extension of the canonical resolution $(X'_j)_{0\le j\le j_{\rm res}}$ of $(X', \mathcal{I}_{Y'})$.

Commutativity with closed embeddings and field extensions for embedded desingularizations follows from the commutativity with closed embeddings and field extensions for principalizations.

In the actual algorithm considered in the paper the moment $X_{i_{\rm res}}$ is detected by some invariant (see section 5.6).

(3)⇒(4) **Canonical desingularization**

Let Y be an algebraic variety over K. By the compactness of Y we find a cover of affine subsets U_i of Y such that each U_i is embedded in an affine space $\mathbf{A}^n$ for $n \gg 0$. We can assume that the dimension n is the same for all U_i by taking if necessary embeddings of affine spaces $\mathbf{A}^{k_i} \subset \mathbf{A}^n$.

Lemma 2.5.3 (see also Jelonek [29], Lemma 4.8.1)**.** *For any closed embeddings $\phi_1, \phi_2 : Y \subset \mathbf{A}^n$ there exist closed embeddings $\psi_1, \psi_2 : \mathbf{A}^n \to \mathbf{A}^{2n}$ such that $\psi_1\phi_1 = \psi_2\phi_2$.*

Proof. The embeddings ϕ_1 and ϕ_2 are defined by two sets of generators $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ respectively. Define three embeddings $\Psi_i : Y \to \mathbf{A}^{2n}$ for $i = 0, 1, 2$ such that

$$\begin{aligned}
\Psi_0(x) &= (g_1(x), \ldots, g_n(x), h_1(x), \ldots, h_n(x)),\\
\Psi_1(x) &= (g_1(x), \ldots, g_n(x), 0, \ldots, 0),\\
\Psi_2(x) &= (0, \ldots, 0, h_1(x), \ldots, h_n(x)).
\end{aligned}$$

Let $i_1, i_2 : \mathbf{A}^n \hookrightarrow \mathbf{A}^{2n}$ be two embeddings defined as $i_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ and $i_2(x_1, \ldots, x_n) = (0, \ldots, 0, x_1, \ldots, x_n)$. Then $\Psi_i = i_i\phi_i$ for $i = 1, 2$. Fix coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $\mathbf{A}^{2n}$. Find polynomials $w_i(x_1, \ldots, x_n)$ and $v_i(y_1, \ldots, y_n)$ such that for $i = 1, \ldots, n$,

$$w_i(h_1, \ldots, h_n) = g_i, \quad v_i(g_1, \ldots, g_n) = h_i.$$

Construct automorphisms Φ_1, Φ_2 of $\mathbf{A}^{2n}$ such that

$$\begin{aligned}
&\Phi_1(x_1, \ldots, x_n, y_1, \ldots, y_n)\\
&\qquad = (x_1, \ldots, x_n, y_1 - v_i(x_1, \ldots, x_n), \ldots, y_n - v_n(x_1, \ldots, x_n)),\\
&\Phi_2(x_1, \ldots, x_n, y_1, \ldots, y_n)\\
&\qquad = (x_1 - w_1(y_1, \ldots, y_n), \ldots, x_n - w_n(y_1, \ldots, y_n), y_1, \ldots, y_n).
\end{aligned}$$

Then for $i = 1, 2$ we have $\Phi_i\Psi_0 = \Psi_i$ and $\Psi_0 = \Phi_i^{-1}\Psi_i = \Phi_i^{-1} i_i\phi_i$. Finally we put $\psi_i := \Phi_i^{-1} i_i$. □

Proposition 2.5.4. *For any affine variety U there is a smooth variety $\widetilde{U}$ along with a birational morphism* $\mathrm{res}: \widetilde{U} \to U$ *subject to the conditions:*

(1) *For any closed embedding $U \subset X$ into a smooth affine variety X, there is a closed embedding $\widetilde{U} \subset \widetilde{X}$ into a smooth variety $\widetilde{X}$ which is a canonical embedded desingularization of $U \subset X$.*
(2) *For any open embedding $V \hookrightarrow U$ there is an open embedding of resolutions $\widetilde{V} \hookrightarrow \widetilde{U}$ which is a lifting of $V \to U$ such that $\widetilde{V} \to \mathrm{res}_U^{-1}(V)$ is an isomorphism over V.*
(3) *If L is a field containing the ground field K, then the resolution $\widetilde{U}_L$ of $U_L := U \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,L$ is equal to $\widetilde{U} \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,L$.*

Proof. (1) Consider a closed embedding of U into a smooth affine variety X (for example $X = \mathbf{A}^n$). The canonical embedded desingularization $\widetilde{U} \subset \widetilde{X}$ of $U \subset X$ defines the desingularization $\widetilde{U} \to U$. This desingularization is independent of the ambient variety X. Let $\phi_1 : U \subset X_1$ and $\phi_2 : U \subset X_2$ be two closed embeddings and let $\widetilde{U}_i \subset \widetilde{X}_i$ be two embedded desingularizations. Find embeddings $\psi_i : X_i \to \mathbf{A}^n$ into the affine space $\mathbf{A}^n$. They define the embeddings $\psi_i\phi_i : U \to \mathbf{A}^n$. By Lemma 2.5.3, there are embeddings $\Psi_i : \mathbf{A}^n \to \mathbf{A}^{2n}$ such that $\Psi_1\psi_1\phi_1 = \Psi_2\psi_2\phi_2 : U \to \mathbf{A}^{2n}$. Since embedded desingularizations commute with closed embeddings of ambient varieties we see that the $\widetilde{U}_i$ are isomorphic over U.

(2) Let $V \to U$ be an open embedding of affine varieties. Assume first that $V = U_f = U \setminus V(f)$, where $f \in K[U]$ is a regular function on U. Let $U \subset X$ be a closed embedding into an affine variety X. Then $U_f \subset X_F$ is a closed embedding into an affine variety $X_F = X \setminus V(F)$ where F is a regular function on F which restricts to f. Since embedded desingularizations commute with smooth morphisms the open embedding $X_F \subset X$ defines the open embedding of embedded desingularizations $(\widetilde{X_F}, \widetilde{U_f}) \subset (\widetilde{X}, \widetilde{U})$ and the open embedding of desingularizations $\widetilde{U_f} \subset \widetilde{U}$.

Let $V \subset U$ be any open subset which is an affine variety. Then there are desingularizations $\mathrm{res}_V : \widetilde{V} \to V$ and $\mathrm{res}_U : \widetilde{U} \to U$. Suppose the natural birational map $\widetilde{V} \to \mathrm{res}_U^{-1}(V)$ is not an isomorphism over V. Then we can find an open subset $U_f \subset V$ such that $\mathrm{res}_V^{-1}(U_f) \to \mathrm{res}_U^{-1}(U_f)$ is not an isomorphism over U_f. But $U_f = V_f$ and by the previous case $\mathrm{res}_V^{-1}(U_f) \simeq \widetilde{U_f} = \widetilde{V_f} \simeq \mathrm{res}_U^{-1}(V)$.

(3) Let $U \hookrightarrow \mathbf{A}^n$ be a closed embedding. It defines a closed embedding $U_L \hookrightarrow \mathbf{A}^n_L$. Then commutativity with ground field extensions follows from the commutativity for embedded desingularizations. $\square$

Let U_i be an open affine cover of X. For any two open subsets U_i and U_j set $U_{ij} := U_i \cap U_j$. For any U_i and U_{ij} we find canonical resolutions $\widetilde{U_i}$ and $\widetilde{U}_{ij}$ respectively. By the proposition $\widetilde{U}_{ij}$ can be identified with an open subset of $\widetilde{U}_i$. We define $\widetilde{X}$ to be a variety obtained by glueing $\widetilde{U_i}$ along $\widetilde{U}_{ij}$. Then $\widetilde{X}$ is a smooth variety and $\widetilde{X} \to X$ defines a canonical desingularization independent of the choice of U_{ij}.

Commutativity of non-embedded desingularization with smooth morphisms

Lemma 2.5.5. *Assume that the ground field K is algebraically closed. Let $\phi_1 : U \hookrightarrow \mathbf{A}^m$ and $\phi_2 : V \hookrightarrow \mathbf{A}^n$ be closed embeddings of affine varieties U and V. Let*

$\phi : U \to V$ be a morphism étale at $0 \in U$. Denote by $\phi_1 \times (\phi_2 \circ \phi) : U \hookrightarrow \mathbf{A}^{m+n}$ the induced embedding of U.

Then there exists an open neighborhood $U_0 \subset U$ of 0 such that $\phi : U_0 \to V$ is étale, a smooth locally closed subvariety $X_0 \subset \mathbf{A}^{m+n}$ containing $U_0 \subset U \subset \mathbf{A}^{m+n}$ as a closed subset and an étale morphism $\Phi : X_0 \to \mathbf{A}^n$ extending the morphism $\phi : U_0 \to V$ such that $U_0 = X_0 \times_{\mathbf{A^n}} V$.

Proof. Let $\overline{x} := x_1, \dots, x_n$ and $\overline{y} := y_1, \dots, y_m$ be coordinates on $\mathbf{A}^n$ and $\mathbf{A}^m$ respectively. Let $g_1 := \phi^*(x_1), \dots, g_n := \phi^*(x_n)$ be generators of the ring $K[V] \subset K[U]$. Write $K[V] = K[x_1, \dots, x_n]/(f_1(\overline{x}), \dots, f_l(\overline{x}))$. Extending the set of generators of $K[V]$ to a set of generators of $K[U]$ gives

$$K[U] = K[x_1, \dots, x_n, y_1, \dots, y_m]/(f_1(\overline{x}), \dots, f_l(\overline{x}), h_1(\overline{x}, \overline{y}), \dots, h_r(\overline{x}, \overline{y})).$$

Since ϕ is étale at 0 the functions $x_1, \dots, x_n$ generate the maximal ideal of

$$\begin{aligned} \widehat{\mathcal{O}}_{0,U} &= K[[x_1, \dots, x_n, y_1, \dots, y_m]]/(f_1(x), \dots, f_l(x), h_1, \dots, h_r) \\ &= K[[x_1, \dots, x_n]]/(f_1, \dots, f_l) = \widehat{\mathcal{O}}_{x,V}. \end{aligned}$$

Choose a maximal subset $\{h_{i_1}, \dots, h_{i_s}\} \subset \{h_1, \dots, h_r\}$ for which $x_1, \dots, x_n, h_{i_1}, \dots, h_{i_s} \in \frac{(x_1, \dots, x_n, y_1, \dots, y_m)}{(x_1, \dots, x_n, y_1, \dots, y_m)^2}$ are linearly independent. Then by the Nakayama Lemma $s = m$ and $(x_1, \dots, x_n, h_{i_1}, \dots, h_{i_m}) = (x_1, \dots, x_n, y_1, \dots, y_m)$ define the set of parameters at x.

The subvariety $X := \{(x, y) \mid h_{i_1} = \ldots = h_{i_m} = 0\} \subset \mathbf{A}^{n+m}$ is smooth at 0 and the restriction $p_{|X} : X \to \mathbf{A}^n$ of the natural projection $p : \mathbf{A}^{n+m} \to \mathbf{A}^n$ is étale at 0. Consequently, $U' \to V$ is étale at 0 where $U' := \mathrm{Spec}(K[x_1, \dots, x_n, y_1, \dots, y_m]/(f_1(x), \dots, f_l(x), h_{i_1}(x, y), \dots, h_{i_m}(x, y))$. Also the natural closed embedding $U \to U'$ is étale at 0. Then U is the irreducible component of U' passing through 0, and we can find a smooth open subset $X_0 \subset X \setminus (U' \setminus U)$ containing 0 for which the morphism $X_0 \to \mathbf{A}^n$ is étale and which contains some open neighborhood $U_0 := U \cap X_0$ of U. □

Now let $\overline{K}$ denote the algebraic closure of K. Let $\phi : Y' \to Y$ be a smooth morphism of relative dimension r. Let $\widetilde{Y} \to Y$ and $\widetilde{Y'} \to Y'$ be the resolution morphisms. By definition ϕ lifts to a unique rational map $\widetilde{\phi} : \widetilde{Y'} \dashrightarrow \widetilde{Y}$. Taking the fiber product with $\mathrm{Spec}\,\overline{K}$ over $\mathrm{Spec}\,K$ defines the smooth morphism $\phi_{\overline{K}} : Y'_{\overline{K}} \to Y_{\overline{K}}$ and its lifting $\widetilde{\phi}_{\overline{K}} : \widetilde{Y'}_{\overline{K}} \dashrightarrow \widetilde{Y}_{\overline{K}}$. For any points $y' \in Y'_{\overline{K}}$ and $y = \phi(y') \in Y_{\overline{K}}$ find their open affine neighborhoods $U' \subset Y'_{\overline{K}}$ and $U \subset Y_{\overline{K}}$ such that the restriction $\phi_{|U'} : U' \to U$ factors through an étale morphism $\psi : U' \to U \times \mathbf{A}^r$ followed by the natural projection $p : U \times \mathbf{A}^r \to U$. Since U' and U are affine we can find closed embeddings $U' \hookrightarrow \mathbf{A}^m$ and $U \hookrightarrow \mathbf{A}^n$. The last one defines an embedding $U \times \mathbf{A}^r \to \mathbf{A}^{n+r}$. By the above lemma by shrinking U' we can find a smooth $X' \supset U'$ and an étale morphism $X' \to \mathbf{A}^{n+r}$ which extends the étale morphism $U' \to U \times \mathbf{A}^r$. Consequently, the induced smooth morphism $X' \to \mathbf{A}^n$ extends the smooth morphism $\phi_{|U'} : U' \to U$. The smooth morphism $(X', U') \to (X, U)$, where $X =: \mathbf{A}^n$, defines a smooth morphism of embedded resolutions $(\widetilde{X'}, \widetilde{U'}) \to (\widetilde{X}, \widetilde{U})$ and in particular the smooth morphism of nonembedded resolutions $\widetilde{U'} \to \widetilde{U}$ which is a unique lifting of $\phi_{|U'} : U' \to U$. Finally, the lifting $\widetilde{\phi}_{\overline{K}} : \widetilde{Y'}_{\overline{K}} \to \widetilde{Y}_{\overline{K}}$ is a smooth morphism. The rational map $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ is a smooth morphism as well. □

3. Marked ideals

3.1. **Equivalence relation for marked ideals.** Let us introduce the following equivalence relation for marked ideals:

Definition 3.1.1. Let $(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ and $(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ be two marked ideals on the smooth variety X. Then $(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) \simeq (X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ if

(1) $E_{\mathcal{I}} = E_{\mathcal{J}}$ and the orders on $E_{\mathcal{I}}$ and on $E_{\mathcal{J}}$ coincide.
(2) $\mathrm{supp}(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) = \mathrm{supp}(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$.
(3) All the multiple test blow-ups $X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \ldots \longleftarrow X_i \xleftarrow{\sigma_r} \ldots \longleftarrow X_r$ of $(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ are exactly the multiple test blow-ups of $(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ and moreover we have
$$\mathrm{supp}(X_i, \mathcal{I}_i, E_i, \mu_{\mathcal{I}}) = \mathrm{supp}(X_i, \mathcal{J}_i, E_i, \mu_{\mathcal{J}}).$$

It is easy to show the lemma:

Lemma 3.1.2. *For any $k \in \mathbf{N}$, $(\mathcal{I}, \mu) \simeq (\mathcal{I}^k, k\mu)$.*

Remark. The marked ideals considered in this paper satisfy a stronger equivalence condition: For any smooth morphisms $\phi : X' \to X$, $\phi^*(\mathcal{I}, \mu) \simeq \phi^*(\mathcal{J}, \mu)$. This condition will follow and is not added in the definition.

3.2. **Ideals of derivatives.** Ideals of derivatives were first introduced and studied in the resolution context by Giraud. Villamayor developed and applied this language to his *basic objects*.

Definition 3.2.1 (Giraud, Villamayor)**.** Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth variety X. By the *first derivative* (originally *extension*) $\mathcal{D}_X(\mathcal{I})$ of $\mathcal{I}$ (or simply $\mathcal{D}(\mathcal{I})$) we mean the coherent sheaf of ideals generated by all functions $f \in \mathcal{I}$ with their first derivatives. Then the *i-th derivative* $\mathcal{D}^i(\mathcal{I})$ is defined to be $\mathcal{D}(\mathcal{D}^{i-1}(\mathcal{I}))$. If $(\mathcal{I}, \mu)$ is a marked ideal and $i \leq \mu$, then we define
$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Recall that on a smooth variety X there is a locally free sheaf of differentials $\Omega_{X/K}$ over K generated locally by $du_1, \ldots, du_n$ for a set of local parameters $u_1, \ldots, u_n$. The dual sheaf of derivations $\mathrm{Der}_K(\mathcal{O}_X)$ is locally generated by the derivations $\frac{\partial}{\partial u_i}$. Immediately from the definition we observe that $\mathcal{D}(\mathcal{I})$ is a coherent sheaf defined locally by generators f_j of $\mathcal{I}$ and all their partial derivatives $\frac{\partial f_j}{\partial u_i}$. We see by induction that $\mathcal{D}^i(\mathcal{I})$ is a coherent sheaf defined locally by the generators f_j of $\mathcal{I}$ and their derivatives $\frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq i$.

Remark. In characteristic p the partial derivatives $\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial u^\alpha}$ (where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$) are well-defined and are called the Hasse-Dieudonné derivatives. They should be used in the definition of the derivatives of marked ideals. One of the major sources of problems is that unlike in characteristic zero
$$\mathcal{D}^i(\mathcal{D}^j(\mathcal{I})) \subsetneq \mathcal{D}^{i+j}(\mathcal{I}).$$

Lemma 3.2.2 (Giraud, Villamayor)**.** *For any $i \leq \mu - 1$,*
$$\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^i(\mathcal{I}), \mu - i)).$$
In particular $\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = V(\mathcal{D}^{\mu-1}(\mathcal{I}))$ *is a closed set ($i = \mu - 1$).*

Proof. It suffices to prove the lemma for $i=1$. If $x\in\operatorname{supp}(\mathcal{I},\mu)$, then for any $f\in\mathcal{I}$ we have $\operatorname{ord}_x(f)\geq\mu$. This implies $\operatorname{ord}_x(Df)\geq\mu-1$ for any derivative D and consequently $x\in\operatorname{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. Now, let $x\in\operatorname{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. Then for any $f\in\mathcal{I}$ we have $\operatorname{ord}_x(f)\geq\mu-1$. Suppose $\operatorname{ord}_x(f)=\mu-1$ for some $f\in\mathcal{I}$. Then $f=\sum_{|\alpha|\geq\mu-1}c_\alpha x^\alpha$ and there is α such that $\alpha=\mu-1$ and $c_\alpha\neq 0$. We find $\frac{\partial}{\partial x_i}$ for which $\operatorname{ord}_x(\frac{\partial x^\alpha}{\partial x_i})=\mu-2$ and thus $\operatorname{ord}_x(\frac{\partial f}{\partial x_i})=\mu-2$ and $x\notin\operatorname{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. $\square$

We write $(\mathcal{I},\mu)\subset(\mathcal{J},\mu)$ if $\mathcal{I}\subset\mathcal{J}$.

Lemma 3.2.3 (Giraud, Villamayor). *Let $(\mathcal{I},\mu)$ be a marked ideal and $C\subset\operatorname{supp}(\mathcal{I},\mu)$ be a smooth center and $r\leq\mu$. Let $\sigma: X\leftarrow X'$ be a blow-up at C. Then*

$$\sigma^{\mathrm{c}}(\mathcal{D}^r_X(\mathcal{I},\mu))\subseteq\mathcal{D}^r_{X'}(\sigma^{\mathrm{c}}(\mathcal{I},\mu)).$$

Proof. First assume that $r=1$. Let $u_1,\ldots,u_n$ denote the local parameters at x. Then the local parameters at $x'\in\sigma^{-1}(x)$ are of the form $u_i'=\frac{u_i}{u_m}$ for $i<m$ and $u_i'=u_i$ for $i\geq m$, where $u_m=u_m'=y$ denotes the local equation of the exceptional divisor.

The derivations $\frac{\partial}{\partial u_i}$ of $\mathcal{O}_{x,X}$ extend to derivations of the rational field $K(X)$. Note also that

$$\frac{\partial u_j'}{\partial u_i}=\frac{\delta_{ij}}{u_m},\quad i<m, 1\leq j\leq n;\qquad \frac{\partial u_j'}{\partial u_m}=-\frac{1}{u_m^2}u_j,\quad j<m;$$
$$\frac{\partial u_m'}{\partial u_m}=1,\qquad \frac{\partial u_j'}{\partial u_m}=0, j>m;\qquad \frac{\partial u_i'}{\partial u_j}=\delta_{ij},\quad i\geq m.$$

This gives

$$\frac{\partial}{\partial u_i}=\frac{1}{u_m}\frac{\partial}{\partial u_i'}=\frac{1}{y}\frac{\partial}{\partial u_i'},\quad 1\leq i<m;$$
$$\frac{\partial}{\partial u_m}=-\frac{1}{y}(u_1'\frac{\partial}{\partial u_1'}+\ldots+u_{m-1}'\frac{\partial}{\partial u_{m-1}'}-u_m'\frac{\partial}{\partial u_m'}),$$
$$\frac{\partial}{\partial u_i'}=\frac{\partial}{\partial u_i},\quad m<i\leq n.$$

We see that any derivation D of $\mathcal{O}_X$ induces a derivation $y\sigma^*(D)$ of $\mathcal{O}_{X'}$. Thus the sheaf $y\sigma^*(\operatorname{Der}_K(\mathcal{O}_X))$ of such derivations is a subsheaf of $\operatorname{Der}_K(\mathcal{O}_{X'})$ locally generated by

$$\frac{\partial}{\partial u_i'}, i<m,\quad y\frac{\partial}{\partial y},\quad\text{and}\quad y\frac{\partial}{\partial u_i'}, i>m.$$

In particular $y\sigma^*(\mathcal{D}_X(\mathcal{I}))\subset\mathcal{D}_{X'}(\sigma^*(\mathcal{I}))$. For any sheaf of ideals $\mathcal{J}$ on X' denote by $y\sigma^*(\mathcal{D}_X)(\mathcal{J})\subset\mathcal{D}_{X'}(\mathcal{J})$ the ideal generated by $\mathcal{J}$ and the derivatives $D'(f)$, where $f\in\mathcal{J}$ and $D'\in y\sigma^*(\operatorname{Der}_K(\mathcal{O}_X))$. Note that for any $f\in\mathcal{J}$ and $D'\in y\sigma^*(\operatorname{Der}_K(\mathcal{O}_X))$, y divides $D'(y)$ and

$$D'(yf)=yD'(f)+D'(y)f\quad\in\quad y\sigma^*(\mathcal{D}_X)(\mathcal{J})+y\mathcal{J}=y\sigma^*(\mathcal{D}_X)(\mathcal{J}).$$

Consequently, $y\sigma^*(\mathcal{D}_X)(y\mathcal{J})\subseteq yy\sigma^*(\mathcal{D}_X)(\mathcal{J})$ and more generally $y\sigma^*(\mathcal{D}_X)(y^\mu\mathcal{J})\subseteq y^\mu y\sigma^*(\mathcal{D}_{X'})(\mathcal{J})$. Then

$$\begin{aligned}y\sigma^*(\mathcal{D}_X(\mathcal{I}))\subseteq y\sigma^*(\mathcal{D}_X)(\sigma^*(\mathcal{I}))&=y\sigma^*(\mathcal{D}_X)(y^\mu\sigma^{\mathrm{c}}(\mathcal{I}))\\&\subseteq y^\mu y\sigma^*(\mathcal{D}_X)(\sigma^{\mathrm{c}}(\mathcal{I}))\subseteq y^\mu\mathcal{D}_{X'}(\sigma^{\mathrm{c}}(\mathcal{I})).\end{aligned}$$

Then

$$\sigma^{\mathrm{c}}(\mathcal{D}_X(\mathcal{I})) = y^{-\mu+1}\sigma^*(\mathcal{D}_X(\mathcal{I})) \subseteq \mathcal{D}_{X'}(\sigma^{\mathrm{c}}(\mathcal{I})).$$

Assume now that r is arbitrary. Then $C \subset \operatorname{supp}(\mathcal{I},\mu) = \operatorname{supp}(\mathcal{D}^i_X(\mathcal{I},\mu))$ for $i \le r$ and by induction on r,

$$\sigma^{\mathrm{c}}(\mathcal{D}^r_X\mathcal{I}) = \sigma^{\mathrm{c}}(\mathcal{D}_X(\mathcal{D}^{r-1}_X(\mathcal{I}))) \subseteq \mathcal{D}_{X'}(\sigma^{\mathrm{c}}\mathcal{D}^{r-1}_X(\mathcal{I})) \subseteq \mathcal{D}^r_{X'}(\sigma^{\mathrm{c}}(\mathcal{I})).$$

$\square$

As a corollary from Lemma 3.2.3 we prove the following.

Lemma 3.2.4. *A multiple test blow-up $(X_i)_{0\le i\le k}$ of $(\mathcal{I},\mu)$ is a multiple test blow-up of $\mathcal{D}^j(\mathcal{I},\mu)$ for $0 \le j \le \mu$ and*

$$[\mathcal{D}^j(\mathcal{I},\mu)]_k \subset \mathcal{D}^j(\mathcal{I}_k,\mu).$$

Proof. Induction on k. For $k=0$ it is evident. Let $\sigma_{k+1} : X_k \leftarrow X_{k+1}$ denote the blow-up with a center $C_k \subseteq \operatorname{supp}(\mathcal{I}_k,\mu) = \operatorname{supp}(\mathcal{D}^j(\mathcal{I}_k,\mu)) \subseteq \operatorname{supp}([\mathcal{D}^j(\mathcal{I},\mu)]_k)$. Then by induction $[\mathcal{D}^j(\mathcal{I},\mu)]_{k+1} = \sigma^{\mathrm{c}}_{k+1}[\mathcal{D}^j(\mathcal{I},\mu)]_k \subseteq \sigma^{\mathrm{c}}_{k+1}(\mathcal{D}^j(\mathcal{I}_k,\mu))$. Lemma 3.2.3 gives $\sigma^{\mathrm{c}}_{k+1}(\mathcal{D}^j(\mathcal{I}_k,\mu)) \subseteq \mathcal{D}^j\sigma^{\mathrm{c}}_{k+1}(\mathcal{I}_k,\mu) = \mathcal{D}^j(\mathcal{I}_{k+1},\mu)$. $\square$

Lemma 3.2.5. *Let $\phi : X' \to X$ be an étale morphism of smooth varieties and $\mathcal{I}$ be a sheaf of ideals on X. Then*

$$\phi^*(\mathcal{D}(\mathcal{I})) = \mathcal{D}(\phi^*(\mathcal{I})).$$

Proof. Since ϕ is étale, for any points $x' \in X'$ and $\phi(x') = x$ we have $\widehat{\mathcal{O}}_{x',X'} \simeq \widehat{\mathcal{O}}_{x,X}$ and $\widehat{\phi}^*(\widehat{\mathcal{D}(\mathcal{I})})_x = (\mathcal{D}(\widehat{\phi^*(\mathcal{I})}))_{x'}$. Since $\widehat{\mathcal{O}}_{x,X}$ is faithfully flat over $\mathcal{O}_{x,X}$ and $(\widehat{\mathcal{D}(\mathcal{I})}) = \mathcal{D}(\mathcal{I})\cdot\widehat{\mathcal{O}}_{x,X}$ we get the equality of stalks $\phi^*(\mathcal{D}(\mathcal{I}))_x = \mathcal{D}(\phi^*(\mathcal{I}))_{x'}$, which determines the equality of sheaves $\phi^*(\mathcal{D}(\mathcal{I})) = \mathcal{D}(\phi^*(\mathcal{I}))$. $\square$

3.3. **Hypersurfaces of maximal contact.** The concept of the *hypersurfaces of maximal contact* is one of the key points of this proof. It was originated by Hironaka, Abhyankhar and Giraud and developed in the papers of Bierstone-Milman and Villamayor.

In our terminology we are looking for a smooth hypersurface containing the supports of marked ideals and whose strict transforms under multiple test blow-ups contain the supports of the induced marked ideals. Existence of such hypersurfaces allows a reduction of the resolution problem to codimension 1.

First we introduce marked ideals which locally admit hypersurfaces of maximal contact.

Definition 3.3.1 (Villamayor [34]). We say that a marked ideal $(\mathcal{I},\mu)$ is of *maximal order* (originally *simple basic object*) if $\max\{\operatorname{ord}_x(\mathcal{I}) \mid x \in X\} \le \mu$ or equivalently $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$.

Lemma 3.3.2 (Villamayor [34]). *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order and $C \subset \operatorname{supp}(\mathcal{I},\mu)$ be a smooth center. Let $\sigma : X \leftarrow X'$ be a blow-up at $C \subset \operatorname{supp}(\mathcal{I},\mu)$. Then $\sigma^{\mathrm{c}}(\mathcal{I},\mu)$ is of maximal order.*

Proof. If $(\mathcal{I},\mu)$ is a marked ideal of maximal order, then $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$. Then by Lemma 3.2.3, $\mathcal{D}^\mu(\sigma^{\mathrm{c}}(\mathcal{I},\mu)) \supset \sigma^{\mathrm{c}}(\mathcal{D}^\mu(\mathcal{I}),0) = \mathcal{O}_X$. $\square$

Lemma 3.3.3 (Villamayor [34]). *If $(\mathcal{I},\mu)$ is a marked ideal of maximal order and $0 \le i \le \mu$, then $\mathcal{D}^i(\mathcal{I},\mu)$ is of maximal order.*

Proof. $\mathcal{D}^{\mu-i}(\mathcal{D}^i(\mathcal{I},\mu)) = \mathcal{D}^{\mu}(\mathcal{I},\mu) = \mathcal{O}_X$. □

In particular $(\mathcal{D}^{\mu-1}(\mathcal{I}),1)$ is a marked ideal of maximal order.

Lemma 3.3.4 (Giraud)**.** *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Let $\sigma : X \leftarrow X'$ be a blow-up at a smooth center $C \subsetneq \operatorname{supp}(\mathcal{I},\mu)$. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I},\mu)(U)$ be a function of multiplicity one on U, that is, for any $x \in V(u)$, $\operatorname{ord}_x(u) = 1$. In particular* $\operatorname{supp}(\mathcal{I},\mu) \cap U \subset V(u)$. *Let $U' \subset \sigma^{-1}(U) \subset X'$ be an open set where the exceptional divisor is described by y. Let $u' := \sigma^{\mathrm{c}}(u) = y^{-1}\sigma^*(u)$ be the controlled transform of u. Then*

(1) $u' \in \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}_{|U'},\mu))$.
(2) *u' is a function of multiplicity one on U'.*
(3) *$V(u')$ is the restriction of the strict transform of $V(u)$ to U'.*

Proof. (1) $u' = \sigma^{\mathrm{c}}(u) = u/y \in \sigma^{\mathrm{c}}(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}))$.

(2) Since u was one of the local parameters describing the center of blow-ups, $u' = u/y$ is a parameter, that is, a function of order one.

(3) follows from (2). □

Definition 3.3.5. We shall call a function

$$u \in T(\mathcal{I})(U) := \mathcal{D}^{\mu-1}(\mathcal{I}(U))$$

of multiplicity one a *tangent direction* of $(\mathcal{I},\mu)$ on U.

As a corollary from the above we obtain the following lemma:

Lemma 3.3.6 (Giraud)**.** *Let $u \in T(\mathcal{I})(U)$ be a tangent direction of $(\mathcal{I},\mu)$ on U. Then for any multiple test blow-up (U_i) of $(\mathcal{I}_{|U},\mu)$ all the supports of the induced marked ideals* $\operatorname{supp}(\mathcal{I}_i,\mu)$ *are contained in the strict transforms $V(u)_i$ of $V(u)$.* □

Remarks. (1) Tangent directions are functions defining locally hypersurfaces of maximal contact.

(2) The main problem leading to complexity of the proofs is that of a noncanonical choice of the tangent directions. We overcome this difficulty by introducing *homogenized ideals.*

Lemma 3.3.7 (Villamayor)**.** *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order whose support is of codimension* 1. *Then all codimension one components of* $\operatorname{supp}(\mathcal{I},\mu)$ *are smooth and isolated. After the blow-up $\sigma : X \leftarrow X'$ at such a component $C \subset \operatorname{supp}(\mathcal{I},\mu)$ the induced support* $\operatorname{supp}(\mathcal{I}',\mu)$ *does not intersect the exceptional divisor of σ.*

Proof. By the previous lemma there is a tangent direction $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ whose zero set is smooth and contains $\operatorname{supp}(\mathcal{I},\mu)$. Then $\mathcal{D}^{\mu-1}(\mathcal{I}) = (u)$ and $\mathcal{I}$ is locally described as $\mathcal{I} = (u^{\mu})$. Suppose there is $g \in \mathcal{I}$ written as $g = c_{\mu}(x,u)u^{\mu} + c_{\mu-1}(x)u^{\mu-1} + \ldots + c_0(x)$, where at least one function $c_i(x) \neq 0$ for $0 \leq i \leq \mu-1$. Then there is a multi-index α such $|\alpha| = \mu-i-1$ and $\frac{\partial^{|\alpha|}c_i}{\partial x^{\alpha}}$ is not the zero function. Then the derivative $\frac{\partial^{\mu-1}g}{\partial u^i \partial x^{\alpha}} \in \mathcal{D}^{\mu-1}(\mathcal{I})$ does not belong to the ideal (u).

The blow-up at the component C locally defined by u transforms $(\mathcal{I},\mu) = ((u^{\mu}),\mu)$ to $(\mathcal{I}',\mu)$, where $\sigma^*(\mathcal{I}) = y^{\mu}\mathcal{O}_X$, and $\mathcal{I}' = \sigma^{\mathrm{c}}(\mathcal{I}) = y^{-\mu}\sigma^*(\mathcal{I}) = \mathcal{O}_X$, where $y = u$ describes the exceptional divisor. □

Remark. Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

3.4. **Arithmetical operations on marked ideals.** In this section all marked ideals are defined for the smooth variety X and the same set of exceptional divisors E. Define the following operations of addition and multiplication of marked ideals:

(1) $(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}}):=(\mathcal{I}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{I}}}+\mathcal{J}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{J}}},\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}}))$
or more generally (the operation of addition is not associative)

$$(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m):=(\mathcal{I}_1^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_1} + \mathcal{I}_2^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_2}+\ldots+\mathcal{I}_m^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_m},\mathrm{lcm}(\mu_1,\ldots,\mu_m)).$$

(2) $(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}}):=(\mathcal{I}\cdot J,\mu_{\mathcal{I}}+\mu_{\mathcal{J}})$.

Lemma 3.4.1. (1) $\mathrm{supp}((\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m))=\mathrm{supp}(\mathcal{I}_1,\mu_1)\cap\ldots\cap$ *supp*$(\mathcal{I}_m,\mu_m)$. *Moreover multiple test blow-ups* (X_k) *of* $(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m)$ *are exactly those which are simultaneous multiple test blow-ups for all* $(\mathcal{I}_j,\mu_j)$, *and for any* k *we have the equality for the controlled transforms* $(\mathcal{I}_j,\mu_{\mathcal{I}})_k$*:*

$$(\mathcal{I}_1,\mu_1)_k+\ldots+(\mathcal{I}_m,\mu_m)_k=[(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m)]_k,$$

(2)

$$\mathrm{supp}(\mathcal{I},\mu_{\mathcal{I}})\cap\mathrm{supp}(\mathcal{J},\mu_{\mathcal{J}})\subseteq\mathrm{supp}((\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})).$$

Moreover any simultaneous multiple test blow-up X_i *of both ideals* $(\mathcal{I},\mu_{\mathcal{I}})$ *and* $(\mathcal{J},\mu_{\mathcal{J}})$ *is a multiple test blow-up for* $(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})$, *and for the controlled transforms* $(\mathcal{I}_k,\mu_{\mathcal{I}})$ *and* $(\mathcal{J}_k,\mu_{\mathcal{J}})$ *we have the equality*

$$(\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}})=[(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})]_k.$$

Proof. (1) To simplify notation we restrict ourselves to the case of two marked ideals. The proof for $n>2$ marked ideals is exactly the same. We have

$$\begin{aligned}&\mathrm{supp}((\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}}))=\mathrm{supp}(\mathcal{I}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{I}}}+\mathcal{J}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{J}}},\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}}))\\ &=\mathrm{supp}(\mathcal{I}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{I}}},\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}}))\cap\mathrm{supp}(\mathcal{J}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{J}}},\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}}))\\ &=\mathrm{supp}(\mathcal{I},\mu_{\mathcal{I}})\cap\mathrm{supp}(\mathcal{J},\mu_{\mathcal{J}}).\end{aligned}$$

Suppose now all multiple test blow-ups of $(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}})$ of length $k\geq 0$ are exactly simultaneous multiple test blow-ups for $(\mathcal{I},\mu_{\mathcal{I}})$ and $(\mathcal{J},\mu_{\mathcal{J}})$ and $[(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}})]_k=(\mathcal{I}_k,\mu_{\mathcal{I}})+(\mathcal{J}_k,\mu_{\mathcal{J}})$. Let σ_{k+1} denote a blow-up with smooth center C_k contained in $\mathrm{supp}[(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}})]_k$. Then

$$\begin{aligned}&[(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}})]_{k+1}=\sigma^{\mathrm{c}}_{k+1}((\mathcal{I}_k,\mu_{\mathcal{I}})+(\mathcal{J}_k,\mu_{\mathcal{J}}))\\ &=(y^{-\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})}\sigma^*_{k+1}(\mathcal{I}_k^{\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})/\mu_{\mathcal{I}}}+\mathcal{J}_k^{\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})/\mu_{\mathcal{J}}}),\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})))\\ &=(y^{-\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})}\sigma^*_{k+1}(\mathcal{I}_k^{\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})/\mu_{\mathcal{I}}}),\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}}))\\ &\quad+(y^{-\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})}\sigma^*_{k+1}(\mathcal{J}_k^{\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}})/\mu_{\mathcal{J}}}),\mathrm{lcm}(\mu_{\mathcal{I}}\mu_{\mathcal{J}}))\\ &=(y^{-\mu_{\mathcal{I}}}\sigma^*_k(\mathcal{I}_k),\mu_{\mathcal{I}})+(y^{-\mu_{\mathcal{J}}}\sigma^*_k(\mathcal{J}_k),\mu_{\mathcal{J}})\\ &=\sigma^{\mathrm{c}}_k(\mathcal{I}_k,\mu_{\mathcal{I}})+\sigma^{\mathrm{c}}_k(\mathcal{J}_k,\mu_{\mathcal{J}})=(\mathcal{I}_{k+1},\mu_{\mathcal{I}})+(\mathcal{J}_{k+1},\mu_{\mathcal{J}}).\end{aligned}$$

(2) If $\mathrm{ord}_x(\mathcal{I}) \geq \mu_{\mathcal{I}}$ and $\mathrm{ord}_x(\mathcal{J}) \geq \mu_{\mathcal{J}}$, then $\mathrm{ord}_x(\mathcal{I}\cdot\mathcal{J}) \geq \mu_{\mathcal{I}} + \mu_{\mathcal{J}}$. This implies that $\mathrm{supp}(\mathcal{I},\mu_{\mathcal{I}}) \cap \mathrm{supp}(\mathcal{J},\mu_{\mathcal{J}}) \subseteq \mathrm{supp}((\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}}))$. Suppose now that all simultaneous multiple test blow-ups of $(\mathcal{I},\mu_{\mathcal{I}})$ and $(\mathcal{J},\mu_{\mathcal{J}})$ of length $k \geq 0$ are multiple test blow-ups for $(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})$ and there is equality

$$(\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}}) = [(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})]_k.$$

Let σ_{k+1} denote the blow-up with a smooth center C_k contained in $\mathrm{supp}(\mathcal{I}_k,\mu_{\mathcal{I}}) \cap \mathrm{supp}(\mathcal{J}_k,\mu_{\mathcal{J}}) \subseteq \mathrm{supp}((\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}}))$. Then

$$\begin{aligned}[(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})]_{k+1} &= \sigma^{\mathrm{c}}_{k+1}((\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}})) \\ = (y^{-(\mu_{\mathcal{I}}+\mu_{\mathcal{J}})}\sigma^*_{k+1}(\mathcal{I}_k\mathcal{J}_k),\mu_{\mathcal{I}}+\mu_{\mathcal{J}})) &= (y^{-\mu_{\mathcal{I}}}\sigma^*_k(\mathcal{I}_k),\mu_{\mathcal{I}})\cdot(y^{-\mu_{\mathcal{J}}}\sigma^*_k(\mathcal{J}_k),\mu_{\mathcal{J}}) \\ = \sigma^{\mathrm{c}}_k(\mathcal{I}_k,\mu_{\mathcal{I}})\cdot\sigma^{\mathrm{c}}_k(\mathcal{J}_k,\mu_{\mathcal{J}}) &= (\mathcal{I}_{k+1},\mu_{\mathcal{I}})\cdot(\mathcal{J}_{k+1},\mu_{\mathcal{J}}).\end{aligned}$$

□

Remark. The operation of multiplication of marked ideals is associative while the operation of addition is not. However we have the following lemma.

Lemma 3.4.2. $((\mathcal{I}_1,\mu_1)+(\mathcal{I}_2,\mu_2))+(\mathcal{I}_3,\mu_3) \simeq (\mathcal{I}_1,\mu_1)+((\mathcal{I}_2,\mu_2)+(\mathcal{I}_3,\mu_3))$.

Proof. It follows from Lemma 3.4.1 that the supports of the two marked ideals are the same. Moreover by the same lemma the supports remain the same after consecutive blow-ups of multiple test blow-ups. □

3.5. **Homogenized ideals and tangent directions.** Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Set $T(\mathcal{I}) := \mathcal{D}^{\mu-1}\mathcal{I}$. By the *homogenized ideal* we mean

$$\mathcal{H}(\mathcal{I},\mu) := (\mathcal{H}(\mathcal{I}),\mu) = (\mathcal{I}+\mathcal{D}\mathcal{I}\cdot T(\mathcal{I})+\ldots+\mathcal{D}^i\mathcal{I}\cdot T(\mathcal{I})^i+\ldots+\mathcal{D}^{\mu-1}\mathcal{I}\cdot T(\mathcal{I})^{\mu-1},\mu).$$

Lemma 3.5.1. *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order.*

(1) *If $\mu = 1$, then $(\mathcal{H}(\mathcal{I}),1) = (\mathcal{I},1)$.*
(2) $\mathcal{H}(\mathcal{I}) = \mathcal{I} + \mathcal{D}\mathcal{I}\cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I}\cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I}\cdot T(\mathcal{I})^{\mu-1} + \mathcal{D}^{\mu}\mathcal{I}\cdot T(\mathcal{I})^{\mu} + \ldots$.
(3) $(\mathcal{H}(\mathcal{I}),\mu) = (\mathcal{I},\mu) + \mathcal{D}(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1) + \ldots + \mathcal{D}^i(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1)^i + \ldots + \mathcal{D}^{\mu-1}(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1)^{\mu-1}$.
(4) *If $\mu > 1$, then $\mathcal{D}(\mathcal{H}(\mathcal{I},\mu)) \subseteq \mathcal{H}(\mathcal{D}(\mathcal{I},\mu))$.*
(5) $T(\mathcal{H}(\mathcal{I},\mu)) = T(\mathcal{I},\mu)$.

Proof. (1) $T(\mathcal{I}) = \mathcal{I}$ and $\mathcal{D}^i(\mathcal{I})T(\mathcal{I})^i \subseteq \mathcal{I}$.
(2) $\mathcal{D}^{\mu-1}(\mathcal{I})T(\mathcal{I}) = T(\mathcal{I})^{\mu}$ and $\mathcal{D}^i(\mathcal{I})T(\mathcal{I})^i \subset T(\mathcal{I})^{\mu}$ for $i \geq \mu$.
(3) By definition.
(4) Note that $T(\mathcal{D}(\mathcal{I})) = T(\mathcal{I})$ and $\mathcal{D}(\mathcal{D}^i(\mathcal{I})T^i(\mathcal{I})^i) \subseteq \mathcal{D}^i(\mathcal{D}(\mathcal{I}))T(\mathcal{D}(\mathcal{I})) + \mathcal{D}^{i-1}(\mathcal{D}\mathcal{I})T(\mathcal{D}(\mathcal{I}))^{i-1} \subseteq \mathcal{H}(\mathcal{D}(\mathcal{I},\mu))$.
(5) $T(\mathcal{I}) = \mathcal{D}^{\mu-1}(\mathcal{I}) \subseteq \mathcal{D}^{\mu-1}(\mathcal{H}(\mathcal{I})) \subseteq \mathcal{H}(\mathcal{D}^{\mu-1}(\mathcal{I})) = \mathcal{H}(T(\mathcal{I})) = T(\mathcal{I})$.

□

Remark. A homogenized ideal features two important properties:

(1) It is equivalent to the given ideal.
(2) It "looks the same" from all possible tangent directions.

By the first property we can use the homogenized ideal to construct resolution via the Giraud Lemma 3.3.6. By the second property such a construction does not depend on the choice of tangent directions.

Lemma 3.5.2. *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Then*

(1) $(\mathcal{I},\mu)\simeq(\mathcal{H}(\mathcal{I}),\mu)$.
(2) *For any multiple test blow-up (X_k) of $(\mathcal{I},\mu)$, $(\mathcal{H}(\mathcal{I}),\mu)_k = (\mathcal{I},\mu)_k + [\mathcal{D}(\mathcal{I},\mu)]_k\cdot[(T(\mathcal{I}),1)]_k+\ldots+[\mathcal{D}^{\mu-1}(\mathcal{I},\mu)]_k\cdot[(T(\mathcal{I}),1)]_k^{\mu-1}$.*

Proof. Since $\mathcal{H}(\mathcal{I})\supset\mathcal{I}$, every multiple test blow-up of $\mathcal{H}(\mathcal{I},\mu)$ is a multiple test blow-up of $(\mathcal{I},\mu)$. By Lemma 3.2.4, every multiple test blow-up of $(\mathcal{I},\mu)$ is a multiple test blow-up for all $\mathcal{D}^i(\mathcal{I},\mu)$ and consequently, by Lemma 3.4.1 it is a simultaneous multiple test blow-up of all $(\mathcal{D}^i(\mathcal{I})\cdot T(\mathcal{I})^i,\mu)=(\mathcal{D}^i(\mathcal{I}),\mu-i)\cdot(T(\mathcal{I})^i,i)$ and

$$\begin{aligned}\operatorname{supp}(\mathcal{H}(\mathcal{I},\mu)_k)&=\bigcap_{i=0}^{\mu-1}\operatorname{supp}(\mathcal{D}^i(\mathcal{I})\cdot T(\mathcal{I})^i,\mu)_k\\&=\bigcap_{i=0}^{\mu-1}\operatorname{supp}(\mathcal{D}^i(\mathcal{I}),\mu-i)_k\cdot(T(\mathcal{I})^i,i)_k\\&\supseteq\bigcap_{i=0}^{\mu-1}\operatorname{supp}(\mathcal{D}^i(\mathcal{I},\mu))_k=\operatorname{supp}(\mathcal{I}_k,\mu).\end{aligned}$$

Therefore every multiple test blow-up of $(\mathcal{I},\mu)$ is a multiple test blow-up of $\mathcal{H}(\mathcal{I},\mu)$ and by Lemmas 3.5.1(3) and 3.4.1 we get (2). □

Lemma 3.5.3. *Let $\phi: X'\to X$ be a smooth morphism of smooth varieties and let $(X,\mathcal{I},\emptyset,\mu)$ be a marked ideal. Then*

$$\phi^*(\mathcal{H}(\mathcal{I}))=\mathcal{H}(\phi^*(\mathcal{I})).$$

Proof. A direct consequence of Lemma 3.2.5. □

Although the following Lemmas 3.5.4 and 3.5.5 are used in this paper only in the case $E=\emptyset$ we formulate them in slightly more general versions.

Lemma 3.5.4. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order. Assume there exist tangent directions $u,v\in T(\mathcal{I},\mu)_x=\mathcal{D}^{\mu-1}(\mathcal{I},\mu)_x$ at $x\in\operatorname{supp}(\mathcal{I},\mu)$ which are transversal to E. Then there exists an automorphism $\widehat{\phi}_{uv}$ of $\widehat{X}_x:=\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X})$ such that*

(1) $\widehat{\phi}^*_{uv}(\mathcal{H}\widehat{\mathcal{I}})_x=(\mathcal{H}\widehat{\mathcal{I}})_x$.
(2) $\widehat{\phi}^*_{uv}(E)=E$.
(3) $\widehat{\phi}^*_{uv}(u)=v$.
(4) $\operatorname{supp}(\widehat{\mathcal{I}},\mu):=V(T(\widehat{\mathcal{I}},\mu))$ *is contained in the fixed point set of ϕ.*

Proof. (0) **Construction of the automorphism $\widehat{\phi}_{uv}$.**

Find parameters $u_2,\ldots,u_n$ transversal to u and v such that $u=u_1,u_2,\ldots,u_n$ and $v,u_2,\ldots,u_n$ form two sets of parameters at x and divisors in E are described by some parameters u_i where $i\geq 2$. Set

$$\widehat{\phi}_{uv}(u_1)=v,\quad \widehat{\phi}_{uv}(u_i)=u_i\quad\text{for}\quad i>1.$$

(1) Let $h:=v-u\in T(\mathcal{I})$. For any $f\in\widehat{\mathcal{I}}$,

$$\begin{aligned}\widehat{\phi}^*_{uv}(f)&=f(u_1+h,u_2,\ldots,u_n)\\&=f(u_1,\ldots,u_n)+\frac{\partial f}{\partial u_1}\cdot h+\frac{1}{2!}\frac{\partial^2 f}{\partial u_1^2}\cdot h^2+\ldots+\frac{1}{i!}\frac{\partial^i f}{\partial u_1^i}\cdot h^i+\ldots.\end{aligned}$$

The latter element belongs to

$$\widehat{\mathcal{I}}+\mathcal{D}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}+\ldots+\mathcal{D}^i\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^i+\ldots+\mathcal{D}^{\mu-1}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^{\mu-1}=\mathcal{H}\widehat{\mathcal{I}}.$$

Hence $\widehat{\phi}^*_{uv}(\widehat{\mathcal{I}})\subset\mathcal{H}\widehat{\mathcal{I}}$. Analogously $\widehat{\phi}^*_{uv}(\mathcal{D}^i\widehat{\mathcal{I}})\subset\mathcal{D}^i\widehat{\mathcal{I}}+\mathcal{D}^{i+1}\widehat{\mathcal{I}}\cdot T(\mathcal{I})+\ldots+\mathcal{D}^{\mu-1}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^{\mu-i-1}=\mathcal{H}\widehat{\mathcal{D}^i I}$. In particular by Lemma 3.5.1, $\widehat{\phi}^*_{uv}(\widehat{T(\mathcal{I})},1)\subset\mathcal{H}(\widehat{T(\mathcal{I})},1)=(\widehat{T(\mathcal{I})},1)$. This gives

$$\widehat{\phi}^*_{uv}(\mathcal{D}^i\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^i)\subset\mathcal{D}^i\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^i+\ldots+\mathcal{D}^{\mu-1}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^{\mu-1}\subset\mathcal{H}\widehat{\mathcal{I}}.$$

By the above $\widehat{\phi}^*_{uv}(\mathcal{H}\widehat{\mathcal{I}})_x\subset(\mathcal{H}\widehat{\mathcal{I}})_x$ and since the scheme is Noetherian, $\widehat{\phi}^*_{uv}(\mathcal{H}\widehat{\mathcal{I}})_x=(\mathcal{H}\widehat{\mathcal{I}})_x$.

(2)(3) Follow from the construction.

(4) The fixed point scheme of $\widehat{\phi}^*_{uv}$ is defined by $u_i=\widehat{\phi}^*_{uv}(u_i)$, $i=1,\ldots,n$, that is, $h=0$. But $h\in\mathcal{D}^{\mu-1}(\mathcal{I})$ is 0 on $\operatorname{supp}(\mathcal{I},\mu)$. $\square$

Lemma 3.5.5 (Glueing Lemma)**.** *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order for which there exist tangent directions $u,v\in T(\mathcal{I},\mu)$ at $x\in\operatorname{supp}(\mathcal{I},\mu)$ which are transversal to E. Then there exist étale neighborhoods $\phi_u,\phi_v:\overline{X}\to X$ of $x=\phi_u(\overline{x})=\phi_v(\overline{x})\in X$, where $\overline{x}\in\overline{X}$, such that*

1. *$\phi_u^*(X,\mathcal{H}(\mathcal{I}),E,\mu)=\phi_v^*(X,\mathcal{H}(\mathcal{I}),E,\mu)$.*
2. *$\phi_u^*(u)=\phi_v^*(v)$.*
   Set $(\overline{X},\overline{\mathcal{I}},\overline{E},\mu):=\phi_u^*(X,\mathcal{H}(\mathcal{I}),E,\mu)=\phi_v^*(X,\mathcal{H}(\mathcal{I}),E,\mu)$.
3. *For any $\overline{y}\in\operatorname{supp}(\overline{X},\overline{\mathcal{I}},\overline{E},\mu)$, $\phi_u(\overline{y})=\phi_v(\overline{y})$.*
4. *Let (X_i) be a multiple test blow-up of $(X,\mathcal{I},\emptyset,\mu)$. Then*
   (a) *The induced multiple test blow-ups $\phi_u^*(X_i)$ and $\phi_v^*(X_i)$ of $(\overline{X},\overline{\mathcal{I}},\overline{E},\mu)$ are the same (defined by the same centers). Set $(\overline{X}_i):=\phi_u^*(X_i)=\phi_v^*(X_i)$ and let $\phi_{ui},\phi_{vi}:\overline{X}_i\to X_i$ be the induced morphisms. Then $\phi_{ui}^*(X_i,\mathcal{H}(\mathcal{I})_i,E_i,\mu)=\phi_{vi}^*(X_i,\mathcal{H}(\mathcal{I})_i,E_i,\mu)$.*
   (b) *Let $V(u)$ and $V(v)$ denote the hypersurfaces of maximal contact on X and $V(u)_i$ and $V(v)_i$ be their strict transforms. Then $\phi_{ui}^{-1}(V(u)_i)=\phi_{vi}^{-1}(V(v)_i)$.*
   (c) *For any $\overline{y}_i\in\operatorname{supp}(\overline{X}_i,\overline{\mathcal{I}}_i,\overline{E}_i,\mu)$, $\phi_{ui}(\overline{y}_i)=\phi_{vi}(\overline{y}_i)$.*

Proof. (0) **Construction of étale neighborhoods $\phi_u,\phi_v:\overline{X}\to X$.**

Let $U\subset X$ be an open subset for which there exist $u_2,\ldots,u_n$ which are transversal to u and v on U such that $u=u_1,u_2,\ldots,u_n$ and $v,u_2,\ldots,u_n$ form two sets of parameters on U and divisors in E are described by some u_i, where $i\geq 2$. Let $\mathbf{A}^n$ be the affine space with coordinates $x_1,\ldots,x_n$. Construct first étale morphisms $\phi_1,\phi_2:U\to\mathbf{A}^n$ with

$$\phi_1^*(x_i)=u_i\quad\text{for all } i\qquad\text{and}\quad\phi_2^*(x_1)=v,\quad\phi_2^*(x_i)=u_i\quad\text{for}\quad i>1.$$

Consider the fiber product $U\times_{\mathbf{A}^n}U$ for the morphisms ϕ_1 and ϕ_2. Let ϕ_u, ϕ_v be the natural projections $\phi_u,\phi_v:U\times_{\mathbf{A}^n}U\to U$ such that $\phi_1\phi_u=\phi_2\phi_v$. Then define $\overline{X}$ to be an irreducible component of $U\times_{\mathbf{A}^n}U$ whose images $\phi_u(\overline{X})$ and $\phi_v(\overline{X})$ contain x. Set

$$\begin{aligned}w_1:=\phi_u^*(u)=(\phi_1\phi_u)^*(x_1)=(\phi_2\phi_v)^*(x_1)=\phi_v^*(v),\\ w_i=\phi_u^*(u_i)=\phi_v^*(u_i)\quad\text{for } i\geq 2.\end{aligned}$$

(1) Let $h := v - u$. By the above the morphisms ϕ_u and ϕ_v coincide on $\phi_u^{-1}(V(h)) = \phi_v^{-1}(V(h))$. If $\overline{y} \in \overline{X}$ is a point such that $\phi_u(\overline{y}) \notin \operatorname{supp}(X, \mathcal{I}, E, \mu)$, then $\overline{y} \notin \operatorname{supp}(\phi_u^*(\mathcal{H}(\mathcal{I})) = \operatorname{supp}(\mathcal{H}(\phi_u^*(\mathcal{I}))$ and we have the equality of stalks $\phi_u^*(\mathcal{H}(\mathcal{I})_{\overline{y}} = \mathcal{H}(\phi_u^*(\mathcal{I})_{\overline{y}} = \mathcal{O}_{\overline{X},\overline{y}}$. On the other hand, $\phi_v(\overline{y}) \notin \operatorname{supp}(X, \mathcal{I}, E, \mu)$ and $\phi_v^*(\mathcal{H}(\mathcal{I})_{\overline{y}} = \mathcal{H}(\phi_v^*(\mathcal{I})_{\overline{y}} = \mathcal{O}_{\overline{X},\overline{y}}$.

Let $\overline{y} \in \overline{X}$ be a point such that $\phi_u(\overline{y}) = y \in \operatorname{supp}(X, \mathcal{I}, E, \mu)$. Then $\phi_v(\overline{y}) = y$. Denote by $(\widehat{\phi}_v)_{\overline{y}}$ and $(\widehat{\phi}_u)_{\overline{y}}$ the induced morphisms of the completions $\widehat{\overline{X}}_{\overline{y}} \to \widehat{X}_y$. We have the equality $\widehat{\phi}_{uv} = (\widehat{\phi}_v)_{\overline{y}}(\widehat{\phi}_u^{-1})_{\overline{y}}$ where $\widehat{\phi}_{uv}$ is given as in the proof of Lemma 3.5.4. Consequently, for any such $\overline{y}$,

$$(\mathcal{H}\widehat{\mathcal{I}})_{\overline{y}} = \widehat{\phi}_{uv}^*(\mathcal{H}\widehat{\mathcal{I}})_{\overline{y}} = (\widehat{\phi}_u)_{\overline{y}}(\widehat{\phi}_v^{-1})_{\overline{y}}^*(\mathcal{H}\widehat{\mathcal{I}})_y,$$

$$\phi_u^*(\mathcal{H}\mathcal{I})_{\overline{y}} = (\widehat{\phi}_u)_{\overline{y}}^*(\mathcal{H}\widehat{\mathcal{I}})_{\overline{y}} = (\widehat{\phi}_v^*)_{\overline{y}}(\mathcal{H}\widehat{\mathcal{I}})_{\overline{y}} = \phi_v^*(\mathcal{H}\mathcal{I})_{\overline{y}}.$$

We get the equality of stalks $\phi_u^*(\mathcal{H}\mathcal{I})_{\overline{y}} = \phi_v^*(\mathcal{H}\mathcal{I})_{\overline{y}}$ for all points $y \in \overline{X}$ and for sheaves $\phi_u^*(\mathcal{H}(\mathcal{I})) = \phi_v^*(\mathcal{H}(\mathcal{I}))$. Also $\phi_u^*(E) = \phi_v^*(E)$ by construction and $\phi_u^*(T(\mathcal{I})) = \phi_v^*(T(\mathcal{I}))$.

(2) Follows from the construction.

(3) Let $h := v - u$. The subset of $\overline{X}$ for which $\phi_1(x) = \phi_2(x)$ is described by $h = 0$. Consequently, $\phi_u = \phi_v$ over $V(\phi_u^*(h))$. In particular these morphisms are equal over $\operatorname{supp}(\mathcal{I}, \mu) = \operatorname{supp}(\mathcal{H}(\mathcal{I}), \mu)$.

(4) Let (X_i) be a multiple test blow-up of $(X, \mathcal{I}, E, \mu)$. Let $C_0 \subset \operatorname{supp}(X, \mathcal{I}, E, \mu)$ be the center of σ_1. By (3), $\phi_u = \phi_v$ over $\operatorname{supp}(\mathcal{I}, \mu)$. Fix a point $\overline{y} \in \operatorname{supp}(\phi_u^*(\mathcal{H}(\mathcal{I})), \mu)$ and let $y = \phi_u(\overline{y}) = \phi_v(\overline{y}) \in \operatorname{supp}(X, \mathcal{I}, E, \mu)$. Find parameters $u_1' = u_1, u_2', \dots, u_n'$ on an affine neighborhood U' of y such that divisors in E are described by some u_i for $i \geq 2$ and C_0 is described by $u_1' = u_2' = \ldots = u_m' = 0$ for some $m \geq 0$.

Let $\overline{U} \subset \phi_u^{-1}(U) \cap \phi_v^{-1}(U) \subset \overline{X}$ be an affine neighborhood of $\overline{y}$.

Let $\overline{J}$ be the ideal of $K[\overline{U}]$ generated by all functions $\phi_u^*(f) - \phi_v^*(f)$, where $f \in K[U']$. Then $(\phi_u^*(h)) \subset \overline{J}$. On the other hand, by definition for any point $\overline{z} \in V(\phi_u^*(h)) \subset V(\overline{J})$ we have the equalities of the completions of stalks of the ideals

$$\widehat{\overline{J}}_{\overline{z},\overline{U}} = (\phi_u^*(u_i') - \phi_v^*(u_i'))_{i=1,\dots,n} = (\phi_u^*(h)) \cdot \widehat{\mathcal{O}}_{\overline{z},\overline{U}},$$

which implies the equalites of stalks of the ideals

$$(\phi_u^*(h))_{\overline{z},\overline{U}} = \overline{J}_{\overline{z},\overline{U}}.$$

Finally, $\overline{J} = (\phi_u^*(h))$ and consequently, for any $i = 1, \dots, n$, we have

$$\phi_u^*(u_i') - \phi_v^*(u_i') \in (\phi_u^*(h)) \subset \phi_u^*(T(\mathcal{I}))(\overline{U}) = \phi_v^*(T(\mathcal{I}))(\overline{U}).$$

For simplicity denote u_i' simply by u_i and U' by U. Let $\overline{\sigma}_1 : \overline{X}_1 \to \overline{X}$ be the blow-up of $\overline{X}$ at $\overline{C}_0 := \phi_u^{-1}(C_0) = \phi_v^{-1}(C_0) \subset \overline{X}$. Then both morphisms ϕ_u and ϕ_v lift to étale morphisms $\phi_{u1}, \phi_{v1} : \overline{X}_1 \to X_1$ by the universal property of a blow-up. Observe that

$$\phi_{u1}^*(T(\mathcal{I})_1, 1) = \phi_{u1}^*(\mathcal{I}(D)^{-1} \cdot \sigma^*(T(\mathcal{I}))) = \phi_{v1}^*(\mathcal{I}(D)^{-1} \cdot \sigma^*(T(\mathcal{I}))) = \phi_{v1}^*(T(\mathcal{I})_1),$$

where D denotes the exceptional divisor of σ_1.

Fix a point $\overline{y}_1 \in \operatorname{supp}(\phi_{u1}^*(\overline{\mathcal{I}}_1), \mu) \subset \operatorname{supp}(\phi_{u1}^*(T(\mathcal{I})_1), 1)$ and $y_1 = \phi_{u1}(\overline{y}_1) \in \operatorname{supp}(\mathcal{I}_1, \mu) \subset \operatorname{supp}(T(\mathcal{I}_1), 1)$ such that $\overline{y} = \overline{\sigma}_1(\overline{y}_1) \in \operatorname{supp}(\phi_u^*(\mathcal{I}), \mu)$ and $y = \sigma_1(y_1) \in \operatorname{supp}(\mathcal{I}, \mu)$. Find parameters $u_1, u_2, \dots, u_n$ at y, by replacing $u_2, \dots, u_n$ if necessary by their linear combinations, such that

(1) The parameters at y_1 are given by $u_{i1} := \frac{u_i}{u_m}$ for $1 \leq i < m$ and $u_{i1} := u_i$ for $i \geq m$.
(2) All divisors in E_1 through y_1 are defined by some u_{i1}.

Then $w_i := \phi_u^*(u_i)$ for $i = 1, \ldots, n$ define parameters at a point $\overline{y}$ such that the parameters at $\overline{y}_1$ are given by $w_{i1} := \frac{w_i}{w_m}$ for $i < m$ and $w'_{i1} := w_i$ for $i \geq m$.

Let $U_{m1} \subset X_1$ be the neighborhood defined by the parameter u_m. The subset $U_{m1} \ni y_1$ is described by all points z for which $(u_m/y_D)(z) \neq 0$, where y_D is a local equation of the exceptional divisor of σ_1. Then a point $\overline{z}$ is in $\operatorname{supp}(\phi_{u1}^*(T(\mathcal{I}_1)), \mu) \cap \phi_u^{-1}(U_{m1})$ iff $(\phi_{u1}^*(u_m)/y_{\overline{D}})(\overline{z}) \neq 0$,where $y_{\overline{D}}$ is a local equation of the exceptional divisor of $\overline{\sigma}_1$. But for any $\overline{z} \in \operatorname{supp}(\phi_{u1}^*(T(\mathcal{I}_1)), 1)$,

$$(\phi_v^*(u_m)/y_{\overline{D}})(x) = (\phi_u^*(u_m)/y_{\overline{D}})(x) + \phi_u^*(h)/y_{\overline{D}}(x) = (\phi_u^*(u_m)/y_{\overline{D}})(x),$$

where $h \in (T(\mathcal{I}))(U)$ and $\phi_u^*(h)/y_{\overline{D}} \in \phi_{u1}^*(T(\mathcal{I})_1)(\phi_{u1}^{-1}(U_m))$. This implies that

$$\operatorname{supp}(\phi_{u1}^*(T(\mathcal{I}_1)), \mu) \cap \phi_{u1}^{-1}(U_{m1}) = \operatorname{supp}(\phi_{v1}^*(T(\mathcal{I}_1)), \mu) \cap \phi_{v1}^{-1}(U_{m1}).$$

Then the exceptional divisor of $\overline{\sigma}_1$ is described on $\overline{U}_{m1} := \phi_{u1}^{-1}(U_{m1}) \cap \phi_{v1}^{-1}(U_{m1})$ by $\phi_{u1}^*(u_m)$ and by $\phi_{v1}^*(u_m)$. Note that

$$\begin{aligned}\phi_{u1}^*(u_{i1}) - \phi_{v1}^*(u_{i1}) &= \phi_{u1}^*(\frac{u_i}{u_m}) - \phi_{v1}^*(\frac{u_i}{u_m}) \\ &= \frac{\phi_{u1}^*(u_i)}{\phi_{u1}^*(u_m)} - \frac{\phi_{u1}^*(u_i)}{\phi_{v1}^*(u_m)} + \frac{\phi_{u1}^*(u_i)}{\phi_{v1}^*(u_m)} - \frac{\phi_{v1}^*(u_i)}{\phi_{v1}^*(u_m)} \\ &= \frac{\phi_{u1}^*(u_i)}{\phi_{u1}^*(u_m)} \frac{(\phi_{v1}^*(u_m) - \phi_{u1}^*(u_m))}{\phi_{v1}^*(u_m)} + \frac{\phi_{u1}^*(u_i) - \phi_{v1}^*(u_i)}{\phi_{v1}^*(u_m)} \\ &\in \phi_{u1}^*(T(\mathcal{I})_1(\overline{U}_{m1}) = \phi_{v1}^*(T(\mathcal{I}))_1(\overline{U}_{m1}).\end{aligned}$$

In particular $u_{i1}(\phi_{v1}(\overline{y}_1)) = \phi_{v1}^*(u_{i1})(\overline{y}) = \phi_{u1}^*(u_{i1})(\overline{y}) = 0$. Thus $\phi_{v1}(\overline{y}_1) = y_1 = \phi_{u1}(\overline{y}_1)$ is the only point in $\sigma_1^{-1}(y)$ for which all u_{i1} are zero. This shows that $\phi_{u1} = \phi_{v1}$ over $\operatorname{supp}(T(\mathcal{I})_1(U_{m1}), 1) \supset \operatorname{supp}(\mathcal{I}_1(U_{m1}), \mu)$ and thus over $\operatorname{supp}(T(\mathcal{I})_1) \supset \operatorname{supp}(\mathcal{I}_1, \mu)$.

Set $v_{11} := \frac{v}{u_m}$. Note that the functions

$$\phi_{u1}^*(u_{11}) = \phi_u^*(u)/\phi_u^*(u_m) = \phi_v^*(v)/\phi_u^*(u_m) \sim \phi_{v1}^*(v_{11}) = \phi_v^*(v)/\phi_v^*(u_m)$$

are proportional (up to an invertible function on $\overline{U}_{m1}$). Also for any parameters u_{i1} defining divisors in E_1 through y_1, we have

$$\phi_{u1}^*(u_{i1}) = \phi_u^*(u_i)/\phi_u^*(u_m) \sim \phi_v^*(u_i)/\phi_v^*(u_m) = \phi_{v1}^*(u_{i1}).$$

Denote U_{m1} and $\overline{U}_{m1}$ by U_1 and $\overline{U}_1$. By induction on k we show that $\phi_{uk} = \phi_{vk}$ over $\operatorname{supp}(T(\mathcal{I})_k, 1) \supset \operatorname{supp}(\mathcal{I}_k, \mu)$ and for any point $\overline{y}_k \in \operatorname{supp}(\phi_{uk}^*(\mathcal{I}_k), \mu) = \operatorname{supp}(\phi_{vk}^*(\mathcal{I}_k), \mu)$ and $y_k = \phi_{uk}(\overline{y}_k) = \phi_{vk}(\overline{y}_k) \in \operatorname{supp}(T(\mathcal{I})_k, 1)$ there are affine neighborhoods $\overline{U}_k \ni \overline{y}$ and $U_k \ni y$ such that there exist parameters $u_{1k}, \ldots, u_{nk}$ and a function v_{1k} on U_k such that

(1) u_{1k} and v_{1k} describe hypersurfaces of maximal contact on U_k and $\phi_{uk}^*(u_{k1}) \sim \phi_{vk}^*(v_{k1})$.
(2) The exceptional divisors in E_k through y_k are described by some parameters u_{ik} and $\phi_{uk}^*(u_{ik}) \sim \phi_{vk}^*(u_{ik})$.
(3) $(\phi_{uk})^*(u_{ik}) - (\phi_{vk})^*(u_{ik}) \in \phi_{u1}^*(T(\mathcal{I})_k(\overline{U}_k) = \phi_{vk}^*(T(\mathcal{I}))_k(\overline{U}_k)$ for $i \geq 1$.

Note then the induced marked ideals $\phi_u^*(X_i, \mathcal{H}(\mathcal{I})_i, E_i, \mu)$ and $\phi_v^*(X_i, \mathcal{H}(\mathcal{I})_i, E_i, \mu)$ are equal because they are controlled transforms of $\phi_u^*(X, \mathcal{H}(\mathcal{I}), E, \mu) = \phi_v^*(X, \mathcal{H}(\mathcal{I}), E, \mu)$ defined for $(\overline{X}_i)$ (see Proposition 2.3.2). $\square$

The above lemma can also be generalized as follows:

Lemma 3.5.6. *Let G be the group of all automorphisms $\widehat{\phi}$ of $\widehat{X}_x$ acting trivially on the subscheme defined by $T(\mathcal{I})$ and preserving E; that is, $\widehat{\phi}^*(f) - f \in T(\mathcal{I})$ for any $f \in K[X_x]$ and $\phi^*(D) = D$ for any $D \in E$. Then*

(1) *$\mathcal{H}(\mathcal{I})$ is preserved by G; i.e., $\widehat{\phi}^*(\mathcal{H}(\mathcal{I}))_x = \mathcal{H}(\mathcal{I})_x$ for any $\widehat{\phi} \in G$.*
(2) *G acts transitively on the set of tangent directions $u \in T(\mathcal{I})$ transversal to E.*
(3) *Any multiple test blow-up (X_i) of $(\mathcal{I}, \mu)$ is G-equivariant.*
(4) *G acts trivially on the subscheme defined by $T(\mathcal{I})_i \subset X_i$.*

Since this lemma is not used in the proof and all details but a few are the same as for the proof of Lemma 3.5.4 we just point out the differences.

Proof. In the proof of property (1) of Lemma 3.5.4 we use the Taylor formula for n unknowns

$$\begin{aligned}\widehat{\phi}^*(f) &= f(u_1 + h_1, \ldots, u_n + h_n)\\ &= f + \frac{\partial f}{\partial u_1} \cdot h_1 + \ldots + \frac{\partial f}{\partial u_n} \cdot h_n + \frac{1}{2!}\frac{\partial^2 f}{\partial u_1^2} \cdot h_1^2 + \frac{\partial^2 f}{\partial u_1 \partial u_2} \cdot h_1 h_2 + \frac{1}{2!}\frac{\partial^2 f}{\partial u_2^2} \cdot h_2^2 + \ldots .\end{aligned}$$

In the proof of property (4) we notice that $\widehat{\phi}$ is described by $\widehat{\phi}^*(u_i) = u_i + h_i$, where $h_i \in T(\mathcal{I})$. By a suitable linear change of coordinates we can assume that

(1) The center $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ of the blow-up σ is described at x by $u_1 = \ldots = u_m = 0$.
(2) The coordinates at a point $x_1 \in \sigma^{-1}(x) \cap \operatorname{supp}(\mathcal{I}_1, \mu)$ are $u_{i1} = \frac{u_i}{u_m}$ for $1 \leq i \leq m$ and $u_{i1} = u_i$ for $i > m$.
(3) The automorphism $\widehat{\phi}$ lifts to an automorphism $\widehat{\phi}_1$ preserving x_1 such that
   (a) For $i < m$, $(\widehat{\phi}_1)^*(u_{i1}) = \frac{u_i + h_i}{u_m + h_m} = \frac{u_{i1} + h_i/u_m}{1 + h_m/u_m} = u_{i1} + g_{i1}$, where $g_{i1} \in T(\mathcal{I})_1$.
   (b) For $i \geq m$, $(\widehat{\phi}_1)^*(u_{i1}) = u_i + h_i = u_{i1} + g_{i1}$, where $u_{i1} = u_i$ and $g_{i1} = h_i \in \sigma^*(T(\mathcal{I})) \subset T(\mathcal{I})_1$. $\square$

3.6. **Coefficient ideals and Giraud Lemma.** The idea of coefficient ideals was originated by Hironaka and then developed in papers of Villamayor and Bierstone-Milman. The following definition modifies and generalizes the definition of Villamayor.

Definition 3.6.1. Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. By the *coefficient ideal* we mean

$$\mathcal{C}(\mathcal{I}, \mu) = (\mathcal{I}, \mu) + (\mathcal{D}\mathcal{I}, \mu - 1) + \ldots + (\mathcal{D}^{\mu-1}\mathcal{I}, 1).$$

Remark. The coefficient ideals $\mathcal{C}(\mathcal{I})$ feature two important properties.

(1) $\mathcal{C}(\mathcal{I})$ is equivalent to $\mathcal{I}$.
(2) The intersection of the support of $(\mathcal{I}, \mu)$ with any smooth subvariety S is the support of the restriction of $\mathcal{C}(\mathcal{I})$ to S:

$$\operatorname{supp}(\mathcal{I}) \cap S = \operatorname{supp}(\mathcal{C}(\mathcal{I})_{|S}).$$

Moreover this condition is persistent under relevant multiple test blow-ups.

These properties allow one to control and modify the part of support of $(\mathcal{I},\mu)$ contained in S by applying multiple test blow-ups of $\mathcal{C}(\mathcal{I})_{|S}$.

Lemma 3.6.2. $\mathcal{C}(\mathcal{I},\mu)\simeq(\mathcal{I},\mu)$.

Proof. By Lemma 3.4.1 multiple test blow-ups of $\mathcal{C}(\mathcal{I},\mu)$ are simultaneous multiple test blow-ups of $\mathcal{D}^i(\mathcal{I},\mu)$ for $0\le i\le\mu-1$. By Lemma 3.2.4 multiple test blow-ups of $(\mathcal{I},\mu)$ define a multiple test blow-up of all $\mathcal{D}^i(\mathcal{I},\mu)$. Thus multiple test blow-ups of $(\mathcal{I},\mu)$ and $\mathcal{C}(\mathcal{I},\mu)$ are the same and $\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu))_k=\bigcap\mathrm{supp}(\mathcal{D}^i\mathcal{I},\mu-i)_k=\mathrm{supp}(\mathcal{I}_k,\mu)$. □

The lemma below will be used in the proof of the following proposition.

Lemma 3.6.3. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order whose support* $\mathrm{supp}(\mathcal{I},\mu)$ *does not contain a smooth subvariety S of X. Assume that S has only simple normal crossings with E. Then*

$$\mathrm{supp}(\mathcal{I},\mu)\cap S\subseteq\mathrm{supp}((\mathcal{I},\mu)_{|S}).$$

Let $\sigma:X'\to X$ be a blow-up with center $C\subset\mathrm{supp}(\mathcal{I},\mu)\cap S$. Denote by $S'\subset X'$ the strict transform of $S\subset X$. Then

$$\sigma^{\mathrm{c}}((\mathcal{I},\mu)_{|S})=(\sigma^{\mathrm{c}}(\mathcal{I},\mu))_{|S'}.$$

Moreover for any multiple test blow-up (X_i) with all centers C_i contained in the strict transforms $S_i\subset X_i$ of S, the restrictions $\sigma_{i|S_i}:S_i\to S_{i-1}$ of the morphisms $\sigma_i:X_i\to X_{i-1}$ to S_i define a multiple test blow-up (S_i) of $(\mathcal{I},\mu)_{|S}$ such that

$$[(\mathcal{I},\mu)_{|S}]_i=(\mathcal{I}_i,\mu)_{|S_i}.$$

Proof. The first inclusion holds since the order of an ideal does not drop but may rise after restriction to a subvariety. Let $x_1,\dots,x_k$ describe the subvariety S of X at a point $p\in C$. Let $p'\in S'$ map to p. We can find coordinates $x_1,\dots,x_k,y_1,\dots,y_{n-k}$ such that the center of the blow-up is described by $x_1,\dots,x_k,y_1,\dots,y_m$ and the coordinates at p' are given by

$$x'_1=x_1/y_m,\dots,x'_k=x_k/y_m,y'_1=y_1/y_m,\dots,y'_m=y_m,y'_{m+1}=y_{m+1},\dots,y'_n=y_n$$

where the strict transform $S'\subset X'$ of S is described by $x'_1,\dots,x'_k$. Then we can write a function $f\in\mathcal{I}(U)$ as $f=\sum c_{\alpha f}(y)x^\alpha$, where $c_{\alpha f}(y)$ are formal power series in y_i. The controlled transform $f'=\sigma^{\mathrm{c}}(f)=y^{-\mu}(f\circ\sigma)$ can be written as

$$f'=\sum c'_{\alpha f}(y){x'}^{\alpha},$$

where $c'_{\alpha f}=y_m^{-\mu+|\alpha|}\sigma^*(c_{\alpha f})$. But then $f_{|S}=(c_{0f})_{|S}$ and

$$\sigma^{\mathrm{c}}(f)_{|S'}=(c'_{0f})_{|S'}=y_m^{-\mu}\sigma^*(c_{0f})_{|S'}=y_m^{-\mu}\sigma^*((c_{0f})_{|S})=y_m^{-\mu}\sigma^*(f_{|S})=\sigma^{\mathrm{c}}(f_{|S}).$$

The last part of the theorem follows by induction:

$$\begin{aligned}\mathrm{supp}(\mathcal{I}_{i+1},\mu)\cap S_{i+1}&=\mathrm{supp}(\sigma^{\mathrm{c}}_{i+1}(\mathcal{I}_i,\mu))\cap S_{i+1}\subseteq\mathrm{supp}(\sigma_{i+1|S_i})^{\mathrm{c}}((\mathcal{I}_i,\mu)_{|S_i})\\&\subseteq\mathrm{supp}((\sigma_{i+1|S_i})^{\mathrm{c}}((\mathcal{I},\mu)_{|S})_i)\\&=\mathrm{supp}((\mathcal{I},\mu)_{|S})_{i+1},\\ [(\mathcal{I},\mu)_{|S}]_{i+1}&=\sigma^{\mathrm{c}}_{i+1|S_i}[(\mathcal{I},\mu)_{|S}]_i=\sigma^{\mathrm{c}}_{i+1|S_i}((\mathcal{I}_i,\mu)_{|S_i})\\&=(\sigma^{\mathrm{c}}_{i+1}(\mathcal{I}_i,\mu))_{|S_{i+1}}=(\mathcal{I}_i,\mu)_{|S_{i+1}}.\end{aligned}$$

□

Proposition 3.6.4. *Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order whose support* $\operatorname{supp}(\mathcal{I}, \mu)$ *does not contain a smooth subvariety S of X. Assume that S has only simple normal crossings with E. Let $E' \subset E$ be the set of divisors transversal to S. Set $E'_{|S} := \{D \cap S \mid D \in E'\}$, $\mu_c := \operatorname{lcm}(1, 2, \ldots, \mu)$, and consider the marked ideal $\mathcal{C}(\mathcal{I}, \mu)_{|S} = (S, \mathcal{C}(\mathcal{I}, \mu)_{|S}, E'_{|S}, \mu_c)$. Then*

$$\operatorname{supp}(\mathcal{I}, \mu) \cap S = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S}).$$

Moreover let (X_i) be a multiple test blow-up with centers C_i contained in the strict transforms $S_i \subset X_i$ of S. Then

(1) *The restrictions $\sigma_{i|S_i} : S_i \to S_{i-1}$ of the morphisms $\sigma_i : X_i \to X_{i-1}$ define a multiple test blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$.*
(2) $\operatorname{supp}(\mathcal{I}_i, \mu) \cap S_i = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i$.
(3) *Every multiple test blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ defines a multiple test blow-up (X_i) of $(\mathcal{I}, \mu)$ with centers C_i contained in the strict transforms $S_i \subset X_i$ of $S \subset X$.*

Proof. By Lemmas 3.6.2 and 3.6.3,

$$\operatorname{supp}(\mathcal{I}, \mu) \cap S = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)) \cap S \subseteq \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S}).$$

Let $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ be local parameters at x such that $\{x_1 = 0, \ldots, x_k = 0\}$ describes S. Then any function $f \in \mathcal{I}$ can be written as

$$f = \sum c_{\alpha f}(y) x^\alpha,$$

where $c_{\alpha f}(y)$ are formal power series in y_i.

Now $x \in \operatorname{supp}(\mathcal{I}, \mu) \cap S$ iff $\operatorname{ord}_x(c_{\alpha,f}) \geq \mu - |\alpha|$ for all $f \in \mathcal{I}$ and $0 \leq |\alpha| < \mu$. Note that

$$c_{\alpha f|S} = \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}\right)_{|S} \in \mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$

and consequently

$$\begin{aligned} \operatorname{supp}(\mathcal{I}, \mu) \cap S &= \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \operatorname{supp}(c_{\alpha f|S}, \mu - |\alpha|) \\ &\supseteq \bigcap_{0 \leq i < \mu} \operatorname{supp}((\mathcal{D}^i \mathcal{I})_{|S}) = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S}). \end{aligned}$$

Assume that all multiple test blow-ups of $(\mathcal{I}, \mu)$ of length k with centers $C_i \subset S_i$ are defined by multiple test blow-ups of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ and moreover for $i \leq k$,

$$\operatorname{supp}(\mathcal{I}_i, \mu) \cap S_i = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i.$$

For any $f \in \mathcal{I}$ define $f = f_0 \in \mathcal{I}$ and $f_{i+1} = \sigma_i^{\mathrm{c}}(f_i) = y_i^{-\mu} \sigma^*(f_i) \in \mathcal{I}_{i+1}$. Assume moreover that for any $f \in \mathcal{I}$,

$$f_k = \sum c_{\alpha f k}(y) x^\alpha,$$

where $c_{\alpha f k|S_k} \in (\sigma^k_{|S_k})^{\mathrm{c}}(\mathcal{D}^{\mu - |\alpha|}(\mathcal{I})_{|S})$. Consider the effect of the blow-up of C_k at a point x' in the strict transform $S_{k+1} \subset X_{k+1}$. By Lemmas 3.6.2 and 3.6.3,

$$\begin{aligned} \operatorname{supp}(\mathcal{I}_{k+1}, \mu) \cap S_{k+1} &= \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1} \cap S_{k+1} \\ &\subseteq \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1|S_{k+1}} = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_{k+1}. \end{aligned}$$

Let $x_1, \ldots, x_k$ describe the subvariety S_k of X_k. We can find coordinates $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$, by taking if necessary linear combinations of $y_1, \ldots, y_{n-k}$, such that the center of the blow-up is described by $x_1,\dots,x_k,y_1,\dots,y_m$ and the coordinates at x' are given by

$$x'_1=x_1/y_m,\dots,x'_k=x_k/y_m,\qquad y'_1=y_1/y_m,\dots,y'_m=y_m,$$
$$y'_{m+1}=y_{m+1},\dots,y'_n=y_n.$$

Note that replacing $y_1,\dots,y_{n-k}$ with their linear combinations does not modify the form $f_k=\sum c_{\alpha fk}(y)x^\alpha$. Then the function $f_{k+1}=\sigma^{\mathrm{c}}(f_k)$ can be written as

$$f_{k+1}=\sum c_{\alpha f,k+1}(y){x'}^{\alpha},$$

where $c_{\alpha fk+1}=y_m^{-\mu+|\alpha|}\sigma^*_{k+1}(c_{\alpha fk})$. Thus

$$\begin{aligned}c_{\alpha fk+1|S_{k+1}}&=(\sigma_{k+1|S_{k+1}})^{\mathrm{c}}(c_{\alpha fk|S_k})\\ &\in(\sigma^{k+1}_{|S_{k+1}})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I})_{|S})=(\sigma^{k+1})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I}))_{|S_{k+1}}\end{aligned}$$

and consequently

$$\begin{aligned}\operatorname{supp}(\mathcal{I}_{k+1},\mu)\cap S_{k+1}&=\bigcap_{f\in\mathcal{I},|\alpha|\le\mu}\operatorname{supp}(c_{\alpha fk+1|S_{k+1}},\mu-|\alpha|)\\ &\supseteq\operatorname{supp}[\mathcal{C}(\mathcal{I},\mu)_{|S}]_{k+1}=\operatorname{supp}(\mathcal{C}(\mathcal{I},\mu)_{k+1})_{|S_{k+1}}.\end{aligned}$$ □

A direct consequence of the above lemma is the following result:

Lemma 3.6.5. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order whose support $\operatorname{supp}(\mathcal{I},\mu)$ does not contain a smooth subvariety S of X. Assume that S has only simple normal crossings with E. Let (X_i) be its multiple test blow-up such that all centers C_i are either contained in the strict transforms $S_i\subset X_i$ of S or are disjoint from them. Then the restrictions $\sigma_{i|S_i}:S_i\to S_{i-1}$ of the morphisms $\sigma_i:X_i\to X_{i-1}$ define a multiple test blow-up (S_i) of $\mathcal{C}(\mathcal{I},\mu)_{|S}$ and*

$$\operatorname{supp}(\mathcal{I}_i,\mu)\cap S_i=\operatorname{supp}[\mathcal{C}(\mathcal{I},\mu)_{|S}]_i.$$

As a simple consequence of Lemma 3.6.4 we formulate the following refinement of the Giraud Lemma.

Lemma 3.6.6. *Let $(X,\mathcal{I},\emptyset,\mu)$ be a marked ideal of maximal order whose support $\operatorname{supp}(\mathcal{I},\mu)$ has codimension at least 2 at some point x. Let $U\ni x$ be an open subset for which there is a tangent direction $u\in T(\mathcal{I})$ and such that $\operatorname{supp}(\mathcal{I},\mu)\cap U$ is of codimension at least 2. Let $V(u)$ be the regular subscheme of U defined by u. Then for any multiple test blow-up (X_i) of X,*

1. *$\operatorname{supp}(\mathcal{I}_i,\mu)$ is contained in the strict transform $V(u)_i$ of $V(u)$ as a proper subset.*
2. *The sequence $(V(u)_i)$ is a multiple test blow-up of $\mathcal{C}(\mathcal{I},\mu)_{|V(u)}$.*
3. *$\operatorname{supp}(\mathcal{I}_i,\mu)\cap V(u)_i=\operatorname{supp}[\mathcal{C}(\mathcal{I},\mu)_{|V(u)}]_i$.*
4. *Every multiple test blow-up $(V(u)_i)$ of $\mathcal{C}(\mathcal{I},\mu)_{|V(u)}$ defines a multiple test blow-up (X_i) of $(\mathcal{I},\mu)$.*

□

Lemma 3.6.7. *Let $\phi:X'\to X$ be a smooth morphism of smooth varieties and let $(X,\mathcal{I},\emptyset,\mu)$ be a marked ideal. Then*

$$\phi^*(\mathcal{C}(\mathcal{I}))=\mathcal{C}(\phi^*(\mathcal{I})).$$

Proof. A direct consequence of Lemma 3.2.5. □

4. Algorithm for canonical resolution of marked ideals

The presentation of the following Hironaka resolution algorithm builds upon Villamayor's and Bierstone-Milman's proofs.

Let $\mathrm{Sub}(E_i)$ denote the set of all subsets of E_i. For any subset in $\mathrm{Sub}(E_i)$ write a sequence $(D_1, D_2, \ldots, 0, \ldots)$ consisting of all elements of the subset in increasing order followed by an infinite sequence of zeros. We shall assume that $0 \leq D$ for any $D \in E_i$. Consider the lexicographic order $\leq$ on the set of such sequences. Then for any two subsets $A_1 = \{D^1_i\}_{i \in I}$ and $A_2 = \{D^2_j\}_{j \in J}$ we write

$$A_1 \leq A_2$$

if, for the corresponding sequences, $(D^1_1, D^1_2, \ldots, 0, \ldots) \leq (D^2_1, D^2_2, \ldots, 0, \ldots)$.

Let $\mathbf{Q}_{\geq 0}$ denote the set of nonnegative rational numbers and let

$$\overline{\mathbf{Q}}_{\geq 0} := \mathbf{Q}_{\geq 0} \cup \{\infty\}.$$

Denote by $\overline{\mathbf{Q}}^{\infty}_{\geq 0}$ the set of all infinite sequences in $\overline{\mathbf{Q}}_{\geq 0}$ with a finite number of nonzero elements. We equip $\overline{\mathbf{Q}}^{\infty}_{\geq 0}$ with the lexicographic order.

Theorem 4.0.1. *For any marked ideal $(X, \mathcal{I}, E, \mu)$ such that $\mathcal{I} \neq 0$ there is an associated resolution $(X_i)_{0 \leq i \leq m_X}$, called canonical, satisfying the following conditions:*

(1) *There exist upper semicontinuous invariants* inv, *ν and ρ defined on* $\mathrm{supp}(X_i, \mathcal{I}_i, E_i, \mu)$ *with values in $\mathbf{Q}_{\geq 0} \times \overline{\mathbf{Q}}^{\infty}_{\geq 0}$, $\overline{\mathbf{Q}}_{\geq 0}$ and* $\mathrm{Sub}(E_i)$ *respectively.*

(2) *The centers C_i of blow-ups are regular and defined by the set where* (inv, ρ) *attains its maximum. They are components of the maximal locus of* inv.

(3) (a) *For any $x \in \mathrm{supp}(X_{i+1}, \mathcal{I}_{i+1}, E_{i+1}, \mu)$ and $\sigma(x) \in C_i$, either* $\mathrm{inv}(x) < \mathrm{inv}(\sigma(x))$ *or* $\mathrm{inv}_{i+1}(x) = \mathrm{inv}(\sigma(x))$ *and $\nu(x) < \nu(\sigma(x))$.*
(b) *For any $x \in \mathrm{supp}(X_{i+1}, \mathcal{I}_{i+1}, E_{i+1}, \mu)$ and $\sigma(x) \notin C_i$,* $\mathrm{inv}(x) = \mathrm{inv}(\sigma(x))$, *$\rho(x) = \rho(\sigma(x))$ and $\nu(x) = \nu(\sigma(x))$.*

(4) *For any étale morphism $\phi : X' \to X$ the induced sequence $(X'_i) = \phi^*(X_i)$ is an extension of the canonical resolution of X' such that for the induced marked ideals $(X'_i, \mathcal{I}'_i, E'_i, \mu)$ and $x' \in \mathrm{supp}(X'_i, \mathcal{I}'_i, E'_i, \mu)$, we have* $\mathrm{inv}(\phi_i(x')) = \mathrm{inv}(x')$, *$\nu(\phi_i(x')) = \nu(x')$ and* $\rho(\phi_i(x')) = \rho(x') \in \mathrm{Sub}(E'_i) \subset \mathrm{Sub}(E_i)$.

Remarks. (1) The main idea of the algorithm of resolving marked ideals of maximal order is to reduce the procedure to the hypersurface of maximal contact (Step 1b).

(2) By Lemma 3.3.4 hypersurfaces of maximal contact can be constructed locally. They are in general not transversal to E and cannot be used for the reduction procedure. We think of E and its strict transforms as an obstacle to existence of a hypersurface of maximal contact (transversal to E). These divisors are often referred to as "old" ones.

(3) In Step 1a we move "old" divisors apart from the support of the marked ideal. In this process we create "new" divisors but these divisors are "born" from centers lying in the hypersurface of maximal contact. Note that the hypersurfaces of maximal contact are constructed on X and then lifted to the blow-ups performed in Step 1a. The "new" divisors are transversal to hypersurfaces of maximal contact. After eliminating "old" divisors from the support in Step 1a

all divisors are "new" and we may reduce the resolving procedure to hypersurfaces of maximal contact (Step 1b).

(4) In Step 2 we resolve general marked ideals by reducing the algorithm to resolving some marked ideals of maximal order (companion ideals).

Proof. Induction on the dimension of X. If X is 0-dimensional, $\mathcal{I} \neq 0$ and $\mu > 0$, then $\operatorname{supp}(X, \mathcal{I}, \mu) = \emptyset$ and all resolutions are trivial.

The invariants inv, ν and ρ will be defined successively in the course of the resolution algorithm using the inductive assumptions and property (2). Let S be a smooth subvariety of X having simple normal crossings with the set of divisors E. Let E' be the subset of E consisting of the divisors transversal to S. We shall often identify the set $E'_{|S} := \{D \cap S \mid D \in E\}$ with E'. In particular $\operatorname{Sub}(E'_{|S})$ can be identified with $\operatorname{Sub}(E') \subset \operatorname{Sub}(E)$.

Step 1. Resolving a marked ideal $(X, \mathcal{J}, E, \mu)$ of maximal order.

The process of resolving the marked ideals of maximal order is controlled by an auxiliary invariant $\overline{\operatorname{inv}}$ defined in Step 1. The invariant inv will then be defined for any marked ideals in Step 2.

Before we start our resolution algorithm for the marked ideal $(\mathcal{J}, \mu)$ of maximal order we shall replace it with the equivalent homogenized ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$. Resolving the ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ defines a resolution of $(\mathcal{J}, \mu)$ at this step. To simplify notation we shall denote $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ by $(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1a. Reduction to the nonboundary case.

For any multiple test blow-up (X_i) of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ we shall identify (for simplicity) strict transforms of E on X_i with E. For any $x \in X_i$, let $s(x)$ denote the number of divisors in E through x and set

$$s_i = \max\{s(x) \mid x \in \operatorname{supp}(\overline{\mathcal{J}}_i)\}.$$

Let $s = s_0$. By assumption the intersections of any $s > s_0$ components of the exceptional divisors are disjoint from $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Each intersection of divisors in E is locally defined by intersection of some irreducible components of these divisors. Find all intersections $H^s_\alpha, \alpha \in A$, of s irreducible components of divisors E such that $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha \neq \emptyset$. By the maximality of s, the supports $\operatorname{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$, where $\alpha' \neq \alpha$.

Step 1aa. Eliminating the components H^s_α contained in $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

Let $H^s_\alpha \subset \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$. If $s \geq 2$, then by blowing up $C = H^s_a$ we separate divisors contributing to H^s_a, thus creating new points all with $s(x) < s$. If $s = 1$, then by Lemma 3.3.7, $H^s_\alpha \subset \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$ is a codimension one component and by blowing up H^s_α we create all new points off $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

For all $x \in H^s_a \subset \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$ set

$$\overline{\operatorname{inv}}(x) = (s(x), \infty, 0, \ldots, 0, \ldots), \quad \nu(x) = 0, \quad \rho(x) = \emptyset.$$

This definition, as we see below, is devised so as to ensure that all $H^s_\alpha \subset \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$ will be blown up first and we reduce the situation to the case where no H^s_α is contained in $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1ab. Moving $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$ and H^s_α apart.

After the blow-ups in Step 1aa we arrive at X_p for which no H^s_α is contained in $\operatorname{supp}(\overline{\mathcal{J}}_p, \overline{\mu})$, where $p = 0$ if there were no such components and $p = 1$ if there were some.

Construct the canonical resolutions of $\overline{\mathcal{J}}_{p|H^s_\alpha} := (H^s_\alpha, \overline{\mathcal{J}}_{p|H^s_\alpha}, (E_p \setminus E)_{|H^s_\alpha}, \overline{\mu})$. By Proposition 3.6.4 each such resolution defines a multiple test blow-up of $(\overline{\mathcal{J}}_p, \overline{\mu})$ (and of $(\overline{\mathcal{J}}, \overline{\mu})$). Since the supports $\operatorname{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$, where $\alpha' \neq \alpha$, these resolutions glue to a unique multiple test blow-up $(X_i)_{i \leq j_1}$ of $(\overline{\mathcal{J}}, \overline{\mu})$ such that $s_{j_1} < s$. To control the glueing procedure and ensure its uniqueness we define for all $x \in \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha$ the invariant

$$\overline{\operatorname{inv}}(x) = (s(x), \operatorname{inv}_{\overline{\mathcal{J}}_{|H^s_\alpha}}(x)), \quad \nu_{\overline{\mathcal{J}}} = \nu_{\overline{\mathcal{J}}_{|H^s_\alpha}}, \quad \rho_{\overline{\mathcal{J}}} = \rho_{\overline{\mathcal{J}}_{|H^s_\alpha}}.$$

The blow-ups will be performed at the centers $C \subset \operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha$ for which the invariant $(\overline{\operatorname{inv}}, \rho)$ attains its maximum. Note that by the maximality condition for any H^s_α the irreducible components of the centers are contained in H^s_α or are disjoint from them. Therefore by Lemma 3.6.5,

$$\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_\alpha} = \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^s_\alpha.$$

By applying this multiple test blow-up we create a marked ideal $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$ with support disjoint from all H^s_α. Summarizing the above we construct a multiple test blow-up $(X_i)_{0 \leq i \leq j_1}$ subject to the conditions:

(1) $(H^s_{\alpha i})_{0 \leq i \leq j_1}$ is an extension of the canonical resolution of $\overline{\mathcal{J}}_{|H^s_\alpha}$.

(2) There are invariants $\overline{\operatorname{inv}}$, $\overline{\mu}$ and ρ defined for $0 \leq i < j_1$ and all $x \in \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^s_{\alpha i}$ such that

$$\overline{\operatorname{inv}}(x) = (s(x), \operatorname{inv}_{\overline{\mathcal{J}}_{i|H^s_{\alpha i}}}(x)), \quad \nu_{\overline{\mathcal{J}}_i} = \nu_{\overline{\mathcal{J}}_{i|H^s_{\alpha i}}}, \quad \rho_{\overline{\mathcal{J}_i}} = \rho_{\overline{\mathcal{J}}_{i|H^s_{\alpha i}}}.$$

(3) The blow-ups of X_i are performed at the centers where the invariant $(\overline{\operatorname{inv}}, \rho)$ attains its maximum.

(4) $\operatorname{supp}(\overline{\mathcal{J}}_{j_1}, \overline{\mu}) \cap H^s_{\alpha j_1} = \emptyset$.

Conclusion of the algorithm in Step 1a. After performing the blow-ups in Steps 1aa and 1ab for the marked ideal $(\overline{\mathcal{J}}, \overline{\mu})$ we arrive at a marked ideal $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$ with $s_{j_1} < s_0$. Now we put $s = s_{j_1}$ and repeat the procedure of Steps 1aa and 1ab for $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$. Note that any $H^s_{\alpha j_1}$ on X_{j_1} is the strict transform of some intersection $H^{s_{j_1}}_\alpha$ of $s = s_{j_1}$ divisors in E on X. Moreover by the maximality condition for all s_i, where $i \leq j_1$ and $\alpha \neq \alpha'$, the set $\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^{s_i}_{\alpha' i}$ is either disjoint from $H^{s_{j_1}}_{\alpha i}$ or contained in it. Thus for $0 \leq i \leq j_1$, all centers C_i have components either contained in $H^{s_{j_1}}_{\alpha i} = H^s_{\alpha i}$ or disjoint from them and by Lemma 3.6.5,

$$\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_{\alpha i}} = \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^s_{\alpha i}.$$

Moreover if we repeat the procedure in Steps 1aa and 1ab the above property will still be satisfied until either $(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_\alpha}$ are resolved as in Step 1ab or H^s_α disappear as in Step 1aa.

We continue the above process until $s_{j_k} = s_r = 0$. Then $(X_j)_{0 \leq j \leq r}$ is a multiple test blow-up of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ such that $\operatorname{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$ does not intersect any divisor in E. Therefore $(X_j)_{0 \leq j \leq r}$ and further longer multiple test blow-ups $(X_j)_{0 \leq j \leq r_0}$ for any $r \leq r_0$ can be considered as multiple test blow-ups of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ since, starting from X_r, the strict transforms of E play no further role in the resolution process since they do not intersect $\operatorname{supp}(\overline{\mathcal{J}}_j, \overline{\mu})$ for $j \geq r$.

Note that in Step 1a all points $x \in \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})$ for which $s(x) > 0$ were assigned their invariants $\overline{\operatorname{inv}}$, ν and ρ. (They are assigned the invariants at the moment they are getting blown up. The invariants remain unchanged when the points are transformed isomorphically.) The invariants are upper semicontinuous by the semicontinuity of the function $s(x)$ and the inductive assumption.

Step 1b. **Nonboundary case.**

Let $(X_j)_{0\leq j\leq r}$ be the multiple test blow-up of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ defined in Step 1a.

Step 1ba. **Eliminating codimension one components**

If $\mathrm{supp}(\overline{\mathcal{J}}_r,\overline{\mu})$ is of codimension 1, then by Lemma 3.3.7 all its codimension 1 components are smooth and disjoint from the other components of $\mathrm{supp}(\overline{\mathcal{J}}_r,\overline{\mu})$. These components are strict transforms of the codimension 1 components of $\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu})$. Moreover the irreducible components of the centers of blow-ups were either contained in the strict transforms or disjoint from them. Therefore E_r will be transversal to all the codimension 1 components. Let $\mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu}))$ be the union of all components of $\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu}))$ of codimension 1. We define the invariants for $x\in\mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}_r,\overline{\mu}))$ to be

$$\overline{\mathrm{inv}}(x)=(0,\infty,0,\ldots,0,\ldots),\quad \nu(x)=0,\quad \rho(x)=\emptyset.$$

This definition, as we see below, is devised so as to ensure that all codimension 1 components will be blown up first.

By Lemma 3.3.7 blowing up the components reduces the situation to the case when $\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu})$ is of codimension ≥ 2.

Step 1bb. **Eliminating codimension ≥ 2 components**

After Step 1ba we arrive at a marked ideal $\mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})$, where $p=r$ if there were no codimension one components and $p=r+1$ if there were some and we blew them up.

For any $x\in\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu})\setminus\mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu}))\subset X$ find a tangent direction $u\in\mathcal{D}^{\overline{\mu}-1}(\overline{\mathcal{J}})$ on some neighborhood U_u of x. Then $V(u)\subset U_u$ is a hypersurface of maximal contact. By the quasicompactness of X we can assume that the covering defined by U_u is finite. Let $U_{ui}\subset X_i$ be the inverse image of U_u and let $V(u)_i\subset U_u$ denote the strict transform of $V(u)$. By Lemma 3.6.6, $(V(u)_i)_{0\leq i\leq p}$ is a multiple test blow-up of $(V(u),\overline{\mathcal{J}}_{|V(u)},\emptyset,\overline{\mu})$. In particular the induced marked ideal for $i=p$ is equal to

$$\overline{\mathcal{J}}_{p|V(u)_p}=(V(u)_p,\overline{\mathcal{J}}_{p|V(u)_p},(E_p\setminus E)_{|V(u)_p},\overline{\mu}).$$

Construct the canonical resolution of $(V(u)_i)_{p\leq i\leq m_u}$ of the marked ideal $\overline{\mathcal{J}}_{p|V(u)_p}$. Then the sequence $(V(u)_i)_{0\leq i\leq m_u}$ is a resolution of $(V(u),\overline{\mathcal{J}}_{|V(u)},\emptyset,\overline{\mu})$ which defines, by Lemma 3.6.6, a resolution $(U_{ui})_{0\leq i\leq m_u}$ of $(U_u,\overline{\mathcal{J}}_{|U_u},\emptyset,\overline{\mu})$. Moreover both resolutions are related by the property

$$\mathrm{supp}(\overline{\mathcal{J}}_{i|U_{ui}})=\mathrm{supp}(\overline{\mathcal{J}}_{i|V(u)_i}).$$

We shall construct the resolution of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ by patching together extensions of the local resolutions $(U_{ui})_{0\leq i\leq m_u}$.

For $x\in\mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})\cap U_{up}$ define the invariants

$$\overline{\mathrm{inv}}(x):=(0,\mathrm{inv}_{\overline{\mathcal{J}}_{p|V(u)_p}}(x)),\quad \nu:=\nu_{\overline{\mathcal{J}}_{p|V(u)_p}}(x),\quad \rho(x):=\rho_{\overline{\mathcal{J}}_{p|V(u)_p}}(x).$$

We need to show that these invariants do not depend on the choice of u.

Let $x\in\mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})\cap U_{up}\cap U_{vp}$. By the Glueing Lemma 3.5.5, for any two different tangent directions u and v we find étale neighborhoods $\phi_u,\phi_v:U^{uv}\to U:=U_u\cap U_v$ and their liftings $\phi_{pu},\phi_{pv}:U^{uv}_p\to U_p:=U_{up}\cap U_{vp}$ such that

(1) $X^{uv}_p:=(\phi_{pu})^{-1}(V(u)_p)=(\phi_{pv})^{-1}(V(v)_p)$.

(2) $(U^{uv}_p,\overline{\mathcal{J}}^{uv}_p,E^{uv}_p,\overline{\mu}):=(\phi_{pu})^*(U_p,\overline{\mathcal{J}}_p,E_p,\overline{\mu})=(\phi_{pv})^*(U_p,\overline{\mathcal{J}}_p,E_p,\overline{\mu})$.

(3) There exists $y \in \operatorname{supp}(U_p^{uv}, \overline{\mathcal{J}}_p^{uv}, E_p^{uv}, \overline{\mu})$ such that $\phi_{pu}(\overline{x}) = \phi_{pv}(\overline{x})$.

By the functoriality of the invariants we have

$$\operatorname{inv}_{\overline{\mathcal{J}}_{p|V(u)_p}}(x) = \operatorname{inv}_{\overline{\mathcal{J}}^{uv}_{p|X_p^{uv}}}(\overline{x}) = \operatorname{inv}_{\overline{\mathcal{J}}_{p|V(v)_p}}(x).$$

Analogously $\nu_{\overline{\mathcal{J}}_{p|V(u)_p}}(x) = \nu_{\overline{\mathcal{J}}_{p|V(v)_p}}(x)$ and $\rho_{\overline{\mathcal{J}}_{p|V(u)_p}}(x) = \rho_{\overline{\mathcal{J}}_{p|V(v)_p},}(x)$. Thus the invariants $\overline{\operatorname{inv}}$, ν and ρ do not depend on the choice of the tangent direction.

Define the center C_p of the blow-up on X_p to be the maximal locus of the invariant $(\overline{\operatorname{inv}}, \rho)$. Note that for any tangent direction u, either $C_p \cap U_{up}$ defines the first blow-up of the canonical resolution of $(V(u)_p, \overline{\mathcal{J}}_{p|V(u)_p}, E_{p|V(u)_p}, \overline{\mu})$ or $C_p \cap U_{up} = \emptyset$ and the blow-up of C_p does not change $V(u)_p \subset U_{up}$.

Blowing up C_p defines X_{p+1} and we are in a position to construct the invariants on X_{p+1} and define the center of the blow-up $C_{p+1} \subset X_{p+1}$ as before.

By repeating the same reasoning for $j = p+1, \dots, m$ we construct the resolution $(X_i)_{p\le i\le m}$ of $(X_p, \overline{\mathcal{J}}_p, E_p \setminus E, \overline{\mu})$ satisfying the following properties.

(1) For any u, the restriction of $(X_i)_{p\le i\le m}$ to $(V(u)_i)_{p\le i\le m}$ is an extension of the canonical resolution of $(V(u)_p, \overline{\mathcal{J}}_{p|V(u)_p}, E_{p|V(u)_p}, \overline{\mu})$.

(2) There are invariants $\overline{\operatorname{inv}}$, $\overline{\mu}$ and ρ defined for all points $x \in \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})$, $p \le i \le m$, such that

$$\overline{\operatorname{inv}}(x) := (0, \operatorname{inv}_{\overline{\mathcal{J}}_{i|V(u)_i}}), \quad \nu(x) := \nu_{\overline{\mathcal{J}}_{i|V(u)_i}}(x), \quad \rho(x) := \rho_{\overline{\mathcal{J}}_{i|V(u)_i}}(x).$$

(3) The blow-ups of X_i are performed at the centers where the invariant $(\overline{\operatorname{inv}}, \rho)$ attains its maximum.

(4) $\operatorname{supp}(\overline{\mathcal{J}}_m, \overline{\mu}) = \emptyset$.

The resolution $(X_i)_{p\le i\le m}$ of $(X_p, \overline{\mathcal{J}}_p, E_p \setminus E, \overline{\mu})$ defines the resolution $(X_i)_{0\le i\le m}$ of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ and of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$.

In Step 1b all points $x \in \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})$ with $s(x) = 0$ were assigned the invariants $\overline{\operatorname{inv}}$, ν and ρ. They are upper semicontinuous by the induction assumption.

Commutativity of the resolution procedure in Step 1 with étale morphisms.

Let $\phi : X' \to X$ be an étale morphism. In Step 1a we find a sequence $i_0 := 0 < i_1 < \ldots < i_k = r \le m$ such that $s_{i_0} > s_{i_1} > \ldots > s_{i_k}$ and for $i_l \le i < i_{l+1}$, we have $s_i = s_{i_l}$. Moreover the resolution process for $(X_i)_{i_l\le i\le i_{l+1}}$ is reduced to resolving $\overline{\mathcal{J}}_{i_l|H^s_{\alpha i_l}}$. In Step 1b we reduce the resolution process for $(X_i)_{i_k\le i\le m}$ to resolving $\overline{\mathcal{J}}_{i_k|V(u)_{i_k}}$.

Let $s'_{j_0} > s'_{j_1} > \ldots > s'_{j_{k'}}$ be the corresponding sequence defined for the canonical resolution $(X'_i)_{0\le j\le m'}$ of

$$(X', \overline{\mathcal{J}}', E', \overline{\mu}) := \phi^*(X, \overline{\mathcal{J}}, E, \overline{\mu}).$$

Let $\phi^*(X_i)_{0\le i\le m}$ denote the resolution of $(\overline{\mathcal{J}'}, \overline{\mu})$ induced by $(X_i)_{0\le i\le m}$. In particular $X'_0 = \phi^*(X_0)$. We want to show the following:

Lemma 4.0.2. (1) *$\phi^*(X_i)_{0\le i\le m}$ is an extension of $(X'_j)_{0\le j\le m'}$.*

(2) *For $x' \in \operatorname{supp}(\phi^*(\mathcal{J}_i))$ we have $\overline{\operatorname{inv}}(x') = \overline{\operatorname{inv}}(\phi_i(x'))$, $\nu(x') = \nu(\phi_i(x'))$ and $\rho(x') = \rho(\phi_i(x'))$.*

Proof. Denote by $s(\phi^*(X_i))$ the maximum number of $\phi^*(E)$ through a point in $\operatorname{supp}(\phi^*(\overline{\mathcal{J}_i}))$. In particular $s(\phi^*(X_i)) \le s_i$ for any index $0 \le i \le m$.

Assume that for the index l we can find an index l' such that

$$(\star) \qquad \phi^*(X_{i_l}, \overline{\mathcal{J}}_{i_l}, E_{i_l}, \overline{\mu}) \simeq (X'_{j_{l'}}, \overline{\mathcal{J}}'_{j_{l'}}, E'_{j_{l'}}, \overline{\mu})$$

(This assumption is satisfied for $l = 0$.)

(1) If $s(\phi^*(X_{i_l})) < s_{i_l}$, then the centers C_i of the blow-ups in the sequence $(X_i)_{i_l \le i \le i_{l+1}}$ are contained in the intersections of s_{i_l} divisors in E and do not hit the images $\phi_i(\phi^*(X)_i)$. Thus $\phi^*(X_i)_{i_l \le i \le i_{l+1}}$ consists of isomorphisms. The property $(\star)$ will be satisfied for $l+1$ (and for the same l').

(2) If $s(\phi^*(X_{i_l})) = s_{i_l} > 0$, then the intersections $(H')^s_{\alpha i}$ of $s = s(\phi^*(X_{i_l})) = s_{i_l}$ divisors are inverse images of H^s_α and the resolution process $\phi^*(X_i)_{i_l \le i \le i_{l+1}}$ is reduced to resolving $\phi^*(\overline{\mathcal{J}}_{i_l | H^s_{\alpha i_l}})$. Moreover by the property $(\star)$,

$$\phi^*(\overline{\mathcal{J}}_{i_l | H^s_{\alpha i_l}}) = \overline{\mathcal{J}}'_{j_{l'} | (H')^s_{\alpha j_{l'}}}$$

for some l' such that $s_{j_{l'}} = s_{i_l}$.

By commutativity of étale morphisms with the canonical resolution in lower dimensions we know that all resolutions $\{(H')^s_{\alpha i}\}_{i_l \le i \le i_{l+1}}$ induced by $\phi^*(X_i)_{i_l \le i \le i_{l+1}}$ are extensions of the canonical resolutions of $\overline{\mathcal{J}}'_{i_{l'} | (H')^s_{\alpha i_{l'}}}$. Moreover the restrictions $\phi_{i | (H')^s_{\alpha i}} : (H')^s_{\alpha i} \to H^s_{\alpha i}$ preserve the invariants inv, ν and ρ. Thus $\phi^*(X_i)_{i_l \le i \le i_{l+1}}$ is an extension of $(X'_i)_{j_{l'} \le i \le j_{l'+1}}$. Moreover for $i_l \le i < i_{l+1}$ and $x' \in \operatorname{supp}(\phi_i^*(\overline{\mathcal{J}}_i, \overline{\mu})) \cap (H')^s_{\alpha i}$, and $x = \phi_i(x')$ we have

$$\overline{\operatorname{inv}}(x') = (s(x'), \operatorname{inv}_{(\phi_i^*(\overline{\mathcal{J}})_{i|(H')^s_{\alpha i}})}(x)) = (s(x), \operatorname{inv}_{(\overline{\mathcal{J}}_{i|H^s_{\alpha i}})}(x)) = \overline{\operatorname{inv}}(x).$$

Analogously $\nu(x') = \nu(x)$ and $\rho(x') = \rho(x)$. The property $(\star)$ is satisfied for $l+1$ (and $l'+1$).

(3) If $s(\phi^*(X_{i_k})) = s_{i_k} = 0$, then the resolution process for $(X_i)_{i_k \le i \le m}$ is reduced to the canonical resolution of $\overline{\mathcal{J}}_{i_k | V(u)_{i_k}}$ on a hypersurface of maximal contact $V(u)_{i_k}$. Also the resolution process of $\overline{\mathcal{J}}'_{j_{k'}} \simeq \phi^*(\overline{\mathcal{J}}_{i_k})$ is reduced to the canonical resolution of $\overline{\mathcal{J}}'_{i_k | V(u)_{i_k}} = \phi^*(\overline{\mathcal{J}}_{i_k | V(u)_{i_k}})$ on the hypersurface of maximal contact $V(u)_{i_k}$. Since the inverse image of a hypersurface of maximal contact is a hypersurface of maximal contact by the same reasoning as before (replacing H^s_α with $V(u)$) we deduce that $\phi^*(X_i)_{i_k \le i \le m}$ is an extension of $(X'_i)_{i_{k'} \le i \le m}$. Moreover for $i_k \le i \le m$ and $x' \in \operatorname{supp}(\overline{\mathcal{J}}'_i, \overline{\mu})$,

$$\overline{\operatorname{inv}}(x') = \overline{\operatorname{inv}}(\phi_i(x')), \quad \nu(x') = \nu(\phi_i(x')), \quad \rho(x') = \rho(\phi_i(x')).$$

The lemma is proven.

Step 2. Resolving marked ideals $(X, \mathcal{I}, E, \mu)$**.**

For any marked ideal $(X, \mathcal{I}, E, \mu)$ write

$$I = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}),$$

where $\mathcal{M}(\mathcal{I})$ is the *monomial part* of $\mathcal{I}$, that is, the product of the principal ideals defining the irreducible components of the divisors in E, and $\mathcal{N}(\mathcal{I})$ is a *nonmonomial part* which is not divisible by any ideal of a divisor in E. Let

$$\operatorname{ord}_{\mathcal{N}(\mathcal{I})} := \max\{\operatorname{ord}_x(\mathcal{N}(\mathcal{I})) \mid x \in \operatorname{supp}(\mathcal{I}, \mu)\}.$$

Definition 4.0.3 (Hironaka, Bierstone-Milman, Villamayor, Encinas-Hauser)**.** By the *companion ideal* of $(\mathcal{I},\mu)$ where $I=\mathcal{N}(\mathcal{I})\mathcal{M}(\mathcal{I})$ we mean the marked ideal of maximal order

$$O(\mathcal{I},\mu)=\begin{cases}(\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})})+(\mathcal{M}(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}<\mu,\\ (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}\geq\mu.\end{cases}$$

Step 2a. **Reduction to the monomial case by using companion ideals.** By Step 1 we can resolve the marked ideal of maximal order $(\mathcal{J},\mu_{\mathcal{J}}):=O(\mathcal{I},\mu)$ using the invariant $\overline{\mathrm{inv}}_{O(\mathcal{I},\mu)}$. By Lemma 3.4.1, for any multiple test blow-up of $O(\mathcal{I},\mu)$,

$$\begin{aligned}\mathrm{supp}(O(\mathcal{I},\mu))_i=&\mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i\ \cap\ \mathrm{supp}[M(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(H\mathcal{I})}]_i\\ =&\ \mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i\ \cap\ \mathrm{supp}(\mathcal{I}_i,\mu).\end{aligned}$$

Consequently, such a resolution leads to the ideal $(\mathcal{I}_{r_1},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}<\mathrm{ord}_{\mathcal{N}(\mathcal{I})}$. This resolution is controlled by the invariants inv, ν and ρ defined for all $x\in\mathrm{supp}(\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})})\cap\mathrm{supp}(\mathcal{I}_i,\mu)_i$,

$$\mathrm{inv}(x)=\left(\frac{\mathrm{ord}_{\mathcal{N}(\mathcal{I})}}{\mu},\overline{\mathrm{inv}}_{O(\mathcal{I},\mu)}(x)\right),\quad \nu(x)=\nu_{O(\mathcal{I},\mu)}(x),\quad \rho(x)=\rho_{O(\mathcal{I},\mu)}(x).$$

Then we repeat the procedure for $(\mathcal{I}_{r_1},\mu)$. We find marked ideals $(\mathcal{I}_{r_0},\mu)=(\mathcal{I},\mu),(\mathcal{I}_{r_1},\mu),\ldots,(\mathcal{I}_{r_m},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_0)}>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}>\ldots>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$. The procedure terminates after a finite number of steps when we arrive at the ideal $(\mathcal{I}_{r_m},\mu)$ with $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}=0$ or with $\mathrm{supp}(\mathcal{I}_{r_m},\mu)=\emptyset$. In the second case we get the resolution. In the first case $\mathcal{I}_{r_m}=\mathcal{M}(\mathcal{I}_{r_m})$ is monomial.

In Step 2a all points $x\in\mathrm{supp}(\mathcal{I},\mu)$ for which $\mathrm{ord}_x(\mathcal{I})\neq 0$ were assigned the invariants inv,μ,ρ. They are upper semicontinuous by the semicontinuity of ord_x and of the invariants $\overline{\mathrm{inv}},\mu,\rho$ for the marked ideals of maximal order.

Step 2b. **Monomial case** $\mathcal{I}=\mathcal{M}(\mathcal{I})$.

Define the invariants

$$\mathrm{inv}(x)=(0,\ldots,0,\ldots),\quad \nu(x)=\frac{\mathrm{ord}_x(\mathcal{I})}{\mu}.$$

Let $x_1,\ldots,x_k$ define equations of the components $D_1^x,\ldots,D_k^x\in E$ through $x\in\mathrm{supp}(X,\mathcal{I},E,\mu)$ and let $\mathcal{I}$ be generated by the monomial $x^{a_1,\ldots,a_k}$ at x. In particular $\nu(x)=\frac{a_1+\ldots+a_k}{\mu}$.

Let $\rho(x)=\{D_{i_1},\ldots,D_{i_l}\}\in\mathrm{Sub}(E)$ be the maximal subset satisfying the properties

(1) $a_{i_1}+\ldots+a_{i_l}\geq\mu$,
(2) for any $j=1,\ldots,l$, $a_{i_1}+\ldots+\check{a}_{i_j}+\ldots+a_{i_l}<\mu$.

Let $R(x)$ denote the subsets in $\mathrm{Sub}(E)$ satisfying the properties (1) and (2). The maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ through x are described by the intersections $\bigcap_{D\in A}D$ where $A\in R(x)$. The maximal locus of ρ determines at most one maximal component of $\mathrm{supp}(\mathcal{I},\mu)$ through each x.

After the blow-up at the maximal locus $C=\{x_{i_1}=\ldots=x_{i_l}=0\}$ of ρ, the ideal $\mathcal{I}=(x^{a_1,\ldots,a_k})$ is equal to $\mathcal{I}'=(x'^{a_1,\ldots,a_{i_j-1},a,a_{i_j+1},\ldots,a_k})$ in the neighborhood corresponding to x_{i_j}, where $a=a_{i_1}+\ldots+a_{i_l}-\mu<a_{i_j}$. In particular the invariant ν drops for all points of some maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Thus the maximal value of ν on the maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ which were blown up is bigger than the maximal value of ν on the new maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Since

the set $\frac{1}{\mu}\mathbf{Z}_{\geq 0}$ of values of ν is discrete the algorithm terminates after a finite number of steps. $\square$

Commutativity of the resolution procedure in Step 2 with étale morphisms. The reasoning is the same as in Step 1. Let $\phi : X' \to X$ be an étale morphism. In Step 2a we find a sequence $r_0 := 0 < r_1 < \ldots < r_k = r$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_0})} > \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})} > \ldots > \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$ and for $r_j \leq i < r_{j+1}$, $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_j})} = \mathrm{ord}_{\mathcal{N}(\mathcal{I}_i)}$. Moreover the resolution process for $(\mathcal{I}_i)_{r_j \leq i \leq r_{j+1}}$ is reduced to resolving the marked ideal of maximal order $O(\mathcal{I}_{r_j})$. Let $\mathrm{ord}_{\mathcal{N}(\mathcal{I}'_{p_0})} > \mathrm{ord}_{\mathcal{N}}(\mathcal{I}'_{p_1}) > \ldots > \mathrm{ord}_{\mathcal{N}(\mathcal{I}'_{p_{k'}})}$ be the corresponding sequence defined for the canonical resolution of $(\mathcal{I}', \mu) = \phi^*(\mathcal{I}, \mu)$.

Lemma 4.0.4. (1) $(\phi^*(X_i))_{0 \leq i \leq m}$ *is an extension of* $(X'_i)_{0 \leq j \leq m'}$.
(2) *For* $x' \in \mathrm{supp}(\phi^*(\mathcal{I}_i))$ *we have* $\mathrm{inv}(x') = \mathrm{inv}(\phi_i(x'))$, $\nu(x') = \nu(\phi_i(x'))$ *and* $\rho(x') = \rho(\phi_i(x'))$.

Proof. Note that all morphisms $\phi_i : \phi^*(X_i) \to X_i$ preserve the order of the nonmonomial part at a point $x \in \mathrm{supp}(\phi_i^*(\mathcal{I}_i))$. Assume that for the index l we can find an index l' such that

$$\phi^*(X_{r_l}, \mathcal{I}_{r_l}, E_{r_l}, \overline{\mu}) \simeq (X'_{p_{l'}}, \mathcal{I}'_{p_{l'}}, E'_{p_{l'}}, \overline{\mu}).$$

(1) If $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_l})} > \mathrm{ord}_{\mathcal{N}(\phi^*_{r_l}\mathcal{I}_{r_l})} = \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{p_{l'}})}$, then the centers of blow-ups of $(X_i)_{r_l \leq i < r_{l+1}}$ are contained in the loci of the points x for which $\mathrm{ord}_x(\mathcal{N}(\mathcal{I}_i)) = \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_l})}$. Therefore they are disjoint from images of $\phi^*(X_i)$. Consequently, $\phi^*(X_i)_{r_l \leq i < r_{l+1}}$ consists of isomorphisms.
(2) If $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_l})} = \mathrm{ord}_{\mathcal{N}(\phi^*_{r_l}\mathcal{I}_{r_l})} = \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{p_{l'}})}$, then $\phi^*_{r_l}(O(\mathcal{I}_{r_l})) = O(\phi^*_{r_l}(\mathcal{I}_{r_l}))$. By commutativity of the canonical resolution in Step 1 we get for any $x' \in \mathrm{supp}(O(\phi_i^*(\mathcal{I}_i))$, and any $r_l \leq i < r_{l+1}$,

$$\begin{aligned}\mathrm{inv}(x') &= (\mathrm{ord}_{\mathcal{N}(\phi^*\mathcal{I}_i)}, \overline{\mathrm{inv}}_{O(\phi^*(\mathcal{I}_i))}(x'))\\ &= (\mathrm{ord}_{\mathcal{N}(\mathcal{I}_i)}, \overline{\mathrm{inv}}_{O(\mathcal{I}_i)}(\phi_i(x'))) = \mathrm{inv}(\phi_i(x')).\end{aligned}$$

Analogously $\nu(x') = \nu(\phi_i(x'))$, $\phi_i(\rho(x')) = \rho(\phi_i(x'))$ and $\phi^*(X_i)_{r_l \leq i < r_{l+1}}$ is an extension of the part of the resolution of $(X'_i)_{p_l \leq i < p_{l+1}}$.
(3) If $\mathcal{I}_{r_k} = \mathcal{M}(\mathcal{I}_{r_k})$ and $\mathcal{I}'_{p_{k'}} = \phi^*(\mathcal{I}_{r_k}) = \mathcal{M}(\phi^*(\mathcal{I}_{r_k}))$ are monomial the resolution process is controlled by the invariant ρ. The set of values of ρ on X' can be identified via ϕ^* with a subset of the set of values of ρ on X: $\mathrm{Sub}(E') \subset \mathrm{Sub}(E)$. By definition ρ and ν commute with smooth morphisms: $\rho(\phi(x')) = \rho(x')$ and $\nu(\phi(x')) = \nu(x')$. The blow-ups on $(X_i)_{r_k \leq i \leq m}$ are performed at the centers where ρ attains its maximum. Thus the induced morphisms on $\phi^*(X_i)_{r_k \leq i \leq m}$ either are blow-ups performed at the centers where ρ attains a maximum or are isomorphisms. Consequently, $\phi^*(X_i)_{r_k \leq i \leq m}$ is an extension of $(X'_i)_{p_{k'} \leq i \leq m'}$. $\square$

4.1. **Summary of the resolution algorithm.** The resolution algorithm can be represented by the following scheme.
Step 2. Resolve $(\mathcal{I}, \mu)$.
Step 2a. Reduce $(\mathcal{I}, \mu)$ to the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$.

If $\mathcal{I} \neq \mathcal{M}(\mathcal{I})$, decrease the maximal order of the nonmonomial part $\mathcal{N}(\mathcal{I})$ by resolving the companion ideal $O(\mathcal{I}, \mu)$. For $x \in \operatorname{supp}(O(\mathcal{I}, \mu))$, set

$$\operatorname{inv}(x) = (\operatorname{ord}_x(\mathcal{N}(\mathcal{I}))/\mu, \overline{\operatorname{inv}}_{O(\mathcal{I},\mu)}).$$

Step 1. Resolve the companion ideal $(\mathcal{J}, \mu_{\mathcal{J}}) := O(\mathcal{I}, \mu)$:
Replace $\mathcal{J}$ with $\overline{\mathcal{J}} := \mathcal{C}(\mathcal{H}(\mathcal{J})) \simeq \mathcal{J}$. (*)

Step 1a. Move apart all strict transforms of E and $\operatorname{supp}(\overline{\mathcal{J}}, \mu)$.

Move apart all intersections H^s_α of s divisors in E (where s is the maximal number of divisors in E through points in $\operatorname{supp}(\mathcal{I}, \mu)$).

Step 1aa. If $\overline{\mathcal{J}}_{|H^s_\alpha} = 0$ for some $H^s_\alpha \subset \operatorname{supp}(\overline{\mathcal{J}})$, blow up H^s_α. For $x \in H^s_\alpha$ set

$$\operatorname{inv}(x) = (\operatorname{ord}_x(\mathcal{N}(\mathcal{J}))/\mu, s, \infty, 0, \ldots), \quad \nu(x) = 0, \quad \rho(x) = \emptyset.$$

Step 1ab. If $\overline{\mathcal{J}}_{|H^s_\alpha} \neq 0$ for any α, resolve all $\overline{\mathcal{J}}_{|H^s_\alpha}$. For $x \in \operatorname{supp}(\overline{\mathcal{J}}, \mu) \cap H^s_\alpha$ set

$$\operatorname{inv}(x) = (\operatorname{ord}_x(\mathcal{N}(\mathcal{J}))/\mu, s, \operatorname{inv}_{\overline{\mathcal{J}}_{|H^s_\alpha}}(x)), \quad \nu(x) = \nu_{\overline{\mathcal{J}}_{|H^s_\alpha}}(x), \quad \rho(x) = \rho_{\overline{\mathcal{J}}_{|H^s_\alpha}}(x).$$

Blow up the centers where $(\operatorname{inv}, \rho)$ is maximal.

Step 1b. If the strict transforms of E do not intersect $\operatorname{supp}(\overline{\mathcal{J}}, \mu)$, resolve $(\overline{\mathcal{J}}, \mu)$.

Step 1ba. If the set $\operatorname{codim}(1)(\operatorname{supp}(\overline{\mathcal{J}}))$ of codimension one components is nonempty, blow it up. For $x \in \operatorname{supp}(\overline{\mathcal{J}}, \mu) = \operatorname{codim}(1)(\operatorname{supp}(\overline{\mathcal{J}})$ set

$$\operatorname{inv}(x) = (\operatorname{ord}_x(\mathcal{N}(\mathcal{J}))/\mu, 0, \infty, 0), \quad \nu(x) = 0, \quad \rho(x) = \emptyset.$$

Step 1bb. Simultaneously resolve all $\overline{\mathcal{J}}_{|V(u)}$ (by induction), where $V(u)$ is a hypersurface of maximal contact. For $x \in \operatorname{supp}(\overline{\mathcal{J}}, \mu) \backslash \operatorname{codim}(1)(\operatorname{supp}(\overline{\mathcal{J}}))$ set

$$\operatorname{inv}(x) = (\operatorname{ord}_x(\mathcal{N}(\mathcal{J}))/\mu, s(x), \operatorname{inv}_{\overline{\mathcal{J}}_{|V(u)}}(x)),$$
$$\nu(x) = \nu_{\overline{\mathcal{J}}_{|V(u)}}(x), \quad \rho(x) = \rho_{\overline{\mathcal{J}}_{|V(u)}}(x).$$

Blow up the centers where $(\operatorname{inv}, \rho)$ is maximal.

Step 2b. Resolve the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$.
(Construct the invariants inv, ρ and ν directly for $\mathcal{M}(\mathcal{I})$.)

Remarks. (1) (*) The ideal $\mathcal{J}$ is replaced with $\mathcal{H}(\mathcal{J})$ to ensure that the invariant constructed in Step 1b is independent of the choice of the tangent direction u. We replace $\mathcal{H}(\mathcal{J})$ with $\mathcal{C}(\mathcal{H}(\mathcal{J}))$ to ensure the equalities $\operatorname{supp}(\mathcal{J}_{|S}) = \operatorname{supp}(\mathcal{J}) \cap S$, where $S = H^s_\alpha$ in Step 1a and $S = V(u)$ in Step 1b.

(2) If $\mu = 1$ the companion ideal is equal to $O(\mathcal{I}, 1) = (\mathcal{N}(\mathcal{I}), \mu_{\mathcal{N}(\mathcal{I})})$, so the general strategy of the resolution of $\mathcal{I}, \mu$ is to decrease the order of the nonmonomial part and then to resolve the monomial part.

(3) In particular if we desingularize Y we put $\mu = 1$ and $\mathcal{I} = \mathcal{I}_Y$ to be equal to the sheaf of the subvariety Y and we resolve the marked ideal $(X, \mathcal{I}, \emptyset, \mu)$. The nonmonomial part $\mathcal{N}(\mathcal{I}_i)$ is nothing but the weak transform $(\sigma^i)^{\mathrm{w}}(\mathcal{I})$ of $\mathcal{I}$.

4.2. **Desingularization of plane curves.** We briefly illustrate the resolution procedure for plane curves.

Let $C \subset \mathbf{A}^2$ be a plane curve defined by $F(x,y) = 0$ (for instance $x^2 + y^5 = 0$). We assign to the curve C the marked ideal $(X, \mathcal{I}_C, \emptyset, 1)$. The nonmonomial part of a controlled transform of the ideal $\mathcal{I}_C$ is the ideal of the strict transform of the curve (in general it is the weak transform of the subvariety). In particular $\mathcal{I}_C = \mathcal{N}(\mathcal{I}_C)$.

In Step 2a we form the companion ideal which is equal to $\mathcal{J} := O(\mathcal{I}_C) = (\mathcal{I}_C, \mu)$, where μ is the maximal multiplicity. Resolving $O(\mathcal{I}_C)$ will eliminate the maximal multiplicity locus of C and decrease the maximal multiplicity of the ideal of the strict transform of C. The maximal multiplicity locus of C is defined by $\mathrm{supp}(\mathcal{I}_C, \mu) = V(\mathcal{D}^{\mu-1}(\mathcal{I}_C))$, which is a finite set of points for a singular curve.

In the example $\mu = 2$ and $\mathcal{J} = O(\mathcal{I}_C) = (\mathcal{I}_C, 2)$, $T(\mathcal{J}) = (\mathcal{D}(\mathcal{I}_C), 1) = ((x, y^4), 1)$, $\mathrm{supp}(\mathcal{I}_C, 2) = V(x, y^4) = \{(0,0)\}$.

In Step 1 we resolve the companion ideal $\mathcal{J} = (\mathcal{I}_C, \mu)$. First replace $\mathcal{J}$ with $\overline{\mathcal{J}} := \mathcal{C}(\mathcal{H}(\mathcal{J}))$. In the example

$$\overline{\mathcal{J}} := \mathcal{C}(\mathcal{H}(\mathcal{J})) = \mathcal{H}(\mathcal{J}) = (x^2, xy^4, y^5).$$

Since at the beginning there are no exceptional divisors we go directly to Step 1b.

Step 1b. For any point p with multiplicity μ we find a tangent direction $u \in T(\mathcal{I}, \mu)$ at p. In particular $u = x$ for $p = (0,0)$. Then assign to p the invariant

$$\mathrm{inv}(p) = (\mu, 0, \mathrm{ord}_p(\mathcal{J}_{|V(u)})/\mu_c, 0, \infty, 0, \ldots).$$

(Recall that $\mu_c := \mathrm{lcm}(1, 2, \ldots, \mu)$.) In general for local coordinates u, v at p we have $\mathcal{J}_{|V(u)} = (v^m, \mu_c)$ and we can write the invariant as

$$\mathrm{inv}(p) = (\mu, 0, m/\mu_c, 0, \infty, 0, \ldots), \quad \nu(p) = 0, \quad \rho(p) = \emptyset.$$

In the example $\mathcal{J}_{|V(u)} = \mathcal{J}_{|V(x)} = (y^5, 2)$ and

$$\mathrm{inv}(p) = (2, 0, 5/2, 0, \infty, 0, \ldots).$$

The resolution of $\mathcal{J}_{|V(u)}$ consists of two steps: Reducing to the monomial case in Step 2a and resolving the monomial case in Step 2b. We blow up all points for which this invariant is maximal. After the blow-ups $\mathcal{J}_{|V(u)}$ is transformed as follows:

$$(v^m, \mu_c) \mapsto (y_{\mathrm{exc}}^{m-\mu_c}, \mu_c).$$

If $\mathrm{supp}(y_{\mathrm{exc}}^{m-\mu_c}, \mu_c) = \emptyset$, then $\mathcal{J}_{|V(u)}$ is resolved and the multiplicity of the corresponding points drops. Otherwise $\sigma^{\mathrm{c}}(\mathcal{J})_{|V(u)} = (y_{\mathrm{exc}}^{m-\mu_c}, \mu_c)$ is monomial for all points with the highest multiplicity. The assigned invariant is

$$\mathrm{inv}(p') = (\mu, 0, 0, 0, 0, \ldots), \quad \nu(p') = (m - \mu_c)/\mu_c, \quad \rho(p') = D_{\mathrm{exc}}.$$

In the example $\sigma^{\mathrm{c}}(y^5, 2) = (y_{\mathrm{exc}}^3, 2)$ and $\mathrm{inv}(p') = (2, 0, 0, \ldots)$ and $\nu(p') = 3/2$. The equation of the strict transform of C at the point with the highest multiplicity changes as follows:

$$x^2 + y^5 = 0 \mapsto x^2 + y_{\mathrm{exc}}^3 = 0. \tag{5}$$

After the next blow-up the invariants for all points with the highest multiplicity are

$$\mathrm{inv}(p'') = (\mu, 0, 0, 0, 0, \ldots), \quad \nu(p'') = (m - 2\mu_c)/\mu_c, \quad \rho(p'') = D'_{\mathrm{exc}}.$$

We continue blow-ups until $m - l\mu_c \le \mu_c$. At this moment $\mathrm{supp}(\sigma^{\mathrm{c}}(\mathcal{J})_{|V(u)}) = \emptyset$ and the marked ideal $\mathcal{J}_{|V(u)}$ is resolved (as in Step 1b). Resolving $\mathcal{J}_{|V(u)}$ is equivalent to resolving $\mathcal{J}$ and results in dropping the maximal multiplicity. In the example after the second blow-up $5-2\cdot 2\leq 2$ and the maximal multiplicity drops to 1:

$$x^2+y_{\mathrm{exc}}^3=0\mapsto x^2+y'_{\mathrm{exc}}=0. \tag{6}$$

After all points with the highest multiplicity are eliminated and the maximal multiplicity of points drops we reconstruct our companion ideals for the controlled transform of $\mathcal{I}_C$. The companion ideal of $\sigma^{\mathrm{c}}(\mathcal{I}_C)$ is equal to $\mathcal{J}':=(\mathcal{I}_{C'},\mu')$, where $\mathcal{I}_{C'}$ is the ideal of the strict transform and μ' is the highest multiplicity. As before $\mathrm{supp}(\mathcal{J}')$ defines the set of points with the highest multiplicity. In our example the curve C' is already smooth and $\mu'=1$. However the process of the embedded desingularization is not finished at this stage. Some exceptional divisors may pass through the points with the highest multiplicity. In the course of resolution of $\mathcal{J}'$ we first move apart all strict transforms of the exceptional divisors and the set of points with multiplicity μ'. This is handled in Step 1a by resolving $\overline{\mathcal{J}'}_{|H^s_\alpha}$. The maximum number of the exceptional divisors passing through points of $\mathrm{supp}(\mathcal{J}')$ can be $s=2$ or $s=1$. If $s=2$, then the assigned invariants are

$$\mathrm{inv}(p')=(\mu',2,\infty,0,\ldots),\quad \nu(p)=0,\quad \rho(p)=\emptyset.$$

The blow-up of the point separates the divisors. If $s=1$, then $H^s_\alpha=D_\alpha$ is a single divisor,

$$\mathrm{inv}(p')=(\mu',1,\mathrm{ord}_{p'}(\mathcal{N}(\overline{\mathcal{J}}'_{|D_\alpha}))/\mu_c,0,\ldots),\quad \nu(p')=0,\quad \rho(p')=\emptyset,$$

where $\mathcal{N}(\overline{\mathcal{J}}'_{|D_\alpha})=((v^m),\mu'_c)$. We resolve this ideal as above: The first blow-up will transform $((v^m),\mu'_c)$ to the monomial ideal $(y''_{\mathrm{exc}})^{m-\mu'_c},\mu'_c)$ with assigned invariant

$$\mathrm{inv}(p'')=(\mu,1,0,0,0,\ldots),\quad \nu(p'')=(m-\mu_c)/\mu_c,\quad \rho(p'')=\{D_\alpha,D''_{\mathrm{exc}}\}.$$

After the next blow-up the invariants at the relevant points are

$$\mathrm{inv}(p''')=(\mu,1,0,0,0,\ldots),\quad \nu(p''')=(m-2\mu'_c)/\mu'_c,\quad \rho(p''')=\{D_\alpha,D'''_{\mathrm{exc}}\}.$$

We continue until $m-l\mu'_c<\mu'_c$.

In our example the second exceptional divisor $y'_{\mathrm{exc}}=0$ passes through the point p'': $x=y'_{\mathrm{exc}}=0$:

$$\begin{gathered}\mathcal{N}(\overline{\mathcal{J}}'_{|D_\alpha})=(x^2,1)\quad\mapsto\quad(y''_{\mathrm{exc}},1)\quad\mapsto\quad(\mathcal{O}_{D'},1),\\ x^2+y'_{\mathrm{exc}}=0\quad\mapsto\quad y''_{\mathrm{exc}}+y'_{\mathrm{exc}}=0\quad\mapsto\quad 1+y'_{\mathrm{exc}}=0,\\ \mathrm{inv}(p'')=(1,1,2,0\ldots),\nu(p'')=0,\quad\mapsto\quad \mathrm{inv}(p''')=(1,1,0,\ldots),\nu(p''')=1.\end{gathered}$$

After the ideals are resolved the strict transforms of all exceptional divisors are moved away from the set of points with highest multiplicity and we arrive at Step 1b. If $\mu'=1$, we stop the resolution procedure. At this moment the invariant for all points of the strict transform of C is constant and equal to $\mathrm{inv}(p)=(1,0,\infty,0,\ldots)$, $\mu(p)=0$. The strict transform of C is now smooth and has simple normal crossings with exceptional divisors. It defines a hypersurface of maximal contact.

If $\mu'>1$, we repeat the procedure for Step 1b described above. After the ideals $\mathcal{J}_{|V(u')}$ are resolved the highest multiplicity drops. The procedure terminates when the invariant is constant along C and equal to

$$\mathrm{inv}(p)=(1,0,\infty,0,\ldots),\qquad \mu(p)=0.$$

5. Conclusion of the resolution algorithm

5.1. **Commutativity of resolving marked ideals with smooth morphisms.** Let $(X,\mathcal{I},\emptyset,\mu)$ be a marked ideal and $\phi: X'\to X$ be a smooth morphism of relative dimension n. Since the canonical resolution is defined by the invariant it suffices to show that $\mathrm{inv}(\phi(x))=\mathrm{inv}(x)$. Let $U'\subset X'$ be a neighborhood of x such that there is a factorization $\phi: U'\stackrel{\phi'}{\to} X\times\mathbf{A}^n\stackrel{\pi}{\to} X$, where ϕ' is étale and π is the natural projection. The canonical resolution $(X_i\times\mathbf{A}^n)$ of $p^*(X,\mathcal{I},E,\mu)$ is induced by the canonical resolution (X_i) of $(X,\mathcal{I},E,\mu)$ and the invariants inv, μ and ρ are preserved by π. Then for $x'\in\mathrm{supp}(X',\mathcal{I}',E',\mu)\cap U'$ we have $\mathrm{inv}(\phi(x'))=\mathrm{inv}(\pi(\phi'(x')))=\mathrm{inv}(\phi'(x'))$. Since ϕ' is étale, the resolution $\phi^*_{|U'}(X_i)={\phi'}^*(X_i\times\mathbf{A}^n)$ is an extension of the canonical resolution of $\mathcal{I}'_{|U'}$ and $\mathrm{inv}(\phi'(x))=\mathrm{inv}(x)$. Finally, $\mathrm{inv}(\phi(x))=\mathrm{inv}(x)$. Analogously $\mu(\phi(x))=\mu(x)$ and $\rho(\phi(x))=\rho(x)$.

5.2. **Commutativity of resolving marked ideals $(X,\mathcal{I},\emptyset,1)$ with embeddings of ambient varieties.** Let $(X,\mathcal{I},\emptyset,1)$ be a marked ideal and $\phi: X\hookrightarrow X'$ be a closed embedding of smooth varieties. Then ϕ defines the marked ideal $(X',\mathcal{I}',\emptyset,1)$, where $\mathcal{I}'=\phi_*(\mathcal{I})\cdot\mathcal{O}_{X'}$ (see remark after Theorem 1.0.3). We may assume that X is a subvariety of X' locally generated by parameters $u_1,\ldots,u_k$. Then $u_1,\ldots,u_k\in\mathcal{I}'(U')=T(\mathcal{I})(U')$ define tangent directions on some open $U'\subset X'$. We run Steps 2a and 1bb of our algorithm through. In Step 2a we assign $\mathrm{inv}(x)=(1,\overline{\mathrm{inv}}_{\mathcal{I}'}(x))$ (since the maximal order of $\mathcal{I}'=\mathcal{N}(\mathcal{I}')$ is equal to 1, and $\mathcal{I}'=O(\mathcal{I}'))$ and in Step 1bb (nonboundary case $s(x)=0$) we assign
(1) $\mathrm{inv}_{\mathcal{I}'}(x)=(1,0,\mathrm{inv}_{\mathcal{I}'_{|V(u_1)}}(x)),\quad \nu_{\mathcal{I}'}(x)=\nu_{\mathcal{I}'_{|V(u_1)}}(x),\quad \rho_{\mathcal{I}'}(x)=\rho_{\mathcal{I}'_{|V(u_1)}}(x)$
passing to the hypersurface $V(u_1)$. By Step 1bb resolving $(X',\mathcal{I}',\emptyset,\mu)$ is locally equivalent to resolving $(V(u),\mathcal{I}'_{|V(u)},\emptyset,\mu)$ with the relation between invariants defined by (1). By repeating the procedure k times and restricting to the tangent directions $u_1,\ldots,u_k$ of the marked ideal $\mathcal{I}$ on X we obtain:
(2) $\mathrm{inv}_{\mathcal{I}'}(x)=(1,0,1,0,\ldots,1,0,\mathrm{inv}_{\mathcal{I}}(x)),\quad \rho_{\mathcal{I}'}(x)=\rho_{\mathcal{I}}(x),\quad \nu_{\mathcal{I}'}(x)=\nu_{\mathcal{I}}(x)$.
Resolving $(X',\mathcal{I}',\emptyset,\mu)$ is equivalent to resolving $(X,\mathcal{I},\emptyset,\mu)$ with the relation between invariants defined by (2).

5.3. **Commutativity of resolving marked ideals with isomorphisms not preserving the ground field.**

Lemma 5.3.1. *Let X,X' be varieties over K and $\mathcal{I}$ be a sheaf of ideals on X. Let $\phi: X'\to X$ be an isomorphism over $\mathbf{Q}$ for which there exists an automorphism ϕ' of $\mathrm{Spec}\,K$ and a commutative diagram*

$$\begin{array}{ccc} X' & \stackrel{\phi}{\to} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,K & \stackrel{\phi'}{\to} & \mathrm{Spec}\,K. \end{array}$$

Then $\phi^*(\mathcal{D}^i(\mathcal{I}))=\mathcal{D}^i(\phi^*(\mathcal{I}))$ for any i.

Proof. It suffices to consider the case $i=1$. The sheaf $\mathcal{D}^i(\mathcal{I})$ is locally generated by functions $f\in\mathcal{I}$ regular on some open subsets U and their derivatives $D(f)$. Then $\phi^*(\mathcal{D}^i(\mathcal{I}))$ is locally generated by $\phi^*(f)$ and $\phi^*Df=\phi^*D(\phi^*)^{-1}\phi^*f$. But for any derivation $D\in\mathrm{Der}_K(\mathcal{O}(U))$, $D':=\phi^*D(\phi^*)^{-1}\in\mathrm{Der}_K(\mathcal{O}(\phi^{-1}(U)))$ defines a K-derivation of $\mathcal{O}(\phi^{-1}(U))$. $\square$

Proposition 5.3.2. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal. Let $\phi : X' \to X$ be as above. For any canonical resolution (X_i) of X the induced resolution $(X_i') := (X_i \times_X X')$ is canonical. Moreover the isomorphism ϕ lifts to isomorphisms $\phi_i : X_i' \to X_i$ such that*

$$\operatorname{inv}(\phi_i(x)) = \operatorname{inv}(x), \quad \nu(\phi_i(x)) = \nu(x), \quad \rho(\phi_i(x)) = \phi_i(\rho(x)).$$

Proof. Induction on dimension of X. First assume that $(X,\mathcal{I},E,\mu) = (X,\mathcal{J},E,\mu)$ is of maximal order as in Step 1. Then, by the lemma, $\phi^*(\mathcal{C}(\mathcal{H}(\mathcal{J}))) = \mathcal{C}(\mathcal{H}(\phi^*(\mathcal{J})))$. The resolution algorithm in Step 1a is reduced to the resolution of the restrictions of marked ideals $\mathcal{I}$ to intersections of the exceptional divisors H^s_α. This procedure commutes with the isomorphism ϕ. Moreover

$$\begin{aligned}\overline{\operatorname{inv}}(\phi(x)) = (s(\phi(x)), \operatorname{inv}_{|\phi(H^s_\alpha)}(\phi(x))) = (s(x), \operatorname{inv}_{|H^s_\alpha}(x)) = \overline{\operatorname{inv}}(x),\\ \nu(\phi(x)) = \nu_{|\phi(H^s_\alpha)}(\phi(x)) = \nu_{|H^s_\alpha}(x) = \nu(x), \quad \rho(\phi(x)) = \phi(\rho(x)),\end{aligned}$$

by the inductive assumption. In Step 1b we reduce the resolution of the marked ideal to its restriction to a hypersurface of maximal contact defined by $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ on an open subset U. This procedure commutes with ϕ. The corresponding marked ideal $(\phi^*(\mathcal{J}),\mu)$ is restricted to the hypersurface of maximal contact on $\phi^{-1}(U)$ defined by $\phi^*(u) \in \phi^*(\mathcal{D}^{\mu-1}(\mathcal{J}))$. The invariants defined in this step commute with ϕ by the inductive assumption. In Step 2a we decompose an arbitrary marked ideal into the monomial and nonmonomial part. Since an isomorphism $\phi : X' \to X$ maps divisors in E' to divisors in E it preserves this decomposition. Consequently, it preserves companion ideals and the invariants $\operatorname{inv}, \rho, \mu$ defined in Step 2a. Also the invariants defined in Step 2b are preserved by ϕ. Therefore ϕ commutes with canonical resolutions.

5.4. **Commutativity with field extensions.** Let $K \subset L$ be an extension of algebraically closed fields. Consider a marked ideal $(X,\mathcal{I},E,\mu)$ over K. Taking the fiber product with $\operatorname{Spec}(L)$ over $\operatorname{Spec} K$ defines a marked ideal $(X^L,\mathcal{I}^L,E^L,\mu)$. The canonical resolution (X_i) of $(X,\mathcal{I},E,\mu)$ defines a resolution $(X_i \times_{\operatorname{Spec} K} \operatorname{Spec} L)$ of $(X^L,\mathcal{I}^L,E^L,\mu)$. We have to show that this resolution is isomorphic to the canonical one (X_i^L).

Lemma 5.4.1. *Let $\mathcal{I}$ be a sheaf of ideals on X. Denote by $\mathcal{I}^L$ the induced sheaf of ideals on X^L. Then we have the equalities $(\mathcal{D}^i(\mathcal{I}^L)) = \mathcal{D}^i((\mathcal{I}))^L$ for any i.*

Proof. It suffices to consider the case $i = 1$. The sheaf $\mathcal{D}(\mathcal{I})$ is locally generated by functions $f \in \mathcal{I}$ regular on some open subsets U and their derivatives $\frac{\partial}{\partial x_i}(f)$. These functions generate $(\mathcal{D}(\mathcal{I}))^L$. But $\frac{\partial}{\partial x_i}(f)$ extend to L-derivatives and $(\mathcal{D}(\mathcal{I}))^L$ is generated locally by the same functions. □

As a simple corollary we obtain the following lemma.

Lemma 5.4.2. $\{x \in X \mid \operatorname{ord}_x(\mathcal{I}) \geq \mu\} \times_{\operatorname{Spec} K} \operatorname{Spec} L = \{y \in X^L \mid \operatorname{ord}_y(\mathcal{I}^L) \geq \mu\}$. *In particular* $\operatorname{supp}(X,\mathcal{I},E,\mu) \times_{\operatorname{Spec} K} \operatorname{Spec} L = \operatorname{supp}(X^L,\mathcal{I}^L,E^L,\mu)$. □

Proposition 5.4.3. *Let (X_i) be the canonical resolution of a marked ideal $(X,\mathcal{I},E,\mu)$ over K. Then $X_i \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is the canonical resolution of $(X^L,\mathcal{I}^L,E^L,\mu)$. The sets of values of the corresponding functions* inv^L, ν^L *and* ρ^L *are the same as the sets of values of* inv, ν *and* ρ *respectively. (We identify*

Sub(E) *with* Sub(E^L).*) Moreover for any* $a \in \overline{\mathbf{Q}}^{\infty}_{\geq 0}$, $b \in \mathbf{Q}_{\geq 0}$, $c \in \mathrm{Sub}(E)$ *we have the following relations between algebraic sets:*

$$\begin{aligned}\mathrm{inv}^{-1}(\geq a) \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L &= (\mathrm{inv}^L)^{-1}(\geq a),\\ \nu^{-1}(\geq b) \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L &= (\nu^L)^{-1}(\geq b),\\ \rho^{-1}(\geq b) \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L &= (\rho^L)^{-1}(\geq b).\end{aligned}$$

Proof. Induction on the dimension of X. Assume that $(X, \mathcal{I}, E, \mu) = (X, \mathcal{J}, E, \mu)$ is of maximal order as in Step 1. Then, by the lemma, $\mathcal{C}(\mathcal{H}(\mathcal{J}^L))) = \mathcal{C}(\mathcal{H}(\phi^*(\mathcal{J}))^L$. The resolution algorithm in Step 1a is reduced to the resolution of the restrictions of marked ideals $\mathcal{I}$ to intersections of the exceptional divisors H^s_α. This procedure commutes with taking the fiber product with $\mathrm{Spec}\, L$. The commutativity of the algorithm in Step 1a and the assertion of the Proposition follow by the inductive assumption. In Step 1b we reduce the resolution of the marked ideal to its restriction to a hypersurface of maximal contact defined by $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ on an open subset U. This procedure commutes with taking the fiber product with $\mathrm{Spec}\, L$. The corresponding marked ideal $(\mathcal{J}^L, \mu)$ is restricted to the hypersurface of maximal contact on (U^L) defined by the same function $u \in \phi^*(\mathcal{D}^{\mu-1}(\mathcal{J}))^L$.

In Step 2a we decompose an arbitrary marked ideal into the monomial and nonmonomial part. Taking the fiber product with $\mathrm{Spec}\, L$ preserves this decomposition. Consequently, it preserves companion ideals and the invariants inv, ρ, μ defined in Step 2a are related by the assertion of the theorem. Analogously in Step 2b the algorithm commutes with taking the fiber product with $\mathrm{Spec}\, L$. $\square$

5.5. **Principalization.** Resolving the marked ideal $(X, \mathcal{I}, \emptyset, 1)$ determines a principalization commuting with smooth morphisms and embeddings of the ambient varieties.

The principalization is often reached at an earlier stage upon transformation to the monomial case (Step 2b). This moment is detected by the invariant inv, which becomes equal to $\mathrm{inv}(x) = (0, \ldots, 0, \ldots)$. (However the latter procedure does not commute with embeddings of ambient varieties.)

5.6. **Weak embedded desingularization.** Let Y be a closed subvariety of the variety X. Consider the marked ideal $(X, \mathcal{I}_Y, \emptyset, 1)$. Its support $\mathrm{supp}(\mathcal{I}_Y, 1)$ is equal to Y. In the resolution process of $(X, \mathcal{I}_Y, \emptyset, 1)$, the strict transform of Y is blown up. Otherwise the generic points of Y would be transformed isomorphically, which contradicts the resolution of $(X, \mathcal{I}_Y, \emptyset, 1)$. At the moment where the strict transform is blown up the invariant along it is the same for all its points and equal to

$$\mathrm{inv}(x) = (1, 0; 1, 0; \ldots; 1, 0; \infty; 0, 0, 0, \ldots),$$

where $(1, 0)$ is repeated n times, where n is the codimension of Y. This value of the invariant can be computed for the generic smooth point of Y. We apply Step 2a ($\mathrm{ord}_x(\mathcal{I}) = 1, \mathcal{I} = O(\mathcal{I})$) and Step 1b (nonboundary case $s(x) = 0$) passing to a hypersurface n times. Each time after running through 2a and 1b we adjoin a couple (1,0) to the constructed invariant. After the n-th time we arrive at Step 1ba where the algorithm terminates and ∞ followed by zeros is added at the end of the invariant.

5.7. **Resolving marked ideals over a non-algebraically closed field.** Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal defined over a field K. Let $\overline{K}$ be the algebraic closure of K. Then the base change $K \mapsto \overline{K}$ defines the $G = \mathrm{Gal}(\overline{K}/K)$-invariant marked ideal $(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu)$ over $\overline{K}$. The canonical resolution $(\overline{X}_i)$ of $(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu)$ is G-equivariant and defines the canonical resolution (X_i) of $(X, \mathcal{I}, E, \mu)$ over K. Moreover we have $\overline{X}_i = X_i \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$.

Commutativity with smooth morphisms over a non-algebraically closed field. If $\phi : X' \to X$ is a smooth morphism, then $\overline{\phi} : \overline{X} \to \overline{X}'$ is a G-equivariant smooth morphism. If $(\overline{X}_i)$ is a G-equivariant resolution of $(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu)$, then $\overline{\phi}^*(\overline{X}_i)$ is a G-equivariant extension of the G-equivariant canonical resolution of $(\overline{X}', \overline{\mathcal{I}}', \overline{E}', \mu)$. Thus $\phi^*(X_i)$ is an extension of the canonical resolution of $(X', \mathcal{I}', E', \mu)$.

Commutativity with field extensions. Let $K \subset L$ denote a (separable) field extension. Let $\overline{K}$, $\overline{L}$ denote the algebraic closures of K and L. Consider a marked ideal $(X, \mathcal{I}, E, \mu)$ over K and its canonical resolution (X_i). Then taking the fiber product with $\operatorname{Spec}(\overline{K})$ over $\operatorname{Spec} K$ defines the canonical resolution $(X_i^{\overline{K}})$ of the marked ideal $(X^{\overline{K}}, \mathcal{I}^{\overline{K}}, E^{\overline{K}}, \mu)$ over $\overline{K}$. By 5.4 taking the fiber product with $\operatorname{Spec}(\overline{L})$ over $\operatorname{Spec} K$ defines the canonical resolution $(X_i^{\overline{L}})$ of a marked ideal $(X^{\overline{L}}, \mathcal{I}^{\overline{L}}, E^{\overline{L}}, \mu)$ over $\overline{L}$. The resolution $(X^{\overline{L}})_i$ is $\mathrm{Gal}(\overline{L}/L)$-equivariant and defines the canonical resolution $(X_i^L) = (X_i \times_{\operatorname{Spec} K} \operatorname{Spec} L)$ of $(X^L, \mathcal{I}^L, E^L, \mu)$.

Commutativity with embeddings of ambient varieties.

Follows from 5.2.

5.8. **Bravo-Villamayor strengthening of the weak embedded desingularization.**

Theorem 5.8.1 (Bravo-Villamayor [13], [11])**.** *Let Y be a reduced closed subscheme of a smooth variety X and $Y = \bigcup Y_i$ be its decomposition into the union of irreducible components. There is a canonical resolution of a subscheme $Y \subset X$, that is, a sequence of blow-ups $(X_i)_{0 \le i \le r}$ subject to conditions (a)-(d) from Theorem* 1.0.2 *such that the strict transforms $\widetilde{Y}_i$ of Y_i are smooth and disjoint. Moreover the full transform of Y is of the form*

$$(\widetilde{\sigma})^*(\mathcal{I}_Y) = \mathcal{M}((\widetilde{\sigma})^*(\mathcal{I}_Y)) \cdot \mathcal{I}_{\widetilde{Y}},$$

where $\widetilde{Y} := \bigcup \widetilde{Y}_i \subset \widetilde{X}$ is a disjoint union of the strict transforms $\widetilde{Y}_i$ of Y_i, $\mathcal{I}_{\widetilde{Y}}$ is the sheaf of ideals of $\widetilde{Y}$ and $\mathcal{M}((\widetilde{\sigma})^*(\mathcal{I}_Y))$ is the monomial part of $(\widetilde{\sigma})^*(\mathcal{I}_Y)$.

Proof. Consider the canonical resolution procedure for the marked ideal $(X, \mathcal{I}_Y, \emptyset, 1)$ (and in general for $(X, \mathcal{I}, E, \mu)$) described in the proof of Theorem 4.0.1. We shall modify the construction of the invariants in the canonical resolution. In Step 1 we define $\overline{\mathrm{inv}}'$, ρ', ν' in the same way as before. In Step 2 we modify the definition of the companion ideal to be

$$O'(\mathcal{I}, \mu) = \begin{cases} (\mathcal{M}(\mathcal{I}), 1) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})} \le 1 \text{ and } \mu = 1 \text{ and } \mathcal{M}(\mathcal{I}) \ne \mathcal{O}_X, \\ O(\mathcal{I}, \mu) & \text{otherwise.} \end{cases}$$

We define invariants as follows. If $\mathrm{ord}_{\mathcal{N}(\mathcal{I})} \le 1$ and $\mu = 1$ and $\mathcal{M}(\mathcal{I}) \ne \mathcal{O}_X$ we set

$$\mathrm{inv}'(x) = (3/2, 0, 0, \ldots), \quad \nu'(x) = \nu_{(\mathcal{M}(\mathcal{I}),1)}(x), \quad \rho'(x) = \rho_{(\mathcal{M}(\mathcal{I}),1)}(x)$$

defined as in Step 2b. Otherwise we put as before

$$\operatorname{inv}(x) = \left(\frac{\operatorname{ord}_{\mathcal{N}(\mathcal{I})}}{\mu}, \overline{\operatorname{inv}}_{O(\mathcal{I},\mu)}(x)\right), \quad \nu(x) = \nu_{O(\mathcal{I},\mu)}(x), \quad \rho(x) = \rho_{O(\mathcal{I},\mu)}(x).$$

Note that the reasoning is almost the same as before. The difference occurs for resolving marked ideals $(\mathcal{I}, 1)$ in Step 2 when we arrive at the moment when $\max\{\operatorname{ord}_x(\mathcal{N}(\mathcal{I}))\} = 1$. Let $\nu^1(x)$ denote the first coordinate of the invariant inv. Note that $\nu^1(x) = 3/2$ for all points of $\operatorname{supp}(\mathcal{I}, 1)$ for which $\mathcal{I}$ is not purely nonmonomial. First resolve its monomial part as in Step 2b (for all points with $\nu^1(x) = 3/2$). The blow-ups are performed at exceptional divisors for which $\rho(x)$ is maximal. We arrive at the purely nonmonomial case ($\nu^1(x) = 1$) and continue the resolution as before.

Let us order the codimensions of the components Y_i in an increasing sequence $r_1 := \operatorname{codim} Y_1 \leq \ldots \leq r_k := \operatorname{codim} Y_k$.

We shall run Steps 1-2 of this procedure with the above modifications until the strict transform of one of the components Y_i is the center of the next blow-up. At this point the invariants are constant along this strict transform and are equal to

$$\operatorname{inv}(x) = (1,0;1,0;\ldots;1,0;\infty;0,0,0,\ldots), \quad \nu'(x) = 0, \quad \rho'(x) = \emptyset$$

where $(1, 0)$ is repeated r_1 times. (These are the values of the invariants for a generic smooth point of $\widetilde{Y}_1$.)

Claim. *Let $(\mathcal{I}, 1)$ be a marked ideal on X such that Y_1 is an irreducible component of $\operatorname{supp}(\mathcal{I})$. Moreover assume that $\mathcal{I} = \mathcal{I}_{Y_1}$ in a neighborhood of a generic point of Y_1. At the moment (of the modified resolution process) for which*

$$\max(\operatorname{inv}(x)) = (1,0;1,0;\ldots;1,0;\infty;0,0,0,\ldots)$$

the controlled transform of $(\mathcal{I}, 1)$ is equal to $\mathcal{I}_{\widetilde{Y}_1}$ in the neighborhood of the strict transform $\widetilde{Y}_1$ of Y_1.

We prove this claim by induction on codimension. Note that when we run Step 2a of the algorithm at some point we arrive at a marked ideal for which $\max(\nu^1(x)) = 1$. At this stage $\mathcal{I} = \mathcal{N}(\mathcal{I})$ is purely nonmonomial and $O'(\mathcal{I}) = (\mathcal{I}, 1)$. Note that starting from this point the controlled transform of $\mathcal{I}$ remains nonmonomial for all points with $\nu^1(x) = 1$. Then we go to Step 1 and construct $\mathcal{C}(\mathcal{H}(\mathcal{I}, 1)) = (\mathcal{I}, 1)$. In Step 1a we run the algorithm arriving at the nonboundary case in Step 1b. At this point we restrict $\mathcal{I}$ to a smooth hypersurface of maximal contact $V(u)$. If we are in Step 1ba this hypersurface is the strict transform of Y_1. Moreover the order of the controlled transform $\sigma^{\mathrm{c}}(\mathcal{I})$ of $\mathcal{I}$ is 1 along the strict transform of Y_1 and thus $\sigma^{\mathrm{c}}(\mathcal{I})$ is the ideal of this strict transform (in the neighborhood of the strict transform).

In Step 1bb we apply the (modified) canonical resolution to the restriction $\mathcal{I}_{|V(u)}$. This restriction $\mathcal{I}_{|V(u)}$ satisfies the assumption of the claim for $Y_1 \subset V(u)$ (we skip indices here). By the inductive assumption the controlled transform $\sigma^{\mathrm{c}}(\mathcal{I})_{|V(u)}$ of $(\mathcal{I}_{|V(u)}, 1)$ is locally equal to the ideal of the strict transform $\widetilde{Y}_1 \subset \widetilde{V(u)}$ of Y_1. Since $u \in \sigma^{\mathrm{c}}(\mathcal{I})$ it follows that $\sigma^{\mathrm{c}}(\mathcal{I}) = \mathcal{I}_{\widetilde{Y}_1}$ (in the neighborhood of $\widetilde{Y}_1$). The claim is proven.

All the strict transforms of codimension r_1 are isolated. We continue the (modified) canonical resolution procedure ignoring these isolated components. We arrive at the moment where some codimension $r_2 > r_1$ component is the center of the blow-up and the invariant inv is equal to

$$\mathrm{inv}(x) = (1,0;1,0;\ldots;1,0;\infty;0,0,0,\ldots), \quad \nu'(x) = 0, \quad \rho'(x) = \emptyset,$$

where $(1,0)$ is repeated r_2 times. Again by the claim the controlled transforms of all codimension $r_2 > r_1$ components coincide with the strict transforms and are isolated. Starting from this moment those components are ignored in the resolution process. We continue for all r_i. At the end we principalize all components if there are any which do not intersect the strict transforms of Y_i.

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