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NONCOMMUTATIVE MAXIMAL ERGODIC THEOREMS

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0. Introduction

The connection between ergodic theory and the theory of von Neumann algebras goes back to the very beginning of the theory of "rings of operators". Maximal inequalities in ergodic theory provide an important tool in classical analysis. In this paper we prove the noncommutative analogue of the classical Dunford-Schwartz maximal ergodic theorem, thereby connecting these different aspects of ergodic theory.

At the early stage of noncommutative ergodic theory, only mean ergodic theorems have been obtained (cf., e.g., [Ja1, Ja2] for more information). The study of individual ergodic theorems really took off with Lance's pioneering work [L]. Lance proved that the ergodic averages associated with an automorphism of a σ-finite von Neumann algebra which leaves invariant a normal faithful state converge almost uniformly. Lance's ergodic theorem was extensively extended and improved by (among others) Conze, Dang-Ngoc [CoN], Kümmerer [Kü] (see [Ja1, Ja2] for more references). On the other hand, Yeadon [Ye] obtained a maximal ergodic theorem in the preduals of semifinite von Neumann algebras. Yeadon's theorem provides a maximal ergodic inequality which might be understood as a weak type $(1, 1)$ inequality. This inequality is the ergodic analogue of Cuculescu's [Cu] result obtained previously for noncommutative martingales. We should point out that in

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contrast with the classical theory, the noncommutative nature of these weak type $(1,1)$ inequalities seems a priori unsuitable for classical interpolation arguments.

Since then the problem of finding a noncommutative analogue of the Dunford-Schwartz maximal ergodic inequalities was left open. The main reason is that all the usual techniques in classical ergodic theory involving maximal functions seem no longer available in the noncommutative case. In fact, this applies for the definition of the maximal function itself. As an example, we consider

$$a_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then there is no 2×2 matrix a such that

$$\langle \xi, a\xi \rangle = \max \big\{ \langle \xi, a_1\xi \rangle,\ \langle \xi, a_2\xi \rangle\ \langle \xi, a_3\xi \rangle \big\}$$

holds for all $\xi \in \ell_2^2$.

However, this obstacle has been overcome recently in the theory of noncommutative martingale inequalities. In fact, most of the classical martingale inequalities have been successfully transferred to the noncommutative setting. These include Burkholder inequalities on conditioned square functions [JX2], Burkholder-Gundy inequalities on square functions [PX1], the Doob maximal inequality [Ju], Rosenthal inequalities on independent random variables [JX4] and boundedness of martingale transforms [Ra]. See the survey [X] for the state of the art regarding this theory.

Let us point out that this new development of noncommutative martingale inequalities is inspired and motivated by interactions with operator space theory. For instance, the formulation of the noncommutative Doob maximal inequality was directly derived from Pisier's theory of vector-valued noncommutative L_p-spaces [P].

Following the well-known analogy between martingale theory and ergodic theory, we show that the techniques developed for noncommutative martingales can be used to prove the noncommutative maximal ergodic inequalities as well.

To state our main results we need some notation. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $L_p(\mathcal{M})$ be the associated noncommutative L_p-space. Let $T : \mathcal{M} \to \mathcal{M}$ be a linear map which might satisfy some of the following properties:

- (0.I) T is a contraction on $\mathcal{M}$: $\|Tx\|_\infty \le \|x\|_\infty$ for all $x \in \mathcal{M}$;
- (0.II) T is positive: $Tx \ge 0$ if $x \ge 0$;
- (0.III) $\tau \circ T \le \tau$: $\tau(T(x)) \le \tau(x)$ for all $x \in L_1(\mathcal{M}) \cap \mathcal{M}_+$;
- (0.IV) T is symmetric relative to τ: $\tau(T(y)^*x) = \tau(y^*T(x))$ for all x, y in the intersection $L_2(\mathcal{M}) \cap \mathcal{M}$.

The properties (0.I), (0.II) and (0.III) will be essential for what follows. If T satisfies these properties, then T naturally extends to a contraction on $L_p(\mathcal{M})$ for all $1 \le p < \infty$ (see Lemma 1.1 below). The extension will still be denoted by T. If T additionally has (0.IV), then its extension is selfadjoint on $L_2(\mathcal{M})$. We consider the ergodic averages of T:

$$M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k\,, \quad n \in \mathbb{N}.$$

The following is one of our main results:

Theorem 0.1. *Let $1 < p < \infty$ and T be a linear map satisfying* (0.I)–(0.III) *above.*

i) *For any $x \in L_p(\mathcal{M})$ with $x \geq 0$ there is an $a \in L_p(\mathcal{M})$ such that*
$$\forall\, n \in \mathbb{N},\ M_n(T)(x) \leq a \quad \textit{and} \quad \|a\|_p \leq C_p \|x\|_p\,,$$
where C_p is a positive constant depending only on p. Moreover, $C_p \leq Cp^2(p-1)^{-2}$ and $(p-1)^{-2}$ is the optimal order of C_p as $p \to 1$.

ii) *If additionally T satisfies* (0.IV)*, then for any $x \in L_p(\mathcal{M})$ with $x \geq 0$ there is an $a \in L_p(\mathcal{M})$ such that*
$$\forall\, n \in \mathbb{N},\ T^n(x) \leq a \quad \textit{and} \quad \|a\|_p \leq C'_p \|x\|_p\,.$$

Part i) above is the noncommutative analogue of the classical Dunford-Schwartz theorem in commutative L_p-spaces (cf. [DS]). Note that the optimal order of the constant C_p above is different from that in the commutative case, which is $(p-1)^{-1}$ as $p \to 1$. Part ii) is the noncommutative analogue of Stein's maximal ergodic inequality (see [St2]). Note that in the case where τ is normalized (i.e. $\tau(1) = 1$), the following weak form of part i) was obtained in [GoG]: Given $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $x \in L_p(\mathcal{M})$ $(x \geq 0)$ there is an $a \in L_{p-\varepsilon}(\mathcal{M})$ such that
$$\forall\, n \in \mathbb{N},\ M_n(T)(x) \leq a \quad \text{and} \quad \|a\|_{p-\varepsilon} \leq C_{p,\varepsilon} \|x\|_p\,.$$

As in the commutative case, Theorem 0.1 also holds for all elements of $L_p(\mathcal{M})$ (not only the positive ones). This requires an appropriate definition of the space $L_p(\mathcal{M}; \ell_\infty)$ in the noncommutative setting (see section 2 for more details). On the other hand, by discretization, we have a similar theorem for semigroups.

The proof of Theorem 0.1, i) relies on Yeadon's weak type (1,1) maximal ergodic inequality already quoted (see also Lemma 1.2 below). As in the commutative case, the main idea is to interpolate this weak type (1,1) inequality with the trivial case $p = \infty$. Additional complications are due to the fact that the weak type (1,1) estimate does not provide a majorant $-a \leq M_n(T)(x) \leq a$ such that a is in weak L_1. In our proof of the noncommutative version of the classical Marcinkiewicz theorem (see Theorem 3.1 below) we first establish an intermediate inequality using noncommutative Lorentz spaces. Then we use the real interpolation method. We should emphasize that contrary to the classical situation, this interpolation theorem is not valid for the spaces $L_p(\mathcal{M}; \ell_\infty)$ themselves but only for their positive cones.

For the proof of part ii) of Theorem 0.1, we adapt Stein's arguments in [St2] to the noncommutative setting.

As usual, the maximal ergodic inequalities in Theorem 0.1 imply the corresponding pointwise ergodic theorems. The arguments are standard in the tracial case. However, the nontracial case requires additional work (see section 7). Our approach to the individual ergodic theorems seems new. In order to ensure pointwise convergence we use the space $L_p(\mathcal{M}; c_0)$ which is the closure of finite sequences in $L_p(\mathcal{M}; \ell_\infty)$ $(p < \infty)$. The main step towards individual ergodic theorems is contained in the following result:

Theorem 0.2. *Let $1 < p < \infty$ and T satisfy* (0.I)–(0.III)*. Let F be the projection onto the fixed point subspace of T considered as a map on $L_p(\mathcal{M})$. Then $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0)$ for any $x \in L_p(\mathcal{M})$. If additionally, T has* (0.IV)*, then $\big(T^n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0)$.*

Let us mention here one application to free group von Neumann algebras for illustration. Let $\mathbb{F}_n$ be a free group on n generators, and let $VN(\mathbb{F}_n)$ be its von Neumann algebra equipped with the canonical normalized trace τ. Let λ be the left regular representation of $\mathbb{F}_n$ on $\ell_2(\mathbb{F}_n)$. Recall that $VN(\mathbb{F}_n)$ is the von Neumann algebra on $\ell_2(\mathbb{F}_n)$ generated by $\{\lambda(g) : g \in \mathbb{F}_n\}$. Let $|\cdot|$ denote the length function on $\mathbb{F}_n$ (relative to a fixed family of n generators). Haagerup [H4] proved that the map T_t defined by $T_t(\lambda(g)) = e^{-t|g|}\lambda(g)$ extends to a completely positive map on $VN(\mathbb{F}_n)$. It is easy to check that T_t possesses all properties (0.I)–(0.IV). Consequently, T_t extends to a positive contraction on $L_p(VN(\mathbb{F}_n))$ for all $1 \le p < \infty$ (still denoted by T_t). It is then clear that (T_t) is a symmetric semigroup on $L_p(VN(\mathbb{F}_n))$ and strongly continuous for $p < \infty$. Thus applying Theorem 0.1, ii) to this semigroup, we obtain the following result formulated in terms of (bilateral) almost uniform convergence:

Theorem 0.3. *Let $1 < p \le \infty$. Then for any $x \in L_p(VN(\mathbb{F}_n))$ with $x \ge 0$ there is an $a \in L_p(VN(\mathbb{F}_n))$ such that*

$$\forall\, t > 0,\ T_t(x) \le a \quad \textit{and} \quad \|a\|_p \le C'_p \|x\|_p\,.$$

Consequently, $\lim_{t\to 0} T_t(x) = x$ *bilaterally almost uniformly* (*and almost uniformly if $p > 2$*) *for any $x \in L_p(VN(\mathbb{F}_n))$.*

The notion of (bilateral) almost uniform convergence is a noncommutative analogue of the notion of almost everywhere convergence. We refer to section 6 for the relevant definitions. Note that when $n = 1$, $VN(\mathbb{F}_1)$ is just $L_\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle and T_t becomes the usual Poisson semigroup. Thus the theorem above is the free analogue of the classical radial maximal inequality and of the radial pointwise convergence theorem about the Poisson integral in the unit disc.

Let us end this introduction with a brief description of the organization of the paper. The first six sections concern solely the semifinite case. After a preliminary section, we give some elementary properties of the vector-valued noncommutative L_p-spaces $L_p(\mathcal{M}; \ell_\infty)$ in section 2. These vector-valued L_p-spaces were first introduced by Pisier [P] for injective von Neumann algebras and then extended to general von Neumann algebras by the first named author in [Ju]. They provide the main tool of this paper.

Section 3 is devoted to the noncommutative analogue of the classical Marcinkiewicz interpolation theorem. This is the most technical result of the paper. It seems reasonable to expect further applications in the noncommutative setting.

Section 4 contains our first maximal ergodic theorems. The main result there is Theorem 0.1, i). This is an immediate consequence of the previous interpolation theorem.

Section 5 deals with the maximal inequalities when assuming the symmetry condition (0.IV). In particular, we prove Theorem 0.1, ii). Our proof requires Stein's interpolation technique using fractional averages (which makes it quite involved).

In Section 6 we study the individual ergodic theorems. In particular, we prove Theorem 0.2 above.

The objective of section 7 is to extend all previous results to the general (nontracial) von Neumann algebras by a reduction argument. This argument is based on an important (unfortunately) unpublished result due to Haagerup [H3]. Let us mention that the arguments for pointwise convergence in Haagerup L_p-spaces are usually more delicate than their semifinite counterparts. However, our new approach presented in section 6 permits us to give a unified treatment of both cases.

Section 8 presents some natural examples to which our theory applies. These include the free products of completely positive semigroups, the Poisson semigroup of a free group (which yields Theorem 0.3 above) and the q-Ornstein-Uhlenbeck semigroups.

The main results of this paper were announced in [JX1].

1. Preliminaries

The noncommutative L_p-spaces used in this paper are most of the time those based on semifinite von Neumann algebras, except those in the last two sections. Thus in this preliminary section we concentrate only on the semifinite noncommutative L_p-spaces. There are numerous references for these spaces. Our main reference is [FK]. The recent survey [PX2] presents a rather complete picture on noncommutative integration and contains a lot of references.

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $\mathcal{S}_+$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x$ denotes the support of x. Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Then $\mathcal{S}$ is a w*-dense $*$-subalgebra of $\mathcal{M}$. Given $0 < p < \infty$, we define

$$\|x\|_p = \big[\tau(|x|^p)\big]^{1/p}, \quad x \in \mathcal{S},$$

where $|x| = (x^*x)^{1/2}$ is the modulus of x. Then $(\mathcal{S},\ \|\cdot\|_p)$ is a normed (or quasi-normed for $p < 1$) space, whose completion is the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$, denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm.

The elements in $L_p(\mathcal{M})$ can be viewed as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts). We recall this briefly. Let $L_0(\mathcal{M}) = L_0(\mathcal{M}, \tau)$ denote the space of all closed densely defined operators on H measurable with respect to $(\mathcal{M}, \tau)$. For a measurable operator x we define its generalized singular numbers by

$$\mu_t(x) = \inf\big\{\lambda > 0 : \tau\big(\mathbb{1}_{(\lambda,\infty)}(|x|)\big) \le t\big\}, \quad t > 0.$$

Let

$$V(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \mu_\varepsilon(x) \le \delta\}.$$

Then $\{V(\varepsilon, \delta) : \varepsilon > 0, \delta > 0\}$ is a system of neighbourhoods at 0 for which $L_0(\mathcal{M})$ becomes a metrizable topological $*$-algebra. The convergence with respect to this topology is called the convergence in measure. Moreover, $\mathcal{M}$ is dense in $L_0(\mathcal{M})$.

The trace τ extends to a positive tracial functional on the positive part $L_0^+(\mathcal{M})$ of $L_0(\mathcal{M})$, still denoted by τ, satisfying

$$\tau(x) = \int_0^\infty \mu_t(x)dt, \quad x \in L_0^+(\mathcal{M}).$$

Then for $0 < p < \infty$,

$$L_p(\mathcal{M}) = \big\{x \in L_0(\mathcal{M}) : \tau(|x|^p) < \infty\big\}$$

and for $x \in L_p(\mathcal{M})$,

$$\|x\|_p^p = \tau(|x|^p) = \int_0^\infty (\mu_t(x))^p\, dt\,.$$

More generally, we can define the noncommutative Lorentz space $L_{p,q}(\mathcal{M})$:

$$L_{p,q}(\mathcal{M},\tau)=\big\{x\in L_0(\mathcal{M}):\|x\|_{p,q}<\infty\big\},$$

where

$$\|x\|_{p,q}=\Big(\int_0^\infty \big(t^{\frac{1}{p}}\mu_t(x)\big)^q\,\frac{dt}{t}\Big)^{1/q}$$

for $q<\infty$ and with the usual modification for $q=\infty$. The positive cone of $L_{p,q}(\mathcal{M})$ is denoted by $L^+_{p,q}(\mathcal{M})$

As the commutative L_p-spaces, the noncommutative L_p-spaces behave well with respect to the complex interpolation method and the real interpolation method (in the semifinite case). Let $0<\theta<1$, $1\le p_0<p_1\le\infty$ and $1\le q_0,q_1,q\le\infty$. Then

$$L_p(\mathcal{M})=\big(L_{p_0}(\mathcal{M}),\ L_{p_1}(\mathcal{M})\big)_\theta \quad \text{(with equal norms)} \tag{1.1}$$

and

$$L_{p,q}(\mathcal{M})=\big(L_{p_0,q_0}(\mathcal{M}),\ L_{p_1,q_1}(\mathcal{M})\big)_{\theta,q} \quad \text{(with equivalent norms)}, \tag{1.2}$$

where $\dfrac{1}{p}=\dfrac{1-\theta}{p_0}+\dfrac{\theta}{p_1}$, and where $(\cdot\,,\cdot)_\theta$, $(\cdot\,,\cdot)_{\theta,q}$ denote respectively the complex and real interpolation methods. Our reference for interpolation theory is [BeL].

Let T be a linear map on $L_p(\mathcal{M})$. T is called *positive* if T preserves the positive cone of $L_p(\mathcal{M})$, i.e., $x\ge 0\implies Tx\ge 0$. The following lemma is elementary and certainly well known. See [Ye], where the extensions of T to all $L_p(\mathcal{M})$ were obtained but not their contractivity. The contractive extensions are, of course, important in ergodic theory. We include a proof for completeness.

Lemma 1.1. *Let T satisfy* (0.I)–(0.III). *Then T extends in a natural way to a positive contraction on $L_p(\mathcal{M})$ for all $1\le p<\infty$. Moreover, T is normal on $\mathcal{M}$. If T additionally has* (0.IV)*, the extension of T on $L_2(\mathcal{M})$ is selfadjoint.*

Proof. It is clear that T extends to $L^+_1(\mathcal{M})$ and $\|Tx\|_1\le\|x\|_1$ for all $x\in L^+_1(\mathcal{M})$. Then by a standard argument, T extends to a bounded map on $L_1(\mathcal{M})$, still denoted by T, which is of norm ≤ 2 and positive too. By duality, $S=T^*:\mathcal{M}\to\mathcal{M}$ is a bounded positive map. Consequently, $\|S\|=\|S(1)\|_\infty$ (see [Pau]). However, one easily checks that $\|S(1)\|_\infty\le 1$. Thus S is contractive, and so is T on $L_1(\mathcal{M})$. Then by complex interpolation, T extends to a contraction on $L_p(\mathcal{M})$ for all $1<p<\infty$.

Note that the positive map S on $\mathcal{M}$ introduced above satisfies the same assumptions as T. Thus applying the result just proved to S instead of T, we see that S can be extended to a contraction on $L_1(\mathcal{M})$. Then a simple calculation shows that the adjoint of this extension of S on $L_1(\mathcal{M})$ is equal to T. Hence T is normal on $\mathcal{M}$. The last part is clear. □

In the sequel, unless explicitly specified otherwise, T will always denote a map on $\mathcal{M}$ with (0.I)–(0.III). The same symbol T will also stand for the extensions of T on $L_p(\mathcal{M})$ given by Lemma 1.1. Let T be such a map. We form its ergodic averages:

$$M_n(T)=\frac{1}{n+1}\sum_{k=0}^{n}T^k\,.$$

$M_n(T)$ will be denoted by M_n whenever no confusion can occur.

By general ergodic theory on Banach spaces (cf. [DS]), one sees that T is mean ergodic on $L_p(\mathcal{M})$ for any $1<p<\infty$; i.e., $M_n(x)$ converges to $\widehat{x}$ in $L_p(\mathcal{M})$ for all

$x \in \mathcal{M}$. On the other hand, in case the trace τ is finite, by a well-known mean ergodic theorem (cf., e.g., [Ja1, Theorem 2.2.1] and the references therein), $M_n(x)$ converges to $\widehat{x}$ with respect to the strong operator topology for every $x \in \mathcal{M}$. This implies (and in fact is equivalent to) that $M_n(x)$ converges to $\widehat{x}$ for any $x \in L_1(\mathcal{M})$.

T induces a canonical splitting on $L_p(\mathcal{M})$ for $1 < p < \infty$:

$$L_p(\mathcal{M}) = \mathcal{F}_p(T) \oplus \mathcal{F}_p(T)^{\perp},$$

where $\mathcal{F}_p(T) = \{x \in L_p(\mathcal{M}) : T(x) = x\}$ and $\mathcal{F}_p(T)^{\perp}$ is the closure of the image $(I - T)(L_p(\mathcal{M}))$. The dual space of $\mathcal{F}_p(T)$ coincides with $\mathcal{F}_{p'}(T)$, where p' is the index conjugate to p. If in addition τ is finite, the previous splitting is also true for $p = 1$ and $p = \infty$. Note then however that $\mathcal{F}_\infty(T)^{\perp}$ is the w*-closure of the image $(I - T)(L_\infty(\mathcal{M}))$.

Let F_p be the contractive positive projection from $L_p(\mathcal{M})$ onto $\mathcal{F}_p(T)$. Then F_2 is the orthogonal projection from $L_2(\mathcal{M})$ onto $\mathcal{F}_2(T)$ and $F_p^* = F_{p'}$ for $1 < p < \infty$. (This is also true for $p = 1$ if τ is finite.) Note that F_p and F_q coincide on $\mathcal{F}_p(T) \cap \mathcal{F}_q(T)$ for two different p, q. This allows us to denote the F_p's by the same symbol F in the sequel. All previous facts are elementary and well known (cf. e.g. [Ye]).

Let us transfer the discussion above to the setting of semigroups. We will say that a semigroup $(T_t)_{t \geq 0}$ of linear maps on $\mathcal{M}$ satisfies one of the conditions (0.I)–(0.IV) if so does T_t for every $t \geq 0$. All semigroups considered in this paper will be assumed to satisfy (0.I)–(0.III). They will be further required to be w*-continuous on $\mathcal{M}$ and such that T_0 is the identity. As before, such semigroups are automatically extended to positive contractive semigroups on $L_p(\mathcal{M})$ for every $1 \leq p \leq \infty$. Note that the w*-continuity of (T_t) on $\mathcal{M}$ implies that (T_t) is strongly (i.e. norm) continuous on $L_p(\mathcal{M})$ for $1 \leq p < \infty$. Put again

$$M_t = \frac{1}{t} \int_0^t T^s \, ds, \quad t > 0.$$

Note that for notational simplicity we will use the same letter M to denote the ergodic averages for a contraction as well as for a semigroup. The precise meaning should be clear in the concrete context. Again, the mean ergodic theorem asserts that $M_t(x)$ converges to $F(x)$ in $L_p(\mathcal{M})$ for all $x \in L_p(\mathcal{M})$ $(1 < p < \infty)$, where F stands for the projection from $L_p(\mathcal{M})$ onto the fixed point space of (T_t), i.e., the space $\{x \in L_p(\mathcal{M}) \ : \ T_t(x) = x, \forall\, t > 0\}$.

The following result due to Yeadon [Ye] will play an important role in this paper. $\mathcal{P}(\mathcal{M})$ denotes the lattice of projections in $\mathcal{M}$. Given $e \in \mathcal{P}(\mathcal{M})$, set $e^{\perp} = 1 - e$.

Lemma 1.2. *Let T satisfy* (0.I)–(0.III). *Let $x \in L_1^+(\mathcal{M})$. Then for any $\lambda > 0$ there is $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\sup_{n \geq 0} \| e\, M_n(T)(x)\, e \|_\infty \leq \lambda \quad \textit{and} \quad \tau(e^{\perp}) \leq \frac{\|x\|_1}{\lambda} \ .$$

The reader can easily recognize that this is a noncommutative analogue of the classical weak type (1,1) maximal ergodic inequality. Yeadon's theorem has a martingale predecessor obtained by Cuculescu [Cu].

2. The spaces $L_p(\mathcal{M}; \ell_\infty)$

A fundamental objective of this paper is to study the noncommutative spaces $L_p(\mathcal{M}; \ell_\infty)$. Given $1 \leq p \leq \infty$, $L_p(\mathcal{M}; \ell_\infty)$ is defined as the space of all sequences

$x = (x_n)_{n\geq 0}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there are $a, b \in L_{2p}(\mathcal{M})$ and $y = (y_n) \subset L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad \forall\, n \geq 0.$$

We then define

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\big\{\|a\|_{2p} \sup_{n\geq 0}\|y_n\|_\infty\, \|b\|_{2p}\big\},$$

where the infimum runs over all factorizations as above. One can (rather easily) check that $\big(L_p(\mathcal{M};\ell_\infty),\ \|\cdot\|_{L_p(\mathcal{M};\ell_\infty)}\big)$ is a Banach space. These spaces are introduced in [P] and [Ju]. (In [P], $\mathcal{M}$ is required to be hyperfinite.) To gain a very first understanding on $L_p(\mathcal{M};\ell_\infty)$, let us consider a positive sequence $x = (x_n)$. Then one can show that $x \in L_p(\mathcal{M};\ell_\infty)$ iff there are an $a \in L_p^+(\mathcal{M})$ and a $y_n \in L_\infty^+(\mathcal{M})$ such that

$$x_n = a^{\frac{1}{2}}\, y_n\, a^{\frac{1}{2}}, \quad \forall\, n \geq 0.$$

We can clearly assume that the y_n are positive contractions. Thus if x has a factorization as above, then $x_n \leq a$ for all n. Conversely, if $x_n \leq a$ for some $a \in L_p^+(\mathcal{M})$, then $x_n^{1/2} = u_n a^{1/2}$ for a contraction $u_n \in \mathcal{M}$, and so $x_n = a^{1/2}u_n^* u_n a^{1/2}$. Thus $x \in L_p(\mathcal{M};\ell_\infty)$. In summary, a positive sequence x belongs to $L_p(\mathcal{M};\ell_\infty)$ iff there is an $a \in L_p^+(\mathcal{M})$ such that $x_n \leq a$ for all n, and moreover,

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\big\{\|a\|_p \ :\ a \in L_p^+(\mathcal{M}) \text{ s.t. } x_n \leq a,\ \forall\, n \geq 0\big\}.$$

Convention. The norm of x in $L_p(\mathcal{M};\ell_\infty)$ will often be denoted by $\big\|\sup_n^+ x_n\big\|_p$.

We should <u>warn</u> the reader that $\big\|\sup_n^+ x_n\big\|_p$ is just a notation for $\sup_n x_n$ and does not make any sense in the noncommutative setting. We find, however, that $\big\|\sup_n^+ x_n\big\|_p$ is more intuitive than $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$.

It is proved in [Ju] that $L_p(\mathcal{M};\ell_\infty)$ is a dual space for every $p > 1$. Its predual is $L_{p'}(\mathcal{M};\ell_1)$ (p' being the index conjugate to p). Let us define this latter space. Given $1 \leq p \leq \infty$, a sequence $x = (x_n)$ belongs to $L_p(\mathcal{M};\ell_1)$ if there are $u_{kn},\ v_{kn} \in L_{2p}(\mathcal{M})$ such that

$$x_n = \sum_{k\geq 0} u_{kn}^*\, v_{kn}$$

for all n and

$$\sum_{k,n\geq 0} u_{kn}^*\, u_{kn} \in L_p(\mathcal{M}), \quad \sum_{k,n\geq 0} v_{kn}^*\, v_{kn} \in L_p(\mathcal{M}).$$

Here all series are required to be convergent in $L_p(\mathcal{M})$ (relative to the w*-topology in the case of $p = \infty$). $L_p(\mathcal{M};\ell_1)$ is a Banach space when equipped with the norm

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \inf\Big\{\big\|\sum_{k,n\geq 0} u_{kn}^*\, u_{kn}\big\|_p^{\frac{1}{2}}\, \big\|\sum_{k,n\geq 0} v_{kn}^*\, v_{kn}\big\|_p^{\frac{1}{2}}\Big\},$$

where the infimum is taken over all (u_{kn}) and (v_{kn}) as above. It is clear that finite sequences are dense in $L_p(\mathcal{M};\ell_1)$ if $p < \infty$. The duality between $L_p(\mathcal{M};\ell_\infty)$ and $L_{p'}(\mathcal{M};\ell_1)$ is given by

$$\langle x,\ y\rangle = \sum_{n\geq 0} \tau(x_n y_n).$$

As previously for $L_p(\mathcal{M};\ell_\infty)$, it is easy to describe the positive sequences in $L_p(\mathcal{M};\ell_1)$. In fact, a positive sequence $x=(x_n)$ belongs to $L_p(\mathcal{M};\ell_1)$ iff $\sum_n x_n \in L_p(\mathcal{M})$. If this is the case,

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \big\|\sum_{n\ge 0} x_n\big\|_p\ .$$

Compare this equality (whose member on the right has the usual sense) with our previous convention for the norm in $L_p(\mathcal{M};\ell_\infty)$. This partly justifies the intuitive notation $\big\|\sup_n^+ x_n\big\|_p$.

We collect some elementary properties of these spaces in the following proposition. We denote by $L_p(\mathcal{M};\ell_\infty^{n+1})$ the subspace of $L_p(\mathcal{M};\ell_\infty)$ consisting of all finite sequences $(x_0,x_1,\cdots,x_n,0,\cdots)$. In accordance with our preceding convention, the norm of x in $L_p(\mathcal{M};\ell_\infty^{n+1})$ will be denoted by $\|\sup^+_{0\le k\le n} x_k\|_p$. Similarly, we introduce the subspace $L_p(\mathcal{M};\ell_1^{n+1})$ of $L_p(\mathcal{M};\ell_1)$.

Proposition 2.1. *Let $1\le p\le\infty$.*

i) *Each element in the unit ball of $L_p(\mathcal{M};\ell_\infty)$ (resp. $L_p(\mathcal{M};\ell_1)$) is a sum of sixteen (resp. eight) positive elements in the same ball.*

ii) *A sequence $x=(x_n)$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ iff*

$$\sup_{n\ge 0}\big\|\sup_{0\le k\le n}{}^+\ x_k\big\|_p<\infty.$$

If this is the case, then

$$\big\|\sup_n{}^+ x_n\big\|_p=\sup_{n\ge 0}\big\|\sup_{0\le k\le n}{}^+\ x_k\big\|_p\ .$$

iii) *Let $x=(x_n)$ be a positive sequence in $L_p(\mathcal{M};\ell_\infty)$. Then*

$$\big\|\sup_n{}^+ x_n\big\|_p=\sup\big\{\sum_n\tau(x_ny_n)\ :\ y_n\in L^+_{p'}(\mathcal{M})\ \text{and}\ \big\|\sum_n y_n\big\|_{p'}\le 1\big\}.$$

iv) *We have the following Cauchy-Schwarz type inequality: for any sequences (x_n) and (y_n) in $L_{2p}(\mathcal{M})$,*

$$\big\|\sup_n{}^+ x_n^*\,y_n\big\|_p\le\big\|\sup_n{}^+ x_n^*x_n\big\|_p^{\frac12}\,\big\|\sup_n{}^+ y_n^*y_n\big\|_p^{\frac12}\ .$$

Proof. i) First note that both $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$ are closed with respect to involution. Thus we need only to consider selfadjoint elements. Let x be a selfadjoint element (i.e., $x_n=x_n^*$ for all n) in the unit ball of $L_p(\mathcal{M};\ell_\infty)$. Write a factorization of x:

$$x_n=a^*y_nb\quad\text{with}\quad \|a\|_{2p}\le 1,\ \|b\|_{2p}\le 1\ \text{and}\ \sup_n\|y_n\|_\infty\le 1.$$

Then by a standard polorization argument,

$$\begin{aligned} x_n &= \frac14\sum_{k=0}^3 i^{-k}\,(a+i^kb)^*\,y_n\,(a+i^kb)=\frac14\sum_{k=0}^3(a+i^kb)^*\,z_n\,(a+i^kb)\\ &= \sum_{k=0}^3\big(\frac{a+i^kb}{2}\big)^*\,z_n^+\,\big(\frac{a+i^kb}{2}\big)\ -\ \sum_{k=0}^3\big(\frac{a+i^kb}{2}\big)^*\,z_n^-\,\big(\frac{a+i^kb}{2}\big)\,, \end{aligned}$$

where

$$z_n=\frac{i^{-k}y_n+(i^{-k}y_n)^*}{2}\,.$$

Hence the assertion concerning $L_p(\mathcal{M};\ell_\infty)$ follows. The one for $L_p(\mathcal{M};\ell_1)$ is proved similarly.

ii) It is trivial that

$$\sup_{n\geq 0}\big\|\sup_{0\leq k\leq n}{}^{+}\, x_k\big\|_p \leq \big\|\sup_n{}^{+} x_n\big\|_p\,.$$

To prove the converse, we introduce the subspace $L_p(\mathcal{M};\ell_1^0)$ of $L_p(\mathcal{M};\ell_1)$, which consists of all finite sequences x admitting a factorization as in the definition above of $L_p(\mathcal{M};\ell_1)$ but with finite families (u_{kn}) and (v_{kn}) only. Note that for $p<\infty$, $L_p(\mathcal{M};\ell_1^0)$ is dense in $L_p(\mathcal{M};\ell_1)$. Now let x be a sequence such that $\sup_{n\geq 0}\big\|\sup^+_{0\leq k\leq n} x_k\big\|_p = 1$. Define $\ell : L_{p'}(\mathcal{M};\ell_1^0)\to\mathbb{C}$ by $\ell(y)=\sum_n \tau(x_ny_n)$. Then ℓ is a continuous linear functional of norm ≤ 1. Thus if $p>1$ (i.e., $p'<\infty$), by the duality result in [Ju] already quoted previously, ℓ can be identified with an element of $L_p(\mathcal{M};\ell_\infty)$. This element must be x, and so we are done in this case. It remains to consider the case $p=1$. Using a standard Hahn-Banach argument as presented in [Ju], we deduce two states φ and ψ on $\mathcal{M}$ such that

$$|\tau(x_nu^*v)|\leq \big(\varphi(u^*u)\big)^{\frac12}\,\big(\psi(v^*v)\big)^{\frac12}\ ,\quad n\geq 0,\ \forall\, u,v\in\mathcal{M}.$$

Since $x_n\in L_1(\mathcal{M})\simeq\mathcal{M}_*$ is a normal functional, we can replace in the inequality above φ and ψ by their normal parts respectively, and so we can assume φ and ψ are already normal. (In fact, in the present case, one can check that the singular parts of φ and ψ are zero.) Identifying φ and ψ with two positive operators a and b in the unit ball of $L_1(\mathcal{M})$, respectively, we rewrite the inequality above as

$$|\tau(x_nu^*v)|\leq \|ua^{\frac12}\|_2\,\|vb^{\frac12}\|_2\ ,\qquad n\geq 0,\ \forall\, u,v\in\mathcal{M}.$$

Then as in [Ju], we find contractions $y_n\in\mathcal{M}$ such that $x_n=b^{1/2}\,y_n\,a^{1/2}$. Therefore, $x\in L_1(\mathcal{M};\ell_\infty)$ and $\|\sup^+_n x_n\|_1\leq 1$.

Note that if additionally x is positive, in the Hahn-Banach argument above, we can use only the positive cone $L^+_{p'}(\mathcal{M};\ell_1^0)$ to get a factorization of x as $x_n = a^{1/2}\,y_n\,a^{1/2}$ with $a\in L_p^+(\mathcal{M})$ and y_n positive contractions. See [Ju] for more details.

iii) For $p>1$ this is already proved in [Ju]. For $p=1$ this is a consequence of ii) and the previous remark.

iv) We use duality. Let (u_{kn}) and (v_{kn}) be two finite families in $L_{2p'}(\mathcal{M})$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}\big|\sum_{k,n}\tau(x_n^*\,y_n\,u_{kn}^*\,v_{kn})\big| &\leq \big(\tau\sum_{k,n}x_n\,v_{kn}^*\,v_{kn}\,x_n^*\big)^{\frac12}\,\big(\tau\sum_{k,n}y_n\,u_{kn}^*\,u_{kn}\,y_n^*\big)^{\frac12}\\ &\leq \big\|\sup_n{}^{+}x_n^*x_n\big\|_p^{\frac12}\,\big\|\sup_n{}^{+}y_n^*y_n\big\|_p^{\frac12}\,\big\|\sum_{k,n}v_{kn}^*v_{kn}\big\|_{p'}^{\frac12}\big\|\sum_{k,n}u_{kn}^*u_{kn}\big\|_{p'}^{\frac12}\,,\end{aligned}$$

whence the desired inequality. □

Remark 2.2. i) From the proof of part ii) above, one sees that the infimum defining the norm $\|\sup_n x_n\|_p$ is attained for any $x\in L_p(\mathcal{M};\ell_\infty)$ $(1\leq p\leq\infty)$. The same proof shows that $L_1(\mathcal{M};\ell_\infty)$ is identified as an isometric subspace of the dual of $L_\infty(\mathcal{M};\ell_1^0)$.

ii) We have a statement similar to Proposition 2.1, ii) for $L_p(\mathcal{M};\ell_1)$. On the other hand, let $L_p(\mathcal{M};c_0)$ be the closure of finite sequences in $L_p(\mathcal{M};\ell_\infty)$ for $1\leq p<\infty$. Then one can show that the dual space of $L_p(\mathcal{M};c_0)$ is equal to $L_{p'}(\mathcal{M};\ell_1)$ isometrically.

iii) Neither $L_p(\mathcal{M};\ell_\infty)$ nor $L_p(\mathcal{M};\ell_1)$ is stable under the operation $(x_n)_n \mapsto (|x_n|)_n$. Thus $\|\sup_n^+ x_n\|_p \neq \|\sup_n^+ |x_n|\,\|_p$ in general.

Remark 2.3. If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra such that the trace τ restricted to $\mathcal{N}$ is semifinite on $\mathcal{N}$, then we have a natural isometric inclusion $L_p(\mathcal{N}) \subset L_p(\mathcal{M})$. This extends to isometric inclusions:

$$L_p(\mathcal{N};\ell_\infty) \subset L_p(\mathcal{M};\ell_\infty) \quad \text{and} \quad L_p(\mathcal{N};\ell_1) \subset L_p(\mathcal{M};\ell_1)\,.$$

Indeed, by the definition of $L_p(\mathcal{M};,\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$, the inclusions above are contractive. On the other hand, the duality result from the preceding proposition and remarks implies immediately that they are both isometric.

Remark 2.4. The definitions of $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$ can be extended to an arbitrary index set. Let I be an index set. Then $L_p(\mathcal{M};\ell_\infty(I))$ and $L_p(\mathcal{M};\ell_1(I))$ are defined similarly as before. For instance, $L_p(\mathcal{M};\ell_\infty(I))$ consists of all families $(x_i)_{i\in I}$ in $L_p(\mathcal{M})$ which can be factorized as $x_i = ay_ib$ with $a, b \in L_{2p}(\mathcal{M})$ and a bounded family $(y_i)_{i\in I} \subset L_\infty(\mathcal{M})$. The norm of $(x_i)_{i\in I}$ in $L_p(\mathcal{M};\ell_\infty)$ is defined as the infimum

$$\inf \|a\|_{2p} \sup_i \|y_i\|_\infty \|b\|_{2p}$$

running over all factorizations as above. As before, this norm is also denoted by

$$\big\| \sup_i{}^+ x_i \big\|_p\ .$$

Again the dual space of $L_p(\mathcal{M};\ell_1(I))$ for $p < \infty$ is $L_{p'}(\mathcal{M};\ell_\infty(I))$. Proposition 2.1 remains true in this general setting.

We end this section with a simple result on complex interpolation of these vector-valued noncommutative L_p-spaces.

Proposition 2.5. *Let $1 \le p_0 < p_1 \le \infty$ and $0 < \theta < 1$. Then we have isometrically*

$$L_p(\mathcal{M};\ell_1) = \big(L_{p_0}(\mathcal{M};\ell_1),\ L_{p_1}(\mathcal{M};\ell_1)\big)_\theta$$

and

$$L_p(\mathcal{M};\ell_\infty) = \big(L_{p_0}(\mathcal{M};\ell_\infty),\ L_{p_1}(\mathcal{M};\ell_\infty)\big)_\theta\ ,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. We use the column and row spaces $L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ and $L_p(\mathcal{M};\ell_2^r(\mathbb{N}^2))$ (cf. [PX1] for the definition). It is known that $\{L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))\}_{1\le p\le\infty}$ form an interpolation scale with respect to complex interpolation. The same is true for the row spaces. Note that by the definition of $L_p(\mathcal{M};\ell_1)$, the bilinear map

$$\begin{array}{cccccl} B\ : & L_p(\mathcal{M};\ell_2^r(\mathbb{N}^2)) & \times & L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2)) & \longrightarrow & L_p(\mathcal{M};\ell_1) \\ & \big(u_{kn}\big)_{k,n\ge 0} & \times & \big(v_{kn}\big)_{k,n\ge 0} & \longmapsto & \big(\sum_k u_{kn}v_{kn}\big)_{n\ge 0} \end{array}$$

is contractive (in fact, $L_p(\mathcal{M};\ell_1)$ is just the quotient space of $L_p(\mathcal{M};\ell_2^r(\mathbb{N}^2)) \times L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ by the kernel of B). Thus by complex interpolation for bilinear maps (cf. [BeL]), we deduce that

$$B : L_p(\mathcal{M};\ell_2^r(\mathbb{N}^2)) \times L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2)) \to \big(L_{p_0}(\mathcal{M};\ell_1),\ L_{p_1}(\mathcal{M};\ell_1)\big)_\theta$$

is contractive. This yields

$$\text{(2.1)} \qquad L_p(\mathcal{M};\ell_1) \subset \big(L_{p_0}(\mathcal{M};\ell_1),\ L_{p_1}(\mathcal{M};\ell_1)\big)_\theta\,, \quad \text{a contractive inclusion.}$$

Similarly, using the complex interpolation for trilinear maps, we obtain the following contractive inclusion

$$L_p(\mathcal{M};\ell_\infty) \subset \big(L_{p_0}(\mathcal{M};\ell_\infty),\ L_{p_1}(\mathcal{M};\ell_\infty)\big)_\theta\,. \tag{2.2}$$

Alternatively, this can be easily proved by using directly the factorization of elements in $L_p(\mathcal{M};\ell_\infty)$.

Dualizing the corresponding inclusion of (2.1) for finite sequences, we get

$$\big(L_{p_0'}(\mathcal{M};\ell_\infty^n),\ \big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^*\big)^\theta \subset L_{p'}(\mathcal{M};\ell_\infty^n),$$

where $(\cdot,\ \cdot)^\theta$ denotes Calderón's second complex interpolation method. However,

$$\big(L_{p_0'}(\mathcal{M};\ell_\infty^n),\ \big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^*\big)_\theta \subset \big(L_{p_0'}(\mathcal{M};\ell_\infty^n),\ \big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^*\big)^\theta\,.$$

It follows that

$$\big(L_{p_0'}(\mathcal{M};\ell_\infty^n),\ \big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^*\big)_\theta \subset L_{p'}(\mathcal{M};\ell_\infty^n).$$

Since $L_{p_0'}(\mathcal{M};\ell_\infty^n)\cap\big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^*$ is dense in the complex interpolation space on the left and

$$L_{p_0'}(\mathcal{M};\ell_\infty^n)\cap\big(L_{p_1}(\mathcal{M};\ell_1^n)\big)^* \subset L_{p_0'}(\mathcal{M};\ell_\infty^n)\cap L_{p_1'}(\mathcal{M};\ell_\infty^n),$$

we deduce that

$$\big(L_{p_0'}(\mathcal{M};\ell_\infty^n),\ L_{p_1'}(\mathcal{M};\ell_\infty^n)\big)_\theta \subset L_{p'}(\mathcal{M};\ell_\infty^n),\quad \text{a contractive inclusion.}$$

Reformulating this for the indices p_0, p_1, we have

$$\big(L_{p_0}(\mathcal{M};\ell_\infty^n),\ L_{p_1}(\mathcal{M};\ell_\infty^n)\big)_\theta \subset L_{p}(\mathcal{M};\ell_\infty^n),\quad \text{a contractive inclusion.}$$

From this we easily get the same inclusion for infinite sequence spaces. Indeed, let $x=(x_k)_{k\ge0}$ be an element in $\big(L_{p_0}(\mathcal{M};\ell_\infty),\ L_{p_1}(\mathcal{M};\ell_\infty)\big)_\theta$ of norm ≤ 1 and let $x^{(n)}=(x_0,x_1,...,x_n,0,0,...)$. Then $x^{(n)}\in\big(L_{p_0}(\mathcal{M};\ell_\infty^{n+1}),\ L_{p_1}(\mathcal{M};\ell_\infty^{n+1})\big)_\theta$ and is of norm ≤ 1. Thus $x^{(n)}\in L_p(\mathcal{M};\ell_\infty^{n+1})$ and is of norm ≤ 1. Consequently,

$$\sup_n \big\|x^{(n)}\big\|_{L_p(\mathcal{M};\ell_\infty)} \le 1.$$

Therefore, by Proposition 2.1, ii), we deduce that x belongs to the unit ball of $L_p(\mathcal{M};\ell_\infty)$, and so by homogeneity, we obtain the converse inclusion of (2.2).

The converse inclusion of (2.1) can be proved similarly. But this time instead of Proposition 2.1, it suffices to use the fact that finite sequences are dense in $\big(L_{p_0}(\mathcal{M};\ell_1),\ L_{p_1}(\mathcal{M};\ell_1)\big)_\theta$. We omit the details. □

Remark. In a forthcoming paper we will show that the interpolation equalities in Proposition 2.5 are no longer true for the real interpolation. This is one of the difficulties we will encounter for proving the Marcinkiewicz type theorem in the next section.

3. An interpolation theorem

The main result of this section is a Marcinkiewicz type interpolation theorem for $L_p(\mathcal{M};\,\ell_\infty)$. It is the key to our proof of the noncommutative maximal ergodic inequalities. We first introduce the following notion. For every integer $n\ge0$ assume given a map $S_n: L_p^+(\mathcal{M})\to L_0^+(\mathcal{M})$. Set $S=(S_n)_{n\ge0}$. Thus S is a map which sends positive operators to sequences of positive operators. We say that S

is of *weak type* (p,p) $(p<\infty)$ if there is a positive constant C such that for any $x\in L_p^+(\mathcal{M})$ and any $\lambda>0$ there is a projection $e\in\mathcal{M}$ such that

$$\tau(e^{\perp})\le\big[C\,\frac{\|x\|_p}{\lambda}\big]^p\quad\text{and}\quad e\big(S_n(x)\big)e\le\lambda\,,\ \ \forall\ n\ge 0.$$

Similarly, we say that S is of *type* (p,p) (this time p may be equal to ∞) if there is a positive constant C such that for any $x\in L_p^+(\mathcal{M})$ there is an $a\in L_p^+(\mathcal{M})$ satisfying

$$\|a\|_p\le C\,\|x\|_p\quad\text{and}\quad S_n(x)\le a,\ \ \forall\ n\ge 0.$$

In other words, S is of type (p,p) iff

$$\|S(x)\|_{L_p(\mathcal{M};\,\ell_\infty)}\le C\,\|x\|_p\,,\quad\forall\ x\in L_p^+(\mathcal{M}).$$

Theorem 3.1. *Let $1\le p_0<p_1\le\infty$. Let $S=\big(S_n\big)_{n\ge0}$ be a sequence of maps from $L_{p_0}^+(\mathcal{M})+L_{p_1}^+(\mathcal{M})$ into $L_0^+(\mathcal{M})$. Assume that S is subadditive in the sense that $S_n(x+y)\le S_n(x)+S_n(y)$ for all $n\in\mathbb{N}$. If S is of weak type (p_0,p_0) with constant C_0 and of type (p_1,p_1) with constant C_1, then for any $p_0<p<p_1$, S is of type (p,p) with constant C_p satisfying*

$$C_p\le C\,C_0^{1-\theta}\,C_1^{\theta}\,\big(\frac{1}{p_0}-\frac{1}{p}\big)^{-2}\,, \tag{3.1}$$

where θ is determined by $1/p=(1-\theta)/p_0+\theta/p_1$ and C is a universal constant.

The reader can easily recognize that this result is a noncommutative analogue of the classical Marcinkiewicz interpolation theorem. Recall that in the classical case the constant C_p is majorized by $C_0^{1-\theta}\,C_1^{\theta}\,\big(1/p_0-1/p\big)^{-1}$, i.e., without the square in (3.1). We will see later that the estimate given by (3.1) is optimal in the noncommutative setting. This difference indicates that though similar in form to the classical Marcinkiewicz interpolation theorem, Theorem 3.1 cannot be proved by the standard argument in the commutative case. The rest of this section is entirely devoted to its proof. In the following S will be fixed as in the theorem above; p will denote a number such that $p_0<p<p_1$, and θ is determined by $1/p=(1-\theta)/p_0+\theta/p_1$. C will stand for a universal constant.

The following lemma is entirely elementary.

Lemma 3.2. *Let (x_{ij}) be a finite matrix of bounded operators on a Hilbert space H. Let (e_i) and (f_i) be two sequences of pairwise disjoint projections in $B(H)$. Then*

$$\|\sum_{i,j}e_ix_{ij}f_j\|_{B(H)}\le\big\|\big(\|e_ix_{ij}f_j\|_{B(H)}\big)_{i,j}\big\|_{B(\ell_2)}\ .$$

Proof. Let $\xi,\eta\in H$. Then

$$\begin{aligned}\langle\xi,\ \sum_{i,j}e_ix_{ij}f_j\eta\rangle\ &=\ \sum_{i,j}\|e_ix_{ij}f_j\|\,\|e_i\xi\|\,\|f_j\eta\|\\ &\le\ \big\|\big(\|e_ix_{ij}f_j\|_{B(H)}\big)_{i,j}\big\|_{B(\ell_2)}\big(\sum_i\|e_i\xi\|^2\big)^{\frac12}\ \big(\sum_j\|f_j\eta\|^2\big)^{\frac12}\\ &\le\ \big\|\big(\|e_i\,x_{ij}\,f_j\|_{B(H)}\big)_{i,j}\big\|_{B(\ell_2)}\,\|\xi\|\,\|\eta\|.\end{aligned}$$

This yields the desired inequality. □

Lemma 3.3. *Let $x \in L_{p_0}^+(\mathcal{M})$ and $\lambda > 0$. Then there is an $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(e^\perp) \le \big[C_0 \lambda^{-1} \|x\|_{p_0}\big]^{p_0},\ \big\| \sup_n{}^+ \big(eS_n(x)e\big)\big\|_p \le C(1-\frac{p_0}{p})^{-1-\frac{1}{p}} \big[C_0\|x\|_{p_0}\big]^{\frac{p_0}{p}} \lambda^{1-\frac{p_0}{p}}\,.$$

Proof. Fix an $x \in L_{p_0}^+(\mathcal{M})$. Set $x_n = S_n(x)$ for $n \in \mathbb{N}$. Since S is of weak type (p_0, p_0), given $k \in \mathbb{Z}$ there is $f_k \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(f_k^\perp) \le \big[C_0 2^{-k} \|x\|_{p_0}\big]^{p_0} \quad \text{and} \quad f_k x_n f_k \le 2^k\,, \quad \forall\, n \in \mathbb{N}.$$

Let

$$g_k = \bigvee_{j \ge k} f_j^\perp\,.$$

Then $(g_k)_{k\in\mathbb{Z}}$ is a decreasing sequence of projections and

$$\tau(g_k) \le \big[C_0 2^{-k+1} \|x\|_{p_0}\big]^{p_0}\,.$$

Thus $\lim_{k\to+\infty} g_k = 0$. Put $g_{-\infty} = \lim_{k\to-\infty} g_k$. Then $g_{-\infty} \ge g_k \ge f_k^\perp$, and so $g_{-\infty}^\perp \le g_k^\perp \le f_k$ for all $k \in \mathbb{Z}$. Put

$$d_k = g_k - g_{k+1} \quad \text{and} \quad e_k = \sum_{j\le k} d_j\,.$$

Then $e_k = g_{-\infty} - g_{k+1}$. We claim that

$$(g_{-\infty}^\perp + e_k)\, x_n\, (g_{-\infty}^\perp + e_k) = e_k x_n e_k.$$

Since $g_{-\infty}^\perp \le f_k$, by the choice of f_k,

$$g_{-\infty}^\perp x_n g_{-\infty}^\perp = g_{-\infty}^\perp \big(f_k x_n f_k\big) g_{-\infty}^\perp \le 2^k g_{-\infty}^\perp\,, \quad k \in \mathbb{Z};$$

thus letting $k \to -\infty$, we get $g_{-\infty}^\perp x_n g_{-\infty}^\perp = 0$. On the other hand,

$$\|g_{-\infty}^\perp x_n e_k\|_\infty \le \|g_{-\infty}^\perp x_n g_{-\infty}^\perp\|_\infty^{\frac12}\ \|e_k x_n e_k\|_\infty^{\frac12} = 0\,,$$

whence

$$g_{-\infty}^\perp x_n e_k = 0 = e_k x_n g_{-\infty}^\perp\,.$$

Therefore our claim is proved.

Now let $0 < s < 1$ such that $sp > p_0$. Set

$$b_k = \sum_{j\le k} 2^{js} d_j\,.$$

Since the d_j are disjoint projections, we have

$$\begin{aligned} \|b_k\|_p &= \big(\sum_{j\le k} 2^{jsp} \tau(d_j)\big)^{\frac1p} \\ &\le \big(\sum_{j\le k} 2^{jsp} \big[C_0 2^{-j+1} \|x\|_{p_0}\big]^{p_0}\big)^{\frac1p} \\ &\le C(sp-p_0)^{-\frac1p} \big(C_0\|x\|_{p_0}\big)^{\frac{p_0}{p}}\, 2^{k(s-\frac{p_0}{p})}\,. \end{aligned}$$

On the other hand, since the support of b_k is equal to e_k, $e_k b_k^{-\frac12}$ can be regarded as a well-defined operator, and we have

$$e_k b_k^{-\frac12} = \sum_{j\le k} 2^{-\frac{js}{2}} d_j\,.$$

Thus

$$b_k^{-\frac{1}{2}}e_kx_ne_kb_k^{-\frac{1}{2}} = \sum_{i,j\le k} 2^{-\frac{is}{2}}2^{-\frac{js}{2}}d_ix_nd_j\,.$$

Since $d_i \le g_{i+1}^{\perp} \le f_{i+1}$, then by the choice of f_i, we get

$$\|d_ix_nd_j\|_\infty \le \|d_ix_nd_i\|_\infty^{\frac{1}{2}}\,\|d_jx_nd_j\|_\infty^{\frac{1}{2}} \le 2^{\frac{i+1}{2}}\,2^{\frac{j+1}{2}}.$$

Therefore, using Lemma 3.2, we deduce

$$\big\|b_k^{-\frac{1}{2}}e_kx_ne_kb_k^{-\frac{1}{2}}\big\|_\infty \le C(1-s)^{-1}\,2^{k(1-s)}\,. \tag{3.2}$$

Note that the sequence $(e_kx_ne_k)_{n\ge 0}$ admits the following factorization:

$$e_kx_ne_k = b_k^{\frac{1}{2}}\,\big[b_k^{-\frac{1}{2}}e_kx_ne_kb_k^{-\frac{1}{2}}\big]\,b_k^{\frac{1}{2}}\,\,.$$

Combining this with the previous inequalities, we obtain

$$\big\|\sup_n{}^{+}\big(e_kx_ne_k\big)\big\|_p \le C(1-s)^{-1}(sp-p_0)^{-\frac{1}{p}}\big(C_0\|x\|_{p_0}\big)^{\frac{p_0}{p}}\,2^{k(1-\frac{p_0}{p})}\,.$$

Thus the choice of $s=(1+p_0)/(1+p)$ yields

$$\big\|\sup_n{}^{+}\big(e_kx_ne_k\big)\big\|_p \le C(1-\frac{p_0}{p})^{-1-\frac{1}{p}}\big(C_0\|x\|_{p_0}\big)^{\frac{p_0}{p}}\,2^{k(1-\frac{p_0}{p})}\,.$$

Given $\lambda>0$ we choose k such that $2^k\le\lambda<2^{k+1}$. Then $e=g_{-\infty}^{\perp}+e_k$ is the desired projection. $\square$

Remark. If we simply use the triangle inequality to majorize $\big\|b_k^{-\frac{1}{2}}e_kx_ne_kb_k^{-\frac{1}{2}}\big\|_\infty$ instead of Lemma 3.2, the estimates in (3.2) become $(1-s)^{-2}$. This does not give the right estimate in (3.1).

The following lemma is a key step towards the proof of Theorem 3.1.

Lemma 3.4. *For any $x\in L_{p_0}^+(\mathcal{M})\cap L_{p_1}^+(\mathcal{M})$,*

$$\big\|\sup_n{}^{+}S_n(x)\big\|_p \le C(1-\frac{p_0}{p})^{-1-\frac{1}{p}}\big(C_0\|x\|_{p_0}\big)^{1-\theta}\big(C_1\|x\|_{p_1}\big)^{\theta}\,.$$

Proof. Fix $x\in L_{p_0}^+(\mathcal{M})\cap L_{p_1}^+(\mathcal{M})$, and set $x_n=S_n(x)$ as before. Let $\lambda>0$. Choose $e\in\mathcal{P}(\mathcal{M})$ as in Lemma 3.3. Then

$$x_n = ex_ne + e^{\perp}x_ne + ex_ne^{\perp} + e^{\perp}x_ne^{\perp}\,.$$

Let us first estimate $\|\sup_n e^{\perp}x_ne^{\perp}\|_p$. Since S is of type (p_1,p_1), there is an $a\in L_{p_1}^+(\mathcal{M})$ such that

$$\|a\|_{p_1}\le C_1\|x\|_{p_1} \quad\text{and}\quad x_n\le a, \quad \forall\, n\in\mathbb{N}.$$

Thus

$$e^{\perp}x_ne^{\perp}\le e^{\perp}ae^{\perp}, \quad \forall\, n\in\mathbb{N}.$$

With r determined by $1/r=1/p-1/p_1$, by the Hölder inequality, we have

$$\|e^{\perp}ae^{\perp}\|_p \le \big(\tau(e^{\perp})\big)^{\frac{1}{r}}\,\|a\|_{p_1} \le \big(C_0\|x\|_{p_0}\lambda^{-1}\big)^{\frac{p_0}{r}}\,C_1\|x\|_{p_1}\,.$$

Therefore

$$\big\|\sup_n{}^{+}\big(e^{\perp}x_ne^{\perp}\big)\big\|_p \le \big(C_0\|x\|_{p_0}\lambda^{-1}\big)^{\frac{p_0}{r}}\,C_1\|x\|_{p_1}\,.$$

For the two mixed terms, by the Cauchy-Schwarz inequality in Proposition 2.1, we have

$$\big\| \sup_n{}^+ \big(e^{\perp} x_n e\big)\big\|_p \le \big\| \sup_n{}^+ \big(e^{\perp} x_n e^{\perp}\big)\big\|_p^{\frac12}\, \big\| \sup_n{}^+ \big(e x_n e\big)\big\|_p^{\frac12}\,,$$

and the same inequality holds for the other mixed term. Hence, we deduce

$$\begin{aligned}\big\| \sup_n{}^+ x_n\big\|_p &\le 2\big(\big\| \sup_n{}^+ \big(e x_n e\big)\big\|_p + \big\| \sup_n{}^+ \big(e^{\perp} x_n e^{\perp}\big)\big\|_p\big)\\ &\le C(1-\frac{p_0}{p})^{-1-\frac1p}\big(C_0\|x\|_{p_0}\big)^{\frac{p_0}{p}}\lambda^{1-\frac{p_0}{p}} + C\big(C_0\|x\|_{p_0}\lambda^{-1}\big)^{\frac{p_0}{r}}\, C_1\|x\|_{p_1}\ .\end{aligned}$$

Choosing λ such that

$$\lambda^{1-\frac{p_0}{p_1}} = \big(C_0\|x\|_{p_0}\big)^{-\frac{p_0}{p_1}}\, C_1\|x\|_{p_1}\,,$$

we obtain the desired inequality. □

The previous lemma can be restated as follows.

Lemma 3.5. *For any* $x\in L^+_{p,1}(\mathcal{M})$,

$$\big\| \sup_n{}^+ S_n(x)\big\|_p \le C(1-\frac{p_0}{p})^{-1-\frac1p}\, C_0^{1-\theta}\, C_1^{\theta}\, \|x\|_{p,1}\,.$$

We will need to interpolate a compatible couple of cones. We refer to [BeL] for the J- and K-methods in interpolation theory for Banach spaces. Let $(B_0,\ B_1)$ be a compatible couple of Banach spaces. Let $A_i\subset B_i$ be a closed cone $(i=0,1)$. Given $0<\theta<1$ and $1\le q\le\infty$ we can define the J-method for the couple (A_0,A_1). More precisely, $(A_0,A_1)_{\theta,q;J}$ consists of all $x\in B_0+B_1$ which admit a decomposition of the following form:

$$x=\int_0^\infty u(t)\,\frac{dt}{t}\quad (\text{convergence in } B_0+B_1)$$

with $u(t)\in A_0\cap A_1$ such that

$$\Big(\int_0^\infty \big[t^{-\theta}\,\max\big(\|u(t)\|_{B_0},\, t\,\|u(t)\|_{B_1}\big)\big]^q\,\frac{dt}{t}\Big)^{\frac1q}<\infty.$$

We define

$$\|x\|_{(A_0,A_1)_{\theta,q;J}} = \inf\Big\{\Big(\int_0^\infty \big[t^{-\theta}\,\max\big(\|u(t)\|_{B_0},\, t\,\|u(t)\|_{B_1}\big)\big]^q\,\frac{dt}{t}\Big)^{\frac1q}\Big\},$$

where the infimum runs over all decompositions of x as above.

It is clear that

$$(A_0,A_1)_{\theta,q;J}\subset (B_0,B_1)_{\theta,q;J}\ ,\quad \text{a contractive inclusion.}$$

But in general the norm in $(A_0,A_1)_{\theta,q;J}$ is not equivalent to that of $(B_0,B_1)_{\theta,q;J}$ when restricted to $(A_0,A_1)_{\theta,q;J}$. However, this is true for a couple of noncommutative L_p-spaces.

Remark 3.6. The following natural inclusion

$$\big(L^+_{p_0,q_0}(\mathcal{M}), L^+_{p_1,q_1}(\mathcal{M})\big)_{\theta,q;J}\subset \big(L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M})\big)_{\theta,q;J}$$

is isometric.

Proof. Let $x \in \big(L^+_{p_0,q_0}(\mathcal{M}), L^+_{p_1,q_1}(\mathcal{M})\big)_{\theta,q;J}$. Let

$$x = \int_0^\infty u(t)\,\frac{dt}{t}$$

be a decomposition of x relative to $\big(L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M})\big)_{\theta,q;J}$ with $u(t)$ in the space $L_{p_0,q_0}(\mathcal{M}) \cap L_{p_1,q_1}(\mathcal{M})$ such that

$$\Big(\int_0^\infty \big[t^{-\theta}\,\max\big(\|u(t)\|_{p_0,q_0},\, t\,\|u(t)\|_{p_1,q_1}\big)\big]^q\,\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Then we must find a similar decomposition of x with all $u(t)$ in $L^+_{p_0,q_0}(\mathcal{M}) \cap L^+_{p_1,q_1}(\mathcal{M})$ without increasing the integral above. Since $x \geq 0$, we can assume all $u(t)$ above to be selfadjoint. Decomposing $u(t)$ into its positive and negative part, we have

$$x = \int_0^\infty u(t)^+\,\frac{dt}{t} - \int_0^\infty u(t)^-\,\frac{dt}{t} \leq \int_0^\infty u(t)^+\,\frac{dt}{t}\,.$$

Therefore there is a contraction $v \in \mathcal{M}$ such that

$$x^{\frac{1}{2}} = v\,\Big[\int_0^\infty u(t)^+\,\frac{dt}{t}\Big]^{\frac{1}{2}},$$

and so

$$x\int_0^\infty \big[v\,u(t)^+\,v^*\big]\,\frac{dt}{t}$$

yields the desired decomposition of x. □

We will need the following result from [Ho], which gives the optimal estimates for the equivalence constants in (1.2). Note that this result is stated in [Ho] for the commutative L_p-spaces only. It is easy to see that the noncommutative result follows immediately.

Lemma 3.7. *Let $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0, q_1, q \leq \infty$. Then the equivalence constants in the following equality*

$$L_{p,q}(\mathcal{M}) = \big(L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M})\big)_{\theta,q;K}$$

are estimated as follows:

$$\begin{aligned} C^{-1}\,\theta^{-\min(\frac{1}{q},\frac{1}{q_0})}\,(1-\theta)^{-\min(\frac{1}{q},\frac{1}{q_1})}\,\|x\|_{p,q} &\leq \|x\|_{\theta,q;K} \\ &\leq C\,\theta^{-\max(\frac{1}{q},\frac{1}{q_0})}\,(1-\theta)^{-\max(\frac{1}{q},\frac{1}{q_1})}\,\|x\|_{p,q}\,. \end{aligned}$$

Lemma 3.8. *The norm of the following inclusion*

$$L_p(\mathcal{M}) \subset \big(L_{p_0,1}(\mathcal{M}), L_{p_1}(\mathcal{M})\big)_{\theta,p;J}$$

is majorized by $C\,(1-\theta)^{1-1/p}$.

Proof. This is an immediate consequence of Lemma 3.7 by duality. □

Our last result in this section concerns the real interpolation of the positive cones $L_p^+(\mathcal{M};\ell_\infty)$ of the spaces $L_p(\mathcal{M};\ell_\infty)$. Together with Lemma 3.5, it constitutes the main technical part of the proof of Theorem 3.1.

Lemma 3.9. *We have*

$$\big(L_{p_0}^+(\mathcal{M};\ell_\infty), L_{p_1}^+(\mathcal{M};\ell_\infty)\big)_{\theta,p;J} \subset L_p^+(\mathcal{M};\ell_\infty)$$

and the inclusion norm is $\le C\theta^{-1+1/p}(1-\theta)^{-1+1/p_1}$.

Proof. Let $x \in \big(L_{p_0}^+(\mathcal{M};\ell_\infty), L_{p_1}^+(\mathcal{M};\ell_\infty)\big)_{\theta,p;J}$ of norm < 1. Choose $u(t)$ in the space $L_{p_0}^+(\mathcal{M};\ell_\infty) \cap L_{p_1}^+(\mathcal{M};\ell_\infty)$ such that

$$x = \int_0^\infty u(t)\,\frac{dt}{t} \quad \text{and} \quad \int_0^\infty \big[t^{-\theta}\, J_t(u(t))\big]^p\,\frac{dt}{t} < 1.$$

Here we have set

$$J_t(y) = \max\big(\|y\|_{L_{p_0}^+(\mathcal{M};\ell_\infty)},\, t\,\|y\|_{L_{p_1}^+(\mathcal{M};\ell_\infty)}\big).$$

In order to prove $x \in L_p^+(\mathcal{M};\ell_\infty)$, we use duality. Let $y = (y_n)_n \in L_{p'}^+(\mathcal{M};\ell_1)$ be of norm ≤ 1. Set $a = \sum_n y_n$. Then $\|a\|_{p'} \le 1$. Let K_t denote the K-functional relative to $(L_{p_0'}(\mathcal{M}),\, L_{p_1'}(\mathcal{M}))$, i.e., $K_t(\cdot)$ is the norm of the space $L_{p_0'}(\mathcal{M}) + tL_{p_1'}(\mathcal{M})$. Since $a \ge 0$, for every $t > 0$ there is a spectral projection $e(t)$ of a such that

$$\|e(t)a\|_{p_0'} + t^{-1}\|e(t)^\perp a\|_{p_1'} \le 2K_{t^{-1}}(a).$$

Then

$$\begin{aligned}\langle x, y\rangle &= \sum_n \tau(x_n y_n) = \int_0^\infty \sum_n \tau[u_n(t)y_n]\,\frac{dt}{t}\\ &= \int_0^\infty \sum_n \tau\big[u_n(t)[e(t)y_ne(t) + e(t)^\perp y_n e(t)^\perp + e(t)y_ne(t)^\perp + e(t)^\perp y_n e(t)]\big]\,\frac{dt}{t}.\end{aligned}$$

Since y_n is positive, we have

$$e(t)y_ne(t)^\perp + e(t)^\perp y_n e(t) \le e(t)y_ne(t) + e(t)^\perp y_n e(t)^\perp\ .$$

Hence $u_n(t) \ge 0$ implies

$$\tau\big[u_n(t)[e(t)y_ne(t)^\perp + e(t)^\perp y_ne(t)]\big] \le \tau\big[u_n(t)[e(t)y_ne(t) + e(t)^\perp y_ne(t)^\perp]\big].$$

Therefore

$$\begin{aligned}\langle x, y\rangle &\le 2\int_0^\infty \sum_n \tau\big[u_n(t)[e(t)y_ne(t) + e(t)^\perp y_ne(t)^\perp]\big]\,\frac{dt}{t}\\ &= 2\int_0^\infty \big[\langle u(t), w(t)\rangle + \langle u(t), v(t)\rangle\big]\,\frac{dt}{t},\end{aligned}$$

where

$$w(t) = \big(e(t)y_ne(t)\big)_{n\ge 0} \quad \text{and} \quad v(t) = \big(e(t)^\perp y_n e(t)^\perp\big)_{n\ge 0}\ .$$

Note that

$$\|w(t)\|_{L_{p_0'}(\mathcal{M};\ell_1)} = \big\|\sum_n e(t)y_ne(t)\big\|_{p_0'} = \|e(t)a\|_{p_0'}\ .$$

Similarly,

$$\|v(t)\|_{L_{p_1'}(\mathcal{M};\ell_1)} = \|e(t)^\perp a\|_{p_1'}\ .$$

It then follows that

$$\begin{aligned}\langle x,y\rangle &\le 2\int_0^\infty \big[\|u(t)\|_{L_{p_0}(\mathcal{M};\ell_\infty)}\,\|w(t)\|_{L_{p_0'}(\mathcal{M};\ell_1)}\\ &\qquad\qquad + \|u(t)\|_{L_{p_1}(\mathcal{M};\ell_\infty)}\,\|v(t)\|_{L_{p_1'}(\mathcal{M};\ell_1)}\big]\,\frac{dt}{t}\\ &\le 4\int_0^\infty J_t(u(t))\,K_{t^{-1}}(a)\,\frac{dt}{t}\\ &\le 4\Big(\int_0^\infty \big[t^{-\theta}\,J_t(u(t))\big]^p\,\frac{dt}{t}\Big)^{\frac1p}\,\Big(\int_0^\infty \big[t^{\theta}\,K_{t^{-1}}(a)\big]^{p'}\,\frac{dt}{t}\Big)^{\frac{1}{p'}}\\ &\le 4\,\|a\|_{(L_{p_0'}(\mathcal{M}),\,L_{p_1'}(\mathcal{M}))_{\theta,p';K}}\,.\end{aligned}$$

By Lemma 3.7, the norm of the following inclusion

$$L_{p'}(\mathcal{M})\subset (L_{p_0'}(\mathcal{M}),\,L_{p_1'}(\mathcal{M}))_{\theta,p';K}$$

is controlled by $C\theta^{-1/p'}(1-\theta)^{-1/p_1'}$. Hence we deduce

$$\langle x,y\rangle\le C\,\theta^{-\frac{1}{p'}}(1-\theta)^{-\frac{1}{p_1'}}\,.$$

Finally, taking the supremum over all positive y in the unit ball of $L_{p'}(\mathcal{M};\ell_1)$, we obtain the announced result. $\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Fix $x\in L_p^+(\mathcal{M})$ such that $\|x\|_p\le 1$. Let $p_0<q<p$. Let η and φ be determined by $1/q=(1-\eta)/p_0+\eta/p_1$ and $(1-\varphi)\eta+\varphi=\theta$. Applying Remark 3.6 and Lemma 3.8 with q in place of p_0, we deduce that $x\in \big(L_{q,1}^+(\mathcal{M}),L_{p_1}^+(\mathcal{M})\big)_{\varphi,p;J}$, and x admits a decomposition

$$x=\int_0^\infty u(t)\,\frac{dt}{t},$$

such that

$$\int_0^\infty \big[t^{-\varphi}\max\big(\|u(t)\|_{q,1}\,,\ t\|u(t)\|_{p_1}\big)\big]^p\,\frac{dt}{t}\le C^p\,(1-\varphi)^{p-1}\,.$$

Set $v(t)=u(C_0^{\eta-1}\,C_1^{1-\eta}\,t)$. Then we again have

$$x=\int_0^\infty v(t)\,\frac{dt}{t}\,.$$

Therefore, the subadditivity of S implies

$$S(x)\le\int_0^\infty S(v(t))\,\frac{dt}{t}\overset{\text{def}}{=}y\,. \tag{3.3}$$

Applying Lemma 3.5 with q instead of p and by the type (p_1,p_1) of S, we deduce

$$\begin{aligned}&\max\big(\|S(v(t))\|_{L_q^+(\mathcal{M};\ell_\infty)}\,,\ t\|S(v(t))\|_{L_{p_1}^+(\mathcal{M};\ell_\infty)}\big)\\ &\quad\le C\,(1-\frac{p_0}{q})^{-1-\frac1q}\max\big(C_0^{1-\eta}\,C_1^{\eta}\,\|v(t)\|_{q,1},\ t\,C_1\|v(t)\|_{p_1}\big)\,.\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^\infty \big[t^{-\varphi} \max\big(\|S(v(t))\|_{L_q^+(\mathcal{M};\ell_\infty)}\,,\ t\|S(v(t))\|_{L_{p_1}^+(\mathcal{M};\ell_\infty)}\big)\big]^p\,\frac{dt}{t}\\
&\le \Big[C\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}\Big]^p \int_0^\infty \big[t^{-\varphi}\max\big(C_0^{1-\eta}\,C_1^{\eta}\,\|v(t)\|_{q,1},\ t\,C_1\|v(t)\|_{p_1}\big)\big]^p\,\frac{dt}{t}\\
&= \Big[C\,C_0^{1-\theta}\,C_1^{\theta}\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}\Big]^p \int_0^\infty \big[t^{-\varphi}\max\big(\|u(t)\|_{q,1},\ t\,\|u(t)\|_{p_1}\big)\big]^p\,\frac{dt}{t}\\
&\le \Big[C\,C_0^{1-\theta}\,C_1^{\theta}\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}(1-\varphi)^{1-\frac{1}{p}}\Big]^p.
\end{aligned}$$

It thus follows that

$$y \in \big(L_q^+(\mathcal{M};\ell_\infty), L_{p_1}^+(\mathcal{M};\ell_\infty)\big)_{\varphi,p;J}$$

and

$$\|y\|_{\varphi,p;J} \le C\,C_0^{1-\theta}\,C_1^{\theta}\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}(1-\varphi)^{1-\frac{1}{p}}\,.$$

Therefore, by Lemma 3.9 (applied with q and φ in place of p_0 and θ, respectively), we deduce that $y \in L_p^+(\mathcal{M};\ell_\infty)$ and

$$\begin{aligned}
\|y\|_{L_p(\mathcal{M};\ell_\infty)} &\le C\,C_0^{1-\theta}\,C_1^{\theta}\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}\varphi^{-1+\frac{1}{p}}(1-\varphi)^{-\frac{1}{p}+\frac{1}{p_1}}\\
&\le C\,C_0^{1-\theta}\,C_1^{\theta}\,(1-\frac{p_0}{q})^{-1-\frac{1}{q}}\,(\frac{1}{q}-\frac{1}{p})^{-1+\frac{1}{p}}\,.
\end{aligned}$$

Choosing q such that

$$\frac{1}{p_0}-\frac{1}{q}=\frac{1}{2}\,(\frac{1}{p_0}-\frac{1}{p})\ ,$$

we get

$$\|y\|_{L_p(\mathcal{M};\ell_\infty)} \le C\,C_0^{1-\theta}\,C_1^{\theta}\,(\frac{1}{p_0}-\frac{1}{p})^{-2}\,.$$

This last inequality, together with (3.3), implies the desired inequality (3.1). Thus we have completed the proof of Theorem 3.1. □

4. Maximal ergodic inequalities

The following is our main maximal ergodic inequality in noncommutative L_p-spaces. Restricted to the positive cone $L_p^+(\mathcal{M})$, it becomes Theorem 0.1, i). Recall that for a map T with (0.I)–(0.III), T also denotes its extensions to $L_p(\mathcal{M})$ given by Lemma 1.1.

Theorem 4.1. *Let T be a linear map with* (0.I)–(0.III). *Let*

$$M_n \equiv M_n(T) = \frac{1}{n+1}\sum_{k=0}^{n} T^k.$$

Then for every $1 < p \le \infty$ we have

$$\big\|\sup_n{}^+ M_n(x)\big\|_p \le C_p\,\|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}). \tag{4.1}$$

Moreover, $C_p \le C\,p^2(p-1)^{-2}$, and this is the optimal order of C_p as $p \to 1$.

Proof. Decomposing an operator into a linear combination of four positive ones, we can assume $x \in L_p^+(\mathcal{M})$. Now consider each M_n as a map defined on $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M})$. Then M_n is positive and additive (and so subadditive too). Let $M = (M_n)_{n\geq 0}$. Yeadon's inequality says that M is of weak type (1,1). On the other hand, M is trivially of type (∞, ∞) with constant 1. Therefore, by Theorem 3.1, M is of type (p, p) for every $1 < p < \infty$ with constant C_p verifying

$$C_p \leq C\left(1 - \frac{1}{p}\right)^{-2}.$$

Thus we have proved (4.1). The optimality of this estimate follows from the optimal order of the best constant in the noncommutative Doob maximal inequality obtained in [JX3] and the following useful lemma due to Neveu, for which the validity in the noncommutative setting was observed by Dang-Ngoc [Da].

Lemma 4.2. *Let $(\mathcal{M}_n)_{n\geq 0}$ be a decreasing sequence of von Neumann subalgebras of $\mathcal{M}$. Assume that for every n there is a normal faithful conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ such that $\tau \circ \mathcal{E}_n = \tau$. Let (α_n) be an increasing sequence in $[0, 1)$ with $\alpha_0 = 0$. Then the map*

$$T = \sum_{n\geq 0} (\alpha_{n+1} - \alpha_n)\mathcal{E}_n$$

satisfies all conditions (0.I)–(0.IV). *Moreover, given any $\varepsilon > 0$ one can choose (α_n) and an increasing sequence (m_n) of positive integers such that*

$$\sum_{n\geq 0} \left\| M_{m_n}(T) - \mathcal{E}_n \right\| < \varepsilon,$$

where the norm is relative to $L_p(\mathcal{M})$ for any $1 \leq p \leq \infty$.

Note that if additionally $\lim_n \alpha_n = 1$, T preserves the trace τ since the $\mathcal{E}_n$ preserve τ. With the help of this lemma, one sees that the noncommutative maximal inequality (4.1) implies the noncommutative Doob maximal inequality proved in [Ju] and $\delta_p \leq C_p$, where δ_p is the best constant in the latter inequality. On the other hand, it was shown in [JX3] that the optimal order of δ_p is $(p-1)^{-2}$ as $p \to 1$. It then follows that the estimate for C_p in (4.1) is optimal. Theorem 4.1 is thus proved. □

Remark. The optimal order of the constant C_p in (4.1) implies that the estimate given in (3.1) is the best possible as $p \to p_0$ (with $p_0 = 1$). Recall that in the commutative case the best C_p in (4.1) is of order $(p-1)^{-1}$ as $p \to 1$. This explains partly the extra (noncommutative) effort in getting (4.1).

We will see in section 6 that Theorem 4.1 implies that the ergodic averages $(M_n(x))_n$ converge bilaterally almost uniformly for any $x \in L_p(\mathcal{M})$. However, for $p > 2$ the bilateral almost uniform convergence can be improved to the almost uniform convergence. This improvement will be a consequence of the following corollary of Theorem 4.1. For the formulation of this result we need further notation from [Mu, DJ1]. Let $2 \leq p \leq \infty$ and I be an index set. We define the space $L_p(\mathcal{M}; \ell_\infty^c(I))$ as the family of all $(x_i)_{i\in I} \subset L_p(\mathcal{M})$ for which there are an $a \in L_p(\mathcal{M})$ and $(y_i)_{i\in I} \subset L_\infty(\mathcal{M})$ such that

$$x_i = y_i a \quad \text{and} \quad \sup_{i\in I} \|y_i\|_\infty < \infty.$$

$\|(x_i)\|_{L_p(\mathcal{M};\ell_\infty^c(I))}$ is then defined to be the infimum $\{\sup_{i\in I}\|y_i\|_\infty \|a\|_p\}$ over all factorizations of (x_i) as above. It is easy to check that $\|\ \|_{L_p(\mathcal{M};\ell_\infty^c(I))}$ is a norm, which makes $L_p(\mathcal{M};\ell_\infty^c(I))$ a Banach space. Note that $(x_i)\in L_p(\mathcal{M};\ell_\infty^c(I))$ iff $(x_i^*x_i)\in L_{\frac{p}{2}}(\mathcal{M};\ell_\infty(I))$. If $I=\mathbb{N}$, $L_p(\mathcal{M};\ell_\infty^c(I))$ is simply denoted by $L_p(\mathcal{M};\ell_\infty^c)$.

Corollary 4.3. *Let T be as in Theorem* 4.1 *and $2<p\le\infty$. Then*

$$\big\|\big(M_n(x)\big)_{n\ge 0}\big\|_{L_p(\mathcal{M};\ell_\infty^c)} \le \sqrt{C_{p/2}}\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M}).$$

Proof. Let $x\in L_p(\mathcal{M})$. By decomposing x into its real and imaginary parts, we can assume x is selfadjoint. Since T is positive, so is M_n for every n. Thus by the classical Kadison inequality [Ka], we have

$$(M_n(x))^2\le M_n(x^2).$$

Thus applying Theorem 4.1 to $x^2\in L_{p/2}(\mathcal{M})$ we get $b\in L_{p/2}^+(\mathcal{M})$ such that

$$\|b\|_{p/2}\le C_{p/2}\,\|x^2\|_{p/2}\quad\text{and}\quad M_n(x^2)\le b,\quad \forall\, n\ge 0.$$

Hence $(M_n(x))^2\le b$. It then follows that for each n there is a contraction $y_n\in\mathcal{M}$ such that $M_n(x)=y_nb^{1/2}$. This gives the desired factorization of $\big(M_n(x)\big)_{n\ge0}$ as an element in $L_p(\mathcal{M};\ell_\infty^c)$ and thus proves the corollary. □

The following maximal inequality for multiple ergodic averages is an easy consequence of Theorem 4.1.

Corollary 4.4. *Let $T_1,\dots,T_d$ be d maps satisfying* (0.I)–(0.III). *Set*

$$M_{n_1,\dots,n_d}=\Big[\prod_{j=1}^d\frac{1}{n_j+1}\Big]\sum_{k_d=0}^{n_d}\cdots\sum_{k_1=0}^{n_1}T_d^{k_d}\cdots T_1^{k_1}\,.$$

Then for any $1<p<\infty$,

$$\big\|\sup_{n_1,\dots,n_d}{}^{+}\ M_{n_1,\dots,n_d}(x)\big\|_p\le C_p^d\,\|x\|_p\ ,\quad\forall\, x\in L_p(\mathcal{M})$$

and for $2<p<\infty$,

$$\big\|\big(M_{n_1,\dots,n_d}(x)\big)_{n_1,\dots,n_d}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{N}^d))}\le C_{p/2}^{\frac{d}{2}}\,\|x\|_p\ ,\quad\forall\, x\in L_p(\mathcal{M}).$$

Proof. The first part is obtained from Theorem 4.1 by iteration. The second is proved in the same way as Corollary 4.3. □

By a standard discretization argument, Theorem 4.1 and the previous corollaries imply the following maximal ergodic inequalities for semigroups.

Theorem 4.5. i) *Let $(T_t)_{t\ge0}$ be a semigroup satisfying the conditions* (0.I)–(0.III). *Let*

$$M_t=\frac1t\int_0^t T_s\,ds,\quad t>0.$$

Then for $1<p<\infty$,

$$\big\|\sup_t{}^{+}M_t(x)\big\|_p\le C_p\,\|x\|_p\ ,\quad\forall\, x\in L_p(\mathcal{M})$$

and for $2<p<\infty$,

$$\big\|\big(M_t(x)\big)_{t>0}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}_+))}\le\sqrt{C_{p/2}}\,\|x\|_p\ ,\quad\forall\, x\in L_p(\mathcal{M}).$$

ii) *Let $(T_t^{(1)})_{t\geq 0}, ..., (T_t^{(d)})_{t\geq 0}$ be d such semigroups. Let*

$$M_{t_1,\dots,t_d} = \frac{1}{t_1\,\cdots\,t_d}\int_0^{t_d} T_{s_d}^{(d)}\,ds_d\,\cdots\int_0^{t_1} T_{s_1}^{(1)}\,ds_1.$$

Then for any $1<p<\infty$,

$$\big\|\sup_{t_1>0,\dots,t_d>0}{}^{+}\,M_{t_1,\,\dots\,,t_d}(x)\big\|_p \leq C_p^d\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M})$$

and for $2<p<\infty$,

$$\big\|\big(M_{t_1,\dots,t_d}(x)\big)_{t_1,\dots,t_d}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}^d))} \leq C_{p/2}^{\frac{d}{2}}\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M}).$$

Proof. We show only the first inequality in i). Recall that the semigroup $(T_t)_{t\geq 0}$ is strongly continuous on $L_p(\mathcal{M})$; i.e., for any $x\in L_p(\mathcal{M})$ the function $t\mapsto T_t(x)$ is continuous from $[0,\infty)$ to $L_p(\mathcal{M})$, and so is the function $t\mapsto M_t(x)$. Thus to prove the first inequality in i) it suffices to consider $M_t(x)$ for t in a dense subset of $(0,\infty)$, for instance, the subset $\{n2^{-m}\ :\ m,n\in\mathbb{N}\}$. Using once more the strong continuity of $(T_t)_{t\geq 0}$, we can replace the integral defining $M_t(x)$ by a Riemann sum. Thus we have approximately

$$\begin{aligned} M_{n2^{-m}}(x) &= \frac{1}{n2^{-m}}\sum_{k=0}^{n-1}\int_{k2^{-m}}^{(k+1)2^{-m}} T_s(x)\,ds \\ &\approx \frac{1}{n}\sum_{k=0}^{n-1} T_{k2^{-m}}(x) = M_{n-1}(T_{2^{-m}})(x). \end{aligned}$$

Thus by Theorem 4.1 applied to $T=T_{2^{-m}}$, we obtain

$$\big\|\sup_n{}^{+} M_{n2^{-m}}(x)\big\|_p \leq C_p\,\|x\|_p\ .$$

Since the subsets $\{n2^{-m}\ :\ n\in\mathbb{N}\}$ are increasing in m, by Proposition 2.1, we get

$$\big\|\sup_{m,n}{}^{+}\,M_{n2^{-m}}(x)\big\|_p \leq C_p\,\|x\|_p\ .$$

This is the desired inequality. □

It is easy to show that the ergodic averages in Theorem 4.5 can be replaced by many other averages. Let us consider, for instance, the Poisson semigroup subordinate to (T_t):

$$P_t = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}T_{t^2/4u}\,du. \tag{4.2}$$

Recall that if A denotes the infinitesimal generator of (T_t), then that of (P_t) is $-(-A)^{1/2}$. More generally, given any $0<\alpha<1$, we can consider a semigroup (P_t) subordinate to (T_t) via the following formula:

$$P_t = \int_0^\infty \varphi(s)T_{t^\beta s}\,ds, \tag{4.3}$$

where $\beta=1/\alpha$ and φ is the function on $\mathbb{R}_+$ defined by

$$\varphi(s)=\int_0^\infty \exp\big[st\cos\theta - t^\alpha\cos(\alpha\theta)\big]\times\sin\big[st\sin\theta - t^\alpha\cos(\alpha\theta)+\theta\big]dt,$$

θ being any number in $[\pi/2,\ \pi]$. When $\alpha=1/2$, (4.3) reduces to (4.2). Note that the infinitesimal generator of (P_t) in (4.3) is $-(-A)^\alpha$ (see [Yo, IX]).

Corollary 4.6. *Let* (T_t) *be a semigroup verifying* (0.I)–(0.III)*. Let* $0<\alpha<1$ *and* (P_t) *be the semigroup subordinate to* (T_t) *as in* (4.3)*. Then for* $1<p\le\infty$*,*

$$\big\|\sup_t{}^+ P_t(x)\big\|_p \le C_{\alpha,p}\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M}) \tag{4.4}$$

and for $2<p<\infty$*,*

$$\big\|\big(P_t(x)\big)_{t>0}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}_+))} \le C'_{\alpha,p}\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M}).$$

Proof. Let us first rewrite (4.3) as

$$P_t = t^{-\beta}\int_0^\infty \varphi(t^{-\beta}s)T_s\,ds = t^{-\beta}\int_0^\infty \varphi(t^{-\beta}s)\,\frac{d}{ds}(sM_s)\,ds.$$

Thus by integration by parts,

$$P_t = t^{-\beta}\int_0^\infty \varphi'(t^{-\beta}s)\,t^{-\beta}s\,M_s\,ds = \int_0^\infty s\varphi'(s)\ M_{t^\beta s}\,ds.$$

Therefore, by Theorem 4.5, i), for any $x\in L_p(\mathcal{M})$,

$$\big\|\sup_t{}^+ P_t(x)\big\| \le C_p\int_0^\infty s|\varphi'(s)|\,ds\,\|x\|_p.$$

It is easy to see that the integral on the right is finite by virtue of the definition of φ. Thus we have proved (4.4). In the same way, we get the second inequality. □

Remark 4.7. Let (T_t) be a semigroup as in Theorem 4.5, i). Using Lemma 1.2 and the preceding discretization argument, one can easily obtain the following weak type (1,1) inequality: for any $x\in L_1^+(\mathcal{M})$ and $\lambda>0$ there is a projection $e\in\mathcal{M}$ such that

$$\sup_{t>0}\big\|eM_t(x)e\big\|_\infty \le \lambda \quad\text{and}\quad \tau(e^\perp)\le\frac{\|x\|_1}{\lambda}\ .$$

Moreover, M_t in the inequality above can be replaced by P_t in (4.3).

5. Maximal inequalities for symmetric contractions

The main result of this section is the following, which is a reformulation of Theorem 0.1, ii) for general elements in $L_p(\mathcal{M})$.

Theorem 5.1. *Let* T *be a linear map on* $\mathcal{M}$ *satisfying* (0.I)–(0.IV)*. Then for any* $1<p<\infty$ *we have*

$$\big\|\sup_n{}^+ T^n(x)\big\|_p \le C'_p\,\|x\|_p\ ,\quad \forall\, x\in L_p(\mathcal{M}), \tag{5.1}$$

where C'_p *is a constant depending only on* p*.*

This is the noncommutative analogue of a classical inequality due to Stein (cf. [St1]; see also [St2, Chapter III]). Stein's approach is via complex interpolation. The main ingredient is the maximal ergodic inequality (4.1), which allows us to deduce similar maximal inequalities for fractional averages. We refer to [Sta] (and the references therein) for more general maximal inequalities based on Rota's dilation theorem. (We are grateful to the referee for bringing [Sta] to our attention.) Let us point out that Rota's theorem is not sufficiently understood in the noncommutative setting and hence Starr's method is not yet available. It is Stein's original approach that suits well to the noncommutative setting. Thus we will follow the same pattern as in [St2, Chapter III]. Throughout the remainder of this section T will be fixed as in Theorem 5.1.

We begin by introducing the fractional averages on the powers of T. Given a complex number α and a nonnegative integer n, set

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

and

$$S_n^\alpha \equiv S_n^\alpha(T) = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}\, T^k\,, \quad M_n^\alpha \equiv M_n^\alpha(T) = (n+1)^{-\alpha}\, S_n^\alpha\ .$$

The M_n^α are the so-called fractional averages of the T^k. Note that $M_n^0 = T^n$ and M_n^1 is the usual ergodic average M_n already considered before. Also if α is a negative integer $-m$, then

$$S_n^{-m} = \Delta_n^m\big((T^k)_{k\ge 0}\big),$$

where Δ_n denotes the first difference map on sequences; i.e.,

$$\Delta_n(a) = a_n - a_{n-1}$$

for every sequence $a = (a_n)$. Then $\Delta_n^m = \Delta_n(\Delta_n^{m-1})$ is defined by induction and is the difference map of order m. Here and in the sequel we adopt the convention that for any sequence $(a_n)_{n\ge 0}$ we put $a_n = 0$ for $n < 0$. Since we will only consider actions of Δ_n^m on the sequence $(T^k)_{k\ge 0}$, we will simply put

$$\Delta_n^m = \Delta_n^m\big((T^k)_{k\ge 0}\big).$$

Thus $M_n^{-m} = (n+1)^m \Delta_n^m$.

We will need a generalization of Theorem 5.1.

Theorem 5.2. *Let T be as in Theorem* 5.1 *with the additional assumption that T is positive as an operator on $L_2(\mathcal{M})$ (i.e., $\langle x,\, Tx\rangle = \tau(x^*Tx) \ge 0$ for all $x \in L_2(\mathcal{M})$). Then for all $\alpha \in \mathbb{C}$ and $p \in (1,\infty)$ we have*

$$\big\|\sup_n{}^+ M_n^\alpha(x)\big\|_p \le C_{\alpha,p}\,\|x\|_p, \quad \forall\, x \in L_p(\mathcal{M})\,, \tag{5.2}$$

where $C_{\alpha,p}$ is a constant depending only on α and p.

It is easy to see that Theorem 5.2 implies Theorem 5.1. Indeed, applying Theorem 5.2 to T^2 with $\alpha = 0$, we get

$$\big\|\sup_n{}^+ T^{2n}(x)\big\|_p \le C'_p\,\|x\|_p \quad \text{and} \quad \big\|\sup_n{}^+ T^{2n+1}(x)\big\|_p \le C'_p\,\|Tx\|_p \le C'_p\,\|x\|_p$$

for every $x \in L_p(\mathcal{M})$. Thus (5.1) follows.

As already said before, our proof of Theorem 5.2 will follow the pattern set up by Stein. The main steps are as follows. First, using Theorem 4.1, we show that (5.2) holds for all complex α whose real part is greater than 1. Then with the help of the discrete Littlewood-Paley function, we deduce (5.2) for $p = 2$ and for all nonpositive integers α. It is this L_2 result which demands the symmetry of T. For interpolation we need to modify slightly this L_2 result into another one, i.e., to prove (5.2) for $p = 2$ again and for all complex α whose real part is of the form $-m + 1/2$ with $m \in \mathbb{N}$. Finally, complex interpolation permits us to conclude the proof.

We will use the following elementary properties of the A_n^α: for all $\alpha \in \mathbb{C}$ and $\beta > -1$,

$$A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}\,, \quad A_n^\alpha - A_n^{\alpha-1} = A_{n-1}^\alpha\,, \quad A_n^\beta \le c_\beta\,(n+1)^\beta\,, \tag{5.3}$$

where c_β is a positive constant depending only on β (noting that $A_n^\beta > 0$ when $\beta > -1$). The reader is referred to [Z, Vol. I, Chapter III.1] for these formulas. The following estimates on A_n^α are also well known.

Lemma 5.3. *Let $\alpha = \beta + i\gamma \in \mathbb{C}$.*

i) *If $\beta = m + r$ with $m \in \mathbb{Z}$ and $0 < r < 1$, then $|A_n^\alpha| \le \exp(c_r\gamma^2)\,|A_n^\beta|$.*
ii) *If $\beta > -1$, then $|A_n^\alpha| \le \exp(c_\beta\gamma^2)\,A_n^\beta$.*

Proof. We have

$$\mathrm{Log}\,\frac{A_n^\alpha}{A_n^\beta} = \sum_{k=1}^n \mathrm{Log}(1 + i\,\frac{\gamma}{\beta+k}).$$

Writing $\dfrac{A_n^\alpha}{A_n^\beta} = e^{u+iv}$ with $u, v \in \mathbb{R}$, we see that

$$u = \sum_{k=1}^n \mathrm{Re}\big(\mathrm{Log}(1 + i\,\frac{\gamma}{\beta+k})\big) \le \frac{\gamma^2}{2}\sum_{k=1}^n \frac{1}{(\beta+k)^2} \le c_r\,\gamma^2.$$

The second part is proved similarly. □

The following is an easy consequence of Theorem 4.1.

Lemma 5.4. *Let $\alpha = \beta + i\,\gamma$ with $\beta > 1$. Then for all $1 < p < \infty$,*

$$\big\|\sup_n{}^+ M_n^\alpha(x)\big\|_p \le C_{p,\beta}\,\exp(c_\beta\gamma^2)\|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

Proof. Without loss of generality, we assume $x \ge 0$. Let $(y_n) \subset L_{p'}^+(\mathcal{M})$. Using Lemma 5.3 and (5.3), we have

$$\begin{aligned}
|\tau(M_n^\alpha(x)y_n)| &\le (n+1)^{-\beta}\sum_{k=0}^n |A_{n-k}^{\alpha-1}|\,\tau[T^k(x)y_n]\\
&\le C_\beta\,\exp(c_\beta\gamma^2)\,(n+1)^{-\beta}\sum_{k=0}^n (n-k+1)^{\beta-1}\,\tau[T^k(x)y_n]\\
&\le C_\beta\,\exp(c_\beta\gamma^2)(n+1)^{-1}\sum_{k=0}^n \tau[T^k(x)y_n] = C_\beta\,\exp(c_\beta\gamma^2)\tau[M_n(x)y_n].
\end{aligned}$$

Therefore,

$$\sum_{n\ge0} |\tau(M_n^\alpha(x)y_n)| \le C_\beta\,\exp(c_\beta\,\gamma^2)\sum_{n\ge0}\tau[M_n(x)y_n].$$

Taking the supremum over all $(y_n) \subset L_{p'}^+(\mathcal{M})$ such that $\|\sum_n y_n\|_{p'} \le 1$ and using Proposition 2.1, iii) and Theorem 4.1, we deduce the assertion. □

Our next step is to prove a similar maximal inequality in $L_2(\mathcal{M})$ for M_n^α with $\alpha = -m + 1/2$ and $m \in \mathbb{N}$. To this end we will need the following inequality on the discrete Littlewood-Paley square function. Let

$$B_k^m = k(k-1)\cdots(k-m+1) \text{ for } m \le k \quad\text{and}\quad B_k^m = 0 \text{ for } m > k.$$

Lemma 5.5. *Let $m \in \mathbb{N}$. Then for every selfadjoint operator $x \in L_2(\mathcal{M})$,*

$$\tau\Big[\sum_{k\ge m} k\big(B_{k-1}^{m-1}\,\Delta_k^m(x)\big)^2\Big] \le C_m\,\tau(x^2).$$

Proof. Note that if x is selfadjoint, so is $\Delta_k^m(x)$ (recalling that T is positive). Moreover, Δ_k^m, considered as an operator on $L_2(\mathcal{M})$, is also selfadjoint by virtue of (0.IV). Fix a unit selfadjoint $x \in L_2(\mathcal{M})$. We have

$$\begin{aligned}\tau\big[\sum_{k\geq m} k\big(B_{k-1}^{m-1}\,\Delta_k^m(x)\big)^2\big] &= \sum_{k\geq m} k\,(B_{k-1}^{m-1})^2\,\|\Delta_k^m(x)\|_2^2 \\ &= \sum_{k\geq m} k\,(B_{k-1}^{m-1})^2\,\langle x,\,(\Delta_k^m)^2(x)\rangle,\end{aligned}$$

where $\langle\,,\,\rangle$ stands for the scalar product on $L_2(\mathcal{M})$. Observe the following easily checked formula:

$$\Delta_k^m = (T-1)^m\,T^{k-m}\ , \quad \forall\, k\geq m.$$

Let $T = \displaystyle\int_0^1 \lambda\,de_\lambda$ be the spectral resolution of T on $L_2(\mathcal{M})$ (recalling that T is a selfadjoint positive contraction on $L_2(\mathcal{M})$). Then

$$\langle x,\,(\Delta_k^m)^2(x)\rangle = \int_0^1 (1-\lambda)^{2m}\,\lambda^{2k-2m}\,d\langle x,\,e_\lambda x\rangle\ .$$

Since $d\langle x,\,e_\lambda x\rangle$ is a probability measure on $[0,1]$, it remains to estimate

$$\begin{aligned}&\sum_{k\geq m} k\,(B_{k-1}^{m-1})^2\,(1-\lambda)^{2m}\,\lambda^{2k-2m} \leq C_m + C_m\,(1-\lambda)^{2m}\sum_{k\geq 2m-1} B_k^{2m-1}\,\lambda^{2k-2m}\\ &\quad\leq C_m + C_m\,\lambda^{2m-2}(1-\lambda)^{2m}\sum_{k\geq 2m-1} B_k^{2m-1}\,(\lambda^2)^{k-2m+1}\\ &\quad\leq C_m + C_m\,\lambda^{2m-2}(1-\lambda)^{2m}\,(1-\lambda^2)^{-2m} \leq C_m.\end{aligned}$$

Therefore the lemma is proved. □

Lemma 5.6. *Let $m\in\mathbb{N}$. Then*

$$\big\|\sup_n{}^+\big((n+1)^m\,\Delta_n^m(x)\big)\big\|_2 \leq C_m\,\|x\|_2, \quad \forall\, x\in L_2(\mathcal{M}).$$

Proof. It suffices to show this for a positive $x\in L_2(\mathcal{M})$. To this end let us first observe the following formula:

$$\sum_{k=m}^{n} B_k^{m+1}\,(a_k - a_{k-1}) = B_n^{m+1}\,a_n - (m+1)\sum_{k=m}^{n-1} B_k^m\,a_k.$$

Applying this to $a_k = \Delta_k^m(x)$, we deduce

$$B_n^{m+1}\Delta_n^m(x) = (m+1)\sum_{k=m}^{n-1} B_k^m\,\Delta_k^m(x) + \sum_{k=m}^{n} B_k^{m+1}\,\Delta_k^{m+1}(x). \tag{5.4}$$

Now let $(y_n) \subset L_2^+(\mathcal{M})$. Using the convexity of the operator-valued function $x \mapsto |x|^2$, we have (recalling that $\Delta_k^m(x)$ is selfadjoint)

$$\begin{aligned}\Big|\tau\big(\frac{1}{n+1}\sum_{k=m}^{n-1} B_k^m\,\Delta_k^m(x)y_n\big)\Big| &\leq \tau\big(\frac{1}{n+1}\sum_{k=m}^{n-1} B_k^m\,|\Delta_k^m(x)|\,y_n\big)\\ &\leq \tau\big[\big(\sum_{k=m}^{n-1}(k+1)^{-1}\big|B_k^m\,\Delta_k^m(x)\big|^2\big)^{\frac12}\,y_n\big]\\ &\leq \tau\big[\big(\sum_{k=m}^{\infty}(k+1)^{-1}\big|B_k^m\,\Delta_k^m(x)\big|^2\big)^{\frac12}\,y_n\big].\end{aligned}$$

Hence, by Lemma 5.5,

$$\begin{aligned}&\Big|\sum_n\tau\big(\frac{1}{n+1}\sum_{k=m}^{n-1} B_k^m\,\Delta_k^m(x)y_n\big)\Big| \leq \tau\big[\big(\sum_{k=m}^{\infty}(k+1)^{-1}\big|B_k^m\,\Delta_k^m(x)\big|^2\big)^{\frac12}\,\sum_n y_n\big]\\ &\leq \big\|\big(\sum_{k=m}^{\infty}(k+1)^{-1}\big|B_k^m\,\Delta_k^m(x)\big|^2\big)^{\frac12}\big\|_2\,\big\|\sum_n y_n\big\|_2\\ &\leq C_m\,\|x\|_2\,\|\sum_n y_n\|_2.\end{aligned}$$

Similarly,

$$\Big|\sum_n\tau\big(\frac{1}{n+1}\sum_{k=m}^{n} B_k^{m+1}\,\Delta_k^{m+1}(x)y_n\big)\Big| \leq C_m\,\|x\|_2\,\big\|\sum_n y_n\big\|_2\ .$$

Combining these inequalities with (5.4), we deduce

$$\Big|\sum_n\tau\big(\frac{1}{n+1}\,B_n^{m+1}\Delta_n^m(x)y_n\big)\Big| \leq C_m\,\|x\|_2\,\big\|\sum_n y_n\big\|_2\ ,$$

whence

$$\big\|\sup_n{}^+\big(\frac{1}{n+1}\,B_n^{m+1}\,\Delta_n^m(x)\big)\big\|_2 \leq C_m\,\|x\|_2\ .$$

This is clearly equivalent to the desired inequality. □

Lemma 5.7. *Let $x=(x_n)\in L_p(\mathcal{M};\ell_\infty)$ and $(z_{n,k})_{n,k}\subset\mathbb{C}$. Then*

$$\big\|\sup_n{}^+\sum_k z_{n,k}x_k\big\|_p \leq \sup_n\big(\sum_k|z_{n,k}|\big)\,\big\|\sup_k{}^+x_k\big\|_p\ .$$

Proof. This is easy. Indeed, given a factorization of x as $x_k = ay_kb$, we have

$$\sum_k z_{n,k}x_k = a\,\big(\sum_k z_{n,k}y_k\big)\,b\ .$$

Thus

$$\begin{aligned}\big\|\sup_n{}^+\sum_k z_{n,k}x_k\big\|_p &\leq \|a\|_{2p}\,\|b\|_{2p}\,\sup_n\big\|\sum_k z_{n,k}y_k\big\|_\infty\\ &\leq \|a\|_{2p}\,\|b\|_{2p}\,\sup_k\|y_k\|_\infty\,\sup_n\sum_k|z_{n,k}|\ .\end{aligned}$$

This implies the desired inequality. □

Lemma 5.8. *Let $\alpha = \beta + i\,\gamma$ be such that $\beta = -m + 1/2$ with $m \in \mathbb{N}$. Then*
$$\big\| \sup_n{}^+ M_n^\alpha(x) \big\|_2 \le C_m \exp(5\gamma^2)\, \|x\|_2, \quad \forall\, x \in L_2(\mathcal{M})\ .$$

Proof. Given $n \in \mathbb{N}$ choose $d_n \in \mathbb{N}$ such that $n/2 + 1 \le d_n < n/2 + 3$. Then by successive use of the Abel summation, we obtain

$$\begin{aligned}
S_n^\alpha &= \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k = \sum_{k=0}^{d_n-1} A_{n-k}^{\alpha-1} T^k + \sum_{k=d_n}^{n} A_{n-k}^{\alpha-1} T^k \\
&= \sum_{k=0}^{d_n-1} A_{n-k}^{\alpha-1} T^k + A_{n-d_n}^{\alpha}\, T^{d_n-1} + \sum_{k=d_n}^{n} A_{n-k}^{\alpha}\, \Delta_k \\
&= \sum_{k=0}^{d_n-1} A_{n-k}^{\alpha-1} T^k + A_{n-d_n}^{\alpha}\, T^{d_n-1} + A_{n-d_n}^{\alpha+1}\, \Delta_{d_n-1} + \sum_{k=d_n}^{n} A_{n-k}^{\alpha+1}\, \Delta_k^2 \\
&\ \vdots \\
&= \sum_{k=0}^{d_n-1} A_{n-k}^{\alpha-1} T^k + \sum_{j=1}^{m} A_{n-d_n}^{\alpha+j-1}\, \Delta_{d_n-1}^{j-1} + \sum_{k=d_n}^{n} A_{n-k}^{\alpha+m-1}\, \Delta_k^m\ .
\end{aligned}$$

Therefore, by the triangle inequality and Lemma 5.7, we get
$$\big\| \sup_n{}^+ M_n^\alpha(x) \big\|_2 \le I \times II,$$
where
$$I = \sum_{j=0}^{m} \big\| \sup_n{}^+ (n+1)^j \Delta_n^j(x) \big\|_2$$
and
$$II = \sup_n \frac{1}{(n+1)^\beta} \max\left\{ \sum_{k=0}^{d_n-1} |A_{n-k}^{\alpha-1}|,\ \max_{1\le j\le m} \frac{|A_{n-d_n}^{\alpha+j-1}|}{d_n^{j-1}}\,,\ \sum_{k=d_n}^{n} \frac{|A_{n-k}^{\alpha+m-1}|}{(k+1)^m} \right\}.$$
By Lemma 5.6, $I \le C_m\, \|x\|_2$. On the other hand, by Lemma 5.3, i) ($c_{1/2} \le 5$ with $r = 1/2$ there) we may estimate II by
$$C_m \exp(5\gamma^2) \sup_n \frac{1}{(n+1)^\beta} \max\left\{ \sum_{k=0}^{d_n-1} |A_{n-k}^{\beta-1}|,\ \max_{1\le j\le m} \frac{|A_{n-d_n}^{\beta+j-1}|}{d_n^{j-1}},\ \sum_{k=d_n}^{n} \frac{|A_{n-k}^{\beta+m-1}|}{(k+1)^m} \right\}.$$
Now using the following easily verified estimate
$$|A_k^\delta| \le C_\delta\, (k+1)^\delta$$
for real δ (see also [Z, Vol. I, Chapter III.1]) and by the choice of d_n, we get
$$\sum_{k=d_n}^{n} \frac{|A_{n-k}^{\beta+m-1}|}{(k+1)^m} \le \frac{C_m}{(n+1)^m} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le C_m\, (n+1)^\beta\ .$$
This gives the desired estimate on the last term in the brackets above. The other two terms can be estimated similarly. Therefore, $II \le C_m$. Putting together all preceding inequalities yields the lemma. □

Now we are in a position to prove Theorem 5.2.

Proof of Theorem 5.2. Write $\alpha = \beta + i\gamma$ with $\beta, \gamma \in \mathbb{R}$. Choose $\theta \in (0,1)$, $q \in (1, \infty)$, $m \in \mathbb{Z}$ and $b > \max(\beta, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q} \quad \text{and} \quad \beta = (1-\theta)a + \theta\, b \quad \text{with } a = m + \frac{1}{2}.$$

Let $x \in L_p(\mathcal{M})$ and $y = (y_n)$ be a finite sequence in $L_{p'}(\mathcal{M})$ with $\|x\|_p < 1$ and $\|y\|_{L_{p'}(\mathcal{M};\ell_1)} < 1$. Define

$$f(z) = u\,\big|x\big|^{\frac{p(1-z)}{2} + \frac{pz}{q}}, \quad z \in \mathbb{C},$$

where $x = u|x|$ is the polar decomposition of x. On the other hand, by Proposition 2.5, there is a function $g = (g_n)_n$ continuous on the strip $\{z \in \mathbb{C} \,:\, 0 \le \mathrm{Re}(z) \le 1\}$ and analytic in the interior such that $g(\theta) = y$ and

$$\sup_{t \in \mathbb{R}} \max\Big\{\|g(i\,t)\|_{L_2(\mathcal{M};\ell_1)}\ ,\ \|g(1+i\,t)\|_{L_{q'}(\mathcal{M};\ell_1)}\Big\} < 1.$$

Now define

$$F(z) = \exp\big(\delta(z^2 - \theta^2)\big) \sum_n \tau\big[M_n^{(1-z)a + zb + i\,\gamma}(f(z))\, g_n(z)\big],$$

where $\delta > 0$ is a constant to be specified. F is a function analytic in the open strip $\{z \in \mathbb{C} \,:\, 0 < \mathrm{Re}(z) < 1\}$. Applying Lemma 5.4 when $m \ge 1$ and Lemma 5.8 when $m \le 0$, we have

$$\begin{aligned} |F(it)| &\le \exp\big(\delta(-t^2 - \theta^2)\big)\, \big\|\big(M_n^{a + i(-ta + tb + \gamma)}(f(it))\big)_n\big\|_{L_2(\mathcal{M};\ell_\infty)}\, \big\|g(i\,t)\big\|_{L_2(\mathcal{M};\ell_1)} \\ &\le C_\alpha \exp\big((-\delta + c_{\beta,b,\gamma})t^2 - \delta\theta^2\big)\, \|f(it)\|_2 \le C_\alpha \exp\big((-\delta + c_{\beta,b,\gamma})t^2 - \delta\theta^2\big)\ . \end{aligned}$$

Similarly, by Lemma 5.4,

$$|F(1+it)| \le C_{\alpha,q} \exp\big((-\delta + c'_{\beta,b,\gamma})t^2 + \delta(1-\theta^2)\big)\ .$$

Choosing δ bigger than $\max(c_{\beta,b,\gamma},\ c'_{\beta,b,\gamma})$, we get

$$\sup_{t \in \mathbb{R}} \max\Big\{|F(i\,t)|\ ,\ |F(1+i\,t)|\Big\} \le C_{\alpha,p}.$$

Therefore, by the maximum principle, $|F(\theta)| \le C_{p,\beta,b,\gamma}$. Namely,

$$\Big|\sum_n \tau\big[M_n^\alpha(x)\, y_n\big]\Big| \le C_{\alpha,p}\ .$$

This yields (5.2), and thus the theorem is proved. □

We end this section with some direct consequences of Theorem 5.1.

Corollary 5.9. *Let T be as in Theorem* 5.1 *and* $2 < p < \infty$. *Then*

$$\big\|\big(T^n(x)\big)_n\big\|_{L_p(\mathcal{M};\ell_\infty^c)} \le \sqrt{C'_p}\, \|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

Proof. Based on Theorem 5.1, this corollary is proved in the same way as Corollary 4.3. □

By iteration, we get the following.

Corollary 5.10. *Let $T_1, ..., T_d$ satisfy* (0.I)–(0.IV). *Then for $1 < p < \infty$,*

$$\big\| \sup_{n_1,...,n_d}{}^{+} \; T_d^{n_d} \cdots T_1^{n_1}(x)\big\|_p \le (C'_p)^d \, \|x\|_p \,, \quad \forall\, x \in L_p(\mathcal{M})$$

and for $2 < p < \infty$,

$$\big\|\big(T_d^{n_d} \cdots T_1^{n_1}(x)\big)_{n_1,...,n_d}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{N}^d))} \le (C'_p)^{\frac{d}{2}} \, \|x\|_p \,, \quad \forall\, x \in L_p(\mathcal{M}).$$

By discretization, the previous maximal inequalities on contractions imply similar ones on semigroups.

Corollary 5.11. i) *Let $(T_t)_{t\ge 0}$ be a semigroup verifying the conditions* (0.I)–(0.IV). *Then for $1 < p < \infty$,*

$$\big\| \sup_{t\ge 0}{}^{+} \, T_t(x)\big\|_p \le C'_p \, \|x\|_p \,, \quad \forall\, x \in L_p(\mathcal{M})$$

and for $2 < p < \infty$,

$$\big\|\big(T_t(x)\big)_t\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}_+))} \le \sqrt{C'_p}\, \|x\|_p \,, \quad \forall\, x \in L_p(\mathcal{M}).$$

ii) *A similar statement holds for d such semigroups.*

6. Individual ergodic theorems

In this section we apply the maximal inequalities proved in the two previous sections to study the pointwise ergodic convergence. To this end we first need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the almost uniform convergence introduced by Lance [L] (see also [Ja1]).

Definition 6.1. Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite normal faithful trace τ. Let $x_n, x \in L_0(\mathcal{M})$.

i) (x_n) is said to converge bilaterally almost uniformly (b.a.u. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0.$$

ii) (x_n) is said to converge almost uniformly (a.u. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0.$$

In the commutative case, both convergences in the definition above are equivalent to the usual almost everywhere convergence by virtue of Egorov's theorem. However they are different in the noncommutative setting. Similarly, we introduce these notions of convergence for functions with values in $L_0(\mathcal{M})$ and for nets in $L_0(\mathcal{M})$.

In order to deduce the individual ergodic theorems from the corresponding maximal ergodic theorems, it is convenient to use the subspace $L_p(\mathcal{M}; c_0)$ of $L_p(\mathcal{M}; \ell_\infty)$. $L_p(\mathcal{M}; c_0)$ is defined as the space of all sequences $(x_n) \subset L_p(\mathcal{M})$ such that there are $a, b \in L_{2p}(\mathcal{M})$ and $(y_n) \subset \mathcal{M}$ verifying

$$x_n = a y_n b \quad \text{and} \quad \lim_{n\to\infty} \|y_n\|_\infty = 0.$$

It is easy to check that $L_p(\mathcal{M}; c_0)$ is a closed subspace of $L_p(\mathcal{M}; \ell_\infty)$ and

$$\big\|\sup_n{}^{+} x_n\big\|_p = \inf\big\{\|a\|_{2p} \sup_{n\ge 0} \|y_n\|_\infty \, \|b\|_{2p}\big\},$$

where the infimum runs over all factorizations of (x_n) as above. We define similarly the subspace $L_p(\mathcal{M}; c_0^c)$ of $L_p(\mathcal{M}; \ell_\infty^c)$. Note that $L_p(\mathcal{M}; c_0)$ (resp. $L_p(\mathcal{M}; c_0^c)$) is just the closure in $L_p(\mathcal{M}; \ell_\infty)$ (resp. $L_p(\mathcal{M}; \ell_\infty^c)$) of finite sequences in $L_p(\mathcal{M})$ for $1 \le p < \infty$. The definition of these spaces is readily extended to any index set instead of $\mathbb{N}$.

The following simple lemma from [DJ1] will be useful for our study of individual ergodic theorems. We include a proof for completeness.

Lemma 6.2. i) *If $(x_n) \in L_p(\mathcal{M}; c_0)$ with $1 \le p < \infty$, then x_n converges b.a.u. to* 0.

ii) *If $2 \le p < \infty$ and $(x_n) \in L_p(\mathcal{M}; c_0^c)$, then x_n converges a.u. to* 0.

Proof. i) Let $(x_n) \in L_p(\mathcal{M}; c_0)$. Then there are $a, b \in L_{2p}(\mathcal{M})$ and $y_n \in \mathcal{M}$ such that

$$x_n = a y_n b \quad \text{and} \quad \|a\|_{2p} < 1,\ \|b\|_{2p} < 1,\ \lim_{n\to\infty} \|y_n\|_\infty = 0.$$

We can clearly assume $a, b \ge 0$. Let e' be a spectral projection of a such that

$$\tau(e'^{\perp}) < \frac{\varepsilon}{2} \quad \text{and} \quad \|e'a\|_\infty \le \big(\frac{2}{\varepsilon}\big)^{\frac{1}{2p}}\ .$$

Similarly, we find a spectral projection e'' of b. Set $e = e' \wedge e''$. Then

$$\tau(e^{\perp}) \le \tau(e'^{\perp}) + \tau(e''^{\perp}) < \varepsilon$$

and

$$\|e x_n e\|_\infty \le \|ea\|_\infty \, \|y_n\|_\infty \, \|be\|_\infty \le \|y_n\|_\infty \, \|e'a\|_\infty \, \|be''\|_\infty \le \big(\frac{2}{\varepsilon}\big)^{\frac{1}{p}} \, \|y_n\|_\infty\ .$$

Thus $\lim_n \|e x_n e\|_\infty = 0$, and so $x_n \to 0$ b.a.u.

ii) The proof of this part is similar and left to the reader. □

Now let T be a linear map satisfying the conditions (0.I)–(0.III). Let $(M_n)_n$ denote the ergodic averages of T. Recall that $\mathcal{F}_p$ denotes the fixed point subspace of T in $L_p(\mathcal{M})$ and F the projection from $L_p(\mathcal{M})$ onto $\mathcal{F}_p$ (see section 1).

Theorem 6.3. *Let T be a map on $\mathcal{M}$ satisfying* (0.I)–(0.III). *Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. Then $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0)$ Moreover, if $p > 2$, $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0^c)$.*

Proof. Let $x \in L_p(\mathcal{M})$. Since $(I - T)\big(L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})\big)$ is dense in $\mathcal{F}_p^{\perp}$, there are $x_k \in (I - T)\big(L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})\big)$ such that

$$\lim_{k\to\infty} \|x - F(x) - x_k\|_p = 0.$$

By Theorem 4.1,

$$\big\|\big(M_n(x) - F(x) - M_n(x_k)\big)_n\big\|_{L_p(\mathcal{M};\ell_\infty)} \le C_p \, \|x - F(x) - x_k\|_p\, .$$

Thus

$$\lim_{k\to\infty} \big(M_n(x_k)\big)_n = \big(M_n(x) - F(x)\big)_n \quad \text{in} \quad L_p(\mathcal{M}; \ell_\infty).$$

Since $L_p(\mathcal{M}; c_0)$ is closed in $L_p(\mathcal{M}; \ell_\infty)$, it suffices to show $\big(M_n(x_k)\big)_n \in L_p(\mathcal{M}; c_0)$ for every k. To this end consider an arbitrary $z \in (I - T)\big(L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})\big)$. Let $y \in L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})$ be such that $z = y - T(y)$. Then

$$M_n(z) = \frac{1}{n+1}\big[y - T^{n+1}(y)\big].$$

Since $z \in L_q(\mathcal{M})$ for any $1 < q < \infty$ we deduce from Theorem 4.1 that $\big(M_n(z)\big)_n$ belongs to $L_q(\mathcal{M}; \ell_\infty)$. Choose a $q \in (1, p)$. Then by Proposition 2.5, for any $m < n$,

$$\begin{aligned}\big\| {\sup_{m \le j \le n}}^+ M_j(z) \big\|_p &\le \sup_{m \le j \le n} \big\| M_j(z) \big\|_\infty^{1-\frac{q}{p}} \big\| {\sup_{m \le j \le n}}^+ M_j(z) \big\|_q^{\frac{q}{p}} \\ &\le \Big[\frac{2\|y\|_\infty}{m+1}\Big]^{1-\frac{q}{p}} \big\| {\sup_{j \ge 1}}^+ M_j(z) \big\|_q^{\frac{q}{p}} .\end{aligned}$$

Let $\vec{z}^{(k)}$ denote the finite sequence $(M_0(x), ..., M_k(x), 0, ...)$. The inequality above shows that the sequence $(\vec{z}^{(k)})_{k \ge 0}$ converges to $(M_n(z))_n$ in $L_p(\mathcal{M}; \ell_\infty)$ as $k \to \infty$. Thus $\big(M_n(z)\big)_n \in L_p(\mathcal{M}; c_0)$, as wanted.

The second part can be similarly proved. Now we use Corollary 4.3 and the analogue for the spaces $L_p(\mathcal{M}; c_0^c)$ of Proposition 2.5. □

The following is an extension of Yeadon's noncommutative individual ergodic theorem [Ye] to all $L_p(\mathcal{M})$ with $1 < p < \infty$.

Corollary 6.4. *Let T be a map satisfying the conditions* (0.I)–(0.III). *Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. Then $(M_n(x))_n$ converges to $F(x)$ b.a.u. for $1 < p \le 2$ and a.u. for $2 < p < \infty$.*

Proof. This is an immediate consequence of Lemma 6.2 and Theorem 6.3. □

Remark. Corollary 6.4 can also be proved by using Yeadon's theorem. This is however not the case for the multiple individual ergodic theorem below. We refer to [Ska] for multiple ergodic theorems for commuting operators.

Remark 6.5. Again using Yeadon's theorem, one can prove that the convergence in Theorem 6.4 is a.u. for $p = 2$.

Proof. Fix $x \in L_2(\mathcal{M})$ and $\varepsilon > 0$. By decomposing x into its real and imaginary parts, we can assume that x is selfadjoint. Let (ε_n) and (δ_n) be two sequences of small positive numbers. Then for each $m \ge 1$ there are $w_m \in L_2(\mathcal{M}) \cap L_\infty(\mathcal{M})$ and $z_m \in L_2(\mathcal{M})$ such that

$$x = F(x) + y_m + z_m \quad \text{with} \quad y_m = w_m - T(w_m) \quad \text{and} \quad \|z_m\|_2 < \delta_m.$$

Since x is selfadjoint, w_m, y_m and z_m can be chosen to be selfadjoint too. We have

$$M_n(x) - F(x) = M_n(y_m) + M_n(z_m)$$

and

$$\|M_n(y_m)\|_\infty \le \frac{2}{n+1} \|w_m\|_\infty.$$

Now we apply Yeadon's weak type (1,1) inequality (Lemma 1.2) to z_m^2. Thus there is a projection e_m such that

$$\sup_n \big\| e_m M_n(z_m^2) e_m \big\|_\infty \le \varepsilon_m^2 \quad \text{and} \quad \tau(e_m^\perp) < \varepsilon_m^{-2}\, \tau(z_m^2) \le \varepsilon_m^{-2}\, \delta_m^2 .$$

By Kadison's Cauchy-Schwarz inequality [Ka], we get

$$\big\| M_n(z_m) e_m \big\|_\infty^2 \le \big\| e_m M_n(z_m^2) e_m \big\|_\infty .$$

Thus

$$\sup_n \big\| M_n(z_m) e_m \big\|_\infty \le \varepsilon_m .$$

Let $e = \bigwedge_m e_m$. Then

$$\tau(e^{\perp}) \leq \sum_{m\geq 1} \varepsilon_m^{-2}\, \delta_m^2 < \varepsilon$$

provided ε_m and δ_m are appropriately chosen. On the other hand, by the preceding inequalities, we deduce

$$\begin{aligned}\big\|e[M_n(x) - F(x)]e\big\|_\infty &\leq \frac{2}{n+1}\|w_m\|_\infty + \|eM_n(z_m)e\|_\infty \\ &= \frac{2}{n+1}\|w_m\|_\infty + \|e[e_m M_n(z_m)e_m]e\|_\infty \leq \frac{2}{n+1}\|w_m\|_\infty + \varepsilon_m.\end{aligned}$$

It then follows that

$$\limsup_{n\to\infty} \big\|e[M_n(x) - F(x)]e\big\|_\infty \leq \varepsilon_m.$$

Since $\lim_m \varepsilon_m = 0$, we get that $\lim_{n\to 0} \|e[M_n(x) - F(x)]e\|_\infty = 0$. Hence, $M_n(x)$ converges to $F(x)$ b.a.u. □

We pass to the multiple version of Theorem 6.3 and Corollary 6.4. Let $T_1, \dots, T_d$ be d maps satisfying (0.I)–(0.III). As before, set

$$M_{n_1,\dots,n_d} = \Big[\prod_{j=1}^{d} \frac{1}{n_j+1}\Big] \sum_{k_d=0}^{n_d} \cdots \sum_{k_1=0}^{n_1} T_d^{k_d} \cdots T_1^{k_1}\,.$$

Let F_j be the projection onto the fixed point subspace of T_j.

Theorem 6.6. *Let $T_1, \dots, T_d$ be d maps satisfying* (0.I)–(0.III). *Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. Then*

$$\big(M_{n_1,\dots,n_d}(x) - F_d \cdots F_1(x)\big)_{n_1,\dots,n_d\geq 1} \in L_p(\mathcal{M}; c_0(\mathbb{N}^d))$$

and if $p > 2$,

$$\big(M_{n_1,\dots,n_d}(x) - F_d \cdots F_1(x)\big)_{n_1,\dots,n_d\geq 1} \in L_p(\mathcal{M}; c_0^c(\mathbb{N}^d)).$$

Consequently, $M_{n_1,\dots,n_d}(x)$ converges b.a.u. to $F_d \cdots F_1(x)$ as $n_1, \dots, n_d$ tend to ∞. Moreover, the convergence is a.u. in the case of $p > 2$.

Proof. This proof is similar to that of Theorem 6.3, modulo an iteration argument. We consider only the typical case $d = 2$. Note that

$$M_{n_1,n_2} = M_{n_2}(T_2)M_{n_1}(T_1).$$

Fix $x \in L_p(\mathcal{M})$ and decompose x as $x = F_1(x) + y_k + u_k$ with

$$y_k \in (I - T_1)\big(L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})\big), \quad u_k \in L_p(\mathcal{M}), \quad \|u_k\|_p \leq \frac{1}{k}.$$

Similarly, we decompose $F_1(x)$ with respect to T_2: $F_1(x) = F_2(F_1(x)) + z_k + v_k$ with

$$z_k \in (I - T_2)\big((L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})\big), \quad v_k \in L_p(\mathcal{M}), \quad \|v_k\|_p \leq \frac{1}{k}.$$

Applying successively $M_{n_1}(T_1)$ to x and $M_{n_2}(T_2)$ to $F_1(x)$, we get

$$M_{n_1,n_2}(x) - F_2F_1(x) = M_{n_1,n_2}(y_k) + M_{n_1,n_2}(u_k) + M_{n_2}(T_2)(z_k) + M_{n_2}(T_2)(v_k)\,.$$

By Corollary 4.4,

$$\big\|\sup_{n_1,n_2}{}^{+}\, M_{n_1,n_2}(u_k)\big\|_p \leq C_p^2\, \|u_k\|_p \leq \frac{C_p^2}{k} \ \to 0 \text{ as } k \to \infty.$$

Similarly,

$$\lim_{k\to\infty}\big\|\sup_{n_2}{}^{+}M_{n_2}(T_2)(v_k)\big\|_p=0.$$

Therefore, as in the proof of Theorem 6.3, we need only to show

$$\big(M_{n_1,n_2}(y_k)\big)_{n_1,n_2\geq 1}\in L_p(\mathcal{M};c_0(\mathbb{N}^2))\quad\text{and}\quad \big(M_{n_2}(z_k)\big)_{n_2\geq 1}\in L_p(\mathcal{M};c_0).$$

Theorem 6.3 implies the latter. The former is proved by the arguments in the proof of Theorem 6.3. Thus the first part of the theorem is proved. The second part for $p>2$ is left to the reader.

Then applying Lemma 6.2 to multiple sequences, we deduce the announced pointwise multiple ergodic convergence. □

We have the following stronger convergence result for symmetric T.

Theorem 6.7. *Let T be a map satisfying* (0.I)–(0.IV)*. Assume further that T is positive as an operator on $L_2(\mathcal{M})$. Let $1<p<\infty$ and $x\in L_p(\mathcal{M})$. Then $(T^n(x)-F(x))_n$ belongs to $L_p(\mathcal{M};c_0)$ and to $L_p(\mathcal{M};c_0^c)$ if additionally $p>2$. Consequently, $T^n(x)$ converges to $F(x)$ b.a.u. for $1<p\leq 2$ and a.u. for $2<p<\infty$.*

Proof. Let us first treat the case $p=2$. Write the spectral decomposition of T:

$$T=\int_0^1\lambda\,d\,e_\lambda\,.$$

Note that for any $x\in\overline{(I-T)(L_2(\mathcal{M}))}$,

$$\lim_{\lambda\to 1}e_\lambda(x)=x\quad\text{in}\quad L_2(\mathcal{M}).$$

Given $x\in L_2(\mathcal{M})$ choose $x_k\in(I-T)(L_2(\mathcal{M}))$ such that $\lim\|x-F(x)-x_k\|_2=0$. Then $\lim_{\lambda\to1}e_\lambda(x_k)=x_k$. Thus replacing x_k by $e_{\lambda_k}(x_k)$ with an appropriate $\lambda_k\in(0,1)$, we can assume that $x_k=e_{\lambda_k}(y_k)$ for some $y_k\in L_2(\mathcal{M})$. Then

$$T^n(x_k)=\int_0^{\lambda_k}\lambda^n\,d\,e_\lambda(y_k)\,,\quad\text{whence}\quad \|T^n(x_k)\|_2\leq\lambda_k^n\,\|y_k\|_2\,.$$

It then follows that $\big(T^n(x_k)\big)_n\in L_2(\mathcal{M};c_0)$ for every k, and so by Theorem 5.1, $\big(T^n(x)-F(x)\big)_n\in L_2(\mathcal{M};c_0)$.

To treat the general case we first claim that

$$\lim_n\|T^n(x)-F(x)\|_p=0,\quad\forall\,x\in L_p(\mathcal{M}).$$

Indeed, the preceding argument shows that this is true for $p=2$. Now let $2<p<\infty$ and $x\in L_1(\mathcal{M})\cap\mathcal{M}$. By interpolation,

$$\|T^n(x)-F(x)\|_p\leq\|T^n(x)-F(x)\|_\infty^{1-\frac{2}{p}}\,\|T^n(x)-F(x)\|_2^{\frac{2}{p}}\,,$$

whence $\lim_n\|T^n(x)-F(x)\|_p=0$. Then our claim in the case $p>2$ follows from the density of $L_1(\mathcal{M})\cap\mathcal{M}$ in $L_p(\mathcal{M})$. The case $p<2$ is proved similarly.

Now we can easily finish the proof of the theorem. Let $x\in L_p(\mathcal{M})$. Fix $k\in\mathbb{N}$. Then Theorem 5.1 and the claim above imply

$$\lim_{k\to\infty}\big\|\big(T^n\big(T^k(x)-F(x)\big)\big)_n\big\|_{L_p(\mathcal{M};\ell_\infty)}=0.$$

Note that $T^n\big(T^k(x) - F(x)\big) = T^{n+k}(x) - F(x)$, and so the sequence $\big(T^n\big(T^k(x) - F(x)\big)\big)_{n\geq 0}$ can be considered as the rest of $\big(T^n(x) - F(x)\big)_{n\geq 0}$ starting from the k-th coordinate. It follows that $\big(T^n(x) - F(x)\big)_{n\geq 0} \in L_p(\mathcal{M}; c_0)$.

In a similar way, using Corollary 5.9, we show that $\big(T^n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0^c)$ for any $x \in L_p(\mathcal{M})$ with $p > 2$. $\square$

Remark. If we remove the additional assumption that T is a positive operator on $L_2(\mathcal{M})$ in Theorem 6.7, then for any $x \in L_p(\mathcal{M})$ the two subsequences $(T^{2n}(x))_n$ and $(T^{2n+1}(x))_n$ still converge b.a.u.; however, their limits are not equal in general.

We end this section with the pointwise ergodic theorems for semigroups. Let $(T_t)_{t\geq 0}$ be a semigroup satisfying (0.I)–(0.III). We denote again by F the projection from $L_p(\mathcal{M})$ onto the fixed point subspace of $(T_t)_{t\geq 0}$.

Theorem 6.8. *Let $(T_t)_{t\geq 0}$ be a semigroup with (0.I)–(0.III). Let $(M_t)_{t>0}$ denote the ergodic averages of $(T_t)_{t\geq 0}$. Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$.*

i) *Then*

a) *$M_t(x)$ converges to $F(x)$ b.a.u. for $1 < p < 2$ and a.u. for $2 \leq p < \infty$ when $t \to \infty$.*

b) *$M_t(x)$ converges to x b.a.u. for $1 < p < 2$ and a.u. for $2 \leq p < \infty$ when $t \to 0$.*

ii) *Assume in addition that $(T_t)_{t\geq 0}$ satisfies (0.IV). Then*

a) *$T_t(x)$ converges to $F(x)$ b.a.u. for $1 < p \leq 2$ and a.u. for $2 < p < \infty$ when $t \to \infty$.*

b) *$T_t(x)$ converges to x b.a.u. for $1 < p \leq 2$ and a.u. for $2 < p < \infty$ when $t \to 0$.*

Proof. The two statements a) can be proved similarly as in the discrete case, using Theorem 4.5. The main step here is to obtain the semigroup analogue of Theorem 6.3 as $t \to \infty$, namely, to show that the family $\big(M_t(x) - F(x)\big)_{t\geq 1}$ belongs to $L_p(\mathcal{M}; c_0([1,\infty)))$ or $L_p(\mathcal{M}; c_0^c([1,\infty)))$. Note however that the a.u. convergence for $p = 2$ in the first statement a) is shown similarly as Remark 6.5, using now Remark 4.7. We leave this part of the proof to the reader and will show the two statements b). (The first of them is the noncommutative analogue of the classical Wiener local pointwise ergodic theorem.)

Let us first consider i), b). Let $x \in L_p(\mathcal{M})$. By Lemma 6.2, it suffices to prove that $\big(M_t(x) - x\big)_{0<t\leq 1}$ belongs to $L_p(\mathcal{M}; c_0((0,1]))$ (with respect to $t \to 0$). By the mean ergodic theorem, we have $M_t(x) \to x$ when $t \to 0$. Thus by a limit argument as in the proof of Theorem 6.3, we may assume $x = M_{t_0}(y)$ for some $0 < t_0 < 1$ and $y \in L_p(\mathcal{M})$. On the other hand, by the density of $L_1(\mathcal{M}) \cap \mathcal{M}$ in $L_p(\mathcal{M})$, we can further assume that $y \in L_1(\mathcal{M}) \cap \mathcal{M}$. Let $0 < s \leq t < t_0$. Then

$$T_s(x) - x = \frac{1}{t_0}\Big[\int_{t_0}^{t_0+s} T_u(y)du - \int_0^s T_u(y)du\Big].$$

It then follows that

$$\|T_s(x) - x\|_\infty \leq \frac{2s\|y\|_\infty}{t_0}$$

and so

$$\|M_t(x) - x\|_\infty \leq \frac{2t\|y\|_\infty}{t_0} \ \to 0 \text{ as } t \to 0.$$

Thus by the interpolation argument used in the proof of Theorem 6.3, we deduce that the family $\big(M_t(x)-x\big)_{0<t\leq 1}$ belongs to $L_p(\mathcal{M}; c_0((0,1]))$. This proves the first part of the statement i), b).

The second part for $p>2$ can be shown in the same way. The case $p=2$ is dealt with similarly to Remark 6.5 by virtue of Remark 4.7.

ii), b) is proved similarly by using Corollary 5.11 and Lemma 6.2. Indeed, for $x=M_{t_0}(y)$ as above, we have already proved

$$\lim_{t\to 0}\|T_t(x)-x\|_\infty = 0.$$

Therefore, $\big(M_t(x)-x\big)_{0<t\leq 1} \in L_p(\mathcal{M}; c_0((0,1]))$. Thus the proof of the theorem is finished. □

Remarks. i) Both statements in Part i) of Theorem 6.8 also hold for $p=1$ because of Remark 4.7. On the other hand, both Theorem 6.7 and Theorem 6.8 admit multiple versions, similar to Theorem 6.6.

ii) Using Corollary 4.6, one sees that the ergodic averages M_t in Theorem 6.8, i) can be replaced by P_t, where (P_t) is a semigroup subordinate to (T_t). This is also true for $p=1$.

7. The nontracial case

So far we have restricted our attention to the semifinite case only. In this section we will extend the previous results to arbitrary von Neumann algebras. Despite the obvious similarity between the statements in the semifinite and the nontracial cases, we want to point out that the situation for type III von Neumann algebras is more complicated. This is due to the fact that for a state φ the equality $\varphi(e\vee f)\leq \varphi(e)+\varphi(f)$ is no longer valid, and therefore many (Egorov type) arguments from the previous section do not apply in this general setting. Our tool for maximal ergodic inequalities in Haagerup noncommutative L_p-spaces is an important unpublished theorem due to Haagerup, which consists in reducing the general case to the tracial one. For clarity, we divide this section into several subsections.

7.1. **Haagerup noncommutative L_p-spaces.** The general noncommutative L_p-spaces used below will be those constructed by Haagerup [H2]. Our reference is [Te]. Throughout this section $\mathcal{M}$ will be a von Neumann algebra equipped with a distinguished normal faithful state φ, unless explicitly stated otherwise. $L_p(\mathcal{M})$ denotes the associated noncommutative L_p-space ($0<p\leq\infty$). Recall that $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ itself and $L_1(\mathcal{M})$ is the predual of $\mathcal{M}$. The duality between $\mathcal{M}$ and $L_1(\mathcal{M})$ is realized via the distinguished tracial functional tr on $L_1(\mathcal{M})$:

$$\langle x,y\rangle = \mathrm{tr}(xy),\quad y\in L_1(\mathcal{M}),\ x\in\mathcal{M}.$$

As a normal positive functional on $\mathcal{M}$, φ corresponds to a positive element in $L_1(\mathcal{M})$. In the sequel this element will always be denoted by D, called the density of φ in $L_1(\mathcal{M})$. Then φ can be recovered from D through the preceding duality:

$$\varphi(x)=\mathrm{tr}(xD)=\mathrm{tr}(Dx),\quad x\in\mathcal{M}.$$

We will often use the density of $D^{\frac{1-\theta}{p}}\mathcal{M}D^{\frac{\theta}{p}}$ in $L_p(\mathcal{M})$ for any $p\in(0,\infty)$ and $0\leq\theta\leq 1$. Moreover, $D^{\frac{1-\theta}{p}}\mathcal{M}_aD^{\frac{\theta}{p}}$ is also dense in $L_p(\mathcal{M})$, where $\mathcal{M}_a$ is the family of all elements in $\mathcal{M}$ analytic with respect to the modular group σ_t^φ of φ (see [JX2, Lemma 1.1]).

An important link between the spaces $L_p(\mathcal{M})$ is the following external product. Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $x \in L_p(\mathcal{M}),\ y \in L_q(\mathcal{M})$. Then $xy \in L_r(\mathcal{M})$ and

$$\|xy\|_r \leq \|x\|_p \, \|y\|_q \ .$$

Namely, the usual Hölder inequality extends to Haagerup L_p-spaces too. In particular, the dual space of $L_p(\mathcal{M})$ is $L_{p'}(\mathcal{M})$ for $1 \leq p < \infty$, and we have

$$\mathrm{tr}(xy) = \mathrm{tr}(yx), \quad x \in L_p(\mathcal{M}),\ y \in L_{p'}(\mathcal{M}).$$

The definition of all vector-valued L_p-spaces extends verbatim to the present setting. These include $L_p(\mathcal{M};\ell_\infty)$, $L_p(\mathcal{M};\ell_\infty^c)$, $L_p(\mathcal{M};c_0)$ and $L_p(\mathcal{M};c_0^c)$. For instance, $L_p(\mathcal{M};\ell_\infty)$ consists of all sequences $x = (x_n)$ in $L_p(\mathcal{M})$ which admit a factorization of the following type: there are $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $(y_n) \subset \mathcal{M}$ such that $x_n = a y_n b$ for all n. The norm of x is then defined as

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\big\{\|a\|_{2p} \sup_n \|y_n\|_\infty \, \|b\|_{2p}\big\},$$

where the infimum runs over all factorizations as above. We adopt the convention introduced in section 2 and denote again this norm by $\big\|\sup_n{}^+ x_n\big\|_p$. Note that $L_p(\mathcal{M};c_0)$ (resp. $L_p(\mathcal{M};c_0^c)$) is again a closed subspace of $L_p(\mathcal{M};\ell_\infty)$ (resp. $L_p(\mathcal{M};\ell_\infty^c)$). Similarly, given an index set I we define the analogues of these spaces for families indexed by I.

All properties in section 2 continue to hold in the present setting. However, for an inclusion $L_p(\mathcal{N}) \subset L_p(\mathcal{M})$ we will now require the existence of a normal state preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Under this hypothesis Remark 2.3 still holds. Also note that for the interpolation result in Proposition 2.5 we use Kosaki's interpolation theorem [Ko].

7.2. **An extension result.** Let T be a map on $\mathcal{M}$. We will assume that T satisfies conditions similar to (0.I)–(0.IV). More precisely, we will consider the following properties of T:

- (7.I) T is a contraction on $\mathcal{M}$;
- (7.II) T is completely positive;
- (7.III) $\varphi \circ T \leq \varphi$;
- (7.IV) $T \circ \sigma_t^\varphi = \sigma_t^\varphi \circ T$ for all $t \in \mathbb{R}$;
- (7.V) T is symmetric with respect to φ; i.e., $\varphi(T(y)^* x) = \varphi(y^* T(x))$ for all $x, y \in \mathcal{M}$.

In order to consider maximal ergodic inequalities in $L_p(\mathcal{M})$, we need first to extend a map T with the properties above to a contraction on $L_p(\mathcal{M})$ for all $1 \leq p < \infty$. The following is the nontracial analogue of Lemma 1.1.

Lemma 7.1. *Let T be a map on $\mathcal{M}$ satisfying* (7.I)–(7.III). *Define*

$$\begin{aligned} T_p \ : \ D^{\frac{1}{2p}}\, \mathcal{M}\, D^{\frac{1}{2p}} \quad &\to \quad D^{\frac{1}{2p}}\, \mathcal{M}\, D^{\frac{1}{2p}}, \\ D^{\frac{1}{2p}}\, x\, D^{\frac{1}{2p}} \quad &\mapsto \quad D^{\frac{1}{2p}}\, T(x)\, D^{\frac{1}{2p}} \ . \end{aligned}$$

Then T_p extends to a positive contraction on $L_p(\mathcal{M})$ for all $1 \leq p < \infty$. Moreover, T is normal. If additionally T verifies (7.V)*, then the extension of T_2 is selfadjoint on $L_2(\mathcal{M})$.*

In fact, the complete positivity assumption can be weakened to positivity. This result comes from [JX5]. Its proof is much more involved than that of Lemma

1.1. The main difficulty is to show the extension property of T_1. This extension is essentially a reformulation of Lemma 1.2 from [H1] into the present setting. We refer to [JX5] for more details. The following observation is easily checked:

Remark. Let T and T_p be as in Lemma 7.1 (T_p also denoting the extension). Let $S = T_1^*$. Then S satisfies (7.I)–(7.III) too. Moreover, $S_p^* = T_{p'}$ for all $1 \le p < \infty$, where S_p is the extension of S on $L_p(\mathcal{M})$, guaranteed by Lemma 7.1. This shows, in particular, that T is normal.

The extension in Lemma 7.1 is symmetric with respect to the injection of $\mathcal{M}$ into $L_p(\mathcal{M})$. We could also consider the left extension: $xD^{1/p} \mapsto T(x)D^{1/p}$ $(x \in \mathcal{M})$. More generally, for any $0 \le \theta \le 1$ we can define

$$\begin{aligned} T_{p,\theta} \ : \ D^{\frac{1-\theta}{p}} \mathcal{M} D^{\frac{\theta}{p}} \quad &\to \quad D^{\frac{1-\theta}{p}} \mathcal{M} D^{\frac{\theta}{p}}, \\ D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}} \quad &\mapsto \quad D^{\frac{1-\theta}{p}} T(x) D^{\frac{\theta}{p}} . \end{aligned}$$

Note that $T_{p,1/2}$ is exactly the T_p defined in Lemma 7.1. Assume in addition that T satisfies (7.IV). Using the equality

$$D^{\frac{1-\theta}{p}} \mathcal{M}_a D^{\frac{\theta}{p}} = \mathcal{M}_a D^{\frac{1}{p}} ,$$

one easily checks that

$$T_{p,\theta}\Big|_{\mathcal{M}_a D^{\frac{1}{p}}} = T_p\Big|_{\mathcal{M}_a D^{\frac{1}{p}}} .$$

Thus $T_{p,\theta}$ does not depend on θ (at least when restricted to analytic elements). Consequently, $T_{p,\theta}$ extends to a contraction on $L_p(\mathcal{M})$. Since we will use this observation later, we formulate it explicitly.

Remark 7.2. Let T satisfy (7.I)–(7.IV). Then $T_{p,\theta}$ does not depend on $\theta \in [0,1]$ and extends to a positive contraction on $L_p(\mathcal{M})$.

Convention. In the sequel, we will denote, by the same symbol T, all the maps T_p and $T_{p,\theta}$ as well as their extensions to the L_p-spaces in Lemma 7.1 and Remark 7.2, whenever no confusion can occur.

Let T be a map on $\mathcal{M}$ with (7.I)–(7.III). We will consider again the ergodic averages:

$$M_n \equiv M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k .$$

All discussions in section 1 concerning the mean ergodic theorem are still valid now. Thus T is ergodic on $L_p(\mathcal{M})$ for all $1 \le p \le \infty$ (relative to the w*-topology for $p = \infty$). We still have the decomposition

$$L_p(\mathcal{M}) = \mathcal{F}_p(T) \oplus \mathcal{F}_p(T)^{\perp},$$

with $\mathcal{F}_p(T) = \{x \in L_p(\mathcal{M}) : T(x) = x\}$. In the previous sections, we used several times the fact that $(I-T)(L_1(\mathcal{M}) \cap \mathcal{M})$ is dense in $\mathcal{F}_p(T)^{\perp}$. Now this fact should be changed to the following: $D^{\frac{1}{2p}} (I-T)(\mathcal{M}) D^{\frac{1}{2p}}$ is dense in $\mathcal{F}_p(T)^{\perp}$. If T further satisfies (7.IV), this dense subspace can be replaced by $D^{\frac{1-\theta}{p}} (I-T)(\mathcal{M}_a) D^{\frac{\theta}{p}}$ for any $\theta \in [0,1]$, which is equal to $(I-T)(\mathcal{M}_a) D^{\frac{1}{p}}$ too. The easy verification of these facts is left to the reader. As before in the tracial case, the projection from $L_p(\mathcal{M})$ onto $\mathcal{F}_p(T)$ will be denoted by F for any $1 \le p \le \infty$. Again F is normal as a map on $L_\infty(\mathcal{M})$.

The discussion above is readily transferred to semigroups. Let (T_t) be a semigroup of maps on $\mathcal{M}$ satisfying (7.I)–(7.III) (i.e., each T_t satisfying (7.I)–(7.III)). Then (T_t) extends to a semigroup of positive contractions on $L_p(\mathcal{M})$ for $1 \le p < \infty$. We will assume that (T_t) is w*-continuous and T_0 is the identity. Then (T_t) is strongly continuous on $L_p(\mathcal{M})$ for every $p < \infty$. Again the fixed point projection of (T_t) is denoted by F. Then the mean ergodic theorem asserts that $M_t(x)$ converges to $F(x)$ as $t \to \infty$ for all $x \in L_p(\mathcal{M})$ (relative to the w*-topology for $p = \infty$), where M_t denotes the ergodic averages of (T_t).

The following extends a well-known result in the commutative case to the present situation.

Remark 7.3. Let T be a map on $\mathcal{M}$ verifying (7.I)–(7.III). Assume in addition that $\varphi \circ T = \varphi$. Then $\mathcal{F}_\infty(T)$ is a von Neumann subalgebra of $\mathcal{M}$, and F is the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{F}_\infty(T)$ such that $\varphi \circ F = \varphi$.

Proof. First note that under the assumptions above, both T and F are unital and F preserves the state φ. Thus F is a normal unital completely positive projection from $\mathcal{M}$ onto $\mathcal{F}_\infty(T)$. Consequently, $\mathcal{F}_\infty(T)$ contains the unit of $\mathcal{M}$ and is closed under involution, and so $\mathcal{F}_\infty(T)$ is a w*-closed operator system. Therefore, it remains to show that $\mathcal{F}_\infty(T)$ is closed under the product of $\mathcal{M}$.

To that end we will use the following formula from [ChE] (formula (3.1) there),

$$(7.1) \qquad F(aF(x)) = F(ax) \quad \text{and} \quad F(F(x)a) = F(xa), \quad \forall\, a \in \mathcal{F}_\infty(T),\ x \in \mathcal{M}.$$

Let us consider the pre-adjoint of F, $F_* : \mathcal{M}_* \to \mathcal{M}_*$. We claim that

$$F_*(x\varphi) = F(x)\varphi, \quad \forall\, x \in \mathcal{M}.$$

Indeed, since $\varphi \circ F = \varphi$, given $y \in \mathcal{M}$, by (7.1) we have

$$\begin{aligned} F_*(x\varphi)(y) &= x\varphi(F(y)) = \varphi(F(y)x) = \varphi[F(F(y)x)] = \varphi[F(F(y)F(x))] \\ &= \varphi[F(yF(x))] = \varphi(yF(x)) = [F(x)\varphi](y). \end{aligned}$$

Now let $a, b \in \mathcal{F}_\infty(T)$. Then $F_*(ab\varphi) = F(ab)\varphi$. On the other hand, for any $x \in \mathcal{M}$,

$$\begin{aligned} F_*(ab\varphi)(x) &= \varphi(F(x)ab) = \varphi[F(F(x)ab)] = \varphi\big[F\big(F[F(x)a]b\big)\big] \\ &= \varphi[F(F(xa)b)] = \varphi[F(xab)] = \varphi(xab) = [ab\varphi](x). \end{aligned}$$

Hence, $F_*(ab\varphi) = ab\varphi$. It thus follows that $F(ab)\varphi = ab\varphi$. Then the faithfulness of φ implies that $F(ab) = ab$, and so $\mathcal{F}_\infty(T)$ is stable under multiplication, as desired. □

7.3. **Maximal ergodic inequalities.** The following is the extension of Theorems 4.1 and 5.1 to the nontracial case.

Theorem 7.4. i) *Let T satisfy* (7.I)–(7.IV). *Let (M_n) denote the ergodic averages of T. Then for any $1 < p < \infty$,*

$$\big\| \sup_n{}^+ M_n(x) \big\|_p \le C_p \,\|x\|_p\,, \quad x \in L_p(\mathcal{M}).$$

ii) *If T further satisfies* (7.V)*, then*

$$\big\| \sup_n{}^+ T^n(x) \big\|_p \le C'_p \,\|x\|_p\,, \quad x \in L_p(\mathcal{M}).$$

Here C_p and C'_p are respectively the constants in (4.1) *and* (5.1).

Remark. Compared with Theorems 4.1 and 5.1 in the tracial case, the assumption in Theorem 7.4 is a little bit stronger, namely, the positivity of T in those theorems is now reinforced to the complete positivity (7.II). It is likely that this is not really needed.

As in the tracial case, Theorem 7.4 immediately yields the following two corollaries.

Corollary 7.5. *Let T satisfy* (7.I)–(7.IV) *and $2 < p < \infty$. Then*

$$\big\|\big(M_n(x)\big)_{n\geq 0}\big\|_{L_p(\mathcal{M};\ell_\infty^c)} \leq \sqrt{C_{p/2}}\, \|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

If additionally T has (7.V), *then*

$$\big\|\big(T^n(x)\big)_{n\geq 0}\big\|_{L_p(\mathcal{M};\ell_\infty^c)} \leq \sqrt{C'_{p/2}}\, \|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

Corollary 7.6. *Let (T_t) be a w^*-continuous semigroup of maps on $\mathcal{M}$ satisfying* (7.I)–(7.IV). *Let*

$$M_t = \frac{1}{t}\int_0^t T_s\, ds, \quad t > 0.$$

Then for any $1 < p < \infty$,

$$\big\|\sup_t{}^+ M_t(x)\big\|_p \leq C_p\, \|x\|_p\ , \quad x \in L_p(\mathcal{M})$$

and for $p > 2$,

$$\big\|\big(M_t(x)\big)_{t>0}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}^+))} \leq \sqrt{C_{p/2}}\, \|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

If additionally each T_t satisfies (7.V), *then*

$$\big\|\sup_t{}^+ T_t(x)\big\|_p \leq C'_p\, \|x\|_p\ , \quad x \in L_p(\mathcal{M})$$

and for $p > 2$,

$$\big\|\big(T_t(x)\big)_{t>0}\big\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{R}^+))} \leq \sqrt{C'_{p/2}}\, \|x\|_p\ , \quad \forall\, x \in L_p(\mathcal{M}).$$

Although they are not stated here, all other inequalities in sections 4 and 5 continue to hold for Haagerup noncommutative L_p-spaces. We omit the details. The rest of this subsection is devoted to the proof of Theorem 7.4. It relies in a crucial way on Haagerup's reduction theorem [H3]. We will need the precise form of Haagerup's construction that we recall very briefly below.

Let G denote the discrete subgroup $\bigcup_{m\geq 1} 2^{-m}\mathbb{Z}$ of $\mathbb{R}$. We consider the crossed product $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} G$. Here the modular automorphism group σ^φ is also regarded as an automorphic representation of G on $\mathcal{M}$. As usual, $\mathcal{M}$ is viewed as a von Neumann subalgebra of $\mathcal{R}$. Let $\widehat{\varphi}$ denote the dual weight of φ. Since G is discrete, $\widehat{\varphi}$ is a normal faithful state on $\mathcal{R}$ and its restriction to $\mathcal{M}$ coincides with φ. Moreover, there is a normal faithful conditional expectation Φ from $\mathcal{R}$ onto $\mathcal{M}$ such that

$$\widehat{\varphi}\circ\Phi = \widehat{\varphi} \quad \text{and} \quad \sigma_t^{\widehat{\varphi}}\circ\Phi = \Phi\circ\sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}.$$

Then Haagerup's reduction theorem can be stated as follows.

Theorem 7.7 (Haagerup)**.** *With the notation above, there is an increasing sequence $(\mathcal{R}_m)_{m\geq 1}$ of von Neumann subalgebras of $\mathcal{R}$ satisfying the following properties:*

i) *each $\mathcal{R}_m$ is equipped with a normal faithful tracial state τ_m;*
ii) *$\bigcup_{m\geq 1}\mathcal{R}_m$ is w^*-dense in $\mathcal{R}$;*
iii) *there is a normal faithful conditional expectation Φ_m from $\mathcal{R}$ onto $\mathcal{R}_m$ such that*

$$\widehat{\varphi}\circ\Phi_m=\widehat{\varphi}\quad \textit{and}\quad \sigma_t^{\widehat{\varphi}}\circ\Phi_m=\Phi_m\circ\sigma_t^{\widehat{\varphi}},\quad t\in\mathbb{R}.$$

We refer to [H3] for the proof. [JX5] reproduces Haagerup's proof and presents several applications of Theorem 7.7.

In the situation above, $L_p(\mathcal{M})$ and $L_p(\mathcal{R}_m)$ can be regarded naturally and isometrically as subspaces of $L_p(\mathcal{R})$. Moreover, the conditional expectation Φ (resp. Φ_m) extends to a positive contractive projection from $L_p(\mathcal{R})$ onto $L_p(\mathcal{M})$ (resp. $L_p(\mathcal{R}_m)$) (see [JX2]; this is also a particular case of Lemma 7.1). On the other hand, $\bigcup_{m\geq 1}L_p(\mathcal{R}_m)$ is dense in $L_p(\mathcal{R})$ for $p<\infty$, and the sequence (Φ_m) is increasing. Thus $(\mathcal{R}_m)$ gives rise to a martingale structure on $\mathcal{R}$, and consequently, given $x\in L_p(\mathcal{R})$ with $1\leq p<\infty$, $\Phi_m(x)$ converges to x in $L_p(\mathcal{R})$ as $m\to\infty$.

Let us also observe that by Remark 2.3 applied to Haagerup spaces, $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{R}_m;\ell_\infty)$ are isometrically subspaces of $L_p(\mathcal{R};\ell_\infty)$.

For the proof of Theorem 7.4 we will further need the following result from [JX5].

Lemma 7.8. *Let T be as in Theorem* 7.4*.*

i) *Then T has an extension $\widehat{T}$ to $\mathcal{R}$ which satisfies* (7.I)–(7.IV) *relative to $(\mathcal{R},\widehat{\varphi})$. Moreover, if T verifies* (7.V)*, so does $\widehat{T}$ relative to $\widehat{\varphi}$.*
ii) *$\widehat{T}(\mathcal{R}_m)\subset\mathcal{R}_m$ and $\tau_m\circ\widehat{T}\leq\tau_m$ for all $m\geq 1$.*

Now we are ready to show Theorem 7.4.

Proof of Theorem 7.4. Fix $1<p<\infty$ and $x\in L_p(\mathcal{M})$. We consider x as an element in $L_p(\mathcal{R})$ and then apply the conditional expectation Φ_m to it: $x_m\stackrel{\rm def}{=}\Phi_m(x)\in L_p(\mathcal{R}_m)$. Note that $\widehat{T}\big|_{\mathcal{R}_m}$ satisfies the conditions (0.I)–(0.III) relative to τ_m. So we can apply Theorem 4.1 to $\widehat{T}$ on $\mathcal{R}_m$ and get

$$\big\|\sup_n{}^+ M_n(\widehat{T})(x_m)\big\|_p\leq C_p\left\|x\right\|_p,\quad \forall\, m\in\mathbb{N}.$$

By the martingale convergence theorem recalled previously,

$$\lim_{m\to\infty}x_m=x\quad\text{in } L_p(\mathcal{R}).$$

Consequently,

$$\lim_{m\to\infty}\widehat{T}^{\,k}(x_m)=x\quad\text{in } L_p(\mathcal{R}),\quad\forall\, k\geq 0.$$

On the other hand, it is clear that the norm of $L_p(\mathcal{R};\ell_\infty^n)$ is equivalent to that of $\ell_\infty^n\big(L_p(\mathcal{R})\big)$ for each fixed n. We then deduce that

$$\lim_{m\to\infty}\big\|\sup_{1\leq k\leq n}{}^+\, M_k(\widehat{T})(x_m)\big\|_p=\big\|\sup_{1\leq k\leq n}{}^+\,|M_k(\widehat{T})(x)|\big\|_p\,.$$

However, since $x\in L_p(\mathcal{M})$,

$$M_k(\widehat{T})(x)=M_k(T)(x).$$

Therefore, we deduce

$$\Big\| \sup_{1\le k\le n}{}^{+} M_k(T)(x)\Big\|_p \le C_p \|x\|_p\,, \quad \forall\, n\in\mathbb{N}.$$

Thus, by Proposition 2.1 and Remark 2.3, we have

$$\Big\| \sup_n{}^{+} M_n(T)(x)\Big\|_p \le C_p \|x\|_p\,.$$

This shows the first part of Theorem 7.4. The second part is proved similarly. □

7.4. **Individual ergodic theorems.** In this subsection we consider individual ergodic theorems in Haagerup's noncommutative L_p-spaces. As mentioned earlier, the situation is more complicated than that in the tracial case. One of the reasons is that the elements in $L_p(\mathcal{M})$ are no longer closed densely defined operators affiliated with $\mathcal{M}$ but affiliated with a larger von Neumann algebra, namely the crossed product $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$. We first need to introduce an appropriate analogue of the almost everywhere convergence for sequences in $L_p(\mathcal{M})$. There are several such generalizations. Here we adopt the almost sure convergence introduced by Jajte [Ja2] (following ideas from [DJ2]). In the L_∞-case, we continue to use Lance's almost uniform convergence.

Definition 7.9. i) Let $x_n, x\in\mathcal{M}$. x_n is said to converge almost uniformly (a.u. in short) to x if for every $\varepsilon>0$ there is a projection $e\in\mathcal{M}$ such that

$$\varphi(e^\perp)<\varepsilon \quad\text{and}\quad \lim_{n\to\infty}\|(x_n-x)e\|_\infty=0.$$

ii) Let $x_n, x\in L_p(\mathcal{M})$ with $p<\infty$. The sequence (x_n) is said to converge almost surely (a.s. in short) to x if for every $\varepsilon>0$ there is a projection $e\in\mathcal{M}$ and a family $(a_{n,k})\subset\mathcal{M}$ such that

$$\varphi(e^\perp)<\varepsilon \quad\text{and}\quad x_n-x=\sum_{k\ge1}a_{n,k}\,D^{\frac1p}\,,\quad \lim_{n\to\infty}\Big\|\sum_{k\ge1}(a_{n,k}\,e)\Big\|_\infty=0,$$

where the two series converge in norm in $L_p(\mathcal{M})$ and $\mathcal{M}$, respectively.

iii) Similarly, we define bilateral almost uniform (b.a.u.) convergence and bilateral almost sure (b.a.s.) convergence. Note that for the latter we use the symmetric injection of $\mathcal{M}$ into $L_p(\mathcal{M})$: $a\mapsto D^{\frac{1}{2p}}\,a\,D^{\frac{1}{2p}}$.

The following nontracial analogue of Lemma 6.2 is obtained in [DJ2]. For the sake of completeness we provide a simplified proof.

Lemma 7.10. i) *If $(x_n)\in L_p(\mathcal{M};c_0)$ with $1\le p<\infty$, then x_n converges b.a.s. to 0.*

ii) *If $2\le p<\infty$ and $(x_n)\in L_p(\mathcal{M};c_0^c)$, then x_n converges a.s. to 0.*

Proof. Suppose $(x_n)\in L_p(\mathcal{M};c_0)$. Then there are $a,b\in L_{2p}(\mathcal{M})$ and $y_n\in\mathcal{M}$ such that

$$x_n=ay_nb \quad\text{and}\quad \|a\|_{2p}<1,\ \|b\|_{2p}<1,\ \lim_{n\to\infty}\|y_n\|_\infty=0.$$

By the density of $D^{\frac{1}{2p}}\mathcal{M}$ in $L_{2p}(\mathcal{M})$, there are $a_k\in\mathcal{M}$ such that

$$a=\sum_{k\ge1}D^{\frac{1}{2p}}a_k \quad\text{and}\quad \big\|D^{\frac{1}{2p}}a_k\big\|_{2p}<2^{-k}\,.$$

Similarly,

$$b = \sum_{k\geq 1} b_k D^{\frac{1}{2p}} \quad \text{and} \quad \big\| b_k D^{\frac{1}{2p}} \big\|_{2p} < 2^{-k} \,.$$

Thus

$$x_n = \sum_{j,k} D^{\frac{1}{2p}} a_j y_n b_k D^{\frac{1}{2p}} \quad \text{converges in } L_p(\mathcal{M}).$$

By the Hölder inequality,

$$\varphi(a_k a_k^*) = \big\| D^{\frac{1}{2}} a_k a_k^* D^{\frac{1}{2}} \big\|_1 \leq \big\| D^{\frac{1}{2p}} a_k a_k^* D^{\frac{1}{2p}} \big\|_p < 2^{-2k} \,.$$

In the same way, $\varphi(b_k^* b_k) < 2^{-2k}$. Now let $\varepsilon > 0$. Then by [Ja1, Corollary 2.2.13], there is a projection $e \in \mathcal{M}$ such that

$$\varphi(e^\perp) < \varepsilon \quad \text{and} \quad \max\big\{ \|e a_k a_k^* e\|_\infty,\ \|e b_k^* b_k e\|_\infty \big\} \leq 8\,\varepsilon^{-1}\, 2^{-k} \,, \quad \forall\, k \geq 1.$$

Therefore,

$$\sum_{j,k\geq 1} \|e a_j y_n b_k e\|_\infty \leq 8\,\varepsilon^{-1} \, \|y_n\|_\infty \big[\sum_k 2^{-k/2} \big]^2 \,,$$

whence the double series $\sum_{j,k} (e a_j y_n b_k e)$ converges absolutely in $\mathcal{M}$ and

$$\lim_{n\to\infty} \sum_{j,k} e a_j y_n b_k e = 0.$$

Hence $x_n \to 0$ b.a.s. The second part is proved similarly. $\square$

Theorem 7.11. i) *Let T be a map on $\mathcal{M}$ satisfying* (7.I)–(7.IV). *Then* $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0)$ *for $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. More generally, let T_1, ..., T_d be d such maps and let*

$$M_{n_1,\,\dots,\,n_d} = M_{n_d}(T_d) \,\cdots\, M_{n_1}(T_1) \,.$$

Let F_k be the projection on the fixed point subspace of T_k. Then

$$\big(M_{n_1,\,\dots,\,n_d}(x) - F_d \,\cdots\, F_1(x)\big)_{n_1,\,\dots,\,n_d \geq 1} \in L_p(\mathcal{M}; c_0(\mathbb{N}^d)),$$

for all $x \in L_p(\mathcal{M})$, $1 < p < \infty$ and

$$\big(M_{n_1,\,\dots,\,n_d}(x) - F_d \,\cdots\, F_1(x)\big)_{n_1,\,\dots,\,n_d \geq 1} \in L_p(\mathcal{M}; c_0^c(\mathbb{N}^d)),$$

for all $x \in L_p(\mathcal{M})$, $2 < p < \infty$.

ii) *If the T_k further verify* (7.V) *and are positive operators on $L_2(\mathcal{M})$, then in the statement above the iterated ergodic averages $M_{n_1,\,\dots,\,n_d}$ can be replaced by the iterated powers $T_d^{n_d} \cdots T_1^{n_1}$.*

Proof. i) Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. By the discussion following Remark 7.2, we can find $y_k \in \mathcal{M}$ and

$$x_k = D^{\frac{1}{2p}} \big(y_k - T(y_k)\big) D^{\frac{1}{2p}}$$

such that

$$\lim_{k\to\infty} \|x - F(x) - x_k\|_p = 0 \,.$$

We have

$$M_n(x_k) = \frac{1}{n+1}\, D^{\frac{1}{2p}} \big[y_k - T^{n+1}(y_k)\big] D^{\frac{1}{2p}}$$

and so $\big(M_n(x_k)\big)_n \in L_p(\mathcal{M}; c_0)$. Then as in the proof of Theorem 6.3, we deduce that $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0)$.

Now assume $2 < p < \infty$. Then by Remark 7.2, the x_k above can be defined by

$$x_k = (y_k - T(y_k))D^{\frac{1}{p}} \quad \text{with } y_k \in \mathcal{M}_a \ .$$

Then $\big(M_n(x_k)\big)_n \in L_p(\mathcal{M}; c_0^c)$, and so $\big(M_n(x) - F(x)\big)_n \in L_p(\mathcal{M}; c_0^c)$.

Iterating the arguments above as in the proof of Theorem 6.6 and using the Haagerup space analogue of Corollary 4.4, we obtain the result for the multiple ergodic averages.

ii) In the case of one contraction, this part is proved in the same way as Theorem 6.7. The general case is dealt with by iteration. We omit the details. □

Corollary 7.12. *With the assumption and notation in Theorem* 7.11 i)*, for any* $1 < p < \infty$ *and* $x \in L_p(\mathcal{M})$*,*

$$\lim_{n_1\to\infty,\, \dots,\, n_d\to\infty} M_{n_1,\, \dots,\, n_d}(x) = F_d\, \cdots\, F_1(x) \quad \textit{b.a.s.};$$

if $p > 2$*, the convergence above is a.s. With the same assumption as in Theorem* 7.11 ii)*, we have*

$$\lim_{n_1\to\infty,\, \dots,\, n_d\to\infty} T_d^{n_d}\, \cdots\, T_1^{n_1}(x) = F_d\, \cdots\, F_1(x) \quad \textit{b.a.s.}$$

for $x \in L_p(\mathcal{M})$ *and* $1 < p < \infty$*. Again the convergence is a.s. for* $p > 2$*.*

Remarks. i) Combining the preceding arguments with those in the tracial case in section 6, we easily show that Theorem 6.8 continues to hold in the present setting of Haagerup L_p-spaces with a.s. convergence in place of a.u. convergence, as in Corollary 7.12 above.

ii) Using Goldstein's maximal weak type $(1, 1)$ inequality ([Go]; see also [Ja2, Theorem 2.2.12]), we can show that the first part of Corollary 7.12 remains true for $p = 1$ and $d = 1$ (i.e., for only one contraction).

iii) Jajte [Ja2] states a multiple individual ergodic theorem in $L_2(\mathcal{M})$ (Theorem 2.3.4 there), which corresponds to the first part of Corollary 7.12 in the case of $p = 2$. His proof uses in an essential way his previous Theorem 2.2.4. Based upon an iteration using Goldstein's maximal weak type $(1, 1)$ inequality, the proof of the latter theorem seems, however, to present a serious gap.

Corollary 7.12 excludes the case $p = \infty$, so does not allow us to recover all previous results by Lance [L], Kümmerer [Kü], etc. This situation can be easily remedied. This is done by virtue of the following simple lemma (see also [DJ2]):

Lemma 7.13. *Let* $1 \le p < \infty$ *and* $x_n \in \mathcal{M}$*. Then*

$$\begin{aligned} (D^{\frac{1}{2p}} x_n D^{\frac{1}{2p}})_n \in L_p(\mathcal{M}; c_0) \quad &\Longrightarrow \quad x_n \to 0 \ \textit{b.a.u.},\\ (x_n D^{\frac{1}{2p}})_n \in L_p(\mathcal{M}; c_0^c) \quad &\Longrightarrow \quad x_n \to 0 \ \textit{a.u.} \end{aligned}$$

Proof. Assume $(D^{\frac{1}{2p}} x_n D^{\frac{1}{2p}})_n \in L_p(\mathcal{M}; c_0)$. Choose a, b, y_n, a_k and b_k exactly as in the proof of Lemma 7.10 (with $D^{\frac{1}{2p}} x_n D^{\frac{1}{2p}} = a y_n b$). Next for each n choose an integer k_n such that

$$\big\| D^{\frac{1}{2p}} x_n D^{\frac{1}{2p}} - \sum_{j,k=1}^{k_n} D^{\frac{1}{2p}} a_j y_n b_k D^{\frac{1}{2p}} \big\|_p < 4^{-n} \ .$$

Set

$$z_n = x_n \, - \, \sum_{j,k=1}^{k_n} a_j y_n b_k.$$

Then

$$\big\| D^{\frac12} z_n D^{\frac12} \big\|_1 \le \big\| D^{\frac{1}{2p}} z_n D^{\frac{1}{2p}} \big\|_p < 4^{-n} \ .$$

Let u_n and v_n be respectively the real and imaginary parts of z_n. Then the inequality above holds with u_n and v_n instead of z_n. Now we apply [H1, Lemma 1.2] already quoted previously and reformulated in our setting as in [JX5]. We then find $u_n', u_n'' \in \mathcal{M}_+$ such that $u_n = u_n' - u_n''$ and

$$\big\| D^{\frac12} u_n D^{\frac12} \big\|_1 = \big\| D^{\frac12} u_n' D^{\frac12} \big\|_1 + \big\| D^{\frac12} u_n'' D^{\frac12} \big\|_1 = \varphi(u_n') + \varphi(u_n'').$$

Similarly, we have v_n' and v_n'' for v_n. Thus

$$\varphi(u_n') + \varphi(u_n'') < 4^{-n} \ , \quad \varphi(v_n') + \varphi(v_n'') < 4^{-n} \ .$$

Now given $\varepsilon > 0$, applying [Ja1, Corollary 2.2.13] to the family

$$\big\{ a_n a_n^*, \ b_n^* b_n, \ u_n', \ u_n'', \ v_n', \ v_n'' \ : \ n \in \mathbb{N} \big\},$$

we get a projection $e \in \mathcal{M}$ such that $\varphi(e^\perp) < \varepsilon$ and

$$\begin{aligned} &\max \big\{ \| e a_n a_n^* e \|_\infty, \, \| e b_n^* b_n e \|_\infty, \, \| e u_n' e \|_\infty, \, \| e u_n'' e \|_\infty, \, \| e v_n' e \|_\infty, \, \| e v_n'' e \|_\infty \big\} \\ &\quad < 16 \, \varepsilon^{-1} \, 2^{-n} \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \| e x_n e \|_\infty &\le \| e z_n e \|_\infty + \big\| \sum_{j,k=1}^{k_n} e a_j y_n b_k e \big\|_\infty \\ &\le \| e(u_n' - u_n'') e \|_\infty + \| e(v_n' - v_n'') e \|_\infty + \sum_{j,k=1}^{k_n} \| e a_j y_n b_k e \|_\infty \\ &\le 64 \, \varepsilon^{-1} \, 2^{-n} + 16 \, \varepsilon^{-1} \, \| y_n \|_\infty \big[\sum_{k \ge 1} 2^{-k/2} \big]^2 \ \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $x_n \to 0$ b.a.u. The proof of the second part on the a.u. convergence is similar and even easier (without appealing to Haagerup's Lemma). Thus we omit the details. □

The first part of the following is well known (cf., e.g., [Ja1]).

Corollary 7.14. *Let $T_1, \ldots, T_d$ satisfy* (7.I)–(7.IV) *and let*

$$M_{n_1, \ldots, n_d} = M_{n_d}(T_d) \cdots M_{n_1}(T_1) \, .$$

Then for any $x \in \mathcal{M}$,

$$\lim_{n_1 \to \infty, \ldots, n_d \to \infty} M_{n_1, \ldots, n_d}(x) = F_d \cdots F_1(x) \quad a.u.$$

If $T_1, \ldots, T_d$ additionally have (7.V)*, then*

$$\lim_{n_1 \to \infty, \ldots, n_d \to \infty} T_d^{n_d} \cdots T_1^{n_1}(x) = F_d \cdots F_1(x) \quad a.u.$$

Proof. This immediately follows from Theorem 7.11 and Lemma 7.13. □

Remark. In the case of $d = 1$, the first part of Corollary 7.14 permits us to recover Lance's theorem. However, compared with Kümmerer's theorem, our hypothesis is stronger since Kümmerer assumed only that T is a positive contraction verifying (7.III). We do not know whether all ergodic theorems in this section hold for such contractions or not. In particular, is Theorem 7.4 true for a positive contraction T satisfying (7.III) (and (7.V))?

Remark. As in the tracial case, all the preceding individual ergodic theorems admit semigroup analogues.

8. Examples

We will give some natural examples to which the results in the previous sections can be applied.

8.1. **Modular groups.** The very first examples are modular automorphism groups. Let φ be a normal faithful state on a von Neumann algebra $\mathcal{M}$. Let σ_t^φ be the modular group of φ. Then $T_t = \sigma_t^\varphi$ satisfies the properties (7.I)–(7.IV). On the other hand, (7.V) is equivalent to $\varphi(\sigma_t^\varphi(y)x) = \varphi(y\sigma_{-t}^\varphi(x))$ for all $x, y \in \mathcal{M}$ and $t \in \mathbb{R}$. Thus applying Corollary 7.6, we get that for $1 < p < \infty$,

$$\big\| \sup_t{}^+ \frac{1}{t}\int_0^t \sigma_s^\varphi(x)ds \big\|_p \le C_p\, \|x\|_p\,, \quad \forall\, x \in L_p(\mathcal{M}).$$

Note that the fixed point subspace $\mathcal{F}_\infty$ of (σ_t^φ) coincides with the centralizer $\mathcal{M}_\varphi$ of φ. Consequently, $\mathcal{F}_p$ coincides with $L_p(\mathcal{M}_\varphi)$, considered as a subspace of $L_p(\mathcal{M})$. Thus applying the results in subsection 7.4, we deduce that the ergodic averages

$$\frac{1}{t}\int_0^t \sigma_s^\varphi(x)ds$$

converge b.a.u. to x (resp. $F(x)$) as $t \to 0$ (resp. $t \to \infty$) for all $x \in L_p(\mathcal{M})$ and $1 \le p \le \infty$. Moreover, the convergence is a.u. in the case of $p \ge 2$. Let us consider a state $\varphi(x) = \lambda x_{11} + \mu x_{22}$ on the matrix algebra $\mathbb{M}_2$ of 2×2 matrices, where $0 < \lambda \ne \mu < 1$. Then we see that $\sigma_t^\varphi(e_{12}) = e^{it(\lambda-\mu)}\, e_{12}$ is not convergent for $t \to \infty$. At least in this case it is obvious that the symmetry condition (7.V) is really necessary.

8.2. **Semi-noncommutative case.** Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\mathcal{N}$ be a von Neumann algebra equipped with a semifinite normal faithful trace ν. We consider the von Neumann algebra tensor product $(\mathcal{M}, \tau) = (L_\infty(\Omega), \mu)\bar{\otimes}(\mathcal{N}, \nu)$. (Note here that μ is understood as a trace on $L_p(\Omega)$ via integration.) Given $p < \infty$ the corresponding noncommutative $L_p(\mathcal{M})$ is just $L_p(\Omega; L_p(\mathcal{N}))$, the usual L_p-space of strongly measurable p-integrable functions on Ω with values in $L_p(\mathcal{N})$. Now let (S_t) be a semigroup on $L_p(\Omega)$ satisfying the conditions (0.I)–(0.III) (with $\mathcal{M} = L_\infty(\Omega)$ there). Then $T_t = I \otimes S_t$ is a semigroup on $L_p(\mathcal{M})$ verifying the same conditions. Moreover, if S_t is symmetric, so is T_t. Thus we can transfer all classical semigroups to this semi-noncommutative setting and obtain the corresponding ergodic theorems. In particular, applying this procedure to the usual Poisson semigroup (P_t) on the unit circle $\mathbb{T}$ or on $\mathbb{R}^n$, by Corollary 4.6, we get

$$\big\| \sup_t{}^+ I \otimes P_t(x) \big\|_p \le C_p\, \|x\|_p, \quad x \in L_p(\mathbb{R}^n; L_p(\mathcal{N})),\ 1 < p < \infty.$$

For $p = 1$ we also have a weak type inequality (see Remark 4.7). These results were also proved by Mei [M] using a different method. Moreover, he obtained the nontangential analogue (for the upper half-plane) of the inequality above. Note that in this discussion, the usual Poisson semigroup on $\mathbb{R}^n$ can be replaced by the Poisson semigroup subordinate to the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^n$.

The situation above readily extends to the nontracial case. Assume that μ is a probability measure and $\mathcal{N}$ a von Neumann algebra equipped with a normal faithful state ψ. Then the tensor product $\mathcal{M}$ is equipped with the tensor state $\varphi = \mu \otimes \psi$. This allows us to apply the ergodic results in section 7 to this semi-noncommutative setting.

8.3. **Schur multipliers.** Let $\mathcal{M} = B(\ell_2)$. Then the associated noncommutative L_p-spaces are the Schatten classes S_p. The elements in S_p are represented as infinite matrices. Let ϕ be a function on $\mathbb{N} \times \mathbb{N}$. Recall that ϕ is a Schur multiplier on S_p if the map $M_\phi : x \mapsto (\phi_{jk} x_{jk})$, defined for finite matrices x, extends to a bounded map on S_p (which is still denoted by M_ϕ).

Let us consider a function $f : \mathbb{N} \to H$, where H is a real Hilbert space, and the associated kernel

$$K(j,k) = \|f(j) - f(k)\|, \quad j,k \in \mathbb{N}.$$

We are interested in the semigroups (T_t) and (P_t) of Schur multipliers, which are determined by

$$T_t(e_{jk}) = e^{-tK(j,k)^2} e_{jk} \quad \text{and} \quad P_t(e_{jk}) = e^{-tK(j,k)} e_{jk}\,,$$

where the e_{jk}'s stand for the canonical matrix units of $B(\ell_2)$. It is well known that these are completely positive contractive semigroups on $B(\ell_2)$. Indeed, let μ be a Gaussian measure on H, i.e., a probability space (Ω, μ) together with a measurable function $w : \Omega \to H$ such that

$$\exp\big(-\|h\|^2\big) = \int_\Omega \exp\big(i\langle h,\, w(\omega)\rangle\big)\, d\mu(\omega)\,, \quad h \in H.$$

Given $\omega \in \Omega, t > 0$ we consider the diagonal matrix $D_t(\omega)$ with the diagonal matrix with entries $\exp\big(i\sqrt{t}\,\langle f(j),\, w(\omega)\rangle\big)$, $j \in \mathbb{N}$. Then it is easy to see that

$$T_t(x) = \int \Omega D_t(\omega) x D_t(\omega)^*\, d\mu(\omega)\,, \quad x \in B(\ell_2). \tag{8.1}$$

Since $D_t(\omega)$ is unitary, this formula shows that T_t is a completely positive contraction on $B(\ell_2)$. In fact, (8.1) is the Stinespring representation of T_t. The semigroup (T_t) satisfies all properties (0.I)–(0.IV) with $\mathcal{M} = B(\ell_2)$ and τ being the usual trace on $B(\ell_2)$. Since (P_t) is the Poisson semigroup subordinate to (T_t) via (4.2), (P_t) has the same properties. Thus these semigroups extend to symmetric positive contractive semigroups on S_p for $1 \le p < \infty$.

Thus we have the maximal inequalities in Theorem 4.1 and Theorem 5.1 for (T_t) as well as (4.4) for (P_t). Note that in this situation the a.u. convergence reduces to the uniform convergence in $B(\ell_2)$.

8.4. **Hamiltonians.** In this subsection, $\mathcal{M}$ is semifinite and equipped with a normal faithful semifinite trace τ. Let $L \in L_0(\mathcal{M})$ be selfadjoint. We consider the Hamiltonian semigroup given by the generator $\operatorname{ad} L$:

$$\operatorname{ad} L(x) = Lx - xL, \quad x \in \mathcal{M}.$$

Note that

$$(\operatorname{ad} L)^2(x) = L^2x + xL^2 - 2LxL.$$

Set

$$T_t = e^{-t(\operatorname{ad} L)^2} \quad \text{and} \quad P_t = e^{-t\,|\operatorname{ad} L|}.$$

It is again well known that these are completely positive contractive semigroups on $\mathcal{M}$ (see [Par, Example 30.1]). Since (P_t) is the Poisson semigroup subordinate to (T_t), it suffices to show this for (T_t). In fact, (T_t) admits a Stinespring representation similar to (8.1):

$$T_t(x) = \mathbb{E}\big[e^{i\sqrt{t}\,gL}xe^{-i\sqrt{t}\,gL}\big], \quad x \in \mathcal{M}, \tag{8.2}$$

where g is a Gaussian variable with mean zero and variance $\sqrt{2}$ and $\mathbb{E}$ denotes the expectation with respect to g. To check this, let us first write the spectral resolution of L:

$$L = \int_{-\infty}^{\infty} \lambda\, de_\lambda.$$

Let $R > 0$ and e be the spectral projection of L corresponding to the interval $[-R, R]$. Consider $x \in \mathcal{M}$ such that $x = exe$. Then

$$\|L^j xL^k\| \le R^{j+k}\|x\|, \quad \forall\, j, k \ge 0. \tag{8.3}$$

A simple induction shows that

$$(\operatorname{ad} L)^n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k C_n^k\, L^k xL^{n-k}.$$

Now consider the formal power series representation

$$\begin{aligned}
\mathbb{E}\big[e^{i\sqrt{t}\,gL}xe^{-i\sqrt{t}\,gL}\big] &= \mathbb{E}\Big[\sum_{j,k=0}^{\infty} \frac{(i\sqrt{t})^j(-i\sqrt{t})^k}{j!\,k!}\, g^{j+k}L^jxL^k\Big]\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \sum_{j+k=2n} \frac{(2n)!}{j!\,k!}(-1)^k L^jxL^k\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!}(\operatorname{ad} L)^{2n}(x) = e^{-t(\operatorname{ad} L)^2}(x).
\end{aligned}$$

Note that the series above are absolutely convergent due to (8.3). Thus (8.2) is proved for all $x \in \mathcal{M}$ such that $x = exe$. However, the left-hand side of (8.2) defines a normal contraction on $\mathcal{M}$ because $\exp(i\sqrt{t}\,g(\omega)L)$ is a unitary in $\mathcal{M}$ for every ω. On the other hand, $\lim_{R\to\infty} \mathbb{1}_{[-R,\,R]}(L) = 1$ weakly in $\mathcal{M}$. By the w*-continuity, we see that (8.2) is true for all $x \in \mathcal{M}$. (8.2) also shows that T_t preserves the trace τ. On the other hand, since $(\operatorname{ad} L)^2$ is positive on $L_2(\mathcal{M})$, T_t is symmetric. Thus the semigroup (T_t) verifies (0.I)–(0.IV).

Remark. Let us consider a particular case where $\mathcal{M} = B(\ell_2)$ and L is a real diagonal matrix with diagonal entries $(\lambda_0, \lambda_1, \cdots)$. Then

$$|\operatorname{ad} L|(x) = \big(|\lambda_j - \lambda_k|x_{jk}\big)_{j,k}.$$

Thus $|\operatorname{ad} L|$ becomes a Schur multiplier and so (T_t) reduces to the semigroup already considered in the previous example with $H = \mathbb{R}$ and $f(j) = \lambda_j$.

8.5. **Free product.** Let $(\mathcal{M}_i, \varphi_i)_{i\in I}$ be a family of von Neumann algebras, each equipped with a normal faithful state φ_i. Let

$$(\mathcal{M}, \varphi) = *_{i\in I}\big(\mathcal{M}_i, \varphi_i\big)$$

be the von Neumann algebra reduced free product (cf. [V] and [VDN]). Recall that φ is a normal faithful state on $\mathcal{M}$. If all φ_i are tracial, so is φ. Now for every $i \in I$ let a w*-continuous semigroup $(T_t^i)_{t\geq 0}$ on $\mathcal{M}_i$ be given satisfying the following conditions:

i) T_t^i is unital;
ii) $\varphi \circ T_t^i = \varphi_i$;
iii) T_t^i is completely positive.

As usual, we always assume $T_0^i = \mathrm{id}_{\mathcal{M}_i}$. Then by [BlD] (see also [Ch]) it follows that for each t the family $\{T_t^i\}_{i\in I}$ defines a completely positive unital map T_t on $\mathcal{M}$, preserving the state φ. T_t is uniquely determined by its action on the monomials:

$$T_t(x_1 \,\cdots\, x_n) = T_t^{i_1}(x_1) \,\cdots\, T_t^{i_n}(x_n)$$

for any x_1, ..., x_n with $x_k \in \mathcal{M}_{i_k}^\circ$ and $i_1 \neq i_2 \neq \cdots \neq i_n$, where $\mathcal{M}_i^\circ = \{x \in \mathcal{M}_i : \varphi_i(x) = 0\}$. T_t is called the free product of the family $\{T_t^i\}_{i\in I}$ and is denoted by $T_t = *_{i\in I}\, T_t^i$. Then it is easy to see that (T_t) is a w*- continuous semigroup on $\mathcal{M}$. Thus this semigroup satisfies the conditions (7.I)–(7.III). By Lemma 7.1, (T_t^i) and (T_t) extend to norm continuous semigroups respectively on $L_p(\mathcal{M}_i)$ and $L_p(\mathcal{M})$ for all $1 \leq p < \infty$.

Recall that the modular group σ_t^φ is the free product of the modular groups $\sigma_t^{\varphi_i}$, $i \in I$ (cf. [Dy]). Thus if each T_t^i satisfies (7.IV), so does T_t. On the other hand, it is clear that the property (I.V) is also stable under free product.

Let us consider one special case. Note that $\mathcal{M}_i = \mathbb{C}1_{\mathcal{M}_i} \oplus \mathcal{M}_i^\circ$. Let $T_t^i : \mathcal{M}_i \to \mathcal{M}_i$ be defined by

$$T_t^i\big|_{\mathbb{C}1_{\mathcal{M}_i}} = \mathrm{id}_{\mathbb{C}1_{\mathcal{M}_i}} \quad \text{and} \quad T_t^i\big|_{\mathcal{M}_i^\circ} = e^{-t}\,\mathrm{id}_{\mathcal{M}_i^\circ}\,, \quad t \geq 0.$$

Then it is easy to check that (T_t^i) verifies the conditions i)–iii) above; moreover, T_t^i is symmetric relative to φ_i. The corresponding free product semigroup (T_t) is uniquely determined by

$$T_t(x_1 \,\cdots\, x_n) = e^{-nt} x_1 \,\cdots\, x_n$$

for any x_1, ..., x_n with $x_k \in \mathcal{M}_{i_k}^\circ$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ with $n \in \mathbb{N}$. This is the free analogue of the classical Poisson semigroup on the unit circle. It plays an important role in [RX].

The fixed point subspace of (T_t) above is simply $\mathbb{C}1_{\mathcal{M}}$. Let us briefly discuss the pointwise convergence in this case. Every element $x \in \mathcal{M}$ admits the following formal development:

$$x = \varphi(x) + \sum_{n\geq 1} \sum_{i_1\neq \cdots \neq i_n} x_1 \,\cdots\, x_n\,,$$

where $x_k \in \mathcal{M}_{i_k}^\circ$. Then

$$T_t(x) = \varphi(x) + \sum_{n\geq 1} e^{-nt} \sum_{i_1\neq \cdots \neq i_n} x_1 \,\cdots\, x_n\,.$$

Thus by the results in subsection 7.4,

$$\lim_{t\to 0} T_t(x) = x \quad \text{and} \quad \lim_{t\to\infty} T_t(x) = \varphi(x) \quad \text{a.u.}$$

A similar result also holds for $x \in L_p(\mathcal{M})$ with $1 < p < \infty$.

8.6. **Group von Neumann algebras.** Let G be a discrete group. Let $VN(G)$ denote the group von Neumann algebra of G. Recall that $VN(G)$ is a von Neumann algebra on $\ell_2(G)$ generated by the left regular representation λ. Let τ_G be the canonical faithful tracial state on $VN(G)$; i.e., τ_G is the vector state given by the unit basis vector δ_e, where e is the identity of G and where $\{\delta_g\}_{g\in G}$ denotes the canonical basis of $\ell_2(G)$.

Now we assume that G is equipped with a length function, denoted by $|\cdot|$. More precisely, $|\cdot|$ is a positive function on G satisfying the following conditions:

i) $|e| = 0$;
ii) $|g^{-1}| = |g|$ for any $g \in G$;
iii) if $d(f,g) = \frac{1}{2}(|f| + |g| - |fg^{-1}|)$, then for all $f, g, h \in G$,

$$d(f,g) \geq \min\{d(f,h),\ d(h,g)\}.$$

Bożejko [Bo1] proved that $g \mapsto e^{-t|g|}$ is a positive definite function on G (see also [Bo2]). Thus the associated Herz-Schur multiplier T_t is a normal completely positive unital map on $VN(G)$. More precisely, T_t is given on polynomials by

$$T_t\big(\sum_g a_g\, \lambda(g)\big) = \sum_g e^{-t|g|}\, a_g\, \lambda(g).$$

Moreover, T_t preserves the trace τ_G. Thus by Lemma 1.1, (T_t) extends to a semigroup on $L_p(VN(G))$ for all $1 \leq p < \infty$. Note that if $G = \mathbb{Z}$, then $VN(G) = L_\infty(\mathbb{T})$ and T_t becomes the usual Poisson semigroup on $\mathbb{T}$.

More generally, it is proved in [Bo1] that for any $0 < \alpha < 1$ the function $g \mapsto e^{-t|g|^\alpha}$ is positive definite on G. It follows that

$$P_t\big(\sum_g a_g\, \lambda(g)\big) = \sum_g e^{-t|g|^\alpha}\, a_g\, \lambda(g)$$

defines a completely positive unital trace-preserving semigroup on $VN(G)$. This last statement also follows from the previous since (P_t) is subordinate to (T_t) by (4.3).

Now let us specify the situation above to free groups. Let G be a free group, say, $G = \mathbb{F}_n$, a free group on n generators $\{g_1, \ldots, g_n\}$ (n can be infinite). Let $|\cdot|$ be the length function with respect to $\{g_1, \ldots, g_n\}$. Then the fact that $e^{-t|\cdot|}$ is a positive definite function on $\mathbb{F}_n$ goes back to Haagerup [H4]. Note that this is also a special case of the free product in the previous example. Indeed, writing $\mathbb{F}_n$ as the reduced free product of n copies of $\mathbb{Z}$, we have

$$\big(VN(\mathbb{F}_n),\ \tau_{\mathbb{F}_n}\big) *_{1\leq k\leq n} \big(L_\infty(\mathbb{T}),\ \tau_{\mathbb{Z}}\big).$$

Then the semigroup on $\mathbb{F}_n$ appears as the free product of n copies of the usual Poisson semigroup on $\mathbb{T}$. Applying our ergodic theorems to this case, we get Theorem 0.3.

More generally, let $\{G_i\}_{i\in I}$ be a family of discrete groups, each equipped with a length function. Let T^i_t be the associated semigroup on G_i defined previously. Let $G = *_{i\in I}\, G_i$ be the reduced free product. Then by [Bo1] (or the previous example), the free product $T_t = *_{i\in I} T_t^i$ yields a symmetric completely positive contractive semigroup on G.

8.7. **q-Ornstein-Uhlenbeck semigroups.** Let $H_{\mathbb{R}}$ be a real Hilbert space and $H_{\mathbb{C}}$ its complexification. For $-1 \le q \le 1$ let $\mathcal{F}_q(H_{\mathbb{C}})$ be the q-Fock space based on $H_{\mathbb{C}}$ constructed by Bożejko and Speicher (see [BS1] and [BS2]). Note that $\mathcal{F}_1(H_{\mathbb{C}})$, $\mathcal{F}_{-1}(H_{\mathbb{C}})$ and $\mathcal{F}_0(H_{\mathbb{C}})$ are respectively the symmetric, anti-symmetric and full (= free) Fock spaces. Given a vector $h \in H_{\mathbb{C}}$, let $c(h)$ denote the associated (left) creation operator on $\mathcal{F}_q(H_{\mathbb{C}})$. $c(h)$ is a bounded operator for $q < 1$ and a closed densely defined operator for $q = 1$. Its adjoint $c(h)^*$ is the annihilation operator associated to h and denoted by $a(h)$. Let

$$g_q(h) = c(h) + a(h), \quad h \in H_{\mathbb{R}}.$$

$g_q(h)$ is a so-called q-Gaussian variable. Note that $g_1(h)$ is a usual Gaussian variable, $g_0(h)$ a semi-circular variable in Voiculescu's sense (cf. [V] and [VDN]), and finally $g_{-1}(h)$ corresponds to a Fermion. The q-von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$ is the von Neumann algebra on $\mathcal{F}_q(H_{\mathbb{C}})$ generated by all q-Gaussians, namely,

$$\Gamma_q(H_{\mathbb{R}}) = \{g_q(h) \ : \ h \in H_{\mathbb{R}}\}'' \subset B(\mathcal{F}_q(H_{\mathbb{C}})).$$

Let Ω be the vacuum vector in $\mathcal{F}_q(H_{\mathbb{C}})$ and τ_q the associated vector state. Then τ_q is faithful and tracial. Hence $\Gamma_q(H_{\mathbb{R}})$ is a type II_1 von Neumann algebra for $q < 1$. ($\Gamma_1(H_{\mathbb{R}})$ is commutative.) Moreover, it is a noninjective factor if $-1 < q < 1$ and $\dim H \ge 2$. We refer to [BKS], [N] and [Ri] for more information.

Now let S be a contraction on $H_{\mathbb{R}}$. Then S extends to a contraction on $H_{\mathbb{C}}$. The second quantization $\Gamma(S)$ is a normal completely positive unital trace-preserving map on $\Gamma_q(H_{\mathbb{R}})$. To give the definition of $\Gamma(S)$, we recall the Wick product. Since Ω is separating for $\Gamma_q(H_{\mathbb{R}})$, the map $x \in \Gamma_q(H_{\mathbb{R}}) \mapsto x(\Omega)$ is injective. Its image is a dense subspace of $\mathcal{F}_q(H_{\mathbb{C}})$ (for Ω is cyclic). It is easy to see that all elementary tensors belong to this image. The inverse map (defined on the image) is called the Wick product, denoted by W. Thus if ξ is a linear combination of elementary tensors, $W(\xi)$ is the unique operator in $\Gamma_q(H_{\mathbb{R}})$ such that $W(\xi)\Omega = \xi$. Note that the collection of all such $W(\xi)$'s forms a w*-dense $*$-subalgebra of $\Gamma_q(H_{\mathbb{R}})$. Then $\Gamma(S)$ is uniquely determined by

$$\Gamma(S)\big(W(h_1 \otimes \cdots \otimes h_n)\big) W(Sh_1 \otimes \cdots \otimes Sh_n), \quad h_1, ..., h_n \in H_{\mathbb{C}}.$$

Applying this construction to $S = e^{-t}\,\mathrm{id}_{H_{\mathbb{R}}}$ for $t \ge 0$, we get a normal completely positive unital trace-preserving map $T_t = \Gamma(e^{-t}\,\mathrm{id}_{H_{\mathbb{R}}})$. The action of T_t on the Wick products is given by

$$T_t\big(W(h_1 \otimes \cdots \otimes h_n)\big) = e^{-nt}\, W(h_1 \otimes \cdots \otimes h_n).$$

Then (T_t) is a semigroup on $\Gamma_q(H_{\mathbb{R}})$ satisfying all conditions (0.I)–(0.IV). This is the q-Ornstein-Uhlenbeck semigroup associated with $H_{\mathbb{R}}$. The negative of its infinitesimal generator is the so-called number operator. The cases $q = 1$ and $q = -1$ correspond respectively to the classical and Fermionic Ornstein-Uhlenbeck semigroup. $(T_t)_t$ in these two special cases have been extensively studied. See [Bo3], [CaL] and [Bi] for related results.

The preceding discussion also applies to the quasi-free case. Then the corresponding von Neumann algebras are of type III. See [Sh] for the case of $q = 0$ and

[Hi] for the general case. In particular, for $q = -1$, we have the classical Araki-Woods factors. In this case, the resulting semigroup is the extension of the previous Fermionic Ornstein-Uhlenbeck semigroup to the type III setting.

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