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BOUNDARY BEHAVIOR OF SLE

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1. Introduction and results

Introduction. Several lattice models from statistical physics such as random walks (RWs), loop-erased random walks (LERWs), self-avoiding random walks (SAWs), and critical FK (Fortuin and Kasteleyn) percolations have been shown or are conjectured to be invariant under conformal mappings. The stochastic Loewner evolution (SLE) was first introduced by O. Schramm as a possible scaling limit for the planar LERW [33]. As a one-parameter family of random growth processes, the SLE curves are the only random non-self-crossing curves with a certain Markovian type property and conformal invariance.

Loewner chains are widely applied in complex analysis. For instance, de Branges used Loewner evolutions to prove the Bieberbach conjecture, which states that the nth coefficient in the power series of a univalent function in the class $\mathcal{S}$ should be no greater than n [6]. In fact, Loewner introduced this concept in the 1920's in order to calculate an estimate on the third coefficient.

G. F. Lawler, O. Schramm, and W. Werner used SLE_6 to determine the two-sided disconnection exponent for Brownian motion. It led to the proof of Mandelbrot's conjecture that the Hausdorff dimension of the planar Brownian frontier is 4/3. See [21], [22], [23], [24], and [25]. The planar Brownian frontier is defined as the boundary of the unbounded component of the complement of the planar Brownian path. In [19], G. F. Lawler expressed the Hausdorff dimension of the planar Brownian frontier in terms of the two-sided disconnection exponent.

Numerous discrete models have been proven or are expected to correspond to SLE_κ for some κ. Using Cardy's formula in Carleson's form, Smirnov proved that the critical site percolation on the triangular grid has a conformal invariant scaling limit. He also showed that the scaling limit is described by SLE_6 [35]. G. F. Lawler, O. Schramm, and W. Werner proved that the scaling limits of LERW and the uniform spanning tree (UST) Peano curve with appropriate boundary conditions are, respectively, SLE_2 and SLE_8 [26]. G. F. Lawler, O. Schramm, and W. Werner also showed that if the scaling limit of planar SAWs exists and is conformally invariant, then it is $\mathrm{SLE}_{8/3}$ [27]. R. Kenyon conjectured that the scaling limit of

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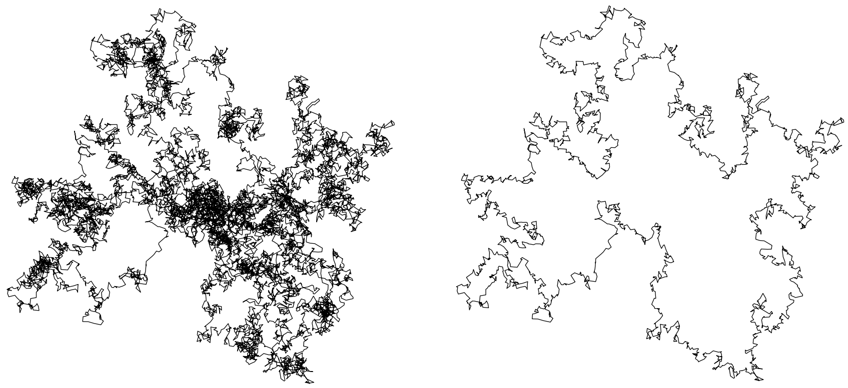

FIGURE 1. Planar Brownian motion and its frontier.

the double domino paths is SLE_4. On the other hand, O. Schramm and S. Sheffield recently proved that the harmonic explorer converges to SLE_4 [34].

From the dimension estimate for the trace and outer boundary of the hull, B. Duplantier conjectured that $\mathrm{SLE}_{\kappa'}$ ($\kappa' = 16/\kappa$) should describe the boundary of the hull of SLE_κ when $\kappa > 4$ (see [7]). In particular, the central charge c (see [8] for the definition) of SLE is invariant under the map $\kappa \mapsto 16/\kappa$:

$$c = 1 - 6\Big(\sqrt{\frac{\kappa}{4}} - \sqrt{\frac{4}{\kappa}}\Big)^2.$$

Duplantier duality has been shown to hold for $\kappa = 8$ and $\kappa = 6$. In the case $\kappa = 8$, the frontier of the UST Peano curve consists of two LERWs, one in the tree and the other in the dual tree. When $\kappa = 6$, the restriction property makes it possible to describe the outer boundary of conditioned SLE_6 in terms of $\mathrm{SLE}_{8/3}$. Using arguments of conformal field theory, B. Duplantier and I. Binder derived the mixed multifractal spectrum $f_{\mathrm{mixed}}(\alpha, \lambda)$ for the scaling and winding (with respect to harmonic measure) in terms of the central charge:

$$f_{\mathrm{mixed}}(\alpha, \lambda) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2}, \tag{1.1}$$

where $b = (25 - c)/12$. See [2] for definitions and [9] for more details of the results.

Definitions. For each $\kappa \geq 0$ and each $z \in \mathbb{H}$, let $g_t(z)$ be the solution of the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z, \tag{1.2}$$

where B_t is a one-dimensional standard Brownian motion on the real line, starting from 0. The solution exists whenever $g_t(z) - \sqrt{\kappa}B_t$ is bounded away from zero. This implies $g_t(z)$ is well-defined up to the first time $\tau(z)$ such that $\lim_{t \uparrow \tau(z)} g_t(z) - \sqrt{\kappa}B_t = 0$. For each $t > 0$, the map g_t is a conformal mapping from the domain $H_t := \{z \in \mathbb{H} : \tau(z) > t\}$ onto $\mathbb{H}$. The process $t \mapsto g_t$ is called *chordal stochastic Loewner evolution* in $\mathbb{H}$ with parameter κ, or SLE_κ. The sets $K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$ are called the *hulls* of the SLE. Chordal SLE_κ is scale invariant in the following sense. For $c > 0$, the process $t \mapsto c^{-1/2} K_{ct}$ has the same law as $t \mapsto K_t$. The process $(t, z) \mapsto c^{-1/2} g_{ct}(\sqrt{c}z)$ has the same law as the process $(t, z) \mapsto g_t(z)$.

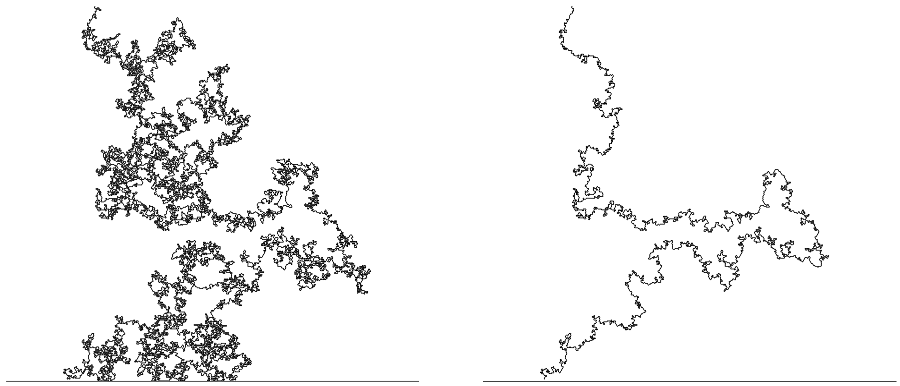

FIGURE 2. The boundary of the hull of SLE_6 describes $\mathrm{SLE}_{8/3}$.

There is (almost surely) a uniquely defined continuous path $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \geq 0$. This path is called the SLE_κ *trace* from 0 to ∞ in $\mathbb{H}$ and is given by

$$\gamma(t) = \lim_{z\to 0} g_t^{-1}(z + \sqrt{\kappa}B_t).$$

Define the backward flow $f_t = g_{-t}$ of g_t for nonnegative t. Then f_t is a conformal map from $\mathbb{H}$ into a subset of $\mathbb{H}$ satisfying

$$\partial_t f_t(z) = \frac{-2}{f_t(z) + \sqrt{\kappa}B_t}, \quad f_0(z) = z. \tag{1.3}$$

For each fixed $t \in \mathbb{R}$, the map $f_t(z)$ has the same distribution as the map $z \mapsto g_t^{-1}(z + \sqrt{\kappa}B_t) - \sqrt{\kappa}B_t$. This follows from the Markov property and translation invariance of Brownian motion. Note that $f_t(z)$ is well-defined on $\mathbb{H}$ for all $t \geq 0$.

Basic properties. In [31], S. Rohde and O. Schramm showed that the SLE_κ trace is almost surely a continuous path for $\kappa \neq 8$. In the special case $\kappa = 8$, this was shown as a consequence of the theorem that the scaling limit of UST Peano curve is chordal SLE_8 [26]. They estimated the derivative expectations for the backward flow to show that g_t^{-1} is almost surely Hölder continuous unless $\kappa = 4$ (when it is not Hölder continuous).

V. Beffara established that the Hausdorff dimension of the SLE_κ trace is almost surely $\min(1 + \kappa/8, 2)$ [1]. S. Rohde and O. Schramm estimated the convergence exponent for the Whitney decomposition of H_1 to obtain an upper bound for the box-counting dimension of the boundary of the SLE_κ hull. They proved that the Hausdorff dimension of ∂K_1 is almost surely at most $1 + 2/\kappa$ for $\kappa > 4$. On the other hand, R. Kenyon conjectured that the Hausdorff dimension of ∂K_1 is almost surely $1 + 2/\kappa$ for $\kappa \geq 4$. (As remarked earlier, this is known for $\kappa = 6, 8$.) For more basic properties of SLE, see [31], [20], and [37].

(Pre-)Schwarzian derivatives. The *logarithmic derivative* or *pre-Schwarzian derivative* Lf of a locally univalent function f is defined by

$$Lf(z) = \frac{f''(z)}{f'(z)}.$$

The *Schwarzian derivative* Sf of a locally univalent function f is defined by

$$Sf(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2.$$

The logarithmic derivative and the Schwarzian derivative satisfy the composition law

$$\begin{aligned} L(f\circ g)(z) &= Lg(z) + Lf(g(z))\cdot g'(z), \\ S(f\circ g)(z) &= Sg(z) + Sf(g(z))\cdot g'(z)^2. \end{aligned} \tag{1.4}$$

Thus, the Schwarzian derivative is Möbius-invariant; that is, $S(T\circ f) = Sf$ for any Möbius transformation T. Also, note that $ST = 0$ if and only if T is a Möbius transformation. Suppose f maps $\mathbb{D}$ conformally into $\mathbb{C}$. Then, by the distortion theorem,

$$|(1-|z|^2)Lf(z) - 2\overline{z}| \le 4 \quad \text{and} \quad (1-|z|^2)^2|Sf(z)| \le 6, \tag{1.5}$$

for $z \in \mathbb{D}$. See p. 9 and p. 13 in [29] for the first estimate and the second estimate, respectively. On $\mathbb{H}$, the second estimate in (1.5) becomes

$$y^2|Sf(z)| \le \frac{3}{2},$$

where $y = \operatorname{Im} z$.

Some analytic criteria for univalence have limited applications because they are far from necessary. The following general criteria involving the (pre-)Schwarzian derivative are useful sufficient conditions and almost necessary in a certain sense. See [29] for more references.

Theorem (Becker univalence criterion)**.** *Suppose f is analytic and locally univalent in $\mathbb{D}$. If*

$$(1-|z|^2)|zLf(z)| \le 1 \tag{1.6}$$

for $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$. The bound 1 *is sharp.*

Theorem (Nehari univalence criterion)**.** *Suppose f is analytic and locally univalent in $\mathbb{D}$. If*

$$(1-|z|^2)^2|Sf(z)| \le 2 \tag{1.7}$$

for $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$. The bound 2 *is sharp.*

These univalence criteria have the following geometric application: If either estimate (1.6) or (1.7) is uniformly bounded away from the sharp constant, then f maps $\mathbb{D}$ conformally onto a quasi-disk. A Jordan curve J is called a *quasi-circle* if

$$\operatorname{diam} J(a,b) \le C|a-b| \text{ for } a, b \in J,$$

where $J(a,b)$ is the smallest arc (in the sense of diameter) of J between a and b. The inner domain of a quasi-circle is called a *quasi-disk*. The asymptotic behavior of the (pre-)Schwarzian derivative measures the degree of non-conformality of the boundary [29].

Beta numbers. In order to study the subsets of rectifiable curves, given a set E in the complex plane and a square Q, P. W. Jones introduced an L^∞ version of the *beta number*

$$\beta(Q) = \beta_E(Q) := \frac{1}{\ell(Q)} \inf_{L \in \mathcal{L}} \sup_{z \in E \cap 3Q} \operatorname{dist}(z, L),$$

where $\mathcal{L}$ is the set of all lines L intersecting Q and an L^∞ *Jones content*

$$\mathcal{J}_\infty(E) = \sum_{Q \in \mathcal{D}} \beta_\infty^2(Q)\ell(Q),$$

where the summation extends over all dyadic squares.

Theorem A (Jones [17])**.** *Suppose E is a subset of $\mathbb{R}^2$. Then E is contained in a rectifiable curve if and only if* $\operatorname{diam}(E)$ *and the L^∞ Jones content $\mathcal{J}_\infty(E)$ are finite.*

A connected set E is called *uniformly wiggly* with constant β_0 if $\beta_E(Q) > \beta_0$ for every Q such that $3^{-1}Q$ intersects E and $\ell(Q) \leq \operatorname{diam}(E)$.

Theorem B (Bishop, Jones [4])**.** *Suppose $E \subset \mathbb{R}^2$ is a closed, connected, uniformly wiggly set with constant β_0. Then* $\dim(E) \geq 1 + C\beta_0^2$*, where C is an absolute constant.*

C. J. Bishop, P. W. Jones, R. Pemantle, and Y. Peres used a stochastic version of this theorem to prove that the dimension of the Brownian frontier is greater than 1 [5]. Also, C. J. Bishop and P. W. Jones showed that large Schwarzian implies large beta numbers. For the precise statement, see [3] or [13]. Here, we state the quasiconformal version. Using this, J. Graczyk and P. W. Jones proved that every subarc of the boundary of the Siegel disk (for rotation numbers of "constant type") has the Hausdorff dimension strictly larger than 1 [15]. A Jordan curve Γ is K-*quasiconformal* (or a K-*quasicircle*) if it is the image of the unit circle by a K-quasiconformal homeomorphism $h : \mathbb{C} \to \mathbb{C}$.

Theorem C (Graczyk, Jones [15])**.** *Suppose that Γ is a K-quasicircle. Assume also that there exist $\Delta > 0$ and $\epsilon > 0$ so that for every $z_0 \in \mathbb{D}$,*

$$\sup_{\rho(z,z_0) \leq \Delta} \big|S(h)\big|\big|1 - |z|\big|^2 \geq \epsilon.$$

Then there exists $\epsilon_0 > 0$ which depends solely on $K, \Delta,$ and ϵ so that

$$\beta_\Gamma > \epsilon_0.$$

Itô's formula. If X is a continuous local martingale and f has two continuous derivatives, then almost surely

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s, \tag{1.8}$$

where $\langle X\rangle_t$ is the quadratic variation or the variance process of X_t, which is defined to be the unique continuous process such that $X_t^2 - \langle X\rangle_t$ is a local martingale. See [10] or [18]. If X and Y are continuous local martingales, then almost surely

$$X_tY_t - X_0Y_0 = \int_0^t Y_s\, dX_s + \int_0^t X_s\, dY_s + \langle X, Y\rangle_t, \tag{1.9}$$

where $\langle X, Y\rangle_t$ is the covariance process of X and Y, defined by

$$\langle X, Y\rangle_t = \frac{1}{4}\Big(\langle X+Y\rangle_t - \langle X-Y\rangle_t\Big).$$

Goluzin's identities. We will use a chordal version of Goluzin's identities to compute the second moment of the (pre-)Schwarzian. Goluzin used a radial version of the identities to obtain the sharpened forms of certain inequalities which constitute the distortion theorems for the class Σ, consisting of all univalent functions

$$g(\zeta) = \zeta + b_0 + b_1\zeta^{-1} + \cdots \quad (|\zeta| > 1),$$

in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Suppose f_t is a radial Loewner chain:

$$\frac{df_t}{dt}(z) = -f_t(z)\frac{1+k(t)f_t(z)}{1-k(t)f_t(z)}, \tag{1.10}$$

where $k(t)$ is a driving function ($|k(t)|=1$). Goluzin's identities state

$$\frac{d}{dt}\log\Big[\frac{e^{-t}}{f_t(z)f_t(w)}\frac{f_t(z)-f_t(w)}{z-w}\Big] = -2\frac{k(t)f_t(z)}{1-k(t)f_t(z)}\frac{k(t)f_t(w)}{1-k(t)f_t(w)} \tag{1.11}$$

and

$$\frac{d}{dt}\log(1-f_t(z)\overline{f_t(w)}) = 2\frac{k(t)f_t(z)}{1-k(t)f_t(z)}\overline{\Big(\frac{k(t)f_t(w)}{1-k(t)f_t(w)}\Big)}. \tag{1.12}$$

See p. 118 in [14]. We will state and prove a chordal version of Goluzin's identities in the proof of Lemma 2.1. For the reader's convenience, we give a proof of (1.11) and (1.12).

Proof of Goluzin's identities. It follows from (1.10) that

$$\begin{aligned}
\frac{d}{dt}\log\Big[\frac{e^{-t}}{f_t(z)f_t(w)}\frac{f_t(z)-f_t(w)}{z-w}\Big] &= -1 + \frac{d}{dt}\log\Big(\frac{1}{f_t(w)} - \frac{1}{f_t(z)}\Big) \\
&= -1 + \frac{\dfrac{1}{f_t(w)}\dfrac{1+k(t)f_t(w)}{1-k(t)f_t(w)} - \dfrac{1}{f_t(z)}\dfrac{1+k(t)f_t(z)}{1-k(t)f_t(z)}}{\dfrac{1}{f_t(w)} - \dfrac{1}{f_t(z)}} \\
&= \frac{\dfrac{1}{f_t(w)}\Big(-1+\dfrac{1+k(t)f_t(w)}{1-k(t)f_t(w)}\Big) - \dfrac{1}{f_t(z)}\Big(-1+\dfrac{1+k(t)f_t(z)}{1-k(t)f_t(z)}\Big)}{\dfrac{1}{f_t(w)} - \dfrac{1}{f_t(z)}} \\
&= \Big(\frac{1}{f_t(w)} - \frac{1}{f_t(z)}\Big)^{-1}\Big(\frac{2k(t)}{1-k(t)f_t(w)} - \frac{2k(t)}{1-k(t)f_t(z)}\Big) \\
&= \frac{f_t(z)f_t(w)}{f_t(z)-f_t(w)}\frac{-2k(t)^2\big(f_t(z)-f_t(w)\big)}{\big(1-k(t)f_t(z)\big)\big(1-k(t)f_t(w)\big)} \\
&= \frac{-2k(t)^2 f_t(z)f_t(w)}{\big(1-k(t)f_t(z)\big)\big(1-k(t)f_t(w)\big)}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}\frac{d}{dt}\log(1-f_t(z)\overline{f_t(w)}) &= \frac{-1}{1-f_t(z)\overline{f_t(w)}}\Big(\frac{df_t}{dt}(z)\overline{f_t(w)} + \overline{\frac{df_t}{dt}(w)}f_t(z)\Big)\\ &= \frac{f_t(z)\overline{f_t(w)}}{1-f_t(z)\overline{f_t(w)}}\Big(\frac{1+k(t)f_t(z)}{1-k(t)f_t(z)} + \frac{1+\overline{k(t)f_t(w)}}{1-\overline{k(t)f_t(w)}}\Big)\\ &= \frac{2f_t(z)\overline{f_t(w)}}{1-f_t(z)\overline{f_t(w)}}\frac{1-|k(t)|^2 f_t(z)\overline{f_t(w)}}{\big(1-k(t)f_t(z)\big)\big(1-\overline{k(t)f_t(w)}\big)}\\ &= \frac{2f_t(z)\overline{f_t(w)}}{\big(1-k(t)f_t(z)\big)\big(1-\overline{k(t)f_t(w)}\big)},\end{aligned}$$

which completes the proof. □

Definition of $\mathcal{BMO}$ and the John-Nirenberg inequality. Just as the Hardy space H^1 is an appropriate substitute for L^1 in many results concerning singular integrals, the BMO space, or the space of functions of bounded mean oscillation, is a natural substitute for L^∞. The BMO space on $\mathbb{R}^n$ is the set of equivalence classes of locally integrable functions f (modulo additive constants) for which the inequality

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le C \tag{1.13}$$

holds for all cubes Q. Here, f_Q denotes the mean value of f over the cube Q. The smallest such C is taken to be BMO norm and is denoted by $\|f\|_{\mathrm{BMO}}$ (see [36]).

Theorem (John and Nirenberg [16]). *There exist positive constants c and C such that for each cube Q, each $f \in \mathrm{BMO}$, and each λ,*

$$|\{x\in Q : |f(x)-f_Q| > \lambda\}| \le C|Q|\exp(-c\lambda/\|f\|_{\mathrm{BMO}}).$$

This notion can be modified in the setting of continuous martingales. A continuous martingale M belongs to the space $\mathcal{BMO}$ of martingales of bounded mean oscillation if there exists C such that for all stopping times τ

$$\mathbb{E}\big(|M_\infty - M_\tau|^2\big|\mathcal{F}_\tau\big) \le C^2, \quad \text{almost surely.}$$

The smallest such C is defined as the $\mathcal{BMO}$ norm of M and is denoted by $\|M\|_{\mathcal{BMO}}$. Let $M^* = \sup_t |M_t|$. For the following results, see pp. 208–211 in [10].

Theorem (Probabilistic analogue of John and Nirenberg inequality). *There exists a positive constant C such that for each $M \in \mathcal{BMO}$,*

$$\mathbb{P}[M^* > \lambda\|M\|_{\mathcal{BMO}}] \le Ce^{-\lambda/e}.$$

Main results. For the SLE related maps, it is well known that $\operatorname{Im} f_t(z)$ is monotone increasing in t for every $z \in \mathbb{H}$. For $z \in \mathbb{H}$ and $u \in \mathbb{R}$, set a stopping time

$$T_u = T_u(z) := \inf\{t \in \mathbb{R} : \operatorname{Im}(f_t(z)) \ge e^u\}. \tag{1.14}$$

It is also well known that for all $z \in \mathbb{H}$, almost surely $T_u \neq \infty$. See [31].

Theorem 1. *For any $u \in \mathbb{R}$ and $x \neq 0$, the normalized pre-Schwarzian $yLf_t(x+iy)$ at $t = T_u$ has the asymptotic second moment*

$$\lim_{y\to 0}\mathbb{E}\big[|yLf_{T_u}(x+iy)|^2\big] = \frac{1}{2(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}. \tag{1.15}$$

Furthermore, for fixed $z = x + iy (x \neq 0) \in \mathbb{H}$,

$$\lim_{t\to\infty} \mathbb{E}\big[|yLf_t(z)|^2\big] = \frac{1}{2(\sqrt{\kappa/4} + \sqrt{4/\kappa})^2}. \tag{1.16}$$

The normalized pre-Schwarzian $yLf_t(z)$, after we subtract a negligible term, is a complex martingale of $\mathcal{BMO}$. To see this, define a random conformal map F_t from $\mathbb{H}$ into a subset of $\mathbb{C} \setminus \mathbb{R}_+$ by $F_t(z) = (f_t(z) + \sqrt{\kappa}B_t)^2$ and set

$$\begin{aligned} L_t = L_t(z) &:= \frac{\kappa}{4+\kappa}\, yLf_t(z) + \frac{4}{4+\kappa}\big(yLF_t(z) - yLF_0(z)\big) \\ &= yLf_t(z) + \frac{4}{4+\kappa}\Big(\frac{yf_t'(z)}{f_t(z) + \sqrt{\kappa}B_t} - \frac{y}{z}\Big) \\ &= \frac{-2}{\sqrt{\kappa/4} + \sqrt{4/\kappa}} \int_0^t \frac{yf_s'(z)}{(f_s(z) + \sqrt{\kappa}B_s)^2}\, dB_s. \end{aligned} \tag{1.17}$$

Theorem 2. *The process L is a $\mathcal{BMO}$ martingale in t and*

$$\|L\|^2_{\mathcal{BMO}} = \frac{1}{2(\sqrt{\kappa/4} + \sqrt{4/\kappa})^2}.$$

Theorem 3. *For any $u \in \mathbb{R}$ and $x \neq 0$, the normalized Schwarzian $y^2 Sf_t(x+iy)$ at $t = T_u$ has the asymptotic second moment*

$$\lim_{y\to 0} \mathbb{E}\big[|y^2 Sf_{T_u}(x+iy)|^2\big] = \frac{9\kappa}{(\kappa+6)(3\kappa+8)}. \tag{1.18}$$

Furthermore, for $x \neq 0$, we have

$$\lim_{y\to 0}\Big(\lim_{t\to\infty} \mathbb{E}\big[|y^2 Sf_t(x+iy)|^2\big]\Big) = \frac{9\kappa}{(\kappa+6)(3\kappa+8)}. \tag{1.19}$$

The normalized Schwarzian $y^2 Sf_t(z)$, after we subtract a negligible term, is a complex martingale of $\mathcal{BMO}$. To see this, set

$$\begin{aligned} S_t = S_t(z) &:= \frac{3\kappa}{3\kappa+8} y^2 Sf_t(z) + \frac{8}{3\kappa+8}\big(y^2 SF_t(z) - y^2 SF_0(z)\big) \\ &= y^2 Sf_t(z) + \frac{12}{3\kappa+8}\Big(\frac{y^2}{z^2} - \frac{y^2 f_t'(z)^2}{(f_t(z) + \sqrt{\kappa}B_t)^2}\Big) \\ &= \frac{24\sqrt{\kappa}}{3\kappa+8} \int_0^t \frac{y^2 f_s'(z)^2}{(f_s(z) + \sqrt{\kappa}B_s)^3}\, dB_s. \end{aligned} \tag{1.20}$$

Theorem 4. *The process S is a $\mathcal{BMO}$ martingale in t. Furthermore, for $x \neq 0$,*

$$\lim_{y\to 0} \|S(x+iy)\|^2_{\mathcal{BMO}} = \frac{9\kappa}{(3\kappa+8)(\kappa+6)}.$$

Recall the definition of the hyperbolic distance or Poincaré metric $d_{\mathbb{H}}$:

$$\cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2y_1 y_2} = -1 + \frac{|z_1 - \overline{z_2}|^2}{2y_1 y_2}, \tag{1.21}$$

where $y_j = \operatorname{Im} z_j$ $(j = 1, 2)$. See p. 136 in [30]. The normalized (pre-)Schwarzian has correlations that decay exponentially in the hyperbolic distance from $z_1 = x_1 + iy_1$ to $z_2 = x_2 + iy_2$. Thus, the normalized (pre-)Schwarzian derivatives are nearly independent if z_1 and z_2 are far away from each other. For $z_1, z_2 \in \mathbb{H}$ and $u \in \mathbb{R}$, set a stopping time $T_u^* = T_u^*(z_1, z_2) := \max\big(T_u(z_1), T_u(z_2)\big)$.

Theorem 5. *The $\mathcal{BMO}$ martingale L_t at $t = \infty$ has exponential decay of correlations:*

$$\left|\mathbb{E}L_\infty(z_1)\overline{L_\infty(z_2)}\right| = \frac{1}{2(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\cosh^{-2}\frac{d_{\mathbb{H}}(z_1,z_2)}{2}. \tag{1.22}$$

Furthermore, given $u \in \mathbb{R}$ and a compact subset K of the real line, L_t at T_u^* has exponential decay of correlations:

$$\limsup_{y_1,y_2\to 0} y_1 y_2 \left|\mathbb{E}L_{T_u^*}(z_1)\overline{L_{T_u^*}(z_2)}\right| \le \frac{1}{2(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\cosh^{-2}\frac{\rho}{2}, \tag{1.23}$$

where the limsup is taken over all $(z_1, z_2) \in \big(K \times (0,1)\big)^2$ such that $d_{\mathbb{H}}(z_1, z_2) \ge \rho$.

Theorem 6. *The $\mathcal{BMO}$ martingale S_t at $t = \infty$ has exponential decay of correlations:*

$$\begin{aligned}\left|\mathbb{E}S_\infty(z_1)\overline{S_\infty(z_2)}\right| \le{}& \frac{9\kappa}{(\kappa+6)(3\kappa+8)}\cosh^{-4}\frac{d_{\mathbb{H}}(z_1,z_2)}{2}\\ &+ C_2(\kappa)\frac{y_1y_2}{|z_1z_2|}\cosh^{-2}\frac{d_{\mathbb{H}}(z_1,z_2)}{2}.\end{aligned} \tag{1.24}$$

Furthermore, given $u \in \mathbb{R}$ and a compact subset K $(0 \notin K)$ of the real line, S_t at T_u^* has exponential decay of correlations:

$$\limsup_{y_1,y_2\to 0} \left|\mathbb{E}S_{T_u^*}(z_1)\overline{S_{T_u^*}(z_2)}\right| \le \frac{9\kappa}{(\kappa+6)(3\kappa+8)}\cosh^{-4}\frac{\rho}{2}, \tag{1.25}$$

where the limsup is taken over all $(z_1, z_2) \in \big(K \times (0,1)\big)^2$ such that $d_{\mathbb{H}}(z_1, z_2) \ge \rho$.

It is likely that this result leads to an estimate on the lower bound for the Hausdorff dimension of the SLE boundary. This should be true because Theorems A, B, C mean that "most often" the boundary is wiggly near $f_t(z)$. Furthermore, the decay of correlations from Theorem 6 means that a statistical version of Theorem C should hold, as it does in [5]. The estimate for the upper bound on the Hausdorff dimension is already established by S. Rohde and O. Schramm [31]. While the Hausdorff dimension of the SLE_κ trace was proved by V. Beffara [1], it remains an open conjecture for the boundary of the hull in case $\kappa > 4$.

Computing the derivative expectation or moment generating function for $|g_t'(z)|$ for an arbitrary complex number z, S. Rohde and O. Schramm proved that g_t^{-1} is almost surely Hölder continuous when $\kappa \ne 4$ in [31]. We reexamine their derivative expectation to derive the conjectured sharp estimate for the Hölder exponent. After oral communication with us, I. Binder and B. Duplantier derived the same formula from the multifractal spectrum of SLE_κ independently.

Theorem 7. *Suppose $\kappa \ne 4$, a sufficiently small $c > 0$, and a bounded set $D \subset \mathbb{H}$ are given. Then almost surely f_1 is h-Hölder continuous in D on the event that $\operatorname{Im} f_1(z) \ge c$ in D, provided*

$$h < h(\kappa) := 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2}+\frac{2}{\mu}},$$

where $\mu = \kappa/4 + 2 + 4/\kappa$.

2. (Pre-)Schwarzian derivatives

Suppose f_t is a general Loewner chain with a driving function U_t. By direct calculation, we observe

$$\partial_t L f_t(z) = \frac{-4 f_t'(z)}{(f_t(z)+U_t)^3}, \quad \partial_t S f_t(z) = \frac{12 f_t'(z)^2}{(f_t(z)+U_t)^4}. \tag{2.1}$$

Consider a Loewner evolution $F_t := (f_t + U_t)^2$ in the slit domain $\mathbb{C}\setminus\mathbb{R}_+$. Suppose a sufficiently small $c > 0$ and a bounded set $D \subset \mathbb{H}$ are given. Here, by bounded set we mean bounded in the sense of the Euclidean metric on $\mathbb{R}^2$ (not the hyperbolic metric on $\mathbb{H}$). Due to the composition law for the (pre-)Schwarzian derivatives, the normalized (pre-)Schwarzian derivatives of F_t and f_t have the same asymptotic behavior in D, as long as $\inf_{z\in D} \operatorname{Im} f_t(z) \geq c$. By (1.4), observe

$$L F_t(z) = L f_t(z) + \frac{f_t'(z)}{f_t(z)+U_t} \tag{2.2}$$

and

$$S F_t(z) = S f_t(z) - \frac{3}{2}\Big(\frac{f_t'(z)}{f_t(z)+U_t}\Big)^2. \tag{2.3}$$

For instance, if $\inf_{z\in D} \operatorname{Im} f_t(z) \geq c$, then, by the Koebe distortion theorem, we have

$$|yLF_t(z) - yLf_t(z)| \leq \frac{|yf_t'(z)|}{|f_t(z)+U_t|} \leq \frac{|yf_t'(z)|}{c} \leq C\operatorname{dist}(f_t(z), \partial f_t(\mathbb{H})). \tag{2.4}$$

Now, consider the SLE backward flow f_t. Use the Itô formula to obtain

$$d\frac{f_t'(z)}{f_t(z)+\sqrt{\kappa}B_t} = (\kappa+4)\frac{f_t'(z)}{(f_t(z)+\sqrt{\kappa}B_t)^3}\,dt - \sqrt{\kappa}\frac{f_t'(z)}{(f_t(z)+\sqrt{\kappa}B_t)^2}\,dB_t \tag{2.5}$$

and

$$d\Big(\frac{f_t'(z)}{f_t(z)+\sqrt{\kappa}B_t}\Big)^2 = (3\kappa+8)\frac{f_t'(z)^2}{(f_t(z)+\sqrt{\kappa}B_t)^4}\,dt - 2\sqrt{\kappa}\frac{f_t'(z)^2}{(f_t(z)+\sqrt{\kappa}B_t)^3}\,dB_t. \tag{2.6}$$

To show (2.5), let $Z_t = f_t'(z)$ and $W_t = 1/(f_t(z)+\sqrt{\kappa}B_t)$. By the Itô formula, $dZ_t = 2Z_tW_t^2\,dt$ and $dW_t = (\kappa+2)W_t^3\,dt - \sqrt{\kappa}W_t^2\,dB_t$. On the other hand, $d\langle Z, W\rangle_t = 0$. It follows from (1.9) that

$$d(Z_tW_t) = (\kappa+4)Z_tW_t^3\,dt - \sqrt{\kappa}Z_tW_t^2\,dB_t.$$

For (2.6), the Itô formula and (2.5) imply that

$$d(Z_tW_t)^2 = 2Z_tW_td(Z_tW_t) + d\langle ZW\rangle_t = (3\kappa+8)Z_t^2W_t^4\,dt - 2\sqrt{\kappa}Z_t^2W_t^3\,dB_t.$$

Combining (2.1), (2.2), (2.3), (2.5), and (2.6), one can easily check the identities in (1.17) and (1.20).

Pre-Schwarzian expectation.

Lemma 2.1. *Suppose f_t is a general Loewner chain with a driving function U_t. Then the total variation of the modified Schwarzian derivative of f_t is*

$$y^2 \int_0^t |\partial_s S f_s(z)|\, ds = \frac{3}{2}\Big(1 - \frac{|y f_t'(z)|^2}{(\operatorname{Im} f_t(z))^2}\Big). \tag{2.7}$$

Proof. Based on a radial version, it is not hard to formulate and prove Goluzin's identities in $\mathbb{H}$:

$$\frac{d}{dt}\frac{f_t'(z) f_t'(w)}{(f_t(z)-f_t(w))^2} = \frac{2 f_t'(z) f_t'(w)}{(f_t(z)+U_t)^2 (f_t(w)+U_t)^2} \tag{2.8}$$

and

$$\frac{d}{dt}\frac{f_t'(z)\overline{f_t'(w)}}{(f_t(z)-\overline{f_t(w)})^2} = \frac{2 f_t'(z)\overline{f_t'(w)}}{(f_t(z)+U_t)^2 (\overline{f_t(w)}+U_t)^2}. \tag{2.9}$$

To verify (2.8), we first note that

$$\frac{d}{dt}\frac{\partial}{\partial z}\frac{\partial}{\partial w}\log\frac{f_t(z)-f_t(w)}{z-w} = \frac{d}{dt}\frac{f_t'(z) f_t'(w)}{(f_t(z)-f_t(w))^2}.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial z}\frac{\partial}{\partial w}\log\frac{f_t(z)-f_t(w)}{z-w} &= \frac{\partial}{\partial z}\frac{\partial}{\partial w}\frac{2}{(f_t(z)+U_t)(f_t(w)+U_t)} \\ &= \frac{2 f_t'(z) f_t'(w)}{(f_t(z)+U_t)^2 (f_t(w)+U_t)^2},\end{aligned}$$

which shows (2.8). In particular, by letting $w \to z$ in the above identity,

$$\frac{d}{dt}\frac{S f_t(z)}{6} = \frac{2 f_t'(z)^2}{(f_t(z)+U_t)^4},$$

which shows the second part of (2.1). For (2.9), one can use a method similar to that above. Alternately, we compute directly:

$$\begin{aligned}&\frac{(f_t(z)-\overline{f_t(w)})^2}{2 f_t'(z)\overline{f_t'(w)}}\frac{d}{dt}\frac{f_t'(z)\overline{f_t'(w)}}{(f_t(z)-\overline{f_t(w)})^2} \\ &\quad = \frac{1}{(f_t(z)+U_t)^2} + \frac{1}{(\overline{f_t(w)}+U_t)^2} \\ &\quad\quad - \frac{1}{f_t(z)-\overline{f_t(w)}}\Big(\frac{-2}{f_t(z)+U_t} - \frac{-2}{\overline{f_t(w)}+U_t}\Big) \\ &\quad = \frac{(\overline{f_t(w)}+U_t)^2 + (f_t(z)+U_t)^2 - 2(f_t(z)-\overline{f_t(w)})}{(f_t(z)+U_t)^2(\overline{f_t(w)}+U_t)^2} \\ &\quad = \frac{(f_t(z)-\overline{f_t(w)})^2}{(f_t(z)+U_t)^2(\overline{f_t(w)}+U_t)^2}.\end{aligned}$$

It follows from (2.1) and (2.9) that

$$y^2\int_0^t |\frac{d}{ds}Sf_s(z)|\,ds = y^2\int_0^t \frac{d}{ds}\frac{6|f_s'(z)|^2}{(f_s(z)-\overline{f_s(z)})^2}\,ds$$
$$= y^2\int_0^t \frac{d}{ds}\frac{6|f_s'(z)|^2}{-4(\operatorname{Im} f_s(z))^2}\,ds$$
$$= \frac{3}{2}(1-\frac{|yf_t'(z)|^2}{(\operatorname{Im} f_t(z))^2}),$$

which completes the proof. □

Proof of Theorem 1. By (2.1), (1.17), and the optional stopping theorem, we get

$$\lim_{y\to 0}\mathbb{E}\Big|yLf_{T_u}(z)+\frac{4}{4+\kappa}\Big(\frac{yf_{T_u}'(z)}{f_{T_u}(z)+\sqrt{\kappa}B_{T_u}}-\frac{y}{z}\Big)\Big|^2$$
$$=\frac{1}{3(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\lim_{y\to 0}\mathbb{E}\int_0^{T_u} y^2|\partial_s Sf_s(z)|\,ds$$
$$=\frac{1}{2(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\Big(1-\lim_{y\to 0}\mathbb{E}\frac{|yf_{T_u}'(z)|^2}{e^{2u}}\Big).$$

The last equality above comes from Lemma 2.1 and the definition of T_u (1.14). We need to prove that $\lim_{y\to 0}\mathbb{E}|yf_{T_u}'(z)|^2=0$. By scale invariance, we may assume $u=0$. For $\kappa\neq 4$, it follows from the derivative expectation or Theorem 3.2 in [31] that

$$\lim_{y\to 0}\mathbb{E}|yf_{T_0}'(z)|^2\le C\lim_{y\to 0} y^{2(2-\frac{1-\sqrt{1-4/\mu}}{2/\mu})}=0,$$

where $\mu=\kappa/4+2+4/\kappa$. (This estimate can be found in (4.4) of this paper.) For $\kappa=4$, the Schwarz lemma implies that $|yf_{T_0}'(z)|\le \operatorname{Im} f_{T_0}(z)=1$, and hence $\mathbb{E}|yf_{T_0}'(z)|^2\le\mathbb{E}|yf_{T_0}'(z)|^{3/2}\le Cy$. On the other hand,

$$\lim_{y\to 0}\mathbb{E}\Big|\frac{yf_{T_u}'(z)}{f_{T_u}(z)+\sqrt{\kappa}B_{T_u}}\Big|^2\le\lim_{y\to 0}\mathbb{E}\Big|\frac{yf_{T_u}'(z)}{\operatorname{Im} f_{T_u}(z)}\Big|^2=0, \tag{2.10}$$

which completes the proof of (1.15).

The second part, the equality (1.16) can be easily proved since $|yf_t'(z)|/\operatorname{Im} f_t(z)$ tends to 0 as $t\to\infty$. Indeed, $\operatorname{Im} f_t(z)$ is increasing in t and it tends to ∞ as $t\to\infty$. On the other hand, it follows from the Koebe distortion theorem that $|yf_t'(z)|$ is comparable to the distance from $f_t(z)$ to the boundary $\partial f_t(\mathbb{H})$.

In the subsection "$\mathcal{BMO}$ norm and ODE" below, we will also prove this second part using a different method. □

Pre-Schwarzian and $\mathcal{BMO}$. In a proper setup, the pre-Schwarzian derivatives of SLE_κ maps are $\mathcal{BMO}$ martingales. (Recall (1.17).) As a consequence, they satisfy the John-Nirenberg inequality.

Proof of Theorem 2. Suppose a stopping time τ is given. By the strong Markov property, we have

$$y^2\int_\tau^\infty \mathbb{E}\big[|\partial_s Sf_s(z)|\big|\mathcal{F}_\tau\big]\,ds = y^2|f_\tau'(z)|^2\int_0^\infty \mathbb{E}\,|\partial_s Sf_s(f_\tau(z)+\sqrt{\kappa}B_\tau)|\,ds. \tag{2.11}$$

The Schwarz lemma implies $y|f_\tau'(z)| \le \operatorname{Im}(f_\tau(z) + \sqrt{\kappa}B_\tau)$. Therefore, it suffices to show that

$$y^2\mathbb{E}\int_0^\infty |\partial_s Sf_s(z)|\, ds$$

is uniformly bounded above by a universal constant C. (This is just a version of Goluzin's theorem.) Since $|yf_t'(z)|/\operatorname{Im} f_t(z) \to 0$ as $t \to \infty$, it follows from (2.7) that the above integral has the constant value $3/2$. $\square$

$\mathcal{BMO}$ **norm and ODE.** The method presented in this subsection turns out to be useful to formulate the Schwarzian expectation. However, the reader may skip this subsection. We will use a different method to prove the second part of Theorem 1,

$$y^2\mathbb{E}\int_0^\infty |\partial_t Sf_t(z)|\, dt = \frac{3}{2}.$$

Claim 1. *Define a function w by*

$$w(x) = \frac{1}{6}(x^2+1)\mathbb{E}\int_0^\infty |\partial_t Sf_t(x+i)|\, dt. \tag{2.12}$$

Then

$$y^2\mathbb{E}\int_0^\infty |\partial_t Sf_t(z)|\, dt = 6\frac{w(x/y)}{1+(x/y)^2}. \tag{2.13}$$

Claim 2. *The function w satisfies the inhomogeneous ODE: $Hw = 1$, where H is the second-order linear differential operator given by*

$$Hv := -\frac{\kappa}{4}(1+x^2)v_{xx} + (\kappa+2)xv_x - \frac{(4+3\kappa/2)x^2 - (4+\kappa/2)}{1+x^2}v.$$

Claim 3. $w(x) = (x^2+1)/4$.

Proof of Claim 1. It follows from the scaling property of SLE that $|f_t(x+iy) + \sqrt{\kappa}B_t|$ and $|f_t'(x+iy)|$ have the same distribution as $y|f_{ty^{-2}}(x/y+i) + \sqrt{\kappa}B_{ty^{-2}}|$ and $|f_{ty^{-2}}'(x/y+i)|$, respectively. Consequently, we observe that $|\partial_t Sf_t(x+iy)|$ and $y^{-4}|(\partial_t Sf)_{ty^{-2}}(x/y+i)|$ are identically distributed. With the substitution $s = ty^{-2}$, we have

$$y^2\mathbb{E}\int_0^\infty |\partial_t Sf_t(x+iy)|\, dt = \mathbb{E}\int_0^\infty |\partial_s Sf_s(\frac{x}{y}+i)|\, ds,$$

which completes the proof. $\square$

Proof of Claim 2. For $z = x+i \in \mathbb{H}$ and any nonnegative number u, we set a stopping time

$$\tau_u = \tau_u(z) := \inf\{t \ge 0 : \operatorname{Im} f_t(z) \ge ye^u\}. \tag{2.14}$$

τ_u is well-defined since $\operatorname{Im} f_t$ is monotone increasing in t. Use a change of variable $u = U_t = \log(\operatorname{Im} f_t(z)/\operatorname{Im} f_0(z))$, $dU_t = 2/|f_t(z)+\sqrt{\kappa}B_t|^2 dt$ to get

$$\begin{aligned}\mathbb{E}\int_0^\infty |\partial_t Sf_t(z)|\, dt &= 12\mathbb{E}\int_0^\infty \frac{|f_t'(z)|^2\, dt}{|f_t(z)+\sqrt{\kappa}B_t|^4} \\ &= 6\mathbb{E}\int_0^\infty \frac{|f_{\tau_u}'(z)|^2\, du}{|f_{\tau_u}(z)+\sqrt{\kappa}B_{\tau_u}|^2}.\end{aligned}$$

Let $v(x,s) := \mathbb{E}\big(|(x+i)f'_{\tau_s}(x+i)|^2/|f_{\tau_s}(x+i)+\sqrt{\kappa}B_{\tau_s}|^2\big)$. We will show that v is a solution to the parabolic PDE: $v_s = -Hv$ with an initial condition $v(x,0)=1$. By (2.5) and the Itô formula, we have

$$\begin{aligned} d\log\frac{f'_t(z)}{f_t(z)+\sqrt{\kappa}B_t} &= \frac{(\frac{1}{2}\kappa+4)}{(f_t(z)+\sqrt{\kappa}B_t)^2}\,dt \\ &\quad - \frac{\sqrt{\kappa}}{f_t(z)+\sqrt{\kappa}B_t}\,dB_t. \end{aligned}$$

Let $X_u = \operatorname{Re}(f_{\tau_u}(z)+\sqrt{\kappa}B_{\tau_u})/\operatorname{Im}(f_{\tau_u}(z)+\sqrt{\kappa}B_{\tau_u})$. Then X_u is an Itô diffusion

$$dX_u = -2X_u du + \sqrt{\frac{\kappa}{2}}\sqrt{1+X_u^2}dB_u, \tag{2.15}$$

with $X_0 = x$. Now, v can be expressed as a Feynman-Kac type integral in terms of X_u:

$$v(x,s) = \mathbb{E}\Big(\exp\Big[(4+\frac{\kappa}{2})\int_0^s \frac{X_u^2-1}{X_u^2+1}du - \sqrt{2\kappa}\int_0^s \frac{X_u}{\sqrt{X_u^2+1}}dB_u\Big]\Big).$$

Therefore, v satisfies the following parabolic equation with an initial condition:

$$v_s = -Hv \text{ and } v(x,0)\equiv 1.$$

Here, we use the following version of the Feynman-Kac formula: If X_t is an Itô diffusion with

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

then

$$v = v(x,s) = \mathbb{E}^x\Big(\exp\Big[\int_0^s p(X_t)\,dB_t - \frac{1}{2}\int_0^s p(X_t)^2\,dt - \int_0^s q(X_t)\,dt\Big]\Big)$$

satisfies

$$v_s = \frac{1}{2}\sigma^2 v_{xx} + (b+p\sigma)v_x - qv.$$

Observe that

$$w(x) = \int_0^\infty v(x,t)\,dt = \int_0^\infty e^{-Ht}v(x,0)\,dt = \int_0^\infty e^{-Ht}\,dt.$$

Using the formula

$$(H+\lambda I)^{-1} = \int_0^\infty e^{-Ht}e^{-\lambda t}\,dt,$$

we deduce that w is a solution to the inhomogeneous ODE: $Hw=1$. □

Proof of Claim 3. The function w is of the form

$$\begin{aligned} w(x) &= c_1(1+x^2)^{\frac{3}{2}+\frac{2}{\kappa}}p_\nu^\mu(x) + c_2(1+x^2)^{\frac{3}{2}+\frac{2}{\kappa}}q_\nu^\mu(x) \\ &\quad + (1+x^2)^{\frac{3}{2}+\frac{2}{\kappa}}p_\nu^\mu(x)\int_x^\infty \frac{4}{\kappa+8}\frac{q_\nu^\mu(t)}{(1+t^2)^{\frac{3}{2}+\frac{2}{\kappa}}}\,dt \\ &\quad - (1+x^2)^{\frac{3}{2}+\frac{2}{\kappa}}q_\nu^\mu(x)\int_x^\infty \frac{4}{\kappa+8}\frac{p_\nu^\mu(t)}{(1+t^2)^{\frac{3}{2}+\frac{2}{\kappa}}}\,dt, \end{aligned}$$

where c_1 and c_2 are two constants and p_ν^μ and q_ν^μ are two independent solutions to the homogeneous ODE with initial conditions:

$$(1+x^2)y''+2xy'-[\nu(\nu+1)-\mu^2/(1+x^2)]y=0,$$
$$p_\nu^\mu(0)=0,\ q_\nu^\mu(0)=\frac{\Gamma(\frac{3}{2}+\nu)\Gamma(\frac{1}{2})}{\Gamma(1+\frac{\mu+\nu}{2})\Gamma(1+\frac{-\mu+\nu}{2})},$$
$$(p_\nu^\mu)'(0)=1,\ (q_\nu^\mu)'(0)=\frac{\Gamma(\frac{3}{2}+\nu)\Gamma(-\frac{1}{2})}{\Gamma(\frac{1-\mu+\nu}{2})\Gamma(\frac{1+\mu+\nu}{2})},$$

where $\nu=4/\kappa$ and $\mu=1-\nu$. To see this, set $w(x)=w_0(x)(1+x^2)^{\frac{3}{2}+\frac{2}{\kappa}}$. Then w_0 is a solution to the inhomogeneous ODE:

$$y''+\frac{2x}{1+x^2}y'+\frac{-\nu(\nu+1)x^2+1-3\nu}{(1+x^2)^2}y=\frac{-\nu}{(1+x^2)^{\frac{5}{2}+\frac{2}{\kappa}}}.$$

Thus, there are constants C_1 and C_2 such that

$$w_0(x)=C_1p_\nu^\mu(x)+C_2q_\nu^\mu(x)-\int_0^x\frac{p_\nu^\mu(x)q_\nu^\mu(t)-q_\nu^\mu(x)p_\nu^\mu(t)}{W(t)}\frac{-\nu}{(1+t^2)^{\frac{5}{2}+\frac{2}{\kappa}}}\,dt,$$

where the Wronskian W is of the form $W(x)=W(0)\exp(-\int_0^x 2t/(1+t^2))\,dt=W(0)/(1+x^2)$. After changing the limits,

$$w_0(x)=c_1p_\nu^\mu(x)+c_2q_\nu^\mu(x)+\int_x^\infty\frac{p_\nu^\mu(x)q_\nu^\mu(t)-q_\nu^\mu(x)p_\nu^\mu(t)}{W(0)}\frac{-\nu}{(1+t^2)^{\frac{3}{2}+\frac{2}{\kappa}}}\,dt,$$

where $W=W[p_\nu^\mu,q_\nu^\mu]$. We expand p_ν^μ and q_ν^μ as hypergeometric functions:

$$p_\nu^\mu(x)=(1+x^2)^{\frac{1}{2}\nu}\,{}_2F_1(\frac{\mu-\nu}{2},-\frac{\mu+\nu}{2};\frac{1}{2}-\nu;\frac{1}{1+x^2})\tag{2.16}$$

and

$$q_\nu^\mu(x)=(1+x^2)^{-\frac{1}{2}\nu-\frac{1}{2}}\,{}_2F_1(\frac{1-\mu+\nu}{2},\frac{1+\mu+\nu}{2};\frac{3}{2}+\nu;\frac{1}{1+x^2})\tag{2.17}$$

on the set $\mathbb{R}_+$ of all positive real numbers. Formally, the classical generalized Legendre function $P_\nu^\mu(z)$ of the first kind and $Q_\nu^\mu(z)$ of the second kind are linear combinations of $p_\nu^\mu(iz)$ and $q_\nu^\mu(iz)$. For special functions, see [11]. In particular, the expansion (2.16) has an extension to all of $\mathbb{R}$ given by

$$p_\nu^\mu(x)=x(1+x^2)^{\frac{\nu-1}{2}}=x(1+x^2)^{\frac{2}{\kappa}-\frac{1}{2}}.\tag{2.18}$$

We will show $w(x)=(x^2+1)/4$. One can easily get $c_1=0$ by the distortion theorem (1.5) and the unboundedness of $p_\nu^\mu(x)(1+x^2)^{\frac{1}{2}+\frac{2}{\kappa}}$ as $x\to\infty$. By the symmetry of SLE, we have $w'(0)=0$. This information makes it possible to determine the value of c_2. Therefore, with the properties that w is even and $w(x)/(1+x^2)$ is bounded, w is the unique solution to the inhomogeneous ODE: $Hw=1$. One can easily check that $w(x)=(x^2+1)/4$ is the desired solution. $\square$

Recall that the Schwarzian derivative, after we subtract a negligible term, is a complex martingale (1.20). Set

$$w(x) := 2(x^2+1)^2\mathbb{E}\int_0^\infty \frac{|f_t'(x+i)|^4}{|f_t(x+i)+\sqrt{\kappa}B_t|^6}\,dt. \tag{2.19}$$

As in (2.13), it follows from the scaling property of SLE that

$$\mathbb{E}\int_0^\infty \frac{|f_t'(z)|^4}{|f_t(z)+\sqrt{\kappa}B_t|^6}\,dt = \frac{1}{2y^4}\frac{w(x/y)}{(1+x^2/y^2)^2}. \tag{2.20}$$

As in the pre-Schwarzian, the function w satisfies the inhomogeneous ODE: $Hw = 1$, where H is the second-order linear differential operator given by

$$Hv = -\frac{\kappa}{4}(1+x^2)v_{xx} + (2+2\kappa)xv_x - \frac{(8+5\kappa)x^2-(8+\kappa)}{1+x^2}v.$$

To see this, let $v(x,s) := \mathbb{E}\big(|(x+i)f_{\tau_s}'(x+i)|^4/|f_{\tau_s}(x+i)+\sqrt{\kappa}B_{\tau_s}|^4\big)$, where the stopping time τ_s is defined by (2.14). Then v can be expressed as a Feynman-Kac type integral

$$v(x,s) = \mathbb{E}\Big(\exp\Big[(8+\kappa)\int_0^s \frac{X_u^2-1}{X_u^2+1}du - 2\sqrt{2\kappa}\int_0^s \frac{X_u}{\sqrt{X_u^2+1}}dB_u\Big]\Big),$$

where $X_u := \mathrm{Re}(f_{\tau_u}(z)+\sqrt{\kappa}B_{\tau_u})/\,\mathrm{Im}(f_{\tau_u}(z)+\sqrt{\kappa}B_{\tau_u})$ is an Itô diffusion (2.15). Therefore, v satisfies the following parabolic equation with an initial condition:

$$v_s = -Hv \text{ and } v(x,0)\equiv 1.$$

On the other hand, with the properties that $w(x)/(1+x^2)^2$ is bounded and w is even, w is the unique solution to the inhomogeneous ODE: $Hw = 1$. It is easy to check that

$$w(x) = \frac{8+3\kappa}{32(6+\kappa)}x^4 + \frac{16+3\kappa}{16(6+\kappa)}x^2 + \frac{24+3\kappa}{32(6+\kappa)}. \tag{2.21}$$

By (2.20) and (2.21), we obtain

$$\mathbb{E}\int_0^\infty \frac{|f_t'(z)|^4}{|f_t(z)+\sqrt{\kappa}B_t|^6}\,dt = \frac{1}{y^4}\frac{3\kappa+8}{64(\kappa+6)} + \frac{1}{y^2(x^2+y^2)}\frac{1}{4(\kappa+6)}. \tag{2.22}$$

For $x\neq 0$, we have

$$\lim_{y\to 0}\mathbb{E}\int_0^\infty \frac{|yf_t'(z)|^4}{|f_t(z)+\sqrt{\kappa}B_t|^6}\,dt = \frac{3\kappa+8}{64(\kappa+6)},$$

which proves the second part of Theorem 3.

Because of (2.22), it is useful to study the Itô derivatives

$$d\frac{|f_t'(z)|^4}{(f_t(z)-\overline{f_t(z)})^2|f_t(z)+\sqrt{\kappa}B_t|^2} \quad\text{and}\quad d\frac{|f_t'(z)|^4}{(f_t(z)-\overline{f_t(z)})^4}$$

in order to derive the Schwarzian expectation, and we now turn to this task.

Schwarzian expectation.

Proof of Theorem 3. Use (1.9),(2.5), and Goluzin's identity (2.9) to derive

$$
\begin{aligned}
&d\,\frac{f_t'(z)\overline{f_t'(z)}}{(f_t(z)-\overline{f_t(z)})^2}\,\frac{f_t'(z)\overline{f_t'(z)}}{(f_t(z)+\sqrt{\kappa}B_t)(\overline{f_t(z)}+\sqrt{\kappa}B_t)}\\
&=\frac{|f_t'(z)|^4}{(f_t(z)-\overline{f_t(z)})^2|f_t(z)+\sqrt{\kappa}B_t|^2}\\
&\times\Big(\frac{\kappa+4}{(f_t(z)+\sqrt{\kappa}B_t)^2}+\frac{\kappa+4}{(\overline{f_t(z)}+\sqrt{\kappa}B_t)^2}+\frac{\kappa}{|f_t(z)+\sqrt{\kappa}B_t|^2}\Big)\,dt\\
&+\frac{2f_t'(z)^2\overline{f_t'(z)^2}}{(f_t(z)+\sqrt{\kappa}B_t)^3(\overline{f_t(z)}+\sqrt{\kappa}B_t)^3}\,dt+\text{martingale}\\
&=\frac{(\kappa+6)f_t'(z)^2\overline{f_t'(z)^2}}{(f_t(z)+\sqrt{\kappa}B_t)^3(\overline{f_t(z)}+\sqrt{\kappa}B_t)^3}\,dt\\
&+\frac{3\kappa+8}{4}d\frac{f_t'(z)^2\overline{f_t'(z)^2}}{(f_t(z)-\overline{f_t(z)})^4}+\text{martingale}.
\end{aligned}
$$

By the optional stopping theorem, the above equality implies that

$$
\begin{aligned}
\mathbb{E}\int_0^{T_u}\frac{|f_s'(z)|^4}{|f_s(z)+\sqrt{\kappa}B_s|^6}\,ds&=\frac{3\kappa+8}{64(\kappa+6)}\Big(\frac{1}{y^4}-\mathbb{E}\frac{|f_{T_u}'(z)|^4}{(\operatorname{Im} f_{T_u}(z))^4}\Big)\\
&+\frac{1}{4(\kappa+6)}\Big(\frac{1}{y^2(x^2+y^2)}-\mathbb{E}\frac{|f_{T_u}'(z)|^4}{(\operatorname{Im} f_{T_u}(z))^2|f_{T_u}(z)+\sqrt{\kappa}B_{T_u}|^2}\Big).
\end{aligned}
$$

However, by (2.1), (1.20), and the optional stopping theorem, we obtain

$$
\begin{aligned}
\lim_{y\to 0}\mathbb{E}\Big|y^2 Sf_{T_u}(z)+\frac{12}{3\kappa+8}\big(\frac{y^2}{z^2}-\frac{y^2 f_{T_u}'(z)^2}{(f_{T_u}(z)+\sqrt{\kappa}B_{T_u})^2}\big)\Big|^2&\\
=\frac{576\kappa}{(3\kappa+8)^2}\lim_{y\to 0}\mathbb{E}\int_0^{T_u}\frac{|yf_s'(z)|^4}{|f_s(z)+\sqrt{\kappa}B_s|^6}\,ds&\\
=\frac{9\kappa}{(\kappa+6)(3\kappa+8)},&
\end{aligned}
$$

unless $x=0$. One can easily check that $\mathbb{E}|yf_{T_u}'(z)/(f_{T_u}(z)+\sqrt{\kappa}B_{T_u})|^4\to 0$ as $y\to 0$. (See the calculation used in (2.10).) This completes the proof of the first part, equation (1.18). The second part, equation (1.19), was proved in (2.22). □

Schwarzian and $\mathcal{BMO}$.

Proof of Theorem 4. Suppose a stopping time τ is given. By the strong Markov property, we have

$$
\begin{aligned}
&y^4\int_\tau^\infty\mathbb{E}\big[\frac{|f_t'(z)|^4}{|f_t(z)+\sqrt{\kappa}B_t|^6}\big|\mathcal{F}_\tau\big]\,dt\\
&\qquad=y^4|f_\tau'(z)|^4\int_0^\infty\mathbb{E}\,\frac{|f_t'(f_\tau(z)+\sqrt{\kappa}B_\tau)|^4}{|f_t(f_\tau(z)+\sqrt{\kappa}B_\tau)+\sqrt{\kappa}B_t|^6}\,dt.
\end{aligned}
$$

The Schwarz lemma implies $y|f_\tau'(z)|\le\operatorname{Im}(f_\tau(z)+\sqrt{\kappa}B_\tau)$. Therefore, it suffices to show that

$$
\mathbb{E}\int_0^\infty\frac{|yf_t'(z)|^4}{|f_t(z)+\sqrt{\kappa}B_t|^6}\,dt
$$

is uniformly bounded above by a universal constant $C = C(z)$. By (2.22), we have

$$\mathbb{E}\big(|S_\infty - S_\tau|^2 \big| \mathcal{F}_\tau\big) \le \frac{9\kappa}{(3\kappa+8)(\kappa+6)} + \frac{y^2}{x^2+y^2}\frac{144\kappa}{(\kappa+6)(3\kappa+8)^2},$$

and S is a $\mathcal{BMO}$ martingale. □

3. Exponential decay of correlations

In many problems, random variables with exponential decay of correlations are the appropriate substitutes for the independent random variables. For example, B. Schmuland and W. Sun proved the law of the iterated logarithm for a random field with exponential decay of correlations. See [32].

Proof of Theorem 5. By (1.17), (1.9), and the chordal version of Goluzin's identity (2.9), we obtain

$$\begin{aligned}\mathbb{E}(L_t(z_1)\overline{L_t(z_2)}) &= \frac{4}{(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\mathbb{E}\int_0^t \frac{y_1y_2 f_s'(z_1)\overline{f_s'(z_2)}}{(f_s(z_1)+\sqrt{\kappa}B_s)^2(\overline{f_s(z_2)}+\sqrt{\kappa}B_s)^2}\,ds\\ &= \frac{2}{(\sqrt{\kappa/4}+\sqrt{4/\kappa})^2}\mathbb{E}\Big[\frac{y_1y_2 f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)-\overline{f_t(z_2)})^2} - \frac{y_1y_2}{(z_1-\overline{z_2})^2}\Big].\end{aligned}$$

Recall the definition of the hyperbolic distance (1.21) or

$$\frac{y_1y_2}{|z_1-\overline{z_2}|^2} = \frac{1}{4}\cosh^{-2}\big(\frac{d_{\mathbb{H}}(z_1,z_2)}{2}\big).$$

On the other hand,

$$\limsup_{t\to\infty}\Big|\mathbb{E}\Big(\frac{y_1y_2 f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)-\overline{f_t(z_2)})^2}\Big)\Big| \le \frac{1}{4}\limsup_{t\to\infty}\mathbb{E}\Big(\Big|\frac{y_1 f_t'(z_1)}{\operatorname{Im} f_t(z_1)}\Big|\Big|\frac{y_2 f_t'(z_2)}{\operatorname{Im} f_t(z_2)}\Big|\Big) = 0,$$

which implies (1.22). It follows from identity (2.7) that $|yf_t'(z)/(\operatorname{Im} f_t(z))|$ is monotone decreasing in t. Hence,

$$\Big|\mathbb{E}\Big(\frac{y_1y_2 f_{T_0^*}'(z_1)\overline{f_{T_0^*}'(z_2)}}{(f_{T_0^*}(z_1)-\overline{f_{T_0^*}(z_2)})^2}\Big)\Big| \le \frac{1}{4}\prod_{j=1}^{2}\sqrt{\mathbb{E}\Big(\Big|\frac{y_j f_{T_0(z_j)}'(z_j)}{\operatorname{Im} f_{T_0(z_j)}(z_j)}\Big|^2\Big)}.$$

In the proof of Theorem 1, we have shown that $\mathbb{E}|yf_{T_0(z)}'(z)|^2$ converges to 0 as $y\to 0$. Estimate (1.23) now follows by scale invariance. □

Proof of Theorem 6. By (1.20) and (1.9), we obtain

$$\mathbb{E}\Big(Sf_t(z_1)\overline{Sf_t(z_2)}\Big) = \frac{576\kappa}{(3\kappa+8)^2}\mathbb{E}\int_0^t \frac{y_1^2y_2^2 f_s'(z_1)^2\overline{f_s'(z_2)^2}}{(f_s(z_1)+\sqrt{\kappa}B_s)^3(\overline{f_s(z_2)}+\sqrt{\kappa}B_s)^3}\,ds.$$

Use Goluzin's identity to derive

$$\begin{aligned}d\frac{f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)-\overline{f_t(z_2)})^2}&\frac{f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)+\sqrt{\kappa}B_t)(\overline{f_t(z_2)}+\sqrt{\kappa}B_t)}\\ &= (\kappa+6)\frac{f_t'(z_1)^2\overline{f_t'(z_2)^2}}{(f_t(z_1)+\sqrt{\kappa}B_t)^3(\overline{f_t(z_2)}+\sqrt{\kappa}B_t)^3}\\ &\quad+\frac{3\kappa+8}{4}d\frac{f_t'(z_1)^2\overline{f_t'(z_2)^2}}{(f_t(z_1)-\overline{f_t(z_2)})^4} + \text{martingale}.\end{aligned}$$

Taking the expectation in the above equation, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^t \frac{y_1^2 y_2^2 f_s'(z_1)^2 \overline{f_s'(z_2)^2}}{(f_s(z_1)+\sqrt{\kappa}B_s)^3(\overline{f_s(z_2)}+\sqrt{\kappa}B_s)^3}\,ds \\
&\quad = \frac{3\kappa+8}{4(\kappa+6)}\frac{y_1^2y_2^2}{(z_1-\overline{z_2})^4} - \frac{1}{\kappa+6}\frac{y_1^2y_2^2}{(z_1-\overline{z_2})^2 z_1\overline{z_2}} \\
&\quad - \frac{3\kappa+8}{4(\kappa+6)}\mathbb{E}\frac{y_1^2y_2^2 f_t'(z_1)^2\overline{f_t'(z_2)^2}}{(f_t(z_1)-\overline{f_t(z_2)})^4} \\
&\quad + \frac{1}{\kappa+6}\mathbb{E}\frac{y_1y_2 f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)-\overline{f_t(z_2)})^2}\frac{y_1y_2 f_t'(z_1)\overline{f_t'(z_2)}}{(f_t(z_1)+\sqrt{\kappa}B_t)(\overline{f_t(z_2)}+\sqrt{\kappa}B_t)}.
\end{aligned}$$

The estimates (1.24) and (1.25) can be easily obtained by the same method used in the proof of Theorem 5. □

4. Hölder continuity

S. Rohde and O. Schramm's estimate. The derivative expectation $\mathbb{E}[|g_t'(1)|^p]$ was computed by G. F. Lawler, O. Schramm, and W. Werner to obtain the crossing exponent for SLE_κ, which is closely related to the Brownian intersection exponent. This computation led them to prove the Mandelbrot conjecture. To obtain more information about the regularity of the backward flows, the derivative expectation or moment generating function for $|g_t'(z)|$ for an arbitrary complex number z has been computed in [31]. Using this, S. Rohde and O. Schramm proved that g_t^{-1} is almost surely Hölder continuous when $\kappa \neq 4$.

For $z \in \mathbb{H}$ and $u \in \mathbb{R}$, set

$$T_u = T_u(z) := \sup\{t \in \mathbb{R} : \mathrm{Im}(g_t(z)) \geq e^u\} \tag{4.1}$$

and

$$X_u = \mathrm{Re}(g_{T_u(z)}(z) - \sqrt{\kappa}B_{T_u}).$$

The stopping time T_u is well-defined since $\mathrm{Im}\, g_t(z)$ is monotone decreasing in t for each $z \in \mathbb{H}$. It is well known that for all $z \in \mathbb{H}$, almost surely $T_u \neq \pm\infty$.

Theorem (S. Rohde, O. Schramm)**.** *Suppose $z = x + iy \in \mathbb{H}$ with $y < 1$ given. For each $b \in \mathbb{R}$, define p and q by*

$$p := 2b + \kappa b(1-b)/2, \quad q := 4b + \kappa b(1-2b)/2. \tag{4.2}$$

Then

$$y^p \mathbb{E}\big[(1+X_0^2)^b |g_{T_0(z)}'(z)|^p\big] = (1 + x^2/y^2)^b y^q. \tag{4.3}$$

Corollary (S. Rohde, O. Schramm)**.** *For $b \in [0, 1+4/\kappa]$, there is a constant $C(\kappa, b)$, depending only on κ and b, such that the following derivative upper bound estimate holds for all $t \in [0,1], y, \delta \in (0,1]$ and $x \in \mathbb{R}$.*

$$\mathbb{P}[|f_t'(x+iy)| \geq \delta y^{-1}] \leq C(\kappa,b)(1+x^2/y^2)^b (y/\delta)^q \theta(\delta, p-q),$$

where

$$\theta(\delta, s) = \begin{cases} \delta^{-s} & s > 0, \\ 1 + |\log\delta| & s = 0, \\ 1 & s < 0. \end{cases}$$

Theorem (S. Rohde, O. Schramm)**.** *For every $\kappa \neq 4$, there exists $\widetilde{h}(\kappa) > 0$ such that for each bounded set $D \subset \mathbb{H}$ and each $t > 0$, almost surely f_t is Hölder continuous with Hölder exponent $\widetilde{h}(\kappa)$ on D,*

$$|f_t(z) - f_t(w)| \leq C|z-w|^{\widetilde{h}(\kappa)}, \qquad z, w \in D,$$

where C is a random constant depending on t and D. Moreover,

$$\lim_{\kappa \searrow 0} \widetilde{h}(\kappa) = \frac{1}{2} \text{ and } \lim_{\kappa \nearrow \infty} \widetilde{h}(\kappa) = 1.$$

Here, by bounded set we mean bounded in the sense of the Euclidean metric on $\mathbb{R}^2$ (not the hyperbolic metric on $\mathbb{H}$). When $\kappa = 0$, the Schramm-Loewner evolution is not a stochastic process anymore. In this case, the backward flow f_t coincides with g_t^{-1} and is given by

$$f_t(z) = \sqrt{z^2 - 4t},$$

which has a Hölder exponent of 1/2. However, it has a local Hölder exponent of 1 except at $z = \pm 2\sqrt{t}$. These two points are mapped into the base point, where the geometry of Loewner evolution is different from any other points. The geometry of the base point will not be taken into consideration with the conditioning on the event that $\operatorname{Im} f_1(z) \geq c$ on D. For example, to compute the size of the hull boundary, S. Rohde and O. Schramm considered the collection W_c of the Whitney squares Q of H_1 such that $\operatorname{dist}(Q, K_1) \leq 1$ and $\sup\{\operatorname{Im} z : z \in Q\} \geq c$. For $\delta > 0$, they introduced

$$S_c(\delta) := \sum_{Q \in W_c} d(Q)^\delta$$

and computed the convergence exponent $\delta(\kappa)$ such that $\mathbb{E}[S_c(\delta)] = \infty$ if and only if $\delta \leq \delta(\kappa)$. It is conjectured that the convergent exponent $\delta(\kappa)$ is the Hausdorff dimension for the boundary of the hull.

From (4.3), we have the following estimate for the derivative expectation: For each bounded set $D \subset \mathbb{H}$,

$$\mathbb{E}|g'_{T_0}(z)|^p \leq C y^{p(1 - \frac{1-\sqrt{1-2p/\mu}}{p/\mu})}, \tag{4.4}$$

where $\mu = \kappa/4 + 2 + 4/\kappa$ and C is a constant depending on D. Without rigorous proof, we will derive the estimate on the distributions for the derivatives from (4.4) in the next subsection. With the conditioning on the event $\operatorname{Im} f_1(z) \geq c$ on D, we will rigorously derive the conjectured sharp estimate on the Hölder exponent in the last subsection. The next subsection may be of help to formulate Theorem 7. However, the reader may wish to skip it.

Distributions for derivatives. Set the distribution $G(\lambda) = \mathbb{P}[|g'_{T_0}(z)/y| > \lambda]$. Then estimate (4.4) says

$$\int_0^\infty p G(\lambda) \lambda^{p-1}\, d\lambda \leq C y^{-\mu + \mu\sqrt{1-2p/\mu}}.$$

Use the substitution $r = 1 - 2p/\mu$ $(0 < r < 1)$ and $\nu = \lambda^{-\mu/2}$ to obtain

$$\int_0^\infty y^\mu \frac{G(\lambda)}{\nu} \nu^{r-1}\, d\nu \leq C \frac{e^{-\sqrt{ar}}}{1-r}, \tag{4.5}$$

where $\sqrt{a} = \mu \log y^{-1}$. We will estimate $y^\mu G(\lambda)/\nu$ by taking the inverse Mellin transform of $e^{-\sqrt{ar}}/(1-r)$ formally on both sides of the above inequality.

Let us recall the definition of the Mellin transform and the Laplace transform. The Mellin transform $M[f]$ is defined as

$$M[f](z) = \int_0^\infty t^{z-1} f(t)\, dt,$$

and its inverse transform $M^{-1}[\phi]$ is defined as

$$M^{-1}[\phi](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \phi(z)\, dz.$$

The transform $\phi(z) = M[f](z)$ exists if the integral

$$\int_0^\infty |f(x)| x^{k-1}\, dx$$

is bounded for some $k > 0$, in which case the inverse $f(t) = M^{-1}[\phi](t)$ exists when $c > k$. The Laplace transform $\mathcal{L}$ is defined by

$$\mathcal{L}[f](s) = \int_0^\infty f(t) e^{-st}\, dt.$$

If f is a piecewise continuous function on every finite interval in $[0, \infty)$ satisfying $|f(t)| \leq Ce^{at}$ for all $t \in [0, \infty)$, then $\mathcal{L}[f](s)$ exists for all $s > a$. The Laplace transform is unique, and its inverse transform $\mathcal{L}^{-1}$ is defined as

$$\mathcal{L}^{-1}[\phi](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tz} \phi(z)\, dz.$$

The error function and the complementary error function are defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \quad \text{erfc}(x) = 1 - \text{erf}(x). \tag{4.6}$$

We will use the following elementary properties of the complementary error function.

(1) It satisfies the identity: $\text{erfc}(-x) = 2 - \text{erfc}(x)$.
(2) It has the tail estimate: For $x > 0$,

$$\frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{x + \sqrt{x^2 + 2}} < \text{erfc}(x) < \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{x + \sqrt{x^2 + \frac{4}{\pi}}}. \tag{4.7}$$

To compute the inverse Mellin transform of $e^{-\sqrt{ar}}/(1-r)$, we need the following lemma.

Lemma 4.1. *The inverse Laplace transform of* $e^{-\sqrt{ar}}/(1-r)$ *on* $0 < r < 1$ *is*

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xz - \sqrt{az}}}{1-z}\, dz = \frac{1}{2} e^x \Big[e^{-\sqrt{a}}\, \text{erfc}\,\big(\sqrt{x} - \frac{1}{2}\sqrt{\frac{a}{x}}\,\big) - e^{\sqrt{a}}\, \text{erfc}\,\big(\frac{1}{2}\sqrt{\frac{a}{x}} + \sqrt{x}\,\big) \Big],$$

where erfc *is the complementary error function.*

Proof. It is well known that

$$\mathcal{L}^{-1}\Big(\frac{e^{-\sqrt{ap}}}{p}\Big)(x) = \text{erfc}\,\big(\frac{1}{2}\sqrt{\frac{a}{x}}\,\big).$$

See p. 264 in [12]. It follows from the basic properties of the Laplace transform that

$$\begin{aligned}\mathcal{L}^{-1}(\frac{e^{-\sqrt{ap}}}{p-1})(x) &= e^x \int_0^x e^{-t}\mathcal{L}^{-1}(e^{-\sqrt{ap}})(t)\,dt \\ &= e^x \int_0^x e^{-t}\frac{d}{dt}\mathcal{L}^{-1}(\frac{e^{-\sqrt{ap}}}{p})(t)\,dt \\ &= e^x \int_0^x e^{-t}\frac{d}{dt}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{t}}\,\big)\,dt \\ &= \frac{1}{2}e^x\Big[e^{-\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}-\sqrt{x}\,\big)+e^{\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}+\sqrt{x}\,\big)\Big].\end{aligned}$$

Suppose $p>1$ and $0<r<1$. Then, using a contour integral, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{xz-\sqrt{az}}}{1-z}\,dz &= \operatorname{Res}_{z=1}\big(\frac{e^{xz-\sqrt{az}}}{z-1}\big)-\frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty}\frac{e^{xz-\sqrt{az}}}{z-1}\,dz \\ &= e^{x-\sqrt{a}}-\frac{1}{2}e^x\Big[e^{-\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}-\sqrt{x}\,\big)+e^{\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}+\sqrt{x}\,\big)\Big] \\ &= \frac{1}{2}e^xe^{-\sqrt{a}}\Big[2-\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}-\sqrt{x}\,\big)\Big]-\frac{1}{2}e^xe^{\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}+\sqrt{x}\,\big) \\ &= \frac{1}{2}e^xe^{-\sqrt{a}}\operatorname{erfc}\big(\sqrt{x}-\frac{1}{2}\sqrt{\frac{a}{x}}\,\big)-\frac{1}{2}e^xe^{\sqrt{a}}\operatorname{erfc}\big(\frac{1}{2}\sqrt{\frac{a}{x}}+\sqrt{x}\,\big).\end{aligned}$$

This completes the proof. □

Fix $\kappa \neq 4$. By scaling, we may assume $D=[-1,1]\times(0,1]$. With $\nu=\lambda^{-\mu/2}$, $\sqrt{a}=\mu\log y^{-1}$, and $e^{-x}=\nu$, we take the inverse Mellin transform on both sides of (4.5) to formally derive the following estimate without rigorous proof:

$$\mathbb{P}\Big[\Big|\frac{g'_{T_0}(z)}{y}\Big|>\lambda\Big]\le C\big(\operatorname{erfc}(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda y)}{\sqrt{\log\lambda}})-\frac{1}{y^{2\mu}}\operatorname{erfc}(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda/y)}{\sqrt{\log\lambda}})\big).$$

From the above estimate, it is not hard to obtain the following estimate. The reader can find the details in the next subsection. Let $c>0$. Then

$$\begin{aligned}&\mathbb{P}\Big[\Big|\frac{f_1'(z)}{y}\Big|>\lambda\,\Big|\,\inf_{z\in D}\operatorname{Im} f_1(z)\ge c\Big] \\ &\qquad\le C\big(\operatorname{erfc}(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda y)}{\sqrt{\log\lambda}})-\frac{1}{y^{2\mu}}\operatorname{erfc}(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda/y)}{\sqrt{\log\lambda}})\big).\end{aligned}$$

With $\lambda=y^{h-2}$ and $y=2^{-n}$, the above estimate gives

$$\mathbb{P}\Big[\Big|\frac{f_1'(z)}{y}\Big|>\lambda\,\Big|\,\inf_{z\in D}\operatorname{Im} f_1(z)\ge c\Big]\le C2^{-n\frac{\mu}{2}\frac{(1-h)^2}{2-h}}. \tag{4.8}$$

Consider the Whitney decomposition $\{Q_{j,n}\}(n\ge 0, 1\le j\le 2^n)$ of D. Denote the center of the Whitney square $Q_{j,n}$ by $z_{j,n}$. By the Borel-Cantelli lemma and the distortion theorem, f_1 is almost surely h-Hölder in D on the event that $\operatorname{Im} f_1(z)\ge c$ in D, if

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2^n}\mathbb{P}\big[\big|f_1'(z_{j,n})\big|>2^{n(1-h)}\,\big|\,\inf_{z\in D}\operatorname{Im} f_1(z)\ge c\big]<\infty.$$

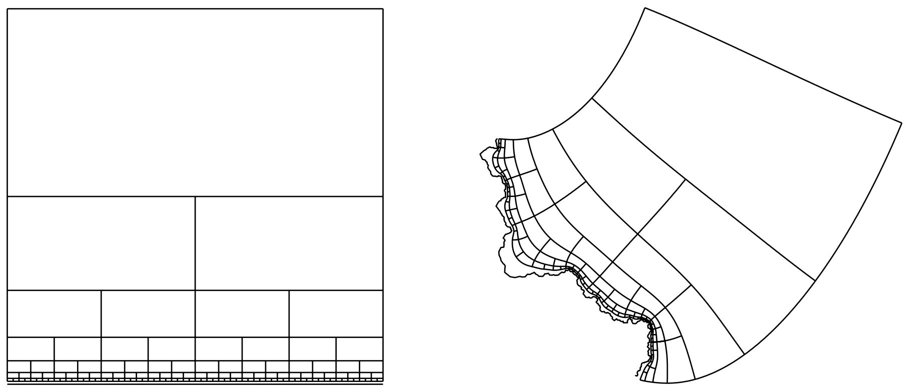

FIGURE 3. Whitney decomposition of $\mathbb{H}$ and its image under SLE.

This is summable if

$$1 - \frac{\mu}{2}\frac{(1-h)^2}{2-h} < 0, \tag{4.9}$$

or

$$h < h(\kappa) = 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2} + \frac{2}{\mu}},$$

where $\mu = \kappa/4 + 2 + 4/\kappa$. This motivates the formulation of Theorem 7.

Remark 4.2. Note that $\lim_{\kappa\to 0} h(\kappa) = 1$, $\lim_{\kappa\to\infty} h(\kappa) = 1$, and $\lim_{\kappa\to 4} h(\kappa) = 0$. Also, the exponent $h(\kappa)$ satisfies the Duplantier duality. I. Binder and B. Duplantier derived the same formula from the multifractal spectrum of SLE_κ independently. Indeed, the exponent $h(\kappa)$ satisfies $f(1/h(\kappa)) = 0$ for the multifractal spectrum f of SLE_κ. Here, $f(\alpha) = f_{\text{mixed}}(\alpha, 0)$ for the mixed multifractal spectrum f_{mixed} in (1.1). Hence, $f(\alpha) = \alpha + b - b\alpha^2/(2\alpha - 1)$, where $b = (25 - c)/12$ and $c = 1 - 6(\sqrt{\kappa/4} - \sqrt{4/\kappa})^2$. However, their result is not rigorous. We note that J. Lind has also obtained Theorem 7 independently. She has shown that $(f_t(\sqrt{z}) + \sqrt{\kappa}B_t)^2$ is almost surely h-Hölder continuous provided $h < h(\kappa)$. See [28] for this.

Conjectured sharp estimate for the Hölder exponent. We will prove Theorem 7 rigorously in this subsection.

Proof of Theorem 7. Fix $\kappa \neq 4$. By scaling, we may assume $D = [-1/4, 1/4] \times (0, 1]$. Consider the Whitney decomposition $\{Q_{j,n}\}$ of D such that $Q_{j,n}$ is a square of side length 2^{-n-1} $(n \geq 0, 1 \leq j \leq 2^n)$. We denote the center of the Whitney square $Q_{j,n}$ by $z_{j,n}$. Recall that f_1 is almost surely h-Hölder in D on the event that $\operatorname{Im} f_1(z) \geq c$ in D provided

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2^n} \mathbb{P}\big[|f_1'(z_{j,n})| > 2^{n(1-h)} \;\big|\; \inf_{z\in D} \operatorname{Im} f_1(z) \geq c\big] < \infty.$$

Take $m \in \mathbb{Z}$ such that $e^m \leq c < e^{m+1}$. Recall the stopping time (4.1) and the fact that f_1' has the same distribution as g_{-1}'. As in the proof of Corollary 3.5 in [31], we may assume $T_0(z_{j,n}) \geq 1$ with probability one. Let E be the event

$E := \{\inf_{z\in D} \operatorname{Im} f_1(z) \geq c\}$. For fixed $z = z_{j,n}$ and $k = m, \cdots, 0$, let E_k be the event $E_k := \{T_k(z) \leq 1 < T_{k+1}(z)\}$. Then

$$\begin{aligned}
&\mathbb{P}\big[y\big|f_1'(z)\big| > \lambda \,\big|\, E\big] \\
&\leq C_1 \sum_{k=m}^{0} \mathbb{P}\big[y\big|g_{T_k(z)}'(z)\big| > C_2\lambda \,\big|\, EE_k\big]\mathbb{P}\big[E_k \,\big|\, E\big] \\
&\leq C \sum_{k=m}^{0} \lambda^{-p} e^{kp} F_p(e^{-k}z),
\end{aligned}$$

where the moment generating function F_p is given by $F_p(z) = y^p\mathbb{E}\big[|g_{T_0(z)}'(z)|^p\big]$.

It follows from the above estimate and the estimate (4.4) that

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2^n} \mathbb{P}\big[\big|f_1'(z_{j,n})\big| > 2^{n(1-h)} \,\big|\, E\big] \leq \sum_{n=0}^{\infty} C2^{n(1+ph-2p+\mu-\mu\sqrt{1-2p/\mu})}.$$

The right-hand side in the above inequality is summable if

$$h < h_\kappa(p) := 2 - \frac{1}{p} + \frac{\mu}{p}(\sqrt{1-2\frac{p}{\mu}} - 1).$$

As a function of p, $h_\kappa(p)$ has the derivative

$$h_\kappa'(p) = \Big(1 - \frac{p}{\sqrt{1-2\frac{p}{\mu}}} - \mu(\sqrt{1-2\frac{p}{\mu}} - 1)\Big)\frac{1}{p^2}$$

and takes the critical value at $p = p_c$, where

$$p_c = -(2 + \frac{1}{\mu}) + (\mu+1)\sqrt{\frac{1}{\mu^2} + \frac{2}{\mu}}.$$

Furthermore, the critical value is

$$\begin{aligned}
h_\kappa(p_c) &= 2 - \frac{1}{p_c} + \frac{1}{p_c}\Big(1 - \frac{p_c}{\sqrt{1-2\frac{p_c}{\mu}}}\Big) \\
&= 2 - \frac{1}{\sqrt{1-2\frac{p_c}{\mu}}} = 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2} + \frac{2}{\mu}} = h(\kappa),
\end{aligned}$$

which gives the conjectured sharp estimate for the Hölder exponent. □

Remark 4.3. For the backward flow of SLE_4 one expects a continuity property of logarithmic type instead of Hölder continuity.

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