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REPRESENTATIONS OF AFFINE HECKE ALGEBRAS AND BASED RINGS OF AFFINE WEYL GROUPS

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It is known that an interesting part of the study of the representation theory of p-adic groups can be reduced to the study of the representation theory of affine Hecke algebras [B, V]. Let (W, S) be an extended affine Weyl group and H_{k,q_0} the corresponding Hecke algebra over a field k with a nonzero parameter $q_0 \in k$. When k is the complex numbers field and q_0 is not a root of unity, a classification of simple representations of H_{k,q_0} was established in [KL2] (Deligne-Langlands-Lusztig classification). For affine type A, a classification of simple representations of H_{k,q_0} was obtained in [AM] for any q_0 and arbitrary sufficiently large k. When k is algebraically closed and has positive characteristic, the representations of H_{k,q_0} were studied by Vignéras, as part of her study of modular representations of p-adic groups [V]. In this paper we shall verify a conjecture of Lusztig [L6, 7(a)] by means of the based ring of an extended affine Weyl group (Theorem 3.3). The conjecture says that if the parameter q_0 is not a root of the corresponding Poincaré polynomial, then the classification established in [KL2] remains valid. The restriction is necessary for the classification; see Remark 3.4 (a).

1. Extended affine Weyl groups and their Hecke algebras

1.1. Let G be a connected reductive group over the field $\mathbf{C}$ of complex numbers with simply connected derived group and T a maximal torus of G. Let $N_G(T)$ be the normalizer of T in G. Then $W_0 = N_G(T)/T$ is a Weyl group, which acts on the character group $X = \mathrm{Hom}(T, \mathbf{C}^*)$ of T. The semi-direct product $W = W_0 \ltimes X$ is called an extended affine Weyl group. We shall denote by S the set of simple reflections of W.

Denote by H_{k,q_0} the Hecke algebra of (W, S) over an arbitrary field k with a nonzero parameter $q_0 \in k$. We shall assume that k contains the square roots of q_0. The following result is due to J. Bernstein; see [L1, Theorem 8.1] for a proof.

(a) The center Z of H_{k,q_0} is a finitely generated k-algebra and H_{k,q_0} is a finitely generated Z-module.

The following result was proved in [KL2, Proof of Prop. 5.13] when k is uncountable, by using an argument of Dixmier.

Proposition 1.2. *Any simple H_{k,q_0}-module is finite dimensional.*

Proof. Let M be a simple H_{k,q_0}-module and $\mathcal{D} = \mathrm{End}_{H_{k,q_0}} M$. Then $\mathcal{D}$ is a division ring. For z in Z, let $f_z : M \to M,\ m \to zm$. Then f_z is in $\mathcal{D}$ and the map

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$f : Z \to \mathcal{D},\ z \to f_z$ is a homomorphism of k-algebras. Let $Y = f(Z)$. By section 1.1 (a), Y is a finitely generated k-algebra. We only need to show that each element in Y is algebraic over k.

Let r be the transcendency degree of Y over k. By the Noether normalization theorem, there are elements $y_1, ..., y_r$ in Y such that Y is integral over $k[y_1, ..., y_r]$.

We need to show that r is zero. Assume that $r \geq 1$. Note that y_1^{-1} is not in Y since $y_1, ..., y_r$ are algebraically independent and Y is integral over $k[y_1, ..., y_r]$. By section 1.1 (a), M is a finitely generated Z-module. Let $v_1, ..., v_g$ be elements in M which generate M as a Z-module. Choose x in Z such that $f_x = y_1$. Since y_1 is invertible in $\mathcal{D}$, we can find u_i in M such that $v_i = xu_i$ for all i. Let $u_i = \sum_j \xi_{ji} v_j,\ \xi_{ji} \in Z$. Set $\eta_{ji} = x\xi_{ji}$ if $j \neq i$, and $\eta_{ii} = 1 - x\xi_{ii}$. Then we have $\det(\eta_{ij})v_i = 0$ for all i. But $\det(\eta_{ij}) = 1 - xz$ for some z in Z. Thus $f_{1-xz} = 1 - f_x f_z = 1 - y_1 f_z = 0$. This implies that y_1 is invertible in Y and leads to a contradiction. Therefore we must have $r = 0$. The proposition is proved. □

2. a-FUNCTION AND BASED RING

In this section we will see that the simple J_k-modules and simple H_{k,q_0}-modules have a nice relationship.

2.1. We refer to [L2, 2.1] and [L3, 2.3] for the definitions of the function $a : W \to \mathbf{N}$ and of the based ring J of W respectively. Following [L3] we denote by $t_w,\ w \in W$ the basis elements of J. For each nonnegative integer i we denote by J^i the subgroup of J generated by all t_w with $a(w) = i$. Then J^i is a two-sided ideal of J and J is the direct sum of all J^i. Set $J_k = J \otimes_{\mathbf{Z}} k$ and $J_k^i = J^i \otimes_{\mathbf{Z}} k$. Thus J_k^i is a direct summand of J_k and is also a k-algebra. By abusing notation we also write t_w for $t_w \otimes 1$.

Let $C_w,\ w \in W$ be the Kazhdan-Lusztig basis of H_{k,q_0} in [KL1, L4] and write $C_w C_u = \sum h_{w,u,v} C_v,\ h_{w,u,v} \in k$. Let D be the set of distinguished involutions of W. The following properties are due to Lusztig; see [L3, 2.4 (a)] and [L4, Prop. 1.7, Prop. 1.6 (i), (ii)].

(a) There is a well-defined homomorphism $\varphi :\ H_{k,q_0} \to J_k$ of k-algebras such that

$$\varphi(C_w) = \sum_{\substack{d \in D \\ u \in W \\ a(d)=a(u)}} h_{w,d,u} t_u, \qquad w \in W.$$

(b) The homomorphism φ in (a) is injective. Thus H_{k,q_0} can be regarded as a subalgebra of J_k by means of φ.

(c) The center $Z(J_k)$ of J_k is a finitely generated k-algebra and J_k is a finitely generated $Z(J_k)$-module.

(d) There is a well-defined right H_{k,q_0}-module structure on J_k^i such that

$$t_w C_u = \sum_{\substack{v \in W \\ a(v)=a(w)}} h_{w,u,v} t_v.$$

In this way, J_k^i becomes a J_k-H_{k,q_0}-bimodule. See [L4, 1.4 (b)].

The following result was proved by Lusztig [L4, Prop. 1.6 (iii)] provided that k is uncountable.

Lemma 2.2. *Any simple J_k-module is finite dimensional.*

Proof. A proof is similar to that for Proposition 1.2. □

2.3. Let E be a J_k-module through the homomorphism φ, it is endowed with an H_{k,q_0}-module structure. We denote the H_{k,q_0}-module by E_φ. **Convention:** For any subset N of E and any subset L of H_{k,q_0}, we often write LN for $\varphi(L)N$. Thus, as a set the notation LN is unambiguous, no matter whether N is regarded as a subset of E or as a subset of E_φ.

For each simple J_k-module E, there is a unique i such that $J_k^iE = E$. We define $a(E)$ to be i. For an integer i, we denote by $H_{k,q_0}^{\geq i}$ (resp. $H_{k,q_0}^{>i}$) the subspace of H_{k,q_0} spanned by all C_w with $a(w) \geq i$ (resp. $a(w) > i$). Both $H_{k,q_0}^{\geq i}$ and $H_{k,q_0}^{>i}$ are two-sided ideals of H_{k,q_0}. For each H_{k,q_0}-module M we then define $a(M)$ to be i if $H_{k,q_0}^{\geq i}M \neq 0$ but $H_{k,q_0}^{>i}M = 0$.

Let M be an H_{k,q_0}-module with $a(M) = i$. We define $\tilde{M}$ to be $J_k^i \otimes_{H_{k,q_0}} M$; here we regard J_k^i as a J_k-H_{k,q_0}-bimodule as in section 2.1 (d). Then $\tilde{M}$ is a J_k-module. There is a natural homomorphism of H_{k,q_0}-modules $p : \tilde{M}_\varphi \to M,\ t_w \otimes m \to C_w m$. We have ([L4, Proof of Lemma 1.9]).

(a) When M is simple, the map p is surjective and $C_w \ker p = 0$ whenever $a(w) \geq a(M)$.

The following assertion is clear.

(b) Let E be a simple J_k-module. Then $H_{k,q_0}^{>a(E)}E_\varphi = 0$. In particular, $a(M) \leq a(E)$ for any simple constituent M of E_φ. Also for any subset N of E or E_φ, $H_{k,q_0}^{\geq a(E)}N$ is spanned by all $C_wN,\ w \in W$ with $a(w) = a(E)$.

Lemma 2.4. *Let E be a simple J_k-module and N a submodule of E_φ such that $C_wN \neq 0$ for some $w \in W$ with $a(w) = a(E)$. Regarding N as a subset of E, then $H_{k,q_0}^{\geq a(E)}N = E$. In particular, $N = E_\varphi$ as H_{k,q_0}-modules.*

Proof. Using section 2.3 (b) we know $a(N) = a(E)$. Thus $\tilde{N} = J_k^{a(E)} \otimes_{H_{k,q_0}} N$. We have a well-defined k-linear map

$$\theta : \tilde{N} \to E,\ t_w \otimes v \to \varphi(C_w)v.$$

Using [L3, 2.4 (c)] we see that θ is a homomorphism of J_k-modules. Since E is a simple J_k-module and $\theta(\tilde{N}) = H_{k,q_0}^{\geq a(E)}N \neq 0$, we must have $H_{k,q_0}^{\geq a(E)}N = E$. The lemma is proved. □

Lemma 2.5. *Let E be a simple J_k-module. Then*

(a) *E_φ has at most one simple constituent M such that $a(M) = a(E)$.*

(b) *If E_φ has a simple constituent M such that $a(M) = a(E)$, then M is a quotient module of E_φ.*

(c) *If E_φ has a simple constituent M such that $a(M) = a(E)$, then M is the unique simple quotient module of E_φ.*

Proof. Assume that E_φ has a simple constituent M such that $a(M) = a(E)$. Let $N_2 \subset N_1$ be two submodules of E_φ such that the quotient module N_1/N_2 is M. Then $C_wN_1 \neq 0$ for some $w \in W$ with $a(w) = a(E)$. By Lemma 2.4 we have $N_1 = E_\varphi$. Since $H_{k,q_0}^{\geq a(E)}$ is a two-sided ideal, using Lemma 2.4 we see that $N_2 = \{v \in E_\varphi \mid H_{k,q_0}^{\geq a(E)}v = 0\}$.

(a) and (b) follow.

Now we argue for (c). Let N be a maximal submodule of E_φ. Using Lemma 2.4 we see that N is a submodule of $N_2 = \{v \in E_\varphi \mid H_{k,q_0}^{\geq a(E)} v = 0\}$. By the argument for (a) and (b), N_2 is a maximal submodule of E_φ. Thus $N = N_2$ and $E_\varphi/N = M$ is the unique simple quotient module of E_φ.

The lemma is proved. $\square$

Corollary 2.6. *Let E be a simple J_k-module. Then E_φ has a simple constituent M with $a(M) = a(E)$ if and only if $C_w E_\varphi \neq 0$ for some w with $a(w) = a(E)$. In this case E_φ has a unique maximal submodule.*

Proof. The "only if" part is obvious. Now we prove the "if" part. Assume that E_φ had no simple constituent M with $a(M) = a(E)$. Let N be a maximal submodule of E_φ. Then E_φ/N is simple. By assumption and section 2.3 (b), we have $H_{k,q_0}^{\geq a(E)} E_\varphi \subset N$. However, $C_w E_\varphi \neq 0$ for some w with $a(w) = a(E)$. By Lemma 2.4 we have $H_{k,q_0}^{\geq a(E)} E_\varphi = E_\varphi$. This is a contradiction. The corollary is proved. $\square$

Lemma 2.7. *Let E and E' be two simple J_k^i-modules. Assume that E_φ (resp. E'_φ) has a simple quotient M (resp. M') such that $a(M) = i$ (resp. $a(M') = i$). Then M is isomorphic to M' if and only if E is isomorphic to E'.*

Proof. Let $\pi : E_\varphi \to M$ be the natural projection. Since $H_{k,q_0}^{\geq i} E_\varphi \neq 0$, by section 2.3 (b) we have $\widetilde{E_\varphi} = J_k^i \otimes_{H_{k,q_0}} E_\varphi$. For simplicity, we shall write $\tilde{E}$ for $\widetilde{E_\varphi}$. There are two well-defined k-linear maps

$$p' : \tilde{E} \to \tilde{M},\ t_w \otimes v \to t_w \otimes \pi(v),$$

$$\theta : \tilde{E} \to E,\ t_w \otimes v \to \varphi(C_w)v.$$

Clearly p' is a homomorphism of J_k-modules. According to the proof of Lemma 2.4, θ is also a homomorphism of J_k-modules. Obviously we have $\pi\theta = pp'$ (see section 2.3 for the definition of $p :\ \tilde{M}_\varphi \to M$).

Since p' is a surjection, the homomorphism p' induces a surjective homomorphism of J_k-modules, $\bar{p}' : \tilde{E}/\ker\theta \to \tilde{M}/p'(\ker\theta)$. As J_k-modules, $\tilde{E}/\ker\theta$ is isomorphic to E, since E is simple and $\theta(\tilde{E}) = H_{k,q_0}^{\geq i} E \neq 0$. Thanks to $\pi\theta = pp'$, we know that $p'(\ker\theta)$ is in the kernel of p. By section 2.3 (a), $\ker p \subsetneq \tilde{M}$, so $\bar{p}'$ is an isomorphism and E is isomorphic to $\tilde{M}/p'(\ker\theta)$.

By section 2.3 (a), $H_{k,q_0}^{\geq i} \ker p = 0$; hence we have $H_{k,q_0}^{\geq i} p'(\ker\theta) = 0$. Thus E can be characterized as the unique simple constituent F of the J_k-module $\tilde{M}$ such that $H_{k,q_0}^{\geq i} F_\varphi \neq 0$.

As a consequence, if M is isomorphic to M', then E must be isomorphic to E'. The lemma is proved. $\square$

Corollary 2.8 ([L4, Corollary 3.6])**.** *Assume that for each simple J_k^i-module E, the H_{k,q_0}-module E_φ has a simple constituent M with $a(M) = i$. Then both of the J_k-modules $\tilde{E}$ and $\tilde{M}$ are isomorphic to E.*

Proof. By Lemma 2.5 (c), M is the unique simple quotient of E_φ. Note that $J_k^r \tilde{E} = 0$ if $r \neq i$ (recall that $\tilde{E}$ stands for $\widetilde{E_\varphi}$). Let $\theta :\ \tilde{E} \to E$ be as in the proof of Lemma 2.7. As in the proof of [L4, Lemma 1.9], one may check that $C_w \ker\theta = 0$ whenever $a(w) \geq i$. If $\ker\theta \neq 0$, then by assumption, $C_w \ker\theta \neq 0$ for some w with $a(w) = i$. This yields a contradiction. Therefore $\ker\theta = 0$ and as J_k-modules,

$\tilde{E}$ is isomorphic to E. By the proof of Lemma 2.7 we know that $\tilde{E}$ and $\tilde{M}$ are isomorphic in this case. The corollary is proved. □

3. Main results

In this section we give our main results.

Denote by W^I the subgroup of W generated by a subset I of S and call it a parabolic subgroup. Let J_k^I be the subspace spanned by all t_w, $w \in W^I$.

Theorem 3.1. *Assume that* $\operatorname{char} k = 0$. *Then as a two-sided ideal, J_k is generated by all J_k^I for all finite parabolic subgroups W^I.*

Proof. According to [L5, Theorem 4.2] and [L5, Theorem 6.7(a2)], for any simple $J_{\mathbf{C}}$-module E, we can find a finite parabolic subgroup W^I of W such that the action of $J_{\mathbf{C}}^I$ on E is nonzero. This implies that as a two-sided ideal, $J_{\mathbf{C}}$ is generated by all $J_{\mathbf{C}}^I$ for all finite parabolic subgroups W^I. With respect to the basis $\{t_w | w \in W\}$, the structure constants of J_k are in $\mathbf{N}$ if $\operatorname{char} k = 0$. The theorem follows. □

When q_0 is not a root of unity, the following result was proved by Lusztig [L4, Theorem 3.4], except for the uniqueness in (a).

Theorem 3.2. *Assume that* $\operatorname{char} k = 0$ *and* $\sum_{w \in W_0} q_0^{l(w)} \neq 0$ *(l is the length function of W). Then*

(a) *for each simple J_k-module E, the H_{k,q_0}-module E_φ has a unique simple constituent M such that $a(M) = a(E)$. For other simple constituents M' of E_φ we have $a(M') < a(E)$. The H_{k,q_0}-module M is the unique simple quotient of E_φ. (The uniqueness is part of* [L2, 9.10, Conjecture A]. *The other part of the conjecture was proved in* [L3].*)*

(b) *Keep the notation in (a). The map $E \to M$ defines a bijection between the isomorphism classes of simple J_k-modules and the isomorphism classes of simple H_{k,q_0}-modules.*

Proof. Let W^I be a finite parabolic subgroup of W. Since $\sum_{w \in W_0} q_0^{l(w)} \neq 0$, it is easy to check that $\sum_{w \in W^I} q_0^{l(w)} \neq 0$. Thus the subalgebra H_{k,q_0}^I of H_{k,q_0} generated by all C_w ($w \in W^I$) is semisimple [G1, Theorem 3.9]. Then the restriction of φ to H_{k,q_0}^I induces an isomorphism $\varphi_I : H_{k,q_0}^I \to J_k^I$ [G2, Lemma 2.1]. The isomorphism φ_I sends C_w ($w \in W^I$) to a linear combination of $t_u, u \in W^I$ with $a(u) \geq a(w)$.

Now for each simple J_k-module E, we can find a finite parabolic subgroup W^I such that $J_k^I E \neq 0$ (Theorem 3.1). Let $N_1 = J_k^I E$ and $N_2 = \{v \in E \mid J_k^I v = 0\}$. Then $E = N_1 \oplus N_2$ and $J_k^I N_1 = N_1$. Moreover, for any v in N_1 and h in H_{k,q_0}^I, we have $\varphi(h)v = \varphi_I(h)v$. Let $u \in W^I$ be such that $t_u N_1 \neq 0$. Then $a(u) = a(E)$ and $h = \varphi_I^{-1}(t_u)$ is a linear combination of $C_w, w \in W^I$ with $a(w) \geq a(E)$. Now we have $hN_1 = \varphi(h)N_1 = \varphi_I(h)N_1 = t_u N_1 \neq 0$. Using section 2.3 (b) we can find an element $w \in W^I$ such that $a(w) = a(E)$ and $C_w N_1 \neq 0$. This implies that $C_w E_\varphi \neq 0$. By Corollary 2.6 and Lemma 2.5, we see that E_φ has a unique simple constituent M such that $a(M) = a(E)$. Moreover, M is the unique simple quotient of E_φ.

Using section 2.3 (b), we know that for other simple constituents M' of E_φ we have $a(M') < a(E)$. Part (a) is proved.

Using section 2.3 (a) and Lemma 2.7 we see that (b) is true. □

Theorem 3.3. *Assume that $k = \mathbf{C}$ and $\sum_{w \in W_0} q_0^{l(w)} \neq 0$. Then the classification of simple H_{k,q_0}-modules in* [KL2] *remains valid.*

Proof. The theorem follows from [L5, Theorem 4.2] and Theorem 3.2 (b). □

Remark 3.4. (a) When $\sum_{w \in W_0} q_0^{l(w)} = 0$, there are simple $J_{\mathbf{C}}$-modules E such that the H_{k,q_0}-modules E_φ have no simple constituents M with $a(M) = a(E)$ [X1, Theorem 7.8].

(b) A weaker result was proved in [X1, Theorem 6.6].

(c) In [Gr], Grojnowski announced a stronger result. The proof seems to not be available yet. The validity of the result will be commented on in a future work.

(d) For type $\tilde{A}_n$, rank 2 cases, the structure of the based ring J is known explicitly [X1, X2, BO]. In these cases we can get a classification of simple H_{k,q_0}-modules for any field k containing square roots of q_0, by means of J_k. The result suggests that an analogue of the Deligne-Langlands-Lusztig classification of simple H_{k,q_0}-modules remains true, provided that k is algebraically closed and the subalgebra $H(W_0)_{k,q_0}$ of H_{k,q_0} generated by all C_w ($w \in W_0$) is semisimple. The details will appear elsewhere.

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