ERRATUM TO “REAL BOUNDS, ERGODICITY AND NEGATIVE SCHWARTZIAN FOR MULTIMODAL MAPS”

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In Part 1 of Theorem C of the paper Real bounds, ergodicity and negative Schwarzian for multimodal maps, see [1], the assumption that \( V \) is nice was, by mistake, omitted. We would like to thank Weixiao Shen for pointing this out. The correct version of Theorem C(1) is as follows:

**Theorem C(1)** (Improved Macroscopic Koebe Principle). Assume that \( f: M \to M \) is contained in \( A^{1+2\text{Y}gmund} \). Then for each \( \xi > 0 \), there exists \( \xi' > 0 \) such that if \( I \) is a nice interval, \( V \) is nice and \( \xi \text{-well-inside} \( I \) and \( x \in I \), \( f^k(x) \in V \) (with \( k \geq 1 \) not necessarily minimal), then the pullback of \( V \) along \( \{x, \ldots, f^k(x)\} \) is \( \xi'\text{-well-inside} \) the return domain to \( I \) containing \( x \).

Here, as before, we define an open interval \( K \) to be nice if no iterate of \( \partial K \) enters \( K \). This implies that if \( K_1 \) and \( K_2 \) are pullbacks of \( K \), then they are either disjoint or nested.

In Lemma 9 (page 762) it was implicitly assumed that \( V \) is disjoint from \( J_n \). It is for this reason that the proof of Theorem C(1) does not work unless we assume \( V \) is nice (or something similar). The proof of Theorem C(1) as stated above is essentially the same as before, using Lemma 6' below instead of Lemma 6; then in Lemma 9 (page 762) we do not need to require that \( k_{n+1} \) is a jump time provided we assume that \( V \) is nice. Making the additional assumption that \( V \) is nice, Proposition 1 (and its proof) and the rest of the paper go through unchanged.

**Lemma 6'**. For each \( \rho > 0 \) sufficiently small, there exists \( \delta_3 > 0 \) such that if \( I \) is a \( \rho \text{-scaled} \) neighbourhood of a nice interval \( V \subset I \), then \( J \) is a \( \delta_3 \text{-scaled} \) neighbourhood of any component \( A \) of \( \phi_{j}^{-k}(V) \) (where \( k \geq 1 \) is arbitrary).

**Proof.** Let \( V_i \), \( i = 0, \ldots, k \) be the component of \( \phi_{j}^{-k}(V) \) containing \( \phi'(A) \). Of course, we may assume that \( k \) is large and that \( V_0, \ldots, V_k \) are disjoint.

**Claim.** There exists \( \alpha > 0 \) such that if \( 0 \leq j < k \) and \( V_{j+1} \) is contained in a neighbourhood of \( V_j \) of size \( (1 + \alpha)|V_j| \), then \( V_j \) is \( \alpha \text{-well-inside} \) \( I \). Similarly, if \( 0 \leq j < k - 1 \) and \( V_j \) lies between \( V_{j+1} \) and \( V_{j+2} \), then \( V_j \) is \( \alpha \text{-well-inside} \) \( I \).

**Proof of Claim.** If \( V_{j+1} \) is contained in a neighbourhood of \( V_j \) of size \( (1 + \alpha)|V_j| \) and \( \alpha \) is small enough, then \( \phi' \) is close to zero on a definite neighbourhood of \( V_j \). So \( V_j \) is contained in the basin of an attracting fixed point with multiplier close to zero. Since \( V_k \) is nice and \( \delta_3 \text{-well-inside} \) \( I \), we easily get that \( V_j \) is \( \delta_3^{j} \text{-well-inside} \)
I, proving the first part of the claim. The second part of the claim follows in the same way by first applying part 1 to $V_j, V_{j+1}$ and then applying it to $V_j, V_{j+2}$ while considering $\phi^2$ instead of $\phi$, completing the proof of the claim.

If both sides of $J$ are small, then $|\phi'|$ is bounded on $J$. There are three possibilities.

(a) $V_1$ lies between $V_2$ and $V_3$, in which case, by the second part of the Claim, $V_1$ is well-inside $I$.

(b) $V_2$ lies between $V_1$ and $V_3$; in this case since $|V_1|/|V_2|$ is not small and by the first part of the Claim, $V_2$ is well-inside $I$.

(c) $V_3$ lies between $V_1$ and $V_2$; then, because $|V_1|/|V_2|$ and $|V_2|/|V_3|$ are not small, $V_3$ is well-inside $I$. In all cases, we get that $V_0$ is well-inside $J$.

So assume one of the sides, say the right side, of $J$ is not small. Let $1 \leq j \leq k$ be the largest integer so that $V_1, \ldots, V_j$ are all not $\alpha$-well-inside $I$. By taking $\alpha > 0$ small, we may assume $j \leq k - 2$. Since $V_{j+1}$ is $\alpha$-well-inside $I$, we may assume that $j \geq 1$ and that there exists $\alpha' > 0$ so that $V_j, V_{j-1}$ are $\alpha'$-well-inside $J \subset I$. By the claim, for each $i = 1, \ldots, j$, $V_i$ has an $\alpha$-small and an $\alpha$-big side, and $V_{j+1}$ is contained in the $\alpha$-big side. Since the right side of $J$ is not small, $V_1, \ldots, V_{j+1}$ lie therefore ordered from left to right. If for each $i = 1, \ldots, j - 1$, $V_{i-1}$ is contained in a $\beta$-scaled neighbourhood of $V_i$, then $V_j$ is in a $(\beta + \beta^2 + \cdots + \beta^j)$-scaled neighbourhood of $V_{j-1}$. So taking $\beta \in (0, 1)$ so small that $\beta/(1 - \beta) < \alpha'/2$, then, because $V_{j-1}$ is $\alpha'$-well-inside $I$, the left component of $I \setminus V_1$ has at least size $\frac{\alpha'}{2} |V_{j-1}| > \frac{\alpha'}{2} |V_1|$, i.e., $V_1$ is well-inside $I$, and $V_0$ is well-inside $J$. Hence we may assume there exists $i \in \{1, \ldots, j - 1\}$ so that $V_{i-1}$ is not contained in a $\beta$-scaled neighbourhood of $V_i$. This and the first part of the Claim imply that $V_i$ is well-inside the convex hull $H_i := [V_{i-1}, V_{i+1}]$ of $V_{i-1}$ and $V_{i+1}$. Because the intervals $V_1, \ldots, V_{i+2}$ lie ordered, it follows that the pullback of $H_i$ along $V_1, \ldots, V_i$ has intersection multiplicity at most 4 and therefore that $V_1$ is well-inside $I$. This again gives that $V_0$ is well-inside $J$. (This method of proof can also be used to provide a slightly shorter proof of Lemma 5.)

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\section*{References}

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