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ON A SHARP LOWER BOUND ON THE BLOW-UP RATE FOR THE L^2 CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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1. Introduction

1.1. **Setting of the problem.** We consider in this paper the L^2 critical nonlinear Schrödinger equation

$$(1.1) \qquad (NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases}$$

with $u_0 \in H^1 = H^1(\mathbf{R}^N)$ in dimension $N \geq 1$. From a result of Ginibre and Velo [7], (1.1) is locally well-posed in H^1 and thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$ and either $T = +\infty$ (we say the solution is global) or $T < +\infty$ and then $\limsup_{t \uparrow T} |\nabla u(t)|_{L^2} = +\infty$ (we say the solution blows up in finite time).

(1.1) admits the following conservation laws in the energy space H^1:

L^2-norm : $\int |u(t,x)|^2 = \int |u_0(x)|^2$;
Energy : $E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{2+\frac{4}{N}}\int |u(t,x)|^{2+\frac{4}{N}} = E(u_0)$;
Momentum : $Im\left(\int \nabla u \overline{u}(t,x)\right) = Im\left(\int \nabla u_0 \overline{u_0}(x)\right)$.

For notational purposes, we shall introduce the following invariant:

$$(1.2) \qquad E^G(u) = E(u) - \frac{1}{2}\left(\frac{|Im(\int \nabla u \overline{u})|}{|u|_{L^2}}\right)^2.$$

It is classical from the conservation of energy and the L^2-norm that the power of the nonlinearity in (1.1) is the smallest power for which blowup may occur, and the existence of blow-up solutions is known from the virial identity: let an initial condition $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$; then the corresponding solution $u(t)$ to (1.1) satisfies

$$(1.3) \qquad u(t) \in \Sigma \quad \text{and} \quad \frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 16E(u_0).$$

Thus if $u_0 \in \Sigma$ with $E(u_0) < 0$, the positive quantity $\int |x|^2|u(t,x)|^2$ cannot exist for whole times and u blows up in finite time.

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Equation (1.1) admits a number of symmetries in the energy space H^1: if $u(t,x)$ is a solution to (1.1), then $\forall(\lambda_0,t_0,x_0,\beta_0,\gamma_0)\in\mathbf{R}^+_*\times\mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{R}$, so is

$$v(t,x)=\lambda_0^{\frac{N}{2}}u(t+t_0,\lambda_0x+x_0-\beta_0t)e^{i\frac{\beta_0}{2}\cdot(x-\frac{\beta_0}{2}t)}e^{i\gamma_0}.$$

The following pseudo-conformal symmetry is not in the energy space H^1 but in the virial space Σ: if $u(t,x)$ solves (1.1), then so does

$$v(t,x)=\frac{1}{|t|^{\frac{N}{2}}}\overline{u}(\frac{1}{t},\frac{x}{t})e^{i\frac{|x|^2}{4t}}.$$

Special solutions play a fundamental role for the description of the dynamics of (1.1). They are the so-called solitary waves of the form $u(t,x)=e^{i\omega t}W_\omega(x)$, $\omega>0$, where W_ω solves

$$\Delta W_\omega+W_\omega|W_\omega|^{\frac{4}{N}}=\omega W_\omega. \tag{1.4}$$

Equation (1.4) is a standard nonlinear elliptic equation, and from [2] and [8], there is a unique positive solution up to translation $Q_\omega(x)$. Q_ω is, in addition, radially symmetric. Letting $Q=Q_{\omega=1}$, then $Q_\omega(x)=\omega^{\frac{N}{4}}Q(\omega^{\frac{1}{2}}x)$ from the scaling property. Therefore, one computes

$$|Q_\omega|_{L^2}=|Q|_{L^2}.$$

Moreover, multiplying (1.4) by $\frac{N}{2}Q_\omega+x\cdot\nabla Q_\omega$ and integrating by parts yields the so-called Pohozaev identity:

$$E(Q_\omega)=\omega E(Q)=0.$$

In particular, none of the three conservation laws in H^1 sees the variation of size of the Q_ω stationary solutions.

For $|u_0|_{L^2}<|Q|_{L^2}$, the solution is global in H^1 from the conservation of energy, the L^2-norm and the Gagliardo-Nirenberg inequality as exhibited by Weinstein in [23]:

$$\forall u\in H^1,\quad E(u)\geq\frac{1}{2}\left(\int|\nabla u|^2\right)\left(1-\left(\frac{\int|u|^2}{\int Q^2}\right)^{\frac{2}{N}}\right). \tag{1.5}$$

In addition, this condition is sharp: for $|u_0|_{L^2}\geq|Q|_{L^2}$, blowup may occur. Indeed, the pseudo-conformal transformation applied to the stationary solution $e^{it}Q(x)$ yields an explicit solution

$$S(t,x)=\frac{1}{|t|^{\frac{N}{2}}}Q(\frac{x}{t})e^{-i\frac{|x|^2}{4t}+\frac{i}{t}} \tag{1.6}$$

which blows up at $T=0$ with $|S(t)|_{L^2}=|Q|_{L^2}$. Note that the blow-up speed for $S(t)$ is

$$|\nabla S(t)|_{L^2}\sim\frac{1}{|t|}.$$

Moreover, from [13], $S(t)$ is the unique minimal mass finite time blow-up solution up to the symmetries.

Most results concerning the blow-up dynamics of (1.1) now concern the perturbative situation when

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \ \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^*\},$$

for some small constant $\alpha^* > 0$. At least two different blow-up behaviors are known to possibly occur:

- There exists in dimension $N = 1, 2$ a family of solutions of type $S(t)$ by a result of Bourgain and Wang, [4], that is, solutions with $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow-up time.
- On the other hand, it has been suspected since the 70s that the blow-up speed of generic initial data is different from the $S(t)$ speed, which indeed is never observed numerically. Let us say that quite a number of both formal and numerical works have been devoted to the derivation of the exact blow-up law for (1.1) and that different laws have been proposed, in particular by Zakharov. Then in the 80s, a combination of refined numerical simulations and formal asymptotic expansions led Landman, Papanicolaou, Sulem and Sulem [9], to propose in dimension $N = 2$ the log-log law

$$|\nabla u(t)|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$$

  as the generic blow-up speed. A numerical confirmation of the log-log law was recently proposed by Akrivis, Dougalis, Karakashian and McKinney in [1]. Further formal arguments to explain the log-log correction to self-similar blowup may also be found in Dyachenko, Newell, Pushkarev and Zakharov [5] and Pelinovsky [18]. We refer to the monograph [21] and references therein for a complete introduction to the history of the problem. Then in 2001, Perelman in [19] established rigorously the existence in dimension $N = 1$ of an even log-log solution and its stability in some space $E \subset H^1$.

The situation has been clarified in the sequence of papers [14], [15], [16]. More precisely, let us consider the following property:

Spectral Property. *Let $N \geq 1$. Consider the two real Schrödinger operators*

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N} + 1\right) Q^{\frac{4}{N}-1} y \cdot \nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N} Q^{\frac{4}{N}-1} y \cdot \nabla Q, \tag{1.7}$$

and the real-valued quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$:

$$H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1, \varepsilon_1) + (\mathcal{L}_2\varepsilon_2, \varepsilon_2). \tag{1.8}$$

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that $\forall \varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0$, then

$$H(\varepsilon,\varepsilon) \geq \tilde{\delta}_1 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)$$

where $Q_1 = \frac{N}{2} Q + y \cdot \nabla Q$ and $Q_2 = \frac{N}{2} Q_1 + y \cdot \nabla Q_1$.

This property has been proved in [14] for dimension $N = 1$ and will always be implicitly assumed in higher dimension $N \geq 2$. Recently, a *numerical* proof is derived in [6]. Let us say that this Spectral Property contains two parts: first count exactly the number of negative eigenvalues of each quadratic form, second prove that the explicit set of orthogonality conditions chosen is enough to ensure the coercivity of the quadratic forms. As it is stated, the first part always holds numerically until $N = 10$, and the second part holds for $N = 2, 3, 4$, which in particular covers the physically relevant case $N = 2$, but fails for $N = 5, 6$. In conclusion, the Spectral Property holds true at least for

$$N = 1, 2, 3, 4,$$

and for $N \geq 5$, it could be true that another set of orthogonality conditions or a relaxed form of the Spectral Property holds true and is enough to have the whole proof go through. An interesting case for further investigation is when $N \to +\infty$; again in this case, the asymptotic form of the ground state is explicit and all computations can be done. One can hope to derive a similar spectral property which will lead to the same results.

We now have:

Theorem 1 ([14], [15], [16], [20]). *Let $N = 1$ or $N \geq 2$, assuming that the Spectral Property holds true. There exist universal constants $\alpha^* > 0$, $C_1^* > 0$, $C_2^* > 0$ such that the following holds true. Given $u_0 \in \mathcal{B}_{\alpha^*}$, let $u(t)$ be the corresponding solution to* (1.1) *with $[0, T)$ its maximum time interval of existence on the right in H^1. Then:*

- *If $E_0^G < 0$, then $u(t)$ blows up on the right and on the left in time, and the following upper bound on the blow-up rate holds for t close enough to T:*

$$|\nabla u(t)|_{L^2} \leq C_1^* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \tag{1.9}$$

- *If $E_0^G = 0$ and u_0 is not a soliton up to scaling, phase and translation invariances, then $u(t)$ blows up on the right or on the left in time with upper bound* (1.9)*. Moreover, if $u_0 \in \Sigma$, then finite time blowup occurs on both sides in time with upper bound* (1.9)*.*
- *The set $\mathcal{O}$ of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that $u(t)$ blows up on the right in finite time with upper bound* (1.9) *is open in H^1.*
- *If $u(t)$ blows up in finite time and* (1.9) *does not hold, then the following sharp lower bound on the blow-up rate holds:*

$$|\nabla u(t)|_{L^2} \geq \frac{C_2^*}{(T-t)\sqrt{E_0^G}}. \tag{1.10}$$

We now address the question of lower bounds on the blow-up rate. On the one hand, observe that outside the open set where the log-log upper bound (1.9) holds, we have the lower bound (1.10), which is conjectured to be sharp as it is the rate of blowup for the explicit blow-up solution $S(t)$.

On the other hand, in the log-log regime, a known lower bound holds from the scaling argument:

$$|\nabla u(t)|_{L^2} \geq \frac{C^*}{\sqrt{T-t}}. \tag{1.11}$$

Even though self-similar blowup is known to generically happen in other situations, it is conjectured from the criticality of the problem never to hold true in H^1 in our setting. We proved this result in [16] for data $u_0 \in \mathcal{B}_{\alpha^*}$:

$$\sqrt{T-t}|\nabla u(t)|_{L^2} \to +\infty \quad \text{as} \quad t \to T.$$

This result is an L^2 property in the sense that there exist explicit self-similar solutions in $\dot{H}^1$, but they never belong to L^2. The proof in [16] relies on classification results of non-L^2 dispersive blow-up solutions. In a first step, we prove that such an object satisfies additional decay properties in Σ. In a second step, we are able to characterize these solutions thanks to L^2 dispersion under this additional control in Σ. Moreover, using the pseudo-conformal transformation as an explicit symmetry of (1.1), this classification result is equivalent to proving the following characterization of solitons: if $u_0 \in \Sigma$ with $E_G^0 = 0$ and u_0 is not a soliton up to the H^1 symmetries, then $u(t)$ blows up for $t > 0$ and $t < 0$ in finite time with upper bound (1.10).

1.2. **Statement of the results.** This paper is devoted to the proof of the sharp lower bound on the blow-up rate:

Theorem 2 (log-log lower bound)**.** *Let $N = 1$ or $N \geq 2$, assuming that the Spectral Property holds true. There exist universal constants $\alpha^* > 0$, $C_3^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then one has the following lower bound on the blow-up rate for t close to T:*

$$|\nabla u(t)|_{L^2} \geq C_3^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{1.12}$$

Comments on the result.

1. Pointwise estimate: At this stage of the theory, it is noteworthy that the log-log upper bound (1.9) is needed for the proof of the log-log lower bound (1.12). In addition, we obtain a slightly stronger result in the proof as we exhibit a *pointwise in time differential inequality* for the size of the solution; see section 5.

2. Exact blow-up speed for the log-log: From the proof, we in fact have an exact equivalent of the blow-up speed in the log-log regime, i.e., for the solutions which satisfy the upper bound (1.9):

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac{1}{2}} \to \frac{1}{\sqrt{2\pi}} \quad \text{as} \quad t \to T. \tag{1.13}$$

This theorem will be obtained as a refinement of techniques developed in [16], Part B, for the proof of nonexistence of self-similar blow-up solutions. This amounts to understanding precisely the mass exchanges between the blow-up part of the solution and the linear radiative dynamic at infinity. This dispersive mechanism in L^2 has been partially exhibited in [16] in a regime where additional decay assumptions on the solution in Σ hold in the vicinity of the blow-up time. For generic initial data, such estimates do not hold: recall that the result in [16] is a classification result of nondispersive solutions in L^2. In this paper, exhibiting a Lyapounov function in the log-log regime involving the L^2-norm, we are able to extend the analysis in [16] to H^1. Nevertheless, we expect that existence of such a Lyapounov property is very specific to (1.1), whereas arguments given in [16] should be more robust.

In addition, we are able to extend the dynamical characterization of solitons in the zero energy manifold to the full energy space H^1:

Theorem 3 (Blowup for H^1 zero energy solutions)**.** *Let $N=1$ or $N\geq 2$, assuming that the Spectral Property holds true. There exists a universal constant $\alpha^*>0$ such that the following holds true. Let $u_0\in\mathcal{B}_{\alpha^*}$ with $E_0^G=0$ and assume that u_0 is not a soliton up to fixed scaling, phase, translation and Galilean invariances. Then u blows up both for $t<0$ and $t>0$, and (1.13) holds.*

Let us observe again that blowup on the right or on the left in time is known from [16]. The main problem that one is confronted with for zero energy solutions is the possibility of a nonlinear vanishing, $|\nabla u(t)|_{L^2}\to 0$ as $t\to+\infty$. This behavior has been ruled out for data $u_0\in\Sigma$ in [16] using the pseudo-conformal transformation as an explicit symmetry of (1.1). Nevertheless, the zero energy set is an important set in the energy space H^1 as zero energy solutions are natural asymptotic profiles for blow-up solutions. In this setting, no information is usually obtained outside the energy space, and thus Theorem 3 is more than a technical improvement of the result obtained in [16], Part B. Moreover, the nonlinear vanishing dynamic is ruled out from the conservation of the L^2-norm and the zero energy assumption, showing in fact that the Hamiltonian information is in this case enough to prove a strong rigidity of the blow-up dynamic.

These results on (1.1) in the sequence of papers [14], [15], [16], [20] and the present paper may be summarized as follows in a general statement.

Theorem 4 (Dynamics of (1.1))**.** *Let $N=1$ or $N\geq 2$, assuming that the Spectral Property holds true. There exist universal constants $\alpha^*>0$, $C^*>0$ such that the following holds true. For $u_0\in H^1$, let $u(t)$ be the corresponding solution to (1.1) with $[0,T)$ its maximum time interval with existence on the right in H^1. Define the set*

$$\mathcal{O}=\{u_0\in\mathcal{B}_{\alpha^*}\quad \textit{with}\quad \int_0^T|\nabla u(t)|_{L^2}dt<+\infty\}.$$

Then:

- *If $u_0\in\mathcal{O}$, then $0<T<+\infty$ and it follows that*
$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}\left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac12}\to\frac{1}{\sqrt{2\pi}}\quad \textit{as}\quad t\to T.$$
- *The set of initial data $u_0\in\mathcal{B}_{\alpha^*}$ with negative energy $E_0^G\leq 0$ and super critical mass $\int Q^2<\int|u|^2$ is included in $\mathcal{O}$.*
- *$\mathcal{O}$ is open in H^1.*
- *If $0<T<+\infty$ and $u_0\in\mathcal{B}_{\alpha^*}$ does not belong to $\mathcal{O}$, then the following lower bound holds:*
$$|\nabla u(t)|_{L^2}\geq\frac{C^*}{(T-t)\sqrt{E_0^G}}.$$

Remark 1. It is an open problem to get a bound from above on the blow-up rate in the regime where (1.10) holds. Recall again that solutions obtained in [4] have the speed $|\nabla u(t)|_{L^2}\sim\frac{1}{T-t}$.

This paper is organized as follows. Section 2 is devoted to recalling key objects and properties obtained in [14], [15], [16], [20]. In section 3, we outline the strategy of the proof of Theorem 2. In section 4, we obtain new dispersive estimates which allow us in section 5 to conclude the proof. Theorem 3 will follow similarly.

Throughout this paper, we will assume in dimension $N \geq 2$ that the Spectral Property holds true (recall that it has been proven to hold true for $N = 1, 2, 3, 4$). In addition, all results in this paper deal with initial conditions $u_0 \in H^1$ in the L^2 vicinity of Q,

$$\int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*$$

for some $\alpha^* > 0$ small enough.

We also fix some notation. $\delta(\alpha^*) > 0$ will denote a constant such that $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Moreover, given a well-localized function f, we set

$$f_1 = \frac{N}{2} f + y \cdot \nabla f \quad \text{and} \quad f_2 = \frac{N}{2} f_1 + y \cdot \nabla f_1.$$

Note that integration by parts yields

$$(f_1, g) = -(f, g_1).$$

We accept one exception to this notation and will note $\varepsilon = \varepsilon_1 + i\varepsilon_2$ in terms of real and imaginary parts.

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2. Recall of dynamical properties of solutions to (1.1)

Our aim in this section is to recall the objects involved in the analysis of the dynamics of solutions to (1.1) which have been exhibited in [14], [15], [16], [20], and we refer to these papers for detailed proofs and explanations. These properties are the starting point of our analysis. We use a suitable finite-dimensional geometrical decomposition of the solution and estimates on the corresponding geometrical parameters. These dispersive type estimates then allow us to understand part of the interactions between the finite-dimensional dynamic and the infinite-dimensional part of the solution.

2.1. **Localized self-similar profiles and associated radiation.** In this subsection, we recall results concerning the construction of regular approximations to self similar profiles close to Q. Let η be a small parameter, $0 < \eta << 1$, to be fixed later. For $b \neq 0$, set

$$R_b = \frac{2}{|b|}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta} R_b,$$

and $B_{R_b} = \{y \in \mathbf{R}^N, \ \ |y| \leq R_b\}$. We introduce a regular radially symmetric cut-off function $\phi_b(x) = 0$ for $|x| \geq R_b$ and $\phi_b(x) = 1$ for $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$, such that

$$|\phi_b'|_{L^\infty} + |\Delta \phi_b|_{L^\infty} \to 0 \quad \text{as} \quad |b| \to 0. \tag{2.1}$$

We also consider the norm on radial functions $\|f\|_{\mathcal{C}^j} = \max_{0 \leq k \leq j} \|f^{(k)}(r)\|_{L^\infty(\mathbf{R}_+)}$. We then claim the following.

Proposition 1 (Localized self-similar profiles)**.** *See Propositions 8 and 9 of* [16]*. There exist universal constants $C>0$, $\eta^*>0$ such that the following holds true. For all $0<\eta<\eta^*$, there exist constants $\varepsilon^*(\eta)>0$, $b^*(\eta)>0$ going to zero as $\eta\to 0$ such that for all $|b|<b^*(\eta)$, there exists a unique radial solution Q_b to*

$$\begin{cases} \Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y\cdot\nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \quad in \quad B_{R_b}, \\ Q_b(0)\in (Q(0)-\varepsilon^*(\eta), Q(0)+\varepsilon^*(\eta)), \quad Q_b(R_b)=0. \end{cases}$$

Moreover, let

$$\tilde{Q}_b(r) = Q_b(r)\phi_b(r).$$

Then the following holds:
(i) Uniform closeness to Q: $\tilde{Q}_b$ is differentiable with respect to b with estimate
(2.2)
$$\|e^{(1-\eta)\frac{\theta(|b|r)}{|b|}}(\tilde{Q}_b - Q)\|_{\mathcal{C}^3} + \|e^{(1-\eta)\frac{\theta(|b|r)}{|b|}}\left(\frac{\partial}{\partial b}\tilde{Q}_b + i\frac{|y|^2}{4}Q\right)\|_{\mathcal{C}^2} \to 0 \quad as \quad b\to 0,$$

where

$$\theta(w) = 1_{0\le w\le 2}\int_0^w \sqrt{1-\frac{z^2}{4}}dz + 1_{w>2}\frac{\theta(2)}{2}w. \tag{2.3}$$

(ii) Equation of $\tilde{Q}_b$: $\tilde{Q}_b$ satisfies

$$\Delta\tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}} = -\Psi_b \tag{2.4}$$

with

$$-\Psi_b = 2\nabla\phi_b\cdot\nabla Q_b + Q_b(\Delta\phi_b) + ibQ_b y\cdot\nabla\phi_b + (\phi_b^{1+\frac{4}{N}} - \phi_b)Q_b|Q_b|^{\frac{4}{N}}, \tag{2.5}$$

and for any polynomial $P(y)$ and integer $k=0,1$,

$$|P(y)\Psi_b^{(k)}|_{L^\infty} \le e^{-\frac{C_P}{|b|}}.$$

(iii) Degeneracy of the energy and the momentum:

$$2E(\tilde{Q}_b) = -Re\int(\Psi_b)_1\overline{\tilde{Q}_b} \quad so\ that \quad |E(\tilde{Q}_b)|\le e^{-\frac{C}{|b|}}, \tag{2.6}$$

$$Im\left(\int\nabla\tilde{Q}_b\overline{\tilde{Q}_b}\right)=0, \quad Im\left(\int y\cdot\nabla\tilde{Q}_b\overline{\tilde{Q}_b}\right) = -\frac{b}{2}|y\tilde{Q}_b|_2^2.$$

Remark 2. One computes:

$$\theta(2) = \int_0^2\sqrt{1-\frac{z^2}{4}}dz = \int_0^{\frac{\pi}{2}} 2\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta = \frac{\pi}{2}. \tag{2.7}$$

We now claim a nondegeneracy property of $\tilde{Q}_b$ in L^2 with respect to b^2:

Lemma 1. *$\tilde{Q}_b$ has supercritical mass, and more precisely:*

$$0 < \frac{d}{d(b^2)}\left(\int|\tilde{Q}_b|^2\right)_{|b^2=0} < +\infty. \tag{2.8}$$

Proof of Lemma 1. This is a refinement of estimates obtained in [16], Part B. Recall from the proof of Proposition 1 (see Appendix B in [16]) that Q_b is built from $P_b = Q_b e^{i\frac{b|y|^2}{4}}$, which satisfies

$$\begin{cases} \Delta P_b - P_b + \frac{b^2|y|^2}{4} P_b + P_b^{1+\frac{4}{N}} = 0, \\ P_b > 0 \ \text{ in } \ B_{R_b}, \\ P_b(0) \in (Q(0) - \varepsilon(\eta), Q(0) + \varepsilon(\eta)), \quad P_b(R_b) = 0. \end{cases}$$

We then set $\tilde{Q}_b(r) = Q_b(r)\phi_b(r)$ and similarly $\tilde{P}_b(r) = P_b(r)\phi_b(r)$. Observe that $\tilde{P}_b$ is a function of b^2 and satisfies

$$\Delta \tilde{P}_b - \tilde{P}_b + \frac{b^2|y|^2}{4}\tilde{P}_b + \tilde{P}_b^{1+\frac{4}{N}} = -\tilde{\Psi}_b,$$

with $-\tilde{\Psi}_b = 2\nabla\phi_b \cdot \nabla P_b + P_b\Delta\phi_b + (\phi_b^{1+\frac{4}{N}} - \phi_b)P_b^{1+\frac{4}{N}}$. Moreover, from the proof of Proposition 1 in [15], $\tilde{P}_b$ is differentiable with respect to b with estimate

$$\|e^{(1-\eta)\frac{\theta(|b|r)}{|b|}}\frac{\partial}{\partial b}\tilde{P}_b\|_{\mathcal{C}^2} \leq C|b|,$$

and thus for b nonzero:

$$H_b = \frac{\partial \tilde{P}_b}{\partial(b^2)} \quad \text{satisfies} \quad \|e^{(1-\eta)\frac{\theta(|b|r)}{|b|}} H_b\|_{\mathcal{C}^2} \leq C. \tag{2.9}$$

In addition, for any b_0 nonzero, the function $T_{b_0}(b) = \left(\frac{b_0}{b}\right)^{\frac{N}{2}} P_b\left(\frac{b_0}{b}x\right)$ is differentiable with respect to b at b_0 with estimate

$$\|e^{(1-\eta)\frac{\theta(|b_0|r)}{|b_0|}}\frac{\partial T_{b_0}}{\partial b}\|_{\mathcal{C}^2(|y|\leq R_{b_0})} \leq \frac{C}{|b_0|}.$$

From this last estimate and the explicit formula for $\tilde{\Psi}_b$, we conclude that $\tilde{\Psi}_b$ is differentiable with respect to b^2 for b nonzero with estimate:

$$\left\|\frac{\partial \tilde{\Psi}_b}{\partial b^2}\right\|_{\mathcal{C}^1} \leq e^{-\frac{C}{|b|}}. \tag{2.10}$$

Observe now:

$$\frac{d}{d(b^2)}\left(\int |\tilde{Q}_b|^2\right) = \frac{d}{d(b^2)}\left(\int |\tilde{P}_b|^2\right) = 2(H_b, \tilde{P}_b).$$

Moreover, from the convergence $\tilde{P}_b \to Q$ as $b \to 0$ given by (2.2) and the uniform bound (2.9), $\lim_{b\to 0}(H_b, \tilde{P}_b) = \lim_{b\to 0}(H_b, Q)$. To conclude the proof of (2.8), it now suffices to prove from a standard argument:

$$\lim_{b\to 0}(H_b, Q) > 0. \tag{2.11}$$

Proof of (2.11). Let

$$L_+ = -\Delta + 1 - (1 + \frac{4}{N})Q^{\frac{4}{N}} \tag{2.12}$$

be the real part of the linearized operator close to Q. Then H_b satisfies

$$-L_+ H_b = -\frac{\partial \tilde{\Psi}_b}{\partial(b^2)} - \frac{b^2|y|^2}{4}H_b - \frac{|y|^2}{4}\tilde{P}_b + \left(1 + \frac{4}{N}\right)(Q^{\frac{4}{N}} - \tilde{P}_b^{\frac{4}{N}})H_b.$$

Take the inner product of this equation by the well-localized direction Q_1. Then we get from the previous estimates:

$$\lim_{b\to 0}(L_+H_b, Q_1) = \frac{1}{4}\lim_{b\to 0}(|y|^2\tilde{P}_b, Q_1)) = -\frac{1}{4}\int |y|^2 Q^2.$$

We conclude from the algebraic relation $L_+Q_1 = -2Q$ that

$$\lim_{b\to 0}(L_+H_b, Q_1) = -2\lim_{b\to 0}(H_b, Q) < 0,$$

and (2.11) is proved. This concludes the proof of Lemma 1.

The modified profiles $\tilde{Q}_b$ built in the previous subsection are not exact self-similar solutions, and indeed the presence of the nonzero error term Ψ_b in (2.4) is necessary due to the nonexistence of H^1 self-similar profiles. Now as exhibited in [19] and [16], these profiles are not sharp enough to investigate the full interaction between the nonlinear concentration on compact sets in space and the linear dispersive dynamic at infinity. We thus introduce the outgoing radiation escaping the soliton core replacing the tail of an $\dot{H}^1$ self-similar solution according to the following lemma:

Lemma 2 (Linear outgoing radiation)**.** *See Lemma 15 in* [16]. *There exist universal constants $C > 0$ and $\eta^* > 0$ such that $\forall 0 < \eta < \eta^*$, there exists $b^*(\eta) > 0$ such that $\forall 0 < b < b^*(\eta)$, the following holds true: let Ψ_b be given by* (2.5)*; then there exists a unique radial solution ζ_b to*

$$\left\{\begin{array}{l}\Delta\zeta_b - \zeta_b + ib(\zeta_b)_1 = \Psi_b\\ \int |\nabla\zeta_b|^2 < +\infty.\end{array}\right. \tag{2.13}$$

Moreover, let θ be given by (2.3)*, and consider*

$$\Gamma_b = \lim_{|y|\to+\infty} |y|^N|\zeta_b(y)|^2; \tag{2.14}$$

then it follows that

$$\left| |y|^{\frac{N}{2}}(|\zeta_b| + |y||\nabla(\zeta_b)|)\right|_{L^\infty(|y|\geq R_b)} \leq \Gamma_b^{\frac{1}{2}-C\eta} < +\infty, \tag{2.15}$$

$$\int |\nabla\zeta_b|^2 \leq \Gamma_b^{1-C\eta}. \tag{2.16}$$

For $|y|$ large, we have more precisely:

$$\forall |y| \geq R_b^2, \quad e^{-2(1-C\eta)\frac{\theta(2)}{b}} \geq |y|^N|\zeta_b(y)|^2 \geq \frac{4}{5}\Gamma_b \geq e^{-2(1+C\eta)\frac{\theta(2)}{b}}, \tag{2.17}$$

$$\forall |y| \geq R_b^2, \quad |\nabla\zeta(y)| \leq \frac{C}{|y|^{1+\frac{N}{2}}}\frac{\Gamma_b^{\frac{1}{2}}}{|b|}. \tag{2.18}$$

For $|y|$ small, we have: $\forall\sigma \in (0,5)$, $\exists\eta^{**}(\sigma)$ such that $\forall 0 < \eta < \eta^{**}(\sigma)$, $\exists b^{**}(\eta)$ such that $\forall 0 < b < b^{**}(\eta)$, and it follows that

$$\left|\zeta_b(y)e^{-\sigma\frac{\theta(b|y|)}{b}}\right|_{\mathcal{C}^2(|y|\leq R_b)} \leq \Gamma_b^{\frac{1}{2}+\frac{1}{10}\sigma}. \tag{2.19}$$

Last, ζ_b is differentiable with respect to b with estimate

$$\left|\frac{\partial\zeta_b}{\partial b}\right|_{\mathcal{C}^1} \leq \Gamma_b^{\frac{1}{2}-C\eta}. \tag{2.20}$$

We refer to [19] or Appendix E of [16] for the proof of Lemma 2, up to estimates (2.18) and (2.20), which are proved in Appendix A of this paper.

Note that an important fact is that ζ_b is not in L^2 but in $\dot{H}^1$. The constant Γ_b given by (2.14) will play a fundamental role in our analysis and is precisely the object which measures the way ζ_b misses L^2. In our analysis, exponentially small terms in b will systematically be estimated in terms of powers of Γ_b thanks to (2.17):

$$e^{-2(1+C\eta)\frac{\theta(2)}{b}} \leq \Gamma_b \leq e^{-2(1-C\eta)\frac{\theta(2)}{b}}. \tag{2.21}$$

2.2. **Geometrical decomposition of the solution.** In this subsection, we introduce the geometrical decomposition of the solutions to (1.1) adapted to the study of dispersion first exhibited in [14], and collect dynamical results obtained in [15], [16], [20].

Let an initial data $u_0 \in \mathcal{B}_{\alpha^*}$ assuming $\alpha^* > 0$ small enough, and assume that the corresponding solution $u(t)$ to (1.1) is defined on $[0, T)$, $0 < T \leq +\infty$. As exhibited in [14], we first observe that we may modify u_0 by a fixed Galilean transform to ensure

$$Im\left(\int \nabla u_0 \overline{u_0}\right) = 0, \tag{2.22}$$

prove the result in this context and then transpose it to the general case.

We first recall a classical lemma of proximity of H^1 functions up to scaling, phase and translation factors to the function Q related to the variational structure of Q. Recall indeed that for $u \in H^1$, solutions to $E(u) = 0$ and $|u|_{L^2} = |Q|_{L^2}$ are exactly $e^{i\gamma_0}\lambda_0^{\frac{N}{2}} Q(\lambda_0(x + x_0))$ for some fixed parameters $(\lambda_0, \gamma_0, x_0)$.

Lemma 3 (Variational characterization of the ground state)**.** *There exists a universal constant $\alpha_1^* > 0$ such that: for all $0 < \alpha' \leq \alpha_1^*$, there exists $\delta(\alpha')$ with $\delta(\alpha') \to 0$ as $\alpha' \to 0$ such that $\forall u \in H^1$, if $\int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha'$ and*

$$E(u) \leq \alpha' \int |\nabla u|^2, \tag{2.23}$$

then there exist parameters $\lambda_0 = \frac{|\nabla Q|_{L^2}}{|\nabla u|_{L^2}}$, $\gamma_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^N$ such that

$$|Q - e^{i\gamma_0}\lambda_0^{\frac{N}{2}} u(\lambda_0(x + x_0))|_{H^1} < \delta(\alpha').$$

For the rest of this section, we assume that there exists a time $t(u_0) \in [0, T)$ such that

$$\forall t \in [t(u_0), T), \quad E_0 \leq \alpha^* \int |\nabla u(t)|^2, \tag{2.24}$$

and $0 < \alpha^* \leq \frac{1}{2}\alpha_1^*$.

Remark 3. In our further analysis, two kinds of assumptions will ensure (2.24): either $\lim_{t \to T} |\nabla u(t)|_{L^2} = +\infty$ and then (2.24) holds for t close to T, or $E_0 \leq 0$ and then $t(u_0) = 0$.

We now sharpen the decomposition of Lemma 3 and introduce a regular geometrical decomposition of $u(t)$ related to its proximity in H^1 to the manifold

$$\mathcal{M} = \{e^{i\gamma}\lambda^{\frac{N}{2}}\tilde{Q}_b(\lambda y + x)\}.$$

Lemma 4 (Nonlinear modulation of the solution with respect to $\mathcal{M}$)**.** *See Lemma 2 in* [15]*. There exist some continuous functions* $(\lambda, \gamma, x, b) : [t(u_0), T) \to (0, +\infty) \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ *such that*

$$\forall t \in [t(u_0), T), \quad \varepsilon(t, y) = e^{i\gamma(t)}\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y) \tag{2.25}$$

satisfies the following:
(i)

$$\big(\varepsilon_1(t), |y|^2\Sigma_{b(t)}\big) + \big(\varepsilon_2(t), |y|^2\Theta_{b(t)}\big) = 0, \tag{2.26}$$

$$\big(\varepsilon_1(t), y\Sigma_{b(t)}\big) + \big(\varepsilon_2(t), y\Theta_{b(t)}\big) = 0, \tag{2.27}$$

$$-\big(\varepsilon_1(t), (\Theta_{b(t)})_2\big) + \big(\varepsilon_2(t), (\Sigma_{b(t)})_2\big) = 0, \tag{2.28}$$

$$-\big(\varepsilon_1(t), (\Theta_{b(t)})_1\big) + \big(\varepsilon_2(t), (\Sigma_{b(t)})_1\big) = 0, \tag{2.29}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$*,* $\tilde{Q}_b = \Sigma_b + i\Theta_b$ *in terms of real and imaginary parts;*

(ii) $|1 - \lambda(t)\dfrac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}| + |\varepsilon(t)|_{H^1} + |b(t)| \leq \delta(\alpha^*)$ *where* $\delta(\alpha^*) \to 0$ *as* $\alpha^* \to 0$*;*

(iii) $t(u_0) = 0$ *if* $E(u_0) \leq 0$.

We now introduce the rescaled time

$$s = \int_{t(u_0)}^{t} \frac{dt'}{\lambda^2(t')} \tag{2.30}$$

so that $s(t(u_0)) = 0$ and $s\{(t(u_0), T)\} = \mathbf{R}_+$. Moreover, we shall note from now on that

$$\tilde{Q}_b = \Sigma + i\Theta, \quad \Psi_b = Re(\Psi) + iIm(\Psi)$$

in terms of real and imaginary parts.

We first observe from a standard argument that $\{\lambda(s), \gamma(s), x(s), b(s)\}$ are $\mathcal{C}^1$ functions of s on $\mathbf{R}_+$, and ε satisfies the following equation for $s \in \mathbf{R}_+$, $y \in \mathbf{R}^N$:

(2.31)
$$b_s\frac{\partial\Sigma}{\partial b} + \partial_s\varepsilon_1 - M_-(\varepsilon) + b\left(\frac{N}{2}\varepsilon_1 + y\cdot\nabla\varepsilon_1\right) = \left(\frac{\lambda_s}{\lambda} + b\right)\Sigma_1 + \tilde{\gamma}_s\Theta + \frac{x_s}{\lambda}\cdot\nabla\Sigma$$
$$+ \left(\frac{\lambda_s}{\lambda} + b\right)\left(\frac{N}{2}\varepsilon_1 + y\cdot\nabla\varepsilon_1\right) + \tilde{\gamma}_s\varepsilon_2 + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 + Im(\Psi) - R_2(\varepsilon),$$

(2.32)
$$b_s\frac{\partial\Theta}{\partial b} + \partial_s\varepsilon_2 + M_+(\varepsilon) + b\left(\frac{N}{2}\varepsilon_2 + y\cdot\nabla\varepsilon_2\right) = \left(\frac{\lambda_s}{\lambda} + b\right)\Theta_1 - \tilde{\gamma}_s\Sigma + \frac{x_s}{\lambda}\cdot\nabla\Theta$$
$$+ \left(\frac{\lambda_s}{\lambda} + b\right)\left(\frac{N}{2}\varepsilon_2 + y\cdot\nabla\varepsilon_2\right) - \tilde{\gamma}_s\varepsilon_1 + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 - Re(\Psi) + R_1(\varepsilon),$$

with $\tilde{\gamma}(s) = -s - \gamma(s)$. The linear operator close to $\tilde{Q}_b$ is $M = (M_+, M_-)$ with

$$M_+(\varepsilon) = -\Delta\varepsilon_1 + \varepsilon_1 - \left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$
$$M_-(\varepsilon) = -\Delta\varepsilon_2 + \varepsilon_2 - \left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

The nonlinear interaction terms are explicitly:

$$R_1(\varepsilon) = (\varepsilon_1 + \Sigma)|\varepsilon + \tilde{Q}_b|^{\frac{4}{N}} - \Sigma|\tilde{Q}_b|^{\frac{4}{N}} - \left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2, \tag{2.33}$$

$$R_2(\varepsilon) = (\varepsilon_2 + \Theta)|\varepsilon + \tilde{Q}_b|^{\frac{4}{N}} - \Theta|\tilde{Q}_b|^{\frac{4}{N}} - \left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1. \tag{2.34}$$

We now claim the following preliminary estimates for this decomposition:

Lemma 5. *For all $s \in \mathbf{R}+$, it follows that:*
(i) Estimates induced by the conservation of energy and momentum:

$$|2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta)| \leq C(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}) + \Gamma_b^{1-C\eta} + C\lambda^2|E_0|, \tag{2.35}$$
$$|(\varepsilon_2, \nabla\Sigma)| \leq C\delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|})^{\frac{1}{2}}.$$

(ii) Estimates on the modulation parameters:

$$|\frac{\lambda_s}{\lambda} + b| + |b_s| \leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + \Gamma_b^{1-C\eta} + C\lambda^2|E_0| \tag{2.36}$$

$$\left|\tilde{\gamma}_s - \frac{1}{|Q_1|_2^2}(\varepsilon_1, L_+Q_2)\right| + \left|\frac{x_s}{\lambda}\right| \leq \delta(\alpha^*)(\int |\nabla\varepsilon|^2 e^{-2(1-\eta)\frac{\theta(b|y|)}{b}} + \int |\varepsilon|^2 e^{-|y|})^{\frac{1}{2}} + C\int |\nabla\varepsilon|^2 + \Gamma_b^{1-C\eta} + C\lambda^2|E_0|. \tag{2.37}$$

Proof of Lemma 5. This lemma is very similar to Lemma 3 in [15]. The main difference is our will to take the quantity $\lambda^2 E_0$ explicitly into account. Note that the algebraic formulas for the computation of the geometrical parameters under orthogonality conditions (2.26), (2.27), (2.28) and (2.29) have been exhibited in Appendix A of [20]. We then estimate the nonlinear interaction terms as in [15] to get (2.36) and (5). In this last step, we use the following estimate, which has been proven in Lemma 5 in [16]: $\forall\kappa > 0$, there exists $C_\kappa > 0$ such that for all $\varepsilon \in H^1$,

$$\int |\varepsilon|^2 e^{-\kappa|y|} \leq C_\kappa\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right). \tag{2.38}$$

This concludes the proof of Lemma 5.

We now recover the virial estimate obtained in [14], [15].

Proposition 2 (Local virial estimate)**.** *See Lemma 7 of* [20]*. There exist universal constants $\delta_0 > 0$, $C > 0$ such that for all $s \geq 0$, it follows that*

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - C\lambda^2 E_0 - \Gamma_b^{1-C\eta}. \tag{2.39}$$

Let us now focus on the monotonicity properties which hold only for the solutions $u_0 \in \mathcal{O}$; that is, we assume that the corresponding solution $u(t)$ to (1.1) blows up in finite time $0 < T < +\infty$ with the upper bound

$$|\nabla u(t)|_{L^2} \leq C_1^* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}.$$

These estimates are much deeper in nature than those of Lemma 5 and correspond to rigidity properties for those solutions to (1.1) obtained in [14], [15], [20].

Proposition 3 (Dynamical controls in $\mathcal{O}$)**.** *See Lemma 7, section 3 of* [20]*. There exist universal constants $\delta_0 > 0$, $d_0 > 0$ such that for all $u_0 \in \mathcal{O}$, there exists some time $s_0 = s_0(u_0) > 0$ such that for all $s \geq s_0$, the following hold:*
(i) Sign of b:

$$b(s) > 0. \tag{2.40}$$

(ii) Almost monotonicity of the norm:

$$\forall s_2 \geq s_1 \geq s_0, \quad \lambda(s_2) < 2\lambda(s_1). \tag{2.41}$$

(iii) Control of the scaling parameter:

$$\lambda(s) \leq e^{-\frac{1}{\Gamma_b^{d_0}}}. \tag{2.42}$$

Let us point out two facts regarding the above result:

- The analysis developed in [14], [15] for the negative energy solutions is global in time in the sense that the geometrical decomposition of Lemma 4 holds for all time, $s \in \mathbf{R}$, and one can also get the sign of the parameter b and the monotonicity type properties for all time. On the contrary for data in $\mathcal{O}$ with positive energy, Proposition 3 holds only asymptotically near the blow-up time, $s \to +\infty$, as exhibited in [20]. Note that the key rigidity property (2.40) still holds true in $\mathcal{O}$, and is unknown outside of it.
- Control of the scaling parameter (2.42) is false in the other blow-up regime governed by the lower bound (1.10) for which

  $$\lambda\sqrt{E_0} \geq C|b|$$

  holds true; see [20]. On the contrary in the log-log regime, the energy type of the term $\lambda^2|E_0|$ will be asymptotically negligible thanks to (2.42).

3. Strategy of the proof of the log-log lower bound

This section is devoted to a brief sketch of the proof of Theorem 2.

We let an initial data $u_0 \in \mathcal{O}$ with zero momentum according to (2.22), that is, $u(t)$, the corresponding solution to (1.1), blows up in finite time $0 < T < +\infty$ with upper bound

$$|\nabla u(t)|_{L^2} \leq C_1^* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}.$$

Let us recall the virial estimate (2.39) and assume for the sake of simplicity that $E_0 \leq 0$ in which case (2.39) implies

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \Gamma_b^{1-C\eta}. \tag{3.1}$$

In previous works [14], [15], this $\dot{H}^1$ estimate is the starting point of the proof of the upper bound (1.9) and points out the importance of the parameter $b(t)$ in the problem. Coupled with the modulation equation for the scaling parameter $\lambda(t)$, it implies controls (2.40) and (2.42) and eventually a pointwise lower bound for b: for s large enough,

$$b(s) \geq \frac{C}{\log(s)}, \tag{3.2}$$

which eventually implies (1.9).

Our goal now for the proof of Theorem 2 is to obtain the converse inequality: for s large enough,

$$b(s) \leq \frac{C}{\log(s)}. \tag{3.3}$$

In particular, $b(s) \to 0$ as $s \to +\infty$. The problem is that such a decay estimate on b cannot directly follow from local-type estimates like (3.1) and (3.4) which take place in $\dot{H}^1 \cap L^2_{loc}$ and would also apply to self-similar solutions which are in this space and have a constant b (see [16] for a further discussion of this fundamental difficulty).

As partially understood in [16], decay properties for $b(s)$ are related to dispersive estimates in L^2. We claim that in the full energy space H^1, dispersive effects of (1.1) in the log-log regime can be seen through the exhibition of an exact Lyapounov functional. We expect that this strategy will apply in different situations in order to treat degenerate nonlinear dispersive problems.

A: Derivation of a Lyapounov functional.

Step 1. Refined virial estimate on compact sets.

As observed in [16], estimate (3.1) may be improved by introducing the radiation ζ_b of Lemma 2. Yet, as this radiation is not in L^2, we consider a localized radiation $\zeta_{b,A}$ which equals ζ_b for $|y| \leq A$ and is zero elsewhere. We then claim that for a suitable choice of cut-off parameter $A = A(t) = \Gamma_b^{-a}$, $a > 0$ small enough, the new variable $\tilde{\varepsilon} = \varepsilon - \zeta_{b,A}$ satisfies

$$b_s \geq \delta_0 \left(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} \right) + \Gamma_b - \frac{1}{\delta_0} \int_A^{2A} |\varepsilon|^2. \tag{3.4}$$

This remarkable inequality is useless compared to (3.1) to estimate local norms and get monotonicity-type results as in [14], [15], [20]. Now the question at the heart of our analysis is to control local information (size of the parameter b and the set where the approximation $\varepsilon \sim \zeta_{b,A}$ is good in a certain sense) by flux terms at $|y| = A$ in L^2, which is provided by (3.4).

Step 2. Linear dispersive information at infinity in space.

Let us remark that estimate (3.4) was already in [16]. We worked there not with a general H^1 solution but with an asymptotic object which was essentially known to be negligible at infinity in L^2 using extra estimates in the virial space Σ. Here on the contrary, the term $\int_A^{2A}|\varepsilon|^2$ in (3.4) is controlled by a flux in L^2 of the solution for $|y|\geq A$ in a regime when parameter b has a fixed sign $b>0$, which is relevant according to (2.40):

$$\left\{\int_{|y|\geq A}|\varepsilon|^2\right\}_s \geq b\int_A^{2A}|\varepsilon|^2-\Gamma_b^{1+z_0}, \tag{3.5}$$

for some $z_0>0$. Again, a key in this estimate is the choice of the cut-off parameter $A(t)$ which allows some decoupling in the interactions together with the fact that Q is exponentially decreasing at infinity.

Multiplying (3.4) by $\delta_0 b>0$ and using (3.5), we obtain the following Lyapounov functional:

$$\left\{\int_{|y|\geq A}|\varepsilon|^2+\delta_0 b^2\right\}_s \geq \delta_0 b\left(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|}\right)+\delta_0 b\Gamma_b. \tag{3.6}$$

B: Dynamical control of b.

Step 1. Asymptotic stability revisited.

Integration in time of dispersive estimate (3.6) yields a direct proof of the asymptotic stability:

$$b(s)+\int|\nabla\varepsilon(s)|^2+\int|\varepsilon(s)|^2e^{-|y|}\to 0 \ \text{ as } \ s\to+\infty.$$

We moreover make precise the size of ε with respect to b and claim a pointwise control:

$$\int|\nabla\varepsilon(s)|^2+\int|\varepsilon(s)|^2e^{-|y|}\leq\Gamma_{b(s)}^{\frac{1}{2}}. \tag{3.7}$$

Step 2. Conservation of the L^2-norm and conclusion.

Consider again (3.6). We have no a priori bound on the left-hand side, that is, the Lyapounov functional itself. We now use (for the first time in the problem) the global information in space given by the conservation of the L^2-norm:

$$\begin{aligned}\int|u|^2 &= \int|\tilde{Q}_b|^2+2Re(\int\varepsilon\overline{\tilde{Q}_b})+\int|\varepsilon|^2\\ &\sim \int Q^2+b^2+\int_{|y|\geq A}|\varepsilon|^2,\end{aligned}$$

where we used (3.7) to control ε on compact sets, and the fact that the *localized profiles* $\tilde{Q}_b$ *have super critical mass* from (2.8).

Injecting this into the Lyapounov property (3.6), we obtain the key estimate:

$$(-b^2)_s\geq\delta_0 b\Gamma_b. \tag{3.8}$$

This yields (3.3), and the proof of the lower bound (1.12) will now follow from

$$-\frac{\lambda_s}{\lambda} \sim b.$$

In other words and after having estimated all the interaction terms, the Lyapounov function of the problem simply reduces to the parameter b itself or equivalently the size of the L^2-norm of the solution on the rescaled ball $|y| \leq 1$.

Let us summarize this proof in a more physical picture: from the fact that the profiles $\tilde{Q}_b$ have supercritical mass, the remainder term ε of lower order is radiated away by escaping the soliton core according to an outgoing radiation. Now the rate at which this expulsion is performed is submitted to the constraint of the conservation of the L^2-norm together with the dispersive estimates (3.4). Differential inequality (3.8) then simply expresses a flux type of computation in L^2 balancing dispersive effects.

4. Derivation of the Lyapounov functional in H^1

This section is devoted to the proof of dispersive estimates (3.4), (3.5), (3.8) needed for the proof of Theorem 2. We let an initial data $u_0 \in H^1$ with zero momentum according to (2.22) and assume that the corresponding solution satisfies (2.24). It thus admits a decomposition as in Lemma 4 for $s \geq 0$. Moreover, we will always assume that $b(s)$ has a fixed sign:

$$\forall s \geq 0, \quad b(s) > 0. \tag{4.1}$$

4.1. **Virial dispersion in the radiative regime.** As exhibited in [14], [15], the key to estimate the solution on compact sets is the virial type of control (2.39):

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - C\lambda^2 E_0 - \Gamma_b^{1-C\eta},$$

which was a consequence of the more precise control:

$$b_s \geq -(\varepsilon_1, (Re(\Psi))_1) - (\varepsilon_2, (Im(\Psi))_1) + \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - C\lambda^2 E_0 - \Gamma_b^{1+z_0}, \tag{4.2}$$

for some universal constant $z_0 > 0$. This estimate is a consequence of the dispersive structure in $\dot{H}^1$ of the solution on sets $|y| \leq C$. Our goal is to improve it, which according to (4.2) amounts to treating the linear term $-(\varepsilon_1, (Re(\Psi))_1) - (\varepsilon_2, (Im(\Psi))_1)$. This is achieved by introducing the radiation of Lemma 2 as in [16], which corresponds to estimates for $|y| \leq \frac{C}{|b|}$.

A first attempt would be to try to estimate through a virial type relation a new variable $\varepsilon - \zeta_b$. The heart of the matter is now that ζ_b is indeed in $\dot{H}^1$, but not in L^2, and we then are no longer able to estimate the main interaction terms. We thus introduce a cut-off version of the radiation: let a radial cut-off function $\chi_A(r) = \chi\left(\frac{r}{A}\right)$ with $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. The choice of the parameter $A(t)$ is a crucial issue in our analysis, and it roughly relies on two constraints: we want A to be large in order first to enter the radiative zone, i.e., $A >> \frac{2}{b}$, and to ensure the slowest possible variations of the L^2-norm in the zone $|y| \geq A$ (see Lemma 7 and its proof). But we also want A not too large, in particular to keep a good control over local L^2-terms of the form $\int_{|y| \leq A} |\varepsilon|^2$; see in particular the proof of Proposition 5 and the pointwise estimate (5.3). A choice which balances these two constraints is:

$$A = A(t) = e^{2a\frac{\theta(2)}{b(t)}} \text{ so that } \Gamma_b^{-\frac{a}{2}} \leq A \leq \Gamma_b^{-\frac{3a}{2}}, \tag{4.3}$$

for some parameter $a > 0$ small enough to be chosen later and which depends on η. Note that there is in fact some optimality in this choice; see Remark 8 in [17]. We then set

$$\tilde{\zeta}_b = \chi_A \zeta_b = \tilde{\zeta}_{re} + i\tilde{\zeta}_{im}.$$

$\tilde{\zeta}_b$ still satisfies the size estimates of Lemma 2 and is moreover in L^2 with an estimate

$$\int |\tilde{\zeta}|^2 \leq \Gamma_b^{1-C\eta}. \tag{4.4}$$

The equation satisfied by $\tilde{\zeta}$ is now

$$\Delta\tilde{\zeta}_b - \tilde{\zeta}_b + ib(\tilde{\zeta}_b)_1 = \Psi_b + F$$

with

$$F = (\Delta\chi_A)\zeta_b + 2\nabla\chi_A \cdot \nabla\zeta_b + iby \cdot \nabla\chi_A\zeta_b. \tag{4.5}$$

We now may consider the new variable

$$\tilde{\varepsilon} = \varepsilon - \tilde{\zeta}.$$

Let us remark the following key points:

• First, $\tilde{\varepsilon}$ is now small in H^1 from (4.4) and fits into the virial type of analysis.

• Second, our initial goal was to remove the linear term $-(\varepsilon_1, (Re(\Psi))_1) - (\varepsilon_2, (Im(\Psi))_1)$, but ζ_b is not an exact solution to (2.13). This induces a linear term of the form $-(\tilde{\varepsilon}_1, (Re(F))_1) - (\tilde{\varepsilon}_2, (Im(F))_1)$, F given by (4.5), which now needs to be treated differently.

The lemma is as follows:

Lemma 6 (Virial dispersion in the radiative regime)**.** *For some universal constants $\delta_1 > 0$, $c > 0$ and $s \geq 0$, it follows that*

$$\{f_1(s)\}_s \geq \delta_1 \left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} \right) + c\Gamma_b - C\lambda^2 E_0 - \frac{1}{\delta_1}\int_A^{2A} |\varepsilon|^2, \tag{4.6}$$

with

$$f_1(s) = \frac{b}{4}|y\tilde{Q}_b|_2^2 + \frac{1}{2}Im\left(\int y \cdot \nabla\tilde{\zeta}\overline{\tilde{\zeta}}\right) + (\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1). \tag{4.7}$$

Remark 4. Let us compare the two dispersive relations (2.39) and (4.6). First observe from H^1 controls on $\tilde{\zeta}$ and ε that $f_1(s) \sim b(s)$. Now the main difference is the presence of the Γ_b term (to the power one) *with the good sign.* The price to pay is the presence of the boundary term $\int_A^{2A} |\varepsilon|^2$, which cannot be directly estimated. Second, it is noteworthy that this estimate still is a tool in $\dot{H}^1$.

Remark 5. The $-\lambda^2 E_0$ term in the virial estimate (6) either has the good sign for $E_0 \leq 0$, or for $E_0 > 0$ and $u_0 \in \mathcal{O}$ will be asymptotically controlled by (2.42).

Remark 6. This lemma is obtained with the following range for the small parameters $\eta, a > 0$ involved respectively in Proposition 1 and (4.3): there exist $\eta^*, a^* > 0$ such that $\forall 0 < \eta < \eta^*$, $\forall 0 < a < a^*$, there exists $b^*(\eta, a)$ such that $\forall |b| \leq b^*(\eta, a)$, and estimates of Lemma 6 hold with universal constants.

Proof of Lemma 6. The proof of this lemma is similar to the proof of Lemma 16 in [16], even though some technical difficulties arise in the pure H^1 setting. We proceed in several steps.

Step 1: Algebraic dispersive relation.
We claim:

$$\begin{aligned}
&\left\{\frac{b}{4}|y\tilde{Q}_b|_2^2 + \frac{1}{2}Im\left(\int y\cdot\nabla\tilde{\zeta}\overline{\tilde{\zeta}}\right) + (\varepsilon_2, \Sigma_1 + (\tilde{\zeta}_{re})_1) - (\varepsilon_1, \Theta_1 + (\tilde{\zeta}_{im})_1)\right\}_s \\
&= H(\varepsilon - \tilde{\zeta}, \varepsilon - \tilde{\zeta}) + \Big(\varepsilon_1 - \tilde{\zeta}_{re}, Re(F_1)\Big) + \Big(\varepsilon_2 - \tilde{\zeta}_{im}, Im(F_1)\Big) - 2\lambda^2 E_0 \\
&+ b_s\left\{(\varepsilon_2 - \tilde{\zeta}_{im}, \frac{\partial(\Sigma + \tilde{\zeta}_{re})_1}{\partial b}) - (\varepsilon_1 - \tilde{\zeta}_{re}, \frac{\partial(\Theta + \tilde{\zeta}_{im})_1}{\partial b})\right\} \\
&- \frac{A_s}{A^2}\left\{(\varepsilon_2 - \tilde{\zeta}_{im}, (y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{re})_1) - (\varepsilon_1 - \tilde{\zeta}_{re}, (y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{im})_1)\right\} \\
&- \tilde{\gamma}_s\left\{(\varepsilon_1 - \tilde{\zeta}_{re}, (\Sigma + \tilde{\zeta}_{re})_1) + (\varepsilon_2 - \tilde{\zeta}_{im}, (\Theta + \tilde{\zeta}_{im})_1)\right\} \\
&- \left(\frac{\lambda_s}{\lambda} + b\right)\left\{(\varepsilon_2 - \tilde{\zeta}_{im}, (\Sigma + \tilde{\zeta}_{re})_2) + (\varepsilon_1 - \tilde{\zeta}_{re}, (\Theta + \tilde{\zeta}_{im})_2)\right\} \\
&- \frac{x_s}{\lambda}\cdot\left\{(\varepsilon_2 - \tilde{\zeta}_{im}, \nabla(\Sigma + \tilde{\zeta}_{re})_1) + (\varepsilon_1 - \tilde{\zeta}_{re}, \nabla(\Theta + \tilde{\zeta}_{im})_1)\right\} \\
&+ (R_1(\varepsilon), (\tilde{\zeta}_{re})_1) + (R_2(\varepsilon), (\tilde{\zeta}_{im})_1) + (\varepsilon_1 - \tilde{\zeta}_{re}, ((1 + \frac{4}{N})(Q^{\frac{4}{N}}\tilde{\zeta}_{re})_1)) \\
&+ (\varepsilon_2 - \tilde{\zeta}_{im}, (Q^{\frac{4}{N}}\tilde{\zeta}_{im})_1) + (\varepsilon_1, \tilde{L}) + (\varepsilon_2, \tilde{K}) + \tilde{H}_b(\varepsilon, \varepsilon) + (\tilde{R}_1(\varepsilon), \Sigma_1) \\
&+ (\tilde{R}_2(\varepsilon), \Theta_1) - \frac{2}{2 + \frac{4}{N}}\int J(\varepsilon),
\end{aligned} \tag{4.8}$$

where $(\tilde{L}, \tilde{K}, \tilde{H}_b(\varepsilon, \varepsilon), \tilde{R}_{1,2}(\varepsilon), J(\varepsilon))$ are residual terms exhibited in Appendix B where (4.8) is proved.

Step 2: Control of the interaction terms.

We now need to control the interaction terms appearing in (4.8). These estimates are similar to the ones derived in Step 2 of the proof of Lemma 16 in [16]. A major difference nevertheless is that we need estimates in the pure H^1 setting. This will be possible from our choice of cut-off parameter A, (4.3), which in particular ensured (4.4) and thus

$$\int |\tilde{\varepsilon}|^2 \le \delta(\alpha^*). \tag{4.9}$$

We now claim the following estimates:
(0) Sobolev type inequality: for $N = 1$ or $N \ge 3$,

$$\forall B \ge 2, \quad \forall v \in H^1, \quad \int_{|y|\le B} |v|^2 \le CB^2(\int |\nabla v|^2 + \int |v|^2 e^{-|y|}); \tag{4.10}$$

this estimate fails in dimension $N = 2$ where a logarithmic correction must be taken into account:

$$\forall B \ge 2, \quad \forall v \in H^1, \quad \int_{|y|\le B} |v|^2 \le CB^2 \log B(\int |\nabla v|^2 + \int |v|^2 e^{-|y|}). \tag{4.11}$$

A similar estimate we will also later need is still in dimension $N = 2$:

$$\forall B \geq 2, \quad \forall v \in H^1, \quad \int_{1 \leq |y| \leq B} \frac{|v|^2}{|y|^2} \leq C \log^2 B (\int |\nabla v|^2 + \int |v|^2 e^{-|y|}). \tag{4.12}$$

(i) Comparison of local L^2-norms of ε and $\tilde{\varepsilon}$:

$$\int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}} + \int |\tilde{\varepsilon}|^2 e^{-|y|} \leq C(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}.$$

(ii) Second-order interaction terms:

$$\int |R(\varepsilon)| e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} \leq C(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}.$$

(iii) Small second-order interaction terms:

$$\int |R(\varepsilon)|(|\tilde{\zeta}| + |y \cdot \nabla \tilde{\zeta}|) \leq \delta(\alpha^*)(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}.$$

(iv) Small second-order scalar products: for any polynomial $P(y)$ and integers $0 \leq k \leq 2$, $0 \leq l \leq 1$, there exists $C > 0$ such that

$$\left(\int |\tilde{\varepsilon}||P(y)|(|\frac{d^k \tilde{\zeta}}{dy^k}| + |\frac{d^l}{dy^l}\frac{\partial \tilde{\zeta}}{\partial b}|)\right)^2 \leq \Gamma_b^C(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}).$$

Moreover, the term induced by the time dependence of A is controlled by

$$\begin{aligned} & \left|\frac{A_s}{A^2}\left\{(\varepsilon_2 - \tilde{\zeta}_{im}, (y \cdot \nabla \chi\left(\frac{y}{A}\right)\zeta_{re})_1) - (\varepsilon_1 - \tilde{\zeta}_{re}, (y \cdot \nabla \chi\left(\frac{y}{A}\right)\zeta_{im})_1)\right\}\right| \\ \leq \quad & \delta(\alpha^*)\left(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}\right) + \Gamma_b^{1+z_0}. \end{aligned}$$

(v) Formally cubic terms: Let $J(\varepsilon)$, $\tilde{R}_{1,2}(\varepsilon)$, $\tilde{H}_b$ be given respectively by (B.5), (B.3), (B.4) and (B.2). Then

$$\begin{aligned} & \left|\int J(\varepsilon)\right| + \left|(\tilde{R}_1(\varepsilon), \Sigma_1)\right| + \left|(\tilde{R}_2(\varepsilon), \Theta_1)\right| + \left|\tilde{H}_b(\varepsilon, \varepsilon)\right| \\ \leq \quad & \delta(\alpha^*)\left(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}\right) + \Gamma_b^{1+z_0}. \end{aligned}$$

(vi) Linear degenerate scalar products:

$$\begin{aligned} & |(\varepsilon_1, \tilde{L})| + |(\varepsilon_2, \tilde{K})| + |(\tilde{\varepsilon}_1, (Q^{\frac{4}{N}}\tilde{\zeta}_{re})_1)| + |(\tilde{\varepsilon}_2, (Q^{\frac{4}{N}}\tilde{\zeta}_{im})_1)| \\ \leq \quad & \delta(\alpha^*)(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}. \end{aligned}$$

(vii) Cut-off χ_A induced estimates:

$$\int |\varepsilon|(|F| + |y \cdot \nabla F|) \leq C\Gamma_b^{\frac{1}{2}}\left(\int_A^{2A} |\varepsilon|^2\right)^{\frac{1}{2}}.$$

See Appendix C for the proof.

Step 3: Estimate of degenerate scalar products.

We claim similarly as in [16] from the conservation of energy, the Galilean invariance and the choice of the orthogonality conditions:

$$(\varepsilon_1, Q)^2 \leq \delta(\alpha^*)(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0} + C(\lambda^2 E_0)^2, \tag{4.13}$$

$$(\varepsilon_2, \nabla Q)^2 \leq \delta(\alpha^*)(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}, \tag{4.14}$$

$$\begin{aligned}(\tilde{\varepsilon}_1, Q)^2 + (\tilde{\varepsilon}_1, |y|^2 Q)^2 + (\tilde{\varepsilon}_1, yQ)^2 + (\tilde{\varepsilon}_2, Q_1)^2 + (\tilde{\varepsilon}_2, Q_2)^2 + (\tilde{\varepsilon}_2, \nabla Q)^2 \\ \leq \delta(\alpha^*)(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0} + C(\lambda^2 E_0)^2,\end{aligned} \tag{4.15}$$

(4.16)

$$\begin{aligned}\left|\tilde{\gamma}_s \left\{(\varepsilon_1 - \tilde{\zeta}_{re}, (\Sigma + \tilde{\zeta}_{re})_1) + (\varepsilon_2 - \tilde{\zeta}_{im}, (\Theta + \tilde{\zeta}_{im})_1)\right\} - \frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1, L_+ Q_2)(\tilde{\varepsilon}_1, Q_1)\right| \\ \leq \delta(\alpha^*)(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}) + \Gamma_b^{1+z_0}.\end{aligned}$$

Proof of (4.13). Let us recall the conservation of the energy:

$$\begin{aligned}2\,(\varepsilon_1, Q) = & -2(\varepsilon_1, \Sigma - Q + b\Theta_1 - Re(\Psi)) - 2\,(\varepsilon_2, \Theta - b\Sigma_1 - Im(\Psi)) \\ + & \int |\nabla\varepsilon|^2 + 2E(\tilde{Q}_b) - 2\lambda^2 E_0 - \frac{2}{2+\frac{4}{N}}\int J(\varepsilon) \\ - & \int (\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1^2 - \int (\frac{4\Theta^2}{N|\tilde{Q}_b|^2} + 1)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2^2 - \int \frac{8\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1\varepsilon_2,\end{aligned} \tag{4.17}$$

with $J(\varepsilon)$ given by (B.5). From the definition of Ψ_b (2.5) and (2.2):

$$\left|2E(\tilde{Q}_b)\right| = \left|Re\int (\Psi_b)_1 \overline{\tilde{Q}_b}\right| \leq e^{-2(1-C\eta)\frac{\theta(2)}{b}} \leq \Gamma_b^{1-C\eta}.$$

We then first estimate using the estimates of Step 2:

$$\begin{aligned}(\varepsilon_1, Q)^2 \leq \delta(\alpha^*)\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}}\right) \\ + \Gamma_b^{1+z_0} + C(\lambda^2 E_0)^2 + \left(\int |\varepsilon||\Psi|\right)^2.\end{aligned}$$

The last term in the above expression is controlled using (4.10), (4.11):

$$\begin{aligned}\left(\int |\varepsilon||\Psi|\right)^2 & \leq \Gamma_b^{1-C\eta}\left(\int_{R_b^-}^{R_b} |\varepsilon|\right)^2 \\ & \leq \Gamma_b^{1-C\eta}\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + \Gamma_b^{1+z_0},\end{aligned}$$

and (4.13) follows.

(4.14) follows directly from the zero momentum assumption (2.22), and (4.15) follows from the orthogonality conditions (2.26), (2.27), (2.28), (2.29) and (2.19).

Proof of (4.16). Observe that

$$\begin{aligned}
& \tilde{\gamma}_s\left\{(\varepsilon_1-\tilde{\zeta}_{re},(\Sigma+\tilde{\zeta}_{re})_1)+(\varepsilon_2-\tilde{\zeta}_{im},(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
= & \left(\tilde{\gamma}_s-\frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)\right)\left\{(\varepsilon_1-\tilde{\zeta}_{re},(\Sigma+\tilde{\zeta}_{re})_1)+(\varepsilon_2-\tilde{\zeta}_{im},(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
+ & \frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)\left\{(\varepsilon_1-\tilde{\zeta}_{re},(\Sigma-Q+\tilde{\zeta}_{re})_1)+(\varepsilon_2-\tilde{\zeta}_{im},(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
+ & \frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)(\tilde{\varepsilon}_1,Q_1).
\end{aligned}$$

We now remark from (5) that

$$\begin{aligned}
& \left|\tilde{\gamma}_s-\frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)\right|\leq\left|\tilde{\gamma}_s-\frac{1}{|Q_1|_2^2}(\varepsilon_1,L_+Q_2)\right|+\frac{1}{|Q_1|_2^2}\left|(\tilde{\zeta}_1,L_+Q_2)\right|\\
\leq & \ \delta(\alpha^*)(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|})^{\frac{1}{2}}+\Gamma_b^{\frac{1}{2}+z_0}.
\end{aligned}$$

(4.16) now follows from the estimates of Step 2.

Step 4. Conclusion of the proof.

Now let $f_1(s)$ be given by (4.7), and remark that using the orthogonality condition (2.29) and the estimates of Step 2 and (4.16), (4.8) yields so far:

$$\begin{aligned}
(4.18)\quad & \{f_1\}_s\geq H(\tilde{\varepsilon},\tilde{\varepsilon})-\frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)(\tilde{\varepsilon}_1,Q_1)-C\lambda^2E_0\\
+ & \ \{(\varepsilon_1,(Re(F))_1)+(\varepsilon_2,(Im(F))_1)\}-\left\{(\tilde{\zeta}_{re},(Re(F)_1))+(\tilde{\zeta}_{im},(Im(F)_1))\right\}\\
- & \ \delta(\alpha^*)(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|}+\lambda^2|E_0|)-\Gamma_b^{1+z_0}.
\end{aligned}$$

The first-order remainder term in (4.18) is estimated according to (vii) of Step 2:

$$|(\varepsilon_1,(Re(F))_1)+(\varepsilon_2,(Im(F))_1)|\leq C\Gamma_b^{\frac{1}{2}}\left(\int_A^{2A}|\varepsilon|^2\right)^{\frac{1}{2}}\leq\delta_2\Gamma_b+\frac{1}{\delta_2}\int_A^{2A}|\varepsilon|^2$$

where $\delta_2>0$ is a small parameter to be fixed later.

Now, as a consequence of the Spectral Property stated in the introduction, we have obtained, using an algebraic cancellation in [16] and the proof of Lemma 8, the following property (true for any function $\tilde{\varepsilon}$ in H^1):

$$\begin{aligned}
& H(\tilde{\varepsilon},\tilde{\varepsilon})-\frac{1}{|Q_1|_2^2}(\tilde{\varepsilon}_1,L_+Q_2)(\tilde{\varepsilon}_1,Q_1)\geq\tilde{\delta}_1(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|})\\
- & \ \frac{1}{\tilde{\delta}_1}\left((\tilde{\varepsilon}_1,Q)^2+(\tilde{\varepsilon}_1,|y|^2Q)^2+(\tilde{\varepsilon}_1,yQ)^2+(\tilde{\varepsilon}_2,Q_1)^2+(\tilde{\varepsilon}_2,Q_2)^2+(\tilde{\varepsilon}_2,\nabla Q)^2\right),
\end{aligned}$$

for some universal constant $\tilde{\delta}_1>0$. Using now estimate (4.15), we rewrite (4.18) for $\alpha^*>0$ small enough:

$$\begin{aligned}
(4.19)\qquad & \{f_1\}_s\geq\frac{\tilde{\delta}_1}{2}(\int|\nabla\tilde{\varepsilon}_1|^2+\int|\tilde{\varepsilon}_1|^2e^{-|y|})-C\lambda^2E_0-\delta_2\Gamma_b-\frac{1}{\delta_2}\int_A^{2A}|\varepsilon|^2\\
& -\left\{(\tilde{\zeta}_{re},(Re(F))_1)+(\tilde{\zeta}_{im},(Im(F))_1)\right\}.
\end{aligned}$$

We now inject in (4.19) the following flux type of computation: for some universal constant $c = c(N) > 0$, the following holds:

$$-\left\{(\tilde{\zeta}_{re}, (Re(F))_1) + (\tilde{\zeta}_{im}, (Im(F))_1)\right\} > c\Gamma_b. \tag{4.20}$$

This estimate is a key point in the proof and has already been used in [16]. For the sake of completeness, we briefly recall the proof below. (4.6) now follows by taking $\delta_2 = \frac{c}{2}$.

Proof of (4.20). We recall a computation from [16]:

$$\left\{(\tilde{\zeta}_{re}, (Re(F))_1) + (\tilde{\zeta}_{im}, (Im(F))_1)\right\} = Re\left(\int \tilde{\zeta}\overline{F_1}\right) = -Re\left(\int_{A\leq|y|\leq 2A} F\overline{\tilde{\zeta}_1}\right)$$
$$= -Re\left(\int_{A\leq|y|\leq 2A} (\Delta\tilde{\zeta} - \tilde{\zeta} + ib\tilde{\zeta}_1)\overline{\tilde{\zeta}_1}\right) = -Re\left(\int_{A\leq|y|\leq 2A} (\Delta\tilde{\zeta} - \tilde{\zeta})\overline{\tilde{\zeta}_1}\right).$$

We now estimate each term separately. The dominant term is

$$\frac{1}{|Vol(S_{N-1})|} Re\left(\int_{A\leq|y|\leq 2A} \tilde{\zeta}\overline{\tilde{\zeta}_1}\right) = Re\left(\int_A^{2A} \tilde{\zeta}\overline{\left(\frac{N}{2}\tilde{\zeta} + r\tilde{\zeta}'\right)} r^{N-1}dr\right)$$
$$= \frac{N}{2}\int_A^{2A} |\tilde{\zeta}|^2 r^{N-1}dr + \left[\frac{1}{2}|\tilde{\zeta}|^2 r^N\right]_A^{2A} - \frac{N}{2}\int_A^{2A} |\tilde{\zeta}|^2 r^{N-1}dr \leq -\frac{1}{4}\Gamma_b,$$

where we used in a crucial way estimate (2.17). We claim that the other term is negligible. Indeed,

$$\begin{aligned}
&\frac{1}{|Vol(S_{N-1})|} Re\left(\int_{A\leq|y|\leq 2A} \Delta\tilde{\zeta}\overline{\tilde{\zeta}_1}\right)\\
= \ & Re\left(\int_A^{2A} \frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}\tilde{\zeta}')\overline{(\frac{N}{2}\tilde{\zeta} + r\tilde{\zeta}')} r^{N-1}dr\right)\\
= \ & Re\left[r^{N-1}\tilde{\zeta}'\overline{(\frac{N}{2}\tilde{\zeta} + r\tilde{\zeta}')}\right]_A^{2A} - Re\left(\int_A^{2A} \tilde{\zeta}'\overline{(\frac{N}{2}\tilde{\zeta}' + \tilde{\zeta}' + r\tilde{\zeta}'')} r^{N-1}dr\right)\\
= \ & Re\left[r^{N-1}\tilde{\zeta}'\overline{(\frac{N}{2}\tilde{\zeta} + r\tilde{\zeta}')}\right]_A^{2A} - (\frac{N}{2}+1)\int_A^{2A} r^{N-1}|\tilde{\zeta}'|^2 dr - \frac{1}{2}\left[r^N|\tilde{\zeta}'|^2\right]_A^{2A}\\
& + \frac{N}{2}\int_A^{2A} r^{N-1}|\tilde{\zeta}'|^2 dr.
\end{aligned}$$

We now use estimates (2.17), (2.15) and (2.18) to get

$$\left|\frac{1}{|Vol(S_{N-1})|} Re\left(\int_{A\leq|y|\leq 2A} \Delta\tilde{\zeta}\overline{\tilde{\zeta}_1}\right)\right| \leq C\frac{\Gamma_b}{bA^2} \leq \Gamma_b^{1+Ca}$$

for b small enough, and the conclusion follows from (4.3). This ends the proof of Lemma 6.

4.2. **L^2 dispersion at infinity in space.** The next step of our analysis is to control the L^2 flux term $\int_A^{2A}|\varepsilon|^2$ which appears in (4.6) by a derivative in time. This is achieved thanks to the computation of the flux of the L^2-norm which escapes at infinity, and the corresponding control will be purely of linear nature under the assumption on the sign of b (4.1).

We introduce a radial nonnegative cut-off function $\phi(r)$ such that $\phi(r)=0$ for $r\leq\frac{1}{2}$, $\phi(r)=1$ for $r\geq 3$, $\frac{1}{4}\leq\phi'(r)\leq\frac{1}{2}$ for $1\leq r\leq 2$, $\phi'(r)\geq 0$. We then set

$$\phi_A(s,r)=\phi\left(\frac{r}{A(s)}\right),$$

$A(s)$ given by (4.3), and thus:

$$\begin{cases}\phi_A(r)=0 \quad\text{for}\quad 0\leq r\leq\frac{A}{2},\\ \frac{1}{4A}\leq\phi_A'(r)\leq\frac{1}{2A}\quad\text{for}\quad A\leq r\leq 2A,\\ \phi_A(r)=1\quad\text{for}\quad r\geq 3A,\\ \phi_A'(r)\geq 0,\quad 0\leq\phi_A(r)\leq 1.\end{cases}$$

We now claim the following dispersive control at infinity in space:

Lemma 7 (L^2 dispersion at infinity in space)**.** *For some universal constant $C>0$ and $s\geq 0$, the following holds:*

$$\left\{\int\phi_A|\varepsilon|^2\right\}_s\geq\frac{b}{400}\int_A^{2A}|\varepsilon|^2-\frac{C}{b^2}\lambda^2E_0-\Gamma_b^{1+z_0}-\Gamma_b^{\frac{a}{2}}\int|\nabla\varepsilon|^2. \tag{4.21}$$

Remark 7. The range of parameters η,a,b is the same as in Lemma 6; see Remark 6.

Proof of Lemma 7. Take the inner product of (2.31) with $\phi_A\varepsilon_1$ and of (2.32) with $\phi_A\varepsilon_2$ and integrate by parts. Note that the supports of $(\tilde{Q}_b,\Psi_b)$ and ϕ_A are disjoint. We thus get a linear identity decoupled from the nonlinear dynamic $|y|\leq\frac{2}{b}$:

$$\begin{aligned}\frac{1}{2}\left\{\int\phi_A|\varepsilon|^2\right\}_s&=\frac{1}{2}\int\frac{\partial\phi_A}{\partial s}|\varepsilon|^2+\frac{b}{2}\int y\cdot\nabla\phi_A|\varepsilon|^2+Im\left(\int\nabla\phi_A\cdot\nabla\varepsilon\overline{\varepsilon}\right)\\ &-\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int y\cdot\nabla\phi_A|\varepsilon|^2-\frac{1}{2}\frac{x_s}{\lambda}\cdot\int\nabla\phi_A|\varepsilon|^2.\end{aligned} \tag{4.22}$$

First observe from the choice of ϕ:

$$10\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{A}\int y\cdot\nabla\phi\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{10}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{40}\int_A^{2A}|\varepsilon|^2. \tag{4.23}$$

The main term in (4.22) is

$$\frac{b}{2}\int y\cdot\nabla\phi_A|\varepsilon|^2\geq\frac{b}{20}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2.$$

We then estimate from (4.3) and (2.39):

$$\begin{aligned}\int\frac{\partial\phi_A}{\partial s}|\varepsilon|^2&=\frac{-A_s}{A^2}\int y\cdot\nabla\phi\left(\frac{y}{A}\right)|\varepsilon|^2=2a\theta(2)\frac{b_s}{Ab^2}\int y\cdot\nabla\phi\left(\frac{y}{A}\right)|\varepsilon|^2\\ &\geq\frac{a\delta_0}{b^2}\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}-C\lambda^2E_0-\Gamma_b^{1-C\eta}\right)\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2.\end{aligned} \tag{4.24}$$

Next:

$$\left|Im\left(\int \nabla\phi_A\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right| = \left|Im\left(\int \frac{1}{A}\nabla\phi\left(\frac{y}{A}\right)\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right|$$

$$\leq \frac{1}{A}\left(\int|\nabla\varepsilon|^2\right)^{\frac12}\left(\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\right)^{\frac12} \leq \frac{40}{bA}\int|\nabla\varepsilon|^2 + \frac{b}{40}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2,$$

$$\leq \frac{b}{100}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2 + \Gamma_b^{\frac{a}{2}}\int|\nabla\varepsilon|^2. \tag{4.25}$$

From (5), we estimate:

$$\left|\frac{x_s}{\lambda}\cdot\int\nabla\phi_A|\varepsilon|^2\right| \leq \frac{C}{A}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2 \leq \Gamma_b^{\frac{a}{2}}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2. \tag{4.26}$$

Similarly from (2.36):

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)\int y\cdot\nabla\phi_A|\varepsilon|^2\right| \tag{4.27}$$

$$\leq C\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2e^{-|y|} + \Gamma_b^{1-C\eta} + C\lambda^2|E_0|\right)\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2.$$

Observe that the right-hand side of (4.27) is controlled thanks to (4.24) in the range of parameters.

Injecting (4.23), (4.24), (4.25), (4.26) and (4.27) into (4.22) yields

$$\left\{\int\phi_A|\varepsilon|^2\right\}_s \geq \frac{b}{100}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2 - \frac{C}{b^2}\lambda^2E_0 - \Gamma_b^{1+z_0} - \Gamma_b^{\frac{a}{2}}\int|\nabla\varepsilon|^2,$$

and (4.21) follows from the definition of ϕ'. This concludes the proof of Lemma 7.

4.3. **L^2 dispersive constraint on the solution.** In this subsection, we derive the dispersive estimate needed for the proof of Theorems 2 and 3. Virial estimate (4.6) corresponds to nonlinear interactions on compact sets; L^2 linear estimate (4.21) measures the interactions with the linear dynamic at infinity. We now couple these two facts through the invariance of the L^2-norm, which is a global information in space.

Proposition 4 (Lyapounov functional in H^1)**.** *For some universal constant $C>0$ and for $s\geq 0$ assuming (4.1), the following holds:*

$$\{\mathcal{J}\}_s \leq -Cb\left(\Gamma_b + \int|\nabla\tilde{\varepsilon}|^2 + \int|\tilde{\varepsilon}|^2e^{-|y|} + \int_A^{2A}|\varepsilon|^2 - \lambda^2E_0\right) + C\frac{\lambda^2}{b^2}E_0, \tag{4.28}$$

with

$$\mathcal{J}(s) = \left(\int|\tilde{Q}_b|^2 - \int Q^2\right) + 2(\varepsilon_1,\Sigma) + 2(\varepsilon_2,\Theta) + \int(1-\phi_A)|\varepsilon|^2 \tag{4.29}$$

$$- \frac{\delta_1}{800}\left(b\tilde{f}_1(b) - \int_0^b\tilde{f}_1(v)dv + b\{(\varepsilon_2,(\tilde{\zeta}_{re})_1) - (\varepsilon_1,(\tilde{\zeta}_{im})_1)\}\right),$$

where

$$\tilde{f}_1(b) = \frac{b}{4}|y\tilde{Q}_b|_2^2 + \frac12 Im\left(\int y\cdot\nabla\tilde{\zeta}\overline{\tilde{\zeta}}\right). \tag{4.30}$$

Remark 8. Here, the range of parameters is more restricted and yields: there exist $\eta^*, a^*, C_0 > 0$ such that $\forall 0 < \eta < \eta^*$, $\forall 0 < a < a^*$ such that $a > C_0\eta$, there exists $b^*(\eta, a)$ such that $\forall |b| \leq b^*(\eta, a)$, and the estimates of Proposition 4 hold with universal constants.

Remark 9. The gain is that we now have a Lyapounov function $\mathcal{J}$ in H^1. Remark that in a regime when ε is small compared to b in a certain sense, $\mathcal{J} \sim \int |\tilde{Q}_b|^2 - \int Q^2 \sim b^2$ from (2.8) and (4.28) forces b to decay.

Proof of Proposition 4. Multiply (4.6) by $\frac{\delta_1 b}{800}$ and sum with (4.21). We get

$$\begin{aligned}\left\{\int \phi_A|\varepsilon|^2\right\}_s + \frac{\delta_1 b}{800}\{f_1\}_s &\geq \frac{\delta_1^2 b}{800}\Big(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}\Big) + \frac{b}{800}\int_A^{2A} |\varepsilon|^2 \\ &+ \frac{c\delta_1 b}{1000}\Gamma_b - \frac{C}{b^2}\lambda^2 E_0 - \Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2, \end{aligned} \tag{4.31}$$

f_1 given by (4.7). We first integrate the left-hand side of (4.31) by parts in time:

$$\begin{aligned} b\{f_1\}_s &= \left\{b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv + b\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\}\right\}_s \\ &- b_s\left\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\right\}, \end{aligned}$$

$\tilde{f}_1$ given by (4.30), and (4.31) now yields

$$\begin{aligned} &\left\{\int \phi_A|\varepsilon|^2 + \frac{\delta_1}{800}\left[b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv + b\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\}\right]\right\}_s \\ \geq\ & \frac{\delta_1^2 b}{800}\Big(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} + \int_A^{2A} |\varepsilon|^2\Big) + \frac{c\delta_1 b}{1000}\Gamma_b - \frac{C}{b^2}\lambda^2 E_0 \\ -\ & \Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2 + \frac{\delta_1}{800}b_s\left\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\right\}. \end{aligned}$$

We now inject the conservation of the L^2-norm:

$$\int |\varepsilon|^2 + \int |\tilde{Q}_b|^2 + 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) = \int |u_0|^2.$$

Writing $\int \phi_A|\varepsilon|^2 = \int |\varepsilon|^2 - \int (1-\phi_A)|\varepsilon|^2$, we compute

$$\left\{\int \phi_A|\varepsilon|^2\right\}_s = -\left\{\left(\int |\tilde{Q}_b|^2 - \int Q^2\right) + 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1-\phi_A)|\varepsilon|^2\right\}_s.$$

Thus, we get

$$\begin{aligned} \{-\mathcal{J}\}_s \geq \frac{\delta_1^2 b}{800}\Big(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} + \int_A^{2A} |\varepsilon|^2\Big) + \frac{c\delta_1 b}{100}\Gamma_b - \frac{C}{b^2}\lambda^2 E_0 \\ - \Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2 + \frac{\delta_1}{800}b_s\left\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\right\}, \end{aligned} \tag{4.32}$$

where $\mathcal{J}$ is given by (4.29). We now have

$$\Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2 \leq \Gamma_b^{\frac{a}{2}}\left(\Gamma_b^{1-C\eta} + \int |\nabla\tilde{\varepsilon}|^2\right) \leq \Gamma_b^{1+\frac{a}{4}} + \Gamma_b^{\frac{a}{2}}\int |\nabla\tilde{\varepsilon}|^2$$

from the assumption $a > C\eta$. Next, we estimate from (2.36):

$$\left|b_s\left\{(\varepsilon_2,(\tilde{\zeta}_{re})_1)-(\varepsilon_1,(\tilde{\zeta}_{im})_1)\right\}\right| \leq \Gamma_b^{\frac{1}{2}-C\eta}\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}+C\lambda^2|E_0|\right)$$
$$\leq \quad \Gamma_b^{\frac{1}{4}}\left(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|}+\lambda^2|E_0|\right)+\Gamma_b^{1+z_0}.$$

Injecting these two estimates into (4.32) yields (4.28). This concludes the proof of Proposition 4.

5. Proof of the main results of the paper

5.1. **Proof of the log-log lower bound.** This subsection is devoted to the proof of Theorem 2.

Let $u_0 \in \mathcal{B}_{\alpha^*}$ and let $u(t)$ be the corresponding solution to (1.1) which blows up in finite time $0 < T < +\infty$. Using the Galilean invariance and the alternative (1.9) and (1.10), we may reduce to the case $u_0 \in \mathcal{O}$ satisfying (2.22) and

$$|\nabla u(t)|_{L^2} \leq C_1^*\left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}.$$

We recall from Proposition 3 that $u(t)$ admits a decomposition as in Lemma 4 for $s \geq s_0$ with

$$\forall s \geq s_0, \quad b(s) > 0, \quad \lambda(s) \leq e^{-\frac{1}{\Gamma_b^{d_0}}} \quad \text{and} \quad \lambda(s) \leq 2\lambda(s_0). \tag{5.1}$$

In particular from the sign condition on $b(s)$, Proposition 4 holds. Parameters $\eta, a > 0$ are fixed so that Proposition 4 holds true for $\alpha^* > 0$ small enough. The proof will now follow in two steps:

(1) The starting point is to obtain from (4.28) *asymptotic* information as $s \to +\infty$. In particular, we obtain another proof of the stability of the blow-up profile of [16]; more precisely:

$$b(s) \to 0 \quad \text{as} \quad s \to +\infty,$$

and for some time $s_1 > 0$ large enough it follows that

$$\forall s \geq s_1, \quad \int|\nabla\varepsilon(s)|^2+\int|\varepsilon(s)|^2e^{-|y|} \leq \Gamma_{b(s)}^{1-Ca}.$$

(2) On the basis of these uniform asymptotic controls, (4.28) now yields a differential inequality for b:

$$b_s \leq -C\Gamma_b,$$

which with the techniques introduced in [14] will yield the log-log lower bound.

Proposition 5 (Asymptotic stability in $\mathcal{O}$ revised)**.** *We have:*

$$b(s) \to 0 \quad \textit{as} \quad s \to +\infty, \tag{5.2}$$

and for some time $s_1 > 0$ large enough it follows that

$$\forall s \geq s_1, \quad \int|\nabla\varepsilon(s)|^2+\int|\varepsilon(s)|^2e^{-|y|} \leq \Gamma_{b(s)}^{1-Ca}. \tag{5.3}$$

Remark 10. From this proposition, $\int |\nabla\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \to 0$ as $s \to +\infty$, which corresponds to the stability of the blow-up profile Q in $\dot H^1$ first proved in [16]. In addition, the last estimate is essentially optimal as the size of the radiation ζ_b in $\dot H^1$ is $\int |\nabla\zeta_b|^2 \sim \Gamma_b$.

Proof of Proposition 5. **Step 1**: Asymptotic stability.

We first claim (5.2):

$$b(s) \to 0 \quad \text{as} \quad s \to +\infty.$$

Note that this could in fact be assumed from results in [16]. Indeed, from (4.28), one has $\forall s \geq s_0$:

$$\int_{s_0}^{s} b\Gamma_b \leq \mathcal{J}(s_0) - \mathcal{J}(s) + C\int_{s_0}^{s} \frac{\lambda^2}{b^2}E_0.$$

Remark then that from its explicit value (4.29),

$$|\mathcal{J}| \leq C.$$

Moreover, from the assumption $u_0 \in \mathcal{O}$, the uniform estimate (2.42) holds and we rewrite: $\forall s \geq s_0$,

$$b \geq \frac{C}{\log|\log(\lambda)|}, \tag{5.4}$$

so that using $\frac{ds}{dt} = \frac{1}{\lambda^2}$ and (5.1):

$$\begin{aligned}\int_{s_0}^{+\infty} \frac{\lambda^2}{b^2}|E_0|ds &\leq C\int_{s_0}^{s} \lambda^2|E_0|(\log|\log(\lambda)|)^2 ds = C|E_0|\int_{t_0}^{T}(\log|\log(\lambda(\tau))|)^2 dt\\ &\leq C|E_0|\int_{t_0}^{T}(\log|\log(|\nabla u(t)|_{L^2})|)^2 dt < +\infty,\end{aligned}$$

where we used the upper bound on the blow-up rate (1.9) in the last step. Remark that this estimate is not needed in the negative energy case. We conclude that

$$\int_{s_0}^{+\infty} b\Gamma_b < +\infty.$$

Since $|b_s| \leq C$ from (2.36), (5.2) follows. From (5.2), we now have with (2.42):

$$|E_0|\lambda^2(s) \leq e^{-e^{\frac{C}{b(s)}}} \leq e^{-\frac{1}{2\Gamma_b^{d_0}}} \leq \Gamma_b^2,$$

and thus (4.28) can be rewritten: $\forall s \geq s_1$,

$$\{\mathcal{J}\}_s \leq -Cb\Gamma_b + C\frac{\lambda^2}{b^2}E_0 \leq -\frac{1}{2}Cb\Gamma_b \leq 0. \tag{5.5}$$

Step 2. Estimate on $\mathcal{J}$.

From energy-type estimates, we claim the following control on $\mathcal{J}$: $\forall s \geq s_1$,

$$\mathcal{J}(s) - f_2(b(s)) \left\{\begin{array}{l} \geq -\Gamma_b^{1-Ca} + \frac{1}{C}\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2 e^{-|y|}\right),\\ \leq CA^2\log A\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2 e^{-|y|}\right) + \Gamma_b^{1-Ca},\end{array}\right. \tag{5.6}$$

where f_2, given by

$$f_2(b) = \left(\int|\tilde{Q}_b|^2 - \int Q^2\right) - \frac{\delta_1}{800}\left(b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv\right),$$

satisfies

$$0 < \frac{df_2}{db^2}_{|b^2=0} < +\infty. \tag{5.7}$$

Remark 11. The main point is that the dominant term in ε in (4.29) is $2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1-\phi_A)|\varepsilon|^2$, which also appears in the conservation of energy. Now the orthogonality conditions on ε, which in our analysis are related to the virial relation that is dispersion on compact sets, are also adapted to the coercive structure of the linearized energy; see also [14]. Indeed, let

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}$$

be the linear operators close to Q; recall that the quadratic form which appears when developing the energy close to Q is $(L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) - \int |\varepsilon|^2$. We then have from [11] (the proof there is for $N = 1$, but the same proof holds in any dimension assuming that the kernel of L_+ is reduced to $\langle \partial_{x_1} Q, \dots, \partial_{x_n} Q\rangle$. This fact has been proved by Maris in [10] under an assumption (H5) which can be checked in [12]):

Lemma 8 ([11])**.** *There exists a universal constant $\delta_2 > 0$ such that the following holds true. Let $\mu_+ < 0$ be the lowest eigenvalue of L_+ and ϕ_+ be a corresponding eigenvector with $\|\phi_+\|_{L^2} = 1$. Then for all $v = v_1 + iv_2 \in H^1$,*

$$(L_+v_1, v_1) + (L_-v_2, v_2) \geq \delta_2 |v|^2_{H^1} - \frac{1}{\delta_2}\left\{(v_1, \phi_+)^2 + (v_1, \nabla Q)^2 + (v_2, Q)^2\right\}. \tag{5.8}$$

We then claim from this estimate the following: there exists a universal constant $\delta_3 > 0$ such that

$$\begin{aligned}(L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) - \int \phi_A |\varepsilon|^2 &\geq \delta_3 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) \\ &- \frac{1}{\delta_3}\left\{(\varepsilon_1, Q)^2 + (\varepsilon_1, |y|^2 Q)^2 + (\varepsilon_1, yQ)^2 + (\varepsilon_2, Q_2)^2\right\}.\end{aligned} \tag{5.9}$$

See Appendix D for the proof of (5.9).

Proof of (5.6). We rewrite (4.29):

$$\begin{aligned}\mathcal{J}(s) - f_2(b(s)) &= 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1-\phi_A)|\varepsilon|^2 \\ &- \frac{b\delta_1}{800}\left\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\right\}.\end{aligned} \tag{5.10}$$

From the estimates on $\tilde{\zeta}$ of Lemma 2, the choice of A (4.3) and (4.10), (4.11), we have

$$\begin{aligned}\left|(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\right| &\leq \Gamma_b^{\frac{1}{2} - C\eta}\left(\int_0^A |\varepsilon|^2\right)^{\frac{1}{2}} \\ &\leq A^2 \log A \Gamma_b^{1-C\eta} + C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) \\ &\leq \Gamma_b^{1-Ca} + C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right).\end{aligned}$$

The other term in (5.10) is estimated from the conservation of energy (4.17):

$$\begin{aligned}
& 2(\varepsilon_1,\Sigma)+2(\varepsilon_2,\Theta)+\int(1-\phi_A)|\varepsilon|^2 \\
= & \int(1-\phi_A)|\varepsilon|^2+2(\varepsilon_1,Re(\Psi))+2(\varepsilon_2,Im(\Psi))+2E(\tilde{Q}_b)-2\lambda^2E_0 \\
- & \frac{2}{2+\frac{4}{N}}\int J(\varepsilon)+\int|\nabla\varepsilon|^2-\int(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1^2 \\
- & \int(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2^2-\int\frac{8\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1\varepsilon_2,
\end{aligned}$$

with $J(\varepsilon)$ given by (B.5), which can be rewritten as

$$\begin{aligned}
& 2(\varepsilon_1,\Sigma)+2(\varepsilon_2,\Theta)+\int(1-\phi_A)|\varepsilon|^2=(L_+\varepsilon_1,\varepsilon_1)+(L_-\varepsilon_2,\varepsilon_2)-\int\phi_A|\varepsilon|^2 \\
& +2(\varepsilon_1,Re(\Psi))+2(\varepsilon_2,Im(\Psi))+2E(\tilde{Q}_b)-2\lambda^2E_0-\frac{2}{2+\frac{4}{N}}\int J(\varepsilon) \\
& -\int\left[(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1)|\tilde{Q}_b|^{\frac{4}{N}}-(\frac{4}{N}+1)Q^{\frac{4}{N}}\right]\varepsilon_1^2 \\
& -\int\left[(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1)|\tilde{Q}_b|^{\frac{4}{N}}-Q^{\frac{4}{N}}\right]\varepsilon_2^2-\int\frac{8\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1\varepsilon_2.
\end{aligned}$$

We first estimate for $s\geq s_1$:

$$\begin{aligned}
& |(\varepsilon_1,Re(\Psi))|+|(\varepsilon_2,Im(\Psi))|+\left|E(\tilde{Q}_b)\right|+\lambda^2|E_0| \\
& \leq\Gamma_b^{1-Ca}+\Gamma_b^a\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).
\end{aligned}$$

The cubic term $\int|J(\varepsilon)|$ and the rest of the quadratic form are controlled similarly as in the proof of (v) of Step 2 of Lemma 6 by $\delta(\alpha^*)\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)+\Gamma_b^{1+z_0}$. We thus obtain

$$\begin{aligned}
& \left|\mathcal{J}(s)-f_2(b(s))-\left\{(L_+\varepsilon_1,\varepsilon_1)+(L_-\varepsilon_2,\varepsilon_2)-\int\phi_A|\varepsilon|^2\right\}\right| \\
\leq & \ \delta(\alpha^*)\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)+\Gamma_b^{1-Ca}.
\end{aligned}$$

We now are in a position to conclude the proof of (5.6). The upper bound follows from (4.10), (4.11), which indeed yields

$$\int(1-\phi_A)|\varepsilon|^2\leq CA^2\log A\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right). \tag{5.11}$$

For the lower bound, we use the elliptic estimate (5.9) together with estimate (4.15) inherited from our choice of orthogonality conditions. This ends the proof of (5.6). To prove (5.7), we first have from (4.30) and the estimates of Lemma 2:

$$\left|\frac{d}{db^2}\left(b\tilde{f}_1(b)-\int_0^b\tilde{f}_1(v)dv\right)_{|b^2=0}\right|<+\infty.$$

We now use in a fundamental way the *supercritical mass property for the modified* *profile* $\tilde{Q}_b$. Indeed, the constant $\delta_1 > 0$ in (4.29) can be chosen as small as we want, so that (2.8) ensures (5.7).

Step 3: Pointwise control of ε by b.

We now turn to the proof of (5.3). Letting $s \geq s_1$, we consider two cases.
Case 1: $b_s(s) \leq 0$. Then (5.3) follows from (2.39) and $a > C\eta$.
Case 2: $b_s(s) > 0$. First observe from $b(s) \to 0$ as $s \to +\infty$ that we may assume without loss of generality that $s \in (s_1^*, s_2^*)$ with

$$b_s(s_1^*) = b_s(s_2^*) = 0 \ \text{ and } \ b_s \geq 0 \ \text{ in } \ [s_1^*, s_2^*],$$

and thus:

$$b(s_1^*) \leq b(s) \leq b(s_2^*). \tag{5.12}$$

Case 1 applies at $s = s_1^*$:

$$\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)(s_1) \leq \Gamma_{b(s_1)}^{1-Ca}. \tag{5.13}$$

Next, from the Lyapounov monotonicity property (5.5):

$$\mathcal{J}(s_2^*) \leq \mathcal{J}(s) \leq \mathcal{J}(s_1^*),$$

and injecting (5.6) into this inequality, we get

$$\begin{aligned} & f_2(b(s)) + \frac{1}{C}\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)(s) \leq \mathcal{J}(s) + \Gamma_{b(s)}^{1-Ca} \\ \leq \quad & f_2(b(s_1^*)) + \Gamma_{b(s)}^{1-Ca} + \Gamma_{b(s_1^*)}^{1-Ca} \leq f_2(b(s_1^*)) + 2\Gamma_{b(s)}^{1-Ca}, \end{aligned}$$

where we used (4.3) and (5.12) for the two last steps. Now from the monotonicity property (5.7) of $f_2(b)$,

$$f_2(b(s_1^*)) \leq f_2(b(s)),$$

and (5.3), follows. This concludes the proof of Proposition 5.

We now are in position to conclude the proof of Theorem 2.

Proof of Theorem 2. We proceed in several steps, following the analysis in [15], and refer to section 5.2 of [15] for further details.

Step 1. Pointwise uniform control of the scaling parameter.

We claim the following uniform estimate for $s \geq s_2$ large enough:

$$Cb(s) \leq \frac{1}{\log|\log\lambda(s)|}, \tag{5.14}$$

for some universal constant $C > 0$. Note that this estimate expresses the converse inequality in (2.42) obtained in [15] for the proof of the log-log upper bound.

Proof of (5.14). From (5.3) and (5.6), we have for $s \geq s_2$:

$$\frac{b^2}{C} \leq \mathcal{J} \leq Cb^2 \tag{5.15}$$

for some universal constant $C > 0$, and thus from (5.5), $g = \sqrt{\mathcal{J}}$ satisfies the following differential inequality:

$$g_s = \frac{(\mathcal{J})_s}{2\sqrt{\mathcal{J}}} \leq -\frac{C}{g} g e^{-\frac{C}{g}} \leq -\frac{1}{2C} g^2 e^{-\frac{2C}{g}}, \quad \text{i.e.} \quad \left(e^{\frac{2C}{g}}\right)_s \geq 1,$$

and thus we have for $s \geq s_2$ large enough:

$$e^{\frac{C}{b(s)}} \geq s, \quad \text{i.e.} \quad b(s) \leq C g(s) \leq \frac{C}{\log(s)}. \tag{5.16}$$

Now observe from (2.42) that control (2.36) may be written with (5.3):

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + \Gamma_b^{1-C\eta} \leq \Gamma_b^{\frac{1}{2}}$$

and thus

$$\frac{b}{2} \leq -\frac{\lambda_s}{\lambda} \leq 2b. \tag{5.17}$$

We integrate this in time on $[s_2, s]$ using $b > 0$ to derive for $s \geq s_2$ large enough:

$$-\log(\lambda(s)) \leq 2\int_{s_2}^{s} b(\tau) d\tau \leq C \int_{s_2}^{s} \frac{1}{\log(\tau)} d\tau \leq Cs.$$

Taking the log of this inequality and injecting (5.16) yields:

$$\log|\log(\lambda(s))| \leq C \log(s) \leq \frac{C}{b(s)},$$

and (5.14) is proved.

Step 2. Conclusion.

We differ here from the strategy of integration on doubling time intervals of the norm used in [15], which was designed to control the oscillations of λ induced by the variations of ε. Here at the level of our analysis, using the pointwise estimate on ε (5.3), we derive a pointwise differential inequality for $\lambda(t)$: $\forall t \geq t_2$,

$$\frac{1}{C} \leq -\left(\lambda^2 \log|\log(\lambda)|\right)_t \leq C, \tag{5.18}$$

for some universal constant $C > 0$.

Indeed, we compute:

$$\begin{aligned} -\left(\lambda^2 \log|\log(\lambda)|\right)_t &= -\lambda\lambda_t \log|\log(\lambda)| \left(2 + \frac{1}{|\log(\lambda)| \quad \log|\log(\lambda)|}\right) \\ &= -\frac{\lambda_s}{\lambda} \log|\log(\lambda)| \left(2 + \frac{1}{|\log(\lambda)| \quad \log|\log(\lambda)|}\right). \end{aligned}$$

Thus from (5.17), we have

$$\frac{b}{4} \log|\log(\lambda)| \leq -\left(\lambda^2 \log|\log(\lambda)|\right)_t \leq 4b \log|\log(\lambda)|,$$

and (5.18) now follows from (5.4) (which is of course equivalent to the log-log upper bound; see [15]) and (5.14).

Integrating (5.18) in time t yields:

$$\forall t \geq t_2, \quad \frac{T-t}{C} \leq \lambda^2(t) \log|\log(\lambda(t))| \leq C(T-t).$$

The lower bound

$$|\nabla u(t)|_{L^2} \geq C\sqrt{\frac{\log|\log(T-t)|}{T-t}}$$

now easily follows for t close enough to T. This concludes the proof of Theorem 2.

5.2. **Time dependence of the geometrical parameters for the log-log dynamics.** In this subsection, we collect the dispersive controls obtained for the log-log dynamics and the exact size of the different parameters in order to make precise the constants in the asymptotics. We thus let $u_0 \in \mathcal{O}$ satisfy (2.22), where $u(t)$ is the corresponding solution to (1.1) with blow-up time $0 < T < +\infty$. At this stage, we have proved that there exist η_0 small enough, $a_0 = C_0\eta_0$ both fixed with $C_0 > 0$ universal such that the results of the previous subsections apply. This yields for t close to T:

$$\frac{1}{C}\sqrt{\frac{\log|\log(T-t)|}{T-t}} \leq |\nabla u(t)|_{L^2} \leq C\sqrt{\frac{\log|\log(T-t)|}{T-t}},$$

$$\frac{1}{C}\sqrt{\frac{T-t}{\log|\log(T-t)|}} \leq \lambda(t) \leq C\sqrt{\frac{T-t}{\log|\log(T-t)|}},$$

$$\frac{1}{C\log|\log(T-t)|} \leq b(t) \leq \frac{C}{\log|\log(T-t)|},$$

$$\frac{1}{C}|\log(T-t)|\log|\log(T-t)| \leq s(t) \leq C|\log(T-t)|\log|\log(T-t)|,$$

for some universal constant $C > 0$.

We claim the following at blow-up time for the geometrical parameters $\lambda(t), b(t)$ and rescaled time $s(t)$:

Proposition 6 (Exact law for the geometrical parameters)**.** *We have as $t \to T$:*

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}\sqrt{\frac{T-t}{\log|\log(T-t)|}} \to \frac{1}{\sqrt{2\pi}},$$

$$\lambda(t)\sqrt{\frac{\log|\log(T-t)|}{T-t}} \to \sqrt{2\pi}, \tag{5.19}$$

$$b(t)\log|\log(T-t)| \to \frac{1}{\pi}, \tag{5.20}$$

$$\frac{s(t)}{|\log(T-t)|\log|\log(T-t)|} \to \frac{1}{2\pi}. \tag{5.21}$$

Remark 12. The quantity $|\nabla Q|_{L^2}$, or equivalently $|Q|_{L^2}$, may be computed explicitly for $N = 1$. It is an open problem to compute it for $N \geq 2$.

Proof of Proposition 6. The strategy is as follows: for any parameter $(\eta, a = C_0\eta)$, $0 < \eta < \eta_0$, we introduce a new decomposition with parameters $(b(\eta), \lambda(\eta), \varepsilon(\eta))$ and rescaled time $s(\eta)$. Such a decomposition is well defined on a time interval $[t(\eta), T)$ from the asymptotic stability. We then apply the previous scheme of proof to estimate these parameters and improve the various constants appearing in their time-dependent laws. The last step is to compare this decomposition with the one for fixed parameters η_0, a_0, and then conclude the proof.

Step 1. New decomposition for t close to T.

From the asymptotic stability of Proposition 5,

$$b(\eta_0, s) \to 0 \quad \text{as} \quad s \to +\infty$$

and for s large enough:

$$\int |\nabla \varepsilon(\eta_0, s)|^2 + \int |\varepsilon(\eta_0, s)|^2 e^{-|y|} \leq \Gamma_{b(\eta_0,s)}^{1-Ca_0}.$$

Now pick a small parameter $0 < \eta < \eta_0$ and let $a = C_0\eta$. We have from the above controls that $u(t)$ admits on $[\bar{t}(\eta), T)$ a decomposition close to $\tilde{Q}_b(\eta)$ as in Lemma 4 with new parameters $b(\eta,t), \lambda(\eta,t), \varepsilon(\eta,t)$ and rescaled time $s(\eta,t)$. Indeed, the existence of the decomposition is only related to the size of $b(\eta_0,t)$ and the local L^2-norm of $\varepsilon(\eta_0,t)$ both of which go to zero as $t \to T$. Applying the previous analysis to this decomposition, we recover all previous estimates on $[t(\eta), T)$ and in particular (5.3):

$$\forall t \in [t(\eta), T), \quad \int |\nabla \varepsilon(\eta,t)|^2 + \int |\varepsilon(\eta,t)|^2 e^{-|y|} \leq \Gamma_{b(\eta,t)}^{1-Ca}. \tag{5.22}$$

Step 2. Estimates for the η-decomposition.

From (2.7) and (2.21), we have

$$e^{-\frac{\pi(1+C\eta)}{b(\eta)}} \leq \Gamma_{b(\eta)} \leq e^{-\frac{\pi(1-C\eta)}{b(\eta)}}.$$

On the one hand, virial estimate (2.39) with uniform control (2.42) yields: $\forall s \geq s(\eta)$,

$$\{b(\eta)\}_s \geq -\Gamma_{b(\eta)}^{1-C\eta}. \tag{5.23}$$

The integration of the differential inequality (5.23) yields for $s \geq s(\eta)$ large enough:

$$\left(e^{\frac{\pi(1-C\eta)}{b(\eta)}}\right)_s \leq 1, \quad \text{i.e.} \quad b(\eta,s) \geq \frac{\pi(1-C\eta)}{\log(s)}. \tag{5.24}$$

On the other hand, for the upper bound, we recall (4.28):

$$\{\mathcal{J}(\eta)\}_s \leq -Cb(\eta)\Gamma_{b(\eta)}.$$

Injecting (5.3) into the explicit form of $\mathcal{J}(\eta)$ (4.29), we have using (5.7):

$$\lim_{s\to+\infty} \frac{\mathcal{J}(\eta,s)}{b^2(\eta,s)} = \lim_{b\to 0} \frac{f_2(\eta,b)}{b^2(\eta)} = C^* > 0$$

for some universal constant $C^* > 0$. The differential inequality (4.28) can now be rewritten as

$$\left(e^{\frac{\pi(1+C\eta)\sqrt{C^*}}{\sqrt{\mathcal{J}(\eta)}}}\right)_s \geq 1, \quad \text{i.e.} \quad \sqrt{\mathcal{J}(\eta)} \leq \frac{\pi(1+C\eta)\sqrt{C^*}}{\log(s)}.$$

Together with (5.24), we thus have obtained: $\forall \eta > 0$, there exists $s(\eta) \in [0, +\infty)$ such that

$$\forall t \in [s(\eta), +\infty), \quad \frac{\pi(1-C\eta)}{\log(s)} \leq b(\eta,s) \leq \frac{\pi(1+C\eta)}{\log(s)}. \tag{5.25}$$

We now turn to the proof of (5.20). We first have from the control of the blow-up speed both from above and below the crude estimate:

$$s(\eta,t)=\int_0^t\frac{d\tau}{\lambda^2(\eta,\tau)}\quad\text{so that}\quad \frac{1}{C}|\log(T-t)|\le s(\eta,t)\le C|\log(T-t)|.$$

Taking the log of this inequality and injecting it into (5.25) yields: $\forall t\in[t(\eta),T)$,

$$\frac{\pi(1-C\eta)}{\log|\log(T-t)|}\le b(\eta,t)\le\frac{\pi(1+C\eta)}{\log|\log(T-t)|}. \tag{5.26}$$

Now estimate (5.17) may be improved using (2.36) and (5.3) to yield: $\forall t\in[t(\eta),T)$,

$$(1-\eta)b(\eta,t)\le-\frac{\{\lambda(\eta)\}_s}{\lambda(\eta)}=-\lambda(\eta)\left\{\lambda(\eta)\right\}_t\le(1+\eta)b(\eta,t),$$

and thus with (5.26):

$$\forall t\in[t(\eta),T),\quad \frac{\pi(1-C\eta)}{\log|\log(T-t)|}\le-\frac{1}{2}\left\{\lambda^2(\eta)\right\}_t\le\frac{\pi(1+C\eta)}{\log|\log(T-t)|}.$$

Integrating this in time t yields: $\forall t\in[t(\eta),T)$,

$$(1-C\eta)\sqrt{2\pi}\le\lambda(\eta,t)\sqrt{\frac{\log|\log(T-t)|}{T-t}}\le(1+C\eta)\sqrt{2\pi}. \tag{5.27}$$

Step 3. Comparison of the decompositions and conclusion.

(5.19) now easily follows. Indeed, for all $\eta>0$, there exists $t(\eta)\in[0,T)$ such that (5.27) holds. Now for each $0<\eta<\eta_0$, we have

$$\int|\nabla u(t)|^2=\frac{1}{\lambda^2(\eta,t)}\int|\nabla\left(\tilde{Q}_{b(\eta,t)}+\varepsilon(\eta,t)\right)|^2$$

so that from the asymptotic stability:

$$|\nabla u(t)|_{L^2}\lambda(\eta,t)\to|\nabla Q|_{L^2}\quad\text{as}\quad t\to T. \tag{5.28}$$

From (5.27), we conclude:

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\to\frac{1}{\sqrt{2\pi}}\quad\text{as}\quad t\to T.$$

Applying again (5.28) with $\eta=\eta_0$, we obtain (5.19). (5.21) now follows from an explicit computation from (5.19):

$$\begin{aligned}s(t)=\int_0^t\frac{d\tau}{\lambda^2(\tau)}&\sim\frac{1}{2\pi}\int_0^t\frac{\log|\log(T-\tau)|}{T-\tau}d\tau\\&\sim\frac{1}{2\pi}|\log(T-t)|\log|\log(T-t)|\quad\text{as}\quad t\to T.\end{aligned}$$

For (5.20), we get, comparing the two decompositions for $t\in[t(\eta),T)$,

$$|b(t)-b(\eta,t)|\le\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)^{\frac{1}{2}}\le\Gamma_{b(t)}^{\frac{1}{4}}\le\frac{1}{|\log(T-t)|^C},$$

and (5.26) yields the result. This ends the proof of Proposition 6.

Let us conclude this subsection with some dispersive controls on ε which clarify the size of the main (larger-order) terms in the problem. For $\nu > 0$ small, we let

$$(\eta, a) = (\nu^2, \nu),$$

which enters the range of parameters given by Remark 8, and define the geometrical decomposition of Lemma 4. We also let $A(t)$ be the cut-off parameter given by (4.3).

Proposition 7 (Dispersive control on ε)**.** *There exist universal constants $C_1, C_2 > 0$ such that for all $\nu > 0$ small enough, there exists $s(\nu) > 0$ such that $\forall s \in [s(\nu), +\infty)$, we have:*
(i) Pointwise control on ε:

$$\int |\nabla \varepsilon(\eta, s)|^2 + \int |\varepsilon(\eta, s)|^2 e^{-|y|} \leq \Gamma_{b(\eta,s)}^{1-C\nu}. \tag{5.29}$$

(ii) Time-averaged control of ε:

$$\int_s^{+\infty} \left(\int |\nabla \tilde{\varepsilon}(\eta, s)|^2 + \int |\varepsilon(\eta, s)|^2 e^{-|y|} + \Gamma_{b(\eta,s)} \right) ds \leq \frac{C_1}{|\log(s)|}, \tag{5.30}$$

$$\frac{C_2}{|\log(s)|} \leq \int_s^{+\infty} \left(\int_A^{2A} |\varepsilon(\eta, s)|^2 \right) ds \leq \frac{C_1}{|\log(s)|}. \tag{5.31}$$

Remark 13. These estimates hold for η_0, $a_0 = C_0 \eta_0$. For further use, we need to work with the full range of parameters of Remark 8. Moreover, in (5.31), we have from the proof a similar control of $\int_{KA}^{K'A} |\varepsilon(s)|^2$ for any constants $K' > K > 0$, the constants C_1, C_2 depending then on K, K'.

Remark 14. Estimate (5.31) together with (5.30) gives a precise localization property in the space of the L^2 mass inherited from the universal radiative structure.

Proof of Proposition 7. We omit in the proof the dependence of the parameters on η.
(i) (5.29) is (5.22).
(ii) (5.30) is a consequence of the dispersive control (4.28) with (5.15): for some universal constant $C > 0$ and for $s \geq s(a)$ large enough, the following holds:

$$Cb \left(\Gamma_b + \int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} + \int_A^{2A} |\varepsilon|^2 \right) \leq - \{\mathcal{J}\}_s$$

and

$$\frac{b^2(s)}{C} \leq \mathcal{J}(s) \leq C b^2(s).$$

We divide this differential inequality by $\sqrt{\mathcal{J}}$ and get

$$C \left(\Gamma_b + \int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} + \int_A^{2A} |\varepsilon|^2 \right) \leq - \left\{ \sqrt{\mathcal{J}} \right\}_s$$

and thus

$$\begin{aligned} \int_s^{+\infty} \left(\int |\nabla \tilde{\varepsilon}(s)|^2 + \int |\tilde{\varepsilon}(s)|^2 e^{-|y|} + \int_A^{2A} |\varepsilon(s)|^2 + \Gamma_{b(s)} \right) ds &\leq C \sqrt{\mathcal{J}(s)} \\ &\leq C b(s) \leq \frac{C}{|\log(s)|}. \end{aligned}$$

The control of the local L^2-norm of ε only follows from (2.19).

We are left with proving the lower bound in (5.31). Recall (4.6):

$$\{f_1(s)\}_s \geq \delta_1 \left(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} \right) + \frac{c}{2}\Gamma_b - \frac{1}{\delta_1} \int_A^{2A} |\varepsilon|^2,$$

where f_1 given by (4.7) satisfies from (5.3):

$$f_1(s) \geq Cb(s)$$

for some $C > 0$ universal. Integrating the previous differential inequality in time together with the lower bound $b(s) \geq \frac{C}{\log(s)}$ yields the result. This ends the proof of Proposition 7.

5.3. **Proof of blowup for H^1 zero energy solutions.** This subsection is devoted to the proof of Theorem 3, which will follow from the existence of the Lyapounov functional $\mathcal{J}$ of Proposition 4 together with the study of flux exchanges in L^2.

Let $u_0 \in \mathcal{B}_{\alpha^*}$ with

$$E(u_0) = Im(\int \nabla u_0 \overline{u_0}) = 0.$$

Our aim is to prove that $u(t)$, the corresponding solution to (1.1), blows up for $t > 0$ and for $t < 0$ at the exact log-log rate. Remark that in [16], this result was proved for $u_0 \in \Sigma$, using the pseudo-conformal symmetry, which allows us to reduce the problem to proving classification results for blow-up solutions satisfying

$$|u(t)|^2 \rightharpoonup \left(\int |u_0|^2 \right) \delta_{x=0}.$$

In our situation where u_0 is not in Σ, this approach fails. Here, we present a direct proof of this fact based essentially on the dispersive estimate (4.28) together with information on the sign of $b(s)$.

Step 1. Recall of previous analysis in [16].

We first briefly recall the analysis in [16]. An elementary but fundamental observation is that Lemma 4 holds on *the whole time interval* $(-T_-, T_+)$ *from* $E_0 = 0$. We thus introduce the same geometrical decomposition as before, i.e. Lemma 4, and the estimates of Lemma 5 still hold true for $s \in \mathbf{R}$. As pointed out in [16], the main difference concerns the results of Proposition 3:

- Virial estimate (2.39) still holds true on the whole time interval existence $(-T_-, T_+)$ or equivalently $s \in \mathbf{R}$: there exist universal constants $\delta_0 > 0$, $C > 0$, such that $\forall s \in \mathbf{R}$:

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \Gamma_b^{1-C\eta}. \tag{5.32}$$

- In addition, if u_0 is not the soliton up to scaling, phase and translation invariances, then there is *at most* one $s_0 \in \mathbf{R}$ such that $b(s_0) = 0$. If s_0 exists, then

$$b(s) > 0 \text{ for } s > s_0 \text{ and } b(s) < 0 \text{ } for \text{ } s < s_0.$$

> In this case, the analysis in [16] ensures that the solution blows up on both sides in time with the upper bound (1.9), and thus from the previous section, the lower bound (1.12) also holds.

The existence of s_0 has been proved in [14] for strictly negative energy solutions and is unknown in our case where a regime with $|\nabla u(t)|_{L^2} \to 0$ is possible a priori. This is the regime we need to rule out.

In other words, to prove Theorem 3, we may argue by contradiction and (by possibly considering $\overline{u}(-t)$, which is also a solution) assume:

$$\forall s \in \mathbf{R}, \quad b(s) > 0. \tag{5.33}$$

In this situation, $u(t)$ blows up on the right in finite time $0 < T < +\infty$ and is globally defined on the left.

To obtain a contradiction, we study the qualitative behavior of the solution as $s \to -\infty$. The existence of the Lyapounov functional $\mathcal{J}$ allows us to recover the asymptotic stability in the following sense:

$$\int_{-\infty}^{0} \left(\int |\nabla \varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \right) ds < +\infty.$$

Using this dispersive control, we then show that the sign of the parameter b near $-\infty$ forces an ingoing radiative behavior for ε which implies:

$$\int |\varepsilon(s)|^2 \to 0 \quad \text{as} \quad s \to -\infty.$$

From the conservation laws, this means that u is a soliton up to the symmetries, which is a contradiction.

Step 2. Asymptotic stability: $\int |\nabla \varepsilon(s)|^2 + \int |\varepsilon|^2(s) e^{-|y|} \to 0$ as $s \to -\infty$.

From the existence of the Lyapounov functional $\mathcal{J}$ of section 4.3, let us prove first:

$$b(s) \to 0 \quad \text{as} \quad s \to -\infty. \tag{5.34}$$

From (5.33), Proposition 4 applies and we have: $\forall s \in \mathbf{R}$,

$$\{\mathcal{J}\}_s \leq -Cb\Gamma_b, \tag{5.35}$$

with $\mathcal{J}$ given by (4.29). The main difference with the analysis of the previous subsection is that the asymptotic control (5.15) no longer holds a priori, and thus (5.35) does not provide a differential inequality for b. Indeed, we will see that in this regime, $\mathcal{J}(s)$ converges to a nonzero constant as $s \to -\infty$ and thus cannot be compared to b^2 from (5.34).

Nevertheless, observe from its definition that

$$\forall s \in \mathbf{R}, \quad |\mathcal{J}(s)| \leq C,$$

and thus the integration in time of (5.35) ensures

$$\int_{-\infty}^{0} b\Gamma_b ds < +\infty.$$

Now from (5.33) and $|b_s| \leq C$ from (2.36), we get (5.34).

We now claim from the virial identity that this implies:

$$\int |\nabla\varepsilon(s)|^2 + \int |\varepsilon|^2(s)e^{-|y|} \to 0 \quad \text{as} \quad s \to -\infty. \tag{5.36}$$

Proof of (5.36). First, take the inner product of (2.31) with $(\varepsilon_1 e^{-|y|})$ and of (2.32) with $(\varepsilon_2 e^{-|y|})$. Then one evaluates: $\forall s$,

$$\left|\left(\int |\varepsilon|^2 e^{-|y|}\right)_s\right| \leq C. \tag{5.37}$$

Arguing by contradiction, we assume that for some sequence $s_n \to -\infty$:

$$\int |\varepsilon|^2(s_n)e^{-|y|} \geq c_0 > 0. \tag{5.38}$$

Then from (5.37), there exists $\tau_0 > 0$ such that

$$\forall n \geq 0, \quad \forall s \in [s_n, s_n + \tau_0], \quad \int |\varepsilon|^2(s)e^{-|y|} \geq \frac{c_0}{2}. \tag{5.39}$$

Integrating the virial identity (2.39) on $[s_n, s_n+\tau_0]$ with the lower bound (5.39), we get

$$0 < \frac{c_0\delta_0\tau_0}{2} \leq \delta_0 \int_{s_n}^{s_n+\tau_0} \int |\varepsilon|^2 e^{-|y|} \leq C(1+\tau_0) \sup_{s\in[s_n,s_n+\tau_0]} b(s),$$

and a contradiction to (5.38) follows as the right-hand side of this inequality goes to zero as $n \to +\infty$ from (5.34), and thus $\int |\varepsilon|^2 e^{-|y|} \to 0$ as $s \to -\infty$. (5.36) now follows from the conservation of energy (4.17).

Step 3. Decay rate of ε in $\dot{H}^1$.

We claim here:

$$\int_{-\infty}^{0} \left(\int |\nabla\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|}\right) ds < +\infty. \tag{5.40}$$

Using again the Lyapounov function, we first compare ε and Γ_b. Note that (5.6) with (5.7) yields the lower bound:

$$\mathcal{J} \geq \frac{1}{C}\left(b^2 + \int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) > 0.$$

Thus from the uniform bound on $\mathcal{J}$ and the monotonicity (5.35) of $\mathcal{J}$, we have

$$\mathcal{J}(s) \to l_- > 0 \quad \text{as} \quad s \to -\infty.$$

From (5.6),

$$CA^2(s)\log A(s)\left(\int |\nabla\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|}\right) \geq \frac{l_-}{2} - Cb^2 \geq \frac{l_-}{4} > 0$$

for s large enough, and thus from (4.3): there exists $s_3 < 0$ such that

$$\forall s < s_3, \quad \int |\nabla\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \geq \Gamma_{b(s)}^{\frac{1}{4}}. \tag{5.41}$$

Now the virial control (5.32) can now be rewritten with the lower bound (5.41):

$$\forall s < s_3, \;\; b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \Gamma_b^{1-C\eta} \geq \frac{\delta_0}{2} \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right),$$

and (5.40) follows.

Step 4. L^2 control on ε and conclusion.

We now prove that the upper bound (5.40) with the sign assumption (5.33) imply a global control of ε in L^2, which will yield a contradiction.

For $D \geq 2$, consider a cut-off function $\chi_D(r)$, which definition depends on the dimension: for $N = 1$ or $N \geq 3$, we let $\chi_D(r) = \chi\left(\frac{r}{D}\right)$ for some smooth cut-off function $\chi(r) = 1$ for $0 \leq r \leq 1$, $\chi(r) = 0$ for $r \geq 2$, $\chi'(r) \leq 0$; for $N = 2$, we impose a slightly different structure to take into account the logarithmic growth in (4.11) and (4.12), and let

$$\chi_D(r) = \begin{cases} 1 \;\; \text{for} \;\; r \leq D, \\ 2\left(\frac{\log D}{\log r} - \frac{1}{2}\right) \;\; \text{for} \;\; D \leq r \leq D^2, \\ 0 \;\; \text{for} \;\; r \geq D^2. \end{cases} \tag{5.42}$$

We now claim:

$$\forall D \geq 2, \;\; \forall s \leq s_3, \;\; \left\{ \int \chi_D |\varepsilon|^2 \right\}_s \leq C \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right), \tag{5.43}$$

for some universal constant $C > 0$. Let us conclude the proof of Theorem 3 assuming (5.43). Indeed, integrate (5.43) in time on $[\tilde{s}, s]$ where $\tilde{s} < s < s_3$. We get using (5.40): $\forall D \geq 2$, $\forall \tilde{s} < s < s_3$,

$$\begin{aligned} \int \chi_D |\varepsilon(s)|^2 &\leq \int \chi_D |\varepsilon(\tilde{s})|^2 + C \int_{\tilde{s}}^{s} \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) \\ &\leq \int \chi_D |\varepsilon(\tilde{s})|^2 + C \int_{-\infty}^{s} \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right). \end{aligned}$$

Letting $\tilde{s} \to -\infty$, we have from the asymptotic stability (5.36):

$$\forall D \geq 2, \;\; \int \chi_D |\varepsilon(\tilde{s})|^2 \to 0 \;\; \text{as} \;\; \tilde{s} \to -\infty,$$

and thus: $\forall D \geq 2$, $\forall s < s_3$,

$$\int \chi_D |\varepsilon(s)|^2 \leq C \int_{-\infty}^{s} \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right).$$

Letting $D \to +\infty$, we conclude:

$$\forall s < s_3, \;\; \int |\varepsilon(s)|^2 \leq C \int_{-\infty}^{s} \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right),$$

and from (5.40):

$$\int |\varepsilon(s)|^2 \to 0 \;\; \text{as} \;\; s \to -\infty.$$

Injecting this into the conservation of the L^2-norm, we get

$$\int |u_0|^2 = \lim_{s\to-\infty}\int |u(s)|^2 = \int Q^2.$$

Here, we use the L^2 conservation to get an estimate on the size of the solution. From $E(u_0)=0$ and the variational characterization of Q, u is a soliton up to some fixed scaling, translation and phase parameters, which is a contradiction.

Proof of (5.43). Take the inner product of (2.31) with $\chi_D\varepsilon_1$ and of (2.32) with $\chi_D\varepsilon_2$ and integrate by parts. We get:

$$\begin{aligned}(5.44)\quad \frac{1}{2}\left\{\int \chi_D|\varepsilon|^2\right\}_s &= \frac{b}{2}\int y\cdot\nabla\chi_D|\varepsilon|^2 + Im\left(\int \nabla\chi_D\cdot\nabla\varepsilon\overline{\varepsilon}\right)\\ &+\int \frac{4|\tilde{Q}_b|^{\frac{4}{N}}}{N|\tilde{Q}_b|^2}\chi_D\left[(\Sigma^2-\Theta^2)\varepsilon_1\varepsilon_2+\Sigma\Theta(\varepsilon_2^2-\varepsilon_1^2)\right]+(\varepsilon_1,\chi_D Im(\Psi))\\ &-(\varepsilon_2,\chi_D Re(\Psi)) - b_s\left\{(\varepsilon_1,\chi_D\frac{\partial\Sigma}{\partial b})+(\varepsilon_2,\chi_D\frac{\partial\Theta}{\partial b})\right\}+\tilde{\gamma}_s\{(\varepsilon_1,\chi_D\Theta)\\ &-(\varepsilon_2,\chi_D\Sigma)\}+\left(\frac{\lambda_s}{\lambda}+b\right)\left\{(\varepsilon_2,\chi_D\Theta_1)+(\varepsilon_1,\chi_D\Sigma_1)-\frac{1}{2}\int y\cdot\nabla\chi_D|\varepsilon|^2\right\}\\ &+\frac{x_s}{\lambda}\cdot\left\{(\varepsilon_1,\chi_D\nabla\Sigma)+(\varepsilon_2,\chi_D\nabla\Theta)-\frac{1}{2}\int\nabla\chi_D|\varepsilon|^2\right\}\\ &+(R_1(\varepsilon),\chi_D\varepsilon_2)-(R_2(\varepsilon),\chi_D\varepsilon_1).\end{aligned}$$

The dominant term in (5.44) is the flux term which has a sign:

$$\forall s<s_3,\quad \frac{b}{2}\int y\cdot\nabla\chi_D|\varepsilon|^2\leq 0,$$

and this will imply the result.

We now estimate all other terms in (5.44). The most dangerous term is the momentum type of term $Im\left(\int\nabla\chi_D\cdot\nabla\varepsilon\overline{\varepsilon}\right)$ for which we need an estimate *independent of* D. It is a consequence of the L^2 estimate:

$$(5.45)\qquad \int|\nabla\chi_D|^2|\varepsilon|^2\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

Indeed, we have in dimension $N=1$ or $N\geq 3$ from (4.10):

$$\int|\nabla\chi_D|^2|\varepsilon|^2=\frac{1}{D^2}\int\left|\nabla\chi(\frac{y}{D})\right|^2|\varepsilon|^2\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

In dimension $N=2$, we have from (5.42) and (4.12):

$$\begin{aligned}\int|\nabla\chi_D|^2|\varepsilon|^2 &= \log^2 D\int_{D\leq|y|\leq D^2}\frac{|\varepsilon|^2}{|y|^2\log^4|y|}\leq\frac{C}{\log^2 D}\int_{1\leq|y|\leq D^2}\frac{|\varepsilon|^2}{|y|^2}\\ &\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).\end{aligned}$$

We conclude from (5.45):

$$\left|Im\left(\int\nabla\chi_D\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right|\leq C\left(\int|\nabla\varepsilon|^2\right)^{\frac12}\left(\int|\nabla\chi_D|^2|\varepsilon|^2\right)^{\frac12}$$
$$\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

The boundary terms are estimated thanks to (5.41):

$$|(\varepsilon_1,\chi_D Im(\Psi))|+|(\varepsilon_2,\chi_D Re(\Psi))|\leq\Gamma_b^{\frac12-C\eta}\left(\int_{|y|\leq\frac4b}|\varepsilon|^2\right)^{\frac12}$$
$$\leq\quad\Gamma_b^{\frac14}\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)^{\frac12}\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)^{\frac32}.$$

For the quadratic terms, we first estimate from the decay estimates on $\tilde{Q}_b$ of Proposition 1 and (4.10):

$$\left|\int\frac{4|\tilde{Q}_b|^{\frac4N}}{N|\tilde{Q}_b|^2}\chi_D\left[(\Sigma^2-\Theta^2)\varepsilon_1\varepsilon_2+\Sigma\Theta(\varepsilon_2^2-\varepsilon_1^2)\right]\right|\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

Next, we have from (C.3):

$$\int|R(\varepsilon)\chi_D\varepsilon|\leq\quad\begin{cases}C(\int|\varepsilon|^3e^{-\left(\frac4N-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}}+\int|\varepsilon|^{2+\frac4N})\ \text{ for }\quad N\leq3,\\ C\int|\varepsilon|^{2+\frac4N}\ \text{ for }\ N\geq4.\end{cases}$$

This estimate is the same as that of $\int|J(\varepsilon)|$ in the proof of (v) in Appendix C, and we estimate:

$$\int|R(\varepsilon)\chi_D\varepsilon|\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

We now rewrite the estimates on the modulation parameters (2.36) and (5) with $E_0=0$ and (5.41):

$$\left|\frac{\lambda_s}{\lambda}+b\right|+|b_s|\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right),\tag{5.46}$$

$$\left|\frac{x_s}{\lambda}\right|+|\tilde{\gamma}_s|\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)^{\frac12}.\tag{5.47}$$

The decay estimate (2.2) on $\tilde{Q}_b$ together with (2.38) yield the control of the scalar product terms. The two last terms involving L^2-type norms are estimated as follows:

$$\left|\frac{\lambda_s}{\lambda}+b\right|\left|\int y\cdot\nabla\chi_D|\varepsilon|^2\right|\leq C|y\nabla\chi_D|_{L^\infty}\left|\frac{\lambda_s}{\lambda}+b\right|\left(\int|\varepsilon|^2\right)$$
$$\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right),$$

where the last step follows from (5.46); we next have from (5.45) and (5.47):

$$\left|\frac{x_s}{\lambda}\cdot\int\nabla\chi_D|\varepsilon|^2\right|\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)^{\frac12}\left(\int|\nabla\chi_D|^2|\varepsilon|^2\right)^{\frac12}\left(\int|\varepsilon|^2\right)^{\frac12}$$
$$\leq C\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right).$$

This concludes the proof of estimate (5.43) and of Theorem 3.

Appendix

A. Proof of Lemma 2

Lemma 2 has been proved in [16], Appendix E, except for (2.18) and (2.20). We now detail the proof of these two estimates.

Recall from Appendix E in [16] that the outgoing radiation is built as

$$\zeta_b = \xi_b e^{-i\frac{b|y|^2}{4}},$$

where ξ_b solves the following linear equation viewed as an ODE on $r \in [0, +\infty)$:

$$\xi_b'' + \frac{N-1}{r}\xi_b' - \xi_b + \frac{b^2 r^2}{4}\xi_b = \tilde{\Psi}_b \quad \text{and} \quad \int \left|\nabla\left(\xi_b e^{-ib\frac{r^2}{4}}\right)\right|^2 < +\infty, \tag{A.1}$$

where $\tilde{\Psi}_b = \Psi_b e^{i\frac{b|y|^2}{4}}$. More precisely, ξ_b can be written for r large:

$$\forall r \geq \frac{4}{b}, \quad \xi_b(r) = \nu_b Z_{out}(r) \quad \text{with} \quad \Gamma_b = |\nu_b|^2,$$

and where Z_{out} is a solution to the homogeneous equation (A.1) with the following asymptotic behavior as $r \to +\infty$:

$$Z_{out}(r) = \phi_b(r) e^{\frac{i}{b}\Theta(br)}, \quad \phi_b(r) = \frac{1}{r^{\frac{N}{2}}\left(1 - \left(\frac{2}{br}\right)^2\right)},$$

where

$$\Theta(r) = \int_2^r ds\sqrt{\frac{s^2}{4} - 1} + b^2\tilde{\Theta}(r), \quad \|r\tilde{\Theta}(r)\|_{\mathcal{C}^2(r\geq 3)} \leq C.$$

Proof of (2.18). Observe from its explicit value that $\Theta(r)$ has the following asymptotic development:

$$\Theta(r) = \frac{r^2}{4} - \log(r) + \Theta_\infty + \hat{\Theta}(r) + b^2\tilde{\Theta}(r) \quad \text{with} \quad \|\hat{\Theta}\|_{\mathcal{C}^i(r\geq 3)} \leq \frac{C}{r^{1+i}}, \quad i = 1, 2,$$

for some universal constants Θ_∞, C. We thus get the following formula for ζ_b:

$$\forall r \geq R_b^2, \quad \zeta_b(r) = \nu_b \phi_b(r) e^{\frac{i}{b}\overline{\Theta}(br)} \quad \text{with} \quad \overline{\Theta}(r) = -\log(r) + \Theta_\infty + \hat{\Theta}(r) + b^2\tilde{\Theta}(r).$$

Taking the derivative in r of this expression yields (2.18) for $r \geq R_b^2$.

Proof of (2.20). Note that going back to the proof of estimate (2.10), we may exhibit the following bound in terms of Γ_b:

$$\left|\frac{\partial \tilde{\Psi}_b}{\partial b}\right|_{\mathcal{C}^1} \leq \Gamma_b^{\frac{1}{2} - C\eta}.$$

Going back to the explicit formula for the computation of ξ_b in terms of $\tilde{\Psi}_b$, see (229) of [16], we differentiate it with respect to b and get (2.20).

B. Proof of the virial identity (4.8)

This appendix is devoted to the proof of the algebraic virial identity (4.8). This relation is a generalization of the one obtained in [16] in a more specific context, and thus the proof is very similar to the one given in Appendix D in [16]. We briefly recall the main steps of the computation.

Take the inner product of (2.32) with $(\Sigma+\tilde{\zeta}_{re})_1$ and of (2.31) with $-(\Theta+\tilde{\zeta}_{im})_1$ and sum the obtained identities to get:

$$
\begin{aligned}
&\left\{\frac{b}{4}|y\tilde{Q}_b|_2^2+\frac{1}{2}Im\left(\int y\cdot\nabla\tilde{\zeta}\overline{\tilde{\zeta}}\right)+(\varepsilon_2,\Sigma_1+(\tilde{\zeta}_{re})_1)-(\varepsilon_1,\Theta_1+(\tilde{\zeta}_{im})_1)\right\}_s\\
&=-\left(M_+(\varepsilon)+b(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2),\Sigma_1+(\tilde{\zeta}_{re})_1\right)\\
&-\left(M_-(\varepsilon)-b(\frac{N}{2}\varepsilon_1+y\cdot\nabla\varepsilon_1),\Theta_1+(\tilde{\zeta}_{im})_1\right)\\
&+\left\{(\varepsilon_2-\tilde{\zeta}_{im},\frac{\partial}{\partial s}(\Sigma+\tilde{\zeta}_{re})_1)-(\varepsilon_1-\tilde{\zeta}_{re},\frac{\partial}{\partial s}(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
&-\tilde{\gamma}_s\left\{(\varepsilon_1+\Sigma,\Sigma_1+(\tilde{\zeta}_{re})_1)+(\varepsilon_2+\Theta,\Theta_1+(\tilde{\zeta}_{im})_1)\right\}\\
&-\left(\frac{\lambda_s}{\lambda}+b\right)\left\{(\varepsilon_2+\theta,(\Sigma+\tilde{\zeta}_{re})_2)-(\varepsilon_1+\Sigma,(\Theta+\tilde{\zeta}_{im})_2)\right\}\\
&-\frac{x_s}{\lambda}\cdot\left\{(\varepsilon_2+\Theta,\nabla(\Sigma+\tilde{\zeta}_{re})_1)-(\varepsilon_1+\Sigma,\nabla(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
&-(Re(\Psi),\Sigma_1+(\tilde{\zeta}_{re})_1)-(Im(\Psi),\Theta_1+(\tilde{\zeta}_{im})_1)\\
&+(R_1(\varepsilon),\Sigma_1+(\tilde{\zeta}_{re})_1)+(R_2(\varepsilon),\Theta_1+(\tilde{\zeta}_{im})_1),
\end{aligned}
$$

where we used the fact that for any function $f=f_{re}+if_{im}$ in terms of real and imaginary parts,

$$\frac{1}{2}\left\{Im\left(\int y\cdot\nabla f\overline{f}\right)\right\}_s=(\partial_s f_{re},(f_{im})_1)-(\partial_s f_{im},(f_{re})_1).$$

To transform the above identity, first observe from the $\tilde{Q}_b$ equation and an integration by parts (see Appendix D in [16]):

$$
\begin{aligned}
&-\left(M_+(\varepsilon)+b(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2),\Sigma_1\right)-\left(M_-(\varepsilon)-b(\frac{N}{2}+y\cdot\nabla\varepsilon_1),\Theta_1\right)\\
&=2(\varepsilon_1,\Sigma+b\Theta_1-Re(\Psi))+2(\varepsilon_2,\Theta-b\Sigma_1-Im(\Psi))\\
&-(\varepsilon_1,Re(\Psi_1))-(\varepsilon_2,Im(\Psi_1)).
\end{aligned}
$$

Inject the conservation of energy (4.17) together with $(Re(\Psi),\Sigma_1)+(Im(\Psi),\Theta_1)=2E(\tilde{Q}_b)$ and the definition of $\tilde{R}_1(\varepsilon)$, $\tilde{R}_2(\varepsilon)$ given by (B.3), (B.4), to get in a first step:

(B.1)

$$
\begin{aligned}
&\left\{\frac{b}{4}|y\tilde{Q}_b|_2^2+\frac{1}{2}Im\left(\int y\cdot\nabla\tilde{\zeta}\overline{\tilde{\zeta}}\right)+(\varepsilon_2,\Sigma_1+(\tilde{\zeta}_{re})_1)-(\varepsilon_1,\Theta_1+(\tilde{\zeta}_{im})_1)\right\}_s\\
&=-2\lambda^2E_0+[H(\varepsilon,\varepsilon)-(\varepsilon_1,Re(\Psi_1))-(\varepsilon_2,Im(\Psi_1))\\
&-(Re(\Psi),(\tilde{\zeta}_{re})_1)-(Im(\Psi),(\tilde{\zeta}_{im})_1)\\
&-\left(M_+(\varepsilon)+b(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2),(\tilde{\zeta}_{re})_1\right)-\left(M_-(\varepsilon)-b(\frac{N}{2}+y\cdot\nabla\varepsilon_1),(\tilde{\zeta}_{im})_1\right)\Bigg]\\
&+b_s\left\{(\varepsilon_2-\tilde{\zeta}_{im},\frac{\partial(\Sigma+\tilde{\zeta}_{re})_1}{\partial b})-(\varepsilon_1-\tilde{\zeta}_{re},\frac{\partial(\Theta+\tilde{\zeta}_{im})_1}{\partial b})\right\}\\
&-\frac{A_s}{A^2}\left\{(\varepsilon_2-\tilde{\zeta}_{im},(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{re})_1)-(\varepsilon_1-\tilde{\zeta}_{re},(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{im})_1)\right\}\\
&-\tilde{\gamma}_s\left\{(\varepsilon_1-\tilde{\zeta}_{re},(\Sigma+\tilde{\zeta}_{re})_1)+(\varepsilon_2-\tilde{\zeta}_{im},(\Theta+\tilde{\zeta}_{im})_1)\right\}\\
&-\left(\frac{\lambda_s}{\lambda}+b\right)\left\{(\varepsilon_2-\tilde{\zeta}_{im},(\Sigma+\tilde{\zeta}_{re})_2)+(\varepsilon_1-\tilde{\zeta}_{re},(\Theta+\tilde{\zeta}_{im})_2)\right\}\\
&-\frac{x_s}{\lambda}\cdot\left\{(\varepsilon_2-\tilde{\zeta}_{im},\nabla(\Sigma+\tilde{\zeta}_{re})_1)+(\varepsilon_1-\tilde{\zeta}_{re},\nabla((\Theta+\tilde{\zeta}_{im})_1))\right\}\\
&+\tilde{H}_b(\varepsilon,\varepsilon)+(R_1(\varepsilon),(\tilde{\zeta}_{re})_1)+(R_2(\varepsilon),(\tilde{\zeta}_{im})_1)+(\tilde{R}_1(\varepsilon),\Sigma_1)+(\tilde{R}_2(\varepsilon),\Theta_1)\\
&-\frac{2}{2+\frac{4}{N}}\int J(\varepsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{L}&=\left\{\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}-\left(1+\frac{4}{N}\right)Q^{\frac{4}{N}}\right\}(\tilde{\zeta}_{re})_1+\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}(\tilde{\zeta}_{im})_1,\\
\tilde{K}&=\left\{\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}-Q^{\frac{4}{N}}\right\}(\tilde{\zeta}_{im})_1+\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}(\tilde{\zeta}_{re})_1,
\end{aligned}
$$

the quadratic form H is the usual one given by (1.8), and $\tilde{H}_b$ is the corrective term

$$\tilde{H}_b(\varepsilon,\varepsilon)=\int V_1(y)|\varepsilon_1|^2+\int V_2(y)|\varepsilon_2|^2+\int V_{12}(y)\varepsilon_1\varepsilon_2, \tag{B.2}$$

for some well-localized potentials

$$V_1(y) = \frac{2}{N}\left(\frac{4}{N}+1\right)\left\{\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\Sigma^3 y\cdot\nabla\Sigma - Q^{\frac{4}{N}-1}y\cdot\nabla Q\right\}$$
$$+\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\Theta^2\left\{\frac{6}{N}\Sigma\Sigma_1-\left(\frac{4}{N}+2\right)\Sigma^2-\Theta^2\right\}$$
$$+\frac{2}{N}\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\Theta\Theta_1\left\{\Theta^2+\left(\frac{4}{N}-1\right)\Sigma^2\right\},$$
$$V_2(y) = \frac{2}{N}\left\{\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\Sigma^3 y\cdot\nabla\Sigma - Q^{\frac{4}{N}-1}y\cdot\nabla Q\right\}$$
$$+\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\Theta^2\left\{\frac{2}{N}\left(\frac{4}{N}-1\right)\Sigma\Sigma_1-\left(\frac{4}{N}+2\right)\Sigma^2-\left(\frac{4}{N}+1\right)\Theta^2\right\}$$
$$+\frac{2}{N}\Theta\Theta_1\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{3\Sigma^2+\left(\frac{4}{N}+1\right)\Theta^2\right\},$$
$$V_{12}(y) = \frac{4}{N}\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{\Theta[\Sigma_1\Theta^2+\left(\frac{4}{N}-1\right)(\Sigma^2\Sigma_1+\Sigma\Theta\Theta_1)-2\Sigma|\tilde{Q}_b|^2]+\Sigma^3\Theta_1\right\}.$$

The nonlinear interaction terms $(\tilde{R}_i)_{i=1,2}(\varepsilon)$ correspond to the formally cubic part of $(R_i)_{i=1,2}(\varepsilon)$ given by (2.33), (2.34), explicitly:

$$\tilde{R}_1(\varepsilon) = R_1(\varepsilon) - \varepsilon_1^2\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{\frac{2}{N}\left(\frac{4}{N}+1\right)\Sigma^3+\frac{6}{N}\Sigma\Theta^2\right\} \tag{B.3}$$
$$-\varepsilon_2^2\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{\frac{2}{N}\Sigma^3+\frac{2}{N}\left(\frac{4}{N}-1\right)\Sigma\Theta^2\right\}$$
$$-\frac{4}{N}\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\varepsilon_1\varepsilon_2\left\{\left(\frac{4}{N}-1\right)\Sigma^2\Theta+\Theta^3\right\},$$
$$\tilde{R}_2(\varepsilon) = R_2(\varepsilon) - \varepsilon_2^2\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{\frac{2}{N}\left(\frac{4}{N}+1\right)\Theta^3+\frac{6}{N}\Theta\Sigma^2\right\} \tag{B.4}$$
$$-\varepsilon_1^2\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\left\{\frac{2}{N}\Theta^3+\frac{2}{N}\left(\frac{4}{N}-1\right)\Theta\Sigma^2\right\}$$
$$-\frac{4}{N}\frac{|\tilde{Q}_b|^{\frac{4}{N}}}{|\tilde{Q}_b|^4}\varepsilon_1\varepsilon_2\left\{\left(\frac{4}{N}-1\right)\Theta^2\Sigma+\Sigma^3\right\}.$$

The cubic term $J(\varepsilon)$ coming from the conservation of energy becomes:

$$
\begin{aligned}
\text{(B.5)} \quad J(\varepsilon) &= |\varepsilon + \tilde{Q}_b|^{\frac{4}{N}+2} - \tilde{Q}_b^{\frac{4}{N}+2} - \left(\frac{4}{N}+2\right)\frac{|\tilde{Q}_b|^{\frac{4}{N}+2}}{|\tilde{Q}_b|^2}(\Sigma\varepsilon_1 + \Theta\varepsilon_2)\\
&- \varepsilon_1^2 \frac{|\tilde{Q}_b|^{\frac{4}{N}+2}}{|\tilde{Q}_b|^4}\left\{\left(\frac{2}{N}+1\right)\left(\frac{4}{N}+1\right)\Sigma^2 + \left(\frac{2}{N}+1\right)\Theta^2\right\}\\
&- \varepsilon_2^2 \frac{|\tilde{Q}_b|^{\frac{4}{N}+2}}{|\tilde{Q}_b|^4}\left\{\left(\frac{2}{N}+1\right)\left(\frac{4}{N}+1\right)\Theta^2 + \left(\frac{2}{N}+1\right)\Sigma^2\right\}\\
&- \varepsilon_1\varepsilon_2 \frac{|\tilde{Q}_b|^{\frac{4}{N}+2}}{|\tilde{Q}_b|^4}\frac{8}{N}\left(\frac{2}{N}+1\right)\Sigma\Theta.
\end{aligned}
$$

It now remains to transform the first two lines of (B.1) according to the following identity, which has been proved in Appendix D of [16]:

$$
\begin{aligned}
& H(\varepsilon - \tilde{\zeta}_b, \varepsilon - \tilde{\zeta}_b) = H(\varepsilon,\varepsilon) - (\varepsilon_1, Re(\Psi_1)) - (\varepsilon_2, Im(\Psi_1))\\
- \;& (Re(\Psi), (\tilde{\zeta}_{re})_1) - (Im(\Psi), (\tilde{\zeta}_{im})_1)\\
- \;& \left(L_+\varepsilon_1 + b(\frac{N}{2}\varepsilon_2 + y\cdot\nabla\varepsilon_2), (\tilde{\zeta}_{re})_1\right) - \left(L_-\varepsilon_2 - b(\frac{N}{2}\varepsilon_1 + y\cdot\nabla\varepsilon_1), (\tilde{\zeta}_{im})_1\right)\\
- \;& (\varepsilon_1 - \tilde{\zeta}_{re}, (Re(F) + (1+\frac{4}{N})Q^{\frac{4}{N}}\tilde{\zeta}_{re})_1) - (\varepsilon_2 - \tilde{\zeta}_{im}, (Im(F) + Q^{\frac{4}{N}}\tilde{\zeta}_{im})_1).
\end{aligned}
$$

This concludes the proof of (4.8).

C. Proof of the estimates of Step 2 of Lemma 6

Note that each estimate $(i)_{1\leq i\leq 7}$ holds for $0<\eta<\eta_i$, $0<a<a_i$ and constant $0<z_0\leq z_0(\eta_i,a_i)$ for $0<b<b^*(\eta,a)$. Taking the infimum on the seven estimates yields the claim. Note that these constants a priori depend on the dimension N. In most instances, we shall argue differently depending on the dimension.

(0) The proof is similar to the one of (2.38). Let us briefly sketch the argument. We argue differently depending on the dimension:

- $N=1$: Assume $v\in\mathcal{C}_0^\infty$ and let $y_0\in[0,1]$ such that $|v(y_0)|^2\leq\int|v|^2e^{-|y|}$. Then writing $v(y)=v(y_0)+\int_{y_0}^y v_y(x)dx$, we get

$$
\begin{aligned}
\int_{|y|\leq B}|v|^2 &\leq C\int_{|y|\leq B}\left(|v(y_0)|^2 + |y-y_0|\left(\int_{y_0}^y |v_y|^2(x)dx\right)dy\right)\\
&\leq CB^2\left(\int|v|^2e^{-|y|} + \int|v_y|^2\right),
\end{aligned}
$$

  and (4.10) follows.
- $N\geq 3$: This estimate follows from the Sobolev injection $|\varepsilon|_{L^{2^*}}\leq C|\nabla\varepsilon|_{L^2}$ or Hardy's inequality.
- $N=2$: In dimension $N=2$, a logarithmic correction has to be taken into account in the result. Assume $v\in\mathcal{C}_0^\infty$ and decompose $v(r,\theta)$ in Fourier series $v(r,\theta)=\sum_{k=-\infty}^{+\infty}v_k(r)e^{ik\theta}$, $v_k(r)=\frac{1}{2\pi}\int_0^{2\pi}v(r,\theta)e^{-ik\theta}d\theta$. For $k\neq 0$, $|v_k(r)|\leq\frac{Cr}{|k|}\left(\int_0^{2\pi}|\nabla v(r,\theta)|^2d\theta\right)^{\frac{1}{2}}$ from which we recover Hardy's inequality

on the nonradial part:

$$\int_{\mathbf{R}^2} \frac{|v - v_0(|x|)|^2}{|x|^2} \leq C \int_{\mathbf{R}^2} |\nabla v|^2. \tag{C.1}$$

Now let $r_0 \in [\frac{1}{2}, 1]$ such that $|v_0(r_0)|^2 \leq 10 \int |v|^2 e^{-|y|}$. For $r \geq 1$, we write $v_0(r) = v_0(r_0) + \int_{r_0}^{r} \partial_\tau v_0(\tau) d\tau$ and estimate

$$\begin{aligned} |v_0(r)|^2 &\leq C|v_0(r_0)|^2 + C\left(\int_{r_0}^{r} \partial_\tau v_0(\tau) d\tau\right)^2 \\ &\leq C|v_0(r_0)|^2 + C\left(\int_{r_0}^{r} \tau |\partial_\tau v_0(\tau)|^2 d\tau\right)\left(\int_{r_0}^{r} \frac{d\tau}{\tau}\right) \\ &\leq C\int |v|^2 e^{-|y|} + C(\int |\nabla v|^2) \log(r). \end{aligned} \tag{C.2}$$

We conclude from (C.1) and (C.2):

$$\begin{aligned} \int_{|y|\leq B} |v|^2 &= \int_{|y|\leq 1} |v|^2 + \int_{1\leq |y|\leq B} |v|^2 \leq CB^2(\int |v|^2 e^{-|y|} + \int |\nabla v|^2) \\ &+ \int_{1\leq \tau\leq B} \tau |v_0(\tau)|^2 d\tau \leq CB^2 \log B(\int |v|^2 e^{-|y|} + \int |\nabla v|^2), \end{aligned}$$

which concludes the proof of (4.11). Similarly,

$$\begin{aligned} &\int_{1\leq |y|\leq B} \frac{|v|^2}{|y|^2} \leq C(\int |\nabla v|^2 + \int_{1\leq r\leq B} \frac{|v_0(r)|^2}{r} dr) \\ &\leq C(\int |\nabla v|^2 + \int |v|^2 e^{-|y|})(1 + \int_1^B \frac{\log r}{r} dr) \\ &\leq C \log^2 B(\int |\nabla v|^2 + \int |v|^2 e^{-|y|}) \end{aligned}$$

and (4.12) is proved.

(i) First observe from (2.15) and (2.19) that:

$$\begin{aligned} \int |\tilde{\varepsilon}|^2 e^{-|y|} &\leq \int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}} + \int |\tilde{\zeta}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}} \\ &\leq \int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}} + \Gamma_b^{1+z_0}. \end{aligned}$$

We now may apply estimate (2.38) to $\tilde{\varepsilon}$ to get the claim.

(ii) From [15], the following holds:

$$|R(\varepsilon)| \leq \begin{cases} C(|\varepsilon|^2 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}} + |\varepsilon|^{1+\frac{4}{N}}) & \text{for} \quad N \leq 3, \\ C \min(|\varepsilon|^2 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}}, |\varepsilon|^{1+\frac{4}{N}}) & \text{for} \quad N \geq 4. \end{cases} \tag{C.3}$$

We now first estimate from (2.15) and (2.19): $\forall N \geq 1$,

$$\int |\varepsilon|^2 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}} e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} \leq \int |\tilde{\varepsilon}|^2 e^{-\left(\frac{4}{N}\right)(1-C\eta)\frac{\theta(b|y|)}{b}} + \Gamma_b^{1+z_0},$$

which is now estimated with (2.38). The other term for $N \leq 3$ is controlled as follows. We first have

$$\int |\varepsilon|^{1+\frac{4}{N}} e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} \leq \int |\tilde{\varepsilon}|^{1+\frac{4}{N}} e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} + \Gamma_b^{1+z_0}$$

from $1+\frac{4}{N}>2$. The conclusion follows from the Sobolev injection for $N=3$. For $N=1,2$, we write

$$\int|\tilde{\varepsilon}|^{1+\frac{4}{N}}e^{-(1-C\eta)\frac{\theta(b|y|)}{b}}\leq C\left(\int|\tilde{\varepsilon}|^{\frac{8}{N}}+\int|\tilde{\varepsilon}|^{2}e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}}\right)$$
$$\leq \quad C\left(\int|\tilde{\varepsilon}|^{2+\frac{4}{N}}+\int|\tilde{\varepsilon}|^{2}e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}}\right),$$

and the conclusion follows from the Gagliardo-Nirenberg inequality and (2.38).
(iii) We use again the estimate (C.3). For $N\leq 3$,

$$\int|\varepsilon|^{2}e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}}(|\tilde{\zeta}|+|y\cdot\nabla\tilde{\zeta}|)$$
$$\leq\Gamma_b^{1+z_0}+\Gamma_b^{\frac{1}{2}(1-C\eta)}\int|\tilde{\varepsilon}|^{2}e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}},$$

and the conclusion follows from (2.38). The other term is estimated for all $N\geq 1$:

$$\begin{aligned}\int|\varepsilon|^{1+\frac{4}{N}}(|\tilde{\zeta}|+|y\cdot\nabla\tilde{\zeta}|)&\leq\Gamma_b^{\frac{1}{2}(1-C\eta)}\int_{|y|\leq 2A}|\varepsilon|^{1+\frac{4}{N}}\\&\leq\Gamma_b^{\frac{1}{2}(1-C\eta)}\int_{|y|\leq 2A}|\tilde{\varepsilon}|^{1+\frac{4}{N}}+\Gamma_b^{1+z_0}\\&\leq\Gamma_b^{\frac{1}{2}(1-C\eta)}A^C\left(\int|\tilde{\varepsilon}|^{2+\frac{4}{N}}\right)^{\frac{1+\frac{4}{N}}{2+\frac{4}{N}}}+\Gamma_b^{1+z_0},\end{aligned}$$

and the conclusion follows from the Gagliardo-Nirenberg inequality.
(iv) Indeed, first estimate from (2.15), (2.20):

$$\left|P(y)(|\frac{d^k\tilde{\zeta}}{dy^k}|+|\frac{d^l}{dy^l}\frac{\partial\tilde{\zeta}}{\partial b}|)\right|_{L^\infty}\leq A^C\Gamma_b^{\frac{1}{2}-C\eta},$$

so that from the Cauchy-Schwarz inequality:

$$\left(\int|\tilde{\varepsilon}||P(y)(|\frac{d^k\tilde{\zeta}}{dy^k}|+|\frac{d^l}{dy^l}\frac{\partial\tilde{\zeta}}{\partial b}|)\right)^2\leq\Gamma_b^{1-C\eta}A^C\int_{|y|\leq 2A}|\tilde{\varepsilon}|^2.$$

The conclusion follows from (4.10).

For the second claim, we argue similarly. First observe from (4.3) that $\left|\frac{A_s}{A}\right|\leq C\left|\frac{b_s}{b^2}\right|$. Next, from (4.10) and (4.3), we have

$$\left|\frac{A_s}{A^2}\left\{(\varepsilon_2-\tilde{\zeta}_{im},(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{re})_1)-(\varepsilon_1-\tilde{\zeta}_{re},(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{im})_1)\right\}\right|$$
$$=\quad\left|\frac{A_s}{A^2}\left\{(\tilde{\varepsilon}_2,(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{re})_1)-(\tilde{\varepsilon}_1,(y\cdot\nabla\chi\left(\frac{y}{A}\right)\zeta_{im})_1)\right\}\right|$$
$$\leq\quad|b_s|A^C\Gamma_b^{\frac{1}{2}-C\eta}(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|})^{\frac{1}{2}},$$

where we used (4.10) in the last step. We now use (2.36) and the following, which is a straightforward consequence of the conservation of energy:

$$\lambda^2|E_0|\leq C(\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|})^{\frac{1}{2}}+\Gamma_b^{\frac{1}{2}-C\eta},$$

to estimate

$$|b_s| \leq C(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|})^{\frac{1}{2}} + \Gamma_b^{\frac{1}{2}-C\eta},$$

and the conclusion follows.
(v) Recall from [15]:

$$|J(\varepsilon)| \leq \left\{ \begin{array}{l} C(|\varepsilon|^3 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}} + |\varepsilon|^{2+\frac{4}{N}}) \ \text{ for } \ N \leq 3, \\ C|\varepsilon|^{2+\frac{4}{N}} \ \text{ for } \ N \geq 4. \end{array} \right.$$

For all $N \geq 1$,

$$\int |\varepsilon|^{2+\frac{4}{N}} \leq \Gamma_b^{1+z_0} + \int |\tilde{\varepsilon}|^{2+\frac{4}{N}},$$

and the conclusion follows from the Gagliardo-Nirenberg inequality. For $N \leq 3$,

$$\int |\varepsilon|^3 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}} \leq \int |\tilde{\varepsilon}|^3 e^{-\left(\frac{4}{N}-1\right)(1-C\eta)\frac{\theta(b|y|)}{b}} + \Gamma_b^{1+z_0}.$$

For $N = 3$, the conclusion follows from the Sobolev embeddings; for $N = 1$, $|\tilde{\varepsilon}|_{L^\infty} \leq \delta(\alpha^*)$ yields the conclusion; for $N = 2$,

$$\int |\tilde{\varepsilon}|^3 e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} \leq (\int |\tilde{\varepsilon}|^4)^{\frac{1}{2}} (\int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}})^{\frac{1}{2}},$$

and the conclusion follows from the Gagliardo-Nirenberg inequality.

Next, let $\tilde{R}(\varepsilon) = \tilde{R}_1(\varepsilon) + i\tilde{R}_2(\varepsilon)$. Then from [15]:

$$\int \left|\tilde{R}(\varepsilon)\right| e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} \leq \left\{ \begin{array}{l} C(|\varepsilon|^5 e^{-(1-C\eta)\frac{\theta(b|y|)}{b}} + |\varepsilon|^3 e^{-3(1-C\eta)\frac{\theta(b|y|)}{b}}) \\ \qquad\qquad\qquad\qquad \text{for } \ N = 1, \\ C(\int |\varepsilon|^3 e^{-(1-C\eta)\frac{\theta(b|y|)}{b}}) \ \text{ for } \ N = 2, \\ C(\int |\varepsilon|^3 e^{-\frac{2}{3}(1-C\eta)\frac{\theta(b|y|)}{b}}) \ \text{ for } \ N = 3, \\ C(\int |\varepsilon|^{\frac{2}{N}+2} e^{-\frac{2}{N}(1-C\eta)\frac{\theta(b|y|)}{b}}) \ \text{ for } \ N \geq 4, \end{array} \right.$$

and the proof follows similarly as before.

Last, the corrective term $\tilde{H}_b$ given by (B.2) is estimated in any dimension from [15]:

$$\left|\tilde{H}_b(\varepsilon, \varepsilon)\right| \leq \delta(\alpha^*) \int |\varepsilon|^2 e^{-\frac{4}{N}(1-\eta)\frac{\theta(b|y|)}{|y|}},$$

and the conclusion follows.
(vi) This estimate follows from the explicit values of the vectors $(\tilde{L}, \tilde{K})$ of Step 1 and the estimate (2.19). For example,

$$\begin{aligned} |(\varepsilon_1, \tilde{L})| &= \left| \left(\varepsilon_1, \{ (\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1)|\tilde{Q}_b|^{\frac{4}{N}} - (1 + \frac{4}{N}) Q^{\frac{4}{N}} \} (\tilde{\zeta}_{re})_1 \right.\right. \\ &\quad \left.\left. + \frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2} |\tilde{Q}_b|^{\frac{4}{N}} (\tilde{\zeta}_{im})_1 \right) \right| \\ &\leq C(\int |\varepsilon|^2 e^{-\frac{2}{N}(1-C\eta)\frac{\theta(b|y|)}{b}})^{\frac{1}{2}} \Gamma_b^{\frac{1}{2}(1-C\eta+\frac{C}{N})} \\ &\leq \delta(\alpha^*)(\int |\nabla \tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2(1-C\eta)\frac{\theta(b|y|)}{b}}) + \Gamma_b^{1+z_0}. \end{aligned}$$

The others follow similarly with the help of (2.19).
(vii) Injecting (2.18), (2.15) into (4.5) yields

$$|F|_{L^\infty} + |y \cdot \nabla F|_{L^\infty} \leq \frac{C}{A^{\frac{N}{2}}} \Gamma_b^{\frac{1}{2}},$$

and thus by the Cauchy-Schwarz inequality:

$$\begin{aligned}\int |\varepsilon|(|F| + |y \cdot \nabla F|) &\leq (|F|_{L^\infty} + |y \cdot \nabla F|_{L^\infty}) \int_{A \leq |y| \leq 2A} |\varepsilon| \\ &\leq C \frac{\Gamma_b^{\frac{1}{2}}}{A^{\frac{N}{2}}} (A^N)^{\frac{1}{2}} (\int_{A \leq |x| \leq 2A} |\varepsilon|^2)^{\frac{1}{2}},\end{aligned}$$

and the conclusion follows.

D. Proof of estimate (5.9)

This appendix is devoted to the proof of the estimate (5.9), which follows in two steps. We will note for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$:

$$(L\varepsilon, \varepsilon) = (L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2).$$

Step 1. Elliptic estimate on L.

We claim that for some universal constant $\delta_3 > 0$: $\forall \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$,

$$(L\varepsilon, \varepsilon) \geq \delta_3 |\varepsilon|_{H^1}^2 - \frac{1}{\delta_3} \left\{ (\varepsilon_1, Q)^2 + (\varepsilon_1, |y|^2 Q)^2 + (\varepsilon_1, yQ)^2 + (\varepsilon_2, Q_2)^2 \right\}. \tag{D.1}$$

This follows from Lemma 8, where we had

$$(L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) \geq \delta_2 |\varepsilon|_{H^1}^2 - \frac{1}{\delta_2} \left\{ (\varepsilon_1, \phi_+)^2 + (\varepsilon_1, \nabla Q)^2 + (\varepsilon_2, Q)^2 \right\}.$$

Arguing as for the proof of Lemma 3 in [14], we exhibit a similar estimate but with different orthogonality conditions.

Indeed, let $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$ satisfy

$$(\varepsilon_1, |y|^2 Q) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_2) = 0,$$

and let the auxiliary function

$$\hat{\varepsilon} = \varepsilon - aQ_1 - b \cdot \nabla Q - icQ.$$

We choose a, b, c so that

$$(\hat{\varepsilon}_1, \phi_+) = (\hat{\varepsilon}_1, \nabla Q) = (\hat{\varepsilon}_2, Q_2) = 0,$$

that is,

$$a = 4\frac{(\varepsilon_1, \phi_+)}{(Q_1, \phi_+)}, \quad b = \frac{(\varepsilon_1, \nabla Q)}{(\nabla Q, yQ)} \quad \text{and} \quad c = \frac{(\varepsilon_2, Q)}{(Q, Q)}.$$

Note from $L_+\phi_+ = \mu_+\phi_+$ and $L_+Q_1 = -2Q$ that $\mu_+(Q_1, \phi_+) = -2(Q, \phi_+) < 0$ from $Q > 0$, $\phi_+ > 0$. Now using the orthogonality conditions on $\varepsilon_1, \varepsilon_2$, we also have

$$a = -\frac{(\hat{\varepsilon}_1, |y|^2 Q)}{(Q_1, |y|^2 Q)}, \quad b = -\frac{(\hat{\varepsilon}_1, yQ)}{(yQ, \nabla Q)}, \quad \text{and} \quad c = -\frac{(\hat{\varepsilon}_2, Q_2)}{(Q, Q_2)}.$$

Therefore, we have for some constant $K > 0$,

$$\frac{1}{K} |\varepsilon|_{H^1}^2 \leq |\hat{\varepsilon}|_{H^1}^2 \leq K |\varepsilon|_{H^1}^2.$$

Moreover, from $(Q, Q_1) = 0$, $L_+(Q_1) = -2Q$, $L_+(\nabla Q) = 0$, $L_-(Q) = 0$, we have

$$(\hat{\varepsilon}_1, Q) = (\varepsilon_1, Q) \quad , \quad (L_+\hat{\varepsilon}_1, \hat{\varepsilon}_1) = (L_+\varepsilon_1, \varepsilon_1) + 4a(\varepsilon_1, Q)$$

and

$$(L_-\hat{\varepsilon}_2, \hat{\varepsilon}_2) = (L_-\varepsilon_2, \varepsilon_2).$$

Applying the elliptic estimate (5.8) to $\hat{\varepsilon}$, we conclude

$$(L\varepsilon, \varepsilon) = (L\hat{\varepsilon}, \hat{\varepsilon}) - 4a(\varepsilon_1, Q) \geq \delta_5 |\hat{\varepsilon}|^2_{H^1} - 4a(\varepsilon_1, Q) \geq \delta_3 |\varepsilon|^2_{H^1} - \frac{1}{\delta_3}(\varepsilon_1, Q)^2,$$

and (D.1) follows.

Step 2. Localization of the elliptic estimate.

The proof of (5.9) now follows from a standard perturbation result. Let us briefly recall it. We sketch the argument for L_+; the same proof applies for L_-. Let an even cut-off function $\xi_A = \xi\left(\frac{y}{A}\right)$, $\xi(y) = 1$ for $|y| \leq \frac{1}{4}$, $\xi(y) = 0$ for $|y| \geq \frac{1}{2}$ so that $\xi_A(1 - \phi_A) = \xi_A$. Recall that $\phi_A = 0$ for $|y| \leq \frac{A}{2}$ and $\phi_A = 1$ for $|y| \geq 3A$. Given $\omega \in H^1$, we let

$$\omega = \xi_A \omega + (1 - \xi_A)\omega = \omega_i + \omega_e. \tag{D.2}$$

We write $L_+ = -\Delta + 1 - V$ with $V = (1 + \frac{4}{N})Q^{1+\frac{4}{N}}$, and note $F_A(\omega, \omega) = (L_+\omega, \omega) - \int \phi_A |\omega|^2$.
First compute:

$$\begin{aligned} F_A(\omega, \omega) &= \int |\nabla \omega_i|^2 - \int V|\omega_i|^2 + \int (1 - \phi_A)|\omega|^2 \\ &+ \int |\nabla \omega_e|^2 - \int V|\omega_e|^2 + 2\int \nabla \omega_i \cdot \nabla \omega_e - 2\int V\omega_i \omega_e. \end{aligned}$$

We first observe from the support localization of ω_e and going back to the proof of (0) in Appendix C that we have

$$\int |\omega_e|^2 e^{-C|y|} \leq \delta(A) \int |\nabla \omega_e|^2 \quad \text{with} \quad \delta(A) \to 0 \quad \text{as} \quad A \to +\infty.$$

This identity with Cauchy-Schwarz yields

$$\left|\int V|\omega_e|^2\right| + \left|\int V\omega_i \omega_e\right| \leq \delta(A)(\int |\omega_i|^2 + \int |\nabla \omega_e|^2).$$

The gradient interaction term is estimated by reinjecting (D.2) and integrating by parts, which yields

$$\int \nabla \omega_i \cdot \nabla \omega_e \geq -\delta(A) \int_{\frac{A}{4}}^{\frac{A}{2}} |\omega|^2.$$

These two estimates yield for $A > A^*$ large enough:

$$\begin{aligned}
F_A(\omega,\omega) &\geq \frac{1}{2}\int|\nabla\omega_e|^2 + \int|\nabla\omega_i|^2 - \int V|\omega_i|^2 + (1-\delta(A))\int(1-\phi_A)|\omega|^2 \\
&\geq \frac{1}{2}\int|\nabla\omega_e|^2 + \int|\nabla\omega_i|^2 - \int V|\omega_i|^2 + (1-\delta(A))\int|\omega_i|^2 \\
&\geq \frac{1}{2}\int|\nabla\omega_e|^2 + \frac{\delta_3}{2}|\omega_i|^2_{H^1} - \frac{1}{\delta_3}\left\{(\omega_i,Q)^2 + (\omega_i,|y|^2Q)^2 + (\omega_i,yQ)^2\right\}, \\
&\geq \frac{\delta_3}{8}\int|\nabla\omega|^2 + \frac{\delta_3}{4}|\omega_i|^2_{L^2} - \frac{2}{\delta_3}\left\{(\omega,Q)^2 + (\omega,|y|^2Q)^2 + (\omega,yQ)^2\right\},
\end{aligned}$$

where we used (D.1). It now suffices to observe that

$$\int|\omega|^2e^{-|y|} \leq C(\int|\omega_i|^2 + \delta(A)\int|\nabla\omega_e|^2)$$

and estimate (5.9) follows.

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