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CURVATURE AND INJECTIVITY RADIUS ESTIMATES FOR EINSTEIN 4-MANIFOLDS

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0. Statement of main results

It is of fundamental interest to study the geometric and analytic properties of compact Einstein manifolds and their moduli. In dimension 2 these problems are well understood. A 2-dimensional Einstein manifold, (M^2, g), has constant curvature, which after normalization, can be taken to be -1, 0 or 1. Thus, (M^2, g) is the quotient of a space form and the metric, g, is completely determined by the conformal structure. For fixed M^2, the moduli space of all such g admits a natural compactification, the Deligne-Mumford compactification, which has played a crucial role in geometry and topology in the last two decades, e.g. in establishing Gromov-Witten theory in symplectic and algebraic geometry.

In dimension 3, it remains true that Einstein manifolds have constant sectional curvature and hence are quotients of space forms. An essential portion of Thurston's geometrization program can be viewed as the problem of determining which 3-manifolds admit Einstein metrics. The moduli space of Einstein metrics on a 3-dimensional manifold is also well understood. As a consequence of Mostow rigidity, the situation is actually simpler than in two-dimensions.

In dimension 4 however, the class of Einstein metrics is significantly more general than that of metrics of constant curvature. For example, almost all complex surfaces with definite first Chern class admit Kähler-Einstein metrics. Still, the existence of an Einstein metric does impose strong constraints on the underlying 4-manifold. Hence, it is natural to look for sufficient conditions for a closed 4-manifold to admit an Einstein metric. Any approach to this existence problem by geometric analytic methods, e.g. by Ricci flow, will lead to the question of how, in limiting cases, solutions to the Einstein equation can develop singularities, or equivalently, how Einstein metrics can degenerate.

On the other hand, most Einstein 4-manifolds have nontrivial moduli spaces. These moduli spaces and their natural compactifications are differentiable invariants of underlying smooth 4-manifolds. Thus, one wants to understand the geometry of such moduli spaces and their compactifications. Here one can normalize the Einstein constant, λ, to be $-3, 0$ or 3, and in the (scale invariant) case, $\lambda = 0$,

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add the additional normalization so that the volume is equal to 1. Specifically, one would like to know the properties of a natural compactification analogous to the Deligne-Mumford compactification in the 2-dimensional case. For this purpose, once again, one must understand how Einstein metrics can degenerate.

In somewhat more concrete terms, we wish to describe the geometric structure of metric spaces, Y, which arise as limits of sequences, (M^4, g_i), of Einstein manifolds (M^4, g_i) with fixed topology and Einstein constant. After passing to a subsequence, if necessary, such limits always exist in a suitable weak geometric sense, the pointed Gromov-Hausdorff sense. These limit spaces can be thought of as Einstein manifolds with singularities, although a priori they might not have any manifold points whatsoever. Indeed, the first result on the existence of manifold points in the collapsed case, that in which $\dim Y < 4$, is a consequence of Theorem 0.8 of the present paper; compare [ChCo3]. For the structure of noncollapsed limit spaces in dimension 4, see [An1], [An3], [Na], [Ti]; for a structure theory in the noncollapsed case in higher dimensions, see [ChCo0]–[ChCo3], [ChCoTi2], [Ch2].

The present paper constitutes the first step in our program, the ultimate goal of which is to obtain a complete understanding of how Einstein metrics on 4-manifolds can degenerate.

According to the finiteness/compactness theory, [Ch1], [GvLP], for manifolds with bounded sectional curvature, there are precisely three mechanisms (which can occur in combination) which can cause a sequence, (M^n, g_i), to degenerate; namely, the diameter can go to ∞, the volumes of unit balls can go uniformly to 0 at all points, or the curvature can go to $\pm\infty$ at certain points. Thus, in our situation, we have the following issues.

1. For $\lambda = -3, 0$, the diameter, $\mathrm{diam}(M^4, g_i)$ need not remain uniformly bounded. What are the noncompact limits?

2. For $\lambda = 0$, the sequence can collapse; i.e., the volumes of all unit balls can go uniformly to 0. Is $\lambda = 0$ the only case in which collapse can take place?

3. For $\lambda = -3, 0, 3$, the sectional curvature need not remain uniformly bounded. In the noncollapsing case, this phenomenon is well understood. How badly can the curvature blow up in the collapsing case?

Note that in each of the above instances, we wish to understand both the structure of limiting objects and the detailed nature of the convergence to the limit.

In this paper, we prove a number of analytic estimates for Einstein 4-manifolds with finite L^2-norm of curvature. In particular, we solve the third problem (see Theorem 0.14 and Theorem 9.1) and the second problem under the additional assumption that the global volume stays bounded below; see Theorem 0.14. Our results shed light on the first problem as well; see Theorems 7.10, 10.3, 10.5; other applications willl be discussed elsewhere.

In view of [An1], [An3], [Na], [Ti], we only need to consider the case of Einstein 4-manifolds which are sufficiently collapsed. However, this case is technically much more difficult than the noncollapsed case and additional new techniques are required.

Suitably formulated, our results continue to hold for 4-manifolds which are sufficiently Ricci pinched, or whose Ricci tensor has a definite 2-sided bound. We will give detailed arguments in the Einstein case; to avoid nonessential complications in the exposition, we indicate the above-mentioned generalizations in a number of remarks.

Three main results. The following theorem states that if an Einstein 4-manifold is sufficiently collapsed and there is a definite bound on the L_2-norm of the curvature, then in the L_2 sense, almost all of the curvature is concentrated very near at most a definite number of points; for a generalization, see the proof of Theorem 10.5.

Theorem 0.1 (Collapse implies L_2 concentration of curvature)**.** *There exists $v > 0$, β, c, such that the following holds. Let M^4 denote a complete Einstein 4-manifold satisfying*

$$|\lambda| \leq 3\,, \tag{0.2}$$

$$\int_{M^4} |R|^2 \leq C\,, \tag{0.3}$$

and for all p and some $s \leq 1$,

$$\frac{\mathrm{Vol}(B_s(p))}{s^4} \leq v\,. \tag{0.4}$$

Then there exist $p_1, \ldots, p_N$, with

$$N \leq \beta \cdot C\,, \tag{0.5}$$

such that

$$\int_{M^4 \setminus \left(\bigcup_i B_s(p_i)\right)} |R|^2 \leq c \cdot \left(\sum_i \frac{\mathrm{Vol}(B_s(p_i))}{s^4} + \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} \right)\,. \tag{0.6}$$

If $\lambda \neq 0$, then $\mathrm{Vol}(M^4) < \infty$ and, in particular,

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} = 0\,. \tag{0.7}$$

If in Theorem 0.1, the manifold, M^4, is compact, then the bound, (0.3), on the L_2-norm of the curvature, can be replaced by a bound on the Euler characteristic; see (1.3). Of course, in this case, the term, $\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4}$, in (0.6) vanishes.

Our next result is an ϵ-regularity theorem whose significant feature is the *absence* of the assumption that the L_2-norm of the curvature is sufficiently small *with respect to the collapsing*; compare (1.12), (1.15).

Theorem 0.8 (ϵ-regularity)**.** *There exists $\epsilon > 0$, c, such that the following holds. Let M^4 denote an Einstein 4-manifold satisfying* (0.2) *and let $r \leq 1$. If $B_s(p)$ has compact closure for all $s \leq r$ and*

$$\int_{B_r(p)} |R|^2 \leq \epsilon\,, \tag{0.9}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq c \cdot r^{-2}\,. \tag{0.10}$$

If $\lambda = 0$ and the assumption, $r \leq 1$, is dropped, then (0.10) *holds.*

Remark 0.11. If in Theorem 0.1, we take $v \leq \epsilon \cdot (\beta \cdot C)^{-1}$, then by Theorem 0.8, the curvature bound, (0.10), is valid for all $B_r(p) \subset M^4 \setminus \bigcup_i B_s(p_i)$.

Remark 0.12. Theorem 0.8 should be compared with the ϵ-regularity theorems for the cases of 4-dimensional Yang-Mills fields, [Uh], and harmonic maps, [Mor], [SchUh]. The Yang-Mills equation and harmonic map equation are uniformly elliptic modulo gauge transformations. In the Einstein equations however, the nonlinearity is much stronger, since the coefficients of the highest derivatives which occur depend on the solutions. So initially, one must decide if the equation is uniformly elliptic modulo gauge transformations, or equivalently, uniformly elliptic in suitable local coordinates. Although harmonic coordinates will suffice for this purpose, one cannot ensure the existence of such local coordinate systems on metric balls of a definite size. In actuality, the topology of the ball, $B_r(p)$, occurring in the statement of Theorem 0.8 need not be that of a Euclidean ball and there may be no global coordinate sytem at all on such $B_r(p)$.

Indeed, it is a *consequence* of the curvature bound in our ϵ-regularity theorem, together with [ChFuGv], that the point, p, does have a neighborhood of a definite size with known topology. Namely, there exists such a neighborhood which is quasi-isometric (with a definite constant) to either a Euclidean ball or to a tube around a nilmanifold; see Theorem 1.7 of [ChFuGv] and Appendix 1 of [ChFuGv].

Remark 0.13. The ϵ-regularity theorems for Yang-Mills and harmonic maps can be proved by Moser iteration. This requires a bound on the Sobolev constant of the domain. Since in these cases the domain is effectively a standard ball, such a bound is available.

In [An1], [An3], [Na], [Ti], the Moser iteration argument was extended to n-dimensional Einstein manifolds yielding a pointwise curvature bound as in (0.10). In these works, in order to apply Moser iteration, it is assumed that the $L_{\frac{n}{2}}$-norm of the curvature is sufficiently small *with respect to the Sobolov constant.* The latter can be bounded in terms of a lower bound on the collapsing.

According to Theorem 0.8, in dimension 4, to obtain the pointwise curvature bound in (0.10), it merely suffices to assume that the L_2-norm of the curvature is sufficiently small. The proof is accomplished by showing that once the L_2-norm of the curvature is sufficiently small, then on a smaller concentric ball of a definite radius, the L_2-norm will *automatically* be so small with respect to the collapsing that the hypothesis of the ϵ-regularity theorem of [An3] will be verified.

As previously mentioned, the proof of Theorem 0.8 is considerably more difficult than those of the earlier ϵ-regularity theorems and employs entirely different techniques. Neither Moser iteration nor the Sobolev inequality enter directly in the argument. For an outline of the proof, including the role of the assumption, $n = 4$, see Section 1.

Next, we state a theorem on the noncollapsing of Einstein 4-manifolds. If $\lambda = \pm 3$, then $|R|^2$ has the pointwise lower bound $|R|^2 \geq 6$. Substituting this into the left-hand side of (0.6) and using (0.5) on the right-hand side gives the following.

Theorem 0.14 (Lower bound on collapse). *There exists $w > 0$, such that if M^4 denotes a complete 4-dimensional Einstein manifold satisfying* (0.3) *and*

$$\lambda = \pm 3\,, \tag{0.15}$$

then for some $p \in M^4$,

$$\mathrm{Vol}(B_1(p)) \geq w \cdot \frac{\mathrm{Vol}(M^4)}{C}\,. \tag{0.16}$$

For $\lambda = 3$, Myers' theorem together with the Bishop-Gromov inequality provides stronger information than (0.16). The interesting case is $\lambda = -3$, in which relation (0.16) can be viewed as a partial replacement for the Heintze-Margulis theorem. The latter gives a lower bound for the collapse, for compact manifolds with negative *sectional* curvature, $-1 \leq K_{M^n} < 0$.

If M^4 is compact, Kähler-Einstein, with $\lambda = \pm 3$, then $\mathrm{Vol}(M^4) = 2\pi^2 \cdot c_1^2(M^4)$, where c_1 denotes the first Chern class. By (1.1) below, we can take $C = 8\pi^2 \cdot \chi(M^4)$. The topological invariants, $c_1^2(M^4)$, $\chi(M^4)$, are positive integers. In dimension 4, Seiberg-Witten theory provides lower volume bounds under more general assumptions, e.g. if M^4 admits a symplectic structure; see [LeBru1], [LeBru2], [Tau], [Wi].

1. Preliminary discussion of proofs

In this section we describe the various techniques which enter in the proofs of our main results and give a brief indication of how they are used.

Underlying the proofs of our results is an extension of the "equivariant good chopping" theorem of [ChGv3], valid for manifolds with locally bounded curvature, i.e., no global curvature bound is assumed. According to this theorem, a compact domain, K, with "rough" boundary can be approximated from the outside by a smooth submanifold, Z^n, with nonempty boundary, for which the norm of the second fundamental form of the boundary, $|II_{\partial Z}|$, is controlled. The term "equivariant" refers to the possibility of choosing Z to be invariant under the group of isometries which leaves K invariant.

The bound on $|II_{\partial Z}|$ involves the reciprocal of the local scale $r_{|R|}(p)$, for p in a suitable neighborhood of ∂K. By definition, $r_{|R|}(p)$ is the supremum of those r, such that if the metric is rescaled, $g \to r^{-2}g$, then the ball, $B_r(p)$, becomes a unit ball on which the norm of the curvature, $|R|$, is bounded by 1.

The chopping theorem can be used to control the boundary term in the Chern-Gauss-Bonnet formula for manifolds with boundary, thereby yielding information on the interior term; compare [ChGv3]. Moreover, for Einstein manifolds with $L_{\frac{n}{2}}$ curvature bounds, the appearance of the local scale can be removed by employing an inequality which bounds $|(r_{|R|})^{-1}|_{L_{n-1}}$ in terms of $(|R|_{L_{\frac{n}{2}}})^{\frac{n-1}{2}}$, where both norms are computed over a neighborhood of ∂K.

The eventual restriction, $n = 4$, in our *main* results stems from the relation between the Chern-Gauss-Bonnet form, P_χ, and the L_2-norm of the curvature in that dimension.

If M^4 is Einstein, then

$$P_\chi = \frac{1}{8\pi^2} \cdot |R|^2 \cdot \mathrm{Vol}(\,\cdot\,)\,, \tag{1.1}$$

where $\mathrm{Vol}(\,\cdot\,)$ denotes the local choice of volume form corresponding to the choice of local orientation used in defining P_χ; see p. 161 of [Be] and (1.2) below.

There is an alternate route to our main results which bypasses chopping, using in its stead the existence of an essentially canonical transgression form, $\mathcal{T}P_\chi$, satisfying $d\,\mathcal{T}P_\chi = P_\chi$ and $|\mathcal{T}P_\chi| \leq c(n) \cdot (r_{|R|}(p))^{-(n-1)}$, on subsets of Riemannian manifolds which are sufficiently collapsed with locally bounded curvature. In actuality, our first proofs used this approach. The construction of $\mathcal{T}P_\chi$ is briefly indicated in Section 12.

The Chern-Gauss-Bonnet form in dimension 4. Let $W, \overset{\circ}{r}, s$ denote the Weyl tensor, traceless Ricci tensor and scalar curvature respectively. By definition, $\overset{\circ}{r} = \mathrm{Ric} - \frac{s}{n} \cdot g$, where $n = \dim M^n$.

For $n = 4$, the Chern-Gauss-Bonnet form satisfies

$$P_\chi = \frac{1}{8\pi^2} \cdot \left[|W|^2 - |\overset{\circ}{r}|^2 + \frac{1}{24} s^2 \right] \cdot \mathrm{Vol}(\,\cdot\,)\,; \tag{1.2}$$

see [Be].

If M^4 is Einstein, then $\overset{\circ}{r} = 0$ and we get (1.1). It follows from (1.1) that if M^4 is closed, then the quantity on the left-hand side of (0.3) (the square of the global L_2-norm of curvature) has the well-known topological interpretation,

$$\frac{1}{8\pi^2} \int_{M^4} |R|^2 = \chi(M^4)\,, \tag{1.3}$$

where $\chi(\,\cdot\,)$ denotes the Euler characteristic. As previously indicated, in the present paper it is also crucial to consider manifolds with nonempty boundary.

Remark 1.4. Relation (1.2) implies that if the Ricci tensor of M^4 is only sufficiently pinched, then $P_\chi \geq \eta \cdot |R|^2 \cdot \mathrm{Vol}(\,\cdot\,)$, where the constant, $\eta > 0$, depends on the pinching. In addition, if M^4 is arbitrary, with Ricci tensor satisfying $|\mathrm{Ric}_{M^4}| \leq 3$, then $|R|^2$ has a definite bound at points at which P_χ is bounded above by any definite (positive or negative) constant times the 4-form $\mathrm{Vol}(\,\cdot\,)$. From these observations and some additional technicalities, it follows that Theorem 0.1 and its consequences, such as Theorem 0.14, are valid for 4-manifolds which are sufficiently Ricci pinched, while Theorem 0.8 and its consequences have extensions for manifolds with bounded Ricci curvature. This will be explained at greater length in subsequent sections.

Collapse. A subset, $U \subset M^n$, such that for all $p \in U$,

$$\sup_{B_1(p)} \mathrm{Ric}_{M^n} \geq -(n-1)\,, \tag{1.5}$$

is called v-*collapsed* if for all $p \in U$,

$$\mathrm{Vol}(B_1(p)) \leq v\,, \tag{1.6}$$

and v-*noncollapsed* if (1.6) holds for no $p \in U$.

Note that the assertion that U is *not* v-collapsed is weaker than the assertion that U is v-noncollapsed.

The local scale $r_{|R|}(p)$. Let M^n denote an arbitrary Riemannian manifold. Let $r_{|R|}(p) > 0$ denote the supremum of those r such that $B_s(p)$ is compact for $s < r$ and

$$\sup_{B_r(p)} |R| \leq r^{-2}\,. \tag{1.7}$$

In particular,

$$|R(p)| \leq (r_{|R|}(p))^{-2}\,. \tag{1.8}$$

The quantity, $r_{|R|}(p)$, will be called the *local scale at p.* Rescaling the metric, $g \to (r_{|R|}(p))^{-2} \cdot g$, converts the ball, $B_{r_{|R|}(p)}(p)$, to a ball of unit radius on which the norm of the curvature is bounded by 1.

Clearly, either $r_{|R|} \equiv \infty$ and $R \equiv 0$, or $r_{|R|}(p)$ is locally Lipschitz, with local Lipschitz constant,

$$\operatorname{Lip} r_{|R|} \leq 1 \,. \tag{1.9}$$

Collapse with locally bounded curvature. Although the term "collapse with locally bounded curvature" does not enter in the statements of Theorems 0.1, 0.8, 0.14, this notion plays a central role in the proofs.

We say that U is *v-collapsed with locally bounded curvature* if for all $p \in U$,

$$\operatorname{Vol}(B_{r_{|R|}(p)}(p)) \leq v \cdot (r_{|R|}(p))^n \,, \tag{1.10}$$

and that U is *(v,a)-collapsed with locally bounded curvature* if, in addition, for all p with $r_{|R|}(p) \geq a$,

$$\operatorname{Vol}(B_a(p)) \leq v \cdot a^n \,.$$

We say that U is *v-collapsed with bounded curvature* if U is v-collapsed and $r_{|R|}(p) \geq 1$ for all $p \in U$.

F-structures and N-structures. Theorem 0.1 of [ChGv2] (see in particular (0.2)) asserts the existence of a constant, $t = t(n) \leq 1$, such that if M^n is v-collapsed with locally bounded curvature, where $v \leq t$, then M^n carries a topological structure called an *F-structure of positive rank*. This concept generalizes that of a torus action for which all orbits have positive dimension. The local action of the F-structure is isometric for a metric close to the given one, provided v is sufficiently small. The orbits of the structure represent the collapsed directions on the *scale of the injectivity radius*.

If M^n admits an F-structure of positive rank, it is not difficult to construct a locally finite covering, $M^n = \bigcup_i U_i$, such that (a finite covering space of) every nonempty intersection, $U_{i_1} \cap \cdots \cap U_{i_j}$, is invariant under the flow of a nonvanishing vector field; see [ChGv2]. Hence, $\chi(U_{i_1} \cap \cdots \cap U_{i_j}) = 0$. Since $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, this implies $\chi(M^n) = 0$, *provided the covering is finite.*

In case the curvature is bounded, $|R| \leq 1$, a description of the geometry of M^n on a *fixed scale*, $r(n)$, is given in [ChFuGv], in terms of an essentially canonical *nilpotent Killing structure of positive rank*, a structure based on nilpotent Lie groups, rather than tori. From now on, we will refer to such a structure as an N-structure.

The discussion of [ChFuGv] has an essentially obvious extension to the case of collapse with locally bounded curvature. This provides a description of the geometry on the scale $r(n) \cdot r_{|R|}(p)$. N-structures produced by the construction of [ChFuGv] and its extension to the case of locally bounded curvature have some significant properties, several of which are relevant in connection with the results on equivariant good chopping proved in Section 3. Structures with these properties will be referred to as *standard*; for details, see Section 2.

Each orbit, $\mathcal{O}_p$, of the N-structure is the union of orbits of an associated F-structure. A subset which is the union of the orbits of its points (with respect to either structure) is called *saturated.* Any (sufficiently regular) saturated subset has vanishing Euler characteristic. The smallest saturated subset containing a given set is called its *saturation.* Without loss of generality, we can assume that $t = t(n)$ above has been chosen such that if M^n is (t,a)-collapsed with locally bounded curvature, then all orbits, $\mathcal{O}_p$, have extrinsic diameter $\leq \frac{1}{8}\min(r_{|R|}(p), a)$.

Equivariant good chopping. For $K \subset M^n$, $r > 0$, put

$$T_r(K) = \{p \in M^n \mid \overline{p,K} < r\}\,,$$

and for $0 \le r_1 < r_2$,

$$A_{r_1,r_2}(K) = T_{r_2}(K) \setminus \overline{T_{r_1}(K)}\,.$$

Let TP_χ denote the integrand in the boundary term of the Chern-Gauss-Bonnet formula for manifolds with boundary. Thus, TP_χ is a homogeneous invariant polynomial of degree $n-1$ in the second fundamental form of the boundary and the curvature, where second fundamental form terms are regarded as having degree 1 and curvature terms as having degree 2.

Let M^n denote a complete Riemannian manifold. Assume $K \subset M^n$ is compact and that $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, for some $r \le 1$. It follows from the equivariant good chopping theorem, Theorem 3.13, that there is a saturated submanifold, Z^n, with smooth boundary, satisfying $T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{3}r}(K)$, such that the boundary term in the Chern-Gauss-Bonnet formula satisfies

$$\left|\int_{\partial Z^n} TP_\chi\right| \le c(n)\cdot r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right)\,.$$

Since $\chi(Z^n) = 0$, by the Chern-Gauss-Bonnet formula, we get

$$\left|\int_{Z^n} P_\chi\right| \le c(n)\cdot r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right)\,. \tag{1.11}$$

Note that in the case of bounded curvature, $|R| \le 1$, the above estimate reduces to the one given in [ChGv3].

In the next two subsections, we indicate how for Einstein manifolds, under additional collapsing assumptions, collapsed regions with locally bounded curvature can be located, and how (absent any additional collapsing assumptions) the right-hand side of (1.11) can be bounded in terms of $|R|_{L_{\frac{n}{2}}}$.

ϵ-Regularity and collapse with locally bounded curvature. In the context of Einstein manifolds, a condition for a set, U, to be $(v,1)$-collapsed with locally bounded curvature is given in Section 5 of [An3]; see Theorem 5.1. By appealing to [ChGv2], an F-structure on (a slight fattening of) U is obtained.

Anderson's results are based on an ϵ-regularity theorem, Theorem 4.4 of [An3]. This ϵ-regularity theorem is valid for arbitrary n, but when specialized to $n = 4$, the assumptions in its hypothesis are stronger than those of Theorem 0.8; compare (0.9) versus (1.12).

Let M^n_H denote the simply connected space of constant curvature H. In what follows, for fixed r, we consider $M^n_{-r^{-2}}$ and $\underline{p} \in M^n_{-r^{-2}}$.

According to Theorem 4.4 of [An3], there exists $\tau = \tau(n) > 0$, such that if

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\cdot \int_{B_r(p)} |R|^{\frac{n}{2}} \le \tau\,, \tag{1.12}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \le c\cdot r^{-2}\cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\cdot \int_{B_r(p)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}\,. \tag{1.13}$$

Without loss of generality, one can assume $c \cdot \tau^{\frac{2}{n}} \leq 4$, so that

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq 4r^{-2}\,. \tag{1.14}$$

Relation (1.12) can be rewritten as

$$\int_{B_r(p)} |R|^{\frac{n}{2}} \leq \tau \cdot \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}\,, \tag{1.15}$$

which specifies directly that the $L_{\frac{n}{2}}$-norm of the curvature is sufficiently small with respect to the collapsing.

Note that by relative volume comparison, [GvLP], the expression inside the parentheses on the right-hand side of (1.13) is a monotonically nondecreasing function of r, which vanishes at $r = 0$.

Now assume

$$|\lambda| \leq n-1\,. \tag{1.16}$$

Let $\underline{p} \in \underline{M}^n_{-1}$ and define $\theta = \theta(n)$ by

$$\theta = \frac{1}{2^n \mathrm{Vol}(B_1(\underline{p}))}\,. \tag{1.17}$$

Recall that collapse with locally bounded curvature is defined via condition (1.10). If U is $(\theta \cdot v)$-collapsed with $v \leq 1$ and if for all $p \in U$,

$$\int_{B_1(p)} |R|^{\frac{n}{2}} \leq \theta \cdot v \cdot \tau\,, \tag{1.18}$$

then U is v-collapsed with locally bounded curvature, where in particular, one can take $v = t = t(n)$, where t-collapse with locally bounded curvature implies the existence of a standard N-structure; see Theorem 5.1 of [An3] and compare also [Ya]. (Both of the above references deal with F-structures.)

For completeness, we give the argument for the case $v = t$. (For v arbitrary, the argument is the same.)

Modulo the choice of normalizing constant, the following definition is taken from (4.21) of [An3]. The notation is as in (1.12)–(1.18).

If $p \in M^n$ satisfies

$$\frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_1(p))} \cdot \int_{B_1(p)} |R|^{\frac{n}{2}} \leq \tau\,, \tag{1.19}$$

put $\rho(p) = 1$. Otherwise, define $\rho(p)$ to be the (largest) solution of

$$\frac{\mathrm{Vol}(B_{\rho(p)}(\underline{p}))}{\mathrm{Vol}(B_{\rho(p)}(p))} \cdot \int_{B_{\rho(p)}(p)} |R|^{\frac{n}{2}} = \tau\,, \tag{1.20}$$

where, since the left-hand side of (1.20) is a nondecreasing function of r, we have $\rho(p) < 1$.

Relation (1.13) gives

$$\frac{1}{2}\rho(p) \leq r_{|R|}(p)\,. \tag{1.21}$$

If $\rho(p)$ is defined by (1.19), then we have $\sup_{B_{\frac{1}{2}}(p)} |R| \leq 4$, and since U is $(\theta \cdot t)$-collapsed, it follows that (1.10) holds with $v = t$.

If $\rho(p)$ is defined by (1.20), then from (1.18) (with $v = t$) we have

$$\frac{\mathrm{Vol}(B_{\rho(p)}(p))}{\mathrm{Vol}(B_{\rho(p)}(\underline{p}))} \leq \theta \cdot t \,,$$

which by (1.17) and relative volume comparison implies that (1.10) holds with $v = t$.

Remark 1.22. The proof of Theorem 4.4 of [An3], as well as that of an earlier ϵ-regularity theorem proved (independently) in [An1], [Na], [Ti], is based on Moser iteration, which leads to an estimate in which the factor, $\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}$, in (1.12), (1.13), is replaced by s^n, with s a suitable Sobolev constant. In the proof, the Sobolev inequality is applied only to functions which are supported in $B_r(p)$.

In [An1], [Na], [Ti], a global Sobolev constant is employed, while [An3] uses the Sobolev constant, $s(r,p)$, of $B_r(p)$; i.e., for f supported in $B_r(p)$,

$$\left(\int_{B_r(p)} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq s(r,p) \cdot \int_{B_r(p)} |df| \,.$$

Modulo this difference, the analytical details of the proofs are identical.

According to Theorem 4.1 of [An3], there exists c such that

$$s(r,p) \leq c \cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\right)^{\frac{1}{n}} \;;$$

compare the closely related estimate for $s(r,p)$ in [ChGvTa]; see also [Cr] and [ChengLiYau]. Estimation of the global Sobolev constant requires a global diameter bound.

Remark 1.23. The application to collapse with locally bounded curvature only uses the bound on $s(r,p)$ for balls which satisfy $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq \theta \cdot t$ (for which the estimate of [ChGvTa] suffices); compare Remark 1.31. To see this, modify the definition of $\rho(p)$ by replacing $\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}$ by $s(r,p)^n$ in (1.19), (1.20) and proceed, mutatis mutandis, as above.

Bounds on $(r_{|R|}(p))^{-1}$. To obtain a bound on $(r_{|R|}(p))^{-1}$, we first recast the discussion in terms of the maximal function. For $c = c(n)$, we get for any $r \leq 1$,

$$(r_{|R|}(p))^{-(n-1)} \leq c \cdot (r^{-(n-1)} + (M_{|R|^{\frac{n}{2}}}(p,r))^{\frac{n-1}{n}}) \,, \tag{1.24}$$

where $M_{|R|^{\frac{n}{2}}}(p,r)$ denotes the maximal function over balls of radius $\leq r$, of the function, $|R|^{\frac{n}{2}}$, evaluated at the point p; for details, see Section 4.

For $\alpha < 1$, the normalized L_α-norm of the maximal function, M_f, of f can be bounded in terms of the L_1-norm of f; see (4.2). From this, together with (1.24), it follows that the integral of the right-hand side of (1.11) can be bounded in terms of $|R|_{L_{\frac{n}{2}}}$. Thus, if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, for some $r \leq 1$, then for any $\Omega \geq \mathrm{Vol}(A_{\frac{1}{4}r,\frac{3}{4}r}(K))$, we get

$$\left|\int_{Z^n} P_\chi\right| \leq c(n) \cdot \Omega \cdot r^{-1} \left(r^{-(n-1)} + \left(\Omega^{-1} \cdot \int_{A_{\frac{1}{4}r,\frac{3}{4}r}(K)} |R|^{\frac{n}{2}}\right)^{\frac{n-1}{n}}\right) . \tag{1.25}$$

(The detailed proof of (1.25) is concluded after (4.4).)

The key estimate; $n = 4$. Let $E \subset M^4$ denote a bounded open subset such that $T_1(E)$ is $(\theta \cdot t)$-collapsed and (1.18) holds for all $p \in T_1(E)$. Replace E by the saturation of $\overline{E}$ for some standard N-structure and apply the discussion leading to (1.25). Since we assume $n = 4$, on the left-hand side of (1.25), P_χ can be replaced by $\frac{1}{8\pi^2}|R|^2$ and the domain of integration can be changed from Z^n to E. The resulting relation provides an estimate for $(|R|_{L_2})^2$, on the set E in terms of $(|R|_{L_2})^{\frac{3}{2}}$ on the set $A_{\frac{1}{4}r,\frac{3}{4}r}(E)$.

The reduction in the exponent, $2 \to \frac{3}{2}$, leads to an iteration argument yielding the following key estimate.

Theorem 1.26. *There exists $\delta > 0$, $c > 0$, such that the following holds. Let M^4 denote a complete Einstein manifold satisfying* (0.2), (0.3), *and let $E \subset M^4$ denote a bounded open subset such that $T_1(E)$ is t-collapsed with*

$$\int_{B_1(p)} |R|^2 < \delta \qquad (\textit{for all } p \in T_1(E))\,. \tag{1.27}$$

Then

$$\int_E |R|^2 \le c \cdot \mathrm{Vol}(A_{0,1}(E))\,. \tag{1.28}$$

Remark 1.29. In the iteration (or self-improvement) argument, the exponent, $\frac{n-1}{n} = \frac{3}{4} < 1$, in (1.25), plays a role analogous to that which this same exponent plays in Moser iteration. There, it enters via the Sobolev inequality (which is not used in the iteration argument occurring in the proof of Theorem 1.26). As previously mentioned, the ϵ-regularity theorem of [An1], [Na], [Ti] and Theorem 4.4 of [An3] are proved by Moser iteration.

Remark 1.30. Suppose in Theorem 1.26, we drop the assumption that E is bounded. If $\lambda \neq 0$, then by (0.3), we have $\mathrm{Vol}(M^4) < \infty$ and by an obvious exhaustion argument, (1.28) continues to hold. If $\lambda = 0$, an exhaustion argument together with scaling leads easily to (0.6).

Implementation of the key estimate. Apart from the ϵ-regularity theorem, Theorem 0.8, all of our results in dimension 4 are relatively simple consequences of the key estimate.

In proving Theorem 0.1, we use (1.18) and a standard covering argument to choose the balls, $B_s(p_i)$, such that we can take the set, E, to be the set $M^4 \setminus \bigcup_i B_s(p_i)$ with suitably rescaled metric. Then the conclusion, (0.6), reduces to (1.28).

In proving Theorem 0.8, we take $\epsilon = \theta \cdot t \cdot \tau$. If

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \ge \theta^2 \cdot t\,,$$

then the ϵ-regularity Theorem 4.4 of [An3] can be applied directly. Otherwise, we can take E in Theorem 1.26 to be (a suitable rescaling of) $B_{\frac{1}{2}r}(p)$. This gives, for c an absolute constant,

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2 \le c\,.$$

If we knew $c \le \tau$, then the hypothesis, (1.12), of the ϵ-regularity theorem (Theorem 4.4 of [An3]) would be verified. Since this does *not* seem to be clear, our argument employs a second nontrivial step. Namely, we show that once we pass to a smaller concentric ball of a definite radius, the hypothesis of Theorem 4.4 of [An3] is verified.

Remark 1.31. The statement of Theorem 4.4 of [An3] requires only that the ball, $B_r(p)$, satisfies (1.12) and makes no assumption concerning a lower bound for $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}$ or for r. However, in all previous applications, e.g. to collapse with locally bounded curvature, in order to *verify* (1.12), a definite lower bound, $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq v$, on the collapsing of $B_r(p)$, has been used; compare Remark 1.23.

Organization of remaining sections. In Section 2, we review N-structures and their construction, in the context of collapsed manifolds with bounded curvature; see [ChFuGv]. We observe that mutatis mutandis, the discussion extends naturally to collapse with locally bounded curvature.

In Section 3, we prove an extension of the equivariant good chopping theorem of [ChGv3], in which no global bound on curvature is assumed. In the conclusion, the bound on the norm of the second fundamental form is in terms of the reciprocal of the local scale $r_{|R|}$.

In Section 4, we consider Einstein manifolds with $L_{\frac{n}{2}}$ curvature bounds. We note the estimate, (1.24), which provides a lower bound on the local scale. This, together with a universal inequality for the maximal function, leads to a stronger chopping theorem in this more restricted context. An immediate application is the integrality of the geometric Euler characteristic in arbitrary dimension; compare [ChGv3].

In Section 5, we prove Theorem 1.26, the key estimate in dimension 4.

In Section 6, we prove Theorem 0.1, our main global result on collapse in dimension 4.

In Section 7, we consider the case of a negative Einstein constant in dimension 4. We prove our results on noncollapse (Theorems 0.14, 7.8). We also show that for Einstein 4-manifolds with negative Einstein constant and finite L_2-norm of curvature, the volume decreases exponentially at infinity.

In Section 8, we prove Theorem 0.8, the ϵ-regularity theorem in dimension 4.

In Section 9, we note some consequences of Theorem 0.8.

In Section 10, we discuss the implications of our main theorems for Gromov-Hausdorff limit spaces, using the language of compactifications of moduli spaces of Einstein metrics on 4-manifolds.

In Section 11, we speculate on some possible extensions of our main results for anti-self-dual metrics, Kähler metrics of constant scalar curavture, entire solutions of the Ricci flow and higher-dimensional Einstein manifolds.

In the Appendix, Section 12, we indicate the construction of the transgression form $\mathcal{T}P_\chi$. In proving Theorem 1.26, the form, $\mathcal{T}P_\chi$, can be used in place of the good chopping theorem.

2. Collapse with locally bounded curvature; N-structures

An N-structure on a manifold, N^n, is a sheaf of nilpotent Lie algebras of vector fields, which are Killing fields for some Riemannian metric and for which certain additional properties hold; for details see [ChFuGv]. The N-structure decomposes the manifold as a disjoint union of compact orbits, each of which has a canonical affine flat structure isomorphic to that of a nilmanifold. The *rank* of the structure is the dimension of an orbit of smallest dimension. If the rank is positive, then any sufficiently nice saturated subset (e.g. a submanifold, U^n, with piecewise smooth boundary) has vanishing Euler characteristic.

Standard N-structures and invariant metrics. Let (N^n, g) denote a v-collapsed, manifold with $v \le t(n)$, and bounded curvature, $|R| \le 1$. The construction given in [ChFuGv] associates to (N^n, g) an essentially canonical N-structure of positive rank and an invariant metric, $\widetilde{g}$, with the following significant properties:

i) (**Local structure on a fixed scale**) There exist $c = c(n)$, $r = r(n)$, such that for all $p \in M^n$, there is an orbit, $\mathcal{O}_q$, with second fundamental form satisfying $|II_{\mathcal{O}_q}| \le c$ and normal injectivity radius at least $3r$, such that $B_r(p)$ is contained in the tube $T_{3r}(\mathcal{O}_q)$.

ii) (**Invariant metric**) There is a metric, $\widetilde{g}$, for which the local nilpotent actions associated to the structure are isometric. In particular, with respect to $\widetilde{g}$, any tubular neighborhood of an orbit is a saturated subset.

Moreover, for all $\epsilon > 0$, k, C, there exist $\eta = \eta(n, \epsilon)$, $\delta = \delta(\epsilon, k, C) > 0$, such that:

iii) (**Orbits with small diameter**) If in addition, $v \le \eta$, then every orbit, $\mathcal{O}$, has extrinsic diameter satisfying $\mathrm{diam}(\mathcal{O}) \le \epsilon$.

iv) (**Closeness of invariant metric**) If $|\nabla^i R| \le C$, for $i = 0, \ldots, k$, and $v \le \delta$, then $\widetilde{g}$ can be chosen such that $|\nabla^i(R - \widetilde{R})| \le \epsilon$, $i = 1, \ldots k-1$.

Remark 2.1. The chopping theorems, Theorems 3.1, 3.13, depend on properties ii)–iv) above. The estimate, (12.2), for the transgression form, $\mathcal{T}P_\chi$, namely, $|\mathcal{T}P_\chi| \le c(n)(r_{|R|})^{-(n-1)}$, requires property i) as well.

Fix $0 < a \le 1$ and put

$$\ell_a = \min(r_{|R|}, a)\,. \tag{2.2}$$

Let N^n be an arbitrary complete Riemannian manifold and let $\mathcal{N}$ denote an N-structure on N^n. We say that $\mathcal{N}$ is *a-standard* if for all p, its restriction to $B_{\ell_a(p)}(p)$ has properties i)–iv) with respect to the rescaled metric, $g \to (\ell_a(p))^{-2} \cdot g$.

We have $\mathrm{Lip}\, \ell_a \le 1$; see (1.9). As a consequence, when restricted to any ball, $B_{\zeta \cdot \ell_a(p)}(p)$, with $\zeta < 1$, the positive function, ℓ_a, satisfies a Harnack inequality with constant, $\frac{1+\zeta}{1-\zeta}$. Thus, *the function, ℓ_a, varies moderately on its own scale.* This implies, for example, that *locally*, coverings by balls of the form, $\{B_{\ell_a(p_i)}(p_i)\}$, behave essentially like coverings by balls of a fixed radius. As a consequence, for the case of locally bounded curvature discussed below, constructions involving such coverings can be reduced to the case of bounded curvature by local scaling arguments.

Existence of standard N-structures. Due to its essential locality, the construction of [ChFuGv] extends directly to the case of sufficiently collapsed manifolds with locally bounded curvature, thereby yielding a structure satisfying the scaled version of i)–iv).

Theorem 2.3. *There exists $t = t(n) > 0$ such that if M^n is complete and $W \subset M^n$ is (t, a)-collapsed with locally bounded curvature, then there exists an a-standard N-structure on a subset containing W.*

Proof. We will require two simple facts pertaining to coverings. These generalize corresponding statements in the case of Ricci curvature bounded below.

Fix a small constant, $\zeta > 0$, and let $\{p_\alpha\}$ denote a maximal set of points such that

$$\overline{p_{\alpha_1}, p_{\alpha_2}} \geq \zeta \cdot \min(\ell_a(p_{\alpha_1}), \ell_a(p_{\alpha_2})) \qquad (\alpha_1 \neq \alpha_2)\,.$$

If the metric on $B_{\ell_a(p)}(p_i)$ is rescaled, $g \to (\ell_a(p_i))^{-2} \cdot g$, the resulting metric has bounded curvature $|R| \leq 1$. By an obvious variant of the corresponding argument in the case of bounded curvature, it follows that $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$ is a covering, with multiplicity $\leq N(n)$.

The covering, $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$, can be partitioned into at most $N(n)$ disjoint subcollections, S_i, of mutually nonintersecting balls, $B_{2\zeta \cdot \ell_a(p_{i,j})}(p_{i,j})$, such that a given member of any such subcollection intersects at most one member of any other such subcollection; see Lemma 2.2 of [ChGv3] and (6.4.1)–(6.4.5) of [ChGv3]. In addition, if

$$B_{2\zeta \cdot \ell_a(p_{i_1,j_1})}(p_{i_1,j_1}) \cap B_{2\zeta \cdot \ell_a(p_{i_2,j_2})}(p_{i_2,j_2}) \neq \emptyset\,,$$

then

$$(1-\zeta) \cdot \ell_a(p_{i_1,j_1}) \leq \ell_a(p_{i_2,j_2}) \leq (1+\zeta) \cdot \ell_a(p_{i_1,j_1})\,.$$

In [Ab], [Ya], procedures are given for regularizing a metric with bounded curvature, $|R| \leq 1$. Let ∇ denote the Riemannian connection of g. Given $\eta > 0$, we can arrange that the regularized metric, $\widehat{g}$, and its connection, $\widehat{\nabla}$, satisfy

$$(1+\eta)^{-2} g \leq \widehat{g} \leq (1+\eta)^2 g\,,$$
$$|\nabla - \widehat{\nabla}|_g \leq c(n)\eta\,,$$
$$|\widehat{\nabla}^k \widehat{R}|_{\widehat{g}} \leq c(n,k,\eta)\,.$$

We extend this to our situation as follows. (If for example, we are dealing with Einstein manifolds, this step is actually unnecessary since Einstein metrics are already regular in the appropriate sense.)

Fix η. On each ball, $B_{r_{|R|}(p_\alpha)}(p_\alpha)$, we apply the regularization procedure of [Ab] to the rescaled metric, $(\ell_a(p_\alpha))^{-2} g$, the norm of whose curvature tensor is bounded by 1. We denote the resulting regularized metric of bounded curvature by $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$.

Next, as in [ChGv3], we regularize the distance function of the metric, $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$, to obtain a smooth function with definite bounds on all covariant derivatives with respect to the metric $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$.

By composing the regularized distance functions with standard bump functions, we obtain a partition of unity, $\{\phi_\alpha\}$, subordinate to the cover $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$.

The metric, $\widetilde{g}$, defined by

$$\widetilde{g} = \sum_\alpha \phi_\alpha \widehat{g_\alpha}\,,$$

satisfies

$$(1+\eta)^{-2} g \leq \widetilde{g} \leq (1+\eta)^2 g\,, \tag{2.4}$$

$$|\nabla - \widetilde{\nabla}|_g \leq c(n)\eta \ell_a^{-1}\,, \tag{2.5}$$

$$|\widetilde{\nabla}^k \widetilde{R}|_{\widetilde{g}} \leq c(n,k,\eta)(\ell_a)^{-(k+2)}\,; \tag{2.6}$$

compare Theorem 1.12 of [ChFuGv].

As constructed above, the metric, $\widetilde{g}$, need not be invariant under the isometries of g. However, this can be arranged by a simple modification of the construction; compare the corresponding remark in [ChGv3] (in the context of bounded curvature). Namely, we replace the points, p_α, by a corresponding maximal collection of orbits, $\mathcal{O}_\alpha$, under the isometry group, and the balls, $B_{\ell_a(p_\alpha)}(p_\alpha)$, by tubular neighborhoods, $T_{2\zeta\cdot\ell_a(p_\alpha)}(\mathcal{O}_{p_\alpha})$.

The remainder of the construction of [ChFuGv] is local in nature. Hence, by straightforward scaling arguments, the case of locally bounded curvature can be reduced to the case of bounded curvature.

Specifically, in [ChFuGv], one constructs a series of almost mutually compatible $\mathrm{O}(n)$-equivariant local fibrations of the inverse image in the frame bundle of (slightly fattened) members of a covering by balls of fixed radius; see Sections 2–5 of [ChFuGv].

Using the subcollections, S_i, the fibrations are modified so as to become mutually compatible; see Section 6 of [ChFuGv]. Flat affine structures are specified on the fibres and the fibrations are modified again so as to make the flat affine structures compatible as well; see Section 7 of [ChFuGv]. Finally, an invariant metric close to the original one is constructed by a local averaging process; see Section 8 of [ChFuGv].

In the case of collapse with locally bounded curvature, one begins with the covering, $\{B_{2\zeta\cdot\ell_a(p_\alpha)}(p_\alpha)\}$, constructed above. Since for the collection of fibrations described above, the selection process is local, it can be carried out in the context of locally bounded curvature by making the local rescaling of the metric $g \to (\ell_a(p_\alpha))^{-2}\cdot g$. The modification process uses the subcollections, S_i, $i = 1, \dots, N(n)$. As above, the fibration corresponding to each ball in S_i is modified at the i-stage so as to fit the now mutually compatible fibrations corresponding to (slight shrinkings of) the at most $i-1$ balls in $S_1, \dots, S_{i-1}$ with which it has nonempty intersection. Again, since the modification process in [ChFuGv] is local, it follows that, by scaling, the case of local bounded curvature can be reduced to the case of bounded curvature.

The remainder of the proof can be completed by using scaling arguments such as those we have just described. $\square$

Remark 2.7. Suppose that the assumption that M^n is (t,a)-collapsed with locally bounded curvature is weakened in the following way. Rather than assuming that for the metric, $(\ell_a(p))^{-2}g$, on each ball, $B_{\ell_a(p)}(p)$, the curvature satisfies $|R| \leq 1$, we assume that the metric on the universal covering space of this ball satisfies definite $C^{1,\alpha}$-bounds, for all $\alpha < 1$, and $L_{2,q}$-bounds in harmonic coordinates, for all $q < \infty$. In this case, the theory of N-structures (and the subsequent theorem on equivariant choppings) continue to hold. As above, a key point is to construct a suitable regularization of the metric; see [Ya].

3. Equivariant good chopping with local curvature bounds

In this section, we prove Theorems 3.1, 3.13, which generalize the equivariant chopping theorem of [ChGv3] to the case of locally bounded curvature.

Let M^n denote a complete Riemannian manifold and $K \subset M^n$ a closed subset. For $N^k \subset M^n$ a smooth submanifold without boundary, we denote by II_{N^k} the second fundamental form of N^k.

For ℓ_a as in (2.2), put

$$S_a(K) = K \cup \left(\bigcup_{p \in \partial K} B_{\ell_a(p)}(p) \right) .$$

Let $t = t(n)$ be as in previous sections.

Theorem 3.1. *There exists $c = c(n) < \infty$ and a smooth manifold with boundary, Z^n, satisfying*

$$K \subset Z^n \subset K \cup S_a(K)\,, \tag{3.2}$$

$$|II_{\partial Z}| \leq c \cdot \ell_a^{-1}\,, \tag{3.3}$$

and for all $k_1, k_2 > 0$,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} \cdot |R|^{k_2} \leq c \cdot \int_{S_a(K)} {\ell_a}^{-(k_1+1)} (r_{|R|})^{-2k_2}\,. \tag{3.4}$$

In particular,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} |R|^{k_2} \leq c \int_{S_a(K)} \left(a^{-(k_1+2k_2+1)} + (r_{|R|})^{-(k_1+2k_2+1)} \right) . \tag{3.5}$$

Moreover, if $S_a(K)$ is (t,a)-collapsed with locally bounded curvature, then Z^n can be chosen to be saturated for some standard N-structure.

Proof. Given the formulation, the proof is a relatively straightforward generalization of that of the chopping theorem of [ChGv3], which deals with the case of the special case in which M^n has bounded curvature $|R| \leq 1$. We will recall the argument in that case, indicate the required modifications and refer to [ChGv3] for additional details.

The main technical result of [ChGv3] asserts the existence of constants $0 < \delta(n) \leq 1$, $0 < \epsilon(n)$, $c(n)$, such that if $f : M^n \to \mathbf{R}$ satisfies $\operatorname{Lip} f \leq L$, then for all $r \leq 1$, there exists $F : M^n \to \mathbf{R}$, with

$$\begin{gathered} F \leq f \leq (1+\delta(n)) r F\,, \\ |\nabla F| \leq 2L\,, \\ |\mathrm{Hess}_F| \leq c(n) L r^{-1}\,, \\ |\nabla F(x)| \geq \epsilon(n) L \qquad \left(x \in F^{-1}([0, \delta(n) r L]) \right) . \end{gathered}$$

In addition, $F \,|\, F^{-1}((-\infty, \delta(n) r L])$ can be chosen to be invariant under the isometries of $f^{-1}((-\infty, rL])$ which fix $f \,|\, f^{-1}((-\infty, rL])$.

The submanifold with boundary, Z, is constructed as a certain sublevel set, $F^{-1}(\underline{y})$, with $\underline{y} \in [0, \delta(n) r]$, where as above, F is associated to the distance function $f = \rho_K(x) = \overline{x, K}$. The second fundamental form is estimated by means of the relation,

$$\langle \nabla_V W, N \rangle = -\frac{\mathrm{Hess}_F(V,W)}{|\nabla F|}\,, \tag{3.6}$$

where $N = \nabla F / |\nabla F|$ and V, W are tangent to $F^{-1}(\underline{y})$.

At the outset of the construction of the function, F, given in [ChGv3], the metric, g, and the function, f, are regularized. This permits subsequent application of a quantitative version due to Yomdin, of the A.P. Morse lemma, yielding on each ball of a cover, $\{B_1(p_\alpha)\}$, an interval of a definite size, on the inverse image of which the gradient has a definite lower bound; for a discussion of Yomdin's theorem, see [Gv2]. The remainder of the construction consists of a sequence of modifications of the regularized function, eventually yielding the function, F, for which the above mentioned intervals can be chosen independently of the particular ball $B_1(p_\alpha)$.

To see the need for regularization, recall that the classical Morse-Sard theorem makes the (qualitative) assertion that if $f : M^{n_1} \to M^{n_2}$ and $f \in C^k$, with $k-1 \geq \max(n_1 - n_2, 0)$, then almost all values are regular.

Since the gradient and Hessian of f with respect to the *original* metric must be controlled, one must employ a regularization of the metric such that the Riemannian connections of the initial and regularized metrics are close; compare the discussion in Section 2 and see below.

In the present case, which is more general than that considered in [ChGv3], given $\mathrm{Lip}\, f \leq L$, we will construct F satisfying the above invariance property and

$$F \leq \ell_a^{-1} f \leq (1+\delta(n)) rF\,, \tag{3.7}$$

$$|\nabla F| \leq 2L\ell_a^{-1}\,, \tag{3.8}$$

$$|\mathrm{Hess}_F| \leq c(n) L r^{-1} \ell_a^{-2}\,, \tag{3.9}$$

$$|\nabla F(x)| \geq \epsilon(n) L \ell_a^{-1} \ \ \big(x \in F^{-1}([0, \delta(n) rL])\big)\,. \tag{3.10}$$

In the application to chopping, we again choose $f = \rho_K$, or, in the equivariant case, the distance function from the saturation of K and realize Z as $F^{-1}(\underline{y})$, with $\underline{y} \in [0, \delta(n) r]$. Then (3.2) is clear, while (3.3) follows from (3.6), (3.9), (3.10). Finally, (3.4) follows from (3.3), (3.8) and the coarea formula applied to the sublevel set $F^{-1}([0, \delta(n) rL])$.

Note that in applying the coarea formula, we multiply the integrand on the left-hand side of (3.4) by $|\nabla F|_{\widetilde{g}}$, which satisfies the bound (3.8). This accounts for why the exponent in (3.4) is $-(k_1+2k_2+1)$, rather than $-(k_1+2k_2)$. The scale invariant relation, (3.4), leads to the exponent, $-(k_1+2k_2)$, in (3.16), which is crucial for our subsequent applications; compare the derivation of (3.16).

We now describe the construction of the function, F.

By scaling, we can assume $\mathrm{Lip}\, f \leq 1$.

We begin by regularizing the metric locally on the scale, ℓ_a, as in Section 2. Given $\eta > 0$, we can arrange that the metric, $\widetilde{g}$, so obtained, satisfies (2.4)–(2.6); compare (1.2)–(1.4) of [ChGv3].

From now on we choose $(1+\eta)^2 = (\frac{6}{5})^2$.

Next we choose a covering, $\{B_{\ell_a(p_\alpha)}(p_\alpha)\}$, and partition it into at most $N(n)$ mutually disjoint subcollections, S_i, of mutually nonintersecting balls, as in the proof of Theorem 2.3, on the existence of standard N-structures. We put $W_i = \bigcup_j B_{\frac{1}{2}\ell_a(p_{i,j})}(p_{i,j})$.

As in Section 2, we construct a partition of unity, $\{\phi_{i,j}\}$, satisfying

$$|\widetilde{\nabla}^k \phi_{i,j}| \leq c(n,k)(\ell_a(p_{i,j}))^{-k}\,;$$

compare (1.20), (1.21) of [ChGv3].

On each ball, $B_{\ell_a(p_{i,j})}(p_{i,j})$, all covariant derivatives of curvature are bounded for the metric $\ell_a^{-2}(p_{i,j})\widetilde{g}$. Hence, as in[ChGv3], for all i,j, we can smooth the restriction of the function, $\ell_a^{-1} f - \frac{1}{8}$, to the ball $B_{\ell_a(p_{i,j})}(p_{i,j})$. Combining these functions by means of the partition of unity, $\{\phi_{i,j}\}$, yields a function, F_0, satisfying

$$F_0 \le \ell_a^{-1} f \le F_0 + \frac{1}{4}\,,$$
$$|\nabla F_0|_{\widetilde{g}} \le \frac{4}{3}\ell_a^{-1}\,,$$
$$|\widetilde{\nabla}^k F_0|_{\widetilde{g}} \le c(n,k)\ell_a^{-k} \qquad (k \ge 2)\,;$$

compare (1.16)–(1.18) of [ChGv3].

Let $p_{i,j} \in S_i$ and consider the function, $(\ell_a(p_{i,j}))^{-1} f$, on the ball, $B_{\ell_a(p_{i,j})}(p_{i,j})$, with rescaled metric, $\ell_a^{-2}(p_{i,j})g$, for which we have

$$|\nabla F_0|_{\ell_a^{-2}(p_{i,j})\widetilde{g}} \le \frac{4}{3}\,, \tag{3.11}$$

$$|\widetilde{\nabla}^k F_0|_{\ell_a^{-2}(p_{i,j})\widetilde{g}} \le c(n,k) \qquad (k \ge 2)\,. \tag{3.12}$$

Since for the metric $\ell_a^{-2}(p_{i,j})\widetilde{g}$, the curvature and its covariant derivatives satisfy the bounds in (2.4)–(2.6), as in [ChGv3], we can apply the quantitative version of the A.P. Morse Lemma due to Yomdin, to obtain by induction: a sequence of functions, $F_{N(n)} \le F_{N(n)-1}, \ldots$, constants, $0 < \delta_{N(n)}(n) < \delta_{N(n)-1}(n), \ldots$, $0 < \epsilon_{N(n)} < \epsilon_{N(n)-1} < \ldots$ and $c(n,k)$, such that for $1 \le m \le N(n)$,

$$F_m \le \ell_a^{-1} f \le F_m + \frac{1}{4} + \frac{1}{4N(n)}\,,$$
$$|\widetilde{\nabla} F_m|_{\widetilde{g}} \le \left(\frac{4}{3} + \frac{m}{3N(n)}\right)\ell_a^{-1}\,,$$
$$|\widetilde{\nabla}^k F_m|_{\widetilde{g}} \le c_m(n,k)\ell_a^{-k}\,,$$

and for $x \in F_m^{-1}([0,\delta_m(n)]) \cap (W_1 \cup \ldots \cup W_m)$,

$$|\widetilde{\nabla} F_m|_{\widetilde{g}} \ge \epsilon_m(n) > 0\,;$$

compare (1.22)–(1.25) of [ChGv3].

If we put $F = F_{N(n)}$, then (3.7)–(3.10) hold. □

Let K be as above. Theorem 3.1 easily implies:

Theorem 3.13. *There exists $c = c(n) < \infty$ and a smooth manifold with boundary, Z^n, satisfying*

$$T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{3}r}(K)\,, \tag{3.14}$$

$$|II_{\partial Z}| \le c \cdot \left(r^{-1} + (r_{|R|})^{-1}\right)\,, \tag{3.15}$$

and for all $k_1, k_2 > 0$,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1}|R|^{k_2} \le \frac{c}{r}\int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(k_1+2k_2)} + (r_{|R|})^{-(k_1+2k_2)}\right). \tag{3.16}$$

Moreover, if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, then Z^n can be chosen to be saturated for some standard N-structure.

Proof. As in the proof of Theorem 3.1 in the equivariant case, we replace K by its saturation. By scaling, we can suppose $r = 1$.

Let ℓ_a be as in Theorem 3.1 and choose $a = \frac{1}{10}$. For each $s \in [\frac{2}{5}, \frac{3}{5}]$, we apply Theorem 3.1 with the set K replaced by $T_s(K)$. This yields a set, Z_s, for which (3.15), (3.14) hold.

To obtain a value of s for which (3.16) holds as well, we will estimate the integral with respect to s of the function which assigns to each s, the right-hand side of the version of (3.4) obtained by replacing K by $T_s(K)$. From this integral estimate, it will follow immediately that (3.16) holds for some $s \in [\frac{2}{5}, \frac{3}{5}]$. This will suffice to complete the proof.

Let $\{p_\alpha\}$ denote a maximal subset of $A_{\frac{1}{3},\frac{2}{3}}(K)$ such that

$$\overline{p_{\alpha_1}, p_{\alpha_2}} \geq \frac{1}{8} \cdot \min(\ell_a(p_{\alpha_1}), \ell_a(p_{\alpha_2})) \qquad (\alpha_1 \neq \alpha_2)\,.$$

By relative volume comparison together with rescaling, $\{B_{\frac{1}{8}\cdot \ell_a(p_i)}(p_i)\}$ is a covering of $A_{\frac{2}{5},\frac{3}{5}}(K)$, with multiplicity $\leq N$, a definite constant.

Let s_α be such that $p_\alpha \in \partial T_{s_\alpha}(K)$. Let $A \subset T_1(K) \times [0,1]$ be defined by

$$A = \{(x,s) \,|\, x \in S_a(T_s(K)\}\,.$$

We claim that

$$A \subset \bigcup_\alpha B_{\frac{1}{8}\cdot \ell_a(p_\alpha)}(p_\alpha) \times [s_i - \frac{1}{2}\ell_a(p_\alpha), s_\alpha]\,,$$

and in addition, that for $x \in B_{\frac{1}{8}\ell_a(p_i)}(p_i)$, we have

$$\ell_a(x) \geq \frac{7}{8} \cdot \ell_a(p_\alpha)\,.$$

This implies

$$\int_A (\ell_a)^{-(k_1+1)} (r_{|R|})^{-2k_2} \leq c \cdot \sum \int_{B_{\frac{1}{8}\ell_a(p_i)}(p_i)} (\ell_a)^{-(k_1)} (r_{|R|})^{-2k_2}\,.$$

As indicated above, the theorem follows.

To verify the claim, let $x \in S(T_s(K)) \cap B_{\frac{1}{8}\ell_a(p_\alpha)}(p_\alpha)$ for some s. Since $\operatorname{Lip} \ell_a \leq 1$, we have $\ell_a(x) \geq \frac{7}{8} \cdot \ell_a(p_\alpha)$. Moreover, $x \in B_{\frac{1}{8}\ell_a(q)}(q) \cap B_{\frac{1}{8}\ell_a(p_i)}(p_\alpha)$, for some $q \in \partial T_s(K)$. Since $\operatorname{Lip} \ell_a \leq 1$, it follows easily that $B_{\frac{1}{8}\ell_a(q)}(q) \subset B_{\frac{1}{2}\ell_a(p_\alpha)}(p_\alpha)$, which implies $s \geq s_\alpha - \frac{1}{2}\ell_a(p_\alpha)$. $\square$

4. $L_{\frac{n}{2}}$-CURVATURE BOUNDS

In this section, for Einstein manifolds with $L_{\frac{n}{2}}$-curvature bounds, we give a pointwise bound on the scale, $r_{|R|}$, in terms of the maximal function $M_{|R|^{\frac{n}{2}}}$. This gives rise to an estimate on the boundary term of the Chern-Gauss-Bonnet formula applied to a good chopping, in terms of $(|R|_{\frac{n}{2}})^{\frac{n-1}{n}}$. We require some preliminaries concerning maximal functions.

Maximal functions. For (X, μ) a metric measure space, with μ a finite Radon measure, and $f \in L_1$, put

$$⨍_A |f| = \frac{1}{\mu(A)} \int_A |f|\,.$$

Define the maximal function for balls of radius at most r by

$$M_f(x,r) = \sup_{s \leq r} ⨍_{B_s(x)} |f|\,.$$

Let $W \subset X$ denote a measurable subset.

Lemma 4.1. *If every ball, $B_s(x)$, with $x \in W$, $s \leq 4r$, satisfies*

$$\mu(B_{2s}(x)) \leq 2^{\kappa} \mu(B_s(x)) \,,$$

then for all $\Omega \geq \mu(W)$, $\alpha < 1$,

$$\left(\frac{1}{\Omega} \int_W (M_f(x,r))^{\alpha} \, d\mu \right)^{\frac{1}{\alpha}} \leq \frac{c(\kappa,\alpha)}{\Omega} \int_{T_{6r}(W)} |f| \, d\mu. \tag{4.2}$$

Proof. Put

$$W_b = \{ x \in W \mid M_f(x,r) \geq b \} \,.$$

By the weak-type $(1,1)$ inequality,

$$\mu(W_b) \leq 2^{3\kappa} \cdot b^{-1} \cdot \int_{T_{6r}(W)} |f| \, d\mu \,.$$

Fix a to be determined below. By writing

$$W = W_a \cup \left(\bigcup_{i=0}^{\infty} (W_{a \cdot 2^{i+1}} \setminus W_{a \cdot 2^i} \right)$$

and using the weak-type $(1,1)$ inequality, we get

$$\int_W (M_f(x,r)^{\alpha}) \, d\mu \leq a^{\alpha} \mu(W) + 2^{3\kappa} \cdot \sum_{i=0}^{\infty} (a \cdot 2^{(i+1)})^{\alpha} \cdot (a \cdot 2^i)^{-1} \cdot \int_{T_{6r}(W)} |f| \, d\mu \,.$$

Choosing

$$a = \frac{1}{\mu(W)} \cdot \int_{T_{6r}(W)} |f| \, d\mu$$

and summing the above geometric series gives (4.2) with $\Omega = \mu(W)$. Since $\alpha < 1$, this implies (4.2) for all $\Omega \geq \mu(W)$ as well. □

Bounding the local scale from below. For the remainder of this paper, we make the convention $\underline{p} \in M^n_{-1}$.

For $s \leq 1$, we have $\mathrm{Vol}(B_s(\underline{p})) \leq c(n) \cdot s^n$. Thus, from (1.19), (1.20), we get for any $0 < s \leq 1$ and $c = c(n)$,

$$\rho(p)^{-1} \leq c \cdot \max((M_{|R|^{\frac{n}{2}}}(p,s))^{\frac{1}{n}}, s^{-1}) \,,$$

which together with (1.21), gives (1.24), namely,

$$(r_{|R|}(p))^{-(n-1)} \leq c \cdot (s^{-(n-1)} + (M_{|R|^{\frac{n}{2}}}(p,s))^{\frac{n-1}{n}}) \,.$$

Chern-Gauss-Bonnet and the proof of relation (1.25). Let $s \leq r \leq 1$. By Theorem 3.13, we can approximate a compact subset, K, from the outside, by a submanifold with boundary, Z, with $K \subset Z \subset T_r(K)$, where the boundary term in the Chern-Gauss-Bonnet formula for Z satisfies the estimate (3.16). With (1.24), this gives for $c = c(n)$,

$$\left| \int_{\partial Z^n} TP_{\chi} \right| \leq c \cdot r^{-1} \cdot \int_{A_{\frac{1}{3}r, \frac{2}{3}r}(K)} \left(s^{-(n-1)} + \left(M_{|R|^{\frac{n}{2}}}(\,\cdot\,, s) \right)^{\frac{n-1}{n}} \right) \,. \tag{4.3}$$

By choosing $s = \frac{1}{512}r$ and employing (4.2) of Lemma 4.1, we get

$$\begin{aligned}(4.4)\qquad &\mathrm{Vol}(A_{0,r}(K))^{-1}\left|\int_{\partial Z^n} TP_\chi\right| \\ &\leq c\cdot\left(r^{-n}+r^{-1}\left(\frac{1}{\mathrm{Vol}(A_{0,r}(K))}\int_{A_{\frac{1}{4}r,\frac{3}{4}r}(K)}|R|^{\frac{n}{2}}\right)^{\frac{n-1}{n}}\right).\end{aligned}$$

From (4.4) and the Chern-Gauss-Bonnet formula, it follows that if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, then (1.25) holds.

Integrality of the geometric Euler characteristic. By an exhaustion argument as in [ChGv3], our generalized chopping theorem implies the following generalization of an application given in that paper.

Theorem 4.5. *Let M^n be a complete Einstein manifold with bounded Ricci curvature, finite volume and finite $L_{\frac{n}{2}}$-norm of curvature. Then*

$$(4.6)\qquad \int_{M^n} P_\chi \in \mathbf{Z}\,.$$

Proof. Let $K_0 \subset K_1 \subset \cdots$ denote an exhaustion of M^n. Apply (4.4) to each K_i, with $r = 1$ and note that the right-hand side goes to 0 as $i \to \infty$. By the Chern-Gauss-Bonnet formula, for i sufficiently large, we have

$$\int_{M^n} P_\chi = \chi(Z_i) \in \mathbf{Z}\,.$$

□

Remark 4.7. In [An3], it is asserted that (5.25) of that paper follows from the chopping theorem of [ChGv3]. However, since no proof is given that the curvature is uniformly bounded outside a compact subset, the chopping theorem of [ChGv3] cannot be applied. In actuality, (5.25) of [An3] is a special case of our Theorem 4.5 (valid in all dimensions) which does not require a global curvature bound. On the other hand, in the 4-dimensional case considered in [An3], the existence of a global curvature bound outside a compact subset does follow from our Theorem 9.1, a main result of the present paper; compare also Remark 10.10.

Remark 4.8. Suppose that the assumption that M^n is Einstein is weakened to

$$(4.9)\qquad |\mathrm{Ric}_{M^n}| \leq n-1\,.$$

Then the discussion can still be carried out. In particular, (4.4) and Theorem 4.5 continue to hold. The crucial point is the *local bounded covering geometry* in the sense of [ChFuGv]. Specifically, each point p has a neighborhood, U_p, containing a ball, $B_{r(n)}(p)$, of a definite size, such that the universal covering space, $\tilde{U}$, has $C^{1,\alpha}$-bounded geometry and $L_{2,p}$-bounded geometry, for all $p < \infty$; compare Remark 2.7. This follows from the existence of metric g_1, which is at bounded distance from g in the C^0-norm (i.e., bi-Lipschitz g with controlled bi-Lipschitz constant) such that g_1 has a definite bound on its curvature. Such a metric, g_1, is constructed in [Ya] via local Ricci flow; see also [Ab].

5. The key estimate in dimension 4

In this section we prove Theorem 1.26, the main new estimate on which our results in dimension 4 are based.

Lemma on sequences.

Lemma 5.1. *Let $0 \le \alpha < 1$. For $i = 0, 1, \ldots$, let a_i, b_i, x_i be nonnegative real numbers satisfying*

$$x_i \le a_i + b_i \cdot x_{i+1}^{\alpha}\,, \tag{5.2}$$

$$\liminf_{i\to\infty} x_i^{\alpha^i} = 1\,. \tag{5.3}$$

Then

$$x_0 \le \max(2a_0, C_1, C_2)\,, \tag{5.4}$$

where

$$C_1 = \limsup_{i\ge 1} \left(\prod_{j=1}^{i-1} (2b_j)^{\alpha^j}\right) \cdot (2a_i)^{\alpha^i}\,, \tag{5.5}$$

$$C_2 = \limsup_{i\ge 0} \prod_{j=0}^{i} (2b_j)^{\alpha^j}. \tag{5.6}$$

Proof. By (5.2), we have

$$x_i \le \max(2a_i, 2b_i \cdot x_{i+1}^{\alpha})\,.$$

Thus, for $i \ge 1$,

$$x_i^{\alpha^i} \le \max\left((2a_i)^{\alpha^i}, (2b_i)^{\alpha^i} \cdot x_{i+1}^{\alpha^{i+1}}\right)\,.$$

By induction, we get for all i,

$$x_0 \le \max(2a_0, C_{1,i}, C_{2,i})\,,$$

where

$$C_{1,i} = \left(\prod_{j=0}^{i-1} (2b_j)^{\alpha^j}\right) \cdot (2a_i)^{\alpha^i}\,,$$

$$C_{2,i} = \left(\prod_{j=0}^{i} (2b_j)^{\alpha^j}\right) \cdot x_{i+1}^{\alpha^{i+1}}\,.$$

In view of (5.3), this suffices to complete the proof. $\square$

If in particular, the sequence, $\{x_i\}$, is bounded, say $x_i \le C$, then the hypothesis, (5.3), is satisfied, and the lemma provides a bound on the initial term, x_0, which is *independent of* C. This is the situation in the application below where the following special case of Lemma 5.1 will suffice.

If for some constant $K \ge 1$,

$$\max(a_i, b_i, x_i) \le c \cdot K^i\,, \tag{5.7}$$

then we have the (nonsharp) bound

$$x_0 \le (2c)^{2(1+\alpha+\alpha^2+\cdots)} \cdot (2K)^{2(1+1\cdot\alpha+2\cdot\alpha^2+\cdots)} < \infty\,. \tag{5.8}$$

Proof of Theorem 1.26; an iteration argument.

Proof. By assumption, $T_1(E)$ is t-collapsed. Thus, we have (1.25) (proved in Section 4). We will apply (1.25) in each step of an iteration argument.

Since $n = 4$ we can use (1.1) to replace P_χ by $\frac{1}{8\pi^2}|R|^2$ on the left-hand side of (1.25), to get for some constant, c (independent of M^4)

$$\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(A_{0,1}(K))} \fint_E |R|^2 \le c \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))}\int_{A_{\frac{1}{4},\frac{3}{4}}(E)} |R|^2\right)^{\frac{3}{4}}\right). \tag{5.9}$$

For $i = 2, 3, \ldots$, put

$$D_i = \{p \in A_{2^{-i},1-2^{-i}}(E) \,|\, r_{|R|}(p) \le 2^{-(i+1)}\},$$
$$F_i = A_{2^{-i},1-2^{-i}}(E) \setminus D_i .$$

We have

$$T_{2^{-(i+1)}}(D_i) \subset A_{2^{-(i+1)},1-2^{-(i+1)}}(E) .$$

Moreover, $\mathrm{Lip}\, r_{|R|} \le 1$ implies

$$\sup_{T_{2^{-(i+1)}}(D_i)} r_{|R|} \le 2^{-i} .$$

Since $A_{0,1}$ is t-collapsed with locally bounded curvature, it follows that $T_{2^{-(i+1)}}(D_i)$ is $(t, 2^{-(i+1)})$-collapsed with locally bounded curvature. Hence, (1.25) implies that (5.9) holds with E replaced by D_i. By splitting the integral on the left-hand side of (5.10) below into a sum of integrals over D_i and F_i, and applying (5.9) to the former, we get

$$\begin{aligned}&\int_{A_{2^{-i},1-2^{-i}}(E)} |R|^2 \\ &\le c \cdot 2^{4i} \cdot \mathrm{Vol}(A_{0,1}(E)) \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))}\int_{A_{2^{-(i+1)},1-2^{-(i+1)}}(E)} |R|^2\right)^{\frac{3}{4}}\right).\end{aligned} \tag{5.10}$$

From (5.9), (5.10) and Lemma 5.1, we obtain (1.28), which concludes the proof of Theorem 1.26. □

Remark 5.11. By using the observation in Remark 1.4, together with Remark 4.8, it follows easily that Theorem 1.26 can be extended to the case in which the Einstein condition is dropped and (0.2) with the assumption $|\lambda| \le 3$ is replaced by (4.9) with the assumption $|\mathrm{Ric}_{M^4}| \le 3$. Indeed, the effect of this change is just to add a definite constant to the right-hand sides of (5.9), (5.10).

Remark 5.12. We are grateful to Fang-Hua Lin for pointing out to us the formal similarity between our iteration argument and the one used in proving Theorem 2.1 in [LiSch], which provides a mean value inequality for subharmonic functions.

6. Collapse implies L_2 concentration of curvature

In this Section we prove Theorem 0.1, one of our main results on collapsing.

Proof of Theorem 0.1. Let C be as in (0.3). From the discussion of Sections 2–4, and a standard covering argument, we get under the assumptions of Theorem 0.1, that there exists $v > 0$, $\beta > 0$, $p_1, \ldots, p_N$, such that $N \leq \beta C$, and such that if the metric is rescaled, $g \to s^{-2}g$, then the hypothesis of Theorem 1.26 holds with $E = M^4 \setminus \bigcup_i B_s(p_i)$.

Theorem 0.1 follows immediately from Theorem 1.26 (the key estimate); see in particular (1.28). □

Remark 6.1. There are many examples of collapsing sequences of Ricci flat Kähler metrics on $K3$ surfaces with fixed lower diameter bound; see e.g. [GsWi]. The behavior of any such collapsing sequence is governed by Theorem 0.1 and Theorem 9.1 (which relies on Theorem 0.8).

7. Negative Einstein constant; noncollapse and exponential decay

We begin this section with the proof of Theorem 0.14 and some related remarks. Next we recall some standard facts which, in combination with Theorem 0.1, yield Theorem 7.8, our noncollapsing result in the Kähler case.

We also prove a result on exponential decay of the volume and of the L_2-norm of the curvature, for Einstein 4-manifolds with $\lambda = -3$.

Proof of Theorem 0.14; noncollapse.

Proof. It suffices to assume that the hypothesis of Theorem 0.1, say for the case $s = 1$, is satisfied. By (1.1), if $\lambda = \pm 3$, then the left-hand side of (0.6) is bounded below by

$$\frac{1}{6}\left(\mathrm{Vol}(M^4) - \sum_i \mathrm{Vol}(B_1(p_i))\right).$$

Now the claim follows from (0.6). □

Remark 7.1. Recall that Theorem 0.14 is a partial replacement in dimension 4 for the Heintze-Margulis theorem, whose proof, while employing the relation between nilpotency and collapse, rests otherwise on considerations which are entirely different from ours.

Remark 7.2. In any dimension, if the Ricci curvature has a positive lower bound, then (absent any further assumptions) a bound on the diameter is provided by Myers' theorem. If $\mathrm{Ric}_{M^n} \geq n-1$, then a relative volume comparison, [GvLP], gives an inequality in which the constant does not depend on C, namely,

$$\mathrm{Vol}(B_1(p)) \geq \frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_\pi(\underline{p}))} \cdot \mathrm{Vol}(M^n) \qquad (\text{for all } p \in M^n).$$

Remark 7.3. We do not know whether in the absence of an a priori lower bound on $\mathrm{Vol}(M^4)$, collapse with Einstein constant ± 3 can occur.

Noncollapse in the Kähler case. In the Kähler case, the volume can be expressed in terms of the Kähler class. Namely,

$$\mathrm{Vol}(M^n) = \frac{1}{(n/2)!}[\omega]^{\frac{n}{2}}(M^n)\,. \tag{7.4}$$

However, for a sequence of Kähler metrics for which the Kähler class degenerates, the volume can go to zero.

If, in addition, M^n is Einstein and the Einstein constant does not vanish, we make the normalization,

$$\mathrm{Ric}_{M^n} = \pm(n-1)g\,. \tag{7.5}$$

Then the first Chern form, $c_1(R)$, satisfies

$$\begin{aligned} c_1(R) &= \frac{1}{2\pi}\mathrm{Ric}_{M^n}(J,\,\cdot) \\ &= \pm\frac{(n-1)}{2\pi}\omega \end{aligned} \tag{7.6}$$

and

$$\mathrm{Vol}(M^n) = \left(\frac{2\pi}{n-1}\right)^{\frac{n}{2}} |c_1^{\frac{n}{2}}(M^n)|\,. \tag{7.7}$$

Thus we obtain:

Theorem 7.8 (Noncollapsing for $c_1 \neq 0$)**.** *Let M^4 denote a Kähler-Einstein manifold with $c_1 \neq 0$, satisfying* (0.15)*. Then for w as in* (0.16)*, M^4 is not ϑ-collapsed, where*

$$\vartheta = \frac{w}{4}\cdot\frac{c_1^2(M^4)}{\chi(M^4)}\,. \tag{7.9}$$

Proof. Theorem 7.8 follows immediately from Theorem 0.14, together with (7.7). □

For more general lower volume bounds which follow from Seiberg-Witten theory, see [LeBru1], [LeBru2], [Tau], [Wi].

Exponential decay of volume. If $\lambda = -3$ and (0.3) holds, then (1.1) implies $\mathrm{Vol}(M^4) < \infty$.

Theorem 7.10 (Exponential decay of volume)**.** *There exist β, $\gamma > 0$, c, such that if M^4 denotes a complete Einstein* 4*-manifold satisfying* (0.3)*,* (0.15)*, then there exist $p_1, \ldots, p_N$, with*

$$N \leq \beta \cdot C\,, \tag{7.11}$$

such that for $r \geq 5$,

$$\mathrm{Vol}(M^4 \setminus \bigcup_i B_r(p_i)) \leq c\cdot C\cdot e^{-\gamma r}\,. \tag{7.12}$$

Proof. Consider the collection of balls, $B_1(p)$, such that

$$\int_{B_1(p)} |R|^2 \geq \frac{1}{6}\cdot\theta\cdot t\cdot\tau\,,$$

where the notation is as in (1.18). By a standard covering argument there exists a disjoint subcollection, $\{B_1(p_i)\}$, such that the original collection is contained in

$\bigcup_i \overline{B_5(p_i)}$. Since the collection, $\{B_1(p_i)\}$, is disjoint and (0.3) holds, there are at most N such balls, where N satisfies (7.11) for suitable β.

Because we assume (0.15), i.e., $\lambda = \pm 3$, it follows from (1.18) that the set, $M^4 \setminus \bigcup_i \overline{B_5(p_i)}$, is t-collapsed with locally bounded curvature.

Let c be as in (1.28) of Theorem 1.26. Let t denote the smallest integer such that $t > 2c$. For $i = 1, \ldots, t$, apply Theorem 1.26 to each of the sets, $M^4 \setminus \bigcup_i \overline{B_{r+i}(p_i)}$, add the inequalities corresponding to (1.28), and use

$$\fint_{M^4 \setminus \bigcup_i \overline{B_{r+i}(p_i)}} |R|^2 \geq 24\,.$$

The theorem follows. $\square$

Remark 7.13. As mentioned to us by M. Gromov, by the theorem of J. Lohkamp asserting the C^0-density of metrics with negative Ricci curvature in the space of all Riemannian metrics, a negative upper bound on the Ricci curvature is not in general sufficient to guarantee exponential decay of the volume.

Remark 7.14. In view of Theorem 1.26, it follows from Theorem 7.10 that the square of the L_2-norm of the curvature decays exponentially as well.

Sufficiently pinched Ricci curvature.

Remark 7.15. Since the results of this section are essentially formal consequences of Theorem 1.26, they extend to the case in which the Einstein condition is dropped and the assumption, $|\lambda| = 3$, is replaced by the assumption that the Ricci tensor is sufficiently pinched: $0 < a \leq |\mathrm{Ric}_{M^4}| \leq 3$. The particular constants in the conclusions depend on the pinching.

8. ϵ-REGULARITY

In this section, we prove Theorem 0.8, the improved ϵ-regularity theorem.

Proof of Theorem 0.8. In order to apply the results of [ChCoTi2] directly in proving Proposition 8.2 below, it will be convenient to make the following reduction.

Let $0 < \eta < \frac{1}{2}$. If $q \in B_{\frac{1}{2}} r(p)$, then $B_{\eta r}(q) \subset B_r(p)$. Clearly, it suffices to prove the theorem for all such $B_{\eta r}(q)$. As a consequence, we may assume that $r \leq \eta$, for some fixed $\eta < \frac{1}{2}$ (and continue to consider $B_r(p)$). An appropriate value of η will be determined in Proposition 8.2.

The argument has two main steps.

Step 1. The first step is the reduction in (8.1) below, which can be viewed as the analog in our context of a critical initial step in Gromov's proof of his celebrated theorem on almost flat manifolds — a short loop with holonomy which is not too big actually has holonomy comparable to its length.

Under the assumptions of Theorem 0.8, either the hypothesis of Theorem 4.4 of [An3] holds or we can assume that after rescaling, $g \to (2r)^{-2} g$, the assumptions of Theorem 1.26 are satisfied, with $E = B_1(p)$, $T_1(E) = B_2(p)$. In the latter case, by Theorem 1.26, there exists a definite constant, c, as in (1.28), such that

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2 \leq c\,. \tag{8.1}$$

Step 2. We will show that if one passes to a smaller concentric ball, whose radius can be estimated from below, then the hypothesis, (1.12), of Theorem 4.4 of [An3] is satisfied. An application of that theorem will complete the proof of Theorem 0.8. (Step 2 does not rely on (4.4); compare however (8.9).

Our claim is a direct consequence of the following proposition, in which a crucial point is the absence from the hypothesis of a lower volume bound. The proposition asserts that if on some interval, certain conditions are verified, then as r decreases, the quantity appearing on the left-hand side of (1.12) (and (8.5) below) decays at a definite rate. (It is this quantity which the hypothesis of Theorem 4.4 of [An3] requires to be $\leq \tau$.)

Proposition 8.2. *For all $C_1 > 0$, there exists $\eta = \eta(C_1) > 0$, such that if*

$$0 < r \leq \eta\,, \tag{8.3}$$

$$\int_{B_r(p)} |R|^2 \leq 4\pi^2\,, \tag{8.4}$$

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\int_{B_r(p)} |R|^2 \leq C_1\,, \tag{8.5}$$

$$\frac{\mathrm{Vol}(B_{\eta r}(p))}{\mathrm{Vol}(B_{\eta r}(\underline{p}))} \leq \frac{1}{4}\,, \tag{8.6}$$

(8.7)

then

$$\frac{\mathrm{Vol}(B_{\eta r}(\underline{p}))}{\mathrm{Vol}(B_{\eta r}(p))}\int_{B_{\epsilon r}(p)} |R|^2 \leq (1-\eta)\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\int_{B_r(p)} |R|^2\,. \tag{8.8}$$

Proof. By scaling, we can suppose $r = 1$, $\mathrm{Ric}_{M^4} \geq -3\eta^2$, $\underline{p} \in \underline{M}^4_{-\eta^2}$.

Under the assumption that for some sufficiently small $\epsilon > 0$, (8.4)–(8.6) hold but (8.8) fails, we will construct a closed submanifold with boundary, $U \subset B_1(p)$, for which the boundary term, $TP_\chi(\partial U)$, in the Chern-Gauss-Bonnet formula satisfies

$$0 < \int_{\partial U} TP_\chi < \frac{1}{2}\,. \tag{8.9}$$

Since, by (1.1), (8.4), we also have

$$0 < \int_U P_\chi < \frac{1}{2}\,, \tag{8.10}$$

the Chern-Gauss-Bonnet formula gives $0 < \chi(U) < 1$, a contradiction.

Assume that for some $\eta > 0$, (8.4)–(8.6) hold but (8.8) fails. Since each of the factors in the quantity on the left-hand side of (8.5) is a nondecreasing function of r, it follows that

$$\frac{\mathrm{Vol}(B_{\frac{1}{4}}(\underline{p}))}{\mathrm{Vol}(B_{\frac{1}{4}}(p))} \geq (1-\eta)\frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_1(p))}\,, \tag{8.11}$$

$$⨍_{B_1(p)\setminus B_{\frac{1}{4}}(p)} |R|^2 \leq c\cdot\eta\,, \tag{8.12}$$

for some absolute constant c (arising from a relative volume comparison in dimension 4).

We can assume $c \cdot \eta^{\frac{1}{2}} < c \cdot \tau$, where τ is the constant in (1.12) and c is chosen so small that the hypothesis of Theorem 4.4 of [An3] is valid for balls $B_r(q) \subset (B_1(p) \setminus B_{\frac{1}{4}}(p))$. Thus, for some absolute constant, c_1, we have the pointwise curvature bound

$$|R| \leq c_1 \cdot \eta^{\frac{1}{2}} \qquad (\text{on } B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{2}}(p)) \,. \tag{8.13}$$

By (8.11), the set, $A_{\frac{1}{4},1}(p)$, is an almost volume annulus. Hence we can make use of the constructions underlying the proof of the "almost volume annulus implies almost metric annulus" theorem of [ChCo0], as well as subsequent related constructions of [ChCoTi2]. (The reduction, $r \leq \eta$, and hence the assumption (8.3) enables us to quote directly from [ChCoTi2]. As in Sections 2, 3 of [ChCoTi2], this normalization leads to the function, $\mathbf{r}$, appearing in (8.14)–(8.17). By the same token, after our rescaling, $g \to r^{-2}g$, we have $\mathrm{Ric}_{M^4} \geq -3\eta^2$ and the above-mentioned metric annulus lies in a metric cone.)

Let $\Psi = \Psi(\eta) > 0$ denote some definite function (independent of M^4) such that $\Psi \to 0$ as $\eta \to 0$.

According to Section 4 of [ChCo0] and Sections 2, 3 of [ChCoTi2], there exists Ψ and a function, $\mathbf{r} : B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{2}}(p) \to [0, 1]$, such that

$$\Delta \mathbf{r}^2 = 8 \,, \tag{8.14}$$

$$|\mathbf{r} - r| < \Psi \,, \tag{8.15}$$

$$⨍_{\mathbf{r}^{-1}(a)} |\nabla \mathbf{r} - \nabla r|^2 < \Psi \,, \tag{8.16}$$

$$|\nabla \mathbf{r}| < c \,. \tag{8.17}$$

In addition, the following holds. For a a regular value of $\mathbf{r}$, denote by $g_{\mathbf{r}^{-1}(a)}$, the induced metric at points of $\mathbf{r}^{-1}(a)$, and by $II_{\mathbf{r}^{-1}(a)}$, the second fundamental form. Then for some subset, $A \subset [\frac{1}{2}, \frac{3}{4}]$, of regular values of $\mathbf{r}$, if $a \in A$, we have

$$\left| 1 - \frac{\mathrm{Vol}(\mathbf{r}^{-1}(a))}{\mathrm{Vol}(\partial B_a(p))} \right| \leq \Psi \,, \tag{8.18}$$

$$⨍_{\mathbf{r}^{-1}(a)} |II_{\mathbf{r}^{-1}(a)} - \frac{1}{\mathbf{r}} g_{\mathbf{r}^{-1}(a)} \otimes \nabla \mathbf{r}|^2 \leq \Psi \,. \tag{8.19}$$

It is important to note that the integral in (8.19) is normalized by volume. To see (8.18), (8.19) observe that the key assumption of [ChCo0], assumption (4.10) of that paper, concerns a *ratio* of volumes and that the quantities appearing in all subsequent estimates are normalized by volume, as are all estimates of Sections 2, 3 of [ChCoTi2] (which depend on the estimates of Section 4 of [ChCo0]). Note in particular that this holds for (4.84) of [ChCo0], which is the L_2 estimate on the Hessian of the function $\mathbf{r}^2$. Estimate (8.18) is just a restatement of (3.11), which is part of the conclusion of Theorem 3.7 of [ChCoTi2].

Since the boundary term, $TP_\chi(\mathbf{r}^{-1}(a))$, in the Chern-Gauss-Bonnet formula contains terms that are of degree 3 in the second fundamental form, $II_{\mathbf{r}^{-1}(a)}$, from (8.13), (8.18), (8.19), it does *not* follow immediately that

$$\left| \int_{\mathbf{r}^{-1}(a)} TP_\chi - \frac{\mathrm{Vol}(\mathbf{r}^{-1}(a))}{\mathrm{Vol}(\partial B_a(\underline{p}))} \right| \leq \Psi \cdot \mathrm{Vol}(\partial B_a(\underline{p})) \,. \tag{8.20}$$

However, since the curvature on the annulus, $B_{\frac{3}{4}}(p)\setminus\overline{B_{\frac{1}{2}}(p)}$, is uniformly bounded, this annulus has local bounded covering geometry in the sense of [ChFuGv]. Thus, there exists $\underline{s}>0$ (independent of M^4) such that for all $q\in B_{\frac{3}{4}}(p)\setminus\overline{B_{\frac{1}{2}}(p)}$, the universal covering space, $\widetilde{B_{\underline{s}}(q)}$, of $B_{\underline{s}}(q)$ has C^∞ bounded covering geometry with respect to the pull-back metric. In particular, the injectivity radius on $\widetilde{B_{\underline{s}}(q)}$ has a definite positive lower bound. By pulling back the function, $\mathbf{r}$, to $\widetilde{B_{\underline{s}}(q)}$, we reduce to the noncollapsed case.

Now, just as in Section 3 of [ChCoTi2], we can argue by contradiction. After passing to a suitable subsequence, a sequence of counterexamples (in manifolds M_i^4) would converge in the C^∞-topology to a portion of an annulus in a flat cone, and the corresponding functions, $\mathbf{r}_i$, would converge in the C^∞-topology to the distance function from the vertex of this flat limit cone. For all $a\in[\frac{1}{2},\frac{3}{4}]$, this implies convergence of second fundamental forms for sequences of level surfaces, $\mathbf{r}_i^{-1}(a)\to\mathbf{r}^{-1}(a)$, a contradiction.

Thus, taking $U=\mathbf{r}^{-1}((0,a])$, we get (8.20), and hence, (8.9). □

By combining (8.1) with Proposition 8.2, the proof of Theorem 0.8 can be reduced to an application of Theorem 4.4 of [An3]. □

Remark 8.21. Already in deriving (8.13), we made use of Theorem 4.4 of [An3] in a situation in which no a priori lower bound on volume is assumed; compare Remark 1.31. This was made possible by the initial reduction given in (8.1).

Bounded Ricci curvature.

Remark 8.22. Theorem 0.8 can be extended to the case in which the Einstein condition is dropped, the assumption $|\lambda|\le 3$ is replaced by $|\mathrm{Ric}_{M^4}|\le a$, provided in the conclusion, and the condition, $|R|\le c$, is replaced by $C^{1,\alpha}$ bounded covering geometry, $\alpha<1$, or $L_{2,p}$-bounded covering geometry, $p<\infty$. Additionally, one can deduce a definite bound on the L_p-norm of curvature for all $p<\infty$.

To see this, note that in view of Remark 5.11, the only point in the argument which requires modification is (8.10) in the proof of Proposition 8.2. Although P_χ need not be a positive multiple of $|R|^2\cdot\mathrm{Vol}(\,\cdot\,)$, this continues to hold up to an error that is bounded by a definite multiple of $|\mathrm{Ric}_{M^4}|^2$. The neighborhood, U, is close in the Gromov-Hausdorff sense to a ball with center the vertex of a flat cone (which might be very collapsed). Thus, the error term is bounded by a definite multiple of the volume of a small neighborhood of the center of U. Thus, the error term is not only small, but small with respect to the area of the boundary. Once again, we get $0<\chi(U)<1$, a contradiction.

9. Consequences of ϵ-regularity

From Theorem 0.8 and a standard covering argument, we get:

Theorem 9.1 (Bound on the number of blowup points)**.** *There exist $c>0$, $\beta>0$, such that if M^4 denotes a complete Einstein 4-manifold satisfying* (0.2), (0.9)*, then there exist $p_1,\dots,p_N$, with*

$$N\le\beta\cdot C\,, \tag{9.2}$$

such that for all q,

$$|R(q)| \leq c \cdot \sup_{p_\alpha} \max \left((\overline{q, p_\alpha})^{-2}, 1 \right) . \tag{9.3}$$

Moreover, if $\lambda = 0$, then

$$|R(q)| \leq c \cdot \sup_{p_\alpha} (\overline{q, p_\alpha})^{-2} . \tag{9.4}$$

Remark 9.5. For $n > 4$, analogs of the above results are conjectured; compare Section 11. But there remains the possibility that if M^n is sufficiently collapsed relative to the size of its diameter, the pointwise norm of the curvature might be arbitrarily large for all $p \in M^n$. Similarly, for $n > 4$, the Gromov-Hausdorff limit of a collapsing sequence with a uniform bound on diameter might conceivably have no points with locally Euclidean neighborhoods.

Remark 9.6. One may ask whether for complete noncompact Ricci flat manifolds satisfying (0.3), the curvature estimate, (9.4), can be improved, i.e., if decay is actually faster than quadratic. This is known to hold if, in addition, the volume growth is Euclidean; see e.g. [BaKaNa], [ChTi1].

Remark 9.7. By employing the theory of collapse with bounded curvature, it can be shown that for M^4 a complete noncompact Ricci flat 4-manifold satisfying (0.3), with sub-Euclidean volume growth, every tangent cone, with the base point deleted, can be written locally as the quotient of a noncollapsed Ricci flat 4-manifold by a group of isometries whose identity component is nilpotent.

Similarly, in the case of negative Einstein constant, $\lambda = -3$, it follows that if M^4 satisfies (0.3), (0.15), then for any sequence, $p_i \to \infty$, there is a subsequence, $\{p_{i_j}\}$, such that (M^4, p_{i_j}) converges in the Gromov-Hausdorff sense to space Y, which is locally the quotient of a smooth Einstein 4-manifold with bounded curvature by a group of isometries, whose identity component is nilpotent.

Note that in certain important special cases, Einstein metrics with Killing fields on 4-manifolds are known to be given by a local ansatz; see e.g. [GibHaw], [CaPe]. We intend to discuss these matters at greater length elsewhere, including issues which are more global in nature.

10. Moduli spaces

In this section, we discuss the implications of our main theorems for compactifications of the moduli spaces of Einstein metrics on a given compact 4-manifold M^4. Much of our discussion also applies to complete Einstein metrics with finite L_2-norm of curvature; compare Remarks 9.6, 9.7. Our approach uses Gromov-Hausdorff convergence of sequences, (M^4, g_k), or in case there is no a priori bound on the diameter, $\underline{N}$-pointed Gromov-Hausdorff convergence.

Remark 10.1. Our results have an obvious extension to Gromov-Hausdorff limits (respectively $\underline{N}$-pointed Gromov-Hausdorff limits) of sequences, (M_k^4, g_k), in which the underlying manifold is not fixed. All that is actually required is a bound, $\chi(M_k^4) \leq C$, and for noncollapsing theorems, a lower bound on volume: $\mathrm{Vol}(M_k^4) \geq v > 0$.

The completion of the moduli space of Einstein metrics on a *fixed* 4-manifold is studied in [An3] using the extrinsic L_2 metric on the moduli space. By way of comparison with the present approach based on $\underline{N}$-pointed Gromov-Haudorff convergence, we note that for (M^4, g_i) a sequence with $\mathrm{diam}(M^4, g_i) \to \infty$, the assumption that (M^4, g_i) converges with respect to the extrinsic L_2 metric provides a strong additional constraint on the sequence; for further discussion see Remark 10.10.

Among other important theorems, [An3] contains the first written results on the collapsing case, notably, collapse with locally bounded curvature and hence, by [ChGv2], the presence of an F-structure away from a definite number of points. One key consequence of our main results is that "locally bounded curvature" can be replaced by "bounded curvature".

In [An3], a qualitative analogy with the case of constant curvature metrics on Riemann surfaces is proposed. The analogy is, of course, not complete, since in dimension 4, the curvature can concentrate and the moduli space can have positive dimension for $\lambda > 0$.

Gromov-Hausdorff compactifications. Let M^4 denote a smooth compact connected 4-manifold. For fixed $\lambda > 0$, let $\mathcal{M}(M^4, \lambda)$ denote the moduli space of isometry classes of Einstein metrics on M^4, with Einstein constant λ. For $\lambda \leq 0$, there is no a priori bound on the diameter, so we consider the moduli space of $\underline{N}$-pointed isometry classes $\mathcal{M}(M, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, \lambda)$. Two $\underline{N}$-pointed Einstein manifolds, $(M, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, g)$, $(M, \underline{m}'_1, \ldots, \underline{m}'_{\underline{N}}, g')$, are $\underline{N}$-pointed isometric if there exists an isometry between (M, g), (M, g'), carrying $\underline{m}_j$ to $\underline{m}'_j$, for all j.

Below, for $\lambda \neq 0$, we make the normalization $|\lambda| = 3$, which can be achieved by scaling. Due to scale invariance of the condition, $\lambda = 0$, in that case, we impose the additional normalization $\mathrm{Vol}((M^4, g)) = 1$.

For $\lambda = 3$, the space $\mathcal{M}(M^4, \lambda)$ carries certain natural metrics. Here, we use the weakest of these, the Gromov-Hausdorff metric. Since 2) below does in fact hold (i.e., at regular points, weak convergence implies strong convergence) this yields the strongest possible results. With respect to the Gromov-Hausdorff metric, bounded subsets of $\mathcal{M}(M^4, \lambda)$ are typically incomplete. For $\lambda \leq 0$, there is no a priori bound on the diameter of an Einstein manifold, (M^4, g), and the diameter of $\mathcal{M}(M^4, \lambda)$ with respect to the Gromov-Hausdorff metric is typically infinite. In this case, we employ the topology of $\underline{N}$-pointed Gromov-Hausdorff convergence. We write $(M_k, g_k, \underline{m}_{k,1}, \ldots, \underline{m}_{k,\underline{N}}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$ whenever $(M_k, g_k, \underline{m}_{k,j}) \stackrel{d_{GH}}{\to} (Y_j, \underline{y}_j)$, for all $1 \leq j \leq \underline{N}$, i.e., whenever $(M_k, g_k, \underline{m}_{k,j})$ converges to $(Y_j, \underline{y}_j)$ in the pointed Gromov-Hausdorff sense. Since we allow $\underline{N} > 1$ and $\overline{m_{k,j_1}, m_{k,j_2}} \to \infty$, for all $j_1 \neq j_2$, an understanding of the possible limiting collections, $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$, provides a global picture of the degenerating sequence.

By using Gromov's compactness theorem, bounded subsets of $\mathcal{M}(M^4, \lambda)$ can be completed by adding suitable compact connected length spaces Y. The completion of such a bounded subset is compact.

For $\lambda \leq 0$, using the pointed version of Gromov's compactness theorem, the completion of $\mathcal{M}(M^4, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, \lambda)$ can be compactified by adding certain collections of noncompact connected pointed length spaces $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$. Namely, we add such a collection whenever there exists a sequence, $(M^4, g_k, \underline{m}_{k,j})$, such that $(M^4, g_k, \underline{m}_{k,1}, \ldots, \underline{m}_{k,\underline{N}}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$.

In the above formulation the two most basic issues are the following; see below for further amplification.

1) Describe explicitly the geometric structure of the collections of pointed spaces, $\{(Y_1,\underline{y}_1),\ldots,(Y_{\underline{N}},\underline{y}_{\underline{N}})\}$, which must be added in the compactification.

2) Show that near regular points, $y\in Y$, the convergence is actually in the strongest possible sense.

In the Kähler case, additional important issues involving the complex structure arise, but will not be discussed here.

In what follows, the notion of *concentration point* of the curvature plays an important role. Let $(M^4,g_k,\underline{m}_{k,1},\ldots,\underline{m}_{k,\underline{N}})\overset{d_{GH}}{\to}\{(Y_1,\underline{y}_1),\ldots,(Y_{\underline{N}},\underline{y}_{\underline{N}})\}$ as above. We say $y_j\in Y_j$ is *not* a concentration point of curvature, if for some subsequence, $(M^4,g_s,\underline{m}_{s,j})\overset{d_{GH}}{\to}(Y_j,\underline{y}_j)$, the point, y_j, is *not* the point limit of a sequence of points, $p_{s,\alpha}$, with $p_{s,\alpha}$ a blowup point of the curvature as in (9.3) of Theorem 9.1. Thus, by Therem 9.1, if y_j is not a concentration point, then near y_j, the space, Y_j, is the Gromov-Hausdorff limit of a sequence of spaces with *bounded curvature*.

Let $\{y_{j,\alpha_j}\}$ denote the set of concentration points in Y_j. Relation (9.2) of Theorem 9.1 implies that the cardinality, N, of the set, $\bigcup_j\{y_{j,\alpha_j}\}$, satisfies $N\le\beta\cdot 8\pi^2\cdot\chi(M^4)$, for some absolute constant, β, independent of M^4.

Noncollapsed limit spaces. Let Y^4 (respectively, $\{(Y_1,y_1),\ldots,(Y_{\underline{N}},\underline{y}_{\underline{N}})\}$ denote some noncollapsed Gromov-Hausdorff limit (respectively, $\underline{N}$-pointed Gromov-Hausdorff limit) of a sequence of compact Einstein metrics on a fixed compact manifold M^4. Then Y^4 (respectively, Y_j^4) is known to be a connected smooth Einstein manifold away from the set of concentration points of the curvature, at which the singularities are of orbifold type.

Away from the concentration points, for all k, the convergence takes place in the C^k topology on compact subsets. In particular, 2) above has a positive answer in these instances; for the above, see [An1], [An3], [Na], [Ti]; compare also [AnCh].

Next, we describe those cases in which noncollapsed limit spaces are known to arise.

For $\lambda=3$, by Myers' theorem, there is an a priori bound on the diameter and the completion of $\mathcal{M}(M^4,3)$ is itself compact. Thus, those Y which arise as limit points are compact as well. In the Kähler case, $\mathrm{Vol}((M^4,g))$ has an a priori lower bound, which implies that $Y=Y^4$ is noncollapsed. Seiberg-Witten theory provides lower volume bounds under more general assumptions, e.g. if M^4 admits a symplectic structure; see [LeBru1], [LeBru2], [Tau], [Wi]. In the general case, the question of whether Y can be collapsed remains open.

Let $\lambda=0$. Recall that in this case, we impose the additional constraint, $\mathrm{Vol}((M^4,g))=1$. Since complete noncompact manifolds with nonnegative Ricci curvature have infinite volume, it follows from relative volume comparison that if $(Y,\underline{y})$ is a pointed limit space, then Y is noncompact if and only if it is everywhere collapsed. Moreover, given a sequence of 2-pointed Einstein metrics, $(M^4,g_{k,j},\underline{m}_{k,j})\overset{d_{GH}}{\to}\{(Y_1,\underline{y}_1),(Y_2,\underline{y}_2)\}$, where possibly, $\overline{m_{i,1},m_{i,2}}\to\infty$, it follows that Y_1 is collapsed if and only if it is noncompact, and this holds if and only if Y_2 is collapsed and noncompact. If $Y_1=Y_2=Y^4$ is compact, and hence noncollapsed, then the above discussion applies. The noncompact collapsed case will be discussed below.

As a particular example, recall that in the case of $K3$ surfaces, both collapsing and noncollapsing behavior can occur.

In case $\lambda = -3$, we have the following basic results.

Theorem 10.2. *Let $\lambda = -3$ and let $\{(M^4, g_i)\}$ satisfy*

$$\operatorname{diam}(M^4, g_i) \to \infty\,. \tag{10.3}$$

If for some $v > 0$,

$$\operatorname{Vol}((M^4, g_i)) \geq v\,, \tag{10.4}$$

then there exists a pointed subsequence, $(M^4, g_k, \underline{m}_k) \overset{d_{GH}}{\to} (Y^4, \underline{y})$, where Y^4 is a noncollapsed complete noncompact orbifold. Outside of a compact subset, the curvature of Y^4 is bounded and (9.4) *of Theorem* 9.1 *holds. If the (M^4, g_i) are Kähler, then* (10.4) *is satisfied.*

Proof. This follows immediately from Theorems 0.14 and 9.1. □

Theorem 10.5. *Let $\lambda = -3$. There exist $\underline{\beta}, c > 0$ with the following properties. Let $(M^4, g_k, \underline{m}_{k,1}, \dots, \underline{m}_{k,\underline{N}}) \overset{d_{GH}}{\to} \{(Y_1^4, \underline{y}_1), \dots, (Y_{\underline{N}}^4, \underline{y}_{\underline{N}})\}$, where $\overline{m_{i,j_1}, m_{i,j_2}} \to \infty$, for all $j_1 \neq j_2$, and Y_j^4 is noncollapsed for all j. Then*

$$\underline{N} \leq \underline{\beta} \cdot \chi(M^4)\,. \tag{10.6}$$

If Y_j^4 is smooth for some j, then

$$\int_{Y_j^4} |R|^2 \geq c\,. \tag{10.7}$$

If $\underline{N}$ is chosen as large as possible such that the above hypotheses are satisfied, then

$$\lim_{k\to\infty} \operatorname{Vol}(M^4, g_k) = \operatorname{Vol}(Y_1^4) + \cdots + \operatorname{Vol}(Y_{\underline{N}}^4)\,. \tag{10.8}$$

Proof. As above, each Y_j^4 is a complete Einstein orbifold with finite volume and the convergence is smooth away from at most a definite number of points at which the curvature concentrates. It follows that

$$\lim_{k\to\infty} \operatorname{Vol}(M^4, g_k) \geq \operatorname{Vol}(Y_1^4) + \cdots + \operatorname{Vol}(Y_{\underline{N}}^4)\,.$$

Relation (10.7) follows from Theorem 4.5.

These statements imply (10.6). In fact, if we assume

$$\underline{N} \geq \frac{16\pi^2}{c} \cdot \chi(M^4) + \beta \cdot 8\pi^2 \cdot \chi(M^4) + 1\,,$$

where $8\pi^2 \cdot \beta \cdot \chi(M^4)$ bounds the number of concentration points, then at least $\frac{16\pi^2}{c} \cdot \chi(M^4)$ of the spaces, Y_j^4, are smooth. Since for such Y_j^4, the convergence $(M^4, g_k, \underline{m}_j) \to (Y_j^4, \underline{y}_j)$ is also smooth on compact subsets, this gives for k sufficiently large,

$$\int_{(M^4, g_k)} |R|^2 > 8\pi^2 \cdot \chi(M^4)\,,$$

which contradicts (1.3).

Now suppose that $\underline{N}$ is chosen as large as possible subject to the conditions $\overline{m_{i,j_1}, m_{i,j_2}} \to \infty$, for all $j_1 \neq j_2$, and Y_j^4 is noncollapsed for all j.

We claim that if $r >> 1$ and k is sufficiently large, then in the L_2-sense, almost all of the curvature on $M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})$ is concentrated on a finite subset of blowup points of cardinality $\leq \beta \cdot \chi(M^4)$. If we grant this momentarily, and note that since $\lambda = -3$, we have the pointwise relation, $|R|^2 > 1$, it follows that

$$\lim_{k\to\infty, r\to\infty} \mathrm{Vol}(M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})) = 0\,,$$

from which (10.8) follows.

In actuality, the claim is a slight generalization of Theorem 0.1. The maximality of $\underline{N}$ implies that $M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})$ collapses as $k \to \infty, r \to \infty$. Also, each noncollapsed Y_j^4 has finite volume and bounded curvature outside a compact subset. Thus, the claim follows by applying the chopping theorem of [ChGv3] near $\bigcup_j \partial B_r(\underline{m}_{k,j})$ and then repeating the proof of Theorem 0.1. $\square$

Remark 10.9. As mentioned at the end of Section 0, the present paper arose from an initial attempt in 1987 to prove Theorem 10.2. What we lacked at that time was the chopping theorem for local bounded curvature, Theorem 3.13 (or alternatively, the transgression form, $\mathcal{T}P_\chi$) and the specific estimate (1.11).

Remark 10.10. In Theorem III of [An3], under the strong additional assumption that the sequence, (M^4, g_k), converges with respect to the extrinsic L_2 metric, it is shown that the limit consists of a finite number of complete orbifolds of finite volume. Moreover, it is asserted that the curvature of each of these is bounded outside of a compact subset. The argument indicated in [An3] for this assertion is not valid. To establish this fact (even assuming convergence with respect to the L_2 metric) Theorem 9.1, one of the main results of the present paper, is required.

Again with the strong additional hypothesis of extrinsic L_2 convergence, Theorem III of [An3] contains statements corresponding to (10.6), (10.7), (10.8). The argument given in [An3] for (10.6) is not correct as stated, since it rests on the presumed applicability of the good chopping theorem of [ChGv3], whereas the curvature of Y_j^4 is not shown to be bounded outsided a compact subset; compare also Remark 4.7. However, Anderson has pointed out that assuming extrinsic L_2 convergence and granted Theorem 5.3 and Remark 5.4 of [An3], a proof can be given.

The question of whether for $\lambda < 0$, *every* sequence, (M^4, g_i), has a subsequence which converges in the extrinsic L_2 metric is raised in ii) of p. 33 of [An3]. It remains open.

Collapsed limit spaces. Let Y (respectively, $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$) denote a collapsed Gromov-Hausdorff limit space (respectively, an $\underline{N}$-pointed Gromov-Hausdorff limit space). As noted above, the cardinality of the set, $\bigcup_j \{y_{j,\alpha_j}\}$, of concentration points, is bounded by $\beta \cdot \chi(M^4)$.

By Theorem 5.1 of [An3], away from the set, $\bigcup_j \{y_{j,\alpha_j}\}$, the approximating spaces, (M^4, g_k), are collapsed with locally bounded curvature and hence, by [ChGv2], admit an F-structure of positive rank. Equivalently, $Y_j \setminus (\bigcup_{\alpha_j} y_{j,\alpha_j})$ is a limit space with locally bounded curvature.

It follows directly from Theorem 9.1 that the following strengthening holds.

Theorem 10.11. *A compact subspace of $Y_j \setminus \bigcup_j \{y_{j,\alpha_j}\}$ is a collapsed limit space with bounded curvature.*

Apart from the qualitative statement in Theorem 10.11, we have quantitative estimates corresponding to (9.3), (9.4), of Theorem 9.1. The theory of collapse with bounded curvature, including the existence of standard N-structures, describes the geometry of Y_j as well as the convergence on a fixed scale. In particular, for k sufficiently large, regions near regular points, $y_j \in Y_j$, regions of (M^4, g_k) fibre over regions of Y_j with nilpotent fibres. These fibrations are almost Riemannian submersions. The existence of these fibrations can be viewed as an appropriate version of 2) above (weak convergence implies stronger convergence) in the collapsing case. For further information on collapse with bounded curvature, see [ChFuGv].

For additional information concerning the collapsing structure at infinity of noncompact limits, see Remark 9.7.

As mentioned above, for $\lambda = 3$, absent a lower volume bound, it is not known whether collapsed limit spaces can occur. For $\lambda \leq 0$, collapsed $\underline{N}$-pointed limit spaces occur whenever $\mathrm{diam}((M^4, g_i)) \to \infty$ (which can happen).

11. Further directions

In this section, we speculate on some possible extensions of our main results.

Anti-self-dual metrics. A metric g on an oriented Riemannian 4-manifold is called anti-self-dual if its self-dual Weyl tensor $W_+(g)$ vanishes. Partial progress on the study of such metrics has been made in [An4], [TiVia1], [TiVia2].

We conjecture that the curvature estimate in Theorem 0.8 still holds for anti-self-dual metrics with constant scalar curvature (and Kähler metrics with constant scalar curvature). Moreover, we believe that there should be versions of Theorems 0.1 and 0.14 for anti-self-dual metrics with constant scalar curvature. One problem here is to establish a local volume estimate in terms of local Sobolev constant; for the global version, see [TiVia1], [TiVia2].

Kähler metrics of constant scalar curvature. We conjecture that the curvature estimate in Theorem 0.8 still holds for Kähler metrics with constant scalar curvature; compare [TiVia1], [TiVia2].

Ricci flow. The normalized Ricci flow on $[0, T) \times M$ is a solution to the equation,

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) - \frac{r}{n} g)\,, \tag{11.1}$$

where $g(t)$ is a family of metrics on M, $n = \dim M$ and r is the average of the scalar curvatures of $g(t)$. Assume that $n = 4$ and $g(t)$ is an entire solution of (11.1), i.e., $T = \infty$.

In view of recent work for the Yang-Mills flow by Hong and the second author, [HoTi], we conjecture that the curvature and injectivity radius estimates in Theorems 0.1, 0.8 and 0.14 still hold for $g(t)$ as t tends to infinity. Of particular interest is the case of shrinking Ricci solitons.

Higher dimensions. Finally, we consider the higher-dimensional case. We conjecture that, for n arbitrary, if the L_2 bound on curvature is replaced by an $L_{\frac{n}{2}}$ bound, then Theorems 0.1, 0.8 continue to hold. Additionally, we conjecture that for n arbitrary, given an L_2 bound on curvature, Theorem 0.14 holds and the conclusions of Theorems 0.1 and 0.8 are valid off suitable subsets of finite $(n-4)$-dimensional Hausdorff measure. (Analogous statements can be conjectured in higher dimensions for entire solutions of the normalized Ricci flow.) Note that in the particular case of special holonomy, the anti-self duality of the curvature tensor implies that $|R|^2 \cdot \mathrm{Vol}(\,\cdot\,)$ is a definite multiple of a certain characteristic form $C(M^n)$; see e.g. [ChTi2]. Although this relation is analogous to (1.1), in fact $C(M^n) \neq P_\chi$. This circumstance makes it unclear how to extend our present approach to the higher-dimensional case of special holonomy.

12. Appendix; The transgression form $\mathcal{T}P_\chi$

Proof based on $\mathcal{T}P_\chi$. The notation in this appendix is as in Section 1.

On a $(\theta \cdot t)$-collapsed manifold with locally bounded curvature, there exists an *essentially canonical* form, $\mathcal{T}P_\chi$, satisfying

$$d\,\mathcal{T}P_\chi = P_\chi, \tag{12.1}$$

$$|\mathcal{T}P_\chi(p)| \leq c(n) \cdot (r_{|R|}(p))^{-(n-1)}\,. \tag{12.2}$$

The bound, (12.2), on $|\mathcal{T}P_\chi|$ can be transformed into one in which the scale, $r_{|R|}$, is absent by means of (1.24).

A proof of the key estimate, Theorem 1.26, based on the form, $\mathcal{T}P_\chi$, proceeds along the same lines as the one based on equivariant good chopping, modulo the following proviso. In deriving the counterpart of (1.25), an argument employing a cutoff function and Stokes' theorem replaces our previous argument.

Construction of $\mathcal{T}P_\chi$. The detailed construction of the form, $\mathcal{T}P_\chi$, is more technical than the proofs of the chopping theorems, Theorems 3.1, 3.13. In addition to properties ii)-iv) of standard N-structures, it relies on property i) as well; compare Remark 2.1. The construction goes roughly as follows.

Consider first the case of a sufficiently collapsed manifold, M^n, with bounded curvature. Suppose also, that for some standard N-structure, and for the saturation of some ball, $B_{r(n)}(p)$, that all orbits, $\mathcal{O}$, have fixed dimension, k, and in addition, the second fundamental forms of these orbits are bounded, $|II_{\mathcal{O}}| \leq c(n)$. Consider the Whitney sum of the connections obtained by orthogonally projecting the Riemannian connection of an invariant metric, $\overline{g}$, onto the sub-bundles which are tangent to and orthogonal to the orbits. Using the fact that the orbits are nilmanifolds (whose Euler characteristic vanishes) and the fact that the connection on the complementary bundle is locally the pull-back of a connection on an $(n-k)$-dimensional quotient, it follows that the Chern-Gauss-Bonnet form of this connection vanishes identically.

More generally, if the N-structure has an atlas consisting of charts of the above type, then the corresponding connections can be glued together to produce a connection, $\widehat{\nabla}$, for which the Chern-Gauss-Bonnet form vanishes identically, $P_\chi^{\widehat{\nabla}} \equiv 0$.

In the general bounded curvature case, the orbits in a given chart have positive dimension, which need not be constant. However, by using the bound on the second fundamental form in property i) of standard N-structures, a connection, $\widehat{\nabla}$, with vanishing Chern-Gauss-Bonnet form can still be constructed. Recall that property i) states that any point lies in a tubular neighborhood, $T_{3r(n)}(\mathcal{O}_q)$, where the second fundamental form, $II_{\mathcal{O}_q}$, satisfies $|II_{\mathcal{O}_q}| \leq c(n)$; see Section 2.

The invariant metric, $\widetilde{g}$, lies at a definite distance from the given metric, g, and the connection, $\widehat{\nabla}$, is constructed from that of $\widetilde{g}$ by an essentially canonical local procedure. Thus, Chern-Weil theory give rise to a standard form, $\mathcal{T}P_\chi$, with

$$d\mathcal{T}P_\chi = P_\chi - P_\chi^{\widehat{\nabla}} = P_\chi\,.$$

Moreover, the above-mentioned bounds imply $|\mathcal{T}P_\chi| \leq c(n)$.

Since, in the case of collapse with bounded curvature, the connection, $\widehat{\nabla}$, is constructed by an essentially canonical local procedure, the construction has a direct extension to the case of sufficient collapse with locally bounded curvature. In this case, by scaling, we get the bound (12.2), for the norm of the $(n-1)$-form $\mathcal{T}P_\chi$.

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