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L^p BOUNDEDNESS OF DISCRETE SINGULAR RADON TRANSFORMS

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1. Introduction

Assume that $d_1 \geq 1$ is an integer and $K \in C^1(\mathbb{R}^{d_1} \setminus \{0\})$ satisfies the differential inequalities

$$|x|^{d_1}|K(x)| + |x|^{d_1+1}|\nabla K(x)| \leq 1 \tag{1.1}$$

for any $x \in \mathbb{R}^{d_1}$, $|x| \geq 1$, and the cancellation condition

$$\left| \int_{|x|\in[1,\lambda]} K(x)\, dx \right| \leq 1, \tag{1.2}$$

for any $\lambda \geq 1$ (i.e., K is a Calderón–Zygmund kernel on $\mathbb{R}^{d_1}$ away from 0). Let $P = (P_1, \ldots, P_{d_2}) : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ denote a polynomial of degree $A \geq 1$ with real coefficients. We define the (translation invariant) discrete singular Radon transform operator T by the formula

$$T(f)(x) = \sum_{n\in\mathbb{Z}^{d_1}\setminus\{0\}} f(x - P(n))K(n), \tag{1.3}$$

for any Schwartz function $f : \mathbb{R}^{d_2} \to \mathbb{C}$. Our main theorem is the following:

Theorem 1.1. *The operator T extends to a bounded operator on $L^p(\mathbb{R}^{d_2})$, $p \in (1, \infty)$, with*

$$||T(f)||_{L^p(\mathbb{R}^{d_2})} \leq C_p||f||_{L^p(\mathbb{R}^{d_2})}.$$

The constant C_p may depend only on the exponent p, the dimension d_1, and the degree A.

Boundedness properties of the corresponding *continuous* singular Radon transforms

$$S(f)(x) = \int_{\mathbb{R}^{d_1}} f(x - \gamma(x, y))K(y)\, dy,$$

as well as boundedness of the associated maximal Radon transforms, have been studied extensively, under very general finite-type conditions on the function $\gamma : \mathbb{R}^{d_2} \times \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$. See [8], [13], [20] for some results in the translation invariant

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setting, as well as [14], [12], [5], [17], [6], [9] for the general case. For more references in the continuous case, we refer the reader to the recent paper of M. Christ, A. Nagel, E. M. Stein, and S. Wainger [6]. We will use the boundedness of the (translation invariant) continuous singular Radon transforms in our proof of Theorem 1.1; see Lemma 6.5 in Section 6. We emphasize, however, that the main difficulties in proving Theorem 1.1 in the discrete setting are of a different nature than in the continuous case.

To illustrate the difference between the discrete and the continuous singular Radon transforms, consider for instance the case $d_1 = 1$, $d_2 = 2$, $P(y) = (y, y^2)$, and $K(y) = 1/y$. For $k \geq 0$ let

$$S_k(f)(x) = \int_{\mathbb{R}} f(x - (y, y^2))y^{-1}\eta(2^{-k}y)\,dy,$$

where η is a smooth function supported in the set $\{y : |y| \geq 1\}$ and equal to 1 in the set $\{y : |y| \geq 2\}$. Let m_k denote the multiplier of the (translation invariant) operator S_k, i.e., $m_k(\xi_1, \xi_2) = \int_{\mathbb{R}} y^{-1}\eta(2^{-k}y)e^{-2\pi i(y\xi_1 + y^2\xi_2)}\,dy$. It is well known (see, for example, [19, Chapter XI]) that m_k is a multiplier on $L^p(\mathbb{R}^2)$, $p \in (1, \infty)$, supported microlocally in the rectangle $\{(\xi_1, \xi_2) : |\xi_1| \leq 2^{-k}, |\xi_2| \leq 2^{-2k}\}$, more precisely $|m_k(\xi)| \leq C(1 + 2^k|\xi_1| + 2^{2k}|\xi_2|)^{-1/2}$. For contrast, consider also the discrete operator

$$T_k(f)(x) = \sum_{n \in \mathbb{Z}\setminus\{0\}} f(x - (n, n^2))n^{-1}\eta(2^{-k}n).$$

It is shown in [21], following ideas in [4], that the multiplier of the operator T_k is approximated, modulo $O(2^{-\delta k})$ errors, $\delta > 0$, by

$$\sum_{q,a} S(a/q)m_k(\xi - a/q),$$

where the sum is taken over rational points in $\mathbb{R}^2$ with denominators $q \leq C2^{\delta' k}$, m_k is the multiplier of the continuous singular Radon transform S_k, and $S(a/q)$ are Gauss sums. This approximation is enough to prove boundedness if $p = 2$ or even when p is close to 2 by using the uniform decay of the Gauss sums. However, when p is close to 1 or ∞, it is significantly more difficult to sum efficiently the contributions of the multiplier corresponding to different denominators q. Our main new idea is to exploit the "almost orthogonality" in L^p of the pieces of the operator corresponding to different denominators q, when the exponent p is a large even integer. By interpolating and taking adjoints, this yields the theorem in the full range of exponents $p \in (1, \infty)$. In general, it is not known whether there is a suitable L^1 theory for singular Radon transforms, both continuous and discrete.

The systematic study of discrete singular Radon transforms was initiated by E. M. Stein and S. Wainger [21], where they conjectured the bound in Theorem 1.1 and proved a result in the restricted range $p \in (3/2, 3)$. Boundedness in the case $p = 2$, $d_1 = 1$ follows from the earlier work of G. I. Arkhipov and K. I. Oskolkov [1]. The range of exponents was later expanded to $p \in (4/3, 4)$ in the special case $d_1 = d_2 = 1$ (E. M. Stein, personal communication) and used by E. M. Stein and S. Wainger [25] in connection with sharp boundedness properties of certain discrete fractional integral operators (see also [23] and [16]). E. M. Stein and S. Wainger [22] also proved L^2 bounds in a certain "quasi-translation invariant" case.

As in the continuous case, a related question concerns boundedness of the maximal operator

$$M(f)(n) = \sup_{N\in[1,\infty)} \frac{1}{N} \sum_{1\leq m\leq N} |f(n-P(m))|.$$

In this case, $d_1 = d_2 = 1$, and P is a polynomial with integer coefficients. J. Bourgain [2], [3], [4] proved L^p boundedness of the maximal operator M in the full range $p \in (1,\infty]$. Some of the techniques in [4] have played a fundamental role in the development of the subject; we will use these techniques implicitly in Section 6. As in the case of singular Radon transforms, weak boundedness in the case $p = 1$ is an open problem.

If the polynomial P in Theorem 1.1 maps $\mathbb{Z}^{d_1}$ to $\mathbb{Z}^{d_2}$, for instance if P has integer coefficients, then boundedness of the operator T in Theorem 1.1 is equivalent to boundedness of the corresponding discrete operator.

Corollary 1.2. *Assume that $P = (P_1, \ldots, P_{d_2}) : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is a polynomial of degree $A \geq 1$ with the property that $P(\mathbb{Z}^{d_1}) \subset \mathbb{Z}^{d_2}$, and assume that K is a Calderón–Zygmund kernel as before. For compactly supported functions f let*

$$T_{\mathrm{dis}}(f)(m) = \sum_{n\in\mathbb{Z}^{d_1}\setminus\{0\}} f(m-P(n))K(n).$$

The operator T_{dis} extends to a bounded operator on $L^p(\mathbb{Z}^{d_2})$, $p \in (1,\infty)$, with

$$||T_{\mathrm{dis}}(f)||_{L^p(\mathbb{Z}^{d_2})} \leq C_p ||f||_{L^p(\mathbb{Z}^{d_2})}.$$

The constant C_p may depend only on the exponent p, the dimension d_1, and the degree A.

The equivalence of Theorem 1.1 and Corollary 1.2 in the case of polynomials that map $\mathbb{Z}^{d_1}$ to $\mathbb{Z}^{d_2}$ was noticed by E. M. Stein (personal communication). To see that Theorem 1.1 implies Corollary 1.2, assume that $f \in L^p(\mathbb{Z}^{d_2})$ is a given function and define $f_{\mathrm{ext}} \in L^p(\mathbb{R}^{d_2})$, $f_{\mathrm{ext}}(y) = f(\lfloor y \rfloor)$, where $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \ldots, \lfloor y_{d_2} \rfloor)$ denotes the integer part of y. It is easy to verify that $T(f_{\mathrm{ext}})(x) = T_{\mathrm{dis}}(f)(\lfloor x \rfloor)$, which shows that Theorem 1.1 implies Corollary 1.2. The proof of the reverse implication is similar.

By interpolating between Corollary 1.2 and the bounds in [25], we prove optimal boundedness properties of two discrete fractional integral operators. Partial results on the boundedness of these discrete fractional integrals were proved by E. M. Stein and S. Wainger [23] and [25] and D. Oberlin [16]. We remark, however, that, by itself, Corollary 1.2 is not sufficient to prove optimal bounds for fractional integrals in the case when the polynomials involved have higher degrees.

Corollary 1.3. *Assume $\lambda \in (0,1)$ and $p, q \in [1,\infty]$. Then the discrete fractional integral operator*

$$I_\lambda(f)(m) = \sum_{n=1}^{\infty} f(m-n^2)n^{-\lambda}, \ f \text{ compactly supported},$$

extends to a bounded operator from $L^p(\mathbb{Z})$ to $L^q(\mathbb{Z})$ if and only if

(i) $1/q \leq 1/p - (1-\lambda)/2$, and

(ii) $p < 1/(1-\lambda)$, $q > 1/\lambda$.

Corollary 1.4. *Assume $\lambda \in (0,1)$ and $p, q \in [1,\infty]$. Then the discrete fractional integral operator*

$$J_\lambda(f)(m_1, m_2) = \sum_{n=1}^{\infty} f(m_1 - n, m_2 - n^2) n^{-\lambda},\ f \text{ compactly supported},$$

extends to a bounded operator from $L^p(\mathbb{Z}^2)$ to $L^q(\mathbb{Z}^2)$ if and only if

(i) $1/q \le 1/p - (1-\lambda)/3$, and

(ii) $p < 1/(1-\lambda)$, $q > 1/\lambda$.

The main new ingredient in the proof of Theorem 1.1 in the full range $p \in (1,\infty)$ is Theorem 1.5 below, which is in fact the main result of this paper. We introduce first some notation. Assume $d \ge 1$ is an integer. For any $\mu \ge 1$ let $Z_\mu = \mathbb{Z} \cap [1,\mu]$. If $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ is a vector and $q \ge 1$ is an integer, then we denote by (a, q) the greatest common divisor of a and q, i.e., the largest integer $q' \ge 1$ that divides q and all the components $a_1, \ldots, a_d$. Clearly, any vector in $\mathbb{Q}^d$ has a unique representation in the form a/q, with $q \in \{1, 2, \ldots\}$, $a \in \mathbb{Z}^d$, and $(a, q) = 1$; such a vector a/q will be called an *irreducible d-fraction*.

Assume that $m : \mathbb{R}^d \to \mathbb{C}$ is a bounded function supported in the cube $[-1/2, 1/2]^d$, with the property that for any $p \in (1,\infty)$,

$$||\mathcal{F}^{-1}(m \cdot \mathcal{F}(g))||_{L^p(\mathbb{R}^d)} \le B_p ||g||_{L^p(\mathbb{R}^d)} \tag{1.4}$$

for any Schwartz function $g : \mathbb{R}^d \to \mathbb{C}$. Here $\mathcal{F}$ denotes the Euclidean Fourier transform acting on distributions on $\mathbb{R}^d$. In other words, m is an L^p multiplier on $\mathbb{R}^d$, $p \in (1,\infty)$, supported in the cube $[-1/2, 1/2]^d$. For any finite set $Y \subset \{1, 2, \ldots\}$ let

$$\mathcal{R}(Y) = \{a/q : q \in Y, a \in \mathbb{Z}^d, (a, q) = 1\}.$$

For any $\varepsilon \in (0, 1]$ let

$$m_{\varepsilon,Y}(\xi) = \sum_{a/q \in \mathcal{R}(Y)} m((\xi - a/q)/\varepsilon).$$

A measurable function $g : \mathbb{R}^d \to \mathbb{C}$ will be called *periodic* if for any $n \in \mathbb{Z}^d$, $g(\xi + n) = g(\xi)$ a.e. $\xi \in \mathbb{R}^d$. A set $A \subset \mathbb{R}^d$ will be called *periodic* if χ_A is a periodic function. Notice that $m_{\varepsilon,Y}$ is a periodic function.

Theorem 1.5. *For any $\delta > 0$ and $p \in (1,\infty)$ there are constants A_δ and $C_{p,\delta}$ with the following property: for any $N \ge A_\delta$ there is a set of integers $Y_N = Y_{N,\delta}$,*

$$Z_N \subset Y_N \subset Z_{e^{N^\delta}}, \tag{1.5}$$

such that for any $\varepsilon \le e^{-N^{2\delta}}$, the operator $T_N = T_{N,\varepsilon}$ defined by the Fourier multiplier m_{ε,Y_N} extends to a bounded operator on $L^p(\mathbb{R}^d)$, with

$$||T_N(f)||_{L^p(\mathbb{R}^d)} \le C_{p,\delta} (\ln N)^{2/\delta} ||f||_{L^p(\mathbb{R}^d)}. \tag{1.6}$$

The constant A_δ may depend only on δ and d; the constant $C_{p,\delta}$ may depend only on δ, d, the exponent p, and the constants B_p in (1.4)*.*

Remarks. (1) Since the multiplier m_{ε,Y_N} is periodic, the kernel of the operator T_N is supported (as a distribution) in $\mathbb{Z}^d$. Therefore, boundedness on $L^p(\mathbb{R}^d)$ of the operator T_N is equivalent to boundedness on $L^p(\mathbb{Z}^d)$ of the discrete operator defined by the same kernel (the equivalence discussed after Corollary 1.2). It will be more convenient to prove boundedness on $L^p(\mathbb{Z}^d)$ of this discrete operator.

(2) Theorem 1.5 follows easily from Lemma 4.1 in Section 4 for $\delta > 1$, simply by taking $Y_N = \{Q : Q|N!\}$, $\mathcal{R}(Y_N) = \{a/N! : a \in \mathbb{Z}^d\}$. However, to prove Theorem 1.1 in the full range of exponents p, we need to be able to let $\delta = \delta(p)$ tend to 0.

(3) It seems natural to ask whether the operator defined by the Fourier multiplier m_{ε,Z_N}, $\varepsilon \le N^{-C}$, extends to a bounded operator on $L^p(\mathbb{R}^d)$, with bound, say, $C_{p,\delta}N^{\delta}$, $1 < p < \infty$. Our orthogonality argument based on Lemma 2.1 in Section 2 does not seem sufficient, due to the interaction between denominators q with many small distinct primes. It is possible, however, to prove variants of Theorem 1.5; for instance one may replace the set $Z_{e^{N^\delta}}$ in (1.5) with $Z_{N^{C/\delta}}$ at the expense of replacing the bound $C_{p,\delta}(\ln N)^{2/\delta}$ in (1.6) with $C_{p,\delta}N^{\delta}$.

We prove first Theorem 1.5 in Sections 2, 3, and 4. The main ingredients are the square function estimate in Lemma 2.1 and the partition of integers in Lemma 3.1. Using this partition of integers, we divide first the set Y_N into a controlled number of disjoint subsets. Each of these subsets has a certain type of orthogonality property, which is roughly related to the super-orthogonality (2.6) (orthogonality in L^{2r}, $r \ge 1$ integer) needed in Lemma 2.2. This super-orthogonality allows us to estimate our operator in terms of square functions, which we then control using the Marcinkiewicz-Zygmund theorem and techniques inspired by [18]. Then we use Theorem 1.5 and estimates on the multiplier of the operator T to prove Theorem 1.1 (Sections 5, 6 and 7).

2. Estimates using square functions

In this section we prove a square function estimate that will be used in the proof of Theorem 1.5. The Fourier transform of a function $f \in L^1(\mathbb{Z}^d)$ is defined by the formula

$$\hat{f}(\xi) = \mathcal{F}_{\mathbb{Z}^d}(f)(\xi) = \sum_{n\in\mathbb{Z}^d} f(n)e^{-2\pi i n\cdot\xi}, \xi \in \mathbb{R}^d.$$

The function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ is periodic. The inverse Fourier transform is given by the formula

$$\mathcal{F}^{-1}_{\mathbb{Z}^d}(h)(n) = \int_{[0,1]^d} h(\xi)e^{2\pi i n\cdot\xi}\,d\xi,$$

for any periodic function h in, say, $L^2_{\text{loc}}(\mathbb{R}^d)$. By Plancherel's theorem, any bounded, periodic function $m : \mathbb{R}^d \to \mathbb{C}$ defines a bounded operator on $L^2(\mathbb{Z}^d)$ by the formula

$$T(f) = \mathcal{F}^{-1}_{\mathbb{Z}_d}(m \cdot \hat{f}).$$

Recall that for any integer $\mu \ge 1$, $Z_\mu = \{1, \dots, \mu\}$. For any integer $q \ge 1$ let

$$P_q = \{a \in \mathbb{Z}^d : (a, q) = 1\},$$

and let $\widetilde{P}_q = P_q \cap [Z_q]^d$.

Let $S_1, S_2, \dots, S_k$ denote sets of integers $S_j = \{q_{j,1}, \dots, q_{j,\beta(j)}\}$, $j \in Z_k$. Assume that for some $\widetilde{Q}$

$$q_{j,s} \in [2, \widetilde{Q}] \cap \mathbb{Z}, j \in Z_k, s \in Z_{\beta(j)}, \tag{2.1}$$

and

$$(q_{j,s}, q_{j',s'}) = 1 \tag{2.2}$$

if $(j, s) \ne (j', s')$. For any $j \in Z_k$ let

$$T_{\{j\}} = \{a_{j,s}/q_{j,s} : s \in Z_{\beta(j)}, a_{j,s} \in P_{q_{j,s}}\} \subset \mathbb{Q}^d$$

denote the set of irreducible d-fractions with denominators in S_j. Furthermore, for any set $A = \{j_1, \ldots, j_{k'}\} \subset Z_k$ let

$$T_A = \{\theta_{j_1} + \ldots + \theta_{j_{k'}} : \theta_{j_l} \in T_{\{j_l\}} \text{ for } l \in Z_{k'}\}.$$

Finally, for any $(s_{j_1}, \ldots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \ldots \times Z_{\beta(j_{k'})}$ let

$$U_{A,s_{j_1},\ldots,s_{j_{k'}}} = \{a_{j_1,s_{j_1}}/q_{j_1,s_{j_1}} + \ldots + a_{j_{k'},s_{j_{k'}}}/q_{j_{k'},s_{j_{k'}}} : a_{j_l,s_{j_l}} \in P_{q_{j_l,s_{j_l}}} \text{ for } l \in Z_{k'}\},$$

that is, the subset of elements of T_A with fixed denominators $q_{j_1,s_{j_1}}, \ldots, q_{j_{k'},s_{j_{k'}}}$. If $A = \emptyset$, then, by definition, $T_A = U_A = \mathbb{Z}^d$. Notice that the sets T_A and $U_{A,s_{j_1},\ldots,s_{j_{k'}}}$ are periodic subsets of $\mathbb{Q}^d$. Let $\widetilde{T}_A = T_A \cap [0,1)^d$ and $\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}} = U_{A,s_{j_1},\ldots,s_{j_{k'}}} \cap [0,1)^d$.

Assume that $r, Q \geq 1$ are integers and fix

$$\varepsilon \leq (8rQ^{2r}\widetilde{Q}^{2rk})^{-1}, \tag{2.3}$$

where $\widetilde{Q}$ is as in (2.1). Let Y denote an arbitrary set. For any $\sigma \in Y$ fix an integer $Q_\sigma \in Z_Q$ with the property that

$$(Q_\sigma, q_{j,s}) = 1, \sigma \in Y, j \in Z_k, s \in Z_{\beta(j)}. \tag{2.4}$$

For any $\theta \in T_{Z_k}$ and $\sigma \in Y$ let $f_\theta^\sigma \in L^2(\mathbb{Z}^d)$ denote a function whose Fourier transform is supported in an ε-neighborhood of the set $\{\theta + a/Q_\sigma : a \in \mathbb{Z}^d\}$, i.e., in the set

$$\bigcup_{a \in \mathbb{Z}^d} \theta + a/Q_\sigma + B(\varepsilon),$$

where $B(\varepsilon) = \{\xi = (\xi_1, \ldots, \xi_d) : |\xi_l| \leq \varepsilon\}$. We assume that $f_\theta^\sigma = f_{\theta+n}^\sigma$ for any $n \in \mathbb{Z}^d$. Let $(\mathbb{Z}^d, dn)$ denote the set of integers with the counting measure. The main estimate in this section is the following lemma.

Lemma 2.1. *With the notation above we have*

$$\begin{aligned} &\int_{\mathbb{Z}^d} (\sum_{\sigma \in Y} |\sum_{\theta \in \widetilde{T}_{Z_k}} f_\theta^\sigma(n)|^2)^r \, dn \\ &\quad \leq C_{k,r} \sum_{A=\{j_1,\ldots,j_{k'}\}} \sum_{s_{j_1},\ldots,s_{j_{k'}}} \int_{\mathbb{Z}^d} (\sum_{\sigma \in Y} \sum_{\theta' \in \widetilde{T}_{cA}} |\sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} f_{\mu+\theta'}^\sigma(n)|^2)^r \, dn, \end{aligned} \tag{2.5}$$

where the sum in the right-hand side is taken over all sets $A = \{j_1, \ldots, j_{k'}\} \subset Z_k$, and all $(s_{j_1}, \ldots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \ldots \times Z_{\beta(j_{k'})}$. The constant $C_{k,r}$ may depend on k, r, and d but not on $\widetilde{Q}, Q, |Y|, \varepsilon$, $q_{j,s}$, or the functions f_θ^σ.

Remarks. (1) We only use the estimate (2.5) with $|Y| = 1$. The vector valued version is needed to prove Lemma 2.1 by induction over k.

(2) The motivation for Lemma 2.1 is the following: the proof of Theorem 1.5 is based on exploiting cancellation to add up the pieces corresponding to numerators a and orthogonality (in L^{2r}) to deal with addition over denominators q. Lemma 2.1 is the main building block in this proof, as it establishes the orthogonality of the pieces corresponding to different denominators q, provided that (2.2) and (2.4) hold. It is important to notice that the terms in the right-hand side of (2.5) do

not contain any sums over denominators q inside the square function; for instance, when $A=\emptyset$, the corresponding term in the right-hand side of (2.5) is

$$\int_{\mathbb{Z}^d}(\sum_{\sigma\in Y}\sum_{\theta\in\widetilde{T}_{Z_k}}|f_\theta^\sigma(n)|^2)^r\,dn.$$

Such quadratic expressions may be controlled using standard Littlewood–Paley theory (see [18]).

The rest of this section is concerned with proving Lemma 2.1. We investigate first orthogonality conditions on families of functions $f_i:\mathbb{Z}^d\to\mathbb{C}$ that guarantee that the quantities

$$\int_{\mathbb{Z}^d}|\sum_i f_i(n)|^{2r}\,dn \quad\text{and}\quad \int_{\mathbb{Z}^d}(\sum_i|f_i(n)|^2)^r\,dn$$

are comparable, where $r\geq 1$ is an integer.

By convention, we write $(x_1,x_2,\dots,x_m)$ to denote the sequence $x_1,\dots,x_m$ (with possible repetitions, the order is relevant) and $\{x_1,x_2,\dots,x_m\}$ to denote the smallest set containing $x_1,x_2,\dots,x_m$ (no repetitions, irrelevant order). We say that a finite sequence $(x_1,x_2,\dots,x_m)$ has *the uniqueness property U* if there is $k\in\{1,2,\dots,m\}$ with the property that $x_i\neq x_k$ for any $i\in\{1,2,\dots,m\}\setminus\{k\}$.

Lemma 2.2. *Let X denote a finite set and $f_{i,l}\in L^2(\mathbb{Z}^d)$, $i\in X$, $l\in Z_r$, a set of functions. Assume that*

$$\int_{\mathbb{Z}^d} f_{i(1),1}(n)\overline{f}_{j(1),1}(n)\cdot\ldots\cdot f_{i(r),r}(n)\overline{f}_{j(r),r}(n)\,dn=0 \tag{2.6}$$

for any sequence $(i(1),j(1),\dots,i(r),j(r))$ in X with the uniqueness property U. Then

$$\int_{\mathbb{Z}^d}\prod_{l=1}^r|\sum_{i\in X}f_{i,l}(n)|^2\,dn\leq C_r\int_{\mathbb{Z}^d}\prod_{l=1}^r(\sum_{i\in X}|f_{i,l}(n)|^2)\,dn. \tag{2.7}$$

The constant C_r may depend on r and d but not on $|X|$ or the functions $f_{i,l}$.

Proof of Lemma 2.2. We have

$$\prod_{l=1}^r|\sum_{i\in X}f_{i,l}(n)|^2=\sum f_{i_0(1),1}(n)\overline{f}_{i_1(1),1}(n)\cdot\ldots\cdot f_{i_0(r),r}(n)\overline{f}_{i_1(r),r}(n),$$

where the sum is taken over all possible choices of $i_0(1),i_1(1),\dots,i_0(r),i_1(r)$ in X (there are a total of $|X|^{2r}$ terms in the sum). For any set $A\subset X$ with $|A|\leq 2r$ let

$$S_A(n)=\sum f_{i_0(1),1}(n)\overline{f}_{i_1(1),1}(n)\cdot\ldots\cdot f_{i_0(r),r}(n)\overline{f}_{i_1(r),r}(n), \tag{2.8}$$

where the sum is taken over the sequences $(i_0(1),i_1(1),\dots,i_0(r),i_1(r))$ which do not have the uniqueness property U and, in addition, have the property that $\{i_0(1),i_1(1),\dots,i_0(r),i_1(r)\}=A$ as sets. By (2.6)

$$\int_{\mathbb{Z}^d}\prod_{l=1}^r|\sum_{i\in X}f_{i,l}(n)|^2\,dn=\sum_A\int_{\mathbb{Z}^d}S_A(n)\,dn. \tag{2.9}$$

Notice that, by the definition of the uniqueness property U, we have $S_A\equiv 0$ unless $|A|\leq r$. Thus the sum in (2.9) is taken over sets $A\subset X$ with $|A|\leq r$.

By expanding the product in the right-hand side of (2.7), we have

$$\int_{\mathbb{Z}^d}\prod_{l=1}^{r}\Big(\sum_{i\in X}|f_{i,l}(n)|^2\Big)\,dn=\sum_{A}\int_{\mathbb{Z}^d}T_A(n)\,dn, \tag{2.10}$$

where

$$T_A(n)=\sum_{\{i(1),\ldots,i(r)\}=A}|f_{i(1),1}(n)|^2\cdot\ldots\cdot|f_{i(r),r}(n)|^2. \tag{2.11}$$

As before, the sum in (2.10) is taken over all sets $A\subset X$ with $|A|\le r$. Thus, it suffices to prove that

$$\int_{\mathbb{Z}^d}|S_A(n)|\,dn\le C_r\int_{\mathbb{Z}^d}T_A(n)\,dn$$

for any $A\subset X$, $|A|\le r$. For any set A the sum S_A in (2.8) contains at most C_r terms. Thus it suffices to prove that

$$\int_{\mathbb{Z}^d}|f_{i_0(1),1}(n)\overline{f}_{i_1(1),1}(n)\cdot\ldots\cdot f_{i_0(r),r}(n)\overline{f}_{i_1(r),r}(n)|\,dn\le 2\int_{\mathbb{Z}^d}T_A(n)\,dn \tag{2.12}$$

whenever the sequence $(i_0(1),i_1(1),\ldots,i_0(r),i_1(r))$ does not have the uniqueness property U and $\{i_0(1),i_1(1),\ldots,i_0(r),i_1(r)\}=A$ (as sets).

To prove (2.12), we claim that there are two functions $\beta,\gamma:\{1,\ldots,r\}\to\{0,1\}$, $\gamma(l)=1-\beta(l)$, with the property that

$$\{i_{\beta(1)}(1),i_{\beta(2)}(2),\ldots,i_{\beta(r)}(r)\}=\{i_{\gamma(1)}(1),i_{\gamma(2)}(2),\ldots,i_{\gamma(r)}(r)\}=A. \tag{2.13}$$

To clarify the role of the functions β and γ, assume that we write the sequences $(i_0(1),i_0(2),\ldots,i_0(r))$ and $(i_1(1),i_1(2),\ldots,i_1(r))$ in a matrix with two rows and r columns. Our goal is to partition the elements of this matrix into two sequences with r elements, in such a way that each sequence contains exactly one entry from each column of the matrix and the sets generated by the two sequences are both equal to A. This guarantees the fact that the corresponding product is controlled by two of the sums of the definition of T_A (see (2.11)). Formally, assuming (2.13), we would have

$$\begin{aligned}\int_{\mathbb{Z}^d}&|f_{i_0(1),1}(n)\overline{f}_{i_1(1),1}(n)\cdot\ldots\cdot f_{i_0(r),r}(n)\overline{f}_{i_1(r),r}(n)|\,dn\\&\le\int_{\mathbb{Z}^d}|f_{i_{\beta(1)}(1),1}(n)|^2\cdot\ldots\cdot|f_{i_{\beta(r)}(r),r}(n)|^2\,dn\\&+\int_{\mathbb{Z}^d}|f_{i_{\gamma(1)}(1),1}(n)|^2\cdot\ldots\cdot|f_{i_{\gamma(r)}(r),r}(n)|^2\,dn\le 2\int_{\mathbb{Z}^d}T_A(n)\,dn,\end{aligned}$$

which gives (2.12).

It remains to construct the functions β and γ. Assume first that $|A|=r$; since the sequence $(i_0(1),i_1(1),\ldots,i_0(r),i_1(r))$ does not have the uniqueness property U, every element in this sequence appears exactly twice. We argue by induction over r to prove the following statement: if every element in the sequence $(i_0(1),i_1(1),\ldots,i_0(r),i_1(r))$ appears exactly twice, then there are two functions $\beta,\gamma:\{1,\ldots,r\}\to\{0,1\}$, $\gamma(l)=1-\beta(l)$, with the property that

$$\begin{aligned}\{i_{\beta(1)}(1),i_{\beta(2)}(2),\ldots,i_{\beta(r)}(r)\}&=\{i_{\gamma(1)}(1),i_{\gamma(2)}(2),\ldots,i_{\gamma(r)}(r)\}\\&=\{i_0(1),i_1(1),\ldots,i_0(r),i_1(r)\}.\end{aligned}$$

The case $r = 1$ is clear. For $r \geq 2$ define $l_1 = 1$, $\beta(1) = 0$ and $\gamma(1) = 1$. If $i_0(1) = i_1(1)$, then the induction hypothesis applies. Otherwise, there is a unique element in the sequence $(i_0(1), i_1(1), \ldots, i_0(r), i_1(r)) \setminus (i_0(1), i_1(1))$ equal to $i_{\gamma(1)}(1)$; this element is, say, $i_\sigma(l_2) = i_{\gamma(1)}(1)$, for some $l_2 \in \{1, \ldots, r\} \setminus \{1\}$ and $\sigma \in \{0, 1\}$. Define $\beta(l_2) = \sigma$ and $\gamma(l_2) = 1 - \sigma$. We can now proceed recursively: assume $l_1, \ldots, l_k$ are distinct numbers in $\{1, \ldots, r\}$ and the functions $\beta(l_1), \ldots, \beta(l_k)$ and $\gamma(l_1), \ldots, \gamma(l_k)$ are constructed in such a way that $i_{\beta(l_j)}(l_j) = i_{\gamma(l_{j-1})}(l_{j-1})$, $j = 2, \ldots, k$. If $i_{\beta(l_1)}(1) = i_{\gamma(l_k)}(l_k)$, then we apply the induction hypothesis to the remaining sequence with $2(r-k)$ terms (if $k = r$, there is nothing left to prove). Otherwise, $k \leq r-1$ and there is a unique element in the remaining sequence equal to $i_{\gamma(l_k)}(l_k)$. We call this element $i_\sigma(l_{k+1})$ and define $\beta(l_{k+1}) = \sigma$ and $\gamma(l_{k+1}) = 1 - \sigma$. At the end of this process we have $i_{\beta(l_1)}(1) = i_{\gamma(l_k)}(l_k)$. If $k \leq r-1$, we apply the induction hypothesis on the remaining sequence. If $k = r$, then we have already achieved the required partition.

Assume now that $|A| = r' \leq r - 1$. In this case every element in the sequence $(i_0(1), i_1(1), \ldots, i_0(r), i_1(r))$ appears at least twice, and some elements appear at least three times. In the sequence $(i_0(1), i_1(1), \ldots, i_0(r), i_1(r))$ we replace every element that appears for the third time with some element $a \notin X$. The new sequence will contain every element in A repeated twice and $2(r - r')$ terms equal to a. Then we pair the terms equal to a (in an arbitrary way) and replace them with pairs of $a_1, \ldots, a_{r-r'}$, where $a_1, \ldots, a_{r-r'}$ are arbitrary distinct elements not in X. The resulting sequence will have every element appearing exactly twice, so the construction in the previous paragraph applies. □

We also need the following elementary estimate:

Lemma 2.3. *For any numbers $a_1, a_2, \ldots, a_\nu \in [0, \infty)$ and integer $r \geq 1$*

$$(a_1 + \ldots + a_\nu)^r \leq C_r\Big(\sum_{1 \leq i \leq \nu} a_i^r + \sum_{1 \leq i_1 < \ldots < i_r \leq \nu} a_{i_1} \cdot \ldots \cdot a_{i_r}\Big). \tag{2.14}$$

The constant C_r may depend on r but not on ν or the numbers a_i.

Proof of Lemma 2.3. We may assume $r \geq 2$. By expanding the products, we see easily that

$$(a_1 + \ldots + a_\nu)^r \leq A_r\Big[\sum_{1 \leq i_1 < \ldots < i_r \leq \nu} a_{i_1} \cdot \ldots \cdot a_{i_r} + \Big(\sum_{1 \leq i \leq \nu} a_i^2\Big)\Big(\sum_{1 \leq i \leq \nu} a_i\Big)^{r-2}\Big]. \tag{2.15}$$

We prove (2.14) with $C_r = (4A_r + 1)^{r-1}$. If

$$(a_1 + \ldots + a_\nu)^r \leq 2A_r \sum_{1 \leq i_1 < \ldots < i_r \leq \nu} a_{i_1} \cdot \ldots \cdot a_{i_r},$$

then (2.14) is clearly verified. Otherwise, by (2.15)

$$(a_1 + \ldots + a_\nu)^2 \leq 2A_r(a_1^2 + \ldots + a_\nu^2). \tag{2.16}$$

Let m denote the smallest integer larger than $4A_r$. We may assume that the numbers $a_1, \ldots, a_\nu$ are ordered, i.e., $a_1 \geq a_2 \geq \ldots \geq a_\nu \geq 0$. By (2.16),

$$(a_1 + \ldots + a_\nu)^2 + (a_1 + \ldots + a_m)(a_{m+1} + \ldots + a_\nu) \leq 4A_r(a_1^2 + \ldots + a_\nu^2).$$

Since the numbers a_i are in decreasing order and $m \geq 4A_r$, we have

$$(a_1 + \ldots + a_m)(a_{m+1} + \ldots + a_\nu) \geq 4A_r(a_{m+1}^2 + \ldots + a_\nu^2).$$

Thus

$$(a_1 + \ldots + a_\nu)^2 \leq 4A_r(a_1^2 + \ldots + a_m^2).$$

By the Hölder inequality

$$(a_1 + \ldots + a_\nu)^r \leq (4A_r)^{r/2}(a_1^2 + \ldots + a_m^2)^{r/2} \leq (4A_r)^{r/2} m^{r/2-1}(a_1^r + \ldots + a_m^r),$$

which proves (2.14). □

We turn now to the proof of Lemma 2.1. We argue by induction over k.

The case $k = 1$. In this case there is only one set of integers $S_1 = \{q_{1,s} = q_s : s \in Z_{\beta(1)}\}$. Let

$$F_s^\sigma = \sum_{a_s \in \widetilde{P}_{q_s}} f^\sigma_{a_s/q_s}.$$

The left-hand side of (2.5) is equal to

$$\begin{aligned}&\int_{\mathbb{Z}^d} (\sum_{\sigma \in Y} |\sum_{s \in Z_{\beta(1)}} F_s^\sigma(n)|^2)^r \, dn \\ &\qquad = \sum_{\sigma_1, \ldots, \sigma_r \in Y} \int_{\mathbb{Z}^d} |\sum_{s \in Z_{\beta(1)}} F_s^{\sigma_1}(n)|^2 \cdot \ldots \cdot |\sum_{s \in Z_{\beta(1)}} F_s^{\sigma_r}(n)|^2 \, dn.\end{aligned}$$

We would like to apply Lemma 2.2 for any $\sigma_1, \ldots, \sigma_r$ fixed. We have to verify the orthogonality property (2.6), i.e.,

$$\int_{\mathbb{Z}^d} F_{s_1}^{\sigma_1}(n) \overline{F}_{t_1}^{\sigma_1}(n) \cdot \ldots \cdot F_{s_r}^{\sigma_r}(n) \overline{F}_{t_r}^{\sigma_r}(n) \, dn = 0 \tag{2.17}$$

for any sequence $(s_1, t_1, \ldots, s_r, t_r)$ with the uniqueness property U. By symmetry, we may assume that $s_1 \neq s_l$ for any $l \in Z_r \setminus \{1\}$ and $s_1 \neq t_l$ for any $l \in Z_r$. Then the Fourier transform of $F_{s_1}^{\sigma_1}$ is supported in the set

$$\bigcup_{a \in \mathbb{Z}^d, a_{s_1} \in \widetilde{P}_{q_{s_1}}} a_{s_1}/q_{s_1} + a/Q_{\sigma_1} + B(\varepsilon).$$

On the other hand, the Fourier transform of $\overline{F}_{t_1}^{\sigma_1} \cdot \ldots \cdot F_{s_r}^{\sigma_r} \cdot \overline{F}_{t_r}^{\sigma_r}$ is supported in the set obtained as the algebraic sum of the supports of each one of the functions (notice that all the sets involved are symmetric under the map $\xi \to -\xi$) . This set is

$$\bigcup_{\mu, \theta'} \theta' + \mu + B((2r-1)\varepsilon),$$

where θ' is some number of the form $a_{t_1}/q_{t_1} + \ldots + a_{s_r}/q_{s_r} + a_{t_r}/q_{t_r}$ and μ is some number of the form $a_1/Q_{\sigma_1} + \ldots + a_r/Q_{\sigma_r}$. By (2.2), (2.4), and (2.3) the support sets are disjoint, so (2.17) follows.

By Lemma 2.2, the left-hand side of (2.5) is dominated by

$$\begin{aligned}&C_r \sum_{\sigma_1, \ldots, \sigma_r \in Y} \int_{\mathbb{Z}^d} (\sum_{s \in Z_{\beta(1)}} |F_s^{\sigma_1}(n)|^2) \cdot \ldots \cdot (\sum_{s \in Z_{\beta(1)}} |F_s^{\sigma_r}(n)|^2) \, dn \\ &\qquad = C_r \int_{\mathbb{Z}^d} [\sum_{s \in Z_{\beta(1)}} (\sum_{\sigma \in Y} |F_s^\sigma(n)|^2)]^r \, dn.\end{aligned}$$

We use now Lemma 2.3. Thus the left-hand side of (2.5) is dominated by

$$\begin{aligned}(2.18)\qquad & C_r \sum_{s\in Z_{\beta(1)}} \int_{\mathbb{Z}^d} (\sum_{\sigma\in Y} |F_s^\sigma(n)|^2)^r\, dn \\ & + C_r \sum_{1\le s_1<\ldots<s_r\le\beta(1)} \int_{\mathbb{Z}^d} (\sum_{\sigma\in Y} |F_{s_1}^\sigma(n)|^2)\cdot\ldots\cdot(\sum_{\sigma\in Y}|F_{s_r}^\sigma(n)|^2)\, dn.\end{aligned}$$

Notice that the first term in (2.18) coincides with the term in the right-hand side of (2.5) corresponding to $A=\{1\}$. We write the second term in (2.18) as

$$\begin{aligned}& C_r \sum_{\sigma_1,\ldots,\sigma_r\in Y} \sum_{1\le s_1<\ldots<s_r\le\beta(1)} \int_{\mathbb{Z}^d} |F_{s_1}^{\sigma_1}(n)|^2\cdot\ldots\cdot|F_{s_r}^{\sigma_r}(n)|^2\, dn \\ & = C_r \sum_{\sigma_1,\ldots,\sigma_r\in Y} \sum_{1\le s_1<\ldots<s_r\le\beta(1)} \int_{\mathbb{Z}^d} |\sum_{a_{s_1}\in\widetilde{P}_{q_{s_1}}} f^{\sigma_1}_{a_{s_1}/q_{s_1}}(n)|^2 \\ & \quad \cdot\ldots\cdot|\sum_{a_{s_r}\in\widetilde{P}_{q_{s_r}}} f^{\sigma_r}_{a_{s_r}/q_{s_r}}(n)|^2\, dn.\end{aligned}$$

We would like to apply again Lemma 2.2, with the set X equal to the disjoint union of the sets $\widetilde{P}_{q_{s_l}}$. The orthogonality property (2.6) is satisfied for the same reason as before, since the Fourier transform of $f^{\sigma_l}_{a_{s_l}/q_{s_l}}$ is supported in the set

$$\bigcup_{a\in\mathbb{Z}^d} a_{s_l}/q_{s_l} + a/Q_{\sigma_l} + B(\varepsilon).$$

Then we use the fact that the numbers q_{s_l} are pairwise relatively prime, since $s_1<\ldots<s_r$. Thus the second term in (2.18) can be dominated by

$$\begin{aligned}& C_r \sum_{\sigma_1,\ldots,\sigma_r\in Y} \sum_{1\le s_1<\ldots<s_r\le\beta(1)} \\ & \int_{\mathbb{Z}^d} (\sum_{a_{s_1}\in\widetilde{P}_{q_{s_1}}} |f^{\sigma_1}_{a_{s_1}/q_{s_1}}(n)|^2)\cdot\ldots\cdot(\sum_{a_{s_r}\in\widetilde{P}_{q_{s_r}}} |f^{\sigma_r}_{a_{s_r}/q_{s_r}}(n)|^2)\, dn \\ & \quad \le C_r \int_{\mathbb{Z}^d} (\sum_{\sigma\in Y} \sum_{1\le s\le\beta(1)} \sum_{a_s\in\widetilde{P}_{q_s}} |f^\sigma_{a_s/q_s}(n)|^2)^r\, dn.\end{aligned}$$

This coincides with the term in the right-hand side of (2.5) corresponding to $A=\emptyset$.

The induction step. We regard Z_{k-1} as a subset of Z_k. Let θ' denote generic elements in $T_{Z_{k-1}}$; for any $\theta'\in T_{Z_{k-1}}$ and $s\in Z_{\beta(k)}$ let

$$F^\sigma_{\theta',s} = \sum_{a_s\in P_{q_{k,s}}} f^\sigma_{\theta'+a_s/q_{k,s}}.$$

We argue as before. The left-hand side of (2.5) is equal to

$$\int_{\mathbb{Z}^d}\Big(\sum_{\sigma\in Y}\Big|\sum_{s\in Z_{\beta(k)}}\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s}(n)\Big|^2\Big)^r\,dn$$

$$=\sum_{\sigma_1,\ldots,\sigma_r\in Y}\int_{\mathbb{Z}^d}\Big|\sum_{s\in Z_{\beta(k)}}\Big(\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma_1}_{\theta',s}(n)\Big)\Big|^2\cdot\ldots\cdot\Big|\sum_{s\in Z_{\beta(k)}}\Big(\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma_r}_{\theta',s}(n)\Big)\Big|^2\,dn.$$

We apply Lemma 2.2 for any $\sigma_1,\ldots,\sigma_r$ fixed. The orthogonality property (2.6) is verified as in the proof of (2.17), since the Fourier transform of the function $\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s}$ is supported in the set

$$\bigcup_{\theta'\in\widetilde{T}_{Z_{k-1}}}\ \bigcup_{a\in\mathbb{Z}^d,a_s\in\widetilde{P}_{q_{k,s}}}a_s/q_{k,s}+a/Q_\sigma+\theta'+B(\varepsilon).$$

Thus the left-hand side of (2.5) is dominated by

$$C_r\int_{\mathbb{Z}^d}\Big[\sum_{s\in Z_{\beta(k)}}\Big(\sum_{\sigma\in Y}\Big|\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s}(n)\Big|^2\Big)\Big]^r\,dn.$$

We use now Lemma 2.3. It follows that the left-hand side of (2.5) is dominated by

$$\begin{aligned}&C_r\sum_{s\in Z_{\beta(k)}}\int_{\mathbb{Z}^d}\Big(\sum_{\sigma\in Y}\Big|\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s}(n)\Big|^2\Big)^r\,dn\\&+C_r\sum_{1\le s_1<\ldots<s_r\le\beta(k)}\int_{\mathbb{Z}^d}\Big(\sum_{\sigma\in Y}\Big|\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s_1}(n)\Big|^2\Big)\\&\qquad\cdot\ldots\cdot\Big(\sum_{\sigma\in Y}\Big|\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}F^{\sigma}_{\theta',s_r}(n)\Big|^2\Big)\,dn.\end{aligned}\tag{2.19}$$

For the first term in (2.19) we can apply the induction hypothesis for any fixed s, with Q replaced by $Q\widetilde{Q}$. This is possible since the functions $F^{\sigma}_{\theta',s}$ are supported in an ε-neighborhood of the set $\{\theta'+a/Q_\sigma q_{k,s}:a\in\mathbb{Z}^d\}$. Also, the inequality (2.3) remains valid if we replace k with $k-1$ and Q with $Q\widetilde{Q}$. By induction, the first term in (2.19) is controlled by the sum in the right-hand side of (2.5) corresponding to sets A with $k\in A$.

For the second term in (2.19), we write it in the form

$$\begin{aligned}C_r\sum_{\sigma_1,\ldots,\sigma_r\in Y}\ \sum_{1\le s_1<\ldots<s_r\le\beta(k)}\int_{\mathbb{Z}^d}&\Big|\sum_{a_{s_1}\in\widetilde{P}_{q_{k,s_1}}}\ \sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}f^{\sigma_1}_{\theta'+a_{s_1}/q_{k,s_1}}(n)\Big|^2\\&\cdot\ldots\cdot\Big|\sum_{a_{s_r}\in\widetilde{P}_{q_{k,s_r}}}\ \sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}f^{\sigma_r}_{\theta'+a_{s_r}/q_{k,s_r}}(n)\Big|^2\,dn.\end{aligned}$$

As before, we apply again Lemma 2.2, with the set X equal to the disjoint union of the sets $\widetilde{P}_{q_{k,s_l}}$. The orthogonality property (2.6) is satisfied for the same reason as before, since the Fourier transform of $\sum_{\theta'\in\widetilde{T}_{Z_{k-1}}}f^{\sigma_l}_{\theta'+a_{s_l}/q_{k,s_l}}$ is supported in the set

$$\bigcup_{\theta'\in\widetilde{T}_{Z_{k-1}}}\ \bigcup_{a\in\mathbb{Z}^d}a_{s_l}/q_{k,s_l}+a/Q_{\sigma_l}+\theta'+B(\varepsilon).$$

Then we use the fact that the numbers q_{k,s_l} are pairwise relatively prime, since $s_1 < \ldots < s_r$. Thus the second term in (2.19) can be dominated by

$$C_r \int_{\mathbb{Z}^d} \Big(\sum_{\sigma\in Y} \sum_{1\le s\le\beta(k)} \sum_{a_s\in\widetilde{P}_{q_{k,s}}} \Big| \sum_{\theta'\in\widetilde{T}_{Z_{k-1}}} f^{\sigma}_{\theta'+a_s/q_{k,s}}(n)\Big|^2\Big)^r \, dn.$$

For this last term we apply the induction hypothesis, with Y replaced by $Y\times\{s, a_s : s\in Z_{\beta_k}, a_s\in\widetilde{P}_{q_{k,s}}\}$, Q replaced by $Q\widetilde{Q}$, and functions $f'^{\sigma'}_{\theta'} = f^{\sigma}_{\theta'+a_s/q_{k,s}}$. It follows that the second term in (2.19) is controlled by the sum in the right-hand side of (2.5) corresponding to sets A with $k\notin A$.

3. A partition of integers

We construct now the set Y_N in Theorem 1.5 and partition it in a way that is suitable for applying Lemma 2.1. With the notation of Theorem 1.5 (assume $N\ge 10$), let N' denote the smallest integer $\ge N^{\delta/2}$ and $V=\{p_1,p_2,\ldots,p_\nu\}$ the set of prime numbers between $N'+1$ and N. Let D denote the smallest integer $\ge 2/\delta$ and $Q_0=[N'!]^D$. For any $k\in Z_D$ let

$$W^k(V)=\{p_{i_1}^{\alpha_{i_1}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}} : p_{i_l}\in V \text{ distinct}, \alpha_{i_l}\in Z_D, l=1,\ldots,k\},$$

and let $W(V)=\bigcup_{k\in Z_D} W^k(V)$ denote the set of products of up to D factors in V, at powers between 1 and D. Notice that for any $m\in Z_N$ there is a unique decomposition $m=w\cdot Q'$, with $w\in W(V)\cup\{1\}$ and $Q'|Q_0$.

We say that a subset $W'\subset W(V)$ has *the orthogonality property* O if there is $k\in Z_D$ and k sets $S_1,S_2,\ldots,S_k$, $S_j=\{q_{j,1},\ldots,q_{j,\beta(j)}\}$, $j\in Z_k$, with the following properties:

(i) $q_{j,s}=p_{j,s}^{\alpha_j}$ for some $p_{j,s}\in V$, $\alpha_j\in Z_D$;

(ii) $(q_{j,s},q_{j',s'})=1$ if $(j,s)\neq(j',s')$;

(iii) for any $w'\in W'$ there are (unique) numbers $q_{1,s_1}\in S_1,\ldots,q_{k,s_k}\in S_k$ with $w'=q_{1,s_1}\cdot\ldots\cdot q_{k,s_k}$.

For simplicity of notation, we say that the set $W'=\{1\}$ has the orthogonality property O with $k=0$. The orthogonality property O is connected to Lemma 2.1. Notice that if a set has the orthogonality property O, then all its elements have the same number of prime factors. The main result in this section is the following decomposition.

Lemma 3.1. *$W(V)$ can be written as a disjoint union of at most $C_D(\ln N)^{D-1}$ sets with the orthogonality property O.*

We emphasize that all the constants in this section may depend on D or $k\in Z_D$ (thus on δ) but not on N or ν, the number of primes in V.

Proof of Lemma 3.1. Notice that any subset of a set with the orthogonality property O has the orthogonality property O as well. Therefore, for any $k\in Z_D$, it suffices to write $W^k(V)$ as a union of at most $C_D(\ln N)^{k-1}$ (not necessarily disjoint) sets with the orthogonality property O. Notice also that it suffices to write the smaller set

$$\widetilde{W}^k(V)=\{p_{i_1}\cdot\ldots\cdot p_{i_k} : p_{i_l}\in V \text{ distinct}\}$$

as a union of at most $C_D(\ln N)^{k-1}$ (not necessarily disjoint) sets with the orthogonality property O. This is because the number of the possible exponents $\alpha_{i_1},\ldots,\alpha_{i_k}\in Z_D$ is D^k.

We prove by induction over k that for any set $V' \subset V$, the set

$$\widetilde{W}^k(V') = \{p_{i_1} \cdot \ldots \cdot p_{i_k} : p_{i_l} \in V' \text{ distinct}\}$$

can be written as a union of at most $C_D(\ln N)^{k-1}$ (not necessarily disjoint) sets with the orthogonality property O.

For $k = 1$ we simply define

$$W_1'^1(V') = V',$$

which has the orthogonality property O and $W_1'^1(V') = \widetilde{W}^1(V')$.

For the general case we start with a dyadic decomposition of the set V:

$$V = V_{1,1} \cup V_{1,2} = V_{2,1} \cup V_{2,2} \cup V_{2,3} \cup V_{2,4} = \ldots = V_{m,1} \cup \ldots \cup V_{m,2^m},$$

where m is the smallest integer with the property that $2^m \geq \nu$, $V_{\mu,s} = V_{\mu+1,2s-1} \cup V_{\mu+1,2s}$ (disjoint union) for any $\mu \in Z_{m-1}$ and $s \in Z_{2^\mu}$, and $||V_{\mu,2s-1}| - |V_{\mu,2s}|| \leq 1$ for any $\mu \in Z_m$ and $s \in Z_{2^{\mu-1}}$. Then $m \leq C(\ln N)$ and $|V_{m,s}| \in \{0,1\}$ for any $s \in Z_{2^m}$. For $\mu \in Z_m$ we define the sets

$$\begin{cases} G_\mu = \bigcup_{s \in Z_{2^{\mu-1}}} V_{\mu,2s-1}; \\ H_\mu = \bigcup_{s \in Z_{2^{\mu-1}}} V_{\mu,2s}. \end{cases}$$

Clearly, the sets G_μ and H_μ are disjoint, $G_\mu \cup H_\mu = V$ for any $\mu \in Z_m$, and, most importantly, for any subset A of V with $|A| \geq 2$ there is $\mu \in Z_m$ with the property that

$$A \cap G_\mu \neq \emptyset \text{ and } A \cap H_\mu \neq \emptyset. \tag{3.1}$$

This last property is easy to verify using the definitions.

To complete the proof in the case $k = 2$, we simply define the sets

$$W_\mu'^2(V') = (G_\mu \cap V') \cdot (H_\mu \cap V') = \{gh : g \in G_\mu \cap V', h \in H_\mu \cap V'\}, \mu \in Z_m.$$

These m sets have the orthogonality property O and, by (3.1),

$$\bigcup_{\mu \in Z_m} W_\mu'^2(V') = \widetilde{W}^2(V').$$

For general k, we apply the induction hypothesis for the sets $G_\mu \cap V'$ and $H_\mu \cap V'$: for any $k_1, k_2 \in \{1, \ldots, k-1\}$ with $k_1 + k_2 = k$ and $\mu \in Z_m$ let

$$\widetilde{W}^{k_1}(G_\mu \cap V') = \bigcup_{s_1} W_{s_1}'^{k_1}(G_\mu \cap V'), 1 \leq s_1 \leq C_D(\ln N)^{k_1-1}$$

and

$$\widetilde{W}^{k_2}(H_\mu \cap V') = \bigcup_{s_2} W_{s_2}'^{k_2}(H_\mu \cap V'), 1 \leq s_2 \leq C_D(\ln N)^{k_2-1}$$

denote the decompositions in sets with the orthogonality property O of the sets $\widetilde{W}^{k_1}(G_\mu \cap V')$ and $\widetilde{W}^{k_2}(H_\mu \cap V')$. Then, we consider the sets

$$W_{s_1}'^{k_1}(G_\mu \cap V') \cdot W_{s_2}'^{k_2}(H_\mu \cap V') = \{gh : g \in W_{s_1}'^{k_1}(G_\mu \cap V'), h \in W_{s_2}'^{k_2}(H_\mu \cap V')\},$$

for all possible choices of $\mu \in Z_m$, $k_1 + k_2 = k$, s_1, and s_2. There are at most $C_D(\ln N)^{k-1}$ such sets, by the induction hypothesis. Also, these sets have the orthogonality property O (by the induction hypothesis) and their union is equal to $\widetilde{W}^k(V')$, by (3.1). This completes the proof of the lemma. $\square$

4. Proof of Theorem 1.5

In this section we complete the proof of Theorem 1.5. With the notation in Section 3, we define

$$Y_N = \{w \cdot Q' : w \in W(V) \cup \{1\}, Q'|Q_0 = [N'!]^D\}.$$

Clearly, $Z_N \subset Y_N$. Also, $w \cdot Q' \leq N^{D^2}[N'!]^D \leq e^{N^\delta}$ if $N \geq A_\delta$. Thus $Y_N \subset Z_{e^{N^\delta}}$. Using Lemma 3.1, we have the decomposition

$$W(V) \cup \{1\} = \bigcup_s W'_s \text{ as a disjoint union,}$$

where s belongs to a set of cardinality $\leq C_\delta(\ln N)^{2/\delta}$ and the sets W'_s have the orthogonality property O. We define

$$Y^s_N = \{w \cdot Q' : w \in W'_s, Q'|Q_0\}.$$

Then $\mathcal{R}(Y_N) = \bigcup_s \mathcal{R}(Y^s_N)$ as a disjoint union. By Lemma 3.1 and Remark (1) following Theorem 1.5, it suffices to prove that if $W' \subset W(V) \cup \{1\}$ has the orthogonality property O and $Y' = \{w' \cdot Q' : w' \in W', Q'|Q_0\}$, then

$$||\mathcal{F}^{-1}_{\mathbb{Z}^d}(m_{\varepsilon,Y'}\hat{f})||_{L^p(\mathbb{Z}^d)} \leq C_{p,\delta}||f||_{L^p(\mathbb{Z}^d)}. \tag{4.1}$$

For this we start with a simpler lemma. As in Theorem 1.5, assume that m is a multiplier on $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, supported in the cube $[-1/2,1/2]^d$, $Q \geq 1$ is an integer, and $\varepsilon \leq (4Q)^{-1}$.

Lemma 4.1. *The operator S_Q defined by the Fourier multiplier*

$$m^Q_\varepsilon(\xi) = \sum_{a\in\mathbb{Z}^d} m((\xi - a/Q)/\varepsilon)$$

extends to a bounded operator on $L^p(\mathbb{Z}^d)$, $1 < p < \infty$, with

$$||S_Q(f)||_{L^p(\mathbb{Z}^d)} \leq C_p||f||_{L^p(\mathbb{Z}^d)}.$$

The constants C_p may depend on p and the constants B_p in (1.4) *but not on Q or ε.*

Lemma 4.1 follows from [11, Corollary 2.1]. Lemma 4.1 proves (4.1) in the case $W' = \{1\}$, $\mathcal{R}(Y') = a/Q_0$, $a \in \mathbb{Z}^d$. For the proof of (4.1) in general, we need a stronger version. Assume that m is as before, $Q \geq 1$ is an integer, $q_1 = p_1^{\alpha_1}, \dots, q_k = p_k^{\alpha_k}$ are powers of primes, with $(q_l, q_{l'}) = 1$ if $l, l' \in Z_k$, $l \neq l'$, and $(Q, q_l) = 1$ if $l \in Z_k$. Assume that $\varepsilon \leq (4Qq_1 \cdot \dots \cdot q_k)^{-1}$.

Corollary 4.2. *The operator $S_{Q,q_1,\dots,q_k}$ defined by the Fourier multiplier*

$$m^{Q,q_1,\dots,q_k}_\varepsilon(\xi) = \sum_{a\in\mathbb{Z}^d} \sum_{a_1\in\widetilde{P}_{q_1}} \cdots \sum_{a_k\in\widetilde{P}_{q_k}} m((\xi - a/Q - a_1/q_1 - \dots - a_k/q_k)/\varepsilon)$$

extends to a bounded operator on $L^p(\mathbb{Z}^d)$, $1 < p < \infty$, with

$$||S_{Q,q_1,\dots,q_k}(f)||_{L^p(\mathbb{Z}^d)} \leq C_{p,k}||f||_{L^p(\mathbb{Z}^d)}.$$

The constants $C_{p,k}$ may depend on p, k, and the constants B_p in (1.4) *but not on Q, $q_1, \dots, q_k$, or ε.*

Proof of Corollary 4.2. A standard counting argument shows that if Q_1 and Q_2 are integers ≥ 1 with $(Q_1, Q_2) = 1$ and if $a \in \mathbb{Z}^d$, then there are unique $a_1, a_2 \in \mathbb{Z}^d$ such that

$$a/(Q_1 Q_2) = a_1/Q_1 + a_2/Q_2 (\operatorname{mod} \mathbb{Z}^d) \text{ and } a_1/Q_1, a_2/Q_2 \in [0,1)^d. \tag{4.2}$$

Since $q_1 = p_1^{\alpha_1}$, this shows easily that

$$m_\varepsilon^{Q, p_1^{\alpha_1}, q_2, \ldots, q_k} = m_\varepsilon^{Q \cdot p_1^{\alpha_1}, q_2, \ldots, q_k} - m_\varepsilon^{Q \cdot p_1^{\alpha_1 - 1}, q_2, \ldots, q_k}.$$

The corollary follows from Lemma 4.1 and induction over k. $\square$

We turn now to the proof of (4.1). By interpolation, it suffices to prove (4.1) for $p = 2r$, where $r \geq 1$ is an integer. Notice that we may assume that $N \geq C_{r,\delta}$. We may also assume that f is the characteristic function of a finite set, i.e., $f : \mathbb{Z}^d \to \{0, 1\}$.

By (4.2), $\mathcal{R}(Y') = \{b/Q_0 + a'/w' : b \in \mathbb{Z}^d, a'/w' \in \widetilde{\mathcal{R}}(W') = \mathcal{R}(W') \cap [0,1)^d\}$. Thus

$$m_{\varepsilon, Y'}(\xi) = \sum_{a'/w' \in \widetilde{\mathcal{R}}(W')} \sum_{b \in \mathbb{Z}^d} m((\xi - a'/w' - b/Q_0)/\varepsilon).$$

Recall that the set W' has the orthogonality property O. Thus there is $k \leq 2/\delta + 1$ and k sets $S_1, S_2, \ldots, S_k$, $S_j = \{q_{j,1}, \ldots, q_{j,\beta(j)}\}$, $j \in Z_k$, with the properties stated in the definition of the orthogonality property O (Section 3). For any $s_1 \in Z_{\beta(1)}, \ldots, s_k \in Z_{\beta(k)}$ let $\beta(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) = 1$ if $q_{1,s_1} \cdot \ldots \cdot q_{k,s_k} \in W'$ and $\beta(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) = 0$ if $q_{1,s_1} \cdot \ldots \cdot q_{k,s_k} \notin W'$.

By (4.2), any fraction $a'/w' \in \widetilde{\mathcal{R}}(W')$ can be written in a unique way in the form $a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k} (\operatorname{mod} \mathbb{Z}^d)$, with $q_{l,s_l} \in S_l$ and $a_{l,s_l} \in \widetilde{P}_{q_{l,s_l}}$, $l \in \{1, \ldots, k\}$ (see the notation in Section 2). Conversely, if $\beta(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) = 1$, then any sum of the form $a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k}$, with $q_{l,s_l} \in S_l$ and $a_{l,s_l} \in P_{q_{l,s_l}}$, $l = \{1, \ldots, k\}$, belongs to $\mathcal{R}(W')$. Therefore

$$\begin{aligned} m_{\varepsilon, Y'}(\xi) = &\sum_{s_1, a_{1,s_1}, \ldots, s_k, a_{k,s_k}} \sum_{b \in \mathbb{Z}^d} \beta(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) \\ &\times m((\xi - b/Q_0 - a_{1,s_1}/q_{1,s_1} - \ldots - a_{k,s_k}/q_{k,s_k})/\varepsilon), \end{aligned} \tag{4.3}$$

where the sum is taken over all $s_l \in Z_{\beta(l)}$ and all $a_{l,s_l} \in \widetilde{P}_{q_{l,s_l}}$. We apply Lemma 2.1 with $Y = \{0\}$ and define

$$\begin{aligned} &\hat{f}^0_{a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k}}(\xi) \\ &\quad = \beta(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) \hat{f}(\xi) \sum_{b \in \mathbb{Z}^d} m((\xi - b/Q_0 - a_{1,s_1}/q_{1,s_1} - \ldots - a_{k,s_k}/q_{k,s_k})/\varepsilon), \end{aligned}$$

for any $s_l \in Z_{\beta(l)}$ and $a_{l,s_l} \in P_{q_{l,s_l}}$. Inequality (2.3) for ε is satisfied since $Q_0 \leq e^{N^\delta}$, $q_{l,s_l} \leq N^D$, and $\varepsilon \leq e^{-N^{2\delta}}$ (recall that $N \geq C_{r,\delta}$). Notice that the sum in the right-hand side of (2.5) has $2^k = C_\delta$ terms. By Lemma 2.1, for (4.1) it suffices to prove that for any set $A = \{j_1, \ldots, j_{k'}\} \subset Z_k$ we have

$$\sum_{s_{j_1}, \ldots, s_{j_{k'}}} \int_{\mathbb{Z}^d} \Big(\sum_{\theta' \in \widetilde{T}_{c_A}} \Big| \sum_{\mu \in \widetilde{U}_{A, s_{j_1}, \ldots, s_{j_{k'}}}} f^0_{\mu + \theta'}(n) \Big|^2 \Big)^r \, dn \leq C_{r,\delta} ||f||^{2r}_{L^{2r}} \tag{4.4}$$

for any characteristic function of a finite set f. The notation is explained in Section 2. Since $\beta(q_{1,s_1}\cdot\ldots\cdot q_{k,s_k})\in\{0,1\}$, we have

$$
\begin{aligned}
&|\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} f^0_{\mu+\theta'}(n)| \\
&\leq |\mathcal{F}^{-1}_{\mathbb{Z}^d}[\hat{f}(\xi)\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d} m((\xi-b/Q_0-\mu-\theta')/\varepsilon)](n)|
\end{aligned}
\tag{4.5}
$$

for any $(s_{j_1},\ldots,s_{j_{k'}})\in Z_{\beta(j_1)}\times\ldots\times Z_{\beta(j'_k)}$ and $\theta'\in\widetilde{T}_{^cA}$. We would like now to replace the L^p multiplier m with a smooth function. For this, we fix a smooth function φ, with $\varphi(\xi)=1$ if $\max_{l=1,\ldots,d}|\xi_l|\leq 1$ and $\varphi(\xi)=0$ if $\max_{l=1,\ldots,d}|\xi_l|\geq 2$. Clearly $m\varphi=m$. The right-hand side of (4.5) is equal to

$$
\begin{aligned}
&|\mathcal{F}^{-1}_{\mathbb{Z}^d}([\hat{f}(\xi+\theta')\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d}\varphi((\xi-b/Q_0-\mu)/\varepsilon)] \\
&\qquad\cdot\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d} m((\xi-b/Q_0-\mu)/\varepsilon))(n)|.
\end{aligned}
$$

By Corollary 4.2, the multiplier

$$
\xi\to\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d} m((\xi-b/Q_0-\mu)/\varepsilon)
$$

defines a bounded operator on $L^{2r}(\mathbb{Z}^d)$, uniformly in $s_{j_1},\ldots,s_{j_{k'}}$. By the Marcinkiewicz–Zygmund theorem applied to the functions $g_{\theta'}$ defined by

$$
\hat{g}_{\theta'}(\xi)=\hat{f}(\xi+\theta')\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d}\varphi((\xi-b/Q_0-\mu)/\varepsilon),
$$

the left-hand side of (4.4) is dominated by

$$
\begin{aligned}
&C_{r,\delta}\sum_{s_{j_1},\ldots,s_{j_{k'}}}\int_{\mathbb{Z}^d} dn \\
&(\sum_{\theta'\in\widetilde{T}_{^cA}}|\mathcal{F}^{-1}_{\mathbb{Z}^d}([\hat{f}(\xi+\theta')\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d}\varphi((\xi-b/Q_0-\mu)/\varepsilon)])(n)|^2)^r.
\end{aligned}
$$

It remains to prove the following bound:

Lemma 4.3. *Let*

$$
\Phi_{s_{j_1},\ldots,s_{j_{k'}}}(\eta)=\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\sum_{b\in\mathbb{Z}^d}\varphi((\eta-b/Q_0-\mu)/\varepsilon).
$$

Then

$$
\sum_{s_{j_1},\ldots,s_{j_{k'}}}\int_{\mathbb{Z}^d}(\sum_{\theta'\in\widetilde{T}_{^cA}}|\mathcal{F}^{-1}_{\mathbb{Z}^d}(\hat{f}(\xi)\Phi_{s_{j_1},\ldots,s_{j_{k'}}}(\xi-\theta'))(n)|^2)^r\,dn\leq C_{r,\delta}||f||^{2r}_{L^{2r}},
\tag{4.6}
$$

for any characteristic function of a finite set f.

Proof of Lemma 4.3. Notice first that in the case $r = 1$ the bound (4.6) follows from Plancherel's theorem. Also, f is the characteristic function of a finite set. Thus $||f||_{L^{2r}}^{2r} = ||f||_{L^2}^2$. Therefore, for (4.6), it suffices to prove that for any $(s_{j_1}, \ldots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \ldots \times Z_{\beta(j'_k)}$ and $n \in \mathbb{Z}^d$

$$\sum_{\theta' \in \widetilde{T}_{cA}} |\mathcal{F}_{\mathbb{Z}^d}^{-1}(\hat{f}(\xi)\Phi_{s_{j_1},\ldots,s_{j_{k'}}}(\xi - \theta'))(n)|^2 \leq C_\delta.$$

Since $||f||_{L^\infty} = 1$, it suffices to prove that

$$||\mathcal{F}_{\mathbb{Z}^d}^{-1}(\xi \to \sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')\Phi_{s_{j_1},\ldots,s_{j_{k'}}}(\xi - \theta'))||_{L^1(\mathbb{Z}^d)} \leq C_\delta, \tag{4.7}$$

for any complex numbers $\alpha(\theta')$ with

$$\sum_{\theta' \in \widetilde{T}_{cA}} |\alpha(\theta')|^2 = 1. \tag{4.8}$$

Let $\psi(x) = \int_{\mathbb{R}^d} \varphi(\xi)e^{2\pi i x\cdot\xi}\, d\xi$ denote the Euclidean inverse Fourier transform of the function φ. An easy calculation using the definition of $\Phi_{s_{j_1},\ldots,s_{j_{k'}}}$ shows that

$$\begin{aligned}
&\mathcal{F}_{\mathbb{Z}^d}^{-1}(\xi \to \sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')\Phi_{s_{j_1},\ldots,s_{j_{k'}}}(\xi - \theta'))(n) \\
&\quad = (\sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')e^{2\pi i n\cdot\theta'}) \cdot \varepsilon^d\psi(\varepsilon n) \cdot (\sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} \sum_{b \in Z_{Q_0}^d} e^{2\pi i n(b/Q_0+\mu)}).
\end{aligned} \tag{4.9}$$

Consider first the sum over b and μ in (4.9). For any integer $Q' \geq 1$ define the function

$$\delta_{Q'}(n) = \begin{cases} Q'^d \text{ if } n/Q' \in \mathbb{Z}^d; \\ 0 \text{ if } n/Q' \notin \mathbb{Z}^d. \end{cases}$$

Clearly, $\sum_{b \in Z_{Q'}^d} e^{2\pi i n\cdot b/Q'} = \delta_{Q'}(n)$. By arguing as in Corollary 4.2, we see that the sum over b and μ in (4.9) can be written as a sum of $2^{k'}$ functions $\delta_{Q'}$. The possible values of Q' are products of Q_0 and $p_{j_l,s_{j_l}}^{\alpha_l}$ or $p_{j_l,s_{j_l}}^{\alpha_l-1}$, $l = 1, \ldots, k'$, where $q_{j_l,s_{j_l}} = p_{j_l,s_{j_l}}^{\alpha_l}$. Thus, for (4.7), it suffices to prove that

$$||(\sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')e^{2\pi i n\cdot\theta'}) \cdot \varepsilon^d\psi(\varepsilon n)\delta_{Q'}(n)||_{L^1(\mathbb{Z}^d)} \leq C_\delta,$$

for any Q' with

$$Q' \in Z_{e^{N^\delta}} \text{ and } (Q', q_{j,s}) = 1 \text{ for any } j \in {}^{c}\!A, s \in Z_{\beta(j)}. \tag{4.10}$$

This is equivalent to proving that

$$||(\sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')e^{2\pi i Q'n\cdot\theta'}) \cdot (Q'\varepsilon)^d\psi((Q'\varepsilon)n)||_{L^1(\mathbb{Z}^d)} \leq C_\delta, \tag{4.11}$$

provided that (4.8) and (4.10) hold. Recall that $\varepsilon \leq e^{-N^{2\delta}}$. Thus $\varepsilon Q' \ll 1$. The function ψ is a Schwartz function on $\mathbb{R}^d$; by Hölder's inequality it suffices to prove that

$$(Q'\varepsilon)^{d/2}||(\sum_{\theta' \in \widetilde{T}_{cA}} \alpha(\theta')e^{2\pi i Q'n\cdot\theta'}) \cdot (1 + (Q'\varepsilon)^2|n|^2)^{-d}||_{L^2(\mathbb{Z}^d)} \leq C_\delta. \tag{4.12}$$

The left-hand side of (4.12) is equal to
(4.13)

$$(Q'\varepsilon)^{d/2}\left[\sum_{\theta'_1,\theta'_2\in\widetilde{T}_{c_A}}\alpha(\theta'_1)\overline{\alpha(\theta'_2)}\int_{\mathbb{Z}^d}(1+(Q'\varepsilon)^2|n|^2)^{-2d}e^{2\pi in\cdot Q'(\theta'_1-\theta'_2)}\,dn\right]^{1/2}.$$

It remains to estimate the integrals over $\mathbb{Z}^d$ in (4.13). If $\theta'_1=\theta'_2$, then

$$\left|\int_{\mathbb{Z}^d}(1+(Q'\varepsilon)^2|n|^2)^{-2d}e^{2\pi in\cdot Q'(\theta'_1-\theta'_2)}\,dn\right|\leq C(Q'\varepsilon)^{-d}. \tag{4.14}$$

If $\theta'_1\neq\theta'_2$, then, by (4.10), $Q'(\theta'_1-\theta'_2)\notin\mathbb{Z}^d$. Let $\gamma=(\gamma_1,\ldots,\gamma_d)$ denote the fractional part of $Q'(\theta'_1-\theta'_2)$. Since the denominators of θ'_1 and θ'_2 are bounded by N^{C_δ}, there is $l\in\{1,\ldots,d\}$ with the property that

$$\gamma_l\in[e^{-C_\delta(\ln N)},1-e^{-C_\delta(\ln N)}]. \tag{4.15}$$

We are looking to estimate $|\sum_{n\in\mathbb{Z}^d}(1+(Q'\varepsilon)^2|n|^2)^{-2d}e^{2\pi in\cdot\gamma}|$. By summation by parts in the variable n_l corresponding to γ_l in (4.15), we have

$$\left|\int_{\mathbb{Z}^d}(1+(Q'\varepsilon)^2|n|^2)^{-2d}e^{2\pi in\cdot Q'(\theta'_1-\theta'_2)}\,dn\right|\leq Ce^{C_\delta(\ln N)}(Q'\varepsilon)^{-(d-1)} \tag{4.16}$$

if $\theta'_1\neq\theta'_2$. We substitute (4.14) and (4.16) in (4.13). It follows that the left-hand side of (4.12) is dominated by

$$C[\sum_{\theta'\in\widetilde{T}_{c_A}}|\alpha(\theta')|^2+(Q'\varepsilon)e^{C_\delta(\ln N)}(\sum_{\theta'\in\widetilde{T}_{c_A}}|\alpha(\theta')|)^2]^{1/2}.$$

Since $|\widetilde{T}_{c_A}|\leq N^{C_\delta}$ and $Q'\varepsilon\leq e^{-N^{2\delta}/2}$, the bound (4.12) follows from (4.8) and Hölder's inequality. This completes the proof of Lemma 4.3. □

5. A transference principle

We turn now to the proof of Theorem 1.1. We may assume, without loss of generality, that the kernel K in Theorem 1.1 is compactly supported. In this section we use the *method of descent* (cf. [19, Chapter XI]) to reduce the proof of Theorem 1.1 to a certain "universal" case.

Lemma 5.1. *Assume that $L:\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ is a linear map, $m:\mathbb{R}^{n_2}\to\mathbb{C}$ is a continuous function, and $p\in[1,\infty]$. Define the function $m_L:\mathbb{R}^{n_1}\to\mathbb{C}$, $m_L(\xi)=m(L\xi)$. Then the norm on $L^p(\mathbb{R}^{n_1})$ of the operator defined by the Fourier multiplier m_L does not exceed the norm on $L^p(\mathbb{R}^{n_2})$ of the operator defined by the Fourier multiplier m,*

$$||m_L||_{\mathcal{M}_p(\mathbb{R}^{n_1})}\leq||m||_{\mathcal{M}_p(\mathbb{R}^{n_2})}.$$

For a proof, see [19, Chapter XI, p. 515] and [7].

In our case, assume that the polynomial $P=(P_1,\ldots,P_{d_2})$ of degree A in (1.3) is given by $P_l(x)=\sum_{1\leq|\alpha|\leq A}B_{l,\alpha}x^\alpha$ (we can clearly ignore the terms of order 0). Let d denote the cardinality of the set $\{\alpha\in\mathbb{Z}_+^{d_1}:1\leq|\alpha|\leq A\}$. Define the "universal" multiplier $\mu:\mathbb{R}^d\to\mathbb{C}$,

$$\mu(\xi)=\sum_{n\in\mathbb{Z}^{d_1}}K(n)e^{-2\pi i\sum_{1\leq|\alpha|\leq A}n^\alpha\xi_\alpha}, \tag{5.1}$$

where $n^\alpha = n_1^{\alpha_1} \cdot \ldots \cdot n_{d_1}^{\alpha_{d_1}}$. The multiplier μ is continuous, by the *a priori* assumption on the kernel K. Define the linear map $L : \mathbb{R}^{d_2} \to \mathbb{R}^d$ by the formula

$$(L\eta)_\alpha = \sum_{l=1}^{d_2} B_{l,\alpha} \eta_l.$$

It is easy to see that the multiplier on $\mathbb{R}^{d_2}$ of the operator T is μ_L, with the notation in Lemma 5.1. By Lemma 5.1, it suffices to prove that the multiplier μ in (5.1) defines a bounded operator on $L^p(\mathbb{R}^d)$. Thus the proof of Theorem 1.1 is reduced to the special case when $P : \mathbb{R}^{d_1} \to \mathbb{R}^d$ is of the form $[P(x)]_\alpha = x^\alpha$.

6. Estimates for exponential sums

In this section we prove an explicit approximation formula for the multiplier μ. Our method is similar to the method of J. Bourgain [4]. As in [4], the main ingredient is a basic lemma of H. Weyl (see, for example, [15, Chapter 4]):

Lemma 6.1. *If $g(x) = \alpha_d x^d + \ldots + \alpha_0$ is a polynomial with $|\alpha_d - a/q| \leq 1/q^2$ and $(a, q) = 1$, then for any $\delta > 0$ there is a constant C_δ such that*

$$\left| \sum_{m=1}^{n} e^{-2\pi i g(m)} \right| \leq C_\delta n^{1+\delta} [q^{-1} + n^{-1} + qn^{-d}]^{1/2^{d-1}},$$

uniformly in n and q.

Recall that $P(x) = [x^\alpha]_{1 \leq |\alpha| \leq A}$. We will need a d_1-dimensional version of Lemma 6.1. For any $R \geq 1$ let $B_R = \{x \in \mathbb{R}^{d_1} : |x_l| < R, l = 1, \ldots, d_1\}$. Assume that $k : B_R \to \mathbb{C}$ is a C^1 function with the property that

$$|k(x)| + R \cdot |\nabla k(x)| \leq 1$$

for any $x \in B_R$.

Lemma 6.2. *Assume that $\epsilon \in (0, 1/10)$ is fixed and $\xi \in \mathbb{R}^d$ has the property that for some α, $1 \leq |\alpha| \leq A$, there are integers a and q, with $(a, q) = 1$, $q \in [R^\epsilon, R^{|\alpha|-\epsilon}]$, and $|\xi_\alpha - a/q| \leq 1/q^2$. Then*

$$| \sum_{n \in \Omega \cap \mathbb{Z}^{d_1}} e^{-2\pi i P(n) \cdot \xi} k(n)| \leq C R^{d_1 - \delta}, \delta > 0,$$

for any open, convex set $\Omega \subset B_R$. The constants C and δ may depend only on d_1, A, and ϵ but not on R, ξ, or the irreducible fraction a/q.

Lemma 6.2 follows from [22, Proposition 3]. Let $\eta : \mathbb{R}^{d_1} \to [0, 1]$ denote a smooth function supported in $\{x : |x| \in [1/2, 2]\}$ with the property that $\sum_{j=0}^\infty \eta(2^{-j}x) = 1$ for any $x \in \mathbb{R}$ with $|x| \geq 1$. Let

$$K_j(x) = \eta(2^{-j}x) K(x), \tag{6.1}$$

and let T_j denote the operator defined by the kernel K_j. The multiplier of the operator T_j is

$$\mu_j(\xi) = \sum_n K_j(n) e^{-2\pi i P(n) \cdot \xi}. \tag{6.2}$$

For later use we define the function

$$\Phi_j(\xi) = \int_{\mathbb{R}^{d_1}} K_j(x) e^{-2\pi i P(x) \cdot \xi} \, dx. \tag{6.3}$$

By Diriclet's principle, for any $\Lambda \geq 1$ and $\xi \in \mathbb{R}$ there are $q \in Z_\Lambda$ and $a \in \mathbb{Z}$ with $(a, q) = 1$, with the property that

$$|\xi - a/q| \leq 1/(q\Lambda).$$

Therefore, given Λ, we can partition the line $\mathbb{R}$ into periodic sets $I(a/q)$, with

$$\bigcup_{m\in\mathbb{Z}} (m + a/q - (2q\Lambda)^{-1}, m + a/q + (2q\Lambda)^{-1}) \subset I(a/q) \subset \bigcup_{m\in\mathbb{Z}} [m + a/q - (q\Lambda)^{-1}, m + a/q + (q\Lambda)^{-1}],$$

parametrized over irreducible fractions $a/q \in [0, 1)$ with $q \in Z_\Lambda$ (a Farey dissection at level Λ). As in Section 4, let φ denote a smooth function with $\varphi(\xi) = 1$ if $\max_{l=1,\dots,d} |\xi_l| \leq 1$ and $\varphi(\xi) = 0$ if $\max_{l=1,\dots,d} |\xi_l| \geq 2$. For any $q \in \{1, 2, \dots\}$ and $a \in \mathbb{Z}^d$ with $(a, q) = 1$ let

$$S(a/q) = \frac{1}{q^{d_1}} \sum_{n\in[Z_q]^{d_1}} e^{-2\pi i P(n)\cdot a/q}. \tag{6.4}$$

Our next proposition gives an explicit approximation of the multipliers μ_j.

Proposition 6.3. *There is a large constant C_d with the property that for any $D_1 \geq 2$, we have*

(6.5)

$$\mu_j(\xi) = \sum_{q=1}^{q\leq(j+1)^{C_d D_1}} \sum_{a\in P_q} S(a/q)\Phi_j(\xi - a/q)\varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A}) + E_j(\xi).$$

The functions Φ_j are defined in (6.3), *and $|E_j(\xi)| \leq C_{D_1}(j + 1)^{-D_1}$.*

Proof of Proposition 6.3. We may assume $j \geq C_{d,D_1}$. We define the "major arcs"

$$A_j(a/q) = \{\xi \in \mathbb{R}^d : |\xi_\alpha - a_\alpha/q| \leq 2^{-(|\alpha|-1/2)j}\}, \tag{6.6}$$

parametrized over irreducible d-fractions (a/q), with $q \in [1, 2^{j/10}]$.

We show first that if ξ does not belong to the union over $q \in [1, 2^{j/10}]$ of the major arcs, then

$$|\mu_j(\xi)| = O(2^{-c_d j}), c_d > 0. \tag{6.7}$$

This agrees well with the formula (6.5), since the main term in (6.5) is supported in the union of the major arcs. Let $\xi = (\xi_\alpha)$, and, for each α, consider a Farey dissection at level $\Lambda_\alpha = 2^{(|\alpha|-1/2)j}$. Thus

$$|\xi_\alpha - a_\alpha/q_\alpha| \leq (q_\alpha \cdot 2^{(|\alpha|-1/2)j})^{-1},$$

for some integers a_α and q_α, with $(a_\alpha, q_\alpha) = 1$ and $q_\alpha \in [1, 2^{(|\alpha|-1/2)j}]$. Since ξ does not belong to the union over $q \in [1, 2^{j/10}]$ of the major arcs, at least one of the denominators q_α is $\geq 2^{j/(10d)}$. The bound (6.7) follows from Lemma 6.2 with $R = 2^{j+1}$, $k = C^{-1}2^{d_1 j}K_j$, and $\epsilon = 1/(20d)$.

Assume now that ξ belongs to the union over $q \in [1, 2^{j/10}]$ of the major arcs. Since the major arcs are pairwise disjoint, $\xi \in A_j(a/q)$ for some irreducible d-fractions (a/q), with $q \in [1, 2^{j/10}]$. Let $\xi = a/q + \beta$. Then

$$
\begin{aligned}
\mu_j(\xi) &= \sum_m K_j(m) e^{-2\pi i P(m)\cdot\xi} \\
&= \sum_{n\in\mathbb{Z}^{d_1}} \sum_{l\in[Z_q]^{d_1}} K_j(nq+l) e^{-2\pi i P(l)\cdot a/q} e^{-2\pi i P(nq+l)\cdot\beta} \\
&= \Big[\sum_{l\in[Z_q]^{d_1}} e^{-2\pi i P(l)\cdot a/q}\Big]\cdot\Big[\sum_{n\in\mathbb{Z}^{d_1}} K_j(nq) e^{-2\pi i P(nq)\cdot\beta}\Big] + O(2^{-j/4}) \\
&= S(a/q)\int_{\mathbb{R}^{d_1}} K_j(x) e^{-2\pi i P(x)\cdot\beta}\,dx + O(2^{-j/4}) \\
&= S(a/q)\Phi_j(\xi - a/q) + O(2^{-j/4}),
\end{aligned}
\tag{6.8}
$$

where $S(a/q)$ are defined in (6.4). In addition, by [24, Proposition 2.1]

$$
\Phi_j(\xi) \le C_{d,A}\Big(1 + \sum_{1\le|\alpha|\le A} 2^{|\alpha|j}|\xi_\alpha|\Big)^{-1/d}. \tag{6.9}
$$

Thus we can insert the cutoff function φ:

$$
\mu_j(\xi) = S(a/q)\Phi_j(\xi - a/q)\varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1\le|\alpha|\le A}) + O(2^{-c_d j}), \tag{6.10}
$$

for any $\xi \in A_j(a/q)$. Finally, we claim that

$$
|S(a/q)| \le Cq^{-\overline{\delta}} \text{ if } (a,q) = 1, \tag{6.11}
$$

for some constant $\overline{\delta} = \delta(d) > 0$. Assuming (6.11), the formula (6.5) follows immediately from (6.9), (6.10), and the disjointness of the major arcs, with the constant C_d equal to $1/\overline{\delta}$.

To prove the bound (6.11), let $a = (a_\alpha)$, and assume that $a_\alpha/q = a'_\alpha/q'_\alpha$, where a'_α/q'_α is an irreducible 1-fraction. Since $(a,q) = 1$, we have $q \le \prod_\alpha q'_\alpha$. If $q'_\alpha \ge q^{1/(10d^2)}$ for some index α with $|\alpha| \ge 2$, then Lemma 6.2 applies directly with $R = 2q$. Otherwise, we would have $q'_\alpha \le q^{1/(10d^2)}$ for any α with $|\alpha| \ge 2$ and $q'_{\alpha_0} \ge q^{1/(2d)}$ for some α_0 with $|\alpha_0| = 1$. In this case, it is easy to see that $S(a/q) = 0$, by summing first the variable corresponding to the index α_0. □

The large constant D_1 will depend on the parameter ε_1 in Lemma 7.1 (thus on the exponent p in Theorem 1.1). We consider now sums over j. For any integer $k \ge 0$ and $\beta \in \mathbb{R}^d$ let

$$
m_k(\beta) = \varphi(\beta/2)\sum_{j=k+1}^{\infty}\Phi_j(\beta).
$$

We may assume that the sum in the definition above is finite, since the kernel K in Theorem 1.1 may be assumed to be compactly supported. The main result in this section is the following lemma:

Lemma 6.4. *Given C_d and $D_1 \geq 2$ as is Proposition 6.3, we have*

(6.12)

$$\sum_{j=k+1}^{\infty} \mu_j(\xi) = \sum_{q=1}^{q \leq k^{C_d D_1}} \sum_{a \in P_q} S(a/q) m_k(\xi - a/q) \varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) + \widetilde{E}_k(\xi)$$

for any integer $k \geq 1$, where $|\widetilde{E}_k(\xi)| \leq C_{d,D_1} k^{-(D_1-1)}$.

Proof of Lemma 6.4. We use formula (6.5). The sum of the error terms $E_j(\xi)$, $j \geq k+1$, can be incorporated into the error term $\widetilde{E}_k$. Let χ_+ denote the characteristic function of the set $[0, \infty)$. By (6.5),

$$\tag{6.13} \begin{aligned} \sum_{j=k+1}^{\infty} \mu_j(\xi) = & \sum_{q=1}^{\infty} \sum_{a \in P_q} S(a/q) \sum_{j \geq k+1} \chi_+((j+1)^{C_d D_1} - q) \\ & \cdot \Phi_j(\xi - a/q) \varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) + \widetilde{E}_k(\xi). \end{aligned}$$

Assume first that $q \leq k^{C_d D_1}$. Then $\chi_+((j+1)^{C_d D_1} - q) = 1$. In addition,

$$\begin{aligned} & \Phi_j(\xi - a/q) \varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) \\ & \qquad = \Phi_j(\xi - a/q) \varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) + O(2^{-j/(4d)}), \end{aligned}$$

by (6.9). Thus, the sum over $q \leq k^{C_d D_1}$ in (6.13) coincides with the main term in (6.12), modulo acceptable errors.

We break up the sum over $q > k^{C_d D_1}$ in (6.13) into dyadic pieces, $q \in [2^s, 2^{s+1}) \cap (k^{C_d D_1}, \infty) \cap \mathbb{Z}$. Since $C_d = 1/\overline{\delta}$, it suffices to prove that

$$\tag{6.14} \begin{aligned} & \sum_{q=2^s}^{2^{s+1}-1} \sum_{a \in P_q} S(a/q) \sum_{j \geq k+1} \chi_+((j+1)^{C_d D_1} - q) \\ & \qquad \cdot \Phi_j(\xi - a/q) \varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) = O(2^{-\overline{\delta} s}), \end{aligned}$$

whenever $2^{s+1} \geq k^{C_d D_1}$. We may assume $j \geq C_{D_1}$. Then the support of the sum over j in (6.14) is contained in the set $\{\xi : |\xi - a/q| \leq (10q^2)^{-1}\}$. These sets are disjoint when q runs over the integers in the dyadic interval $[2^s, 2^{s+1}-1]$. By (6.11), $S(a/q) = O(2^{-\overline{\delta} s})$. Therefore it remains to prove that

$$\sum_{j \geq k+1} \chi_+((j+1)^{C_d D_1} - q) \Phi_j(\xi - a/q) \varphi([2^{(|\alpha|-1/4)j}(\xi_\alpha - a_\alpha/q)]_{1 \leq |\alpha| \leq A}) = O(1)$$

for any irreducible d-fraction a/q. This follows easily from (6.9) and Lemma 6.5 below with $p = 2$. $\square$

Our last estimate in this section concerns the multipliers m_k.

Lemma 6.5. *The operators defined by the multipliers m_k are bounded on $L^p(\mathbb{R}^d)$, $p \in (1, \infty)$, uniformly in $k \geq 0$, i.e.,*

$$||m_k||_{\mathcal{M}_p(\mathbb{R}^d)} \leq B_p.$$

Proof of Lemma 6.5. This is equivalent to boundedness on $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, of the translation invariant continuous singular Radon transform

$$S(f)(x) = \int_{\mathbb{R}^{d_1}} f(x - P(y))(\sum_{j \geq k+1} K_j(y))\, dy.$$

Since $\sum_{j\geq k+1} K_j$ is a Calderón–Zygmund kernel that satisfies bounds similar to (1.1) and (1.2), uniformly in k, it is known that the operator S extends to a bounded operator on $L^p(\mathbb{R}^d)$. A simple proof that applies in our translation invariant case can be found in [19, Chapter XI, p. 513]. □

7. Proof of Theorem 1.1

Theorem 1.1 will follow easily from the following lemma.

Lemma 7.1. *Given $\varepsilon_1 \in (0,1]$, for any $\lambda \in (0,\infty)$ there are two linear operators $A_{\lambda,\varepsilon_1}$ and $B_{\lambda,\varepsilon_1}$ with $T = A_{\lambda,\varepsilon_1} + B_{\lambda,\varepsilon_1}$,*

$$||A_{\lambda,\varepsilon_1}||_{L^2\to L^2} \leq C_{\varepsilon_1}/\lambda, \tag{7.1}$$

and

$$||B_{\lambda,\varepsilon_1}||_{L^r\to L^r} \leq C_{r,\varepsilon_1}\lambda^{\varepsilon_1} \tag{7.2}$$

for any $r \in [2,\infty)$.

We show first how to use Lemma 7.1 to prove Theorem 1.1. A similar interpolation argument was used in [10]. By duality it suffices to prove the theorem for $p \in [2,\infty)$. By general interpolation theory it suffices to prove that

$$||T||_{L^{p,1}\to L^{p,\infty}} \leq C_p, \quad p \in [4,\infty),$$

where $L^{p,\alpha}$ denote the Lorentz spaces on $\mathbb{Z}^d$. This is equivalent to proving that for any $\mu \in (0,\infty)$ and any finite set F

$$\mu^p|\{n : |T(\chi_F)(n)| > \mu\}| \leq C_p|F|,$$

where χ_F denotes the characteristic function of the set F and $|F|$ denotes its cardinality. We use Lemma 7.1 with $\lambda = \mu^{(p-2)/2}$, $r = 2p$, and $\varepsilon_1 = 1/(p-2) > 0$. Since $p \in [4,\infty)$, we have

$$\begin{aligned}
&|\{n : |T(\chi_F)(n)| > \mu\}| \\
&\qquad \leq |\{n : |A_{\lambda,\varepsilon_1}(\chi_F)(n)| > \mu/2\}| + |\{n : |B_{\lambda,\varepsilon_1}(\chi_F)(n)| > \mu/2\}| \\
&\qquad \leq \frac{4}{\mu^2}||A_{\lambda,\varepsilon_1}(\chi_F)||_{L^2}^2 + \frac{2^r}{\mu^r}||B_{\lambda,\varepsilon_1}(\chi_F)||_{L^r}^r \\
&\qquad \leq C\lambda^{-2}\mu^{-2}|F| + C_{r,\varepsilon_1}\lambda^{r\varepsilon_1}\mu^{-r}|F| \leq C_p\mu^{-p}|F|,
\end{aligned}$$

as desired.

Proof of Lemma 7.1. By Lemma 6.4 with $k = 1$ and Lemma 6.5, the decomposition is trivial in the case $\lambda \leq C_{\varepsilon_1}$. Assume $\lambda \geq 1$ and fix k the smallest integer $\geq \lambda^{\varepsilon_1}$. Let

$$B^1_{\lambda,\varepsilon_1} = \sum_{j=1}^{k} T_j,$$

where the operators T_j are defined by the kernels K_j in (6.1). By (1.1), we have $||K_j|_{\mathbb{Z}^{d_1}}||_{L^1(\mathbb{Z}^{d_1})} \leq C$; therefore,

$$||B^1_{\lambda,\varepsilon_1}||_{L^r \to L^r} \leq C\lambda^{\varepsilon_1}. \tag{7.3}$$

It remains to decompose the sum T_j over $j \geq k+1$. For this we use Lemma 6.4, with $D_1 = \varepsilon_1^{-1} + 1$. Let $A^1_{\lambda,\varepsilon_1}$ denote the operator defined by the error term $\widetilde{E}_k$. By Plancherel's theorem

$$||A^1_{\lambda,\varepsilon_1}||_{L^2 \to L^2} \leq C_{\varepsilon_1}/\lambda. \tag{7.4}$$

For the operator defined by the Fourier multiplier in the main term in the right-hand side of (6.12), let N denote the integer part of $k^{C_d D_1}$. Fix $\delta = (4C_d D_1)^{-1}$ and let $Y_N = Y_{N,\delta}$ denote the set constructed in Theorem 1.5, $Z_N \subset Y_N \subset Z_{e^{N^\delta}}$. Let $A^2_{\lambda,\varepsilon_1}$ denote the operator defined by the Fourier multiplier

$$-\sum_{q\in Y_N\setminus Z_N}\sum_{a\in P_q} S(a/q)m_k(\xi - a/q)\varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A}).$$

By (6.11), Lemma 6.5, the fact that δ is small enough, and Plancherel's theorem

$$||A^2_{\lambda,\varepsilon_1}||_{L^2 \to L^2} \leq C_{\varepsilon_1}/\lambda. \tag{7.5}$$

It remains to decompose the operator $R_{\lambda,\varepsilon_1}$ defined by the Fourier multiplier

$$\sum_{q\in Y_N}\sum_{a\in P_q} S(a/q)m_k(\xi - a/q)\varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A}).$$

Let U_N denote the operator defined by the Fourier multiplier

$$\sum_{q\in Y_N}\sum_{a\in P_q} m_k(\xi - a/q)\varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A}).$$

By Theorem 1.5 with $\varepsilon = 10\cdot 2^{-3k/4}$ and Lemma 6.5,

$$||U_N||_{L^r \to L^r} \leq C_{r,\varepsilon_1}(\ln\lambda)^{2/\delta}. \tag{7.6}$$

The support hypothesis in Theorem 1.5 is satisfied since $\delta = (4C_d D_1)^{-1}$ is small enough. Let J denote the smallest integer $\geq 2^{k/2}$ and V_J the averaging operator

$$V_J(f)(x) = \frac{1}{J^{d_1}}\sum_{n\in[Z_J]^{d_1}} f(x - P(n)).$$

Let $B^2_{\lambda,\varepsilon_1} = U_N \circ V_J$ and $A^3_{\lambda,\varepsilon_1} = R_{\lambda,\varepsilon_1} - U_N \circ V_J$. By (7.6)

$$||B^2_{\lambda,\varepsilon_1}||_{L^r \to L^r} \leq C_{r,\varepsilon_1}\lambda^{\varepsilon_1}. \tag{7.7}$$

The multiplier corresponding to the operator $A^3_{\lambda,\varepsilon_1}$ is

$$\begin{aligned}\xi \to &\sum_{q\in Y_N}\sum_{a\in P_q} S(a/q)m_k(\xi - a/q)\varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A})\\ &\quad - v_J(\xi)\sum_{q\in Y_N}\sum_{a\in P_q} m_k(\xi - a/q)\varphi([2^{(|\alpha|-1/4)k}(\xi_\alpha - a_\alpha/q)]_{1\leq|\alpha|\leq A}),\end{aligned} \tag{7.8}$$

where v_J is the multiplier of the averaging operator V_J. An estimate similar to (6.8), using the fact that $J \leq C2^{k/2}$ and $q \leq Ce^{N^\delta} \leq Ce^{k^{1/2}}$, shows that if $|\xi_\alpha - a_\alpha/q| \leq 2\cdot 2^{-(|\alpha|-1/4)k}$, then

$$v_J(\xi) = S(a/q) + O(2^{-ck}), c > 0.$$

Therefore, the absolute value of the multiplier in (7.8) is bounded by $C_{\varepsilon_1} 2^{-ck}$, $c > 0$. By Plancherel's theorem

$$||A^3_{\lambda,\varepsilon_1}||_{L^2\to L^2} \le C_{\varepsilon_1} 2^{-ck} \le C_{\varepsilon_1}/\lambda. \tag{7.9}$$

We define $A_{\lambda,\varepsilon_1} = A^1_{\lambda,\varepsilon_1} + A^2_{\lambda,\varepsilon_1} + A^3_{\lambda,\varepsilon_1}$ and $B_{\lambda,\varepsilon_1} = B^1_{\lambda,\varepsilon_1} + B^2_{\lambda,\varepsilon_1}$. Clearly, $T = A_{\lambda,\varepsilon_1} + B_{\lambda,\varepsilon_1}$. The bound (7.1) follows from (7.4), (7.5), and (7.9). The bound (7.2) follows from (7.3) and (7.7). □

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