QUASISYMMETRIC GROUPS

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1. Introduction

1.1. The main results. Let T denote the unit circle and D the unit disc. Suppose that \( f : T \to T \) is a homeomorphism. Let \( \hat{f} : D \to D \) be a homeomorphism too. We say that \( \hat{f} \) extends \( f \) if \( \hat{f} \) and \( f \) agree on \( T \). All mappings in this paper are sense preserving (see the remark at the end of introduction).

Definition 1.1. We say that a homeomorphism \( f : T \to T \) is \( K \)-quasisymmetric if there exists a \( K \)-quasiconformal map \( \hat{f} : D \to D \) that extends \( f \).

This is one of a number of equivalent ways to define quasisymmetric maps of the unit circle \( T \). Let \( S \) be a Riemann surface and let \( f \) be an element of the mapping class group of the surface \( S \). We say that \( f \) is \( K \)-quasisymmetric if there exists a \( K \)-quasiconformal map \( \hat{f} : S \to S \) that represents \( f \). If \( S = D \), then this agrees with the above definition.

Definition 1.2. Let \( G \) be a subgroup of the group of homeomorphisms of \( T \). We say that \( G \) is a \( K \)-quasisymmetric group if every element of \( G \) is \( K \)-quasisymmetric.

We will also say that a subgroup \( G \) of the mapping class group of a Riemann surface \( S \) is \( K \)-quasisymmetric if every element from \( G \) can be represented by a \( K \)-quasiconformal map of \( S \) onto itself.

Remark. Unless specified differently a quasiconformal map (or a quasiisometry; see Section 2) is assumed to be a selfmap of \( D \).

By \( \mathcal{M} \) we denote the Lie group of Möbius transformations that preserve the unit disc \( D \) (therefore, those transformations preserve \( T \) as well). If \( u \in \mathcal{M} \), we consider \( u \) as a homeomorphism of \( T \). The corresponding group that acts on \( D \) is denoted by \( \hat{\mathcal{M}} \), and the corresponding element is denoted by \( \hat{u} \). A subgroup \( \mathcal{F} \) of \( \mathcal{M} \) is called a Möbius group. If \( \mathcal{F} \) is discrete, we say that \( \mathcal{F} \) is a Fuchsian group. The corresponding group that acts on \( D \) is denoted by \( \hat{\mathcal{F}} \). The following are the main results of this paper.

Theorem 1.1. Let \( G \) be a discrete \( K \)-quasisymmetric group. Then there exists a \( K_1 \)-quasisymmetric map \( \varphi : T \to T \) and a Fuchsian group \( \mathcal{F} \) such that \( G = \varphi \mathcal{F} \varphi^{-1} \). The constant \( K_1 \) is a function of \( K \); that is, \( K_1 = K_1(K) \).

Hinkkanen proved (see [11], [16] and Proposition 1.2 below) that the same is true for quasisymmetric groups that are not discrete. This gives the next theorem.
Theorem 1.2. Let \( G \) be a \( K \)-quasisymmetric group. Then there exists a \( K_2 \)-quasisymmetric map \( \varphi : T \to T \) and a Möbius group \( F \) such that \( G = \varphi F \varphi^{-1} \). The constant \( K_2 \) is a function of \( K \); that is, \( K_2 = K_2(K) \).

It is a classical result of Sullivan and Tukia (see [23], [25]) that every quasiconformal group that acts on the Riemann sphere \( S^2 \) is quasiconformally conjugated to a Möbius group (subgroup of the group of Möbius transformations of \( S^2 \)). Tukia showed (see [26]) that this is no longer true for higher dimensional spheres. Martin [19] and Freedman and Skora [7] gave various important examples of this nature. Theorem 1.2 settles in positive the case of the one dimensional sphere \( T \).

The notion of convergence groups was introduced by Gehring and Martin (see the Gehring and Palka [11] for the origins of the theory of quasiconformal groups). The theory of quasiconformal groups is closely related to the theory of convergence groups (see [9], [10], [8], [27]). For example, see [2] for connections with the geometric group theory and for further references. In particular, quasiconformal groups are convergence groups. One of the central results in geometric group theory is that every convergence group of the circle homeomorphism is a conjugate of a Möbius group. This theorem was proved by Gabai [8]. Prior to that Tukia [27] proved this result for many cases. Hinkkanen [15] proved the result for non-discrete groups. This theorem was independently proved by Casson and Jungreis [3] by different methods (see [8], [3], [27] for references to other important papers on this subject).

We have the following proposition.

Proposition 1.1. Let \( G \) be a \( K \)-quasisymmetric group. Then there exists a homeomorphism \( \varphi : T \to T \) and a Möbius group \( F \) such that \( G = \varphi F \varphi^{-1} \).

To prove Theorem 1.2, one needs to show that in Proposition 1.1 one can find a Möbius group and a homeomorphism \( \varphi \) so that \( \varphi \) is quasiconformal. The methods used in [8], [25] to produce \( \varphi \) are constructive and explicit. Therefore, one can suspect that by repeating and modifying their construction while keeping in mind that \( \mathcal{G} \) is a quasisymmetric group, the resulting homeomorphism would be quasisymmetric. However, this does not appear to be the case. Nevertheless we will make frequent use of Proposition 1.1.

It follows from the theorem on uniformly quasiconformal groups that Theorem 1.2 is true if and only if every \( K \)-quasisymmetric group can be extended to a \( K_1 \)-quasiconformal group of \( D \), \( K_1 = K_1(K) \). In [5] it was shown that there is no general algorithm that would produce such an extension. The proof of Theorem 1.1 is also explicit and for a given discrete quasisymmetric group we construct such a quasiconformal extension (this extension can be recovered easily from our proof).

We will assume that the reader is familiar with some elementary facts from the theory of convergence and quasiconformal groups. In particular, we will freely use the notion of hyperbolic, parabolic, and elliptic elements of a quasisymmetric group. The order of an elliptic element \( e \in \mathcal{G} \) is the smallest integer \( n \in \mathbb{N} \), such that \( e^n = id \in \mathcal{G} \).

1.2. Elementary and non-discrete quasisymmetric groups. Let \( \varphi : T \to T \) be a homeomorphism. Let \( a, b \in T \), \( a \neq b \), and let \( l \) be the geodesic that connects them. Let \( l' \) be the geodesic that connects the points \( \varphi(a) \) and \( \varphi(b) \). We say that \( l' \) is a push forward of the the geodesic \( l \) and write \( \varphi_\ast l = l' \).
Hinkkanen proved Theorem 1.2 for various cases. In particular, the following proposition is a subcollection of results proved in [13] and [16] that are going to be used in this paper.

**Proposition 1.2.** Let $\mathcal{G}$ be a $K$-quasisymmetric group and suppose that $\mathcal{G}$ is either

1. a discrete elementary group or
2. a non-discrete group.

Then there exist $\tilde{K} = \tilde{K}(K)$ and a $\tilde{K}$-quasisymmetric map $\varphi$ such that $\varphi$ conjugates $\mathcal{G}$ to a Fuchsian group.

In fact, Hinkkanen proves the above proposition for Abelian and non-discrete groups. But this readily implies the case of discrete elementary groups as follows. We know that every discrete elementary quasisymmetric group is conjugated to a Fuchsian group. The list of discrete elementary Fuchsian groups is short (see [17]). Up to a conjugacy (in $\mathcal{M}$) the only non-Abelian discrete elementary group is generated by a hyperbolic element $u \in \mathcal{M}$ (with the fixed points $i, -i$) and the elliptic transformation $e_0 \in \mathcal{M}$, $e_0(z) = -z$, $z \in \mathbb{T}$. Note that $u$ and $e_0$ satisfy the relation

\[
(1.1) \quad u^{-1} = e_0 \circ u \circ e_0.
\]

Every element in this group is either hyperbolic (from the same cyclic group generated by $u$) or it is an elliptic transformation of order two that permutes the two fixed points of $u$ (there are infinitely many of these elliptic elements).

By Proposition 1.1 any elementary discrete quasisymmetric group $\mathcal{G}$ (that is not cyclic) is generated by a hyperbolic element $h$ and the corresponding elliptic element $e$ (that permutes the fixed points of $h$). By conjugating $h$ by a suitable quasisymmetric map, we may assume that $h = u \in \mathcal{M}$. So the group $\mathcal{G}$ is generated by $u$ and $e$, where $u$ is a M"{o}bius map and $e$ is some quasisymmetric map of order two. We can assume that $u$ fixes the points $i, -i$. Since the group $\mathcal{G}$ is topologically conjugated to a M"{o}bius group, we have the relation

\[
(1.2) \quad u^{-1} = e \circ u \circ e.
\]

Denote by $R$ the interval $(i, -i)$ and by $L$ the interval $(-i, i)$ (we take the standard counterclockwise orientation on $\mathbb{T}$). By replacing $e$ if necessary by a map of the form $u^{-k} \circ e \circ u^k$, $k \in \mathbb{Z}$, we can assume that $e(-1)$ belongs to the interval $(u^{-1}(1), u(1)) \subset R$. Let $q : \mathbb{T} \to \mathbb{T}$ be the earthquake map that is the identity on $L$ and such that $q(1) = e(-1)$ (such $q$ is unique and it is quasisymmetric). The map $q$ commutes with $u$ (which means that it conjugates $u$ to itself). By replacing $e$ if necessary by $q^{-1} \circ e \circ q$, we can assume that $e(-1) = 1$.

Since the maps $u, e, e_0$ satisfy (1.1) and (1.2), we conclude that $e = e_0$ on $O(1)$ and $O(-1)$, where $O(1)$ is the orbit of the point 1 under the action of the cyclic group generated by $u$ (similarly for $O(-1)$). Set $f(z) = (e_0 \circ e)(z)$ for $z \in R$, and set $f(z) = z$ for $z \in L$. We have that $f$ fixes every point from $O(1)$ and $O(-1)$. This readily implies that $f$ is quasisymmetric (we already know that $f$ is locally quasisymmetric on both $R$ and $L$). It follows that $f$ conjugates the group $\mathcal{G}$ to the M"{o}bius group generated by $u$ and $e_0$.

We also note the following elementary proposition.

**Proposition 1.3.** Let $\mathcal{G}$ be a $K$-quasisymmetric group and suppose that every finitely generated subgroup of $\mathcal{G}$ is $\tilde{K}$-quasiconformally conjugated to a Fuchsian group. Then so is $\mathcal{G}$. 675
From now on in this paper we assume that every quasisymmetric group is discrete (unless specified otherwise).

1.3. A brief outline. The proof of Theorem 1.1 is divided into several steps. However, there are two main intermediate results which constitute the heart of this paper. The first is Theorem 1.3. This theorem “takes care” of elliptic elements of order three or more.

Remark. This case turned out to be combinatorially very complicated in the proof of Proposition 1.1 about convergence groups. Note that so-called triangle groups must contain at least one elliptic element of order at least three.

Theorem 1.3 will be repeated and proved as Theorem 7.1 in Section 7.

Theorem 1.3. For an arbitrary $K$-quasisymmetric group $G$ there exists a $K_1$-quasisymmetric group $G_1$, $K_1 = K_1(K)$, with the following properties.

1. $G_1$ does not contain elliptic elements of order three or more.
2. If $G_1$ is $K'$-quasisymmetrically conjugated to a Fuchsian group, $K' = K'(K)$, then there exists $K'' = K''(K)$ such that $G$ is $K''$-quasisymmetrically conjugated to a Fuchsian group.

Let $F$ be a Fuchsian group and $\varphi$ a homeomorphism such that $\varphi F \varphi^{-1} = G$. Denote by $E' \subset D$ the set of fixed points of all elliptic elements of $F$ that are of order three or more. Let $\hat{\varphi}$ denote the barycentric extension of $\varphi$ and set $E = \hat{\varphi}(E')$; $S = D - E$. Set $G_1' = \hat{\varphi}F\hat{\varphi}^{-1}$. Then the group $G_1'$ is a $K_1$-quasisymmetric group on $S$, $K_1 = K_1(K)$, which means that every element of $G_1'$ is isotopic as a map of $S$ (rel $\partial S$) to a $K_1$-quasiconformal map. By covering the surface $S$ by the unit disc, we can lift $G_1'$ to the group $G_1$ that is a $K_1$-quasisymmetric group on $D$. The group $G_1$ is not isomorphic to $G_1'$ but it naturally projects to $G_1'$. The kernel of this projection is the group of covering transformations of $S$. This is the outline of the proof of Theorem 1.3. Note that we put no restriction on the choice of the homeomorphism $\varphi$ (in particular, it does not have to be quasisymmetric). The key to proving this theorem is certain analytical properties of the barycentric extension of Douady and Earle (see [4], [1]) that are of a different nature than the standard conformal naturality requirement this extension satisfies (which is also important of course).

The second main intermediate result is Lemma 4.2. Once we eliminate elliptic elements of order three or more (by Theorem 1.3), we can prove Lemma 4.2. This lemma shows that the subgroup $G_z$ of $G$ that is generated by small elements with respect to a point $z \in D$ is cyclic (see the definition in Section 4). Here by small elements we mean elements that are close to the identity (in $C^0$ topology) when seen from the point $z \in D$. This is a quasisymmetric version of the corresponding results about small elements of Fuchsian groups. These results for the Fuchsian case are corollaries of theorems like the Jorgensen inequality or the Margulis lemma (which of course holds in the context of discrete lattices in Lie groups).

Remark. Note that in the Fuchsian case in order to prove the Jorgensen inequality or the Margulis lemma, one does not have to assume that a given Fuchsian group does not have elliptic elements of order three or more. It is possible to prove Lemma 4.2 (the quasisymmetric case) without that assumption as well, but that would require a more delicate proof that would involve the commutator trick (see
Chapter 4 in [24] that is used to prove the Margulis lemma in the context of Lie groups. Although quasiconformal maps do not make a Lie group, they still can be endowed with a manifold structure, and one can modify the ideas from the proof of the Margulis lemma to this case. It is interesting to try to investigate this for quasiconformal groups in higher dimensions, which are not necessarily conjugates of Möbius groups. In the general case, the corresponding group $G_z$ should be almost Abelian.

We use the following rules in connection with the notation. If $C$ and $D$ stand for some abstract constants, then the labeling $C = C(D)$ means that the constant $C$ is a function of $D$. Sometimes $C$ is a function that is itself a function of $D$; that is, we say that $C$ depends on $D$. In other words, for a fixed $D$ one can choose such a $C$. What is important here is that saying that $C = C(D)$ also says that the constant $C$ does not depend on any other parameter. Typically, the constant $C$ will depend on the constant $D$ in a concrete way. Usually it will be bounded above or below in some way that depends on $D$. This will always be clear from the context but we will not necessarily make a note of it, since we do not need it.

From Section 4 onwards $K$ will always stand for the quasisymmetry constant of a $K$-quasisymmetric group. Recall from Proposition 1.2 that $\tilde{K}$ is the quasisymmetry constant of a map $\psi$ that conjugates an elementary $K$-quasisymmetric group to a Fuchsian group. We allow $\tilde{K}$ to be as large as necessary (but always as a function of $K$) so that we can choose appropriate $\tilde{K}$-quasiconformal extensions of the map $\psi$. From Section 4 onwards, the constant $\tilde{K}$ will have this meaning. All other constants will be valid throughout the subsection in which they were introduced. If we refer to a particular constant from a previous section, we will do this in a clear way and no confusion should arise.

In Sections 2 and 3 we prove various technical lemmas about quasiisometries, quasisymmetric groups, quasiconformal maps, and their geometry in $\mathbb{D}$. Some of these results might be known. In Section 4 we study small elements and show how to remove them. In Sections 5 and 6 we prove Theorem 1.1 for torsion-free groups. In Section 7 we show how to eliminate elliptic elements of order three or more. In Section 8 we deal with groups whose only elliptic elements are of order two.

After reading the introduction, the reader can go straight to the very end of this paper to consult the (very short) subsection where we give the summary of the proof of Theorem 1.1. This gives clear guidelines of what is the logical order of the proof.

Remark. It is a part of the definition of quasisymmetric (and quasiconformal maps in general) that they are sense-preserving. Originally, quasiconformal groups are defined to be sense-preserving (see [11], [14]). However, one can naturally extend this definition to sense-reversing maps. A sense-reversing map is quasiconformal if its complex conjugate is quasiconformal in the ordinary sense. Hinkkanen (in [16]) considered these generalized quasisymmetric groups, and his results are valid in this case as well.

We do not state Theorem 1.2 for generalized quasisymmetric groups, but we make the following observation. It appears that all methods that we use go through for sense-reversing maps as well. The only place where we use that our maps are sense-preserving is when listing elementary Fuchsian groups. If one allows sense-reversing Möbius transformations, then this list would have a few more members.
Consequently this implies that there are a few more cases of elementary discrete generalized quasisymmetric groups. Hinkkanen has dealt with this issue in [16], and it seems that his work covers all technical aspects that arise in dealing with these additional elementary groups.

2. Quasiisometries of the unit disc

2.1. Quasiisometric continuation. In this subsection we state several results about quasiisometries of the unit disc. Some of these results are classical. In this paper, $d$ stands for the hyperbolic metric on $D$, and $d(z, w)$ always denotes the hyperbolic distance between the points $z, w \in D$. By $\Delta(z, r)$ we denote the hyperbolic disc centered at $z \in D$ and with the hyperbolic radius $r > 0$.

**Definition 2.1.** Let $\hat{f} : D \to D$ be a homeomorphism. We say that $\hat{f}$ is a $(L, a)$-quasiisometry if

$$L^{-1}d(z, w) - a < d(\hat{f}(z), \hat{f}(w)) < Ld(z, w) + a,$$

for some $L, a > 0$.

**Remark.** The assumption that $\hat{f}$ is a homeomorphism of $D$ is often weakened in the literature by assuming that $\hat{f}$ is only a surjective map of $D$ onto itself. To simplify the notation, we will say that $\hat{f}$ is an $L$-quasiisometry if $L = \max\{L, a\}$.

It is well known that every $L$-quasiisometry has a continuous extension to $\overline{D}$. Let $f : T \to T$ be the corresponding map (the restriction of the extended map $\hat{f}$). There exists $K(L)$ such that $f$ is a $K(L)$-quasisymmetric map. On the other hand, if $\hat{f} : D \to D$ is a $K$-quasiconformal map, there is $a(K) > 0$ such that $\hat{f}$ is a $(K, a(K))$-quasiisometry (see [10]).

Let $l$ be a geodesic in $D$ and let $\gamma : l \to D$ be a map such that

$$L^{-1}d(z, w) < d(\gamma(z), \gamma(w)) < Ld(z, w),$$

for every $z, w \in l$ and some $L > 0$. We say that the map $\gamma$ is an $L$ bilipschitz quasigeodesic. Sometimes we will say that the corresponding curve $\gamma(l)$ is a bilipschitz quasigeodesic if it is clear what the mapping is. Let $l_1$ be the geodesic with the same endpoints as $\gamma(l)$. The main property of $\gamma$ is that there is $D(l) > 0$ such that $d(z, l_1) < D(l)$, for every $z \in \gamma(l)$.

Let $\hat{f}$ be a $L$-quasiisometry and $l \subset D$ a geodesic. The restriction of $\hat{f}$ on $l$ does not have to be a bilipschitz quasigeodesic. Nevertheless, it is easy to construct an $L'(L)$ bilipschitz quasigeodesic $\gamma : l \to D$ such that for some $D_0 = D_0(L)$ we have $d(\hat{f}(z), \gamma(z)) < D_0$, for every $z \in l$. This implies that every point of $\hat{f}(l)$ is within a bounded hyperbolic distance of the corresponding geodesic with the same endpoints. This observation is the key ingredient of the proof of the following well-known proposition.

**Proposition 2.1.** Let $\hat{f} : \overline{D} \to \overline{D}$ be a homeomorphism. Let $f : T \to T$ be the restriction of $\hat{f}$. We have the following.

1. Suppose that $f$ is the identity and that $\hat{f}$ is an $L$-quasiisometry. There exists $D = D(L) > 0$ such that $d(\hat{f}(z), z) < D$, for every $z \in D$.

2. Suppose that for every $z_0 \in D$, there exists an $L$-quasiisometry $\hat{f}_{z_0}$ which extends $f$ and such that $d(\hat{f}_{z_0}(z_0), \hat{f}(z_0)) < D$, for some $D = D(L) > 0$. Then there exists $L' = L'(L)$ such that $\hat{f}$ is $L'$-quasiisometry.
Definition 2.2. Let \((M, d_1)\) and \((N, d_2)\) be two metric spaces. Let \(\delta : \mathbb{R}^+ \to \mathbb{R}^+\) be a function, \(\epsilon \to \delta(\epsilon)\). We say that a map \(F : M \to N\) is \(\delta(\epsilon)\)-continuous if \(d_2(F(z), F(w)) < \epsilon\), whenever \(d_1(z, w) < \delta(\epsilon)\), for every \(z, w \in M\). We also say that such \(F\) is uniformly continuous. A homeomorphism \(F\) is \(\delta(\epsilon)\)-continuous if both \(F\) and \(F^{-1}\) are.

If we say that a map defined on a subset of the unit disc is uniformly continuous, that always refers (unless specified otherwise) to the corresponding hyperbolic metric on \(D\).

Let \(x, y, z \in \mathcal{T}\), and let \(S\) be the geodesic triangle with vertices \(x, y, z\). Denote the geodesics that represent the sides of \(S\) by \(a_{x,y}, a_{y,z}, a_{z,x}\). Suppose that \(f\) is a \(K\)-quasymmetric map. Suppose that there is an \(L_1 = L_1(K)\) with the following properties. There exist \(L_1\)-quasimetrics \(f_{x,y}, f_{y,z}, f_{z,x}\) that extend \(f\) and such that \(f_{x,y}(a_{x,y}) = f(a_{x,y}), f_{y,z}(a_{y,z}) = f(a_{y,z}), f_{z,x}(a_{z,x}) = f(a_{z,x})\).

Lemma 2.1. With the assumptions as above, the following holds. There exist \(L'_1 = L'_1(K)\) and an \(L'_1\)-quasimetric \(\hat{f}\) that extends \(f\) such that the restriction of \(\hat{f}\) on \(a_{x,y}, a_{y,z}\) and \(a_{z,x}\) agrees with \(f_{x,y}, f_{y,z}\), and \(f_{z,x}\), respectively. Moreover, suppose that the restriction of \(\hat{f}\) is \(\delta(\epsilon)\)-continuous on the sides of the triangle \(S\). Then there exists a function \(\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+\) (\(\delta_1(\epsilon)\) depends only on \(\delta(\epsilon)\)), such that \(\hat{f}\) is \(\delta_1(\epsilon)\)-continuous on \(S\).

Proof. By pre-composing and post-composing \(f\) by Möbius transformations, we may assume that \(x = i, y = -1, z = 1\) and that \(f(i) = i, f(-1) = -1, f(1) = 1\). On each halfspace determined by one of the geodesics \(a_{x,y}, a_{y,z}\), and \(a_{z,x}\) that do not contain the triangle \(S\), set \(\hat{f}\) equal to the corresponding quasimetric. It remains to define \(\hat{f}\) on \(S\) (see Figure 1).

Let \(m_1, m_2, m_3\) be the middle points (in the Euclidean sense) of the geodesic \(a_{x,y}, a_{y,z}\), and \(a_{z,x}\), respectively. Denote by \(S_0\) the corresponding geodesic triangle with vertices \(m_1, m_2, m_3\), and denote by \(s_1, s_{-1}, s_i\) the sides of \(S_0\) that face the points \(i, -1, 1\), respectively. Let \(s'_1, s'_{-1}, s'_i\) denote the geodesic arcs which connect the points that are the images of the endpoints of \(s_1, s_{-1}, s_i\) under the corresponding maps \(\hat{f}_{x,y}, \hat{f}_{y,z}, \hat{f}_{z,x}\). Let \(S'_0\) be the corresponding triangle (note that \(s'_1, s'_{-1}, s'_i\) also face the points \(i, -1, 1\), respectively). We define \(\hat{f}\) on each of the sides of \(S_0\) so that \(\hat{f}\) maps each side of \(S\) to the corresponding side of \(S'_0\) and so that \(\hat{f}\) is the restriction of the unique Möbius transformation (of \(C\)) that maps the middle points of \(s_i, s_{-1}, s_1\) onto the corresponding middle points of \(s'_1, s'_{-1}, s'_i\), respectively.

We define \(\hat{f}\) inside \(S_0\) to be any homeomorphism that extends the values of \(\hat{f}\) on \(\partial S_0\) (this homeomorphism maps \(S_0\) onto \(S'_0\)). We can arrange that this homeomorphism is \(\delta_1(\epsilon)\)-continuous for some function \(\delta_1\) that is a function of \(\delta\) (after supplying the regions \(S_0\) and \(S'_0\) with their own Poincaré metrics, one can take this homeomorphism to be either the Euclidean harmonic extension or the barycentric extension or any other classical extension that is a homeomorphism). Note that the hyperbolic diameters of both \(S_0, S'_0\) are bounded above by a constant that is a function of \(K\).

Denote by \(S_i, S_{-1}, S_1\) the corresponding subtriangles of \(S\) such that \(S\) is a disjoint union of \(S_0, S_i, S_{-1}\), and \(S_1\). For a point \(w \in S_i\) let \(\alpha\) be the geodesic arc that contains \(w\), where \(\alpha\) is parallel to \(s_i\) and \(\alpha\) connects the two sides (other than \(s_i\)) of the triangle \(S_i\). Two geodesic arcs are parallel if they are subarcs of two
concentric circles in $C$. By $\alpha'$ we denote the geodesic arc that connects the points that are the images of the endpoints of $\alpha$ under two of the corresponding maps $\hat{f}_{x,y}$, $\hat{f}_{y,z}$, $\hat{f}_{z,x}$. We define $\hat{f}$ on $\alpha$ so that $\hat{f}$ maps $\alpha$ to $\alpha'$ and so that $\hat{f}$ is the restriction of the unique Möbius transformation (of $C$) that maps the (Euclidean) middle point of $\alpha$ onto the corresponding middle point of $\alpha'$. We repeat the same process for the remaining points in $S$. It follows from this construction that $\hat{f}$ is a $L'_1$-quasiisometry, for some $L'_1 = L'_1(K)$. Also, it follows from the geometric nature of our extension that for a fixed function $\delta(\epsilon)$ there exists a function $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\hat{f}$ is $\delta_1$-continuous.

Let $\Lambda$ be a countable collection of geodesics in $D$ that is locally finite and so that no two geodesics intersect transversally in $D$ (they may have a common endpoint in $T$). Clearly $\Lambda$ is a closed subset of $D$. Such a collection $\Lambda$ is also called a discrete geodesic lamination. Suppose that $f$ is $K$-quasisymmetric and that $f\circ_l \in \Lambda$, $f^{-1}\circ_l \in \Lambda$, for each $l \in \Lambda$. In addition, suppose that for each $l \in \Lambda$, there exists an $L_2$-quasiisometry $\hat{f}_l$ that extends $f$ and such that $\hat{f}_l(l) = f\circ_l, f^{-1}\circ_l \in \Lambda$. Here $L_2 = L_2(K)$.

**Lemma 2.2.** With the assumptions as above, the following holds. There exist $L_3 = L_3(K)$ and an $L_3$-quasiisometry $\hat{f}$ that extends $f$ such that the restriction of $\hat{f}$ on each $l \in \Lambda$ agrees with the restriction of $\hat{f}_l$. Let $\beta_i, i \in [1, N], N = N(K) \in \mathbb{N}$, be mutually disjoint geodesic arcs that do not intersect $\Lambda$ transversally but whose endpoints belong to certain geodesics from $\Lambda$. Then we can choose the above map $\hat{f}$ so that $\hat{f}(\beta_i)$ is a geodesic arc.
In addition, assume that the above \( \hat{f} \) is \( \delta(\epsilon) \)-continuous on \( \Lambda \) (here \( \hat{f} \) agrees with \( f_{\hat{f}} \)). Then \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous, where the function \( \delta_1 \) is itself a function of \( \delta(\epsilon) \) and \( K \).

**Proof.** Each geodesic lamination can be completed to a maximal geodesic lamination. Let \( \Lambda' \) be a maximal geodesic lamination that contains \( \Lambda \). Then \( D \) is a disjoint union of \( \Lambda' \) and a collection of open geodesic triangles whose sides are in \( \Lambda' \) (if \( \Lambda' \) foliates the whole unit disc, then these triangles do not exist). Let \( s_n, n \in \mathbb{N} \), be the sequence of geodesics from \( \Lambda' - \Lambda \) so that the closure of the union of \( \Lambda \) and \( \bigcup_{n \in \mathbb{N}} s_n \) is equal to \( \Lambda' \).

We first define \( \hat{f} \) on \( s_1 \). For \( z \in s_1 \) let \( p \) be the geodesic that is orthogonal to \( s_1 \). Let \( \alpha \) be the maximal geodesic subarc of \( p \) that contains \( z \) and that does not intersect \( \Lambda \). There are three possibilities: (i) the endpoints of \( \alpha \) belong to two different geodesics from \( \Lambda \); (ii) one endpoint belongs to a geodesic from \( \Lambda \) and the other is in \( T \); (iii) both endpoints belong to \( T \). In any case the value of \( \hat{f} \) is determined at these two endpoints by the maps \( \hat{f}_l \), \( l \in \Lambda \). Denote by \( \alpha' \) the corresponding geodesic arc whose endpoints are determined by the images of the endpoints of \( \alpha \). Then \( \alpha' \) does not transversally intersect \( \Lambda \). Define \( \hat{f}(z) \) to be the intersection between \( \alpha' \) and \( \hat{f}_s \). This is well defined because \( f \) is a homeomorphism and \( f_s \Lambda = \Lambda \). Now, repeat this process for \( s_2 \), but instead of \( \Lambda \) we use \( \Lambda \cup s_1 \), and so on. By this we define \( \hat{f} \) on every \( s_n \), and it follows from the construction that \( \hat{f} \) is well defined on \( \Lambda' \). On the remaining geodesic triangles we define \( \hat{f} \) by using Lemma 2.1. It follows from Proposition 2.1 that \( \hat{f} \) is an \( L_2' \)-quasiisometry, \( L_2' = L_2(K) \). Because of the geometric construction we made, it follows that \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous on \( D \). One can directly compute \( \delta_1(\epsilon) \) in terms of \( \delta(\epsilon) \) and the Hölder continuity constants of the map \( f \) (which are explicit functions of \( K \) by the theorem of Mori).

Let \( \beta, i \in [1, N] \), be a geodesic arc from the statement of this lemma. Let \( \beta' \) be the geodesic arc with the same endpoints as the curve \( \hat{f}(\beta) \). We have that \( \beta' \) and \( \hat{f}(\beta) \) are a finite hyperbolic distance apart (this upper bound depends only on \( K \)) since \( \hat{f} \) is a quasiisometry. Because of that, one can find an \( L_2'' \)-quasiisometry \( I : D \to D \) \( (L_2'' = L_2''(K)) \) which extends the identity map of \( T \), so that \( I \) pointwise fixes each \( l \in \Lambda \) and \( I(\hat{f}(\beta)) = \beta' \). The map \( I \circ \hat{f} \) is an \( L_3 \)-quasiisometry, \( L_3 = L_3(K) \). Since \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous, we see that if \( \beta \) is a very short geodesic arc, then \( \hat{f}(\beta) \) is very close to the corresponding arc \( \beta' \). This implies the existence of \( \delta_1(\epsilon) \) such that \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous. Moreover, \( \delta_1 \) depends only on \( \delta \).

Repeat this process \( N = N(K) \) times for all the arcs \( \beta_i \) to obtain the resulting quasiisometry. \( \square \)

2.2. **Quasisymmetric groups.** Let \( \hat{G} \) be an \( L \)-quasiisometric group on \( D \). This means that \( \hat{G} \) is a group whose elements are homeomorphisms of \( D \), each of them being \( L \)-quasiisometric. The restrictions of elements from \( \hat{G} \) on the unit circle \( T \) form a group which we denote by \( G \). Clearly, \( G \) is a \( K \)-quasisymmetric group, \( K = K(L) \). Our aim is to show that there is \( K' = K'(L) \) such that \( G \) is \( K' \)-quasisymmetrically conjugated to a Fuchsian group. We will show this under extra assumptions. Suppose that there exist \( \rho > 0 \) and a function \( \delta : \mathbb{R}^+ \to \mathbb{R}^+ \) with the following properties. For every \( \hat{f} \in \hat{G} \setminus \text{id} \) we have that \( d(z, \hat{f}(z)) > \rho \), for every \( z \in D \). Also, each \( \hat{f} \) is \( \delta(\epsilon) \)-continuous. These assumptions on the group \( \hat{G} \) are
valid throughout this subsection. Our aim is to show that there is a quasiconformal

Let $\rho_0 > 0$. We define a discrete set $E = E(\rho_0) \subset D$ as follows. For any point $z \in D$ denote by $O(z)$ its orbit under $G$. Let $\Sigma$ denote the Fuchsian group that is the covering group of a closed Riemann surface of genus at least two. Let $w_n, n \in \mathbb{N}$, denote the orbit of the point $w_1 = 0$ under $\Sigma$. Set $z_1 = w_1 = 0$, and let $E_1 = O(z_1)$. Choose the smallest $n_0 > 1$ so that $d(O(z_1), O(w_{n_0})) > \rho_0$. If such $n_0$ does not exist, then set $E = E_1$. Otherwise, let $z_2 = w_{n_0}$ and set $E_2 = E_1 \cup O(z_2)$. Similarly, let $n_1$ be the smallest number so that $d(O(w_{n_1}), E_2) > \rho_0$. If such $n_1$ does not exist, then set $E = E_2$. Otherwise, let $z_3 = w_{n_1}$ and $E_3 = E_2 \cup O(z_3)$, and so on. Let $E = \bigcup E_i$. The main properties of the set $E$ are as follows.

1. $E$ is invariant under $\hat{G}$.
2. There exists $\rho_1 = \rho_1(\rho_0, \rho, \delta(e))$ such that in every geodesic ball of radius $\rho_1$ there is at least one point from $E$.
3. Every geodesic ball of radius $\frac{\rho_1}{10}$ contains at most one point from $E$.

Remark. The one and only reason we choose $\Sigma$ to be the covering group of a closed surface is to achieve that the sequence $w_n$ is well distributed in $D$, which then yields condition (2) above.

If we let $\rho_0 \to 0$, then we can choose $\rho_1$ so that $\rho_1 \to 0$ (this comes from the fact that $\hat{f}$ is $\delta(e)$-continuous). Because of that we can choose $\rho_0 = \rho_0(\rho, \delta(e))$ small enough so that the corresponding $\rho_1 < \frac{\rho}{100}$. We choose such $\rho_0$ and $\rho_1$ (see the remark below).

Set $S = D - E$. Denote by $d_S$ the hyperbolic metric on $S$. It follows from the assumptions that the metrics $d$ and $d_S$ are comparable, except on a microscale very near the points from $E$. This means that if we take the corresponding densities $\text{dens}(d)$ and $\text{dens}(d_S)$ that define the metrics $d$ and $d_S$, respectively, then there exists a constant $X > 0$ so that for $z \in S = D - E$ we have

$$\frac{1}{X} < \frac{\text{dens}(d)(z)}{\text{dens}(d_S)(z)} < X,$$

and the constant $X$ depends only on the $d(z, S)$. For $z, w \in E$ and a simple curve (simple arc) $\gamma \subset S$ with the endpoints $z, w$, let $\alpha(z, w, \gamma)$ denote the geodesic (with respect to $d_S$) homotopic to $\gamma$ (homotopic in $S$). We say that $\gamma$ is $\delta_1(e)$-continuous if there is a $\delta_1(e)$-continuous homeomorphism between $\gamma$ and $\alpha(z, w, \gamma)$ (here the uniform continuity is with respect to the metric $d$). We assume that the inverse of this homeomorphism is also $\delta_1(e)$-continuous. Here $\delta_1 : (0, \infty) \to (0, \infty)$.

Let $z, w \in E$ and let $\gamma \subset S$ be a curve that connects them. We say that the homotopy class $[\gamma]$ (of $\gamma$) is admissible if for every $\hat{f} \in \hat{G}$ we have that the length of the curve $\alpha(\hat{f}(z), \hat{f}(w), \hat{f}_*\gamma)$ is at most $\rho_2 = \frac{\rho}{10}$. Here we measure the length of $\alpha(\hat{f}(z), \hat{f}(w), \hat{f}_*\gamma)$ in the hyperbolic metric $d$ of the unit disc (the length of $\alpha(\hat{f}(z), \hat{f}(w), \hat{f}_*\gamma)$ in the metric $d_S$ is infinite).

Numerate points in $E$ by $z_n, n \in \mathbb{N}$. Fix $i, j \in \mathbb{N}$. Suppose that $[\gamma]$ is an admissible homotopy class (with the corresponding endpoints $z_i, z_j$) and let $\alpha(z_i, z_j, \gamma)$ be the corresponding geodesic. Then $\alpha(z_i, z_j, \gamma)$ has length at most $\rho_2$. Let $F_{i,j,\gamma}$ be the orbit of $\alpha(z, w, \gamma)$ under $\hat{G}$. By $F_{i,j}$ we denote the union of all $F_{i,j,\gamma}$, where
[\gamma] \text{ ranges over all corresponding admissible homotopy classes} (\text{we note that there are only finitely many such homotopy classes for a fixed pair } i, j). \text{ The union of all the sets } F_{i,j} \text{ is denoted by } F.

Remark. \text{ By taking } \rho_0 \text{ and } \rho_1 \text{ a little bit smaller (but keeping } \rho_2 \text{ as it is), we can arrange that every connected component of the set } \mathbf{D} - F \text{ is a polygon whose diameter is bounded above by a constant that is a function of } \delta(\epsilon) \text{ and } \rho. \text{ This is the final choice of } \rho_0 \text{ and } \rho_1.

By definition each curve in } F \text{ is } \delta(\epsilon)\text{-continuous } (\delta(\epsilon) \text{ is the function that was fixed at the beginning of the subsection). Each curve from } F \text{ is homotopic to a geodesic (in metric } d_S) \text{ whose } d \text{ length is less than } \rho_2. \text{ From } \delta(\epsilon)\text{-continuity of curves from } F, \text{ it follows that there cannot be very many curves from } F \text{ ending at a given point of } E. \text{ This implies that there is } N = N(\rho, \delta(\epsilon)) \in \mathbb{N} \text{ such that for each point } z \in E \text{ there are at most } N \text{ curves from } F \text{ that end at } z. \text{ Because of that, the fact the each curve in } F \text{ is uniformly continuous, and since each } \hat{f} \in \widehat{G} \text{ is uniformly continuous, we conclude that there exists } N_1 = N(\rho, \delta(\epsilon)) \in \mathbb{N} \text{ such that in each hyperbolic disc of radius } \rho_0 \text{ there are at most } N_1 \text{ points that are intersection points between different curves from } F. \text{ Now, we slightly deform the curves (within their homotopy classes in } S \text{) from } F \text{ so that the hyperbolic distance (distance } d) \text{ between any two intersection points is greater than } \rho' = \rho'(\rho, \delta(\epsilon)) \text{ and such that every intersection point is contained in exactly two curves from } F. \text{ In addition, we can arrange that the new set of curves (which we also denote by } F \text{) is also invariant under } \widehat{G}. \text{ This is because } \rho_2 \text{ was chosen to be small enough so that the distance between a curve from } F \text{ and any other curve in its orbit is bounded below by a positive constant that is a function of our original parameters } \rho \text{ and } \delta(\epsilon).

Denote by } E_1 \text{ the union of } E \text{ and the set of all intersection points (after the deformation). It follows from the construction that each connected component of } \mathbf{D} - F \text{ is a polygon with at most } N_2 = N_2(\rho, \delta(\epsilon)) \text{ sides. We can now add new curves (that connect points from } E_1 \text{) to the set } F \text{ (the new set is also called } F). \text{ The new set of curves is denoted by } F_1. \text{ Clearly, } F_1 \text{ retains the same essential properties of } F, \text{ except that } F_1 \text{ is no longer invariant under } \widehat{G}. \text{ However, for each } \gamma \in F_1 \text{ there is a unique curve in } F_1 \text{ that is homotopic to } \hat{f}_*\gamma.

Remark. \text{ Note that if } T \text{ is a triangle from this partition, then } \hat{f}_*T \text{ is well defined. If } \hat{f}_*T = T, \text{ then } f \text{ is the identity. This follows from the fact that } \hat{f} \text{ has no fixed points inside the unit disc. We will not make any use of this fact.}

Because the distance between any two points in } E_1 \text{ is bounded below by } \rho', \text{ there are constants } \rho_3, \rho_4 > 0 \text{ that are functions of } \rho \text{ and } \delta(\epsilon) \text{ such that the hyperbolic length } l_d(\gamma) \text{ of } \gamma \in F_1 \text{ satisfies } \rho_3 < l_d(\gamma) < \rho_4. \text{ Here } l_d \text{ denotes the length with respect to } d. \text{ We now define the new group } \widetilde{G} \text{ of homeomorphisms of } \mathbf{D}. \text{ For } f \in \widehat{G} \text{ we denote by } \hat{f} \text{ the homeomorphism from } \widehat{G} \text{ that extends } f. \text{ We define } \hat{f} \text{ to be equal to } \hat{f} \text{ on the set } E_1. \text{ On each curve } \gamma \text{ from } F_1 \text{ we define } \hat{f} \text{ to be the affine map with respect to the natural parameters on each of the two curves } \gamma \text{ and } \hat{f}_*\gamma = \hat{f}(\gamma).
The natural parameter is taken with respect to the metric $d$ (both curves have finite $d$ length). It also follows that $\tilde{f}$ respects the composition (the composition of two affine maps is an affine map). It remains to define $\tilde{f}$ on the interior of the corresponding triangles. We denote the set of all triangles by $T$ and $g$:

It remains to define the natural parameter is taken with respect to the metric $d$. By the same arguments as above, for every $z \in E_1$ we can choose a small disc $D_z$ (not necessarily geodesic) of the hyperbolic diameter $\sim \rho_5$ such that the set $S_2 = D - \bigcup_{z \in E_1} D_z$ is a connected Riemann surface homotopic to $S_1$. Moreover, we can arrange that $S_2$ is invariant under the action of $\tilde{G}$. We now fix such $\rho_5 = \rho_5(\delta(\epsilon))$. The fact that $\tilde{G}$ is a group follows readily from the construction. Clearly, $\tilde{G}$ is a $K_1$-quasiconformal group on $S_2$, $K_1 = K_1(\rho, \delta(\epsilon))$. Therefore, there exist a Riemann surface $S'_2$, a $K_2$-quasiconformal map $\psi : S_2 \to S'_2$, $K_2 = K_2(K_1) = K_2(\rho, \delta(\epsilon))$, and a discrete conformal group $\tilde{F}$ that acts on $S'_2$, such that $\tilde{G} = \psi \tilde{F} \psi^{-1}$. Note that the restriction of the map $\psi$ to $T$ is a quasisymmetric map (see the remark below).

Since one can realize $S'_2$ as the unit disc minus many (but countably many) small topological discs that lie inside $D$, we can apply the results from [12] and conclude that there exists a conformal map $\pi : S'_2 \to S''_2$, where $S''_2$ is obtained as the unit disc minus countably many geometric discs (these discs are now hyperbolic discs in the unit disc). The map $\pi$ extends continuously to the quasisymmetric map of the unit circle (see the remark below). Using [12] again, we know that every conformal map of $S''_2$ onto itself must be a restriction of a Möbius transformation. This shows that $\pi \tilde{F} \pi^{-1}$ acts as a Fuchsian group on the unit circle. This implies that the restriction of $\tilde{F}$ on the unit circle is a quasisymmetric conjugate of a Fuchsian group.

Remark. Here we use the following result of Kozlovski, Shen, and van Strien about quasiconformal mappings (see appendix in [18] for the formulation and proof). This theorem was inspired by the work of Heinonen and Koskela [13] (also see [22]). Let
Lemma 2.3. With the assumptions and notation as above, we have that the group \( G \) of the form \( M \) be the corresponding component of the set \( U - D \) be a homeomorphism of \( F \) Let \( K \) imply that the following properties. The hyperbolic distance between \( D \) is bounded above by \( C_1 > 0 \). Then the restriction of \( F \) on \( T \) (which has also been proved to exist) is \( K(C, C_1, \hat{K}) \)-quasisymmetric.

Lemma 3.1. With the assumptions and notation as above, we have that the \( G \) is \( K_2 \)-quasisymmetrically conjugated to a Fuchsian group. Here \( K_2 = K_2(\rho, \delta(\epsilon)) \).

3. Quasiconformal continuation and barycentric extensions

3.1. Quasiconformal continuation. Suppose that \( f : T \to T \) is a \( K \)-quasisymmetric map. Let \( S = \{ S_i \}, i \in \mathbb{N} \), be a collection of subsets of \( \overline{D} \) with the following properties. Each \( S_i \) is either a Jordan region or a Jordan curve. Each \( S_i \) is a closed connected subset of \( \overline{D} \) such that there are no bounded (in \( D \)) components of \( \overline{D} - S_i \). This implies that the interior (if any) of \( S_i \) is simply connected. Also, each \( S_i \) touches the unit circle at, at most two points. In addition, there exists \( R > 0 \) such that

\[
d(S_i, S_j) > R, \quad i \neq j.
\]

Let \( K' = K'(K) \). Suppose that for each \( i \in \mathbb{N} \), there is a \( K' \)-quasiconformal map \( \hat{f}_i \) that extends \( f \) and such that \( \hat{f}_i(S_i) \subseteq S \).

Lemma 3.1. With the assumption as above, the following holds. There exists (large enough) \( R = R(K) \), so that if (3.1) holds for \( R \), then there exist \( K_1 = K_1(K) \) and a \( K_1 \)-quasiconformal map \( \hat{f} \) that extends \( f \) and such that \( \hat{f} = \hat{f}_i \) on each \( S_i \).

Remark. By a more involved argument, one can prove that in the above lemma we can always take \( R = 1 \). Also, the assumption that each \( S_i \) touches the unit circle at, at most two points is not essential. However, this will be the case in all applications of this lemma.

Proof. Let \( \tilde{f} \) be any \( K \)-quasiconformal extension of \( f \). Let \( \hat{f}_i(S_i) = S_j \) for some \( j \in \mathbb{N} \). The distance \( d(\tilde{f}(z), \hat{f}_i(z)), z \in S_i \), is bounded by a constant depending only on \( K \). Choose \( R = R(K) \) large enough so that the following holds. For each \( i \in \mathbb{N} \), there exists an open connected set \( U_i \subseteq D \) that contains both \( S_j \) and \( \hat{f}(S_i) \) and such that \( U_i \cap U_j \) is an empty set, \( i \neq j \). Moreover, every point in \( \partial U_i \) is at the hyperbolic distance at least 1 from any point from \( S_j \cup \hat{f}(S_i) \). In addition, we can choose the set \( U_i \) so that \( U_i \) touches the unit circle at the same points that \( S_j \) does and so that the boundary of \( U_i \) is a Jordan curve. Note that the above assumptions imply that \( U_i \) is simply connected.

Now, one can construct a \( K'' \)-quasiconformal map \( I_i : D \to D, K'' = K''(K) \), so that \( I_i \circ \tilde{f} = \hat{f}_i \) on \( S_i \) and \( I_i \) is the identity on \( D - U_i \). To do this, let \( \Omega_i \) be one of at most two components of the open set \( U_i - \hat{f}(S_i) \). In either case, \( \Omega_i \) has two boundary components (these two components meet at \( T \)). One of them is a piece of the boundary of \( \hat{f}(S_i) \), and another one is a piece of the boundary of \( U_i \). Let \( \Omega'_i \) be the corresponding component of the set \( U_i \setminus S_j \), so that \( \partial \Omega'_i \) contains the same piece of the boundary of \( U_i \) as does \( \partial \Omega_i \). Let \( g : \partial \Omega_i \to \partial \Omega'_i \) be the homeomorphism that is the identity on the part of the boundary coming from \( U_i \) and \( g = \hat{f}_i \circ \tilde{f}^{-1} \).
Lemma 3.2. Let \( \rho \) be Lipschitz in \( D \), one-parameter family of diffeomorphism \( I \). \( I_\rho \) between \( I \) satisfies the properties stated in the lemma. Note that the family \( K \) shows that there exists \( K_1 = K_1(K) \), \( \hat{f} \) is a \( K_1 \)-quasiconformal map.

\[ \lim_{i \to \infty} I_\rho \circ \cdots \circ I_1 \circ \hat{f} \]

We have that for some constant \( K_1 = K_1(K) \), \( \hat{f} \) is a \( K_1 \)-quasiconformal map. \( \square \)

Definition 3.1. Let \( E \subset D \) be a discrete set, and let \( \rho > 0 \). We say that \( E \) is a \( \rho \)-discrete set if the hyperbolic distance between any two points in \( E \) is at least \( \rho \).

Lemma 3.2. Let \( E \) be a \( \rho \)-discrete set. Let \( C : E \to D \), where \( E' = C(E) \) is a \( \rho \)-discrete set, and \( d(C(z), z) < D \), for some \( D > 0 \), and every \( z \in E \). Then, there exist \( K_2 = K_2(\rho, D) \) and a \( K_2 \)-quasiconformal map \( I \) that extends the identity such that \( I(z) = C(z) \), \( z \in E \).

Proof: Suppose first that the constant \( D = D_0 \) satisfies \( 0 < D_0 < \frac{\rho}{2} \). Let \( \alpha_z \) be the geodesic arc that connects \( z \) and \( C(z) \). Since \( d(C(z), z) < D_0 \), no two such arcs would intersect. The positive orientation on \( \alpha_z \) is from \( z \) toward \( C(z) \). Let \( X_z \) be the vector field on \( \alpha_z \) so that for \( w \in \alpha_z \) we have that \( X_z(w) \) is the positive tangent vector to \( \alpha_z \) at \( w \). We take \( X_z(w) \) to have the length

\[ \frac{l_d(\alpha_z)}{D_0}, \]

where \( l_d(\alpha_z) \) is the length with respect to the hyperbolic metric \( d \). Choose \( D_0 = D_0(\rho) \) small enough such that we can define a vector field \( X \) on \( D \) which agrees with each \( X_z \) and satisfies the following. We can choose \( X \) so that there exists the one-parameter family of diffeomorphism \( I_t : D \to D \), \( t \in [0, D_0] \), such that

\[ \frac{\partial I_t}{\partial t} = X. \]

For this it is enough to arrange that \( X \) is a smooth vector field and uniformly Lipschitz in \( D \) (which we can do since \( E \) is \( \rho \)-discrete and for \( D_0 \) small enough).

Moreover, since \( E \) is \( \rho \)-discrete, we can choose \( X \) uniformly (uniform on \( D \)) smooth, so that the map \( I_{D_0} \) is \( K'_2 \)-quasiconformal, for some \( K'_2 = K'_2(\rho) \). This conclusion comes from the standard estimates on the solutions of ODE’s. Note that we can choose such \( X \) for some \( D_0 > 0 \), where \( D_0 \) does not depend on the particular set \( E \). We have \( D_0 = D_0(\rho) \). It follows from the construction that the time \( D_0 \) map \( I_{D_0} \) satisfies the properties stated in the lemma. Note that the family \( I_t \) is continuous in the variable \( t \) as well.

The general case goes as follows. We can join \( z \) and \( C(z) \) by a \( C^\infty \) curve \( \gamma_z \), whose length is equal to some fixed \( L(\rho) > 0 \) for each \( z \in E \). We can also arrange the following. Let \( z_t \) be the point on \( \gamma_z \) such that the length of the piece of \( \gamma_z \) between \( z \) and \( z_t \) is equal to \( t \in [0, L(\rho)] \). Let \( E_t \) be the set of all \( z_t \). We can choose the curves \( \gamma_z \) so that each set \( E_t \) is \( \rho_t \)-discrete, \( \rho_t = \rho_t(\rho) \). Now, chop up the interval \( [0, L(\rho)] \) into \( N \) small intervals, \( N = N(\rho, D) \in \mathbb{N} \), such that on each of them we can apply the above construction. This proves the lemma. \( \square \)

Let \( E \) be a \( \rho \)-discrete set, and let \( f_t : D \to D \), \( t \in [0, k_0] \), be a continuous family (in \( t \)) of homeomorphisms (of \( D \)) with the following properties. For every \( t \), \( f_t \) is
the identity on $T$. We have $\tilde{f}_0 = id$. For every $\epsilon > 0$ there exists $\delta(\epsilon) = \delta > 0$ such that for every $t, s \in [0, k_0]$ and $z \in E$, we have that $d(\tilde{f}_t(z), \tilde{f}_s(z)) < \epsilon$, whenever $|t - s| < \delta$. In addition, we assume that the set $E_t = \tilde{f}_t(E)$ is $\rho$-discrete for every $t$.

Set $S_t = D - E_t$.

**Lemma 3.3.** With the above assumptions, we have that there exists a $K_3$-quasiconformal map $\hat{f}_t : S_0 \to S_t$, where $\hat{f}_t$ is isotopic to $\tilde{f}_t$ (isotopic in $S_0$) for every $t \in [0, k_0]$. The constant $K_3$ is a function of $\rho$, the function $\delta(\epsilon)$, and $k_0$.

**Proof.** First, we break the interval $[0, k_0]$ into small enough intervals so that on each of these small intervals we use the construction from the first part of the proof of the previous lemma. Precisely, let $n \in \mathbb{N}$, $n = n(\rho, \delta(\epsilon), k_0)$, be such that for each interval $[s_i, s_{i+1}]$ we can construct a $K_2$-quasiconformal map $\hat{f}_i$, $1 \leq i \leq n$, that maps $E_{s_i}$ to $E_{s_{i+1}}$ (the map $\hat{f}_i$ is from the first part of the proof of the previous lemma). Here $s_i = \frac{i\rho}{n}$. We can do that since $\tilde{f}_i$ is $\delta(\epsilon)$-continuous. Also, the map $\hat{f}_i$ is very close to the identity in the $C^0$ sense uniformly in the metric $d$ on $D$.

In fact, we can choose $n$ large enough so that the maps $\hat{f}_i$ and $\hat{f}_{s_{i+1}} \circ \hat{f}_{s_i}^{-1}$ are isotopic. This can be proved by contradiction as follows. Suppose that for every $n \in \mathbb{N}$ we can produce an example where the above conclusion fails. Then because all the maps involved are uniformly continuous and the set $E$ is $\rho$-discrete for some fixed $\rho > 0$, we can pass onto the limit. The limit of the maps $\hat{f}_{s_{i+1}} \circ \hat{f}_{s_i}^{-1} \circ \hat{f}_i^{-1}$ is the identity map. This is a contradiction since we assumed that none of the maps in the sequence are homotopic to the identity.

By doing this, we produce a continuous family of $K_3$-quasiconformal mappings $\hat{f}_t$, $t \in [0, k_0]$, such that for every $z \in E$, we have $\hat{f}_t(z) = \hat{f}_i(z)$. Moreover, $\hat{f}_t(z) \circ (\hat{f}_i)^{-1}$ is isotopic to the identity on $S$.

**Lemma 3.4.** Let $E \subset D$ be a $\rho$-discrete set and suppose that $0 \in E$. Set $S = D - E$. For $x \in T$ let $s_x$ be the geodesic in $D$ with endpoints $x$ and $-x$. Let $\hat{f} : D \to D$ be a homeomorphism that extends the identity map and that fixes every point from $E$. Suppose that there is $L > 0$ such that the restriction of $\hat{f}$ on $s_x$ is homotopic (in $S$) to a $L$-bilipschitz quasigeodesic $\gamma_x : s_x \to D$. Then there exists $K_4 = K_4(\rho, L)$ such that $\hat{f}$ is isotopic (in $S$) to a $K_4$-quasiconformal map.

**Proof.** Denote by $d_s$ the hyperbolic distance on $S$. Since $E$ is $\rho$-discrete, we have that $d_s$ is comparable with $d$, except on a microscale near the points from $E$. Let $\alpha_x$ be the geodesic in the metric $d_s$ that is homotopic to $s_x$ in $S$. Here $\alpha_x$ is really a union of geodesics in $S$ that connect the corresponding points from $E$. By $\beta_x$ we denote the geodesic (with respect to $d_s$) that is homotopic to $\hat{f}(s_x)$. From the assumption of the lemma and the comparability between $d$ and $d_s$, we conclude that both curves $\alpha_x$ and $\beta_x$ can be parametrized as $L_1$ bilipschitz quasigeodesics in $D$, $L_1 = L_1(\rho, L)$. This implies that for every point $z \in \alpha_x$, there is a point $w \in \beta_x$ such that $d_s(z, w) < M$, $M = M(\rho, L)$. Suppose that $w \in \beta_x$ is such that $d_s(z, w) = d_s(z, \beta_x)$. Define $\tilde{f} : S \to S$ by setting $\tilde{f}(z) = w$. The restriction of $\tilde{f}$ on each $\alpha_x$ is the so-called nearest point retraction map.

Fixing an (arbitrarily small) neighborhood of the set $E$, it is easy to show that there exists $L_1 = L_1(\rho, L)$ such that $\tilde{f}$ is an $L_1$ bilipschitz map (with respect to the metric $d_s$) outside that neighborhood. We will not compute this, but we will suggest how to do it. An easy way to see this is by passing onto the universal
cover of $S$. There (meaning in the universal cover = unit disc), for each $z$ and the corresponding $w = \tilde{f}(z)$, $\alpha_z$ and $\beta_z$ can be realized as two non-intersecting geodesics and the distance between the corresponding lifts of $z$ and $w$ is less than $M$. Since those two geodesics do not intersect, the bilipschitz constant of the nearest point retraction map (away from the cusps) depends only on the distance $M$. Since $\tilde{f}$ is bilipschitz on the unit disc minus a fixed neighborhood of the set $E$, it is quasiconformal as well on the set. Now, one can change the map $\tilde{f}$ in the neighborhood of the set $E$, so that the new map is isotopic to $\tilde{f}$ and quasiconformal as well on $D$. □

The following lemma is elementary.

**Lemma 3.5.** Let $f$ be a $K$-quasisymmetric map and let $l$ be a geodesic in $D$. There exists a $\tilde{K}$-quasiconformal and $L$-bilipschitz map $\tilde{f}$ which extends $f$, $\tilde{K} = K(K)$, $L = L(K)$, such that $\tilde{f}(l) = f(l)$.

**Proof.** Let $\tilde{f}$ be the barycentric extension of $f$. Then $\tilde{f}$ is $L$-bilipschitz. Let $\gamma = \tilde{f}(l)$ and $l' = f(l)$. We have that $\gamma$ is a bilipschitz quasigeodesic with the same endpoints as $l'$. Denote by $\Omega_1$ and $\Omega_2$ the two regions obtained by removing $\gamma$ from $D$. By $H_1$ and $H_2$ we denote the corresponding halfspaces obtained by removing $l'$. Let $p : \gamma \to l$ be any bilipschitz map such that $d(p(z), z) < P$, $P = P(K)$. Denote by $\tilde{q}_1$ the barycentric extension of the map $q_1 : \partial \Omega_1 \to \partial H_1$ (here one has to map $\Omega_1$ and $H_1$ onto the unit by the Riemann maps to define the barycentric extensions). The map $q_1$ is defined to be the identity on the $T$ part of the boundary of $\Omega_1$ (this part of the boundary also borders $H_1$). We set $\tilde{q}_1 = p$ on $\gamma$. We do the analogous thing for $\Omega_2$ to obtain the map $\tilde{q}_2$. Define $\tilde{g} : D \to D$ so that $\tilde{g} = \tilde{q}_1$ on $\Omega_1$ and $\tilde{g} = \tilde{q}_2$ on $\Omega_2$. It follows from standard estimates about the boundary behavior of conformal maps and standard estimates on the barycentric extensions that the map $\tilde{f} = \tilde{g} \circ \tilde{f}$ satisfies the properties stated in the lemma. □

3.2. The barycentric extension. Let $\varphi : T \to T$ be a homeomorphism. The barycentric extension of Douady and Earle is a homeomorphism $\tilde{\varphi} : D \to D$ that extends $\varphi$ and which maps the point $0$ to the barycenter $\tilde{\varphi}(0)$ of the corresponding probability measure $\varphi_*(\sigma_0)$. Here $\sigma_0$ is the normalized Lebesgue measure on $T$ and $\varphi_*(\sigma_0)$ means the push forward of $\sigma_0$ by $\varphi$. It is also customary to write $\tilde{\varphi}(0) = \text{Bar}(\varphi_*, \sigma_0)$. By requiring that $\tilde{\varphi}$ be conformally natural, the above uniquely determines this extension. See [1], [1], [21] for definitions and properties of the barycentric extension.

If $\varphi$ is quasisymmetric, then $\tilde{\varphi}$ is known to be quasiconformal. There are many conformally natural extensions that have this property. One important thing about this particular extension is that one can say what happens with $\tilde{\varphi}$ when $\varphi$ degenerates. The theory of barycentric extensions can be developed not only for homeomorphisms of $T$ but also for limits of homeomorphisms and for certain probability measures on $T$. This case is of interest to us. In fact, if $\mu$ is a probability measure on $T$ that does not contain strong atoms, then the barycenter $\tilde{\mu}(0) \in D$ is well defined. An atom for a probability measure is said to be strong if it has mass at least $\frac{1}{2}$.

However, the only results we need in this direction are known and they all follow from [1] (they were hinted in [1]). One of the results that will be used later is the following. Let $\mu_n$ be a sequence of probability measures on $T$ such that none of
these measures has atoms. Suppose that $\mu$ is a probability measure that has no atoms of mass $\geq \frac{1}{2}$. If $\mu_n \to \mu$ (in the sense of measures), then the barycenters $\hat{\mu}_n(0) \to \hat{\mu}(0) \in D$. This observation was already stated in [4], but it was developed in detail in Sections 3 and 4 of [1].

Another result that we need is to show that under certain assumptions the barycentric extension $\hat{\varphi}$ is quasiconformal in a neighborhood of 0, even though the homeomorphism (or a limit of homeomorphisms) $\varphi$ is not. The following lemma follows from the so-called four sines law, and both the statement and the proof of this lemma can be found in [1].

**Lemma 3.6.** Let $\varphi : T \to T$ be a homeomorphism, and let $u : R \to R$ be an increasing function such that $\varphi(e^{it}) = e^{i u(t)}$, where $t \in R$. Suppose that there are $0 < s_0 < \pi$ and $C > 0$, so that $u(t+s) - u(t) > C$, for every $0 < s_0 < s < \pi$ and every $t \in R$. Then, there exist $K = K(s_0, C)$ and $D = D(s_0, C)$ such that the barycentric extension $\hat{\varphi}$ is $K$-quasiconformal in the hyperbolic disc $\Delta(0, D)$.

**Proof.** Following [21], we have

$$1 - |\text{Belt}(\hat{\varphi})(0)|^2 > \frac{\delta^3}{16}.\tag{3.2}$$

Here $\text{Belt}(\hat{\varphi})(0)$ is the complex dilatation of $\hat{\varphi}$ at 0, and

$$\delta = \frac{1}{2\pi^2} \int_0^\pi \sin(s) \left( \int_0^\pi v(t,s) dt \right) ds = \frac{1}{2\pi^2} \int_0^{\pi} \sin(s) v(t,s) ds dt.\tag{3.3}$$

Here

$$v(t,s) = \sin(\theta_1(t,s)) + \sin(\theta_2(t,s)) + \sin(\theta_3(t,s)) + \sin(\theta_4(t,s)),$$

for $\theta_1(t,s) = u(t+s) - u(t)$; $\theta_2(t,s) = u(t+2\pi) - u(t+s+\pi)$; $\theta_3(t,s) = u(t+s+\pi) - u(t+\pi)$; $\theta_4(t,s) = u(t+\pi) - u(t+s)$. Note that $\theta_1(t,s) + \theta_2(t,s) + \theta_3(t,s) + \theta_4(t,s) = 2\pi$, and $\theta_i(t,s) \geq 0$, $i = 1, 2, 3, 4$.

It follows (see Lemma 4.4 in [1]) from basic trigonometry that

$$v(t,s) = 4\sin\left(\frac{\theta_1 + \theta_2}{2}\right) \sin\left(\frac{\theta_1 + \theta_4}{2}\right) \sin\left(\frac{\theta_2 + \theta_4}{2}\right) \geq 0.\tag{3.4}$$

The above formula holds for any choice of three out of the four $\theta_i$’s. Let $s_1 = \pi - s_0$. For $0 < s < s_1$, we have that both $\theta_2(t,s)$ and $\theta_4(t,s)$ are greater or equal to $C$. Let $P_0 = \{s,t : 0 < s < s_1, 0 < t < \pi\}$. Let $s_2 = \frac{\pi}{N}$. Since every interval of length at least $\pi - s_1 = s_0$ is mapped by $u$ onto a set that contains some interval of length $C$, we conclude that for some $t_0 \in (\frac{\pi}{N}, \pi - s_0)$, we have

$$\theta_1(t_0, s_2) > \frac{C}{N},$$

where $N = N(s_0)$ is the minimal positive integer such that $s_2 > \frac{\pi}{N}$. Set $s_3 = \frac{\pi}{N}$. Then for every $(t,s) \in P_1 = \{(t,s) : 0 < t_0 - s_3 < t < t_0$ and $s_2 + s_3 < \frac{5\pi}{16} < s < \pi\}$, we have

$$\theta_1(t,s) > \frac{C}{N}.\tag{3.5}$$

Since the area of $P_1$ is greater than $\frac{\pi}{N} s_3$ and since $\theta_2(t,s), \theta_4(t,s) > C$, we conclude from (3.3), (3.4), and (3.5) that there is $\delta_0 = \delta_0(s_0, C) > 0$, such that $\delta \geq \delta_0$. The rest follows from (3.2).
We need to show that $\widehat{\psi}$ is quasiconformal at the points close to the origin. We do that by bringing them back to the origin (by M"obius transformations) and then using the same argument as above. Let $a_z \in \mathcal{M}$ be such that $\widehat{a_z}(0) = z$ and $\widehat{a_z}$ preserves the geodesic that contains both 0 and $z$. Let $\varphi_z = \varphi \circ a_z$. Then, one can choose $D = D(s_0, C) > 0$ small enough such that for $z \in \Delta(0, D)$ the map $\varphi_z$ satisfies the assumption of this lemma, where instead of $s_0$ we take $s_0 + \frac{\pi - 2\pi}{2}$. By repeating the same argument, we conclude the proof of the lemma.  

4. SMALL ELEMENTS IN DISCRETE QUASISYMMETRIC GROUPS

4.1. Discrete groups generated by small elements are cyclic. Let $\mathcal{G}$ be a $K$-quasisymmetric group. For each $f \in \mathcal{G}$, let $\psi$ be a $K$-quasisymmetric map and let $u \in \mathcal{M}$, such that $f = \psi \circ u \circ \psi^{-1}$. Let $\widehat{\psi}$ denote a $\widehat{K}$-quasiconformal extension of $\psi$, $K = \widehat{K}(K)$. Set $\hat{f} = \hat{\psi} \circ \hat{u} \circ \hat{\psi}^{-1}$. The $\hat{f}$ is a $K^2$-quasiconformal map that extends $f$. For $z \in \mathbb{D}$ and for $f \in \mathcal{G}$, if $f$ is not elliptic, let

$$P_f(z, \widehat{\psi}) = d(\hat{\psi}^{-1}(z), \hat{u}(\hat{\psi}^{-1}(z))).$$

Set

$$P_f(z) = \sup_{\widehat{\psi}} P_f(z, \widehat{\psi}),$$

where the supremum is being taken with respect to all such $\widehat{K}$-quasiconformal maps that fix 1, i, −1, and all corresponding $u \in \mathcal{M}$. Because of the compactness, there are $u \in \mathcal{M}$ and $\widehat{\psi}$ such that the supremum above is attained. Since we only consider $P_f$ for non-elliptic $f \in \mathcal{G}$, it is proper to say that $P_f(z)$ is the $\widehat{K}$ distance (or just the distance) between $f$ and the identity when seen from the point $z$.

Remark. Here we consider only non-elliptic elements because of the nature of the subsequent applications. However, one can naturally define the notion of being small for elliptic elements.

Recall that if $u$ is a hyperbolic M"obius transformation, then we define its length as

$$\min_{z \in \mathbb{D}} d(z, \hat{u}(z)).$$

This minimum is attained for every point $z \in \mathbb{D}$ that belongs to the axis of the transformation $\hat{u}$. Let $f \in \mathcal{G}$ be hyperbolic. Let $L_f(\psi)$ be the length of the corresponding hyperbolic transformation $u$. Set $L_f = \sup_{\psi} L_f(\psi)$, where the supremum is being taken with respect to all such $\widehat{K}$-quasiconformal maps and all corresponding $u \in \mathcal{M}$. We say that $L_f$ is the $\widehat{K}$ length (or just the length). For $\epsilon > 0$ we say that $f$ is $\epsilon$-small if $L_f \leq \epsilon$.

Remark. When we say that an element $f \in \mathcal{G}$ is $\epsilon$-small, that implies that we are talking about a hyperbolic element.

Let $z \in \mathbb{D}$. We say that $z$ is moved by the hyperbolic distance $d \geq 0$ under $\hat{u}$ if $d(z, \hat{u}(z)) = d$. Suppose that $f$ is either hyperbolic or parabolic (the same is true for the corresponding $u \in \mathcal{M}$ that is a conjugate of $f$). Then the set of points in $\mathbb{D}$ that are moved for some fixed hyperbolic distance under $\hat{u}$ is either the geodesic (or an equidistant line) that connects the fixed points of $u$ in the case when $u$ is hyperbolic or it is a horocircle if $u$ is parabolic. If $\widehat{\psi}$ is a fixed $\widehat{K}$-quasiconformal extension of a map $\psi$ that conjugates $f$ to some $u \in \mathcal{M}$, then we will call the image (under
\[\hat{\psi}\) of the corresponding set, respectively, the quasigeodesic, the quasiequidistant, the quasihorocircle, all of them with respect to \(\hat{\psi}\). Let \(R\) be a symmetric crescent around the axis of a hyperbolic \(u \in \mathcal{M}\). Set \(U = \hat{\psi}(R)\). We say that \(U\) is a quasicrescent. If \(l\) is the quasiequidistant that borders \(U\) from either side, then we say that \(U\) is determined by \(l\). Similarly, let \(H\) be a horoball that touches \(T\) at \(x\), which is a fixed point of the parabolic transformation \(u\). Set \(U = \hat{\psi}(H)\). We say that \(U\) is a quasihoroball. If \(l\) is the quasihorocircle that borders \(U\), then we say that \(U\) is determined by \(l\).

**Lemma 4.1.** With the notation as above, the following holds. Let \(f \in \mathcal{G}\) such that \(f\) is non-elliptic.

1. For \(v \in \mathcal{M}\) set \(\mathcal{G}_v = v\mathcal{G}v^{-1}\). Then \(P_f(z) = P_g(\hat{\psi}(z))\) and \(L_f = L_g\), where \(g = v \circ f \circ v^{-1}\).
2. Let \(l\) be a quasiequidistant (quasihorocircle) and \(z \in l\) such that \(P_f(z) \leq \epsilon\). Then there is a function \(c(\epsilon) = c(K)(\epsilon) > 0\), \(c(\epsilon) \to 0\) when \(\epsilon \to 0\), such that for any \(w \in U\) \((U\) is the quasicrescent (quasihoroball) that corresponds to \(l\)), we have \(P_f(w) \leq c(\epsilon)\).
3. There exists \(c_1(\epsilon) = c_1(K)(\epsilon) > 0\) such that \(c_1(\epsilon) \to \infty\) when \(\epsilon \to 0\) and such that if \(P_f(z) \leq \epsilon\), then \(P_f(w) \leq 2\epsilon\) for every \(w \in D\), with \(d(z, w) \leq c_1(\epsilon)\).
4. Let \(f \in \mathcal{G}\), \(L_f \leq \epsilon\). Then there is \(c_2(\epsilon) = c_2(K)(\epsilon) > 0\), \(c_2(\epsilon) \to 0\) when \(\epsilon \to 0\), such that if \(g\) is a conjugate of \(f\) in \(\mathcal{G}\), then \(L_g \leq c_2(\epsilon)\).
5. Let \(f \in \mathcal{G}\) be parabolic. Then for every \(\epsilon > 0\) there is a quasihoroball \(U\) for \(f\) such that \(P_f(z) \leq \epsilon\), for \(z \in U\). Also, for every \(r > 0\), there exists a quasihoroball \(U\) such that \(P_f(z) > r\), \(z \in D - U\).

**Proof.** Item (1) follows from the definitions of \(P_f\) and \(L_f\). We prove (2). Let \(U\) be a quasicrescent with respect to a \(K\)-quasiconformal map \(\hat{\psi}_1\), and let \(u_1 \in \mathcal{M}\) be the corresponding hyperbolic transformation such that \(U = \hat{\psi}_1(R)\) for some symmetric crescent \(R\) that corresponds to \(u_1\). Let \(w \in U\). Let \(u_0 \in \mathcal{M}\) and \(\psi_0\) be such that \(P_f(w, \psi_0) = d(\psi_0^{-1}(w), \psi_0(\psi_0^{-1}(w))) = P_f(w)\). Let \(R'\) be the symmetric crescent with respect to \(\psi_0\) such that \(w\) belongs to one of the two boundary equidistant lines of \(R'\). Then, if \(\psi_0^{-1}(z') \in R'\), we have that \(\hat{\psi}_0^{-1}(z)\) is a bounded hyperbolic distance away (the upper bound depends only on \(K\) because the distance between \(\hat{\psi}_1\) and \(\psi_0\) depends only on \(K\)) from the equidistant that contains \(\psi_0^{-1}(w)\). This shows that

\[P_f(w) \leq qP_f(z, \hat{\psi}_0) \leq qP_f(z),\]

for some \(q = q(K) > 0\). If \(\hat{\psi}_0^{-1}(z)\) does not belong to \(R'\), we have \(P_f(w) \leq P_f(z)\). Either way, this yields (2).

Items (3), (4), and (5) are proved in a very similar way, and they, as does (2), all follow from the basic estimates on the distortion of the hyperbolic metric under \(K\)-quasiconformal maps. We omit the details. \(\square\)

For a given \(K\)-quasisymmetric group \(\mathcal{G}\) and \(\epsilon > 0\), let \(\mathcal{G}\)\(_{\epsilon}\) be the group generated by all \(f \in \mathcal{G}\) (\(f\) is not elliptic) such that \(P_f(z) < \epsilon\).

**Lemma 4.2.** There exists \(\epsilon(K) > 0\) with the following properties. Let \(\mathcal{G}\) be a \(K\)-quasisymmetric group so that every elliptic element in \(\mathcal{G}\) (if any) is of order two. Then the group \(\mathcal{G}_{\epsilon}(K)\) is either cyclic or it contains only the identity.
Remark. In fact, our proof works even if we assume that the order of any elliptic element in \( \mathcal{G} \) is less than some fixed constant \( C \geq 2 \). However, then the constant \( \epsilon(K) \) would depend on that constant \( C \) as well. Although \( \mathcal{G}_z \) is generated by hyperbolic and/or parabolic elements, the group \( \mathcal{G}_z \) could contain elliptic elements. In our case, the order of every such element is two.

Proof. We show that the group \( \mathcal{G}_z \) is Abelian. Note that every Abelian Fuchsian group is cyclic. We give a proof by contradiction. Assume that there is a sequence \( z_n \in \mathbb{D} \) we have that neither of the groups \( \mathcal{G}_{z_n} \) is Abelian. Here \( n \in \mathbb{N} \). Therefore, the group \( \mathcal{G}_{z_n} \) has at least two generators \( f_n, g_n \in \mathcal{G} \) that do not commute (here we used the fact that every Abelian Fuchsian group, and therefore every quasisymmetric group as well, is cyclic). We have

\[
P_{f_n}(z_n), P_{g_n}(z_n) \to 0,
\]
as \( n \to \infty \), and neither \( f_n \) nor \( g_n \) is elliptic. Set \( g_n \circ f_n \circ g_n^{-1} = f'_n \in \mathcal{G}_n \). Since \( f_n \) and \( g_n \) do not commute, we have that \( f'_n \) and \( f_n \) do not commute. Also, by Lemma 4.1 we have that \( P_{f'_n}(z_n) \to 0 \) as \( n \to \infty \). By replacing \( g_n \) by \( f'_n \) if necessary, we can assume that both \( f_n \) and \( g_n \) are either hyperbolic or both are parabolic. Denote by \( \mathcal{G}_n \subset \mathcal{G}_{z_n} \) the group generated by \( f_n, g_n \). Since \( \mathcal{G}_n \) is a sequence of \( K \)-quasisymmetric groups, we have that the geometric limit \( G \) (of the sequence \( G_n \)) is a \( K \)-quasiconformal group.

Remark. Here by the geometric limit we mean the following. A homeomorphism \( h : T \to T \) belongs to \( G \) if for every \( n \in \mathbb{N} \) we can choose \( h_n \in \mathcal{G}_n \) such that \( h_n \to h \) in the \( C^0 \) sense on \( T \). Clearly \( G \) is a \( K \)-quasisymmetric group.

Our aim is to show that the geometric limit of \( G_n \) is a non-elementary and non-discrete \( K \)-quasisymmetric group. However, since this limit cannot contain elliptic elements of arbitrarily high order, we will obtain a contradiction (see [17]). Here we use the fact that a non-discrete \( K \)-quasisymmetric group is a conjugate of a subgroup of \( \mathcal{M} \). For each \( n \in \mathbb{N} \) fix \( K \)-quasisymmetric normalized maps \( \psi_n \) and \( \phi_n \) such that there are \( u_n, v_n \in \mathcal{M} \), so that \( f_n = \psi_n \circ u_n \circ \psi_n^{-1} \) and \( g_n = \phi_n \circ v_n \circ \phi_n^{-1} \). By \( \overline{\psi}_n \) and \( \overline{\phi}_n \) we denote a choice of \( K \)-quasiconformal extensions of \( \psi_n \) and \( \phi_n \), respectively.

First we consider the case when both \( f_n, g_n \) are hyperbolic. By \( l_n \) and \( s_n \) we denote the corresponding quasigeodesics of \( f_n \) and \( g_n \), with respect to \( \overline{\psi}_n \) and \( \overline{\phi}_n \), respectively. Suppose first that there is \( d_0 > 0 \) such that \( 0 \leq d(l_n, s_n) < d_0 \), for every \( n \) (this includes the case when \( l_n \) and \( s_n \) intersect transversally). By \( (x_n, y_n) \) and \( (x'_n, y'_n) \) we denote the ordered pairs of the repelling and the attracting fixed points of \( f_n \) and \( g_n \), respectively. Since they do not commute, we have that either \( x'_n \) or \( y'_n \) is disjoint from the set \( \{x_n, y_n\} \).

Remark. In fact, since the group \( \mathcal{G} \) is discrete, it is known (see [20]) that non-commuting hyperbolic elements cannot fix the same point.

From Lemma 4.1 we have that the lengths of both \( f_n \) and \( g_n \) tend to \( 0 \) as \( n \to \infty \). Conjugating the whole group \( G_n \) by a M"obius transformation (which does not change the \( K \) lengths of hyperbolic elements nor the \( K \) distance from the identity), we can assume that \( x_n = -i, y_n = i \), and either that \( x'_n = -1 \) or \( x'_n = 1 \). Choose \( x'_n = -1 \). Since \( d(l_n, s_n) < d_0 \), we have that \( y'_n \), after passing to a subsequence if necessary, tends to a point \( y \in T \), and \( y \neq x'_n = -1 \). Denote this
we conclude that the geometric limit of the sequence $G_{m_n}$, and let $F$ be a Fuchsian group that is conjugate to $G$. We have that the cyclic group generated by $f_{m_n}$ tends to a one-parameter, hyperbolic, $K$-quasisymmetric group with the fixed points $-i, i$. We have a similar conclusion for $g_{m_n}$. Since $F$ contains two one-parameter, hyperbolic groups that do not have the same fixed points, we have that $F$ is neither discrete nor elementary. But the only elliptic elements $F$ may contain are of order two, which is a contradiction.

Now, suppose that $d(l_n, s_n) \to \infty$. Let $p_n$ be the quasiequidistant (for $f_n$ and with respect to $\hat{\psi}_n$) that contains $z_n$. If the hyperbolic distance between $p_n$ and $s_n$ stays bounded (including the case when $p_n$ and $s_n$ intersect), as $n \to \infty$, then we choose a quasiequidistant $p'_n$ (for $f_n$) with the following properties: $p'_n$ is between $p_n$ and $l_n$, the distances $d(p'_n, l_n)$ and $d(p'_n, s_n)$ both tend to $\infty$, and for every $z \in p'_n$ we have that $P_{g_n}(z) \to 0$, $n \to \infty$. We can make this choice because $d(l_n, s_n) \to \infty$ and by Lemma 4.1. Note that $P_{f_n}(z) \to 0$ for every $z \in p'_n$, because $p'_n$ is between $p_n$ and $l_n$ (this follows from Lemma 4.1). Now, let $q_n$ be a quasiequidistant (for $g_n$ and with respect to $\hat{\phi}_n$) such that $p'_n \cap q_n$ is non-empty and such that the interior of the quasicrescents that correspond to $p'_n$ and $q_n$, respectively, have empty intersection. Let $w_n$ be any point in $p'_n \cap q_n$. We have that both $P_{f_n}(w_n)$ and $P_{g_n}(w_n)$ tend to 0. By Lemma 4.1 we can assume that $w_n = 0$.

Let $n \to \infty$. Since $d(l_n, 0) \to \infty$, we conclude that both fixed points of $f_n$ tend to a single point $x \in T$. Similarly, both fixed points for $g_n$ tend to a single point $y \in T$. We want to show that $x \neq y$. By passing to a subsequence if necessary, we may assume that $\hat{\psi}_n$ and $\hat{\phi}_n$ converge to $\hat{K}$-quasiconformal maps $\hat{\psi}$ and $\hat{\phi}$. The crescents that correspond to $p'_n$ and $q_n$ converge to quasihorocircles, with respect to $\hat{\psi}$ and $\hat{\phi}$, respectively. Note that both of these horocircles contain the origin 0. Since the interiors of the corresponding quasihoroballs have no points in common, we conclude that $x \neq y$. Similarly as above, after choosing a proper subsequence, we conclude that the geometric limit of $G_n$ (or a properly chosen subsequence), contains two one-parameter parabolic groups that do not have the same fixed point. This implies that this limit group is neither discrete nor elementary. But the only elliptic elements it may contain are of order two, which is a contradiction.

The case when both $f_n$ and $g_n$ are both parabolic is almost identical to the case when $d(l_n, s_n) \to \infty$ above. The only difference is that instead of quasiequidistants we use quasihorocircles.

We have shown that there exists $\epsilon(K) > 0$ such that the corresponding group $G_z$ is cyclic. \qed

The above results are analogues of the corresponding results for Fuchsian groups which state that two non-commuting (hyperbolic or parabolic) elements of a Fuchsian group cannot both move a given point for a very small distance. Some of the main results of this sort for Fuchsian groups are the Jørgensen inequality and the Margulis lemma. This lemma is one of the central results in the theory of Lie groups, and it generalizes these results for Fuchsian groups to discrete lattices in Lie groups (see [21]).

**Lemma 4.3.** Let $\mathcal{F}$ be a Fuchsian group. Suppose that $u, v \in \mathcal{F}$ are elliptic elements of order at least three. Then there is $\epsilon > 0$ such that for $z, w \in D$, $z \neq w$, that are the fixed points of $\tilde{u}$ and $\tilde{v}$, respectively, we have $d(z, w) > \epsilon$. The constant $\epsilon$ is universal (it does not depend on the choice of $\mathcal{F}$ or the elements $u, v \in \mathcal{F}$).
4.2. Removing small hyperbolic elements.

**Theorem 4.1.** There exists $\epsilon(K) > 0$ with the following properties. For an arbitrary $K$-quasisymmetric group $G$ that does not contain any elliptic elements of order three or more, there exist $K_1$-quasisymmetric groups $G_i$, $i \in \mathbb{N}$, $K_1 = K_1(K)$, such that the following hold.

1. None of the groups $G_i$ contain any $\epsilon(K)$-small hyperbolic elements nor any elliptic elements of order three or more.

2. If every $G_i$ is $K'$-quasisymmetrically conjugated to a Fuchsian group, $K' = K'(K)$, then there exists $K'' = K''(K)$ such that $G$ is $K''$-quasisymmetrically conjugated to a Fuchsian group.

**Proof.** Let $G$ be an arbitrary $K$-quasisymmetric group. We will construct the groups $G_i$. Providing that these groups are quasisymmetrically conjugated to the corresponding Fuchsian groups, we construct a homomorphism $E$ of $G$ into the group of quasiconformal selfmaps of $D$ such that $E$ is an isomorphism onto its image.

Let $\epsilon > 0$, and let $h_1, \ldots, h_\infty \in G$, $i \in I$, be the list of all primitive, mutually non-conjugate, $\epsilon$-small hyperbolic elements. Here $I$ is either the set $1, 2, \ldots, n$, for some $n \in \mathbb{N}$, or $I = \mathbb{N}$, depending on whether there are finitely or infinitely many $h_i$. Recall that $h_i$ is a primitive element means that $h_i$ is not a power of another element in $G$. Set $[h_i] = \bigcup_{k \in \mathbb{Z}} \bigcup_{f \in G} f^{-1} \circ h^k \circ f$. In each class $[h_i]$ fix one representative that is primitive, say $p_i$. Let $x_i, y_i$ denote, respectively, the repelling and the attracting fixed points of $h_i$. Let $\text{Stab}_i$ be the subgroup of $G$ whose elements fix the set $\{x_i, y_i\}$. This is an elementary group and therefore there exist an elementary Fuchsian group $\text{Stab}_i'$ and a $\tilde{K}$-quasisymmetric map $\psi_i : T \to T$, which conjugates $\text{Stab}_i'$ and $\text{Stab}_i$.

We can choose $\psi_i$ so that $\psi_i(x_i) = x_i$, $\psi_i(y_i) = y_i$. Let $\hat{\psi}_i$ be some $\tilde{K}$-quasiconformal extension of $\psi_i$. Let $E_i(1)$ be the symmetric crescent of hyperbolic width 1 around the geodesic that connects $x_i$ and $y_i$, and set $U_i(1) = \hat{\psi}_i(E_i(1))$.

Now fix $i \in I$. For every $h \in \text{Stab}_i$ put $\hat{h} = \hat{\psi}_i \circ \hat{u}_i \circ \hat{\psi}_i^{-1}$, where $u_i \in \text{Stab}_i'$ such that $h = \psi_i \circ u_i \circ \psi_i^{-1}$. For $f, g \in G$, we say that $f \sim g$, with respect to the pair $(x_i, y_i)$, if $f(x_i) = g(x_i)$ and $f(y_i) = g(y_i)$. Denote the corresponding equivalence class by $[f]_i$. Note that for $f \sim h$, where $h \in \text{Stab}_i$, we have already defined the extension $f$ because each such $f$ must be in the corresponding stabilizer $\text{Stab}_i$. In every other equivalence class choose one representative, say $f$, and let $\hat{f}$ be a fixed $K$-quasiconformal extension. For every other $g \in [f]_i$, there exists
$h \in \text{Stab}_i$ such that $f^{-1} \circ g = h$. Set $\hat{g} = \hat{f} \circ \hat{h}$. Therefore, for $f \in \mathcal{G}$ and for fixed $i \in I$, we have defined the extension $\hat{f}$, which is $K^3$-quasiconformal. Set $[U_i(1)] = \bigcup_{f \in \mathcal{G}} \hat{f}(U_i(1))$. Now repeat this process for each $i$. Note that formally we should denote the extension $\hat{f}$ by $\hat{f}_i$, because a separate extension is defined for every $i \in I$. We avoid doing this to simplify the notation, and no confusion should arise.

By Lemma 4.1 and Lemma 4.2 there exists $\epsilon_1(K) > 0$ small enough that if we set $\epsilon = \epsilon_1(K)$, then for every $f \in \mathcal{G}$ we have

$$d(\hat{f}(U_j(1)), \hat{g}(U_k(1))) > 1,$$

whenever $\hat{f}(U_j(1))$ and $\hat{g}(U_k(1))$ do not touch the unit circle $T$ at the same points (otherwise, for every $\epsilon > 0$, there would be $z \in D$ such that $\mathcal{G}_z$ is not cyclic). Note that this choice of $\epsilon$ yields that $f(x_i) = g(x_i)$ implies $f(y_i) = g(y_i)$, which we already know to be true.

Let $R = R(K)$ be the constant from Lemma 3.1. By the same argument as above, we can choose $\epsilon_2(K) > 0$ and the symmetric crescent $E_i$ ($E_i$ contains the crescent $E_i(1)$) with the following properties. Set $U_i = \hat{v}_i(E_i)$ and $[U_i] = \bigcup_{f \in \mathcal{G}} \hat{f}(U_i)$. If $f \in \text{Stab}_i$, then

$$\epsilon_2(K) < d(z, \hat{f}(z)),$$

for any $z \in \partial U_i$. If $f, g \in \mathcal{G}$ are such that $f(x_j) \neq g(x_k)$ (or equivalently $f(y_j) \neq g(y_k)$), where $j, k \in I$ are any two numbers ($j, k$ may be equal), then

$$d(\hat{f}(U_j), \hat{g}(U_k)) > R.$$

Set $\Omega = D - \bigcup [U_i]$. We have that $\Omega$ is a disconnected set, but any connected component of $\Omega$ is simply connected. This follows from (4.2). Since $\mathcal{G}$ is not an elementary group, there are infinitely many connected components of $\Omega$. If $\Omega_0$ is a connected component of $\Omega$, then $\Omega_0$ is uniquely determined by its boundary points that lie in $T$. Therefore, if $f \in \mathcal{G}$, we can properly define the connected component $f, \Omega_0$ of $\Omega$. We denote by $\Omega_j, j \in I'$, mutually non-conjugated components such that every other component is in one of the classes $[\Omega_j] = \bigcup_{f \in \mathcal{G}} f(\Omega_0)$. Here $I'$ is the range for $j$, and it may be either a finite set or $I' = N$ (see Figure 2).

For $f \in \mathcal{G}$, we first define $E(f)$ on each $[U_i]$. Let $U_i'$ be a component of $[U_i]$ ($U_i'$ touches $T$ at $x', y'$). Set $x_i'' = f(x_i), y_i'' = f(y_i)$. Let $g', g'' \in \mathcal{G}$ be such that $g'(x_i) = x_i'$, $g'(y_i) = y_i'$, and $g''(x_i) = x_i''$, $g''(y_i) = y_i''$ (such $g', g'' \in \mathcal{G}$ are not uniquely determined). Then, there exists $h \in \text{Stab}_i$ such that $g'' \circ h \circ g'^{-1} = f$. Set

$$\left(g'' \circ \hat{h} \circ \hat{g}^{-1}\right)(z) = E(f)(z),$$

for $z \in U_i'$. This defines $E(f)$ on $\bigcup [U_i] = D - \Omega$. It follows from the definition of the corresponding extensions of $g', g''$ that $E(f)$ is well defined.

**Lemma 4.4.** $E(f)$ is a homeomorphism of $\bigcup [U_i]$. Also,

1. $E(f) \circ E(g) = E(f \circ g)$; $E(id) = id$
2. for every $f \in \mathcal{G}$, there exist $K_1 = K_1(K)$ and a $K_1$-quasiconformal map $\tilde{f}$ which extends $f$ such that $\tilde{f} = E(f)$ on $\bigcup [U_i] = D - \Omega$.

**Proof.** Item (1) and the fact that $E(f)$ is a homeomorphism follow from the definition of $E$ and the discussion above. On each component of $[U_i]$, the map $E(f)$ is a
restriction of a $\tilde{K}^8$-quasiconformal map $\tilde{g}'' \circ \tilde{h} \circ \tilde{g}^{-1}$. Combining this with (4.2), it follows from Lemma 3.1 that there exists the map $\tilde{f}$ with the stated properties. □

Next, we define $E$ on the rest of $D$. In each class $[\Omega_j], \ j \in \Gamma'$, fix one representative, say $\Omega_j$. For $f \in G, \ f_*\Omega_j = \Omega_j$, let $e(f)$ be the restriction of the map $E(f)$ on $\partial \Omega_j$. Let $St_j$ be the group of all maps $e(f)$. The group $St_j$ is isomorphic to the subgroup of $G$ that contains all $f$ so that $f_*\Omega_j = \Omega_j$. From Lemma 4.4 it follows that $St_j$ is a $K_1$-quasisymmetric group on $\Omega_j$. By conjugating the group $St_j$ via the boundary values of the Riemann map that maps $\Omega_j$ to the unit disc, we obtain the $K_1$-quasisymmetric group $G_j$. We get from (4.1), from Lemma 4.1, and from the definition of $\Omega$ that there is an $\epsilon_3(K) > 0$ such that $G_j$ does not contain $\epsilon_3(K)$-small hyperbolic elements, for some $\epsilon_3(K) > 0$. Note that for a hyperbolic $f \in G$ such that $e(f) \in St_j$, we have that $e(f)$ is hyperbolic. It follows from the definition that the length of $f$ is greater than or equal to the corresponding length of $e(f)$. This follows from the fact that a conformal map of $D$ to a subset of $D$ is a contraction with respect to the hyperbolic metric on $D$. By repeating this for every $j$, we obtain the collection of groups from the statement of this theorem. It follows from the construction that none of them has elliptic elements of order three or more (because $G$ does not have any such elliptic elements).

If we assume that $G_j$ is $K'$-quasiconformally conjugated to a Fuchsian group, then there exist $K_2 = K_2(K)$, a Fuchsian group $St'_j$ ($St'_j$ acts on $D$ of course), and a $K_2$-quasiconformal map $\psi_j : D \to \Omega_j$ such that the induced map $\psi_j : T \to \partial \Omega_j$ conjugates the groups $St'_j$ and $St_j$. 

**Figure 2.**
Now fix $i \in I$. For every $c(f) \in S_{t_{i}}$, we put $\hat{f} = \hat{\psi}_{j} \circ \hat{u}_{i} \circ \hat{\psi}_{j}^{-1}$, where $u \in S_{t_{i}}$, such that $c(f) = \psi_{j} \circ u \circ \psi_{j}^{-1}$. For $f, g \in G$, we say that $f \sim g$ with respect to $\Omega_{j}$, if $f_{*}(\Omega_{j}) = g_{*}(\Omega_{j})$. Denote the corresponding equivalence class by $[f]_{j}$. Note that for $f \sim h$, where $e(h) \in S_{t_{i}}$, we have already defined the extension $\hat{f}$, because for each such $f$, the corresponding map $\varepsilon(f)$ must be in the stabilizer $S_{t_{i}}$.

In every other equivalence class, choose one representative, say $f$, and let $\hat{f}$ be the $K_{1}$-quasiconformal map from Lemma 4.4 (note that $\hat{f}$ agrees with $E(f)$ on $\bigcup U_{i}$).

For every other $g \in [f]_{j}$, there exists $h \in S_{t_{i}}$ such that $f^{-1} \circ g = h$. Set $\hat{g} = \hat{f} \circ h$. Therefore, for $f \in G$, we have defined the extension $\hat{f}$ which is $K_{3}$-quasiconformal, $K_{3} = K_{3}(K)$.

Let $\Omega_{j}$ be a component of $[\Omega_{j}]$. Set $\Omega'_{j} = \phi_{*}\Omega'_{j}$. Let $g', g'' \in G$ be such that $g'_{*}\Omega_{j} = \Omega'_{j}$ and $g''_{*}\Omega_{j} = \Omega''_{j}$ (such $g', g'' \in G$ are not uniquely determined). Then, there is an $h \in G$, $h_{*}\Omega_{j} = \Omega_{j}$, such that $g'' \circ h \circ g'^{-1} = f$. Set $E(f)(z) = (g'' \circ h \circ g'^{-1})(z)$, for $z \in \Omega'_{j}$. This defines $E(f)$ on $\Omega$. It is clear from the above discussion that $E$ satisfies all the required properties. In particular there exists $K_{1}' = K_{1}'(K)$ such that $E(G)$ is a $K_{1}'$-quasiconformal group, which implies that $G$ is $K''$-quasisymmetrically conjugated to a Fuchsian group, for some $K'' = K''(K)$. This completes the proof of Theorem 4.1.

5. Hyperbolic quasisymmetric groups

Throughout this section $G$ denotes a $K$-quasisymmetric group all of whose elements are hyperbolic. We also assume that for some $0 < \epsilon(K) < \epsilon(K)$, $G$ does not contain any $\epsilon(K)$-small elements. The constant $\epsilon(K)$ is from Theorem 4.1. We show that such a group is a quasisymmetric conjugate of a Fuchsian group.

5.1. The pants-annuli decomposition of a quasisymmetric group. Recall that if $\mathcal{F}$ is a Fuchsian group, then $\mathcal{F}$ acts on $T$, and the corresponding group that acts on $D$ is denoted by $\hat{\mathcal{F}}$. We first assume that $G$ is finitely generated. Let $\mathcal{F}$ be a Fuchsian group and $\varphi : T \to T$ a homeomorphism such that $G = \varphi \mathcal{F} \varphi^{-1}$. Then $\mathcal{F}$ is finitely generated, and therefore $D/\hat{\mathcal{F}} = S$ is a topologically finite Riemann surface. This means that $S$ is biholomorphic to a closed Riemann surface of finite genus with at most finitely many discs removed.

Remark. The only reason why we temporarily assumed that $G$ is finitely generated is that we can more clearly describe the pants-annuli decomposition of the surface $S$ (see [24]). A similar decomposition is valid for all Riemann surfaces, but we choose to work with the finitely generated case and when needed, we apply Proposition 1.3.

Consider the induced hyperbolic metric on $S$ (this metric agrees with the hyperbolic metric that $S$ inherits from $D/\hat{\mathcal{F}} = S$). We recall the standard decomposition of a topologically finite Riemann surface and the corresponding lift into the universal cover. Let $S_{0}$ be the convex core of $S$ (if $S$ is closed, then $S_{0} = S$). $S$ is a union of $S_{0}$, the simple closed geodesics (in the future just geodesics) that represent the boundary of $S_{0}$, and the annuli (each annulus is bounded by a geodesic which is also a boundary component of $S_{0}$ and a boundary component of $S$). Now, cut up
the surface \( S_0 \) into pairs of pants (recall that a pair of pants is an open Riemann surface which is biholomorphic to the Riemann sphere minus three discs). We have that \( S \) is a disjoint union of the pairs of pants, the annuli, and the geodesics that border the pairs of pants (some of them also border the annuli if they exist). Denote by \( \tilde{P}_i, 1 \leq i \leq n_1 \), the pairs of pants; by \( \tilde{A}_i, 1 \leq i \leq n_2 \), the annuli; and by \( \tilde{l}_i, 1 \leq i \leq n_3 \), the geodesics that border the pairs of pants in this decomposition of \( S \). Here, \( n_1, n_2, n_3 \in \mathbb{N} \) depend on the Euler number of \( S \) and the number of free boundary components of \( S \). We denote by \([\tilde{P}_i], [\tilde{A}_i]\), and \([\tilde{l}_i]\) the totality of the corresponding lifts (the \( i \)'s run throughout the corresponding ranges) in \( D \).

Each connected component in \( D \) of a fixed \( [\tilde{P}_i] \) is a convex subset of \( D \), whose relative boundary (in \( D \)) is contained in the collection of geodesics which are the lifts of geodesics on \( S \) that border \( \tilde{P}_i \) (this is a subset of the union of \([\tilde{l}_i]\), \( 1 \leq i \leq n_3 \), which we denote by \( \bigcup [\tilde{l}_i] \)). We also denote by \( \tilde{P}_i \) the closure of a fixed single lift of a given pair of pants \( \tilde{P}_i \) in \( S \). Here \( \tilde{P}_i \) is the closure of a chosen fundamental region for the corresponding pair of pants. We choose it so that \( \tilde{P}_i \) is a right-angled octagon. Four sides of this octagon are contained in four geodesics from \( \bigcup [\tilde{l}_i] \), and the other four sides are geodesic arcs that are orthogonal to the first four sides, so that all together they complete the boundary of the right-angled octagon \( \tilde{P}_i \). Each \([\tilde{P}_i]\) is equal to the union \( \bigcup \tilde{u}(\tilde{P}_i) \), where \( \tilde{u} \in \mathcal{F} \) (this union is disjoint except that the points which belong to certain sides of \( \tilde{P}_i \) may occur more than once in this union). Note that a fixed (connected) component of \([\tilde{P}_i]\) contains many copies of \( \tilde{P}_i \) (see Figure 3).

A connected component of a fixed \( [\tilde{A}_i] \) is a hyperbolic halfspace bounded by a lift of the geodesic in \( S \) that borders \( \tilde{A}_i \). We also denote by \( \tilde{A}_i \) the closure of a fixed single lift of a given annulus \( \tilde{A}_i \subset S \). Again, \( \tilde{A}_i \) is the closure of a chosen fundamental domain of the corresponding annulus. We choose it so that \( \tilde{A}_i \) is a right-angled parallelogram. One of its sides is contained in a geodesic from \( \bigcup [\tilde{l}_i] \), its opposite side is an arc contained in \( T \), and the remaining two sides are geodesic rays that connect the first two sides to complete the boundary of the right-angled parallelogram \( \tilde{A}_i \). Note that each \([\tilde{A}_i]\) is the union of \( \tilde{u}(\tilde{A}_i) \), where \( \tilde{u} \in \mathcal{F} \) (this union is disjoint except that points in certain sides of \( \tilde{A}_i \) appear twice). Note that a fixed (connected) component of \([\tilde{A}_i]\) contains many copies of \( \tilde{A}_i \).

A connected component of a fixed \( [\tilde{l}_i] \) is a geodesic in \( D \). We also denote by \( \tilde{l}_i \) a fixed connected component (or just a component) of \([\tilde{l}_i]\) (note that this is much more than a fundamental domain of the corresponding closed geodesic on \( S \)). In this case we also have that \([\tilde{l}_i]\) is equal to the union \( \bigcup_{\tilde{u} \in \mathcal{F}} \tilde{u}(\tilde{l}_i) = \bigcup_{u \in \mathcal{F}} u_\infty(\tilde{l}_i) \), but this union is not disjoint. Note that to each \( \tilde{l}_i \) corresponds an infinite cyclic subgroup of the group \( \mathcal{F} \), which consists of the elements of \( \mathcal{F} \) that fix its endpoints. We denote the generator of this subgroup by \( \tilde{h}_l \). Also, if \( u \) is an element of \( \mathcal{F} \) which fixes the endpoints of \( \tilde{l}_i \), then \( u \) belongs to this cyclic subgroup. \( \mathcal{F} \) has only hyperbolic elements; therefore, no element of \( \mathcal{F} \) can permute the endpoints of \( \tilde{l}_i \). Note that no two geodesics from \( \bigcup [\tilde{l}_i] \) intersect.

In the remainder of this section \( \tilde{P}_i, \tilde{A}_i \) will always refer to a choice of single lifts (described above) of the corresponding pair of pants and annulus, while \( \tilde{l}_i \) will stand for a fixed component in \([\tilde{l}_i]\). Now, fix \( \tilde{l}_i \). Since \( \varphi \) is a homeomorphism, we define
the geodesic $l_i$ by setting $l_i = \varphi_*(\widetilde{l}_i)$. We set $[l_i] = \bigcup_{f \in \mathcal{G}} f \ast (l_i) = \bigcup_{u \in \mathcal{F}} \varphi_*(u \ast (\widetilde{l}_i))$. Also, to each $l_i$ corresponds an infinite cyclic subgroup of $\mathcal{G}$ whose elements fix its endpoints. We denote the generator of this subgroup by $h_{l_i}$. We have
\[ h_{l_i} = \varphi \circ \widetilde{h}_{l_i} \circ \varphi^{-1}. \]

Note that if $f(l_i) = l_i$, for some $f \in \mathcal{G}$, then because $\mathcal{G}$ contains only hyperbolic elements, we have that $f$ belongs to the cyclic subgroup generated by $h_{l_i}$. Similarly as above, no element of $\mathcal{G}$ can permute the endpoints of $l_i$. Again, no two geodesics in $\bigcup [l_i]$ intersect.

5.2. Extending the action of $\mathcal{G}$ to the unit disc. Our aim is to extend the action of $\mathcal{G}$ to $D$. We will define a homomorphism $E$ (which is an isomorphism onto its image) of the group $\mathcal{G}$ into the group of quasiisometries of $D$, so that for each $f \in \mathcal{G}$, $E(f)$ extends $f$ to $D$. The image of the group $\mathcal{G}$ under $E$ is denoted simply by $E(\mathcal{G})$. We do this in steps. First we define $E(f)$ for every $f \in \mathcal{G}$, on every $[l_i]$, $1 \leq i \leq n_3$, and then on every $[P_i]$ and $[A_i]$ (see below for the definition of $[P_i]$ and $[A_i]$).

Fix $i \in [1, n_3]$, and let $l_i$ be a fixed component of the set $[l_i]$. Recall that $h_{l_i}$ is the generator of the cyclic group that fixes the endpoints of $l_i$. Let $\psi$ be a $K$-quasisymmetric map and $u \in \mathcal{M}$ such that $\psi \circ u \circ \psi^{-1} = h_{l_i}$. Let $\widehat{\psi}$ be a $K'$-quasiconformal and bilipschitz extension of $\psi$. Set $\widehat{\psi} \circ \widehat{u} \circ \widehat{\psi}^{-1} = \widehat{h}_{l_i}$. Since $\widehat{\psi}$ is bilipschitz, we conclude that $\widehat{h}_{l_i}$ is bilipschitz, where the corresponding constant is a function of $K$. For $f, g \in \mathcal{G}$, we say $f \sim g$ if $f_*(l_i) = g_*(l_i)$. Denote by $[f]$, the corresponding equivalence class. Fix one $f \in [f]$. Let $\tilde{f}$ be a $K'_1$-quasiconformal map, $K'_1 = K'_1(K)$, which extends $f$ and which maps $l_i$ onto the geodesic $f_\ast l_i$ (such exists by Lemma 3.5). We also assume that $\tilde{f}$ is bilipschitz, where the corresponding
constant is a function of $K$. For an arbitrary $g \in [f]$, there is $k \in \mathbb{Z}$, such that $f^{-1} \circ g = h_k^k$. Set $\tilde{g} = \tilde{f} \circ \tilde{h}_i^k$. Therefore, for each $f \in \mathcal{G}$, we have defined a $K_1$-quasiconformal extension, $K_1 = K_1(K)$. Now, repeat this for every $i$.

Let $f$ be an arbitrary element of $\mathcal{G}$, and let $l_i'$ be a component from $[l_i]$ ($l_i'$ is not necessarily equal to $l_i$), which is fixed). Let $l_i'' = f_*(l_i')$. Choose $g', g'' \in \mathcal{G}$ such that $g'(l_i) = l_i'$ and $g''(l_i) = l_i''$ (such $g', g''$ are not unique). Since $(g''^{-1} \circ f \circ g')_*(l_i) = l_i$, there exists $k \in \mathbb{Z}$ such that $g''^{-1} \circ f \circ g' = h_k^k$. We set

$$E_1(f)(z) = \left(\tilde{g} \circ \tilde{h}_i^k \circ \tilde{g}^{-1}\right)(z)$$

for $z \in l_i'$. Repeat this for every $i$. In this way we have defined the mapping $E_1(f)$ for every point in every geodesic from any of the $[l_i]$, $1 \leq i \leq n$. Since no two geodesics from $\bigcup [l_i]$ intersect, $E_1(f)$ is well defined. Also, this definition does not depend on the choice of $g', g''$ because of the way we have defined the corresponding extensions.

We need to slightly modify $E_1(f)$ (to obtain $E(f)$), $f \in \mathcal{G}$, so that $E(f)$ is $\delta(\epsilon)$-continuous on $\bigcup [l_i]$, where $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ is a function that is itself a function of $K$. We have already shown that the restriction of $E_1(f)$ on every geodesic from $\bigcup [l_i]$ is bilipschitz. Let $\epsilon_0, \epsilon_1 > 0$ such that $0 < \epsilon_1 < \epsilon_0 < \frac{1}{2}$. Let $l, l'$ be two geodesics from $\bigcup [l_i]$ such that $d(l, l') < \epsilon_0$. Let $\alpha \subset l$ be a subarc of $l$ such that for $z \in (l - \alpha)$ we have that the distance between $z$ and $l'$ is greater than $\epsilon_1$. We similarly define $\alpha' \subset l'$. Denote by $h, h' \in \mathcal{G}$ the corresponding elements that fix the endpoints of $l$ and $l'$, respectively ($h, h'$ generate the corresponding cyclic groups). Then there exists $L_0 = L_0(\epsilon_0, K)$ such that the lengths of $h$ and $h'$ are greater than $L_0$. Moreover, we have that $L_0 \to \infty$, for $\epsilon_0 \to 0$ (if not, we would have that the geodesic $h, l' \in \bigcup [l_i]$ intersects the geodesic $l'$). This implies that we can choose $0 < \epsilon_1 < \epsilon_0$, both of them being functions of $K$, such that $E_1(h)(\alpha)$ is disjoint from $\alpha$. Similarly, $E_1(h')(\alpha')$ is disjoint from $\alpha'$. For $\epsilon_1$ small enough, the hyperbolic length of both $\alpha$ and $\alpha'$ is greater than 1. Now fix $\epsilon_0, \epsilon_1$ with the above properties.

For $f \in \mathcal{G}$, let $\alpha_f = E_1(f)(\alpha)$, $\alpha'_f = E_1(f)(\alpha')$. Since $f$ is $K$-quasisymmetric and since the value of $E_1(f)$ on $l$ is a restriction of a bilipschitz map (see (5.1)) of $\mathbb{D}$ onto itself, it follows that there exists $\epsilon_2 = \epsilon_2(K)$ such that for any $z \in (f, l - \alpha_f)$ we have that $d(z, f, l') > \epsilon_2$. We have the analogous conclusion for $\alpha'_f$.

We now define a new map $E(f)$ that agrees with $E_1(f)$ on $\bigcup [l_i] - \bigcup_{f \in \mathcal{G}} \alpha'_f$. Let $I_f : \alpha_f \to \alpha'_f$ be the affine map that maps each endpoint of $\alpha_f$ to the closer of the two endpoints of $\alpha'_f$. If $\alpha_f$ is very long, then so is $\alpha'_f$, and $I_f$ is nearly an isometry. In any case $I_f$ is bilipschitz (with the constant that is a function of $K$) regardless of the length of $\alpha_f$ (since the hyperbolic length of $\alpha_f$ is at least one, we conclude that the length of $\alpha_f$ is not too small). For $f \in \mathcal{G}$ and $z \in \alpha' = \alpha'_d$ set

$$E(f)(z) = (I_f \circ E_1(f) \circ I_{\alpha_d}^{-1})(z).$$

Let $g \in \mathcal{G}$, and let $z \in \alpha'_f$, for some $f \in \mathcal{G}$. Set

$$E(g)(z) = E(g \circ f) \circ E(f)^{-1}.$$
By this, we have defined the extension $E(f)$, $f \in \mathcal{G}$, which is a modification of $E_1(f)$. It follows from the construction that $E(f)$ is $\delta(\epsilon)$-continuous, where $\delta(\epsilon)$ is a function of $K$.

**Lemma 5.1.** Let $f, g \in \mathcal{G}$ and let $id$ denote the identity mapping. There exist $L = L(K)$, $\delta : \mathbb{R}^+ \to \mathbb{R}^+$, $\rho = \rho(K)$, and an $L$-quasiisometry $\tilde{f}$ which extends $f$ such that

1. $E(f)$ is a homeomorphism of every geodesic from $\bigcup[l_i]$;
2. $E(f)$ is $\delta(\epsilon)$-continuous on $\bigcup[l_i]$, where $\delta(\epsilon)$ is a function of $K$;
3. for every $z \in \bigcup[l_i]$, $f \in \mathcal{G}$, we have $d(E(f)(z), z) > \rho$;
4. $E(f \circ g) = E(f) \circ E(g)$, and $E(id) = id$;
5. $\tilde{f}$ agrees with $E(f)$ on $\bigcup[l_i]$;
6. $\tilde{f}$ is $\delta(\epsilon)$-continuous on $D$.

**Proof.** It follows from (5.1) that $E_1(f \circ g) = E_1(f) \circ E_1(g)$ and $E_1(id) = id$. The modification preserves this property. We have already proved that $E(f)$ is a uniformly continuous homeomorphisms of $\bigcup[l_i]$. The existence of the quasiisometry $\tilde{f}$ follows from Lemma 2.2.

Let $l \in \bigcup[l_i]$, and let $h \in \mathcal{G}$ be the corresponding hyperbolic element. Let $z \in l$. If $E(f)(z)$ is very close to $z$, then $z$ cannot be in $l$ since $h$ cannot be $\epsilon(K)$-small. This implies that $f$ does not preserve $l$. Let $\tilde{l}' = f_*l$ be the geodesic from $\bigcup[l_i]$ that contains $E(z)$. Since $l$ and $\tilde{l}'$ are disjoint, we have that the corresponding (repelling and attracting, respectively) endpoints of $l$ and $\tilde{l}'$ are very close to each other (because $z$ and $E(z)$ are very close). But this implies again that the length of $f$ is very small, which is a contradiction (recall that we assume that $\mathcal{G}$ does not contain any $\bar{c}(K)$-small hyperbolic elements). This shows the existence of $\rho$. \[\square\]

Next, we define $E(f)$ on the rest of the unit disc. To do this, we first need to define an appropriate extension $\tilde{\varphi} : D \to D$ of the map $\varphi$. Since no two geodesics from $\bigcup[\tilde{l}_i]$ intersect and since $\varphi_*((\tilde{l}_i)) = l_i$, we can choose a homeomorphism $\tilde{\varphi}$ so that $\tilde{\varphi}((\tilde{l}_i)) = l_i$ and so that the equality

\[\tilde{\varphi} \circ \tilde{h}_{l_i} = h_{l_i} \circ \tilde{\varphi}\]

holds on $\tilde{l}_i$, for every $\tilde{l}_i$. Because of (5.2) one can arrange that for every geodesic arc $\alpha$ that is a side of some octagon $\tilde{P}_i' \in [\tilde{P}_i]$, we have that $\tilde{\varphi}(\alpha)$ is also a geodesic arc (if $\alpha$ is contained in one of the geodesics from $\bigcup[\tilde{l}_i]$, then this is already the case). This directly follows from the fact that the endpoints of $\alpha$ belong to two geodesics from $\bigcup[l_i]$, and no two such sides can intersect (except that they can meet at the endpoints which are always in $\bigcup[\tilde{l}_i]$). Similarly, if $\alpha$ is a side in the parallelogram $\tilde{A}_i' \in [\tilde{A}_i]$, we can arrange that $\tilde{\varphi}(\alpha)$ is a geodesic arc, geodesic ray, or a circular arc in $T$, depending on which one of these is $\alpha$.

We set $\tilde{\varphi}((\tilde{P}_i)) = P_i$ and $\tilde{\varphi}((\tilde{A}_i)) = A_i$, for every $\tilde{P}_i$ ($i \in [1, n_1]$) and $\tilde{A}_i$ ($i \in [1, n_2]$). Note that each $P_i$ is an octagon and each $A_i$ is a parallelogram ($P_i$ and $A_i$ are not necessarily right-angled anymore). Also, $\tilde{\varphi}((\tilde{P}_i)) = [P_i]$ and $\tilde{\varphi}((\tilde{A}_i)) = [A_i]$. Since $D$ is a union of $\bigcup[\tilde{P}_i]$ and $\bigcup[\tilde{A}_i]$, we have that the same is true for $[P_i]$ and $[A_i]$.

**Lemma 5.2.** Let $f \in \mathcal{G}$ and fix $P_i$, for some $i \in [1, n_1]$. Then we can choose $L_1 = L_1(K)$, $\rho_1 = \rho_1(K)$, and an $L_1$-quasiisometry $\tilde{f}$, which extends $f$ such that the following hold.
(1) \( \hat{f} \) agrees with \( E(f) \) on \( \bigcup [l_i] \).
(2) \( \hat{f}(P_i) = \hat{\varphi}(\hat{u}(P_i)) \), where \( u = \varphi^{-1} \circ f \circ \varphi \).
(3) Let \( f \in \mathcal{G} \) be such that for the above-defined extension \( \hat{f} \) we have \( \hat{f}(P_i) \cap P_i = \beta \), where \( \beta \) is a side of \( P_i \). Then, \( (\hat{f}^{-1} \circ (\hat{f}^{-1})) (z) = z \), for \( z \in \beta \).
(4) \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous, where \( \delta_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function of \( K \).
(5) For every \( z \in D \), we have \( d(f(z), z) > \rho_1 \).

Also, for a fixed \( A_i, i \in \{1, n_2\} \), we can choose \( L_1 = L_1(K) \), \( \rho_1 = \rho_1(K) \), and a \( L_1 \)-quasiisometry \( \hat{f} \), which extends \( f \), such that the following hold.

(1) \( \hat{f} \) agrees with \( E(f) \) on \( \bigcup [l_i] \).
(2) \( \hat{f}(A_i) = \hat{\varphi}(\hat{u}(A_i)) \), where \( u = \varphi^{-1} \circ f \circ \varphi \).
(3) Let \( f \in \mathcal{G} \) be such that for the above-defined extension \( \hat{f} \) we have \( \hat{f}(A_i) \cap A_i = \beta \), where \( \beta \) is a side of \( A_i \). Then, \( (\hat{f}^{-1} \circ (\hat{f}^{-1})) (z) = z \), for \( z \in \beta \).
(4) \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous, where \( \delta_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function of \( K \).
(5) For every \( z \in D \), we have \( d(f(z), z) > \rho_1 \).

**Proof.** Let \( \hat{f} \) be the \( L \)-quasiisometry from Lemma 5.1. Here \( \hat{f} \) already agrees with \( E(f) \) on \( \bigcup [l_i] \). This implies that the octagons \( \hat{f}(P_i) \) and \( \hat{\varphi}(\hat{u}(P_i)) \) have the corresponding four sides in common. Now, by Lemma 2.2 we can choose an \( L_1 \)-quasiisometry \( \hat{f} \), which agrees with \( f \) on \( \bigcup [l_i] \), such that for a fixed \( P_i \) we have \( \hat{f}(P_i) = \hat{\varphi}(\hat{u}(P_i)) \) and such that \( \hat{f} \) is \( \delta_1(\epsilon) \)-continuous.

Let \( f \in \mathcal{G} \) be such that for the above-defined extension \( \hat{f} \) we have \( \hat{f}(P_i) \cap P_i = \beta \), where \( \beta \) is a side of \( P_i \). First we define the extension \( H_1 \) of \( f^{-1} \) which satisfies (1), (2), and (4) of this lemma. Then we post-compose \( H_1 \) with a homeomorphism \( H_2 \) of \( D \) which pointwise fixes the set \( \bigcup [l_i] \) and all the sides of \( P_i \) except \( \beta \). Since the hyperbolic distance between \( H_1 \) and \( (\hat{f})^{-1} \) is bounded above (by a bound which is a function of \( K \)), we can choose \( H_2 \) to be a quasiisometry and uniformly continuous, so that \( H_2 \circ H_1 \) satisfies (3).

Item (5) follows from Lemma 5.1. We proceed similarly to define the \( \hat{f} \) that corresponds to a fixed \( A_i \). \( \square \)

Let \( f \in \mathcal{G} \) and let \( z \in D \). If \( z \in [l_i] \), then we have already defined \( E(f)(z) \). Suppose \( z \in [P_i] \). Then there exists \( P_i' \subset [P_i] \) that contains \( z \). If \( z \) belongs to a side of the octagon \( P_i' \), then there will exist at least two and at most three (this can happen only in the case when \( z \) is in some \( [l_i] \) octagons from this partition that contain \( z \). Recall that four sides of the octagon \( P_i' \) lie in \( \bigcup [l_i] \). Since \( E(f) \) is defined there, \( E(f) \) uniquely determines the octagon \( P_i'' = E(f)(P_i') \) whose four sides are the images of the four sides of \( P_i' \) under \( E(f) \). In the same way we can find the elements \( g', g'' \in \mathcal{G} \) such that \( g''(P_i) = P_i'' \) and \( g''(P_i) = P_i'' \) (such \( g', g'' \) are unique). We define \( E(f)(z) \) by

\[
E(f)(z) = \left( g'' \circ g^{-1} \right) (z),
\]

where \( \hat{g}' \) and \( \hat{g}'' \) are the extensions from Lemma 5.2 of \( g' \) and \( g'' \), respectively (these extensions correspond to the fixed \( P_i \)). It follows from Lemma 5.2 that if \( z \) belongs to a side of \( P_i' \), then the definition of \( E(f)(z) \) does not depend on the choice of the octagon which contains \( z \). This, together with Lemma 5.1, shows that \( E(f) \)
is continuous and therefore a homeomorphism on every \([P_i]\). We similarly define \(E(f)\) on every \([A_i]\).

**Lemma 5.3.** Let \(f \in \mathcal{G}\). Then \(E(f)\) is an \(L_2\)-quasiisometry of \(D\), \(L_2 = L_2(K)\), and \(E : \mathcal{G} \to E(\mathcal{G})\) is an isomorphism. \(E(f)\) is \(\delta_2(\epsilon)\)-continuous, where \(\delta_2(\epsilon)\) is a function of \(K\). Also, there exist \(p_1 = p_1(K)\) such that \(d(E(f)(z), z) > p_1\).

**Proof.** The fact that \(E(f)\) is an isomorphism and \(\delta_2(\epsilon)\)-continuous follows from the definition of \(E\). It also follows from the definition of \(E\) that there exists \(L_2 = L_2^p(K)\) with the following properties. For every \(z \in D\), there is an \(L_2^p\)-quasiisometry \(\tilde{f}\) that extends \(f\) such that \(E(f)(z) = \tilde{f}(z)\) (at every step of the way we have always defined \(E\) to be a restriction of some quasiisometric map). The rest follows from Proposition 2.1, Lemma 5.1, and Lemma 5.2. □

**Theorem 5.1.** Let \(\mathcal{G}\) be a \(K\)-quasisymmetric group that does not contain any \(\tilde{c}(K)\)-small elements for some \(0 < \tilde{c}(K) < c(K)\) (\(c(K)\) is the constant from Theorem 4.1). Then there exist a Fuchsian group \(F\), \(K_3 = K_3(K)\), and a \(K_3\)-quasisymmetric mapping \(\varphi : T \to T\) such that \(\mathcal{G} = \varphi F\varphi^{-1}\).

**Proof.** If \(\mathcal{G}\) is finitely generated, we have established above that \(E(\mathcal{G})\) is an \(L_2\)-quasiisometric group (this means that every element of \(E(\mathcal{G})\) is an \(L_2\)-quasiisometry) so that \(d(E(f)(z), z) > p_1\) and \(E(f)\) is \(\delta_2(\epsilon)\)-continuous. By Lemma 2.3 there exists a \(K_3\)-quasisymmetric mapping \(\varphi : T \to T\) such that \(\mathcal{G} = \varphi F\varphi^{-1}\), for some Fuchsian group \(F\). The case of infinitely generated \(\mathcal{G}\) follows from Proposition 1.3. □

6. Torsion-Free Quasisymmetric Groups

Let \(\mathcal{G}\) be a torsion-free \(K\)-quasisymmetric group and assume that \(\mathcal{G}\) does not contain any \(\tilde{c}(K)\)-small elements, for some \(0 < \tilde{c}(K) < c(K)\). Here \(c(K)\) is the constant from Theorem 4.1. We show in this section that such a group is a quasisymmetric conjugate of a Fuchsian group. Our aim is to construct a homomorphism \(E\) (which is an isomorphism onto its image) of \(\mathcal{G}\) into the group of quasiconformal self-mappings of \(D\). We first assume that \(\mathcal{G}\) is finitely generated. Since \(\mathcal{G}\) is topologically conjugate to a Fuchsian group, we conclude that there are \(p_1, ..., p_n \in \mathcal{G}\), \(n \in \mathbb{N}\), mutually non-conjugate parabolic elements such that every other parabolic element of \(\mathcal{G}\) is contained in some conjugacy class \([p_i] = \bigcup_{k \in \mathbb{Z}} (\bigcup_{f \in \mathcal{G}} f^{-1} \circ p_i^k \circ f)\). Note that this implies that each \(p_i\) is primitive; that is, it is not a power of another element from \(\mathcal{G}\). Also, if \(f \in \mathcal{G}\) fixes the fixed point of some \(p_i\), then \(f\) belongs to the cyclic group generated by \(p_i\). In each class \([p_i]\) fix one representative that is primitive, say \(p_i\), and let \(x_i \in T\) be the point fixed by \(p_i\). There exists a parabolic Möbius transformation \(u_i\) and a \(K\)-quasisymmetric map \(\psi_i : T \to T\) such that \(\psi_i \circ u_i = p_i \circ \psi_i\) (also \(\psi_i(x_i) = x_i\)). Let \(\hat{\psi}_i\) be some \(K\)-quasiconformal extension of \(\psi_i\). Let \(H_i\) be a horoball that touches \(T\) at \(x_i\) and set \(U_i = \hat{\psi}_i(H_i)\).

Now, fix \(i \in [1, n]\). Let \(f = p_i^k\), for some \(k \in \mathbb{Z}\). Let \(\tilde{f}\) be a \(K^2\)-quasiconformal extension of \(f\) defined by \(\tilde{f} = \tilde{p}_i^k = \hat{\psi}_i \circ \tilde{u}_i^k \circ \hat{\psi}_i^{-1}\). For \(f, g \in \mathcal{G}\), we say that \(f \sim g\) with respect to \(x_i\), if \(f(x_i) = g(x_i)\). Denote the corresponding equivalence class by \([f]_i\). Note that for \(f \sim p_i\) we have already defined the extension \(\tilde{f}\) because each such \(f\) must be in the corresponding cyclic group. In every other equivalence class \([f]_i\), choose one representative, say \(f\), and let \(\tilde{f}\) be a fixed \(K\)-quasiconformal extension. For every other \(g \in [f]_i\), there exist \(k \in \mathbb{Z}\) such that \(f^{-1} \circ g = p_i^k\). Set \(\tilde{g} = \tilde{f} \circ \tilde{p}_i^k\).
Therefore, for \( f \in \mathcal{G} \) we have defined the extension \( \hat{f} \) which is \( \hat{K}^3 \)-quasiconformal. Set \( [U_i] = \bigcup_{f \in \mathcal{G}} \hat{f}(U_i) \). Now, repeat this process for each \( i \) (see Figure 4).

Let \( R = R(K) \) be the constant from Lemma 3.1. By Lemma 4.1 and Lemma 4.2, we can choose the horoballs \( H_i \), \( i \in [1, n] \), so that if \( f, g \in \mathcal{G} \) are such that \( f(x_j) \neq g(x_k) \), where \( j, k \in [1, n] \) are any two numbers \( (j, k \text{ might be equal}) \), then

\[
\text{d}(\hat{f}(U_j), \hat{g}(U_k)) > R.
\]

(6.1)

Let \( U' \) be a component of \( \bigcup [U_i] \), and let \( f \in \mathcal{G} \) be the parabolic element that fixes \( U' \) (such exists by construction). Since \( R \) is fixed and by choosing the horoballs properly, we can also arrange (see Lemma 4.1, item (5)) that

\[
\text{d}(\hat{f}(U_j), \hat{g}(U_k)) > R.
\]

(6.2)

Let \( U' \) be a component of \( \bigcup [U_i] \), and let \( f \in \mathcal{G} \) be the parabolic element that fixes \( U' \) (such exists by construction). Since \( R \) is fixed and by choosing the horoballs properly, we can also arrange (see Lemma 4.1, item (5)) that

\[
\text{d}(\hat{f}(U_j), \hat{g}(U_k)) > R.
\]

(6.1)

Let \( U' \) be a component of \( \bigcup [U_i] \), and let \( f \in \mathcal{G} \) be the parabolic element that fixes \( U' \) (such exists by construction). Since \( R \) is fixed and by choosing the horoballs properly, we can also arrange (see Lemma 4.1, item (5)) that

\[
\text{d}(\hat{f}(U_j), \hat{g}(U_k)) > R.
\]

(6.2)

for \( z \) from the quasihorocircle \( \partial U' \) and some constant \( \epsilon_1(K) > 0 \). Set \( \Omega = D - \bigcup [U_i] \).

It follows from (6.1) that \( \Omega \) is a simply connected region whose boundary consists of \( \mathcal{T} \) and \( \partial \bigcup [U_i] \).

For \( f \in \mathcal{G} \) we first define \( E(f) \) on each \( [U_i] \). Let \( U'_j \) be a component of \( [U_i] \) \( (U'_j \) touches \( \mathcal{T} \) at \( x'_j \). Set \( x''_j = f(x'_j) \). Let \( g', g'' \in \mathcal{G} \) be such that \( g'(x_j) = x'_j \) and \( g''(x_j) = x''_j \) (such exists by construction). Then, there is \( k \in \mathbb{Z} \) such that \( g'' \circ p_k \circ g'^{-1} = f \). Set

\[
\left( g'' \circ p_k \circ g'^{-1} \right)(z) = E(f)(z),
\]

for \( z \in U'_j \). Repeat this construction for every \( i \in [1, n] \).

This defines \( E(f) \) on \( \bigcup [U_i] = D - \Omega \). From the definition of the extensions \( \hat{g}' \) and \( \hat{g}'' \), it follows that \( E(f)(z) \) is well defined; that is, it does not depend on the choice of \( g', g'' \).

**Lemma 6.1.** \( E(f) \) is a homeomorphism of \( \bigcup [U_i] \) such that the following hold.

1. \( E(f) \circ E(g) = E(f \circ g) \); \( E(id) = id \).
2. For every \( f \in \mathcal{G} \) there exist \( K' = K'(K) \) and a \( K' \)-quasiconformal map \( \bar{f} : D \rightarrow D \), which extends \( f \), such that \( f = E(f) \) on \( \bigcup [U_i] = D - \Omega \).

**Proof.** Item (1) and the fact that \( E(f) \) is a homeomorphism follow from the definition of \( E \) and the discussion above. On each component of a given \( [U_i] \), the map \( E(f) \) is a restriction of a \( \hat{K}^3 \)-quasiconformal map \( \hat{g}'' \circ p_k \circ \hat{g}^{-1} \). Combining this with (6.1), it follows from Lemma 3.1 that there exists a map \( \hat{f} \) with the stated properties.

Let \( \mathcal{G}_1 \) be the group (isomorphic to the group \( \mathcal{G} \)) whose elements are the mappings \( e(f) : \partial \Omega \rightarrow \partial \Omega \), where \( e(f) \) is the restriction of \( E(f) \). Let \( \phi : \Omega \rightarrow D \) be the Riemann map. Then \( \phi \mathcal{G}_1 \phi^{-1} \) is a \( K' \)-quasiconsymmetric group, where \( K' \) is from Lemma 6.1. Note that the group \( \phi \mathcal{G}_1 \phi^{-1} \) contains only hyperbolic elements. Moreover, we have assumed that \( \mathcal{G} \) does not contain any \( \epsilon(K) \)-small elements. Then from (6.2) it follows that there is \( \epsilon(K) > 0 \) such that \( \phi \mathcal{G}_1 \phi^{-1} \) does not contain any \( \epsilon(K) \)-small elements. From Theorem 5.1 we conclude that the group \( \phi \mathcal{G}_1 \phi^{-1} \) is \( K_1 \)-quasisymmetrically conjugated to a Fuchsian group, \( K_1 = K_1(K) \). Conjugating the action of this Fuchsian group by the Riemann map, we get that for each \( f \in \mathcal{G} \) there is a \( K_1 \)-quasiconformal map \( e(f) : \Omega \rightarrow \Omega \), which extends \( e(f) \) to \( \Omega \) and such that the map \( e(f) \rightarrow e(\hat{f}) \) is an isomorphism of the group \( \mathcal{G}_1 \) onto a subgroup of
the group of quasiconformal selfmaps of $\Omega$. Set $E(f) = \hat{c}(f)$ on $\Omega$, for each $f \in \mathcal{G}$. Therefore, we have constructed an isomorphism of $\mathcal{G}$ onto a subgroup of the group of quasiconformal selfmaps of $\mathbb{D}$. Moreover, each $E(f)$ is $K_1'$-quasiconformal, where $K_1' = K_1'(K)$ is the maximum of the set $\{K_1, \tilde{K}_8\}$.

**Theorem 6.1.** Let $\mathcal{G}$ be a torsion-free $K$-quasisymmetric group that does not contain any $\hat{c}(K)$-small elements, for some $0 < \hat{c}(K) < c(K)$. Then there exist $K_2 = K_2(K)$ and a $K_2$-quasisymmetric map $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$ for some Fuchsian group $\mathcal{F}$.

**Proof.** If $\mathcal{G}$ is finitely generated, then the existence of a $K_2$-quasisymmetric map $\varphi$ and a Fuchsian group $\mathcal{F}$ follows from the fact that $E(\mathcal{G})$ is a $K_1'$-quasiconformal group. The infinitely generated case follows from Proposition 1.3. □

### 7. Elliptic elements of order greater than 2

#### AND THE BARYCENTRIC EXTENSION

**7.1. Quasisymmetric groups with elliptic elements of order at least three.**

For integer $m \geq 3$ let $u(z) = \exp\left(\frac{2\pi i}{m}\right)z$ be the rotation, $z \in \mathbb{T}$ (exp will sometime stand for the exponential function). Denote by $O_u$ the orbit of the point $z = 1$. Then, irrespective of the order $m$, we can choose three points $x_i, x_2, x_3 \in O_u$ such that $\frac{\pi}{m} < \sigma(x_i, x_j) < \pi - \frac{\pi}{m}$, for every $i \neq j$ ($\frac{\pi}{m}$ is not the best bound). Here $\sigma$ denotes the standard spherical metric on $\mathbb{T}$. By $l_\sigma$ we will denote the spherical length.

Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a homeomorphism, normalized by $\varphi(x_i) = x_i$, $i = 1, 2, 3$, such that the cyclic group generated by $\varphi \circ u \circ \varphi^{-1}$ is a $K$-quasisymmetric group (in this section the symbol $\varphi$ always refers to a homeomorphism with these properties).
From Proposition 1.2 we have that \( \varphi = \psi \circ \phi \), where \( \phi \) fixes pointwise the set \( O_u \) and \( \phi \) commutes with \( u \) (or, equivalently, \( \phi \) conjugates \( u \) to itself) and \( \psi \) is a \( \tilde{K} \)-quasisymmetric map. Clearly \( \psi(x_i) = x_i, i = 1, 2, 3 \).

Set \( l_1 = l_1(K) = \sup \lambda_\sigma(\psi(\alpha)) \), where the supremum is taken with respect to all arcs \( \alpha \subset T \) of length at most \( 2\pi - \frac{\pi}{13} \) and all \( \tilde{K} \)-quasisymmetric maps \( \psi : T \to T \) that fix three points, say \( x_1, x_2, x_3 \), such that \( \frac{\pi}{13} < \sigma(x_i, x_j) < \pi - \frac{\pi}{13}, i \neq j \). Since this family of \( \tilde{K} \)-quasisymmetric maps is a normal family, we have \( 0 < l_1 < 2\pi \). On the other hand, let \( \beta \subset T \) be an arc of length at least \( 2\pi - \frac{\pi}{13} \). Let \( l_2 = \inf_{m \geq 3} l_\sigma(\phi(\beta)) \), where the infimum is taken with respect to all such \( \beta \) and all homeomorphisms \( \phi \) which commute with the rotation \( u(z) = \exp\left(\frac{2\pi i}{m}\right)z \). Clearly \( l_2 > \pi \). This proves the next lemma.

**Lemma 7.1.** With the notation as above, we have the following. Let \( \alpha \) be an arc of spherical length at least \( l_1 \). Then the spherical length \( l_\sigma(\varphi^{-1}(\alpha)) \) is at least \( l_2 \). Here \( l_2 \) is a fixed constant (does not depend on \( K \)).

Set \( l' = l_2 - \left( \frac{l_2}{2}, \pi \right) \). Recall that for \( z \in D \), \( a_z \) is the hyperbolic Möbius transformation such that \( a_z(0) = z \) and which preserves the geodesic that contains both 0 and \( z \). Let \( d_0 > 0 \) be small enough such that for every \( z \in \Delta(0, d_0) \) we have \( l_\sigma(a_z(\alpha)) > l'_2 \), for every arc \( \alpha \) of length at least \( l_2 \).

Denote by \( \hat{\varphi} \) the barycentric extension of the homeomorphism \( \varphi \) defined above.

**Lemma 7.2.** With the notation as above, the following hold.

1. There exists \( r_0 = r_0(K) > 0 \) so that for every \( z \in \Delta(0, d_0) \) we have \( d(\hat{\varphi}(z), 0) < r_0 \).
2. \( \hat{\varphi} \) is \( K_1 \)-quasiconformal in \( \Delta(0, d_1) \). Here, \( d_1 = d_1(K), K_1 = K_1(K) \).

**Proof.** Let \( r_0 > 0 \) be such that for every \( z \in D \), \( d(z, 0) > r_0 \), there exists an arc \( \alpha \subset T \) of spherical length at least \( l_1 \) such that \( 0 < l_\sigma(a_z(\alpha)) < 1 - \frac{\pi}{l_2} \). Clearly \( r_0 < \infty \). Let \( w = \hat{\varphi}(z) \) for \( z \in \Delta(0, d_0) \). Then the barycentric extension of the map \( a_w^{-1} \circ \varphi \circ a_z \) is equal to \( \hat{a}_w^{-1} \circ \hat{\varphi} \circ \hat{a}_z \). Moreover \( (\hat{a}_w^{-1} \circ \hat{\varphi} \circ \hat{a}_z)(0) = 0 \). This implies (see [3] and [4])

\[
(7.1) \quad \int_T (a_\alpha^{-1} \circ \varphi \circ a_z)(\zeta)|d\zeta| = 0.
\]

Suppose that there exists \( z \in \Delta(0, d_0) \) such that \( d(\hat{\varphi}(z), 0) = d(w, 0) > r_0 \). There exists an arc \( \alpha \subset T \), \( l_\sigma(\alpha) \geq l_1 \), with the property that \( l_\sigma(a_w^{-1}(\alpha)) < 1 - \frac{\pi}{l_2} \). Let \( y \) be any point in \( a_w^{-1}(\alpha) \). Then for \( \beta = (\varphi \circ a_z)^{-1}(\alpha) \) we have

\[
(7.2) \quad \left| y \int_\beta |d\zeta| - \int_\beta (a_\alpha^{-1} \circ \varphi \circ a_z)(\zeta)|d\zeta| \right| < l_\sigma(\beta) \left( 1 - \frac{\pi}{l_2} \right) < 2(l_2 - \pi).
\]

Here we used that \( l_\sigma(\beta) > l_2 \). This follows from Lemma 7.1 and the definition of \( l_2 \). But then (7.2) contradicts (7.1). This proves (1).

Let \( v : R \to R \) be an increasing function such that \( \varphi(z) = \varphi(e^{vt}) = e^{ivt} \). Since \( \phi \) commutes with a rotation \( u \) of order at least three and since \( \psi \) is a normalized \( \tilde{K} \)-quasiconformal map (with fixed points \( x_1, x_2, x_3 \)), we conclude that there exists \( C = C(K) \) such that \( v(t + s) - v(t) > C \), for \( \frac{2\pi}{3} < s < \pi \), and every \( t \in R \). Now (2) follows from Lemma 3.5. \( \square \)
Let \( \mu \) be an element of the unit ball of the Banach space \( L^\infty(D) \) of essentially bounded measurable functions on \( D \). Set \( k_0 = \frac{K+1}{K-1} \). For \( t \in [0, k_0] \), let \( \tilde{\eta}_t : D \to D \) be the quasiconformal map with the complex dilatation \( t \mu \) and which fixes the points \( x_1, x_2, x_3 \) as above. Let \( \eta_t : T \to T \) be the map such that \( \tilde{\eta}_t \) extends \( \eta_t \). Set \( \tilde{\varphi}_t = \eta_t \circ \tilde{\varphi} \). By \( \tilde{\varphi}_t \) we denote the barycentric extension of \( \tilde{\varphi}_t \).

**Lemma 7.3.** With the notation as above, we have the following. Let \( h : [0, k_0] \to D \) be the curve \( h(t) = (\tilde{\eta}^{-1}_t \circ \tilde{\varphi}_t)(0), t \in [0, k_0] \). Then for every \( \epsilon > 0 \) there exists \( \delta(K, \epsilon) > 0 \) such that \( d(h(t), h(s)) < \epsilon \), for \( |t-s| < \delta(K, \epsilon) \). In other words, \( h(t) \) is \( \delta(K, \epsilon) \)-continuous.

**Proof.** It is enough to prove the statement of the lemma for the curve \( e(t) = \tilde{\varphi}_t(0) \). This follows from the fact that the family \( \tilde{\eta}_t \) is uniformly Hölder continuous with respect to both \( t \in [0, k_0] \) and the variable in any fixed compact set in \( D \) and that bound depends only on \( k_0 \) and that fixed compact set in \( D \). It follows from the proof below that \( \tilde{\varphi}_t(0) \) does not leave the disc \( \Delta(0, r_0) \).

Proof by contradiction. Suppose that for some fixed \( \epsilon > 0 \) there exist sequences \( \{ \varphi_n \} \) and \( \{ \mu_n \}, n \in \mathbb{N} \), with the above properties and such that for each \( n \) there are \( t_n, s_n \in [0, k_0] \), \( |t_n-s_n| < \frac{1}{2^n} \), so that

\[
(7.3) \quad d(e_n(t_n), e_n(s_n)) \geq \epsilon.
\]

Here, \( e_n : [0, k_0] \to D \) denotes the curve \( e_n(t) = \tilde{\varphi}_{n,t}(0) \), where \( \varphi_{n,t} = \eta_{n,t} \circ \varphi_n \), and \( \eta_{n,t} \) is the normalized (fixing \( x_1, x_2, x_3 \)) quasisymmetric map that can be extended to the normalized quasiconformal map with the complex dilatation \( t \mu_n \). Let \( n \to \infty \).

Then we can assume that \( t_n, s_n \) tend to \( t_0 \in [0, k_0] \). After passing to a subsequence if necessary, the mappings \( \eta_{n,t_n}, \eta_{n,s_n} \) converge to a \( K \)-quasisymmetric map \( \eta_\infty \) which fixes \( x_1, x_2, x_3 \). Write \( \varphi_n = \psi_n \circ \phi_n \). Here \( \phi_n \) fixed pointwise the set \( O_n \) and it commutes with \( u_n \), and \( \psi_n \) is a \( K \)-quasisymmetric map which fixes the points \( x_1, x_2, x_3 \in O_n \) satisfying \( \frac{1}{n} < \alpha(x_i, x_j) < \pi - \frac{1}{n} \) for every \( i \neq j \). Here, for every \( n \in \mathbb{N}, u_n \) is the rotation of order \( m_n \geq 3 \). After passing to a subsequence if necessary, \( \psi_n \) converges to a normalized \( K \)-quasisymmetric map \( \psi_\infty \).

Consider the sequence \( \phi_{n, s_0} \sigma_0 \) of the corresponding probability measures on \( T \). Here \( \sigma_0 \) denotes the normalized Lebesgue measure on \( T \) (we have already used \( \sigma \) to denote the ordinary (non-normalized) Lebesgue measure on \( T \)). We have the induced sequences \( \varphi_{n,t_n,s_0} \sigma_0 \) and \( \varphi_{n,s_n,s_0} \sigma_0 \) of probability measures. After passing to a subsequence if necessary, \( \phi_{n,s_0} \sigma_0 \) converges to a probability measure \( \phi_\infty \) on \( T \). Since \( \phi_n \) commutes with a standard rotation of order at least three, we conclude that if \( \phi_\infty \) has atoms, then \( \phi_\infty \) has at least three atoms of the same mass, and therefore the mass of every atom is at most \( \frac{1}{3} \).

**Remark.** The above observation is the key observation of this section. It is not valid if the order of the rotation is allowed to be two, because then \( \phi_\infty \) can contain strong atoms.

This implies that both \( \varphi_{n,t_n,s_0} \sigma_0 \) and \( \varphi_{n,s_n,s_0} \sigma_0 \) converge to the probability measure \( \varphi_\infty = (\eta_\infty \circ \psi_\infty) \cdot \phi_\infty \). We have seen that every atom (if any) of \( \varphi_\infty \) has mass at most \( \frac{1}{3} \). Therefore, the sequences \( \tilde{\varphi}_{n,t_n}(0) = \text{Bar}(\varphi_{n,t_n,s_0}) \) and \( \tilde{\varphi}_{n,s_n}(0) = \text{Bar}(\varphi_{n,s_n,s_0}) \) both converge to the point \( \tilde{\varphi}_\infty(0) \). Here \( \text{Bar} \) stands for the barycenter of the corresponding measure. Note that since the measure \( \varphi_\infty \) has no strong atoms, we have that \( \tilde{\varphi}_\infty(0) \in D \). On the other hand, from Lemma 7.2 we know that the points \( \tilde{\varphi}_{n,t_n}(0) \) and \( \tilde{\varphi}_{n,s_n}(0) \) are contained in \( \Delta(0, r_1) \), \( r_1 = r_1(K) \). Therefore
the sequences \( \{ \tilde{\varphi}_{n,t_{n}}(0) \} \), \( \{ \tilde{\varphi}_{n,s_{n}}(0) \} \), after passing to a subsequence, converge to the same point in the closure \( \Delta(0,r_{1}) \). But this contradicts (7.3).

\[ \square \]

7.2. Removing elliptic elements of order at least three. We are in a position to prove the following theorem.

**Theorem 7.1.** For an arbitrary \( K \)-quasisymmetric group \( G \) there exists a \( K_{1} \)-quasisymmetric group \( G_{1} \), \( K_{1} = K_{1}(K) \), with the following properties.

1. \( G_{1} \) does not contain elliptic elements of order three or more.
2. If \( G_{1} \) is \( K_{1}' \)-quasisymmetrically conjugated to a Fuchsian group, \( K' = K'(K) \), then there exists \( K'' = K''(K) \) such that \( G \) is \( K'' \)-quasisymmetrically conjugated to a Fuchsian group.

**Proof.** Let \( G \) be an arbitrary \( K \)-quasisymmetric group. Let \( F \) be a Fuchsian group and let \( \varphi : T \to T \) be a homeomorphism, so that \( \varphi F \varphi^{-1} = G \). Denote by \( E' \) the set of all points in \( D \) such that \( z \in E' \) if \( \tilde{\varphi}(z) = z \), for some elliptic element \( \tilde{\varphi} \in \tilde{\mathcal{F}} \) of order at least three. We have by Lemma 4.3 that \( E' \) is a \( \rho_{0} \)-discrete set for some universal constant \( \rho_{0} > 0 \). Let \( \tilde{\varphi} \) be the barycentric extension of \( \varphi \). Set \( E = \tilde{\varphi}(E') \) and \( S = D - E \). Clearly, \( E \) is a discrete subset of \( D \), and \( S \) is a Riemann surface.

The group of homeomorphisms \( \tilde{G} = \tilde{\varphi}^{-1} \tilde{\mathcal{F}} \tilde{\varphi}^{-1} \) naturally acts on \( S \). Denote by \( \tilde{G}' \) the subgroup of the mapping class group of \( S \) induced by \( \tilde{G} \). We have that \( \tilde{G}' \) is isomorphic to \( G \) (because \( G \) is). We show that \( \tilde{G}' \) is a \( K_{1} \)-quasisymmetric group.

Fix \( f \in \tilde{G} \) and let \( \tilde{f} = \tilde{\varphi} \circ \tilde{v} \circ \tilde{\varphi}^{-1} \) where \( v \in \mathcal{F} \) such that \( \varphi \circ v \circ \varphi^{-1} = f \).

We need to show that \( \tilde{f} \) is isotopic (as a selfmap of \( S \)) to a \( K_{1} \)-quasiconformal map. Let \( \eta : T \to T \) be a \( K \)-quasisymmetric map such that \( \eta \circ f \circ \eta^{-1} = \omega \) for some \( \omega \in \mathcal{M} \). Because of the conformal naturality of the barycentric extension, we conclude that \( \tilde{f} \) is isotopic (rel \( \partial S \)) to the map \( A^{-1} \circ \tilde{\varphi} \circ A \). Here \( A = B \circ \tilde{\varphi}^{-1} \), where \( B \) is the barycentric extension of \( \eta \circ \varphi \). So, it is enough to show that \( A \) is a \( K_{1}' \)-quasisymmetric map of \( S \), \( K_{1}' = K'(K) \) (note that \( A \) does not have to map \( S \) onto itself). Let \( \mu' \in L^{\infty}(D) \) be the complex dilatation of \( \eta \) and set \( \mu = \frac{\mu'}{\mu_{1}} \).

For \( t \in [0,k_{0}] \), let \( \tilde{\eta}_{t} : D \to D \) be the quasiconformal map with the complex dilatation \( t \mu \) and which fixes the points \( 1, -1, 0 \). Let \( \tilde{\eta}_{t} : T \to T \) be the map such that \( \tilde{\eta}_{t} \) extends \( \eta_{t} \). Here \( t \in [0,k_{0}] \), \( k_{0} = \frac{K-1}{K+1} \). Set \( \varphi_{t} = \eta_{t} \circ \varphi \), and let \( \tilde{\varphi}_{t} \) be the barycentric extension of \( \varphi_{t} \). Finally, let \( A_{t} = \tilde{\varphi}_{t} \circ \tilde{\varphi}^{-1} \) and \( E_{t} = A_{t}(E) \). Note that \( A = A_{t_{0}} \). From Lemma 7.2 (item (2)) and Lemma 4.3 we have that \( E_{t} \) is a \( \rho(K) \)-discrete set, \( t \in [0,k_{0}] \). Consider the map \( \tilde{\eta}_{t}^{-1} \circ A_{t} \). Note that the restriction of \( \tilde{\eta}_{t}^{-1} \circ A_{t} \) on \( T \) is the identity. After properly pre-composing and post-composing the map \( \tilde{\eta}_{t}^{-1} \circ A_{t} \) by Möbius transformations, it follows from Lemma 7.3 that this map satisfies the assumptions of Lemma 3.3. This proves that \( f \) is isotopic (rel \( \partial S \)) to a \( K_{1} \)-quasiconformal map.

Now, we cover the surface \( S \) by the unit disc and lift the action of \( \tilde{G}' \) to the unit disc. This lifted group \( \tilde{G}'' \) is a \( K_{2} \)-quasisymmetric group of \( T \). Moreover, \( \tilde{G}'' \) does not have any elliptic elements of order three or more. Note that there is a natural homomorphism \( N : \tilde{G}'' \to \tilde{G}' \), and the kernel \( Ker_{N} \) of this homomorphism is the group of covering transformations from the covering of \( S \). Define \( \tilde{G}_{1} = \tilde{G}'' (G_{1}) \) (the group from the statement of this theorem). We have that \( \tilde{G}_{1} \) satisfies (1).

Assume that \( \tilde{G}_{1} = \tilde{G}'' \) is \( K' \)-quasisymmetrically conjugated to a Fuchsian group \( \tilde{F}'' \). Let \( M \) be the normal subgroup of \( \tilde{F}'' \) which corresponds to \( Ker_{N} \) under this conjugation. Then \( D/M \) is a Riemann surface which is biholomorphic to the disc \( D \)
minus a discrete set of points. Let $S' = D/M$. Note that there is a homomorphism of $\mathcal{F}'$ onto a conformal group $\mathcal{F}'$ and the kernel of this homomorphism is $M$. Here $\hat{\mathcal{F}}'$ acts on $S'$, and if we see $S'$ as a subset of $D$ (which we can because of the biholomorphic type of $S'$), then $\hat{\mathcal{F}}'$ acts on $D$ as well. Therefore, $\hat{\mathcal{F}}'$ is a Fuchsian group. It follows that the above conjugation induces a homeomorphism between $S$ and $S'$ such that the restriction of this homeomorphism on $T$, $\varphi : T \to T$, is a $K''$-quasisymmetric map, $K'' = K''(K)$, which conjugates the group $\mathcal{G}$ to the Fuchsian group $\mathcal{G}'$. This proves (2). \hfill \square

8. Elliptic elements of order 2 and the proof of Theorem 1.1

Throughout the next two subsections we assume that $\mathcal{G}$ is a $K$-quasisymmetric group that has no $\epsilon(K)$-small hyperbolic elements and that $\mathcal{G}$ does not contain any elliptic elements of order three or more. Here $\epsilon(K)$ is the constant from Theorem 4.1.

8.1. Quasicenter of an elliptic element of order two. Denote by $\mathcal{E}$ the set of elliptic elements of $\mathcal{G}$ of order two. We adopt a few notions introduced by Gabai (see [8]). Let $l$ be an oriented geodesic in $D$ with the endpoints $a$ and $b$ (positive orientation is from $a$ to $b$). $H^r$, $H^l$ denote, respectively, the right and the left halfspaces determined by $l$. This is chosen so that the arc $[a, b] \subset T$ borders $H^r$ (we use the standard counterclockwise orientation on $T$). Let $e \in \mathcal{E}$. A pair of points $x, y \in T$ is said to be an orbit of $e$ if $e(x) = y$. We say that $e$ is to the left of $l$ if there is a pair of points that is an orbit of $e$ and that belong to the open arc $(h, a) \subset T$. Similarly we define the notion to the right. We say that $e$ is on $l$ if the pair $a, b$ is an orbit of $e$ (it is clear that every $e \in \mathcal{E}$ has to be in one and only one category). If $e$ were a Möbius transformation, then the center of the map $\hat{e}$ would have been in $H^r$ or in $H^l$ on $l$, depending on whether $e$ is to the right of $l$ or to the left of $l$ or on $l$, respectively. Let $h \in \mathcal{G}$ be a hyperbolic element, and let $a_h, b_h$ denote its repelling and attracting fixed points, respectively. Let $l_h$ be the oriented geodesic with endpoints $a_h, b_h$ (the positive orientation is from $a_h$ to $b_h$). $H^r_h$ and $H^l_h$ denote, respectively, the right and the left halfspaces determined by $l_h$. We say that $e$ is to the left of $l_h$ or to the right of $h$ or on $h$ if $e$ is to the left of $l_h$ or to the right of $l_h$ or on $l_h$, respectively.

Let $L \geq K$ and $e \in \mathcal{E}$. We say that a point $z \in D$ is an $L$ quasicenter of $e$ if there exists an $L$-quasiconformal map $\hat{e} : D \to D$ which extends $e$ and such that $\hat{e} \circ e$ is the identity map. Since $\mathcal{G}$ is $K$-quasisymmetric, there exists at least one $K^2$ quasicenter for every $e \in \mathcal{E}$.

**Lemma 8.1.** With the notation as above, there exists $D = D(K) > 0$ such that the following hold. Let $l$ be an oriented geodesic and $e \in \mathcal{E}$. If $z \in D$ is an $L$ quasicenter of $e$, the following hold.

1. If $e$ is to the right (left) of $l$, then $z$ either belongs to $H^r$ ($H^l$) or $d(l, z) < D$.
2. If $e$ is on $l$, then $d(l, z) < D$.
3. For any $r > 0$ there exists $L' = L/(L, r) \geq L$, so that if $z \in D$ is an $L$ quasicenter for $e$, then every point in $\Delta(z, r)$ is an $L'$ quasicenter for $e$.
4. There exists $r(L) > 0$ such that every $L$ quasicenter of $e \in \mathcal{E}$ is contained in a fixed hyperbolic disc of radius $r(L)$. 
Proof. Let \( \hat{e} : D \to D \) be as above; that is, \( \hat{e} \) extends \( e \) and \( \hat{e} \circ \hat{e} = \text{id} \). Then we can choose a \( \hat{K} \)-quasiconformal map \( \hat{f} \) such that \( \hat{e} = \hat{f}^{-1} \circ \hat{e}_0 \circ \hat{f} \), where \( \hat{e}_0(z) = -z \), for \( z \in D \), is the standard order two rotation. Note that \( \hat{f}(z_0) = 0 \), where \( z_0 \) is the \( L \) quasicenter that corresponds to \( \hat{e} \). Let \( \gamma = \hat{f}(l) \), and let \( l' \) be the geodesic with the same endpoints as \( \gamma \). Then 0 is to the left of \( l' \) or to the right of \( l' \) if and only if \( e \) is to the left of \( l \) or to the right of \( l \) on \( l' \), respectively. Therefore \( z_0 \) is to the left of \( \hat{f}^{-1}(l') \) or to the right of \( \hat{f}^{-1}(l') \), respectively, if and only if \( e \) is to the left of \( l \) or to the right of \( l \) on \( l' \), respectively. Since \( \hat{f} \) is \( \hat{K} \)-quasiconformal, (1) and (2) follow.

If \( z \in \Delta(z_0, r) \), let \( \hat{g} : D \to D \) be a \( K' \)-quasiconformal map, \( K' = K'(L, r) \), which maps \( z_0 \) to \( z \) and which is the identity on \( T \). Then, \( z \) is the fixed point of the map \( \hat{e}' = \hat{g}^{-1} \circ \hat{e} \circ \hat{g} \). This proves (3).

Let \( z, z' \) be two \( L \) quasicenters of \( e \), and let \( \hat{f} \) and \( \hat{f}' \) be the corresponding \( \hat{K} \)-quasiconformal maps that conjugate \( e \) to \( e_0 \), where \( z, z' \) are, respectively, the fixed points of \( \hat{f}^{-1} \circ \hat{e}_0 \circ \hat{f} \) and \( \hat{f}'^{-1} \circ \hat{e}_0 \circ \hat{f}' \). Therefore, there is a \( \hat{K}^2 \)-quasiconformal map which maps \( z \) to \( z' \). This proves the last part. \( \square \)

Recall that we assume that \( G \) has no \( \epsilon(K) \)-small hyperbolic elements nor elliptic elements of order three or more.

**Lemma 8.2.** Let \( G \) be a \( K \)-quasisymmetric group. Let \( e : E \to D \) be the map that associates to each \( e \in E \) an \( L \) quasicenter \( c(e) \). \( L = L(K) \). There exists \( N = N(K) \in \mathbb{N} \) such that in each geodesic ball of radius 1 there are at most \( N \) points from \( c(E) \). In addition, if we assume that \( E \) is a \( \rho(K) \)-discrete set, \( \rho(K) > 0 \), then for each \( f \in G \), there exists a \( \hat{K} = \hat{K}(K) \)-quasiconformal map \( \hat{f} \) which extends \( f \) and such that \( \hat{f}(c(e)) = c(f \circ e \circ f^{-1}) \), for every \( e \in E \).

Proof. We prove the first part by contradiction. Assume that \( G_n \) is a sequence of \( K \)-quasisymmetric groups such that for each \( n \in \mathbb{N} \), we have at least \( n \) mutually different \( e_1^n, ..., e_n^n \in E \) and the corresponding \( L \) quasicenters \( c_n(e_1), ..., c_n(e_n) \) are in the hyperbolic disc \( \Delta(0, 1) \). For each \( 1 \leq i \leq n \), by \( f_i^n \) we denote a \( \hat{K} \)-quasiconformal map that conjugates \( e_i^n \) to the standard order two rotation \( \hat{e}_0 \) and such that there is a \( \hat{K} \)-quasiconformal extension \( \hat{f}_i^n \), with \( \hat{f}_i^n(0) = c(e_i^n) \). The family of \( \hat{K} \)-quasiconformal maps which map 0 into \( \Delta(0, 1) \) is a normal family. We conclude that for each \( \epsilon > 0 \), there is an \( n \) large enough and a choice of two maps \( \hat{f}_i^n \) and \( \hat{f}_j^n \), \( i \neq j \), such that \( \hat{f}_i^n \circ \hat{f}_j^n \) is \( \epsilon \) close to the identity map in the \( C^0 \) topology. In particular, we have that \( e_i^n \circ (e_j^n)^{-1} \) is \( \epsilon \) close to the identity in \( C^0 \) topology. The composition of two non-identical elements of order two is always a non-identity hyperbolic element; that is, \( e_i^n \circ e_j^n \neq \text{id} \). By choosing \( \epsilon \) small enough, we obtain a contradiction, because none of the groups \( G_n \) has \( \epsilon(K) \)-small hyperbolic elements.

If \( f \in G \) and \( \hat{f} \) a \( K \)-quasiconformal extension of \( f \), then it follows from the proof of the previous lemma that \( d(\hat{f}(c(e)), c(f \circ e \circ f^{-1})) < r(K) \). Set \( \hat{E} = \bigcup_{f \in G} \hat{f}^{-1}(c(f \circ e \circ f^{-1})) \). Since \( E \) is \( \rho(K) \)-discrete and \( \hat{f} \) is \( K \)-quasiconformal, it follows that \( \hat{E} \) is \( \rho_1(K) \)-discrete, for some \( \rho_1(K) > 0 \). The proof of this lemma follows by applying Lemma 3.2. \( \square \)
8.2. Removing the elliptic elements of order two. Let \( e \in \mathcal{E} \). We can find a \( \hat{K} \)-quasisymmetric map that conjugates \( e \) to the map \( e_0(z) = -z \). By conjugating the whole group \( \mathcal{G} \) by this quasisymmetric map, we may assume that \( \mathcal{G} \), if \( \mathcal{E} \) is not empty, always contains the transformation \( e_0 \).

Now, fix \( x \in \mathbf{T} \). Let \( G_x \) be the subgroup of \( \mathcal{G} \) which contains all elements from \( \mathcal{G} \) fixing the set \( \{x, -x\} \). Each \( G_x \) contains at least \( e_0 \), and it is an elementary group. There is a \( \hat{K} \)-quasisymmetric map \( \psi_x : \mathbf{T} \to \mathbf{T} \) with the following properties. \( \psi_x \) conjugates \( G_x \) to a Fuchsian group \( F_x \) which is an elementary group that fixes the same set \( \{x, -x\} \). Denote by \( \tilde{\psi}_x \) its \( \hat{K} \)-quasiconformal extension. This Fuchsian group may be assumed to contain \( e_0 \) and we may assume that the conjugation map conjugates \( e_0 \) to itself. Also, by Lemma 3.5 we may assume that the conjugation map preserves the origin and the geodesic \( s_x \) that connects \( x \) and \( -x \). An element of \( F_x \) is either a hyperbolic transformation which preserves the points \( x \), \( -x \), or it is an elliptic element of order two that permutes \( \{x, -x\} \).\( \mathcal{G} \)

Removing the elliptic elements of order two.

By conjugating the map preserves the origin and the geodesic \( s_x \) that connects \( x \) and \( -x \). An element of \( F_x \) is either a hyperbolic transformation which preserves the points \( x \), \( -x \), or it is an elliptic element of order two that permutes \( x \) and \( -x \). Denote by \( E''_x \) the set of fixed points (in \( \mathbf{D} \)) of all elliptic transformations from \( F_x \) (note that \( E''_x \subset s_x \)).

Set \( E'_x = \tilde{\psi}_x^{-1}(E''_x) \). Let \( c : E'_x \to s_x \) be the induced map that associates to each \( e \in \mathcal{G}_x \) the corresponding \( \hat{K}^2 \) quasicenter \( c(e) \).

Repeat this process for every \( x \in \mathbf{T} \) (by choosing the appropriate quasisymmetric map \( \psi_x \) for every fixed \( x \in \mathbf{T} \)). Set \( E' = \bigcup_{x \in \mathbf{T}} E'_x \). Note that the origin \( 0 \) belongs to all \( E'_x \subset s_x \) (and all \( E'_x \subset s_x \)). Let \( c : E' \to \mathbf{D} \) be the induced map that associates to each \( e \in \mathcal{E} \), and in particular \( e \in G_x \), the corresponding \( \hat{K}^2 \) quasicenter \( c(e) \in E' \).

Remark. Here we use the fact that every \( e \in \mathcal{E} \) must be contained in \( E'_x \), for some \( x \in \mathbf{T} \). That is, for every \( e \in \mathcal{E} \), there exists \( x \in \mathbf{T} \) such that \( x, -x \) is an orbit of \( e \). In particular, if \( h = e \circ e_0 \) is the corresponding hyperbolic transformation, then \( x, -x \) are the fixed points of \( h \). This is obviously true for Fuchsian groups, and Proposition 1.1 implies it for quasisymmetric groups.

If \( z, w \in E' \) both belong to a fixed \( E'_x \), for some \( x \in \mathbf{T} \), then there exists \( \rho'(K) > 0 \) such that \( d(z, w) \geq \rho'(K) \). This follows from the fact that \( \mathcal{G} \) contains no \( \epsilon(K) \)-small hyperbolic elements and that \( \tilde{\psi}_x \) is \( \hat{K} \)-quasiconformal. For each \( x \in \mathbf{T} \), let \( T_x \) be a diffeomorphism of \( s_x \) onto itself which preserves each \( x \), \( -x \) and \( 0 \) and such that

\[
\text{d}(t_1, t_2) - \frac{\rho'(K)}{3} < \text{d}(T_x(t_1), T_x(t_2)) < \text{d}(t_1, t_2) + \frac{\rho'(K)}{3}.
\]

It follows, with the aid of Lemma 8.2 (since \( E' \) is the set of \( \hat{K}^2 \) quasicenters of \( \mathcal{E} \), it follows from Lemma 8.2 that \( E' \) is not too dense) that for each \( x \) we can choose \( T_x \) which satisfies (8.1) and such that the set

\[
E = \bigcup_{x \in \mathbf{T}} T_x(E'_x)
\]
is \( \rho(K) \)-discrete, for some \( \rho(K) > 0 \). We use the same notation for the induced map \( c : E \to \mathbf{D} \) which associates to each \( e \in \mathcal{E} \) the corresponding \( L'' \) quasicenter (since we moved points from \( E' \) only a finite distance, we have by Lemma 8.2 that \( L'' = L''(K) \)). By \( E_x \) we denote the subset of \( E \) that is contained in \( s_x \).

For a given group \( \mathcal{G} \), we have constructed the set \( E \) of \( L'' \) quasicenters. This set is fixed from now on. Let \( \mathcal{F} \) be a Fuchsian group, and let \( \varphi : \mathbf{T} \to \mathbf{T} \) be a homeomorphism such that \( \mathcal{G} = \varphi \mathcal{F} \varphi^{-1} \). We may assume that \( \mathcal{F} \) contains the transformation \( e_0 \) and that \( \varphi \) conjugates \( e_0 \) to itself. For each \( y \in \mathbf{T} \), let \( F_y \) be the
subgroup of $\mathcal{F}$ containing elements that fix the set $\{y, -y\}$. The set of centers of elliptic elements (which are all of order two) that lie on $s_y$ is denoted by $\bar{E}_y$. Set $\bar{E} = \bigcup_{y \in \mathcal{T}} \bar{E}_y$. Let $\tilde{c}: \bar{E} \to \mathcal{D}$ be the map that associates the center to each elliptic element from $\mathcal{F}$. Let $\varphi(y) = x$. Note that $\varphi$ induces the map from $\bar{E}_y$ onto $E_x$ by $c \circ \varphi \circ \tilde{c}^{-1}$. It follows from (8.2) (the definition of $E$) that this map is orientation preserving, where the orientation for sets $\bar{E}_y$ and $E_x$ comes from the way they lie in $s_y$ and $s_x$, respectively.

We define an appropriate extension $\hat{\varphi}$ of the above homeomorphism $\varphi$. For each $y \in \mathcal{T}$, the restriction of $\hat{\varphi}$ to $s_y$ is a homeomorphism which maps $s_y$ onto $s_x$, where $x = \varphi(y)$ ($\hat{\varphi}$ maps a radius onto a radius). Also $\hat{\varphi}(0) = 0$, $\hat{\varphi}(y) = x$, $\hat{\varphi}(-y) = -x$. Furthermore, we can choose $\hat{\varphi}$ with the following property. If $w = \tilde{c}(u)$, for some elliptic $u \in \mathcal{F}$ of order two, then $\hat{\varphi}(w) = z$, where $z = c(\varphi \circ u \circ (\varphi)^{-1})$.

Set $S = \mathcal{D} - E$, and let $\mathcal{G}_1$ be the subgroup of the mapping class group of $S$ defined as follows. To $f \in \mathcal{G}$ we assign the isotopy class of the map $\hat{\varphi} \circ \tilde{u} \circ \hat{\varphi}^{-1}: S \to S$, where $u = \varphi^{-1} \circ f \circ \varphi$. The group of the corresponding isotopy classes is $\mathcal{G}_1$. The map from $\mathcal{G}$ onto $\mathcal{G}_1$ is an isomorphism.

**Lemma 8.3.** There exists $K_2 = K_2(K)$ such that the group $\mathcal{G}_1$ is $K_2$-quasisymmetric.

**Proof.** Fix $f \in \mathcal{G}$. Let $\tilde{f} = \hat{\varphi} \circ \tilde{u} \circ \hat{\varphi}^{-1}$, where $u = \varphi^{-1} \circ f \circ \varphi$. Our aim is to show that $\tilde{f}$ is isotopic in $S$ to a $K_2(K)$-quasiconformal map.

$E$ is $\rho(K)$-discrete and $E$ is the set of $L''$ quasicenters. Let $\tilde{f}$ be a $K$-quasiconformal map from Lemma 8.2; that is, for every $z \in E$, $z = c(e)$, we have $\tilde{f}(z) = c(f \circ e \circ f^{-1})$.

Fix $x \in \mathcal{T}$, and set $\gamma_x = \tilde{f}(s_x)$. Note that $\hat{\varphi}^{-1}(\gamma_x) = \gamma'_y$ is a geodesic in $\mathcal{D}$ and $\tilde{u}(s_y) = \gamma'_y$, where $\varphi(y) = x$ and $\varphi$ conjugates $u$ to $f$. Let $v \in \mathcal{F}$ be an elliptic element of order two. Since the position of $\tilde{c}(v)$ (the center of $\tilde{v}$) with respect to $\gamma'_y$ respects whether $v$ is to the right of, to the left of, or on $\gamma'_y$, we have that the same is true for the curve $\gamma_x$ and the corresponding quasicenter from $E$. Also, if the endpoints of $\gamma_x$ are not an orbit of $e_0$, then $\gamma_x$ intersects each geodesic $s_y$ at most once (this is because the corresponding statement is true for $\gamma'_y$) (see Figure 5).

Denote by $[\gamma_x]$ the homotopy class of $\gamma_x$ in $S$. For $y \in \mathcal{T}$ let $a_y = \gamma_x \cap s_y$ if this intersection is not empty. For $s_y$, where this intersection is non-empty, denote by $I_y \subset s_y$ the maximal open geodesic arc which contains $\gamma_x \cap s_y = a_y$ and such that $I_y$ does not contain any points from $E$ (except for $a_y$, if $a_y \in E$). If $s_y$ contains no points from $E$ other than 0, then $I_y$ is one of the two geodesic rays that end at 0 and which constitute $s_y$.

Let $\alpha: s_x \to \mathcal{D}$ be a curve in $\mathcal{D}$ that intersects each $s_x$, $x \in \mathcal{T}$, at most once, and set $b_y = \alpha \cap s_y$. We have that $\alpha \in [\gamma_x]$ if and only if the map $\alpha$ is isotopic to the restriction of $\tilde{f}$ on $s_x$, which is equivalent to the following two conditions being satisfied.

1. If $a_y \in E$, then $a_y = b_y$.
2. If $a_y$ does not belong to $E$, then $b_y \in I_y$.

Now we show that for each $x \in \mathcal{T}$ we can choose an $L_1$ bilipschitz quasigeodesic $\alpha: s_x \to \mathcal{D}$, $\alpha \in [\gamma_x]$, $L_1 = L_1(K)$. If $\gamma_x$ is one of the geodesics that contain the origin, then $\alpha = \gamma_x$. If not, then $\gamma_x$ intersects each $s_y$ at most once. Let $l$ be the
geodesic with the same endpoints as the curve $\gamma_x$. Then, by Lemma 8.1 we have that
\[
d(l, I_y) < r(K)
\]
and
\[
d(l, a_y) < r(K), \quad a_y \in E,
\]
for each $I_y$ and for each $a_y \in E$. Since the set $E$ is $\rho(K)$-discrete, it follows from (8.3) that we can choose an $L_1'$ bilipschitz quasigeodesic $\beta : l \to D$, $L_1' = L_1'(K)$, such that the curve $\beta$ is in $[\gamma_x]$.

Let $\alpha' : s_x \to \beta$ be defined as follows (here $\beta = \beta(l)$). For $z \in s_x$, let $\alpha'(z)$ be the point $\beta \cap s_y$, where $y \in T$ is the unique point such that $s_y$ contains the point $\tilde{f}(z)$. Either $\tilde{f}(z) = \alpha'(z)$ or $\tilde{f}(z)$ and $\alpha'(z)$ belong to the same $I_y$. The map $\alpha'$ does not have to be a quasiisometry, but we have $\alpha'(z) = \tilde{f}$ for $z \in E_x$. Since $\tilde{f}$ is $K$-quasiconformal (recall that $\tilde{f}$ is the map from Lemma 8.2 that corresponds to the set $E$) and since $\beta = \alpha'(s_x)$ is a bilipschitz quasigeodesic, we can construct $\alpha : s_x \to \beta = \alpha'(s_x)$, so that $\alpha$ is $L_1$ bilipschitz quasigeodesic.

Consider the map $\tilde{f}^{-1} \circ \tilde{f} : S \to S$. This map is the identity on $T$, and it fixes every point in $E$. Also, for each $s_x$, the restriction of the map $\tilde{f}^{-1} \circ \tilde{f}$ on $s_x$ is isotopic (rel $\partial S$) to an $L_1$ bilipschitz quasigeodesic. It follows from Lemma 3.4 that $\tilde{f}$ is isotopic (rel $\partial S$) to a $K_2$-quasiconformal map, for every $f \in G$, and the group $G_1$ is $K_2$-quasisymmetric. \qed
Note that $\mathcal{G}_1$, as a group that acts on the Riemann surface $S$, does not have any elliptic elements; that is, $\mathcal{G}_1$ is a torsion-free group.

8.3. Proof of Theorem 1.1. Let $\mathcal{G}$ be an arbitrary $K$-quasisymmetric discrete group. We want to prove that such a group is a quasisymmetric conjugate of a Fuchsian group.

By Theorem 7.1 we can assume that $\mathcal{G}$ has no elliptic elements of order three or more. Then, we can apply Theorem 4.1 to the group $\mathcal{G}$ and it follows from Theorem 4.1 that we can further assume that $\mathcal{G}$ does not contain any $\epsilon(K)$-small hyperbolic elements, where $\epsilon(K) > 0$ is the constant from Theorem 4.1.

Since $\mathcal{G}$ has no $\epsilon(K)$-small hyperbolic elements and no elliptic elements of order three, it follows from Lemma 8.3 that the corresponding group $\mathcal{G}_1$ is a $K_2$-quasisymmetric, torsion-free group. But $\mathcal{G}_1$ acts on the Riemann surface $S = \mathbb{D} - \mathcal{E}$.

We now cover the surface $S$ by the unit disc, and in the same way as in the proof of Theorem 7.1 we produce a new $K_2$-quasisymmetric, torsion-free group $\mathcal{G}_2$ (that acts on $\mathbb{D}$) such that $\mathcal{G}_2$ naturally projects to $\mathcal{G}_1$. Moreover, since $\mathcal{G}$ does not have any $\epsilon(K)$-small hyperbolic elements and since the set $\mathcal{E}$ is discrete (this implies that there are no very short closed geodesics on $S$), we can find $\tilde{\epsilon}(K) > 0$ such that $\mathcal{G}_2$ does not contain any $\tilde{\epsilon}(K)$-small hyperbolic elements. As in the proof of Theorem 7.1, we have that $\mathcal{G}_2$ is quasisymmetrically conjugated to a Fuchsian group if and only if $\mathcal{G}$ is.

So, $\mathcal{G}_2$ is a $K_2$-quasisymmetric, torsion-free group, that does not contain any $\tilde{\epsilon}$-small hyperbolic elements. We can now apply Theorem 6.1. This concludes the proof of Theorem 1.1.

References


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