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ON THE RESTRICTION OF DELIGNE-LUSZTIG CHARACTERS

MARK REEDER

1. Introduction

This paper was motivated by the following restriction problem for representations of finite orthogonal groups.

Let $\mathfrak{F}$ be an algebraic closure of a finite field $\mathfrak{f}$ of cardinality q, a power of a prime $p > 2$. Let $G = SO(V)$ be the special orthogonal group of a $(2n+1)$-dimensional $\mathfrak{F}$-space V with nondegenerate quadratic form Q. Assume V and Q are defined over $\mathfrak{f}$, and let F denote the corresponding Frobenius endomorphisms of V and G. Fix $v \in V^F$ with $Q(v) \neq 0$ and let H be the stabilizer of v in G.

Let $\pi \in \mathrm{Irr}(G^F), \sigma \in \mathrm{Irr}(H^F)$ be complex irreducible cuspidal representations of the respective groups G^F and H^F of $\mathfrak{f}$-rational points. The problem is to compute the multiplicity

$$\langle \pi, \sigma \rangle_{H^F} = \dim \mathrm{Hom}_{H^F}(\pi, \sigma)$$

of σ in the restriction of π to H^F.

Using unpublished work of Bernstein and Rallis (independently) on p-adic orthogonal groups, it can be shown that

$$\langle \pi, \sigma \rangle_{H^F} = 0 \quad \text{or} \quad 1.$$

In this paper, we compute $\langle \pi, \sigma \rangle_{H^F}$ exactly, when π and σ are irreducible cuspidal Deligne-Lusztig representations [8]. We do not rely on the above-mentioned work of Bernstein and Rallis. Our calculation follows from a qualitative study of restrictions of Deligne-Lusztig characters for general simple algebraic groups, to be described later in this introduction.

To state our multiplicity result for orthogonal groups, we first recall the inducing data. Let $T \subset G$, $S \subset H$ be F-stable anisotropic tori in G and H. There are unique partitions $\lambda = (j^{\lambda_j}),\ \mu = (j^{\mu_j})$ of n (here λ_j, μ_j are the number of parts equal to j) such that

$$T^F \simeq \prod_j (\mathfrak{f}^1_{2j})^{\lambda_j}, \qquad S^F \simeq \prod_j (\mathfrak{f}^1_{2j})^{\mu_j},$$

where, for any $d > 1$, $\mathfrak{f}_d = \mathfrak{F}^{F^d}$ is the extension of $\mathfrak{f}$ in $\mathfrak{F}$ of degree d, and $\mathfrak{f}^1_{2j}$ is the kernel of the norm mapping $\mathfrak{f}^\times_{2j} \to \mathfrak{f}^\times_j$. The number of parts $\sum_j \mu_j$ is even if H is split, and odd if H is nonsplit.

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Let $\chi \in \mathrm{Irr}(T^F)$ and $\eta \in \mathrm{Irr}(S^F)$ be irreducible characters of T^F and S^F which are **regular** in the sense that χ and η have trivial stabilizers in the respective Weyl groups $W_G(T)^F$ and $W_H(S)^F$. We may write

$$\chi = \otimes_j \chi_j, \qquad \eta = \otimes_j \eta_j,$$

where

$$\chi_j = \chi_{j1} \otimes \cdots \otimes \chi_{j\lambda_j} \in \mathrm{Irr}\left((\mathfrak{f}_{2j}^1)^{\lambda_j}\right),$$

each χ_{jk} is a character of $\mathfrak{f}_{2j}^1$, and likewise for η. Let $\Gamma_{2j} \simeq \mathbb{Z}/2j\mathbb{Z}$ be the Galois group of $\mathfrak{f}_{2j}/\mathfrak{f}$.

Definition 1.1. We say that χ and η **intertwine** if $\eta_{jk'}$ is a Γ_{2j}-conjugate of χ_{jk} for some $1 \leq j \leq n$, $1 \leq k \leq \lambda_j$, $1 \leq k' \leq \mu_j$.

Note that χ and η can intertwine even if $T \not\simeq S$. However, if λ and η have no common parts, that is, if $\lambda_j \mu_j = 0$ for all j, then χ and η do not intertwine.

By Deligne-Lusztig induction, we have virtual representations $R^G_{T,\chi}$ of G^F and $R^H_{S,\eta}$ of H^F, respectively. By the regularity assumptions on χ and η, these are actually irreducible characters, up to sign. In fact, we have

$$(-1)^{\mathrm{rk}\, G} R^G_{T,\chi} \in \mathrm{Irr}(G^F), \qquad (-1)^{\mathrm{rk}\, H} R^H_{S,\eta} \in \mathrm{Irr}(H^F).$$

These two irreducible characters are cuspidal, since T and S are anisotropic. We prove:

Theorem 1.2. *Let T and S be anisotropic F-stable maximal tori in G and H, respectively, and let $\chi \in \mathrm{Irr}(T^F)$, $\eta \in \mathrm{Irr}(S^F)$ be regular characters. Then*

$$(-1)^{\mathrm{rk}\, G + \mathrm{rk}\, H} \langle R^G_{T,\chi}, R^H_{S,\eta} \rangle_{H^F} = \begin{cases} 0 & \text{if } \eta, \chi \text{ intertwine}, \\ 1 & \text{if } \eta, \chi \text{ do not intertwine}. \end{cases}$$

If T and S are arbitrary F-stable maximal tori, but χ and η are still regular, then the multiplicity is either zero or a power of two; see (9.9) below.

The multiplicity result 1.2 is used in [12] to verify some cases of the conjectures of [11] describing restrictions from p-adic SO_{2n+1} to SO_{2n} in terms of symplectic local root numbers and the parameterization of depth-zero supercuspidal L-packets given in [7].

As already mentioned, Theorem 1.2 follows from a qualitative result, in a general setting, on multiplicities of Deligne-Lusztig representations.

Let G be a a connected simple algebraic group defined over $\mathfrak{f}$, and let H be a connected reductive $\mathfrak{f}$-subgroup of G. Fix F-stable maximal tori $T \subset G$ and $S \subset H$, along with arbitrary characters $\chi \in \mathrm{Irr}(T^F)$ and $\eta \in \mathrm{Irr}(S^F)$.

From this data Deligne and Lusztig [8] construct virtual characters $R^G_{T,\chi}$ and $R^H_{S,\eta}$ on G^F and H^F, respectively. Let $\langle\ ,\ \rangle_{H^F}$ be the canonical pairing on virtual characters of H^F. We are interested in the multiplicity

$$\langle R^G_{T,\chi}, R^H_{S,\eta} \rangle_{H^F},$$

where $R^G_{T,\chi}$ is viewed as a virtual character of H^F, by restriction.

Let B and B_H be Borel subgroups of G and H, respectively, and let δ be the minimum codimension of a B_H-orbit in G/B. The invariant δ is called the **complexity** of the H-variety G/B. The theory of complexity was first studied for reductive groups over fields of characteristic zero (cf. [1] and the references therein). In that setting, it is proved in [1] that δ governs the growth of multiplicities in restrictions of algebraic representations. We will show that δ also governs the growth of multiplicities in restrictions of Deligne-Lusztig representations.

Because we are in nonzero characteristic, we need to make an assumption. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H.

Assumption 1.3. *There is an* $\mathrm{Ad}(H)$*-stable decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$*, and a nondegenerate symmetric bilinear form* B *on* $\mathfrak{m}$*, invariant under* $\mathrm{Ad}(H)$*.*

This assumption holds if p is a good prime for $\mathfrak{g}$ and the Killing form of $\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$ [24, I.5.3]. For $G = SO_{N+1}, H = SO_N$, our assumption holds for $p > 2$.

For an integer $\nu \geq 1$, let $N_\nu^T : T^{F^\nu} \to T^F$ be the norm map, and let

$$\chi^{(\nu)} = \chi \circ N_\nu^T, \qquad \eta^{(\nu)} = \eta \circ N_\nu^S.$$

Under Assumption 1.3, we prove the following.

Theorem 1.4. *There is a polynomial of degree at most* δ*:*

$$M(t) = At^\delta + \cdots \in \mathbb{Q}[t],$$

whose coefficients depend on χ *and* η*, and an integer* $m \geq 1$ *such that*

$$\langle R^G_{T,\chi^{(\nu)}}, R^H_{S,\eta^{(\nu)}} \rangle_{H^{F^\nu}} = M(q^\nu)$$

for all positive integers $\nu \equiv 1 \mod m$*. The degree* δ *is optimal: if* q *is sufficiently large, there exist* χ, η *such that the leading coefficient* A *is nonzero.*

We also give an explicit formula for the leading term A in Theorem 1.4 (see Proposition 7.4). For $G = SO_{N+1}$, $H = SO_N$, we have $\delta = 0$, and our explicit formula for A leads to Theorem 1.2 (see Section 9). Even if $\delta > 0$ one can sometimes use Theorem 1.4 to compute exact multiplicities, by exploiting the polynomial nature of $M(t)$. In Section 10 we illustrate this for $G = SO_7$, $H = G_2$, where $\delta = 1$.

Our formula for A also allows us to show, for general G and H, and "very regular" χ (see section 8), that the multiplicity

$$\langle R^G_{T,\chi}, St_H \rangle_{H^F}$$

of the Steinberg representation St_H is a monic polynomial in q of degree δ, while the multiplicity of the trivial representation

$$\langle R^G_{T,\chi}, 1_H \rangle_{H^F}$$

is a polynomial in q of degree strictly less than δ. In particular, for $G = SO_{2n+1}$ and $H = SO_{2n}$, we have

$$\langle R^G_{T,\chi}, St_H \rangle_{H^F} = 1, \qquad \langle R^G_{T,\chi}, 1_H \rangle_{H^F} = 0,$$

for very regular χ.

To prove Theorem 1.4 we use a method introduced by Thoma [27] for the study of the restriction of irreducible representations from $GL_n(\mathfrak{f})$ to $GL_{n-1}(\mathfrak{f})$ (where again $\delta = 0$). In that situation, the Green's functions giving the character on unipotent elements were explicitly known. Hagedorn [13], in his 1994 Ph.D. thesis, showed how some of Thoma's methods could be generalized to Deligne-Lusztig characters for other pairs of classical groups, where the Green's functions are less explicit. The abstract results of Hagedorn gave me the courage to attempt such calculations for general groups and to obtain closed multiplicity formulas for orthogonal groups.

It is a pleasure to thank Dick Gross for initiating the work in [12] which led to this paper, for helpful remarks on an earlier version, and for aquainting me with Hagedorn's thesis.

The referee read the original version of this paper with care and insight, made valuable comments and simplified some of the arguments. In particular, the proof of Lemma 3.1 given below is due to the referee and is much shorter than the original one.

Some general notation: The cardinality of a finite set X is denoted by $|X|$. Equivalence classes are generally denoted by $[\,\cdot\,]$, sometimes with ornamentation. If g is an element of a group G, we write $\mathrm{Ad}(g)$ for the conjugation map $\mathrm{Ad}(g) : x \mapsto gxg^{-1}$, and also write ${}^gT := gTg^{-1}$ for a subgroup $T \subset G$. The center of G is denoted $Z(G)$ and the centralizer of $g \in G$ is denoted $C_G(g)$.

We write $\langle\ ,\ \rangle_H$ for the pairing on the space of class functions on a finite group H, for which the irreducible characters of H are an orthonormal basis. If $G, G' \supset H$ are finite overgroups of H and ψ, ψ' are class functions on G, G' respectively, then $\langle \psi, \psi' \rangle_H$ is understood to mean $\langle \psi|_H, \psi'|_H \rangle_H$, where $|_H$ denotes restriction to H.

2. Remarks on maximal tori

Let G be a connected reductive algebraic $\mathfrak{F}$-group. We assume G is defined over $\mathfrak{f}$ and has Frobenius F. If T is a maximal torus in G we denote its normalizer in G by $N_G(T)$ and write $W_G(T) = N_G(T)/T$ for the Weyl group of T in G. If T is F-stable, we have

$$W(T)^F = N_G(T)^F/T^F,$$

by the Lang-Steinberg theorem.

The reduction formula for Deligne-Lusztig characters (recalled in section 4 below) involves a sum over the following kind of subset of G^F. Fix an F-stable maximal torus $T \subset G$, and let s be a semisimple element in G^F. We must sum over the set

$$N_G(s,T)^F := \{\gamma \in G^F : \ s^\gamma \in T\}.$$

Note that $N_G(s,T)^F$, if nonempty, is a union of $G_s^F \times N_G(T)^F$ double cosets, where $G_s := C_G(s)^\circ$ is the identity component of the centralizer $C_G(s)$ of s in G.

To say that $s^\gamma \in T$ is to say that ${}^\gamma T \subset G_s$, so determining the $G_s^F \times N_G(T)^F$ double cosets in $N_G(s,T)^F$ amounts to determining the G_s^F-conjugacy classes of F-stable maximal tori in G_s which are contained in a given G^F-conjugacy class. Such classes of tori are parameterized by twisted conjugacy classes in Weyl groups of G_s and G.

The aim of this section is to parameterize the $G_s^F \times N_G(T)^F$ double cosets in $N_G(s,T)^F$ in terms of the fiber of a natural map between twisted conjugacy classes in the Weyl groups of G_s and G. This parameterization will be fundamental to our later calculations with Deligne-Lusztig characters.

We begin by recalling the classification of F-stable maximal tori in G. See [5, chap. 3] for more details in what follows. Fix an F-stable maximal torus T_0 in G contained in an F-stable Borel subgroup of G, and abbreviate $N_G = N_G(T_0)$, $W_G = W_G(T_0)$.

Let $\mathcal{T}(G)$ denote the set of all F-stable maximal tori in G. Then $\mathcal{T}(G)$ is a finite union of G^F-orbits. For any $T \in \mathcal{T}(G)$, let

$$[T]_G := \{{}^{\gamma}T : \ \gamma \in G^F\}$$

denote the G^F-orbit of T. There is $g \in G$ such that $T = {}^gT_0$. Since T is F-stable, we have $g^{-1}F(g) \in N_G$. This gives an element

$$w := g^{-1}F(g)T_0 \in W_G.$$

The map $\mathrm{Ad}(g)t = gtg^{-1}$ is an $\mathfrak{f}$-isomorphism

$$\mathrm{Ad}(g) : (T_0, wF) \longrightarrow (T, F),$$

where the second component denotes the action of Frobenius under an $\mathfrak{f}$-structure.

For any finite group A with F-action, we let $H^1(F, A)$ denote the set of F-conjugacy classes in A. These are the orbits of the action of A on itself via $(a, b) \mapsto abF(a)^{-1}$. Let $[b] \in H^1(F, A)$ denote the F-conjugacy class of an element $b \in A$.

For g, T, w as above, the F-conjugacy class of w is independent of the choice of g. Hence we have a well-defined class

$$\mathrm{cl}(T, G) := [w] \in H^1(F, W_G).$$

For each $\omega \in H^1(F, W_G)$, the set

$$\mathcal{T}_\omega(G) := \{T \in \mathcal{T}(G) : \ \mathrm{cl}(T, G) = \omega\}$$

is a single G^F-orbit in $\mathcal{T}(G)$, and all G^F-orbits are of this form. Thus, the partition of the set of F-stable maximal tori into G^F-orbits is given by

$$\mathcal{T}(G) = \coprod_{\omega \in H^1(F, W_G)} \mathcal{T}_\omega(G).$$

Let $s \in G^F$ be semisimple, and let T_s be an F-stable maximal torus of G_s contained in an F-stable Borel subgroup of G_s, and let W_{G_s} be the Weyl group of T_s in G_s. The partition of $\mathcal{T}(G_s)$ into G_s^F-orbits is given, as above, by

$$\mathcal{T}(G_s) = \coprod_{\upsilon \in H^1(F, W_{G_s})} \mathcal{T}_\upsilon(G_s).$$

If $T \in \mathcal{T}(G)$, the set of F-stable maximal tori in G_s which are G^F-conjugate to T is a finite union (possibly empty) of G_s^F-orbits. We want to describe this union in terms of F-conjugacy classes in W_{G_s}. That is, given $\omega \in H^1(F, W_G)$, we have

$$\mathcal{T}_\omega(G) \cap \mathcal{T}(G_s) = \coprod_{\upsilon \in M_\omega} \mathcal{T}_\upsilon(G_s) \tag{2.1}$$

for some subset $M_\omega \subseteq H^1(F, W_{G_s})$, and our task is to find M_ω.

The first point is that T_s is generally not contained in an F-stable Borel subgroup of G. Let $g \in G$ be such that ${}^gT_s = T_0$, and let $\dot{y}_s := gF(g)^{-1}$ have image $y_s \in W_G$. Then

$$\mathrm{cl}(T_s, G) = [y_s] \in H^1(F, W_G),$$

and $\mathrm{Ad}(g)$ is an $\mathfrak{f}$-isomorphism

$$\mathrm{Ad}(g) : (T_s, F) \longrightarrow (T_0, y_sF).$$

Now T_0 is also a maximal torus in $\mathrm{Ad}(g)G_s$, whose Weyl group

$$W'_{G_s} := \mathrm{Ad}(g)W_{G_s}$$

is a subgroup of W_G, stable under $\mathrm{Ad}(y_s) \circ F$.

Define $j_{G_s}: H^1(F, W_{G_s}) \to H^1(F, W_G)$ to be the composition of maps

$$j_{G_s}: H^1(F, W_{G_s}) \xrightarrow{\mathrm{Ad}(g)} H^1(y_sF, W'_{G_s}) \xrightarrow{\text{incl}} H^1(y_sF, W_G) \xrightarrow{\tau_{y_s}} H^1(F, W_G), \tag{2.2}$$

where the middle map is induced by the inclusion $W'_{G_s} \hookrightarrow W_G$ and τ_{y_s} is the twisting bijection given by $\tau_{y_s}[x] = [xy_s]$.

Now let T be an arbitrary F-stable maximal torus in G_s. Write $T = {}^hT_s$, with $h \in G_s$, so that $h^{-1}F(h) \in \mathrm{cl}(T, G_s)$. For $g \in G$ as above, we have $T = {}^{hg^{-1}}T_0$. Since

$$gh^{-1}F(hg^{-1}) = g(h^{-1}F(h))g^{-1} \cdot gF(g)^{-1},$$

it follows that

$$\mathrm{cl}(T, G) = j_{G_s}\left(\mathrm{cl}(T, G_s)\right). \tag{2.3}$$

This proves:

Lemma 2.1. *For each $\omega \in H^1(F, W_G)$ and $T \in \mathcal{T}_\omega(G_s)$, we have*

$$\mathcal{T}_\omega(G) \cap \mathcal{T}(G_s) = \coprod_{\upsilon \in j_{G_s}^{-1}(\omega)} \mathcal{T}_\upsilon(G_s).$$

We can also parameterize the G_s^F-orbits in $[T]_G \cap \mathcal{T}(G_s)$ via the mapping

$$N_G(s, T)^F := \{\gamma \in G^F : \ s \in {}^\gamma T\} \longrightarrow [T]_G \cap \mathcal{T}(G_s), \qquad \gamma \mapsto {}^\gamma T. \tag{2.4}$$

Note that G_s^F acts on $N_G(s, T)^F$ by left multiplication, and that (2.4) factors through the quotient

$$\bar{N}_G(s, T)^F := G_s^F \backslash N_G(s, T)^F. \tag{2.5}$$

The action of $N_G(T)^F$ on $N_G(s, T)^F$ by right multiplication commutes with the G_s^F-action, hence factors through an action on $\bar{N}_G(s, T)^F$, where T^F acts trivially. This gives an action of $W_G(T)^F$ on $\bar{N}_G(s, T)^F$, whose orbits are the $G_s^F \times N(T)^F$-double cosets in $N(s, T)^F$.

Lemma 2.2. *The mapping* (2.4)*, sending $\gamma \mapsto {}^\gamma T$, induces a bijection*

$$\bar{N}_G(s, T)^F / W_G(T)^F \xrightarrow{\sim} G_s^F \backslash \big([T]_G \cap \mathcal{T}(G_s)\big)$$

with the property that the stabilizer in $W_G(T)^F$ of the class $\bar{\gamma} \in \bar{N}(s, T)^F$ is isomorphic, via $\mathrm{Ad}(\gamma)$, to $W_{G_s}({}^\gamma T)^F$.

Proof. The bijectivity is straightforward and left to the reader. Let $w \in W_G(T)^F$, and let $\dot{w} \in N_G(T)^F$ be a representative of w. Then

$$\bar{\gamma} \cdot w = \bar{\gamma} \quad \Leftrightarrow \quad G_s^F \gamma \dot{w} = G_s^F \gamma \quad \Leftrightarrow \quad \mathrm{Ad}(\gamma)\dot{w} \in N_{G_s}({}^\gamma T).$$

This implies the assertion about the stabilizer. □

Combining Lemmas 2.1 and 2.2, we get an explicit formula for $|\bar{N}_G(s, T)^F|$.

Corollary 2.3. *Let $\omega \in H^1(F, W_G)$ and $T \in \mathcal{T}_\omega(G)$. Then the set $N_G(s, T)^F$ is nonempty if and only if the fiber $j_{G_s}^{-1}(\omega)$ is nonempty, in which case we have*

$$|\bar{N}_G(s, T)^F| = \sum_{\upsilon \in j_{G_s}^{-1}(\omega)} \frac{|W_G(T)^F|}{|W_{G_s}(T_\upsilon)^F|},$$

where, for each $\upsilon \in j_{G_s}^{-1}(\omega)$, the torus T_υ is chosen arbitrarily in $\mathcal{T}_\upsilon(G_s)$.

3. On the centralizer of a semisimple element

Let $s \in G^F$ be semisimple. In the previous section we parameterized the set of G_s^F-conjugacy classes of maximal tori in G_s which are contained in a given G^F-conjugacy class, in terms of fibers of the map $j_{G_s} : H^1(F, W_{G_s}) \to H^1(F, W_G)$. To compute this map j_{G_s} concretely, we must find an element $y_s \in W_G$ such that $\mathrm{cl}(T_s, G) = [y_s]$, where $T_s \in \mathcal{T}(G_s)$ is contained in an F-stable Borel subgroup of G_s. This amounts to finding the $\mathfrak{f}$-isomorphism class of the connected centralizer G_s.

An elegant formula for y_s was given by Carter [6], using the Brauer complex. Here we explain a different method that is suited to our later computations; namely we show how the class $[y_s]$ can be determined from the effect of F on a "diagonalized" G-conjugate of s. Unfortunately, both the present method, as well as that of [6] require that $C_G(s)$ be connected. That is, we must assume that $G_s = C_G(s)$. This holds for any semisimple $s \in G$ if G has simply-connected derived group. Our method generalizes that of Gross [10], who determined $C_G(s)$ when this group is a torus (over an arbitrary field).

Let Φ denote the set of roots of T_0 in G. Let ϑ denote the automorphisms of Φ and W_G induced by F. For $\alpha \in \Phi$ with corresponding reflection $s_\alpha \in W_G$, we have

$$\alpha \circ F = q\vartheta^{-1} \cdot \alpha, \qquad \vartheta(s_\alpha) = s_{\vartheta \cdot \alpha}.$$

Here is our recipe for finding $\mathrm{cl}(T_s, G)$. Let $t \in T_0$ be a G-conjugate of s, and let

$$\Phi_t = \{\alpha \in \Phi : \ \alpha(t) = 1\}.$$

Since t has a conjugate in G^F, there is $w \in W$ (not necessarily unique) such that

$$F(t) = t^w. \tag{3.1}$$

Choose such a w arbitrarily. From (3.1) it follows that

$$w\vartheta \cdot \Phi_t = \Phi_t. \tag{3.2}$$

Now choose any positive system $\Phi_t^+ \subset \Phi_t$. Then (3.2) implies that $w\vartheta \cdot \Phi_t^+$ is another positive system in Φ_t. Being the Weyl group of Φ_t, the group W_{G_t} acts simply transitively on positive systems in Φ_t, so there is a unique $x \in W_{G_t}$ such that

$$w\vartheta \cdot \Phi_t^+ = x \cdot \Phi_t^+. \tag{3.3}$$

Setting $y = x^{-1}w$, we see that w can be factored uniquely as

$$w = xy, \tag{3.4}$$

where $x \in W_{G_t}$ and $y\vartheta \cdot \Phi_t^+ = \Phi_t^+$.

Since $C_G(t)$ is connected, the group W_{G_t} is the full stabilizer of t in W_G. This means that a different choice of w satisfying (3.1) will change x, but not y.

Lemma 3.1. *With y constructed as above, we have*

$$\mathrm{cl}(T_s, G) = [y] \in H^1(F, W_G).$$

Proof. The following proof was provided by the referee; it is shorter than the original proof. Choose $g \in G$ such that $\dot{y} = g^{-1}F(g) \in N_G$ is a representative of y. Then

$$yF(t) = t^x = t,$$

which implies that ${}^g t \in G^F$. Since $C_G(s)$ is connected, any element of G^F which is G-conjugate to s is in fact G^F-conjugate to s. Hence, by multiplying g on the left by an element of G^F, we may assume that $s = {}^g t$.

By definition of y, there is an $\mathrm{Ad}(\dot{y})F$-stable Borel subgroup $B_t \subset G_t$ containing T_0. Hence ${}^g B_t$ is an F-stable Borel subgroup of G_s, containing the F-stable maximal torus $T'_s := {}^g T_0$. Since T'_s is G_s^F-conjugate to T_s, it follows that

$$\mathrm{cl}(T_s, G) = [g^{-1}F(g)] = [y],$$

as claimed. □

4. Deligne-Lusztig characters

Let $T \in \mathcal{T}(G)$ be an F-stable maximal torus in G, and let $\chi \in \mathrm{Irr}(T^F)$. The Deligne-Lusztig character $R^G_{T,\chi}$ has the following reduction formula [8]: For u unipotent in G^F_s, we have

$$R^G_{T,\chi}(su) = \sum_{\bar{\gamma} \in \bar{N}_G(s,T)^F} \chi(\gamma^{-1} s \gamma) Q^{G_s}_{\gamma T \gamma^{-1}}(u). \tag{4.1}$$

The summation is over the set $\bar{N}_G(s,T)^F$ defined in (2.5), and for any reductive $\mathfrak{f}$-group H, and $S \in \mathcal{T}(H)$, the **Green function** Q^H_S on the unipotent set of H^F is defined by

$$Q^H_S(u) = R^H_{S,1}(u).$$

In this section we describe the summation over $\bar{N}_G(s,T)^F$ in (4.1) in terms of fibers of the map j_{G_s} studied in the previous two sections.

Breaking the sum (4.1) into $W_G(T)^F$-orbits, we have

$$R^G_{T,\chi}(su) = \sum_{\upsilon \in j^{-1}_{G_s}(\omega)} Q^{G_s}_{T_\upsilon}(u) \sum_{\bar{\gamma} \in \mathcal{O}_\upsilon} \chi(\gamma^{-1} s \gamma), \tag{4.2}$$

where $\omega = \mathrm{cl}(T, G)$, T_υ is any torus in $\mathcal{T}_\upsilon(G_s)$, and $\mathcal{O}_\upsilon$ is the $W_G(T)^F$-orbit in $\bar{N}_G(s,T)^F$ corresponding to $\upsilon \in j^{-1}_{G_s}(\omega)$ as in Lemma 2.2.

By the stabilizer assertion in Lemma 2.2, the inner sum in (4.2) can be written as follows. For any $\gamma \in N_G(s,T)^F$ and $\chi' \in \mathrm{Irr}(T^F)$, the value at s of the transported character

$${}^\gamma\chi' := \chi' \circ \mathrm{Ad}(\gamma^{-1}) \in \mathrm{Irr}({}^\gamma T^F)$$

depends only on the image $\bar{\gamma} \in \bar{N}_G(s,T)^F$. We have

$$\sum_{\bar{\gamma} \in \mathcal{O}_\upsilon} \chi(\gamma^{-1} s \gamma) = \frac{1}{|W_{G_s}(T_\upsilon)^F|} \sum_{x \in W_G(T)^F} {}^{\gamma x}\chi(s), \tag{4.3}$$

where γ on the right side of (4.3) is an arbitrary element of $N_G(s,T)^F$ such that $\bar{\gamma} \in \mathcal{O}_\upsilon$.

In our later computations with $R^G_{T,\chi}$ it will be useful to let s vary in G^F in such a way that G_s is unchanged. Let $Z(G_s)$ denote the center of G_s. For $\upsilon \in H^1(F, W_{G_s})$, the function

$$\chi_\upsilon := \sum_{\bar{\gamma} \in \mathcal{O}_\upsilon} {}^\gamma\chi \tag{4.4}$$

is well-defined on $Z(G_s)^F$, and we have

$$R^G_{T,\chi}(zu) = \sum_{v \in j^{-1}_{G_s}(\omega)} Q^{G_s}_{T_v}(u)\chi_v(z), \quad \text{if } G_z = G_s. \tag{4.5}$$

5. Multiplicity as a polynomial

In this section we begin the proof of Theorem 1.4 and will show that the multiplicity is given by a polynomial function. Let G be a connected reductive algebraic group over $\mathfrak{f}$. Let $H \subset G$ be a connected reductive $\mathfrak{f}$-subgroup of G, and let S be an F-stable maximal torus of H.

5.1. **Summation on H^F.** Suppose we are given a function $f : H^F \to \mathbb{C}$, invariant under conjugation by H^F, with the property that if $h \in H^F$ has Jordan decomposition $h = su$, then $f(h) = 0$ unless the conjugacy class $\operatorname{Ad}(H^F) \cdot s$ meets S. Our first aim is to express the sum of f over H^F as a sum of rational functions in q over an index set which does not depend on q.

Let H^{ss} and H^{upt} be the sets of semisimple and unipotent elements of H. Let $\mathcal{S}(H^F)$ and $\mathcal{U}(H^F)$ be the sets of $\operatorname{Ad}(H^F)$-orbits in $(H^{\mathrm{ss}})^F$ and $(H^{\mathrm{upt}})^F$, respectively.

By the vanishing assumption on f, we have

(5.1)
$$\begin{aligned}\frac{1}{|H^F|}\sum_{h\in H^F} f(h) &= \frac{1}{|H^F|}\sum_{s\in(H^{\mathrm{ss}})^F}\ \sum_{u\in(H_s^{\mathrm{upt}})^F} f(su)\\ &= \frac{1}{|H^F|}\sum_{s\in S^F}\frac{|\operatorname{Ad}(H^F)\cdot s|}{|\operatorname{Ad}(H^F)\cdot s\cap S|}\sum_{[u]\in\mathcal{U}(H_s^F)}|\operatorname{Ad}(H_s^F)\cdot u| f(su).\end{aligned}$$

The map $\gamma \mapsto s^\gamma$ induces a bijection

$$C_H(s)^F\backslash N_H(s,S)^F \xrightarrow{\sim} \operatorname{Ad}(H^F)\cdot s \cap S,$$

so that

$$|\operatorname{Ad}(H^F)\cdot s\cap S| = \frac{|N_H(s,S)^F|}{|C_H(s)^F|}.$$

Recalling that

$$\bar{N}_H(s,S)^F = H_s^F\backslash N_H(s,S)^F,$$

we get

$$\frac{1}{|H^F|}\sum_{h\in H^F} f(h) = \sum_{s\in S^F}\frac{1}{|\bar{N}_H(s,S)^F|}\sum_{[u]\in\mathcal{U}(H_s^F)}\frac{1}{|C_{H_s}(u)^F|}f(su). \tag{5.2}$$

5.2. **A partition of S.** To this point, the overgroup G has not played a role. Now G is used to partition the sum over S^F in (5.2), as follows. Let $I(S)$ be an index set for the set of subgroups

$$\{G_s :\ s \in S\}.$$

Note that each element of $I(S)$ is determined by a subset of the roots of S in G; hence $I(S)$ is finite. For $\iota \in I(S)$ let G_ι be the corresponding connected centralizer, and let

$$S_\iota := \{s \in S :\ G_s = G_\iota\}.$$

Thus, S is finitely partitioned as

$$S = \coprod_{\iota \in I(S)} S_\iota.$$

The F-action on S induces a permutation of $I(S)$, and we let $I(S)^F$ be the F-fixed points in $I(S)$. Note that if S_ι^F is nonempty, then $\iota \in I(S)^F$.

For $\iota \in I(S)$, we set

$$H_\iota := (H \cap G_\iota)^\circ,$$

which is none other than H_s for any $s \in S_\iota$.

Note that if $s \in S_\iota$, then $s \in S \cap C_H(s)$, which implies that

$$s \in H_\iota \subset G_\iota. \tag{5.3}$$

Returning to our sum (5.2), we now have

$$\frac{1}{|H^F|} \sum_{h \in H^F} f(h) = \sum_{\iota \in I(S)^F} \sum_{[u] \in \mathcal{U}(H_\iota^F)} \sum_{s \in S_\iota^F} \frac{1}{|\bar{N}_H(s,S)^F|} \frac{f(su)}{|C_{H_\iota}(u)^F|}. \tag{5.4}$$

5.3. **Restriction of Deligne-Lusztig characters.** We now consider the function f arising in our multiplicity formula. Let $H,\ S$ be as above, let T be an F-stable maximal torus of G, and let $\chi \in \mathrm{Irr}(T^F),\ \eta \in \mathrm{Irr}(S^F)$ be arbitrary characters.

Using the function $f : H^F \to \mathbb{C}$ given by

$$f(h) = \overline{R_{T,\chi}^G(h)} \cdot R_{S,\eta}^H(h), \tag{5.5}$$

we have

$$\langle R_{T,\chi}^G, R_{S,\eta}^H \rangle_{H^F} = \frac{1}{|H^F|} \sum_{h \in H^F} f(h). \tag{5.6}$$

The map

$$j_{G_s} : H^1(F, W_{G_s}) \longrightarrow H^1(F, W_G)$$

defined in (2.2) depends only on G_s, so we set

$$j_{G_\iota} := j_{G_s}, \qquad \text{for any } s \in S_\iota.$$

We have an analogous map

$$j_{H_\iota} : H^1(F, W_{H_\iota}) \longrightarrow H^1(F, W_H).$$

Likewise, the sets $\bar{N}_G(s,T)^F$ and $\bar{N}_H(s,S)^F$ depend only on ι, so we now write

$$\bar{N}_G(\iota,T)^F := \bar{N}_G(s,T)^F, \qquad \bar{N}_H(\iota,S)^F := \bar{N}_H(s,S)^F,$$

for $s \in S_\iota^F$.

Using (4.5) for G and H, along with (5.4), our multiplicity formula becomes

$$\langle R_{T,\chi}^G, R_{S,\eta}^H \rangle_{H^F} = \sum_{\substack{\iota \in I(S)^F \\ [u] \in \mathcal{U}(H_\iota^F)}} \sum_{\upsilon,\ \varsigma} \frac{\overline{Q_{T_\upsilon}^{G_\iota}(u)} Q_{S_\varsigma}^{H_\iota}(u)}{|\bar{N}_H(\iota,S)^F||C_{H_\iota}(u)^F|} \sum_{s \in S_\iota^F} \overline{\chi_\upsilon(s)} \eta_\varsigma(s), \tag{5.7}$$

where the middle sum runs over $\upsilon \in j_{G_\iota}^{-1}(\mathrm{cl}(T,G))$ and $\varsigma \in j_{H_\iota}^{-1}(\mathrm{cl}(S,H))$. The character sums χ_υ and η_ς are as defined in (4.4).

5.4. **Green functions.** We digress from our multiplicity formula (5.7) to recall the polynomial nature of Green functions Q_T^G, defined on the unipotent set of G^F, for a connected reductive $\mathfrak{f}$-group G with Frobenius F and F-stable maximal torus T in G.

For $u = 1$, we have

$$Q_T^G(1) = \epsilon_G(w)[G^F : T^F]_{p'}, \tag{5.8}$$

where $[G^F : T^F]_{p'}$ is the maximal divisor of the index $[G^F : T^F]$ which is prime to p, $w \in \mathrm{cl}(T, G)$ and $\epsilon_G : W_G \to \{\pm 1\}$ is the sign character of W_G. Note that $\epsilon_G(w) = (-1)^{\mathrm{rk}\, G + \mathrm{rk}\, T}$ [5, 7.5.2].

For $u \neq 1$, the Green functions $Q_T^G(u)$ can be expressed as polynomials which are known explicitly by tables for exceptional groups [3], [18] and for classical groups by recursive formulas [19] which can be implemented on a computer [9]. It will suffice for us to know the leading terms of these Green polynomials, which can be expressed in a uniform way.

Let $\mathcal{B}_G$ be the variety of Borel subgroups of G, and let $\mathcal{B}_G^u$ be the variety of u-fixed points in $\mathcal{B}_G$. The irreducible components of $\mathcal{B}_G^u$ all have the same dimension, and we set

$$d_G(u) := \dim \mathcal{B}_G^u.$$

Steinberg proved that

$$2d_G(u) = \dim C_G(u) - \overline{\mathrm{rk}}\, G, \tag{5.9}$$

where $\overline{\mathrm{rk}}\, G$ is the absolute rank of G.

Assume that p is a good prime for G. For each unipotent class $[u] \in \mathcal{U}(G^F)$ and twisted conjugacy class $[w] = \varsigma \in H^1(F, W_G)$, there is a polynomial

$$Q_{w,u}(t) = Q_{\varsigma,u}(t) \in \mathbb{Z}[t],$$

of degree at most $d_G(u)$, such that

$$Q_T^G(u) = Q_{w,u}(q)$$

if $\mathrm{cl}(T, G) = [w]$ (see [20] and the references therein).

The coefficient of $t^{d_G(u)}$ in $Q_{w,u}(t)$ is

$$\mathrm{tr}[w, H^{2d_G(u)}(\mathcal{B}_G^u)],$$

where w acts on the ℓ-adic cohomology of $\mathcal{B}_G^u$ via the Springer construction (see [21], [14], [16]).

If we take $u = 1$, then $d_G(1) = N$ is the number of positive roots of G and

$$Q_{w,1}^G(t) = \epsilon_G(w)t^N \ + \text{ lower powers of } t, \tag{5.10}$$

which is easily seen to be consistent with (5.8).

Suppose now that we replace F by F^ν for some $\nu \geq 1$. The G^{F^ν}-class of T is then represented by

$$(w\vartheta)^\nu \cdot \vartheta^{-\nu} \in W_G,$$

where ϑ is the automorphism of W_G induced by F. Suppose $\nu \equiv 1 \mod m$, where m is a positive integer divisible by the exponent of the finite group $W_G \rtimes \langle \vartheta \rangle$. This implies that $F^\nu = F$ on W_G and that $(w\vartheta)^\nu \cdot \vartheta^{-\nu} = w$ for all $w \in W_G$. It follows that $H^1(F, W_G) = H^1(F^\nu, W_G)$ and that the class $\mathrm{cl}(T, G)$ is the same with respect to F or F^ν.

Likewise, the class of u in G^F or G^{F^ν} is determined by the G-conjugacy class $C \subset G$ containing u, together with a class in $H^1(F, A_G(C))$ or $H^1(F^\nu, A_G(C))$, where $A_G(C)$ is the component group of the centralizer of some F-fixed element in C. As in the preceding paragraph, we may take m sufficiently divisible so that $F^\nu = F$ on $A_G(C)$ and that the class of u in G^F or G^{F^ν} corresponds to the same class in $H^1(F, A_G(C))$. We may choose m so that this holds for every C, since there are finitely many unipotent classes.

Let $Q^G_{T,\nu}$ be the Green function for T on G^{F^ν}. For m sufficiently divisible as in the previous two paragraphs and $\nu \equiv 1 \bmod m$ we have

$$Q^G_{T,\nu}(u) = Q_{w,u}(q^\nu).$$

(Note the difficulty with the exceptional class in E_8 is avoided since our conditions on m imply that ν is odd; see [20, Remark 6.2].)

5.5. **A progression of powers of Frobenius.** The indices of and terms of the summations in (5.7) depend on F, and we wish to remove this dependence for infinitely many powers of F, in order to represent the sum in (5.7) as the value of a rational function.

There is a positive integer m such that F^m acts trivially on the finite set $I(S)$ and the divisibility conditions on m from the previous section hold when G is replaced by G_ι or H_ι for every $\iota \in I(S)$.

In particular, m is divisible by the orders of the component groups $A_\iota(u)$ of the centralizers in H_ι of all unipotent elements $u \in H_\iota^F$ for every $\iota \in I(S)^F$, and F^m is the identity automorphism on $A_\iota(u)$ for all such ι and u. This implies that for each $\iota \in I(S)^F$ and $[u] \in \mathcal{U}(H_\iota^F)$, there is a polynomial $P_{\iota,u}(t) \in \mathbb{Z}[t]$, of degree equal to $\dim C_{H_\iota}(u)$, such that

$$|C_{H_\iota}(u)^{F^\nu}| = P_{\iota,u}(q^\nu) \tag{5.11}$$

for all $\nu \equiv 1 \bmod m$. Moreover, each polynomial $P_{\iota,u}(t)$ is of the form $|A_\iota(u)|$ times a monic polynomial in $\mathbb{Z}[t]$.

The above conditions on m also ensure that the indices in the outer two summations in (5.7), as well as the quantity $|\bar{N}_H(\iota, S)^F|$ are unchanged if F is replaced by F^ν for $\nu \equiv 1 \bmod m$.

To handle the inner sum, we add more conditions: in the next section we will define certain subgroups Z_J of S, indexed by subsets $J \subset I(S)^F$. We also insist that m be divisible by $|Z_J/Z_J^\circ|$ and that F^m act trivially on Z_J/Z_J° for each $J \subset I(S)^F$.

5.6. **Character sums.** In order to interpret the inner sum of (5.7) as a rational function, we shall replace each summand S_ι^F by the group Z_ι^F, where

$$Z_\iota := Z(G_\iota) \cap S. \tag{5.12}$$

It is easy to check that

$$\begin{aligned} &Z_\iota \subset Z(H_\iota),\\ &S_\iota \subset Z_\iota \subset S,\\ &G_\iota = C_G(Z_\iota)^\circ. \end{aligned} \tag{5.13}$$

Let χ_υ and η_ς be the character sums appearing in (5.7). Our aim is to express the sum

$$\frac{1}{|Z_\iota^F|}\sum_{s\in S_\iota^F}\overline{\chi_\upsilon(s)}\eta_\varsigma(s) \tag{5.14}$$

as the value of a rational function.

Define a partial ordering on $I(S)$ by

$$\iota' \leq \iota \qquad \Leftrightarrow \qquad G_\iota \subseteq G_{\iota'}.$$

Equivalently, we have

$$\iota' \leq \iota \qquad \Leftrightarrow \qquad Z_{\iota'} \subseteq Z_\iota.$$

Let

$$Y_\iota := Z_\iota - S_\iota$$

be the complement of S_ι in Z_ι.

Lemma 5.1. *For every $\iota \in I(S)$ we have*

$$Y_\iota = \bigcup_{\iota'<\iota} Z_{\iota'}.$$

Proof. Let $s \in Y_\iota$. Then $s \in S_{\iota'}$ for some $\iota' \in I(S)$, with $\iota' \neq \iota$, so $s \in Z_{\iota'}$. Since $Y_\iota \subset Z_\iota$, we have

$$G_\iota = C_G^\circ(Z_\iota) \subset C_G^\circ(Y_\iota) \subseteq G_s = G_{\iota'},$$

so $\iota' < \iota$.

Conversely, let $s \in Z_{\iota'}$, with $\iota' < \iota$. Note that $s \in Z_\iota$. If $s \notin Y_\iota$, then $s \in S_\iota$. This implies that

$$G_{\iota'} = C_G(Z_{\iota'})^\circ \subset G_s = G_\iota,$$

contradicting $\iota' < \iota$. This proves the lemma. □

For a subset $J \subseteq I(S)$, let

$$Z_J = \bigcap_{\iota'\in J} Z_{\iota'}.$$

There is a polynomial $f_J \in \mathbb{C}[t]$ of degree $\dim Z_J$ such that

$$f_J(q^\nu) = |Z_J^{F^\nu}|, \qquad \text{for all } \nu \equiv 1 \bmod m.$$

For $\nu \geq 1$, let

$$N_\nu^{T_\upsilon} : T_\upsilon^{F^\nu} \longrightarrow T_\upsilon^F, \qquad N_\nu^{S_\varsigma} : S_\varsigma^{F^\nu} \longrightarrow S_\varsigma^F$$

be the norm mappings. These are surjective. Set

$$\chi_\upsilon^{(\nu)} := \chi_\upsilon \circ N_\nu^{T_\upsilon}, \qquad \eta_\varsigma^{(\nu)} := \eta_\varsigma \circ N_\nu^{S_\varsigma}.$$

Assume that $\iota \in J \subset I(S)^F$. Then Z_J is F-stable and

$$Z_J^F \subset Z_\iota^F \subset Z(G_\iota)^F \cap Z(H_\iota)^F.$$

Both χ_υ and η_ς are defined on the latter group (see (4.4)), so we may restrict them to Z_J^F. Our conditions on m at the end of section 5.5 ensure that the restricted norm mapping

$$N_\nu^{S_\varsigma} : Z_J^{F^\nu} \longrightarrow Z_J^F$$

is also surjective. This implies, for all integers $\nu \equiv 1 \bmod m$, that

$$\langle \chi_\upsilon^{(\nu)}, \eta_\varsigma^{(\nu)} \rangle_{Z_J^{F^\nu}} = \langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_J^F}.$$

Hence for each J, we have

$$\sum_{z\in Z_J^{F^\nu}} \overline{\chi_\upsilon^{(\nu)}(z)}\eta_\varsigma^{(\nu)}(z) = \langle \chi_\upsilon, \eta_\varsigma\rangle_{Z_J^F} \cdot f_J(q^\nu). \tag{5.15}$$

Let $I(\iota, S) := \{\iota' \in I(S) :\ \iota' < \iota\}$. It now follows from Möbius inversion that the rational function

$$\Theta_{\iota,\upsilon,\varsigma}(t) := \langle \chi_\upsilon, \eta_\varsigma\rangle_{Z_\iota^F} + \sum_{J\subseteq I(\iota,S)^F} (-1)^{|J|}\langle \chi_\upsilon, \eta_\varsigma\rangle_{Z_J^F}\frac{f_J(t)}{f_\iota(t)} \tag{5.16}$$

has the property that

$$\Theta_{\iota,\upsilon,\varsigma}(q^\nu) = \frac{1}{|Z_\iota^{F^\nu}|}\sum_{s\in S_\iota^{F^\nu}} \overline{\chi_\upsilon^{(\nu)}(s)}\eta_\varsigma^{(\nu)}(s),$$

for all $\nu \equiv 1 \bmod m$. Since $\dim Z_J \leq \dim Z_\iota$ for all $J \subseteq I(\iota, S)^F$, we have

$$\deg \Theta_{\iota,\upsilon,\varsigma} \leq 0. \tag{5.17}$$

5.7. **Multiplicity as a polynomial.** We return to our multiplicity formula (5.7). We have shown that

$$\langle R^G_{T,\chi}, R^H_{S,\eta}\rangle_{H^F} = \sum_\alpha \Psi_\alpha(q)\Theta_\alpha(q), \tag{5.18}$$

where α runs over quadruples $\alpha = (\iota, u, \upsilon, \varsigma)$, with

$$\iota \in I(S)^F, \quad [u] \in \mathcal{U}(H_\iota^F), \quad \upsilon \in j^{-1}_{G_\iota}(\mathrm{cl}(T, G)), \quad \varsigma \in j^{-1}_{H_\iota}(\mathrm{cl}(S, H)), \tag{5.19}$$

$\Theta_\alpha(t) = \Theta_{\iota,\upsilon,\varsigma}(t)$ is the rational function defined in (5.16) and $\Psi_\alpha(t)$ is the rational function defined by

$$\Psi_\alpha(t) = f_\iota(t) \cdot \frac{\overline{Q^{G_\iota}_{\upsilon,u}(t)}Q^{H_\iota}_{\varsigma,u}(t)}{|\bar{N}_H(\iota, S)^F||P_{\iota,u}(t)|}. \tag{5.20}$$

Here $Q^{G_\iota}_{\upsilon,u}(t)$ and $Q^{H_\iota}_{\varsigma,u}(t)$ are the Green polynomials from section 5.4 and $P_{\iota,u}(t)$ is the polynomial from (5.11).

If F is replaced by F^ν with $\nu \equiv 1 \bmod m$, where m is as in section 5.5, the summation indices α are unchanged, so that the rational function

$$M(t) := \sum_\alpha \Psi_\alpha(t)\Theta_\alpha(t) \tag{5.21}$$

has the property that

$$\langle R^G_{T,\chi^{(\nu)}}, R^H_{S,\eta^{(\nu)}}\rangle_{H^{F^\nu}} = M(q^\nu), \tag{5.22}$$

for all $\nu \equiv 1 \bmod m$. In particular, $M(q^\nu)$ is an integer for all $\nu \equiv 1 \bmod m$.

We next observe that the numerator of each term in $M(t)$ belongs to $\mathbb{Z}[t]$, and the denominator of each term in $M(t)$ is an integer times a monic polynomial in $\mathbb{Z}[t]$. Hence there is $a \in \mathbb{Z}$ such that

$$aM(t) = \frac{f(t)}{g(t)},$$

where $f(t)$ and $g(t)$ are in $\mathbb{Z}[t]$ and $g(t)$ is monic. We can therefore write

$$aM(t) = p(t) + r(t),$$

where $p(t) \in \mathbb{Z}[t]$ and $r(t)$ is a rational function of negative degree. On the other hand,

$$r(q^\nu) = aM(q^\nu) - p(q^\nu) = a\langle R^G_{T,\chi^{(\nu)}}, R^H_{S,\eta^{(\nu)}}\rangle_{H^{F^\nu}} - p(q^\nu) \in \mathbb{Z}$$

for all $\nu \equiv 1 \bmod m$. Since $r(q^\nu) \to 0$ as $\nu \to \infty$, we must have $r(t) \equiv 0$, so

$$M(t) = \frac{1}{a} p(t).$$

This shows that $M(t)$ is a polynomial, as claimed.

6. Complexity and the degree of $M(t)$

From now on, the algebraic group G is simple. That is, the center $Z(G)$ is finite and contains every normal subgroup of G. Recall that the complexity δ is the minimum codimension of a B_H-orbit in G/B. In this section we will complete the proof of the first assertion of Theorem 1.4 by showing that δ is an upper bound on the degree of the multiplicity polynomial $M(t)$ defined in (5.21).

6.1. **A formula for the complexity.** In this section we show that δ has the simplest conceivable formula. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. We are assuming that $\mathfrak{g}$ is simple. We also invoke Assumption 1.3. That is, we assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, stable under $\mathrm{Ad}(H)$, and that there is a nondegenerate $\mathrm{Ad}(H)$-invariant symmetric form B on $\mathfrak{m}$. Hence Ad restricts to a homomorphism

$$\mathrm{Ad} : H \longrightarrow SO(\mathfrak{m}).$$

Lemma 6.1. *Assume that $H \neq G$. Then*

$$\ker[\mathrm{Ad} : H \to SO(\mathfrak{m})] = Z(G) \cap H.$$

Proof. Containment "$\supseteq$" is clear. We prove containment "$\subseteq$". Set

$$N := \ker[\mathrm{Ad} : H \to SO(\mathfrak{m})]$$

and let $\mathfrak{n}$ be the Lie algebra of N. We have

$$\mathfrak{n} = \ker[\mathrm{ad} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})],$$

so $\mathfrak{n}$ is an ideal in $\mathfrak{h}$. But $[\mathfrak{n}, \mathfrak{m}] = 0$, so $\mathfrak{n}$ is in fact an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and not equal to $\mathfrak{h}$, we have $\mathfrak{n} = 0$. Hence N is a finite normal algebraic subgroup of H. By [4, 22.1], N is central in H, hence $\mathrm{Ad}(N)$ acts trivially on $\mathfrak{h}$, as well as on $\mathfrak{m}$. It follows that N is central in G. This completes the proof. ☐

Let B and B_H be Borel subgroups of G and H, respectively. Let U and V be their respective unipotent radicals. After conjugating, we may assume that

$$B_H = SV, \qquad B = TU$$

with

$$S \subset T, \qquad V \subset U.$$

Proposition 6.2. *The complexity δ is given by*

$$\delta = \begin{cases} \dim G/B - \dim B_H & \text{if } H \neq G, \\ 0 & \text{if } H = G. \end{cases}$$

Proof. If $H = G$, the fact that $\delta = 0$ is clear from the Bruhat decomposition. Assume from now on that $H \neq G$. We must show that B_H has an orbit in G/B with finite stabilizers. Let w be the element of $W_G(T)$ such that ${}^wB \cap B = T$. Then every element of UwB/B can be uniquely expressed as uwB for $u \in U$. For $v \in V,\ s \in S$, we have

$$vs \cdot uwB = v(sus^{-1})wB.$$

By uniqueness of expression, vs fixes uwB if and only if $v = usu^{-1}s^{-1}$. It follows that the projection $B_H \to S$ gives an isomorphism from the B_H-stabilizer of uwB to the S-stabilizer of $u^{-1}V$ in the quotient variety U/V. We will show there exists $u \in U$ such that the latter stabilizer is finite.

Denote the Lie algebras of U, V, T, S by $\mathfrak{u}, \mathfrak{v}, \mathfrak{t}, \mathfrak{s}$. The tangent space to U/V at eV is $\mathfrak{u}/\mathfrak{v}$. We have

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{t}/\mathfrak{s} \oplus \mathfrak{u}/\mathfrak{v} \oplus \bar{\mathfrak{u}}/\bar{\mathfrak{v}},$$

where $\bar{\mathfrak{u}} = \mathrm{Ad}(w)\mathfrak{u}$ is the opposite nilradical of $\mathfrak{u}$ and $\bar{\mathfrak{v}}$ is the opposite nilradical of $\mathfrak{v}$.

Since $\ker[\mathrm{Ad} : S \to GL(\mathfrak{g}/\mathfrak{h})]$ is finite by Lemma 6.1, it follows that $\ker[\mathrm{Ad} : S \to GL(\mathfrak{u}/\mathfrak{v})]$ is finite. This latter kernel is the set of common zeros of the roots $\Phi(S, U/V)$ of S in $\mathfrak{u}/\mathfrak{v}$ (see [4, 8.17]). We have

$$\mathfrak{u}/\mathfrak{v} = (\mathfrak{u}/\mathfrak{v})^S \oplus \sum_{\alpha \in \Phi(S,U/V)} (\mathfrak{u}/\mathfrak{v})_\alpha.$$

A vector in $\mathfrak{u}/\mathfrak{v}$ whose α-component is nonzero for every $\alpha \in \Phi(S, U/V)$ will therefore have finite stabilizer in S. Proposition 6.2 now follows from a basic result:

Lemma 6.3. *Let k be an algebraically closed field. Suppose a k-torus S acts on a smooth irreducible affine k-variety X, fixing a point $x \in X$, so that S acts on the tangent space T_xX at x. If there exists $v \in T_xX$ having finite stabilizer $S_v \subset S$, then there exists $y \in X$ having finite stabilizer $S_y \subset S$.*

This lemma can be proved as follows. Since the torus S acts completely reducibly on the coordinate ring $k[X]$, the argument of Lemme 1 in [15] shows that there is an S-equivariant morphism $\varphi : X \to T_xX$ such that $\varphi(x) = 0$, and whose differential $d\varphi_x : T_xX \to T_xX$ is bijective. The set $\mathcal{U}$ of points in T_xX with finite stabilizers is open, and nonempty by hypothesis. Since φ is dominant, the preimage $\varphi^{-1}(\mathcal{U})$ is nonempty. If $y \in \varphi^{-1}(\mathcal{U})$, then $S_y \subseteq S_{\varphi(y)}$, and the latter stabilizer is finite.

Lemma 6.3 can also be proved using a T-equivariant embedding of X in a linear representation of T. $\square$

6.2. **Degree of $\Psi_\alpha(t)$.** We return now to our rational function

$$\Psi_\alpha(t) = f_\iota(t) \cdot \frac{\overline{Q^{G_\iota}_{\upsilon,u}(t)}Q^{H_\iota}_{\varsigma,u}(t)}{|\bar{N}_H(\iota, S)^F||P_{\iota,u}(t)|}.$$

We have

$$\deg|P_{\iota,u}(t)| = \dim C_{H_\iota}(u), \qquad \deg|f_\iota(t)| = \dim Z_\iota. \tag{6.1}$$

From section 5.4 and equation (5.9) we find that

$$\begin{aligned}\deg \Psi_\alpha(t) &\le \dim Z_\iota + d_{G_\iota}(u) + d_{H_\iota}(u) - \dim C_{H_\iota}(u) \\ &= \dim Z_\iota + \tfrac{1}{2}\left[\dim C_{G_\iota}(u) - \dim C_{H_\iota}(u) - \overline{\mathrm{rk}}\, G - \overline{\mathrm{rk}}\, H\right].\end{aligned} \tag{6.2}$$

The fixed point spaces $\mathfrak{g}^s, \mathfrak{h}^s, \mathfrak{m}^s$ are the same for any $s \in S_\iota$; we denote them by $\mathfrak{g}_\iota, \mathfrak{h}_\iota, \mathfrak{m}_\iota$. Thus we have an $\mathrm{Ad}(H)$-stable decomposition

$$\mathfrak{g}_\iota = \mathfrak{h}_\iota \oplus \mathfrak{m}_\iota$$

and

$$\begin{aligned} \dim C_{G_\iota}(u) - \dim C_{H_\iota}(u) &= \dim \mathfrak{m}_\iota^u \\ &\leq \dim \mathfrak{m}_\iota \\ &= \dim C_{G_\iota}(1) - \dim C_{H_\iota}(1). \end{aligned} \tag{6.3}$$

Define

$$\begin{aligned} \delta_\iota &:= \dim Z_\iota + \dim \mathcal{B}_{G_\iota} - \dim \mathcal{B}_{H_\iota} - \dim S \\ &= \dim Z_\iota + \tfrac{1}{2}[\dim \mathfrak{m}_\iota - \overline{\mathrm{rk}}\, G - \overline{\mathrm{rk}}\, H]. \end{aligned} \tag{6.4}$$

For example, if ι_0 is the minimal element of $I(S)$, then

$$G_{\iota_0} = G, \quad Z_{\iota_0} = Z(G) \cap S \quad \text{and } \mathfrak{m}_{\iota_0} = \mathfrak{m}. \tag{6.5}$$

Since $Z(G)$ is finite, Proposition 6.2 implies that

$$\delta_{\iota_0} = \tfrac{1}{2}[\dim \mathfrak{m} - \overline{\mathrm{rk}}\, G - \overline{\mathrm{rk}}\, H] = \delta, \qquad \text{if } H \neq G. \tag{6.6}$$

Lemma 6.4. *We have* $\deg \Psi_\alpha(t) \leq \delta_\iota$*, with equality only if* $u = 1$.

Proof. The inequality follows from (6.2) and (6.3), and the last assertion follows from section 6.1. □

We now seek a bound on $\deg \Psi_\alpha$ which is independent of ι. We will show that $\delta_\iota \leq \delta$, and that equality holds only in rather special circumstances.

Let $\mathfrak{m}'_\iota$ be the sum of the eigenspaces of $\mathrm{Ad}(s)$ in $\mathfrak{m}$ with eigenvalues $\neq 1$, for any $s \in S_\iota$. Since $\det \mathrm{Ad}(H) = 1$ on $\mathfrak{m}$, the dimension $\dim \mathfrak{m}'_\iota$ is even. We have $\mathfrak{m} = \mathfrak{m}_\iota \oplus \mathfrak{m}'_\iota$, the form B is nondegenerate on $\mathfrak{m}'_\iota$, and $\mathrm{Ad} : H \to SO(\mathfrak{m})$ restricts to a homomorphism $\mathrm{Ad}_\iota : H_\iota \to SO(\mathfrak{m}'_\iota)$.

Lemma 6.5. *For every* $\iota \in I(S)$ *we have* $\delta_\iota \leq \delta$*. Moreover, if* $H \neq G$*, then the following are equivalent:*

(1) $\delta_\iota = \delta$*;*
(2) $\dim(Z_\iota) = \frac{1}{2} \dim \mathfrak{m}'_\iota$*;*
(3) $\mathrm{Ad}_\iota(Z_\iota)$ *is a maximal torus in* $SO(\mathfrak{m}'_\iota)$*.*

When these hold, the derived group of H_ι *acts trivially on* $\mathfrak{m}'_\iota$*.*

Proof. If $H = G$, then $\delta = 0$ and

$$\delta_\iota = \dim Z_\iota - \dim S \leq 0.$$

From now on assume $H \neq G$. From (6.4) and (6.6), we have

$$\delta - \delta_\iota = \tfrac{1}{2} \dim \mathfrak{m}'_\iota - \dim Z_\iota. \tag{6.7}$$

Now, the group

$$N_\iota := \ker[\mathrm{Ad}_\iota : Z_\iota \longrightarrow SO(\mathfrak{m}'_\iota)]$$

is finite. Indeed, since $Z_\iota \subset Z(G_\iota)$, it follows that N_ι centralizes $\mathfrak{m}_\iota$, as well as $\mathfrak{m}'_\iota$. Hence we have

$$N_\iota \subseteq \ker[\mathrm{Ad} : H \longrightarrow SO(\mathfrak{m})] = Z(G) \cap H,$$

the latter equality being from Lemma 6.1. Hence $N_\iota \subset Z(G)$, and the latter is finite since G is simple.

Since $\mathrm{Ad}(Z_\iota^\circ)$ is a torus in $SO(\mathfrak{m}'_\iota)$ and $\frac{1}{2}\dim\mathfrak{m}'_\iota$ is the dimension of a maximal torus in $SO(\mathfrak{m}'_\iota)$, this proves that both sides of (6.7) are ≥ 0 and that (1)–(3) are equivalent.

For the last assertion, recall that $Z_\iota \subset Z(H_\iota)$. If (1)–(3) hold, then $\mathrm{Ad}_\iota(H_\iota)$ centralizes a maximal torus in $SO(\mathfrak{m}'_\iota)$, hence is contained in that torus. □

With this lemma, the first assertion of Theorem 1.4 has been proved.

6.3. **A remark on the multiplicity formula.** The formula (5.21), as written, contains more terms than are necessary. For, if we write

$$\Psi_\alpha(t)\Theta_\alpha(t) = P_\alpha(t) + R_\alpha(t),$$

where $P_\alpha(t)$ is a polynomial and $\deg R_\alpha(t) < 0$, then

$$M(t) = \sum_\alpha P_\alpha(t) \qquad \text{and} \qquad \sum_\alpha R_\alpha(t) = 0,$$

since $M(t)$ is a polynomial. From (5.17) and Lemma 6.4 we have $\deg P_\alpha \leq \delta_\iota$, where $\alpha = (\iota, u, \upsilon, \varsigma)$. It follows that

$$\langle R^G_{T,\chi}, R^H_{S,\eta}\rangle_{H^F} = M(q) = \sum_\alpha P_\alpha(q), \tag{6.8}$$

where the sum is over just those $\alpha = (\iota, u, \upsilon, \varsigma)$ such that $\delta_\iota \geq 0$.

7. The leading term of $M(t)$

We have shown that the multiplicity polynomial $M(t)$ has the form

$$M(t) = At^\delta + (\text{lower powers of } t).$$

In this section we find an explicit and effective formula for the leading term A of $M(t)$. Recall from (5.21) that

$$M(t) = \sum_\alpha \Psi_\alpha(t)\Theta_\alpha(t),$$

where α runs over quadruples $(\iota, u, \upsilon, \varsigma)$ as in (5.19),

$$\Psi_\alpha(t) = f_\iota(t) \cdot \frac{\overline{Q^{G_\iota}_{\upsilon,u}(t)}\, Q^{H_\iota}_{\varsigma,u}(t)}{|\bar{N}_H(\iota,S)^F||P_{\iota,u}(t)|}$$

and

$$\Theta_\alpha(t) = \langle \chi_\upsilon, \eta_\varsigma\rangle_{Z_\iota^F} + \sum_{J\subseteq I(\iota,S)^F} (-1)^{|J|}\langle \chi_\upsilon, \eta_\varsigma\rangle_{Z_J^F}\frac{f_J(t)}{f_\iota(t)}.$$

By Lemmas 6.4 and 6.5, only quadruples α with $u = 1$ and $\delta_\iota = \delta$ contribute to the leading term; henceforth we assume α is of this form. As a power series in t, we then have

$$\Psi_\alpha(t) = A_\alpha t^\delta + (\text{lower degree terms}),$$

where

$$A_\alpha = [Z_\iota^F : Z_\iota^{\circ F}] \cdot \frac{(-1)^{\mathrm{rk}(G_\iota)+\mathrm{rk}(T)+\mathrm{rk}(H_\iota)+\mathrm{rk}(S)}}{|\bar{N}_H(\iota,S)^F|}. \tag{7.1}$$

At first glance, each function $\Theta_\alpha(t)$ could contribute many terms to A, coming from various $\iota' < \iota$ with $\dim Z_{\iota'} = \dim Z_\iota$, since Z_ι may be disconnected. We now show that in fact $\Theta_\alpha(t)$ contributes only one term.

Lemma 7.1. *If $\delta_\iota = \delta$ and $\iota' < \iota$, then* $\dim Z_{\iota'} < \dim Z_\iota$.

Proof. If $H = G$, we have $\delta_\iota = \dim Z_\iota - \dim S \leq 0 = \delta$ with equality iff $Z_\iota = S$. The lemma holds since S is connected.

Now assume $H \neq G$. Suppose $\delta_\iota = \delta$ and $\iota' < \iota$, yet $\dim Z_{\iota'} = \dim Z_\iota$. Then

$$Z_\iota^\circ \subseteq Z_{\iota'} \subset Z_\iota. \tag{7.2}$$

From Lemma 6.5, the image $\mathrm{Ad}_\iota(Z_\iota)$ is a maximal torus in $SO(\mathfrak{m}'_\iota)$. It follows that

$$\mathrm{Ad}_\iota(Z_\iota^\circ) = \mathrm{Ad}(Z_\iota).$$

Thus, for each $z \in Z_\iota$ there is $z_0 \in Z_\iota^\circ$ such that

$$z_1 := zz_0^{-1} \in \ker[\mathrm{Ad}_\iota : Z_\iota \to SO(\mathfrak{m}'_\iota)].$$

By Lemma 6.1, we have $z_1 \in Z(G) \cap H$. Hence

$$z = z_0 z_1 \in Z_\iota^\circ \cdot (Z(G) \cap H). \tag{7.3}$$

We have shown that

$$Z_\iota = Z_\iota^\circ \cdot (Z(G) \cap H). \tag{7.4}$$

Now S_ι is stable under multiplication by $Z(G) \cap H$. Moreover, S_ι is open in Z_ι, so S_ι meets some connected component of Z_ι in an open dense set. But then (7.4) implies that S_ι meets *every* connected component of Z_ι in an open dense set.

Likewise, $S_{\iota'}$ meets some component of $Z_{\iota'}$ in an open dense set. By (7.2), every such component of $Z_{\iota'}$ is also a component of Z_ι. Therefore S_ι and $S_{\iota'}$ meet a common component of Z_ι in a dense open set. This implies that $S_\iota \cap S_{\iota'}$ is nonempty; hence $\iota = \iota'$, contradicting $\iota' < \iota$. □

As an aside, we mention the following consequence of (7.4) which simplifies our eventual formula for A when G is adjoint.

Lemma 7.2. *Suppose G is simple adjoint. If $\delta_\iota = \delta$, then Z_ι is connected.*

Return now to $\Theta_\alpha(t)$. For each $J \subseteq I(\iota, S)$, the subgroup Z_J is contained in some $Z_{\iota'}$ with $\iota' < \iota$. Lemma 7.1 implies that

$$\deg f_J(t) < \deg f_\iota(t),$$

which shows that the leading term of $\Theta_\alpha(t)$ has the following simple form.

Corollary 7.3. *Let $\alpha = (\iota, 1, \upsilon, \varsigma)$ be a quadruple appearing in $M(t)$ with $\delta_\iota = \delta$. Then*

$$\Theta_\alpha(\infty) = \langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_\iota^F}.$$

From (7.1) and Corollary 7.3 we get the following expression for the leading term A.

Proposition 7.4. *The leading term A of $M(t)$ in Theorem* 1.4 *is given by $A = \sum_\iota A_\iota$, where ι runs over those $\iota \in I(S)^F$ with $\delta_\iota = \delta$, and*

$$A_\iota = (-1)^{\mathrm{rk}(G_\iota)+\mathrm{rk}(T)+\mathrm{rk}(H_\iota)+\mathrm{rk}(S)} \cdot \frac{|Z_\iota^F / Z_\iota^{\circ F}|}{|\bar{N}_H(\iota, S)|} \cdot \sum_{\upsilon,\varsigma} \langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_\iota^F}.$$

In the last summation, υ and ς run over $j_{G_\iota}^{-1}(\mathrm{cl}(T, G))$ and $j_{H_\iota}^{-1}(\mathrm{cl}(S, H))$, respectively.

As a simple illustration of Proposition 7.4, we show how it reduces to the Deligne-Lusztig inner-product formula [8, thm. 6.8] when $G = H$. For $\iota \in I(S)$, we have then

$$\delta_\iota = \dim Z(G_\iota) - \dim S \leq 0 = \delta$$

with equality iff $G_\iota = S = H_\iota$. This means ι is the maximal element of $I(S)^F$, and $M(t) = A = A_\iota$ is the inner product $\langle R^G_{T,\chi}, R^G_{S,\eta}\rangle_{G^F}$.

By Proposition 7.4, if T is not G^F conjugate to S, then $j^{-1}_{G_\iota}(\mathrm{cl}(T,G)) = \varnothing$, so $A = 0$. Otherwise we may take $S = T$, and the fiber of j_{G_ι} over $\mathrm{cl}(S,G)$ is the singleton $\{\upsilon\}$ corresponding to the class of S in itself. We have

$$\chi_\upsilon = \sum_{w \in W_G(S)^F} {}^w\chi, \qquad \eta_\upsilon = \sum_{w \in W_G(S)^F} {}^w\eta, \qquad \bar{N}_G(\iota,S)^F = W_G(S)^F,$$

and the result:

$$A = \frac{\langle \chi_\upsilon, \eta_\upsilon\rangle_{S^F}}{|W_G(S)^F|}$$

is the original Deligne-Lusztig formula for $\langle R^G_{S,\chi}, R^G_{S,\eta}\rangle_{G^F}$.

8. Optimality

Recall that G is simple. In this section we show that the degree δ is optimal. We may assume that $H \neq G$. Let $T \subset G$ and $S \subset H$ be arbitrary F-stable maximal tori. We will show that for sufficiently large q, there are characters $\chi \in \mathrm{Irr}(T^F)$ and $\eta \in \mathrm{Irr}(S^F)$ such that the leading coefficient A is nonzero. In fact, we can take η to be the trivial character.

For each $\iota \in I(S)^F$ with $A_\iota \neq 0$, the fiber $j^{-1}_{G_\iota}(\mathrm{cl}(T,G))$ is nonempty. This means that Z_ι is G^F-conjugate to a subgroup $\tilde{Z}_\iota \subset T$. There are only finitely many of these subgroups $\tilde{Z}_\iota$. Recall from (6.5) that $\iota_0 \in I(S)^F$ is the minimal element, for which $Z_{\iota_0} = Z(G) \cap S$. If $\dim Z_\iota = 0$ and $\delta_\iota = \delta$, then Lemma 7.1 implies that $\iota = \iota_0$. Hence, if $\iota \neq \iota_0$, the torus $T/\tilde{Z}^\circ_\iota$ has strictly smaller dimension than that of T, so that $\left|\mathrm{Irr}\left(T^F/\tilde{Z}^{\circ F}_\iota\right)\right|$ is a polynomial in q of degree strictly less than $\dim T$. Hence for sufficiently large q there are characters $\chi \in \mathrm{Irr}(T^F)$ which are trivial on $Z^F_{\iota_0}$ and nontrivial on every $\tilde{Z}^F_\iota$ for $\iota \neq \iota_0$. We call these χ **very regular**. For very regular χ and ι such that $A_\iota \neq 0$, we have

$$\langle \chi_\upsilon, 1\rangle_{Z^F_\iota} = \begin{cases} 1 & \text{if } \iota = \iota_0, \\ 0 & \text{if } \iota \neq \iota_0. \end{cases} \tag{8.1}$$

It follows that for χ very regular, and $\eta = 1$, the coefficient A of t^δ in $M(t)$ is given by

$$A = \epsilon_G(x)\epsilon_H(y), \tag{8.2}$$

where $x \in \mathrm{cl}(T,G)$ and $y \in \mathrm{cl}(S,H)$.

Let ϑ be the automorphism of W_H induced by F and let ψ be the character of an irreducible representation of $\langle\vartheta\rangle \ltimes W_H$. For each $y \in W_H$, choose an F-stable torus S_y in H such that $y \in \mathrm{cl}(S_y, H)$. We have then a class function R^H_ψ of H^F defined by

$$R^H_\psi = \frac{1}{|W_H|}\sum_{y \in W_H} \psi(\vartheta y) R^H_{S_y,1}.$$

For example, the trivial (1) and sign (ϵ_H) characters of W_H extend to $\langle\vartheta\rangle \ltimes W_H$ (trivially on ϑ). It is known (cf. [5, 7.6]) that $R_1^H = 1_H$ and $R_{\epsilon_H}^H = St_H$ are the trivial and Steinberg characters of H^F, respectively. For very regular χ, (8.2) implies that

$$\langle R^G_{T,\chi}, R^H_\psi\rangle_{H^F} = \epsilon_G(x)\langle\epsilon_H,\psi\rangle_{W_H}\cdot q^\delta + \text{ (lower powers of } q). \tag{8.3}$$

In particular, we have

$$\langle\epsilon_G(x)R^G_{T,\chi}, St_H\rangle_{H^F} = q^\delta + \text{ (lower powers of } q),$$

while $\langle\epsilon_G(x)R^G_{T,\chi}, 1_H\rangle_{H^F}$ has degree $< \delta$. This last result is to be expected, in view of the results in [2] and [17].

9. Restriction from SO_{2n+1} to SO_{2n}.

We return to the situation at the beginning of the introduction. So $p > 2$ and (V, Q) is a $(2n+1)$-dimensional quadratic $\mathfrak{F}$-space, defined over $\mathfrak{f}$, with Frobenius F. Fix $v \in V^F$ with $Q(v) \neq 0$, and let U be the orthogonal space of v in V. We take

$$G = SO(V), \qquad H = G_v = SO(U),$$

with $\mathfrak{f}$-structure on both groups induced from that on V. Assumption 1.3 holds: we may identify the quadratic spaces $(\mathfrak{m}, B) = (U, Q)$.

Let T, S be F-stable maximal tori in G and H respectively, and let $\chi \in \mathrm{Irr}(T^F)$, $\eta \in \mathrm{Irr}(S^F)$ be characters, which for the moment are arbitrary.

We have

$$\delta = \dim\mathcal{B}_G - \dim\mathcal{B}_H - \dim\overline{\mathrm{rk}}\, H = n^2 - (n^2-n) - n = 0.$$

From now on, we only consider $\iota \in I(S)^F$ with $\delta_\iota = 0$. Since G is adjoint, each such Z_ι is connected, by Lemma 7.2. Proposition 7.4 gives the multiplicity formula

$$(-1)^{\mathrm{rk}\,T+\mathrm{rk}\,S}\langle R^G_{T,\chi}, R^H_{S,\eta}\rangle_{H^F} = \sum_{\substack{\iota\in I(S)^F\\ \delta_\iota=0}} \frac{(-1)^{\mathrm{rk}(G_\iota)+\mathrm{rk}(H_\iota)}}{|\bar{N}_H(\iota,S)|}\cdot\sum_{\upsilon,\varsigma}\langle\chi_\upsilon,\eta_\varsigma\rangle_{Z_\iota^F}, \tag{9.1}$$

where υ and ς run over $j^{-1}_{G_\iota}(\mathrm{cl}(T,G))$ and $j^{-1}_{H_\iota}(\mathrm{cl}(S,H))$, respectively.

The connectedness of Z_ι implies that -1 is not an eigenvalue of any $s \in S_\iota$. The last assertion of Lemma 6.5 implies that $s \in S_\iota$ has distinct eigenvalues on V/V^s. It follows that

$$G_\iota = SO(V^s)\times Z_\iota, \qquad H_\iota = SO(U^s)\times Z_\iota. \tag{9.2}$$

Note that $\dim V^s$ is odd, say $\dim V^s = 2a+1$.

The decompositions (9.2) imply that if two F-stable maximal tori in G_ι^F are G^F-conjugate, then they are G_ι^F-conjugate, and likewise for H. In other words, we have

$$|j^{-1}_{G_\iota}(\mathrm{cl}(T,G))|\cdot|j^{-1}_{H_\iota}(\mathrm{cl}(S,H))| \leq 1.$$

Hence the inner sum of (9.1) has at most one term.

To make this precise, we recall that tori in orthogonal groups are described by pairs (λ,λ') of partitions. We write partitions as $\lambda = (j^{\lambda_j})$, meaning that

λ has λ_j parts equal to j, and set $|\lambda| = \sum_j j\lambda_j$. We have pairs of partitions (ν, ν'), (λ, λ'), (μ, μ') such that

$$\begin{aligned} Z_\iota^F &\simeq \prod_j (\mathfrak{f}_j^\times)^{\nu_j} \times (\mathfrak{f}_{2j}^1)^{\nu'_j}, \qquad |\nu| + |\nu'| = n - a, \\ T^F &\simeq \prod_j (\mathfrak{f}_j^\times)^{\lambda_j} \times (\mathfrak{f}_{2j}^1)^{\lambda'_j}, \qquad |\lambda| + |\lambda'| = n, \\ S^F &\simeq \prod_j (\mathfrak{f}_j^\times)^{\mu_j} \times (\mathfrak{f}_{2j}^1)^{\mu'_j}, \qquad |\mu| + |\mu'| = n. \end{aligned} \tag{9.3}$$

We have $|j_{G_\iota}^{-1}(\mathrm{cl}(T, G))| \cdot |j_{H_\iota}^{-1}(\mathrm{cl}(S, H))| = 1$ precisely when

$$\nu_j \leq \lambda_j, \mu_j \qquad \text{and} \qquad \nu'_j \leq \lambda'_j, \mu'_j \tag{9.4}$$

for all j. We assume (9.4) holds from now on. Note that if T and S are anisotropic, then $\lambda_j = \mu_j = \nu_j = 0$ for all j.

We count the number of ι in the sum (9.1) giving rise to a fixed pair of partitions (ν, ν'). For $s \in S_\iota^F$, consider the components $(s_{j1}, \ldots, s_{j\mu_j}; s'_{j1}, \ldots, s'_{j\mu'_j})$ of s in the j^{th} block

$$(\mathfrak{f}_j^\times)^{\mu_j} \times (\mathfrak{f}_{2j}^1)^{\mu'_j} \subset S^F.$$

Then ι is determined by the pair of subsets

$$\{k \in [1, \mu_j] : \ s_{jk} = 1\}, \quad \{k' \in [1, \mu'_j] : \ s'_{jk'} = 1\}.$$

It follows that there are

$$\binom{\mu}{\nu}\binom{\mu'}{\nu'}$$

elements $\iota \in I(S)^F$ giving rise to (ν, ν'), where

$$\binom{\mu}{\nu} := \prod_j \binom{\mu_j}{\nu_j}, \qquad \binom{\mu'}{\nu'} := \prod_j \binom{\mu'_j}{\nu'_j}.$$

From equations (2.3) and $|j_{H_\iota}^{-1}(\mathrm{cl}(S, H))| = 1$ we have

$$|\bar{N}_H(\iota, S)^F| = \frac{|W_H(S)^F|}{|W_{H_\iota}(S)^F|} = \binom{\mu}{\nu}\binom{\mu'}{\nu'} \prod_j (\nu_j!)(\nu'_j!)(2j)^{\nu_j}(2j)^{\nu'_j}. \tag{9.5}$$

Using Lemma 3.1 for G_ι and H_ι, we find that

$$(-1)^{\mathrm{rk}\, G_\iota + \mathrm{rk}\, H_\iota} = (-1)^{\mathrm{rk}\, G + \mathrm{rk}\, H + \sum \nu'_j}. \tag{9.6}$$

(One can also arrive at (9.6) by decomposing U^F into irreducible $\mathfrak{f}Z_\iota^F$-modules, and calculating discriminants.)

Finally, we must calculate the pairing $\langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_\iota^F}$. We may conjugate T and S to arrange that $Z_\iota \subset T \cap S$. Then

$$\chi_\upsilon = \frac{1}{|W_{G_\iota}(T)^F|} \sum_{x \in W_G(T)^F} ({}^x\chi)|_{Z_\iota}, \qquad \eta_\varsigma = \frac{1}{|W_{H_\iota}(S)^F|} \sum_{y \in W_H(S)^F} ({}^y\eta)|_{Z_\iota}.$$

We now assume that χ and η are **regular**, in the sense that they have trivial stabilizers in $W_G(T)^F$ and $W_H(S)^F$, respectively.

On the j^{th} block $(\mathfrak{f}_j^\times)^{\mu_j} \times (\mathfrak{f}_{2j}^1)^{\mu'_j}$ of S^F, we have

$$\eta = \eta_{j1} \otimes \cdots \otimes \eta_{j\mu_k} \otimes \eta'_{j1} \otimes \cdots \otimes \eta'_{j\mu'_k}.$$

Likewise, on the j^{th} block $(\mathfrak{f}_j^\times)^{\lambda_j} \times (\mathfrak{f}_{2j}^1)^{\lambda'_j}$ of T^F, we have

$$\chi = \chi_{j1} \otimes \cdots \otimes \chi_{j\lambda_j} \otimes \chi'_{j1} \otimes \cdots \otimes \chi'_{j\lambda'_j}.$$

Define

$$\begin{aligned} I_j &= \{k \in [1, \mu_j] : \ \eta_{jk} \in \Gamma_j \cdot \{\chi_{j\ell}, {\chi_{j\ell}}^{-1}\}, \quad \text{for some } \ell \in [1, \lambda_j]\}, \\ I'_j &= \{k' \in [1, \mu'_j] : \ \eta'_{jk'} = \Gamma_{2j} \cdot \chi'_{j\ell}, \quad \text{for some } \ell \in [1, \lambda'_j]\}. \end{aligned} \tag{9.7}$$

For every pair of subsets

$$\{k_1, \cdots, k_{\nu_j}\} \subset I_j, \qquad \{k'_1, \cdots, k'_{\nu'_j}\} \subset I'_j,$$

each of the

$$(\nu_j)!(\nu'_j)!(2j)^{\nu_j}(2j)^{\nu'_j}$$

conjugates of the character

$$\eta_{jk_1} \otimes \cdots \otimes \eta_{jk_{\nu_j}} \otimes \eta'_{jk'_1} \otimes \cdots \otimes \eta'_{jk'_{\nu_j}}$$

contributes exactly once to the pairing $\langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_\iota^F}$, by the regularity assumption (1.3). It follows that

$$\langle \chi_\upsilon, \eta_\varsigma \rangle_{Z_\iota^F} = \prod_j \binom{|I_j|}{\nu_j} \binom{|I'_j|}{\nu'_j} (\nu_j)!(\nu'_j)!(2j)^{\nu_j}(2j)^{\nu'_j}. \tag{9.8}$$

Set

$$e := (-1)^{\operatorname{rk} G + \operatorname{rk} T + \operatorname{rk} H + \operatorname{rk} S}.$$

Inserting (9.5), (9.6) and (9.8) into (9.1), and summing over all (ν, ν') satisfying (9.4), we get

$$\begin{aligned} e \cdot \langle R^G_{T,\chi}, R^H_{S,\eta} \rangle_{H^F} &= \sum_{\nu,\nu'} (-1)^{\sum \nu'_j} \prod_j \binom{|I_j|}{\nu_j} \binom{|I'_j|}{\nu'_j} \\ &= \prod_j \left[\left(\sum_{\nu_j=0}^{|I_j|} \binom{|I_j|}{\nu_j} \right) \cdot \left(\sum_{\nu'_j=0}^{|I'_j|} (-1)^{\nu'_j} \binom{|I'_j|}{\nu'_j} \right) \right] \\ &= \begin{cases} 2^r & \text{if } I'_j \text{ is empty for all } j, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{9.9}$$

where

$$r = \sum_j |I_j|.$$

If either T or S is anisotropic, then $r = 0$. This proves Theorem 1.2.

10. Restriction from SO_7 to G_2

The previous situation had $\delta = 0$. We now consider a case where $\delta = 1$. The simplest such case is $G = G_2$, $H = SL_3$, which we leave to the reader.

Here we take $G = SO_7$, $H = G_2$, embedded in G via the irreducible 7-dimensional representation V of G_2. We have

$$\delta = 9 - 8 = 1.$$

We assume $p \geq 5$.

We will calculate the multiplicities $\langle R^G_{T,\chi}, R^H_{S,\eta}\rangle_{H^F}$, using the formula of section 6.3. We do not need any detailed knowledge of Green functions, beyond the general facts about their degrees and leading terms that we have already used.

Let α, β be simple roots of a maximal $\mathfrak{f}$-split torus S_0 in H, with α short. The nonzero weights of H in V are the short roots of S_0. We view the maximal $\mathfrak{f}$-split tori T_0 and S_0 as

$$T_0 = \{(x, y, z) \in \mathfrak{F}^3 : \ xyz \neq 0\}, \qquad S_0 = \{(x, y, z) \in \mathfrak{F}^3 : \ xyz = 1\},$$

in such a way that the coordinate functions e_1, e_2, e_3 on T_0 restrict to the roots $2\alpha + \beta, -\alpha, -\alpha - \beta$ on S_0. In this realization, the simple co-roots of S_0 in H are

$$\check{\alpha}(t) = (t, t^{-2}, t), \qquad \check{\beta}(t) = (1, t, t^{-1}),$$

and the corresponding simple reflections r_α, r_β in the Weyl group W_H act by

$$r_\alpha \cdot (t_1, t_2, t_3) = (t_3^{-1}, t_2^{-1}, t_1^{-1}), \qquad r_\beta \cdot (t_1, t_2, t_3) = (t_1, t_3, t_2).$$

Since S_0 contains regular elements in T_0, it follows that W_H is a subgroup of W_G. If W_G is realized as the group of the cube, then W_H is the subgroup preserving a diagonal of the cube; as coset representatives for W_G/W_H we may take the identity and each coordinate sign change.

Let T, S be F-stable maximal tori in G and H, corresponding to the conjugacy classes of $x \in W_G$ and $y \in W_H$, respectively. Let $\chi \in \mathrm{Irr}(T^F)$, $\eta \in \mathrm{Irr}(S^F)$.

We will use the refined multiplicity formula of section 6.3. We first tabulate the pairs (ι, u) in H, with $\iota \in I(S_0)$ and $u \in H_\iota$, for which

$$\dim Z_\iota + d_{G_\iota}(u) + d_{H_\iota}(u) - \dim C_{H_\iota}(u) \geq 0. \tag{10.1}$$

We find four types as shown:

type	ι	u
a	$(1,1,1)$	1, u_0
b	$(1, t, t^{-1})$, $t^q = t \neq \pm 1$	1
c	$(1, t, t^{-1})$, $t^q = t^{-1} \neq \pm 1$	1
d	regular	1

The middle column shows a typical element in S_0 for each type of ι. There can be more than one ι of the same type. Here $u_0 \in H^F$ is a long root element, which has Jordan partition $1^3 2^2$ on V.

From equation (5.18), we have

$$M(q) = M_a(q) + M_b(q) + M_c(q) + M_d(q),$$

where each term on the right is the sum

$$\sum \Psi_\alpha \Theta_\alpha$$

over those α whose ι component is H-conjugate to an ι of the corresponding type a, b, c, d in the table above. Taking the polynomial part of each sum, as in section 6.3, we have

$$M(q) = P_a(q) + P_b(q) + P_c(q) + P_d(q). \tag{10.2}$$

We now calculate each of the four terms on the right side of (10.2).

Type a: The maps j_{G_ι} and j_{H_ι} are the identity. The centralizers of u_0 in G and H are both connected. Let ϕ_G, ϕ_H be the Springer representations of W_G and W_H corresponding to u_0, let ρ_G, ρ_H be the reflection representations, and let ϵ_G, ϵ_H be the sign representations.

From [5, 13.3], we find that ϕ_G is the unique two-dimensional representation of $W_G = S_3 \ltimes \{\pm 1\}^3$ which is irreducible on S_3 and nontrivial on $\{\pm 1\}^3$, and ϕ_H is the one-dimensional character of W_H given by $\phi_H(r_\alpha) = -1$, $\phi_H(r_\beta) = +1$.

In the following calculation we write $R_1 \sim R_2$ for rational functions R_i such that $\deg(R_1 - R_2) < 0$. We have

$$M_a(q) = \frac{Q_T^G(1) Q_S^H(1)}{|H^F|} + \frac{Q_T^G(u_0) Q_S^H(u_0)}{|C_H(u_0)^F|}.$$

The u_0 term has degree zero, so we may replace it by its leading term:

$$M_a(q) \sim \frac{Q_T^G(1) Q_S^H(1)}{|H^F|} + \phi_G(x)\phi_H(y).$$

Since

$$|T^F| = \det(q - x) = q^3 - \rho_G(x) q^2 + \cdots,$$
$$|S^F| = \det(q - y) = q^2 - \rho_H(y) q + \cdots,$$

it follows that

$$\begin{aligned}\epsilon_G(x)\epsilon_H(y) \frac{Q_T^G(1) Q_S^H(1)}{|H^F|} &= \frac{[G^F : T^F]_{p'} \cdot [H^F : S^F]_{p'}}{|H^F|} \\ &= \frac{(q^6 - 1)(q^4 - 1)(q^2 - 1)}{q^6 |T^F||S^F|} \\ &\sim q + \rho_G(x) + \rho_H(y).\end{aligned}$$

We get

$$P_a(q) = \epsilon_G(x)\epsilon_H(y)\left[q + \rho_G(x) + \rho_H(y)\right] + \phi_G(x)\phi_H(y). \tag{10.3}$$

Type b: For $\iota \in I(S)^F$ of type b, the elements of S_ι^F are H-conjugate to elements of the form $s = (1, t, t^{-1}) \in S_0$ for some $t \in \mathfrak{f}^\times$, $t^2 \neq 1$. We have

$$G_\iota = SO_3 \times GL_2, \qquad H_\iota = GL_2.$$

The stabilizer W'_{H_ι} of s in W_H is generated by the reflection

$$r_0 := r_\alpha r_\beta r_\alpha r_\beta r_\alpha \in W_H.$$

The number of $\iota \in I(S)^F$ of type b is given by

$$\nu_b(y) := \begin{cases} 3 & \text{if } y = 1, \\ 1 & \text{if } [y] = [r_0], \\ 0 & \text{otherwise,} \end{cases} \tag{10.4}$$

where $[\cdot]$ denotes a conjugacy class in W_H. We have

$$\frac{\nu_b(y)}{|\bar{N}_H(\iota, S)^F|} = \nu_b(y) \cdot \frac{|C_{W'_{H_\iota}}(y)|}{|C_{W_H}(y)|} = \frac{1}{2}$$

for $y \in \{1, r_0\}$.

The roots of T_0 vanishing on s are $e_1, e_2 + e_3$. The corresponding reflections $r_1, r_2 \in W_G$ generate the stabilizer W'_{G_ι} of s in W_G. For $s \in S_\iota^F$, the element y_s of section 2 is $y_s = 1$. Hence, the mapping

$$j_{G_\iota} : H^1(F, W'_{G_\iota}) \to H^1(F, W_G)$$

of (2.2) is induced by the inclusion $W'_{G_\iota} \hookrightarrow W_G$; its image consists of the four classes in W_G represented by elements in W'_{G_ι}, as shown in the following table:

$H^1(F, W'_{G_\iota})$	1	r_1	r_2	r_1r_2
$H^1(F, W_G)$	$[1^3, -]$	$[1^2, 1]$	$[12, -]$	$[2, 1]$

Set

$$W_b := W'_{G_\iota} \times W'_{H_\iota}.$$

Using section 7 and taking (10.4) into account, it follows that

$$P_b(q) = \tfrac{1}{2}\epsilon_G(x)\epsilon_H(y)\langle\chi_x, \eta_y\rangle_{Z_\iota^F} \tag{10.5}$$

if (x, y) is $(W_G \times W_H)$-conjugate to an element of W_b, and $P_b(q) = 0$ otherwise. Here we have written χ_x instead of χ_υ, where $\{\upsilon\} = j_{G_\iota}^{-1}([x])$, and likewise for η_y.

Type c: For $\iota \in I(S)^F$ of type c, the elements of S_ι^F are H-conjugate to elements of the form $s = (1, t, t^{-1}) \in S_0$ for some $t \in \mathfrak{f}_2^1$, $t^2 \neq 1$. We have

$$G_\iota = SO_3 \times U_2, \qquad H_\iota = U_2.$$

The groups

$$W'_{G_\iota} = \langle r_1, r_2\rangle, \qquad W'_{H_\iota} = \langle r_0\rangle$$

are as in type b. The number of $\iota \in I(S)^F$ of type c is given by

$$\nu_c(y) := \begin{cases} 3 & \text{if } y = -1,\\ 1 & \text{if } [y] = [-r_0],\\ 0 & \text{otherwise.}\end{cases} \tag{10.6}$$

As before, we have

$$\frac{\nu_c(y)}{|\bar{N}_H(\iota, S)^F|} = \nu_c(y)\cdot\frac{|C_{W'_{H_\iota}}(y)|}{|C_{W_H}(y)|} = \frac{1}{2},$$

now for $y \in \{-1, -r_0\}$.

Let r be the reflection about $e_2 - e_3$. For $s \in S_\iota^F$, the element y_s of section 2 is $y_s = r$. Hence, the mapping

$$j_{G_\iota} : H^1(F, W'_{G_\iota}) \to H^1(F, W_G)$$

is induced by the map $x \mapsto xr$, as shown in the following table:

$H^1(F, W'_{G_\iota})$	1	r_1	r_2	r_1r_2
$H^1(F, W_G)$	$[12, -]$	$[2, 1]$	$[1, 1^2]$	$[-, 1^3]$

Set

$$W_c := W'_{G_\iota}r \times W'_{H_\iota}r.$$

Using section 7 and taking (10.6) into account, it follows that

$$P_c(q) = \tfrac{1}{2}\epsilon_G(x)\epsilon_H(y)\langle\chi_x, \eta_y\rangle_{Z_\iota^F} \tag{10.7}$$

if (x, y) is $(W_G \times W_H)$-conjugate to an element of W_c, and $P_c(q) = 0$ otherwise. Again, we have written χ_x instead of χ_υ, where $\{\upsilon\} = j_{G_\iota}^{-1}([x])$, and likewise for η_y.

Type d: In this case S_ι contains regular elements in G, so $H_\iota = Z_\iota = S$, and $G_\iota = C_G(S)$ is a maximal torus in G. Hence $P_d(q) \neq 0$ only if T is G^F-conjugate to $C_G(S)$. The mapping $S \mapsto C_G(S)$ is given, in terms of conjugacy classes in W_H and W_G, in the following table:

S	1	r_α	r_β	-1	$(r_\alpha r_\beta)^2$	$r_\alpha r_\beta$
$C_G(S)$	$[1^3, -]$	$[2, 1]$	$[12, -]$	$[-, 1^3]$	$[3, -]$	$[-, 3]$

Denote the embedding

$$W_H \hookrightarrow W_G \qquad \text{by } y \mapsto \tilde{y}.$$

Let

$$W_d = \{(\tilde{y}, y):\ y \in W_H\} \subset W_G \times W_H.$$

For $(x, y) \in W_d$, we may assume that $T = C_G(S)$. Then

$$\Theta_\alpha(q) \sim \sum_{\substack{w \in W_G(T)^F \\ v \in W_H(S)^F}} \langle {}^w\chi, {}^v\eta\rangle_{S^F} = |W_H(S)^F| \sum_{w \in W_G(T)^F} \langle {}^w\chi, \eta\rangle_{S^F}.$$

Since

$$\Psi_\alpha(t) = \frac{1}{|W_H(S)^F|},$$

it follows that

$$P_d(q) = \sum_{w \in W_G(T)^F} \langle {}^w\chi, \eta\rangle_{S^F} = \langle \chi_x, \eta\rangle_{S^F} \tag{10.8}$$

if (x, y) is $(W_G \times W_H)$-conjugate to an element of W_d, and $P_d(q) = 0$ otherwise.

10.1. **Cuspidal multiplicities.** From now on we assume that our tori T, S are anisotropic, and that the characters χ and η are regular. We will make the multiplicities computed above more precise.

The elliptic classes in $W(B_n)$ are those of the form $[-, \lambda]$, where λ is a partition of n. So in W_G we have three elliptic classes:

$$[-, 3], \quad [-, 12], \quad [-, 1^3].$$

The first of these is the Coxeter class and the last is $\{-1\}$.

In W_H we also have three elliptic classes, represented by the powers

$$\mathsf{cox}, \quad \mathsf{cox}^2, \quad \mathsf{cox}^3 = -1$$

of a Coxeter element $\mathsf{cox} = r_\alpha r_\beta$. Via the embedding $W_H \hookrightarrow W_G$, the elements $-1, \mathsf{cox}$ of W_H are also the -1 and Coxeter elements of W_G.

Let $x \in \mathrm{cl}(T, G)$, $y \in \mathrm{cl}(S, H)$. Note that $P_b(q) = 0$. Combining formulas (10.3), (10.7), (10.8), we get the multiplicity formula

$$-\langle R^G_{T,\chi}, R^H_{S,\eta}\rangle_{H^F} = q + \rho_G(x) + \rho_H(y) - \phi_G(x)\phi_H(y) + a(x, y) \tag{10.9}$$

where

$$a(x, y) = \begin{cases} \frac{1}{2}\langle \chi_{-1}, \eta_{-1}\rangle_{Z_c^F} - \langle \chi_{-1}, \eta\rangle_{Z_d^F} & \text{if } x = y = -1, \\ -\langle \chi_{\mathsf{cox}}, \eta\rangle_{Z_d^F} & \text{if } x = y = \mathsf{cox}, \\ 0 & \text{otherwise.} \end{cases} \tag{10.10}$$

Here we have written Z_c for Z_ι, when ι has type c, and likewise for $Z_d = S$. The numbers

$$\langle \chi_{-1}, \eta_{-1}\rangle_{Z_c^F}, \qquad \langle \chi_{-1}, \eta_{-1}\rangle_{Z_d^F}, \qquad \langle \chi_{\mathsf{cox}}, \eta\rangle_{Z_d^F}$$

are calculated explicitly in (10.11), (10.13) and (10.16) below.

We calculate $\rho_G(x) + \rho_H(y) - \phi_G(x)\phi_H(y)$ from the character tables of W_G and W_H and then set $A(x) = a(x, x)$ for $x \in \{-1, \mathsf{cox}\}$ to arrive at the multiplicities in Table 1.

Note that S_{-1} has regular characters only for $q \geq 5$ (which we have already assumed), and T_{-1} has regular characters only for $q \geq 7$. For $q = 5$ we get Table 2,

$y\backslash x$	$[-,1^3]=[-1]$	$[-,12]$	$[-,3]=[\mathsf{cox}]$
$[\mathsf{cox}^3]=[-1]$	$q-7+A(-1)$	$q-3$	$q-1$
$[\mathsf{cox}^2]$	$q-2$	$q-2$	$q-2$
$[\mathsf{cox}]$	$q-4$	q	$q+2-A(\mathsf{cox})$

TABLE 1. $-\langle R^G_{T_x,\chi}, R^H_{S_y,\eta}\rangle_{H^F}$ for $q \geq 7$

$y\backslash x$	$[-,1^3]=[-1]$	$[-,12]$	$[-,3]=[\mathsf{cox}]$
$[\mathsf{cox}^3]=[-1]$	-	2	4
$[\mathsf{cox}^2]$	-	3	3
$[\mathsf{cox}]$	-	5	6 or 7

TABLE 2. $-\langle R^G_{T_x,\chi}, R^H_{S_y,\eta}\rangle_{H^F}$ for $q = 5$

where the dichotomy in the $(\mathsf{cox},\mathsf{cox})$ entry arises from the fact that $A(\mathsf{cox}) = 0$ or 1, depending on χ and η (see below).

The rest of this section is devoted to the explicit calculation of $A(-1)$ and $A(\mathsf{cox})$.

A(−1): We identify

$$T^F_{-1} = (\mathfrak{f}^1_2)^3, \quad S^F_{-1} = \{(x,y,z) \in T^F_{-1} :\ xyz = 1\},$$

$$\chi = \chi_1 \otimes \chi_2 \otimes \chi_3, \qquad \eta = \mathrm{Res}_{S^F_{-1}}(\eta_1 \otimes \eta_2 \otimes \eta_3), \qquad \text{with} \quad \chi_i,\ \eta_i \in \mathrm{Irr}(\mathfrak{f}^1_2).$$

Recall that

$$A(-1) = \tfrac{1}{2}\langle \chi_{-1}, \eta_{-1}\rangle_{Z_c} - \langle \chi_{-1}, \eta\rangle_{Z_d}.$$

We have $Z^F_c = \{(1,t,t^{-1}) :\ t \in \mathfrak{f}^1_2\}$ and

$$\langle \chi_{-1}, \eta_{-1}\rangle_{Z^F_c} = 2\sum_{i<j}\left\langle \frac{\chi_i}{\chi_j} + \frac{\chi_j}{\chi_i} + \chi_i\chi_j + \frac{1}{\chi_i\chi_j}, \frac{\eta_1}{\eta_2} + \frac{\eta_2}{\eta_3} + \frac{\eta_3}{\eta_1}\right\rangle_{\mathfrak{f}^1_2}. \tag{10.11}$$

For ι of type d, we have $Z_d = S_{-1}$. The W_G-orbit of χ breaks up into four W_H-orbits:

$$\begin{aligned} \tilde{\mathcal{O}}_0 &= W_H \cdot (\chi_1 \otimes \chi_2 \otimes \chi_3),\\ \tilde{\mathcal{O}}_1 &= W_H \cdot (\bar\chi_1 \otimes \chi_2 \otimes \chi_3),\\ \tilde{\mathcal{O}}_2 &= W_H \cdot (\chi_1 \otimes \bar\chi_2 \otimes \chi_3),\\ \tilde{\mathcal{O}}_3 &= W_H \cdot (\chi_1 \otimes \chi_2 \otimes \bar\chi_3). \end{aligned} \tag{10.12}$$

These restrict to W_H-orbits $\mathcal{O}_0, \dots, \mathcal{O}_3$ in $\mathrm{Irr}(S^F_{-1})$. Even though the orbits $\tilde{\mathcal{O}}_i$ consist of W_G-regular characters, the characters in $\mathcal{O}_i$ need not be W_H-regular. Moreover, it can happen that $\mathcal{O}_i = \mathcal{O}_j$ for $i \neq j$. In any case, formula (10.8) gives

$$\langle \chi_{-1}, \eta\rangle_{Z^F_d} = |\{i \in [0,3] :\ \eta \in \mathcal{O}_i\}|. \tag{10.13}$$

We illustrate with $q = 7$. The unique regular W_G-orbit in $\mathrm{Irr}(T^F_{-1})$ contains the character $\chi = \zeta \otimes \zeta^2 \otimes \zeta^3$, where ζ is a faithful character of $\mathfrak{f}^1_2 \simeq \mu_8$. There are two W_H-orbits of regular characters η, η' in $\mathrm{Irr}(S^F_{-1})$, distinguished as follows: η

belongs to the restrictions $\mathcal{O}_1 = \mathcal{O}_3$, and $\eta' = \mathrm{Res}_{S_{-1}^F}(\zeta \otimes 1 \otimes \zeta^{-2})$ does not extend to a regular character of T_{-1}^F. Formulas (10.11) and (10.13) give

$$\begin{aligned} \langle \chi_{-1}, \eta_{-1} \rangle_{Z_c^F} &= 12, \qquad & \langle \chi_{-1}, \eta_{-1} \rangle_{Z_d^F} &= 2, \\ \langle \chi_{-1}, \eta'_{-1} \rangle_{Z_c^F} &= 10, \qquad & \langle \chi_{-1}, \eta'_{-1} \rangle_{Z_d^F} &= 0. \end{aligned} \tag{10.14}$$

From Table 1 we get

$$\begin{aligned} -\langle R^G_{T_{-1},\chi}, R^H_{S_{-1},\eta} \rangle_{H^F} &= 7 - 7 + \tfrac{1}{2} \cdot 12 - 2 = 4, \\ -\langle R^G_{T_{-1},\chi}, R^H_{S_{-1},\eta'} \rangle_{H^F} &= 7 - 7 + \tfrac{1}{2} \cdot 10 - 0 = 5. \end{aligned} \tag{10.15}$$

A(cox): Here the only relevant type is d. We identify

$$T^F_{\mathsf{cox}} = \mathfrak{f}_6^1 = \ker[\mathfrak{f}_6^\times \xrightarrow{\text{norm}} \mathfrak{f}_3^\times], \qquad S^F_{\mathsf{cox}} = \left(\mathfrak{f}_6^1\right)^{q+1},$$

$W(T_{\mathsf{cox}})^F = W(S_{\mathsf{cox}})^F$ is cyclic of order six, and acts on T^F_{cox} and S^F_{cox} via $\mathrm{Gal}(\mathfrak{f}_6/\mathfrak{f})$. If $\chi \in \mathrm{Irr}(T^F_{\mathsf{cox}})$ and $\eta \in \mathrm{Irr}(S^F_{\mathsf{cox}})$ are both regular, and η appears in the restriction of some Galois conjugate of χ, then the restriction map $\mathrm{Irr}(T^F_{\mathsf{cox}}) \longrightarrow \mathrm{Irr}(S^F_{\mathsf{cox}})$ maps $W_G \cdot \chi$ bijectively onto $W_H \cdot \eta$. Formula (10.8) gives

$$A(\mathsf{cox}) = \langle \chi_{\mathsf{cox}}, \eta \rangle_{Z_d^F} = \begin{cases} 1 & \text{if } \eta = \mathrm{Res}({}^w\chi) \text{ for some } w \in \mathrm{Gal}(\mathfrak{f}_6/\mathfrak{f}), \\ 0 & \text{otherwise.} \end{cases} \tag{10.16}$$

Hence the $(\mathsf{cox}, \mathsf{cox})$ entry in Table 1 is made precise:

(10.17)

$$-\langle R^G_{T_{\mathsf{cox}},\chi}, R^H_{S_{\mathsf{cox}},\eta} \rangle_{H^F} = \begin{cases} q+1 & \text{if } \eta = \mathrm{Res}({}^w\chi) \text{ for some } w \in \mathrm{Gal}(\mathfrak{f}_6/\mathfrak{f}), \\ q+2 & \text{otherwise.} \end{cases}$$

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Department of Mathematics, Boston College, Chestnut Hill, Massachusetts 02467
E-mail address: reederma@bc.edu