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COMPATIBILITY OF LOCAL AND GLOBAL LANGLANDS CORRESPONDENCES

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Introduction

This paper is a continuation of [HT]. Let L be an imaginary CM field and let Π be a regular algebraic (i.e., Π_∞ has the same infinitesimal character as an algebraic representation of the restriction of scalars from L to $\mathbb{Q}$ of GL_n) cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to Π and an arbitrary rational prime l (and an isomorphism $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$) a continuous semisimple representation

$$R_l(\Pi) : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

which is characterised as follows. For every finite place v of L not dividing l,

$$\imath R_l(\Pi)|_{W_{L_v}}^{\mathrm{ss}} = \mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)^{\mathrm{ss}},$$

where rec denotes the local Langlands correspondence and ss denotes the semisimplification (see [HT] for details). In [HT] it is also shown that Π_v is tempered for all finite places v.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \not| \, l$. In particular, we prove the following theorem.

Theorem A. *If $v \not| \, l$, then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the l-adic representation attached to $\imath^{-1}\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$.*

As $R_l(\Pi)$ is semisimple and $\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$ is indecomposable if Π_v is square integrable, we obtain the following corollary.

Corollary B. *If Π_v is square integrable at a finite place $v \not| \, l$, then the representation $R_l(\Pi)$ is irreducible.*

We also obtain some results in the case $v|l$, which we will describe in section one.

Using base change it is easy to reduce to the case that Π_v has an Iwahori fixed vector. We descend Π to an automorphic representation π of a unitary group G which locally at v looks like GL_n and at infinity looks like $U(n-1,1) \times U(n,0)^{[L:\mathbb{Q}]/2-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety X associated to

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G with Iwahori level structure at v. More precisely, for some l-adic sheaf $\mathcal{L}$, the π^p-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that X has strictly semistable reduction and use the results of [HT] to calculate the cohomology of the (smooth, projective) strata of the reduction of X above p as a virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_v^{\mathbb{Z}}$-module (where Frob_v denotes Frobenius). This description and the temperedness of Π_v shows that the π^p-isotypic component of the cohomology of any strata is concentrated in the middle degree. This implies that the π^p-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at E_1, which allows us to calculate the action of inertia at v on $H^{n-1}(X, \mathcal{L})$.

In the special case that Π_v is a twist of a Steinberg representation and Π_∞ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

After we had posted the first version of this paper, Boyer [Bo] announced an alternative proof. He does not make a base change to put himself into the semistable reduction case. Thus he works directly with the vanishing cycles rather than with the Rapoport-Zink weight spectral sequence. His methods are necessarily more complicated, but he proves more. For instance he gets very complete information about the cohomology of Drinfeld's covers of the Lubin-Tate space.

1. The main theorem

We write F^{ac} for an algebraic closure of a field F. Let l be a rational prime and $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$ an isomorphism.

Suppose that p is another rational prime. Let $K/\mathbb{Q}_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of K, $\wp_K$ the unique maximal ideal of $\mathcal{O}_K$, v_K the canonical valuation $K^\times \twoheadrightarrow \mathbb{Z}$, $k(v_K)$ the residue field $\mathcal{O}_K/\wp_K$ and $|\ |_K$ the absolute value normalised by $|x|_K = (\#k(v_K))^{-v_K(x)}$. We will let Frob_{v_K} denote the geometric Frobenius element of $\mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$. We will let I_{v_K} denote the kernel of the natural surjection $\mathrm{Gal}\,(K^{ac}/K) \twoheadrightarrow \mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$. We will let W_K denote the preimage under $\mathrm{Gal}\,(K^{ac}/K) \twoheadrightarrow \mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$ of $\mathrm{Frob}_{v_K}^{\mathbb{Z}}$ endowed with a topology by decreeing that I_K with its usual topology is an open subgroup of W_K. Local class field theory provides a canonical isomorphism $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$, which takes uniformisers to lifts of Frob_{v_K}.

Let Ω be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a *Weil-Deligne representation* of W_K over Ω we mean a finite-dimensional Ω-vector space V together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \mathrm{End}\,(V)$ which satisfies

$$r(\sigma) N r(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K N.$$

We sometimes denote a Weil-Deligne representation by (V, r, N) or simply (r, N). For a finite extension K'/K of p-adic fields, we define

$$(V, r, N)|_{W_{K'}} = (V, r|_{W_{K'}}, N).$$

We call (V, r, N) *Frobenius semisimple* if r is semisimple. If (V, r, N) is any Weil-Deligne representation we define its *Frobenius semisimplification* $(V, r, N)^{F\text{-ss}} = (V, r^{\mathrm{ss}}, N)$ as follows. Choose a lift ϕ of Frob_{v_K} to W_K. Let $r(\phi) = su = us$, where $s \in GL(V)$ is semisimple and $u \in GL(V)$ is unipotent. For $n \in \mathbb{Z}$ and $\sigma \in I_K$

set $r^{\mathrm{ss}}(\phi^n\sigma) = s^n r(\sigma)$. This is independent of the choices and gives a Frobenius semisimple Weil-Deligne representation. We will also set $(V,r,N)^{\mathrm{ss}} = (V, r^{\mathrm{ss}}, 0)$.

One of the main results of [HT] is that, given a choice of $(\# k(v_K))^{1/2} \in \Omega$, there is a bijection rec (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of $GL_n(K)$ over Ω to isomorphism classes of n-dimensional Frobenius semisimple Weil-Deligne representations of W_K, and that this bijection is natural in a number of respects. (See [HT] for details.)

Suppose that $l \neq p$. We will call a Weil-Deligne representation of W_K over $\mathbb{Q}_l^{ac}$ *bounded* if for some (and hence all) $\sigma \in W_K - I_K$, all the eigenvalues of $r(\sigma)$ are l-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of W_K over $\mathbb{Q}_l^{ac}$ and continuous representations of $\mathrm{Gal}\,(K^{ac}/K)$ on finite-dimensional $\mathbb{Q}_l^{ac}$-vector spaces as follows. Fix a lift $\phi \in W_K$ of Frob_{v_K} and a continuous homomorphism $t : I_K \twoheadrightarrow \mathbb{Z}_l$. Send a Weil-Deligne representation (V,r,N) to (V,ρ), where ρ is the unique continuous representation of $\mathrm{Gal}\,(K^{ac}/K)$ on V such that

$$\rho(\phi^n\sigma) = r(\phi^n\sigma)\exp\bigl(t(\sigma)N\bigr)$$

for all $n \in \mathbb{Z}$ and $\sigma \in I_K$. Up to a natural isomorphism this functor is independent of the choices of t and ϕ. We will write $\mathrm{WD}(V,\rho)$ for the Weil-Deligne representation corresponding to a continuous representation (V,ρ). If $\mathrm{WD}(V,\rho) = (V,r,N)$, then we have $\rho|_{W_K}^{\mathrm{ss}} \cong r^{\mathrm{ss}}$. (See [Ta], §4 and [D], §8 for details.)

Now suppose that $l = p$. Let K_0 denote the maximal subfield of K which is unramified over $\mathbb{Q}_p$. Recall the filtered K-algebra B_{DR} and the K_0-algebra B_{st} with endomorphisms N and ϕ, where ϕ is $\mathrm{Frob}_{\mathbb{Q}_p}^{-1}$-semilinear and $\phi N = pN\phi$. (See [Fo].) A continuous representation (V,ρ) of $\mathrm{Gal}\,(K^{ac}/K)$ on a finite-dimensional $\mathbb{Q}_l^{ac}$ vector space is called *de Rham* if

$$(V \otimes_{\mathbb{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(K^{ac}/K)}$$

is free over $\mathbb{Q}_l^{ac} \otimes_{\mathbb{Q}_l} K$ of rank $\dim_{\mathbb{Q}_l^{ac}} V$. The category of de Rham representations is closed under tensor operations. One can also define various invariants of de Rham representations (V,ρ). Firstly, for each embedding $\tau : K \hookrightarrow \mathbb{Q}_l^{ac}$, we will let $\mathrm{HT}_\tau(V,\rho)$ denote the $\dim_{\mathbb{Q}_l^{ac}} V$ element multiset of integers which contains j with multiplicity

$$\begin{aligned}&\dim_{\mathbb{Q}_l^{ac}} \mathrm{gr}^{\,j}(V \otimes_{\tau,K} B_{\mathrm{DR}})^{\mathrm{Gal}\,(K^{ac}/K)}\\ &\qquad = \dim_{\mathbb{Q}_l^{ac}} \mathrm{gr}^{\,j}(V \otimes_{\mathbb{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(K^{ac}/K)} \otimes_{(\mathbb{Q}_l^{ac}\otimes_{\mathbb{Q}_l} K),1\otimes\tau} \mathbb{Q}_l^{ac}.\end{aligned}$$

(These are referred to as the Hodge-Tate numbers of V with respect to τ.) Moreover if (V,ρ) is de Rham, then we can find a finite Galois extension L/K such that

$$(V \otimes_{\mathbb{Q}_l} B_{\mathrm{st}})^{\mathrm{Gal}\,(K^{ac}/L)}$$

is a free $\mathbb{Q}_l^{ac} \otimes_{\mathbb{Q}_l} L_0$-module of rank $\dim_{\mathbb{Q}_l^{ac}} V$, where $L_0/\mathbb{Q}_p$ is the maximal unramified subextension of $L/\mathbb{Q}_p$ (see [B]). Choose such an extension L and choose $\tau : L_0 \hookrightarrow \mathbb{Q}_l^{ac}$. Then set

$$W = (V \otimes_{\tau,L} B_{\mathrm{st}})^{\mathrm{Gal}\,(K^{ac}/L)} = (V \otimes_{\mathbb{Q}_l} B_{\mathrm{st}})^{\mathrm{Gal}\,(K^{ac}/L)} \otimes_{(\mathbb{Q}_l^{ac}\otimes_{\mathbb{Q}_l} L_0),1\otimes\tau} \mathbb{Q}_l^{ac}.$$

For $\sigma \in W_K$ lying above Frob_K^m set

$$r(\sigma) = \rho(\sigma) \otimes (\sigma\phi^{m[k(v_K):\mathbb{F}_l]}) \in GL(W).$$

Finally set

$$\mathrm{WD}(V,\rho) = (W, r, 1 \otimes N).$$

This is a Weil-Deligne representation of W_K, which up to isomorphism is independent of the choices of L and τ. In fact WD is a functor from de Rham representations of $\mathrm{Gal}\,(K^{ac}/K)$ over $\mathbb{Q}_l^{ac}$ to Weil-Deligne representations of W_K over $\mathbb{Q}_l^{ac}$. (These constructions are due to Fontaine. See [Fo] for details.)

In both the cases $l \neq p$ and $l = p$ the functor WD commutes with restriction to open subgroups and tensor operations. Recall the following standard conjecture.

Conjecture 1.1. *Suppose that X/K is a proper smooth variety purely of dimension n. Suppose also that $\sigma \in W_K$ and that $\Gamma \in CH^n(X \times_K X)$ is an algebraic correspondence. Then the alternating sum of the trace*

$$\sum_{i=0}^{2n} (-1)^i \operatorname{tr}\big(\sigma\Gamma^* \big| \mathrm{WD}\big(H^i(X \times_K K^{ac}, \mathbb{Q}_l^{ac})\big)\big)$$

lies in $\mathbb{Q}$ and is independent of l. Here σ induces the automorphism by the right action on $X \times_K K^{ac}$, and Γ^* is the endomorphism defined by $\mathrm{pr}_{1,*} \circ([\Gamma]\cup) \circ \mathrm{pr}_2^*$.

For $l \neq p$ this conjecture was proven by T. Saito [Sa2]. In the introduction to that paper he expresses the conviction that the case $l = p$ can be handled by the same methods. This was carried out by Ochiai [O] in the case that Γ is the diagonal $X \subset X \times_K X$.

Now we turn to some global considerations. Let L be a finite, imaginary CM extension of $\mathbb{Q}$. Let $c \in \mathrm{Aut}(L)$ denote complex conjugation. Suppose that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- Π_∞ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from L to $\mathbb{Q}$ of GL_n;
- for some finite place x of L the representation Π_x is square integrable.

(In this paper 'square integrable' (resp. 'tempered') will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

$$R_{l,\imath}(\Pi) = R_l(\Pi) : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

such that for each finite place $v \nmid l$ of L,

$$\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)^{\mathrm{ss}} = (\imath R_l(\Pi)|_{W_{L_v}}^{\mathrm{ss}}, 0).$$

Moreover it is shown that Π_v is tempered for all finite places v of L, which completely determines the N in $\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$ (see Lemma 1.4 below). If $n = 1$ both these assertions are true without the assumptions that $\Pi \circ c \cong \Pi^\vee$ and (for the first assertion) $v \nmid l$.

The main theorem of this paper identifies $\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}}$. More precisely we prove the following.

Theorem 1.2. *Keep the above notation and assumptions. Then for each finite place $v \nmid l$ of L there is an isomorphism*

$$\imath\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}} \cong \mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$$

of Weil-Deligne representations over $\mathbb{C}$. *If Conjecture* 1.1 *is true, then this holds even for* $v|l$.

As $R_l(\Pi)$ is semisimple and $\mathrm{rec}\big(\Pi_v^\vee|\det|^{\frac{1-n}{2}}\big)$ is indecomposable if Π_v is square integrable, we have the following corollary.

Corollary 1.3. *If* Π_v *is square integrable for a finite place* $v\nmid l$ *or if conjecture* 1.1 *is true, then the representation* $R_l(\Pi)$ *is irreducible. (Recall that we are assuming that* Π_v *is square integrable for some finite place* v.*)*

In the rest of this section we consider some generalities on Galois representations and Weil-Deligne representations. First consider Weil-Deligne representations over an algebraically closed field Ω of characteristic zero and the same cardinality as $\mathbb{C}$. If (W,r) is a finite-dimensional representation of W_K with open kernel and if $s\in\mathbb{Z}_{\geq 1}$ we will write $\mathrm{Sp}_s(W)$ for the Weil-Deligne representation

$$\big(W^s,\ r|\mathrm{Art}_K^{-1}|_K^{s-1}\oplus\cdots\oplus r|\mathrm{Art}_K^{-1}|_K\oplus r,\ N\big),$$

where $N: r|\mathrm{Art}_K^{-1}|_K^{i-1}\xrightarrow{\sim} r|\mathrm{Art}_K^{-1}|_K^{i}$ for $i=1,...,s-1$. This defines $\mathrm{Sp}_s(W)$ uniquely (up to isomorphism). If W is irreducible, then $\mathrm{Sp}_s(W)$ is indecomposable and every indecomposable Weil-Deligne representation is of the form $\mathrm{Sp}_s(W)$ for a unique s and a unique irreducible W. If π is an irreducible cuspidal representation of $GL_g(K)$, then $\mathrm{rec}(\pi)=(r,0)$ with r irreducible. Moreover for any $s\in\mathbb{Z}_{\geq 1}$ we have (in the notation of section I.3 of [HT]) $\mathrm{rec}(\mathrm{Sp}_s(\pi))=\mathrm{Sp}_s(r)$.

If $q\in\mathbb{R}_{>0}$, then by a *Weil* q*-number* we mean $\alpha\in\mathbb{Q}^{ac}$ such that for all $\sigma:\mathbb{Q}^{ac}\hookrightarrow\mathbb{C}$ we have $(\sigma\alpha)(c\sigma\alpha)=q$. We will call a Weil-Deligne representation (V,r,N) of W_K *strictly pure of weight* $k\in\mathbb{R}$ if for some (and hence every) lift ϕ of Frob_{v_K}, every eigenvalue α of $r(\phi)$ is a Weil $(\#k(v_K))^k$-number. In this case we must have $N=0$. We will call (V,r,N) *mixed* if it has an increasing filtration Fil_i^W with $\mathrm{Fil}_i^W V=V$ for $i\gg 0$ and $\mathrm{Fil}_i^W V=(0)$ for $i\ll 0$, such that the i-th graded piece is strictly pure of weight i. If (V,r,N) is mixed, then there is a unique choice of filtration Fil_i^W, and $N(\mathrm{Fil}_i^W V)\subset\mathrm{Fil}_{i-2}^W V$. Finally we will call (V,r,N) *pure of weight* k if it is mixed with all weights in $k+\mathbb{Z}$ and if for all $i\in\mathbb{Z}_{>0}$,

$$N^i:\mathrm{gr}_{k+i}^W V\xrightarrow{\sim}\mathrm{gr}_{k-i}^W V.$$

If W is strictly pure of weight k, then $\mathrm{Sp}_s(W)$ is pure of weight $k-(s-1)$. (It is generally conjectured that if X is a proper smooth variety over a p-adic field K, then $\mathrm{WD}\big(H^i(X\times_K K^{ac},\mathbb{Q}_l^{ac})\big)$ is pure of weight i in the above sense.)

Lemma 1.4. (1) *(V,r,N) is pure if and only if $(V,r,N)^{F\text{-ss}}$ is.*

(2) *If L/K is a finite extension, then (V,r,N) is pure if and only if $(V,r,N)|_{W_L}$ is pure.*

(3) *An irreducible smooth representation π of $GL_n(K)$ has $\sigma\pi$ tempered for all $\sigma:\Omega\hookrightarrow\mathbb{C}$ if and only if $\mathrm{rec}(\pi)$ is pure of some weight.*

(4) *Given (V,r) with r semisimple, there is, up to equivalence, at most one choice of N which makes (V,r,N) pure.*

(5) *If (V,r,N) is a Frobenius semisimple Weil-Deligne representation which is pure of weight k and if $W\subset V$ is a Weil-Deligne subrepresentation, then the following are equivalent:*

(a) *$\bigwedge^{\dim W} W$ is pure of weight $k\dim W$,*

(b) *W is pure of weight k,*

(c) *W is a direct summand of V.*

(6) *Suppose that* (V, r, N) *is a Frobenius semisimple Weil-Deligne representation which is pure of weight* k*. Suppose also that* $\mathrm{Fil}^j V$ *is a decreasing filtration of* V *by Weil-Deligne subrepresentations such that* $\mathrm{Fil}^j V = (0)$ *for* $j \gg 0$ *and* $\mathrm{Fil}^j V = V$ *for* $j \ll 0$*. If for each* j

$$\bigwedge^{\dim \mathrm{gr}^j V} \mathrm{gr}^j V$$

is pure of weight $k \dim \mathrm{gr}^j V$*, then*

$$V \cong \bigoplus_j \mathrm{gr}^j V$$

and each $\mathrm{gr}^j V$ *is pure of weight* k*.*

Proof. The first two parts are straightforward (using the fact that the filtration Fil_i^W is unique). For the third part recall that an irreducible smooth representation $\mathrm{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{Sp}_{s_t}(\pi_t)$ (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the $\mathrm{Sp}_{s_i}(\pi_i)$ are all equal.

Suppose that (V, r, N) is Frobenius semisimple and pure of weight k. As a W_K-module we can write V uniquely as $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where $(V_i, r|_{V_i}, 0)$ is strictly pure of weight $k + i$. For $i \in \mathbb{Z}_{\geq 0}$ let $V(i)$ denote the kernel of $N^{i+1} : V_i \to V_{-i-2}$. Then $N : V_{i+2} \hookrightarrow V_i$ is injective and $V_i = NV_{i+2} \oplus V(i)$. Thus

$$V = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j V(i),$$

and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual W_K-module $[V(i)] = [V_i] - [V_{i+2} \otimes |\mathrm{Art}_K^{-1}|_K]$. Thus if r is semisimple, then (V, r) determines (V, r, N) up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If W is a direct summand it is certainly pure of the same weight k and $\bigwedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if W is pure of weight k, then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j W(i),$$

where $W(i) = W \cap V(i)$. As a W_K-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_j \mathrm{Sp}_{s_j}(X_j),$$

where each X_j is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum_j k_j (\dim \mathrm{Sp}_{s_j}(X_j)) = 0$. On the other hand, as V is pure we see that $k_j \leq 0$ for all j. We conclude that $k_j = 0$ for all j and hence that W is pure of weight k.

The final part follows from the fifth part by a simple inductive argument. □

In view of parts (3) and (4) of this lemma, Theorem 1.2 will follow from the following two results, which we prove in the rest of this paper. (Recall that for each $\sigma \in \mathrm{Aut}(\mathbb{C})$ the representation $\sigma\Pi$ is again cuspidal automorphic [Cl] and hence for each finite place v the representation $\sigma\Pi_v$ is tempered.)

Theorem 1.5. *Keep the notation and assumptions of Theorem* 1.2. *Then for each finite place v of L,*

$$\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$$

is pure.

Proposition 1.6. *Keep the notation and assumptions of Theorem* 1.2. *Assume also that Conjecture* 1.1 *is true. Suppose finally that l' is a rational prime and that $\imath' : \mathbb{Q}_{l'}^{ac} \xrightarrow{\sim} \mathbb{C}$. Then for each finite place v of L,*

$$\imath\mathrm{WD}(R_{l,\imath}(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{\mathrm{ss}} \cong \imath'\mathrm{WD}(R_{l',\imath'}(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{\mathrm{ss}}.$$

Now let L denote a number field. Write $|\ |_L$ for

$$\prod_x |\ |_{L_x} : \mathbb{A}_L^\times/L^\times \longrightarrow \mathbb{R}_{>0}^\times,$$

and write Art_L for

$$\prod_x \mathrm{Art}_{L_x} : \mathbb{A}_L^\times/L^\times \twoheadrightarrow \mathrm{Gal}\,(L^{ac}/L)^{\mathrm{ab}}.$$

We will call a continuous representation

$$R : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

pure of weight k if for all but finitely many finite places x of L the representation R is unramified at x and every eigenvalue α of $R(\mathrm{Frob}_x)$ is a Weil $(\#k(x))^k$-number.

If

$$R : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_1(\mathbb{Q}_l^{ac})$$

is de Rham at all places $v|l$ of L, then there exist

- a CM (possibly totally real) field $L_0 \subset L$;
- an integer k;
- integers $n(\tau)$ for each $\tau\colon L_0 \hookrightarrow \mathbb{Q}_l^{ac}$, such that $n(\tau) + n(\tau c) = k$ for all τ;
- a continuous character $\chi : \mathbb{A}_L^\times \to (\mathbb{Q}^{ac})^\times \subset (\mathbb{Q}_l^{ac})^\times$ such that

$$\chi|_{L^\times} = \prod_{\tau : L \hookrightarrow \mathbb{Q}_l^{ac}} \tau^{n(\tau|_{L_0})};$$

with the following properties:

- $R \circ \mathrm{Art}_L(x) = \chi(x) \prod_{\tau : L \hookrightarrow \mathbb{Q}_l^{ac}} \tau(x_l)^{-n(\tau|_{L_0})}$;
- for all finite places v of L we have $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)}) = (\chi|_{L_v^\times} \circ \mathrm{Art}_{L_v}^{-1}, 0)$.

(See for instance [Se].) Then we see that $|\chi|^2 = |\ |_L^{-k}$ (because both are characters $\mathbb{A}_L^\times/(L_\infty^\times)^0 \to \mathbb{R}_{>0}^\times$ which agree on $L^\times$) and so for every finite place v of L the Weil-Deligne representation $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ is strictly pure of weight k. In particular R is pure of weight k.

We have the following lemma.

Lemma 1.7. *Suppose that M/L is a finite extension of number fields. Suppose also that*

$$R : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

is a continuous semisimple representation which is pure of weight k and such that the restriction $R|_{\mathrm{Gal}\,(L_x^{ac}/L_x)}$ is de Rham for all $x|l$. Suppose that

$$S : \mathrm{Gal}\,(M^{ac}/M) \longrightarrow GL_{an}(\mathbb{Q}_l^{ac})$$

is another continuous representation with $S^{\mathrm{ss}} \cong R|^a_{\mathrm{Gal}\,(M^{ac}/M)}$ for some $a \in \mathbb{Z}_{>0}$. Suppose finally that w is a place of M above a finite place v of L. Then if $\mathrm{WD}(S|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})$ *is pure of weight k, then* $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ *is also pure of weight k.*

Proof. Write

$$R|_{\mathrm{Gal}\,(M^{ac}/M)} = \bigoplus_i R_i,$$

where each R_i is irreducible. Then $\det R_i$ is de Rham at all places $x|l$ of M and is pure of weight $k \dim R_i$. Thus the top exterior power $\bigwedge^{\dim R_i} \mathrm{WD}(R_i|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})$ is also pure of weight $k \dim R_i$. Lemma 1.4(6) tells us that

$$\begin{aligned} \mathrm{WD}(S|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}} &\cong \Big(\bigoplus_i \mathrm{WD}(R_i|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}} \Big)^a \\ &\cong \big(\mathrm{WD}(R|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}}\big)^a, \end{aligned}$$

and that $\mathrm{WD}(R|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}}$ is pure of weight k. Applying Lemma 1.4(1) and (2), we see that $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ is also pure of weight k. $\square$

2. Shimura varieties

In this section we recall some facts about the Shimura varieties considered in [HT] and prove Proposition 1.6.

In this section,

- let E be an imaginary quadratic field, F^+ a totally real field and set $F = EF^+$;
- let p be a rational prime which splits as $p = uu^c$ in E;
- let $w = w_1, w_2, ..., w_r$ be the places of F above u;
- let B be a division algebra with centre F such that
  - $\dim_F B = n^2$,
  - $B^{\mathrm{op}} \cong B \otimes_{F,c} F$,
  - at every place x of F either B_x is split or a division algebra,
  - if n is even, then the number of finite places of F^+ above which B is ramified is congruent to $1 + \frac{n}{2}[F^+ : \mathbb{Q}]$ modulo 2.

Pick a positive involution $*$ on B with $*|_F = c$. Let $V = B$ as a $B \otimes_F B^{\mathrm{op}}$-module. For $\beta \in B^{*=-1}$ define a pairing

$$\begin{array}{rcl} (\ ,\ ) : V \times V & \longrightarrow & \mathbb{Q} \\ (x_1, x_2) & \longmapsto & \mathrm{tr}_{F/\mathbb{Q}}\, \mathrm{tr}_{B/F}(x_1 \beta x_2^*). \end{array}$$

Also define an involution $\#$ on B by $x^\# = \beta x^* \beta^{-1}$ and a reductive group $G/\mathbb{Q}$ by setting, for any $\mathbb{Q}$-algebra R, the group $G(R)$ equal to the set of

$$(\lambda, g) \in R^\times \times (B^{\mathrm{op}} \otimes_{\mathbb{Q}} R)^\times$$

such that

$$gg^{\#} = \lambda.$$

Let $\nu : G \to \mathbb{G}_m$ denote the multiplier character sending (λ, g) to λ. Note that if x is a rational prime which splits $x = yy^c$ in E, then

$$\begin{array}{ccc} G(\mathbb{Q}_x) & \xrightarrow{\sim} & (B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times \\ (\lambda, g) & \longmapsto & (g_y, \lambda). \end{array}$$

We can and will assume that

- if x is a rational prime which does not split in E, then $G \times \mathbb{Q}_x$ is quasi-split;
- the pairing $(\ ,\ )$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariants $(1, n-1)$ at one embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and invariants $(0, n)$ at all other embeddings $F^+ \hookrightarrow \mathbb{R}$.

(See section I.7 of [HT] for details.)

Let U be an open compact subgroup of $G(\mathbb{A}^\infty)$. Define a functor $\mathfrak{X}_U$ from the category of pairs (S, s), where S is a connected locally Noetherian F-scheme and s is a geometric point of S, to the category of sets, sending (S, s) to the set of isogeny classes of four-tuples $(A, \lambda, i, \overline{\eta})$, where

- A/S is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
- $i : B \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\operatorname{Lie} A \otimes_{(E \otimes_{\mathbb{Q}} \mathcal{O}_S), 1 \otimes 1} \mathcal{O}_S$ is locally free over $\mathcal{O}_S$ of rank n and the two actions of F^+ coincide;
- $\lambda : A \to A^\vee$ is a polarisation such that for all $b \in B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$;
- $\overline{\eta}$ is a $\pi_1(S, s)$-invariant U-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which take the standard pairing $(\ ,\ )$ on V to a $(\mathbb{A}^\infty)^\times$-multiple of the λ-Weil pairing on VA_s.

Here $VA_s = \left(\varprojlim_N A[N](k(s))\right) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If s and s' are both geometric points of a connected locally Noetherian F-scheme S, then $\mathfrak{X}_U(S, s)$ and $\mathfrak{X}_U(S, s')$ are in canonical bijection. Thus we may think of $\mathfrak{X}_U$ as a functor from connected locally Noetherian F-schemes to sets. We may further extend it to a functor from all locally Noetherian F-schemes to sets by setting

$$\mathfrak{X}_U\left(\coprod_i S_i\right) = \prod_i \mathfrak{X}_U(S_i).$$

If U is sufficiently small (i.e., for some finite place x of $\mathbb{Q}$ the projection of U to $G(\mathbb{Q}_x)$ contains no element of finite order except 1), then $\mathfrak{X}_U$ is represented by a smooth projective variety X_U/F of dimension $n-1$. The inverse system of the X_U for varying U has a natural right action of $G(\mathbb{A}^\infty)$.

We will write $\mathcal{A}_U$ for the universal abelian variety over X_U. The action of $G(\mathbb{A}^\infty)$ on the inverse system of the X_U extends to an action by quasi-isogenies on the inverse system of the $\mathcal{A}_U$. More precisely if $g^{-1}Vg \subset U$, then we have a map $g : X_V \to X_U$ and there is also a quasi-isogeny of abelian varieties over X_V,

$$g : \mathcal{A}_V \longrightarrow g^*\mathcal{A}_U.$$

(By a quasi-isogeny from A to B we mean an element of $\operatorname{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$, some integer multiple of which is an isogeny.)

Now suppose l is a rational prime and ξ is an irreducible algebraic representation of G over $\mathbb{Q}_l^{ac}$. This defines a lisse $\mathbb{Q}_l^{ac}$-sheaf $\mathcal{L}_\xi = \mathcal{L}_{\xi,l}$ over each X_U and the action of $G(\mathbb{A}^\infty)$ extends to these sheaves. (See section III.2 of [HT] for details.) We write

$$H^i(X, \mathcal{L}_{\xi,l}) = \varinjlim_U H^i(X_U \times_F F^{ac}, \mathcal{L}_{\xi,l}).$$

It is a semisimple admissible representation of $G(\mathbb{A}^\infty)$ with a commuting continuous action of $\mathrm{Gal}\,(F^{ac}/F)$. Thus we can write

$$H^i(X, \mathcal{L}_{\xi,l}) = \bigoplus_\pi \pi \otimes R^i_{\xi,l}(\pi),$$

where π runs over irreducible admissible representations of $G(\mathbb{A}^\infty)$ over $\mathbb{Q}_l^{ac}$, and $R^i_{\xi,l}(\pi)$ is a finite-dimensional continuous representation of $\mathrm{Gal}\,(F^{ac}/F)$ over $\mathbb{Q}_l^{ac}$. We recall the following results from [HT] (section VI.2 and corollary V.6.2).

Lemma 2.1. *Keep the above notation and assumptions. Suppose that $R^i_{\xi,l}(\pi) \neq (0)$ and $R^j_{\xi,l}(\pi') \neq (0)$. If $\pi_x \cong \pi'_x$ for all but finitely many finite places x of $\mathbb{Q}$, then $\pi_x \cong \pi'_x$ for all finite places x of $\mathbb{Q}$ which split in E.*

Lemma 2.2. *Keep the above notation and assumptions and choose $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Suppose that $R^i_{\xi,l}(\pi) \neq (0)$. Then there is an automorphic representation Π of $GL_n(\mathbb{A}_F)$ which occurs in the discrete spectrum and a character ψ of $\mathbb{A}_E^\times/E^\times$ such that for all but finitely many finite places x of $\mathbb{Q}$ we have $(\psi_x, \Pi_x) = \mathrm{BC}(\imath\pi_x)$ in the notation of section VI.2 of* [HT]. *(This characterises (ψ, Π) completely.)*

If moreover Π is cuspidal, then the following hold.

1. *$i = n - 1$.*
2. *$\Pi^\vee = \Pi^c$.*
3. *Π_∞ has the same infinitesimal character as an algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n.*
4. *ψ_∞ has the same infinitesimal character as an algebraic representation of the restriction of scalars from E to $\mathbb{Q}$ of GL_1.*
5. *Π_x is square integrable at some finite place of F. (In fact at any place for which B_x is not split.)*
6. *$R^{n-1}_{\xi,l}(\pi)^{\mathrm{ss}} \cong R_{l,\imath}(\Pi)^a \otimes R_{l,\imath}(\psi)$ for some $a \in \mathbb{Z}_{>0}$.*

We will write $(\psi, \Pi) = \mathrm{BC}_\imath(\pi)$. We will call $\mathrm{BC}_\imath(\pi)$ *cuspidal* if Π is cuspidal.

For an irreducible algebraic representation ξ, in [HT], section III.2, integers $t_\xi, m_\xi \geq 0$ and an element $\varepsilon_\xi \in \mathbb{Q}[S_{m_\xi}]$ (where S_{m_ξ} is the symmetric group of m_ξ letters) are defined. We also set for each integer $N \geq 2$,

$$\varepsilon(m_\xi, N) = \prod_{x=1}^{m_\xi} \prod_{y \neq 1} \frac{[N]_x - N^y}{N - N^y} \in \mathbb{Q}[(N^{\mathbb{Z}_{\geq 0}})^{m_\xi}],$$

where $[N]_x$ denotes the endomorphism generated by multiplication by N on the x-th factor, and y runs from 0 to $2[F^+ : \mathbb{Q}]n^2$ but excluding 1. Also set

$$a_\xi = a_{\xi,N} = \varepsilon_\xi \varepsilon(m_\xi, N)^{2n-1} \in \mathbb{Q}[(N^{\mathbb{Z}_{\geq 0}})^{m_\xi} \rtimes S_{m_\xi}].$$

If we think of $(N^{\mathbb{Z}_{\geq 0}})^{m_\xi} \rtimes S_{m_\xi} \subset \mathrm{End}\,(\mathcal{A}_U^{m_\xi}/X_U)$, then $a_\xi \in \mathrm{End}\,(\mathcal{A}_U^{m_\xi}/X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$. Write proj for the map $\mathcal{A}_U^{m_\xi} \to X_U$ and proj_i for the composition of the i^{th} inclusion

$\mathcal{A}_U \hookrightarrow \mathcal{A}_U^{m_\xi}$ with proj. Then

- $R^j \mathrm{proj}_* \mathbb{Q}_l^{ac} = \bigoplus_{i_1+\ldots+i_{m_\xi}=j} \bigotimes_{a=1}^{m_\xi} \wedge^{i_a} R^1 \mathrm{proj}_{a\,*} \mathbb{Q}_l^{ac}$;
- $\varepsilon(m_\xi, N)$ acts as an idempotent on $R^j \mathrm{proj}_* \mathbb{Q}_l^{ac}$;
- $\varepsilon(m_\xi, N) R^j \mathrm{proj}_* \mathbb{Q}_l^{ac} = \begin{cases} (0) & (j \neq m_\xi), \\ \bigotimes_{a=1}^{m_\xi} R^1 \mathrm{proj}_{a\,*} \mathbb{Q}_l^{ac} & (j = m_\xi); \end{cases}$
- $\varepsilon_\xi \varepsilon(m_\xi, N) R^{m_\xi} \mathrm{proj}_* \mathbb{Q}_l^{ac}(t_\xi) \cong \mathcal{L}_\xi$;
- a_ξ acts as an idempotent on each $H^j(\mathcal{A}_U^{m_\xi} \times_F F^{ac}, \mathbb{Q}_l^{ac}(t_\xi))$;
- $a_\xi H^j(\mathcal{A}_U^{m_\xi} \times_F F^{ac}, \mathbb{Q}_l^{ac}(t_\xi)) \cong \begin{cases} (0) & (j < m_\xi), \\ H^{j-m_\xi}(X_U \times_F F^{ac}, \mathcal{L}_\xi) & (j \geq m_\xi). \end{cases}$

Note that a_ξ commutes with the action of $G(\mathbb{A}^\infty)$. Moreover if $\imath : \mathbb{Q}_l^{ac} \cong \mathbb{Q}_{l'}^{ac}$, then the same a_ξ will work for ξ and $\imath\xi$. (Note that in [HT] the exponent $2n-1$ in the definition of a_ξ is erroneously missed. This gives an idempotent on $R^j \mathrm{proj}_* \mathbb{Q}_l^{ac}(t_\xi)$ with the desired properties, but we do not know that it is an idempotent on $H^j(\mathcal{A}_U^{m_\xi} \times_F F^{ac}, \mathbb{Q}_l^{ac}(t_\xi))$. We only know that this group has a filtration of length $2n-1$ (coming from the composite functor spectral sequence) on each of whose graded pieces $\varepsilon(m_\xi, N)$ is an idempotent.)

Lemma 2.3. *Keep the above notation and assumptions. Let $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$ and $\imath' : \mathbb{Q}_{l'}^{ac} \xrightarrow{\sim} \mathbb{C}$. Suppose that $R^i_{\xi,l}(\pi) \neq (0)$ and that $\mathrm{BC}_\imath(\pi)$ is cuspidal. Suppose also that w is a place of F above a split place of E such that B_w is split. If we assume Conjecture* 1.1*, then, writing ξ' for $(\imath')^{-1}\imath\xi$ and π' for $(\imath')^{-1}\imath\pi$,*

$$\imath \mathrm{WD}(R^{n-1}_{\xi,l}(\pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})^{\mathrm{ss}} \cong \imath' \mathrm{WD}(R^{n-1}_{\xi',l'}(\pi')|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})^{\mathrm{ss}}.$$

Proof. Choose an open compact subgroup $U \subset G(\mathbb{A}^\infty)$ such that $\pi^U \neq (0)$. Also choose $e \in \mathbb{Q}_l^{ac}[U \backslash G(\mathbb{A}^\infty)/U]$ such that e is an idempotent on each $H^j(X_U \times_F F^{ac}, \mathcal{L}_\xi)$ and

$$eH^j(X_U \times_F F^{ac}, \mathcal{L}_\xi) = \pi^U \otimes R^j_{\xi,l}(\pi),$$

and such that $e' = (\imath')^{-1}\imath e$ is an idempotent on each $H^j(X_U \times_F F^{ac}, \mathcal{L}_{\xi'})$ and

$$e'H^j(X_U \times_F F^{ac}, \mathcal{L}_{\xi'}) = (\pi')^U \otimes R^j_{\xi',l'}(\pi').$$

Then, by Conjecture 1.1, Lemma 2.2 and the above discussion, for $\sigma \in W_{F_w}$ we have

$$\begin{aligned}
&(\dim \pi^U)\, \mathrm{tr}\big(\sigma \big| \mathrm{WD}(R^{n-1}_{\xi,l}(\pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})\big) \\
&= \sum_j (-1)^{n-1-j}\, \mathrm{tr}\big(\sigma e \big| \mathrm{WD}\big(H^j(X_U \times_{F_w} F_w^{ac}, \mathcal{L}_\xi)\big)\big) \\
&= \sum_j (-1)^{n-1+m_\xi-j}\, \mathrm{tr}\big(\sigma e a_\xi \big| \mathrm{WD}\big(H^j(\mathcal{A}_U^{m_\xi} \times_{F_w} F_w^{ac}, \mathbb{Q}_l^{ac}(t_\xi))\big)\big) \\
&= \sum_j (-1)^{n-1+m_{\xi'}-j}\, \mathrm{tr}\big(\sigma e' a_{\xi'} \big| \mathrm{WD}\big(H^j(\mathcal{A}_U^{m_{\xi'}} \times_{F_w} F_w^{ac}, \mathbb{Q}_{l'}^{ac}(t_{\xi'}))\big)\big) \\
&= (\dim (\pi')^U)\, \mathrm{tr}\big(\sigma \big| \mathrm{WD}(R^{n-1}_{\xi',l'}(\pi')|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})\big).
\end{aligned}$$

The lemma follows. ∎

Corollary 2.4. *Keep the above notation. Let $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Suppose that $R^i_{\xi,l}(\pi) \neq (0)$ and that $\mathrm{BC}_\imath(\pi) = (\psi, \Pi)$ is cuspidal. Suppose also that w is a place of F above* *a split place of E such that B_w is split. If we assume Conjecture* 1.1, *then*

$$\imath\mathrm{WD}(R_{l,\imath}(\Pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})^{\mathrm{ss}} = \mathrm{rec}\big(\Pi_w^\vee|\det|^{\frac{1-n}{2}}\big)^{\mathrm{ss}}.$$

Proof. If $v \nmid l$, this is part of theorem VII.1.9 of [HT]. Thus suppose $v|l$. Choose $l' \neq l$ and $\imath' : \mathbb{Q}_{l'}^{ac} \xrightarrow{\sim} \mathbb{C}$. It suffices to show that

$$\imath\mathrm{WD}(R_{l,\imath}(\Pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})^{\mathrm{ss}} \cong \imath'\mathrm{WD}(R_{l',\imath'}(\Pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})^{\mathrm{ss}}.$$

But this follows from Lemmas 2.2 and 2.3. □

We can now complete the proof of Proposition 1.6. Recall that L is an imaginary CM field and that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- Π_∞ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from L to $\mathbb{Q}$ of GL_n;
- for some finite place x of L the representation Π_x is square integrable.

Recall also that v is a place of L above a rational prime p. Choose a CM field L' which is a quadratic extension of L in which x and v split. Also choose primes x' above x and v' above v of L', with $x' \nmid v'(v')^c$. (This is only important in the case that x and v lie above the same place of the maximal totally real subfield of L.)

Choose an imaginary quadratic field E not contained in L', in which p and the rational prime below x split. Let $p = uu^c$ in E, let $F = EL'$ and let F^+ denote the maximal totally real subfield of F. Also choose places z (resp. w) of F above x' (resp. v') such that $w|_E = u$. Denote by Π_F the base change of Π to $GL_n(\mathbb{A}_F)$. Note that $\Pi_{F,w}$ is square integrable and hence Π_F is cuspidal.

Choose a division algebra B with centre F as at the start of this section and satisfying

- B_y is split for all places $y \nmid zz^c$ of F.

Also choose $*$, β and G as at the start of this section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

- a character $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$,
- an irreducible algebraic representation ξ of G over $\mathbb{C}$,
- an irreducible representation π of $G(\mathbb{A}^\infty)$ over $\mathbb{C}$,

such that

- $R^{n-1}_{\imath\xi}(\imath\pi) \neq (0)$,
- $R^{n-1}_{\imath'\xi}(\imath'\pi) \neq (0)$,
- $\mathrm{BC}_\imath(\imath\pi) = \mathrm{BC}_{\imath'}(\imath'\pi) = (\psi, \Pi)$.

Proposition 1.6 now follows from the previous corollary.

3. Integral models for Iwahori level structure

We keep the notation of the last section. By a *geometric point* of a scheme X, we mean a map $s\colon \mathrm{Spec}\,k \to X$, where k is a field, such that the algebraic closure of the residue field of $s(\mathrm{Spec}\,k)$ is isomorphic to k. We call it a *closed geometric point* if its image $s(\mathrm{Spec}\,k)$ is a closed point of X.

Choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of B with $\mathcal{O}_B^* = \mathcal{O}_B$. Let $\mathcal{O}_{F,w}$ be the integer ring of F_w, ϖ_w a uniformiser for $\mathcal{O}_{F,w}$, and $\mathcal{O}_{B,w} = \mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,w}$. Also fix an isomorphism $\mathcal{O}_{B,w}^{\mathrm{op}} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1,0,0,...,0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \left(\prod_{i=2}^{r} (B_{w_i}^{\mathrm{op}})^\times\right) \times GL_n(F_w) \times \mathbb{Q}_p^\times.$$

For $m = (m_2, ..., m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set

$$U_p^w(m) = \prod_{i=2}^{r} \ker\big((\mathcal{O}_{B,w_i}^{\mathrm{op}})^\times \to (\mathcal{O}_{B,w_i}^{\mathrm{op}}/w_i^{m_i})^\times\big) \subset \prod_{i=2}^{r} (B_{w_i}^{\mathrm{op}})^\times.$$

Let B_n denote the Borel subgroup of GL_n consisting of upper triangular matrices, and let $\mathrm{Iw}_{n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo w to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:

$$\mathrm{Ma}(m) = U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^\times,$$
$$\mathrm{Iw}(m) = U_p^w(m) \times \mathrm{Iw}_{n,w} \times \mathbb{Z}_p^\times.$$

If U^p is an open compact subgroup of $G(\mathbb{A}^{\infty,p})$, we will write U_0 (resp. U) for $U^p \times \mathrm{Ma}(m)$ (resp. $U^p \times \mathrm{Iw}(m)$).

We recall that in section III.4 [HT] an integral model of X_{U_0} over $\mathcal{O}_{F,w}$ is defined (for U^p sufficiently small). We denote the integral model also by X_{U_0}, by a slight abuse of notation (similarly for the integral model of X_U defined later). It represents a functor $\mathfrak{X}_{U_0}$ from locally Noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $\mathfrak{X}_{U_0}$ is initially defined on the category of connected locally Noetherian $\mathcal{O}_{F,w}$-schemes with a geometric point to sets. It sends (S,s) to the set of prime-to-p isogeny classes of $(r+3)$-tuples $(A, \lambda, i, \overline{\eta}^p, \alpha_i)$, where

- A/S is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\mathrm{Lie}\,A \otimes_{(\mathcal{O}_{E,u} \otimes_{\mathbb{Z}_p} \mathcal{O}_S),1\otimes 1} \mathcal{O}_S$ is locally free of rank n and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda : A \to A^\vee$ is a prime-to-p polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$;
- $\overline{\eta}^p$ is a $\pi_1(S,s)$-invariant U^p-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which take the standard pairing $(\ ,\ )$ on V to a $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing on $V^p A_s$;
- for $2 \leq i \leq r$, $\alpha_i : (w_i^{-m_i}\mathcal{O}_{B,w_i}/\mathcal{O}_{B,w_i})_S \xrightarrow{\sim} A[w_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_B$-actions.

Then X_{U_0} is smooth and projective over $\mathcal{O}_{F,w}$ ([HT], page 109). As U^p varies, the inverse system of the X_{U_0}'s has an action of $G(\mathbb{A}^{\infty,p})$.

Given an $(r+3)$-tuple as above we will write $\mathcal{G}_A$ for $\varepsilon A[w^\infty]$, a Barsotti-Tate $\mathcal{O}_{F,w}$-module. Over a base in which p is nilpotent it is one dimensional and *compatible*; i.e., the two actions of $\mathcal{O}_{F,w}$ on $\mathrm{Lie}\,\mathcal{G}_A$ coincide (see [HT]). If $\mathcal{A}_{U_0}$ denotes the universal abelian scheme over X_{U_0}, we will write $\mathcal{G}$ for $\mathcal{G}_{\mathcal{A}_{U_0}}$.

Write $\overline{X}_{U_0}$ for the special fibre $X_{U_0} \times_{\mathrm{Spec}\,\mathcal{O}_{F,w}} \mathrm{Spec}\,k(w)$. For $0 \leq h \leq n-1$, we let $\overline{X}_{U_0}^{[h]}$ denote the reduced closed subscheme of $\overline{X}_{U_0}$ whose closed geometric points s are those for which the maximal étale quotient of $\mathcal{G}_s$ has $\mathcal{O}_{F,w}$-height at most h, and let

$$\overline{X}_{U_0}^{(h)} = \overline{X}_{U_0}^{[h]} - \overline{X}_{U_0}^{[h-1]}$$

(where we set $\overline{X}_{U_0}^{[-1]} = \emptyset$). Then $\overline{X}_{U_0}^{(h)}$ is nonempty and smooth of pure dimension h (lemma III.4.3, corollary III.4.4 of [HT]), and on it there is a short exact sequence

$$(0) \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{\mathrm{et}} \longrightarrow (0),$$

where $\mathcal{G}^0$ is a formal Barsotti-Tate $\mathcal{O}_{F,w}$-module and $\mathcal{G}^{\mathrm{et}}$ is an étale Barsotti-Tate $\mathcal{O}_{F,w}$-module with $\mathcal{O}_{F,w}$-height h.

Lemma 3.1. *If $0 \leq h \leq n-1$, then the Zariski closure of $\overline{X}_{U_0}^{(h)}$ contains $\overline{X}_{U_0}^{(0)}$.*

Proof. This is 'well known', but for lack of a reference we give a proof. Let x be a closed geometric point of $\overline{X}_{U_0}^{(0)}$. By lemma III.4.1 of [HT] the formal completion of $\overline{X}_{U_0} \times \operatorname{Spec} k(w)^{ac}$ at x is isomorphic to the equicharacteristic universal deformation ring of $\mathcal{G}_x$. According to the proof of proposition 4.2 of [Dr] this is isomorphic to $\operatorname{Spf} k(w)^{ac}[[T_1, ..., T_{n-1}]]$ and we can choose the T_i and a formal parameter S on the universal deformation of $\mathcal{G}_x$ such that

$$[\varpi_w](S) \equiv \varpi_w S + \sum_{i=1}^{n-1} T_i S^{\#k(w)^i} + S^{\#k(w)^n} \pmod{S^{\#k(w)^n+1}}.$$

Thus we get a morphism

$$\operatorname{Spec} k(w)^{ac}[[T_1, ..., T_{n-1}]] \longrightarrow \overline{X}_{U_0}$$

lying over $x\colon \operatorname{Spec} k(w)^{ac} \to \overline{X}_{U_0}$, such that, if k denotes the algebraic closure of the field of fractions of $k(w)^{ac}[[T_1, ..., T_{n-1}]]/(T_1, ..., T_{n-h-1})$, then the induced map

$$\operatorname{Spec} k \longrightarrow \overline{X}_{U_0}$$

factors through $\overline{X}_{U_0}^{(h)}$. Thus x is in the closure of $\overline{X}_{U_0}^{(h)}$, and the lemma follows. $\square$

Now we will define an integral model for X_U. It will represent a functor $\mathfrak{X}_U$ defined as follows. Again we initially define it as a functor from the category of connected locally Noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally Noetherian schemes to sets. The functor $\mathfrak{X}_U$ will send (S, s) to the set of prime-to-p isogeny classes of $(r+4)$-tuples $(A, \lambda, i, \overline{\eta}^p, \mathcal{C}, \alpha_i)$, where $(A, \lambda, i, \overline{\eta}^p, \alpha_i)$ is as in the definition of $\mathfrak{X}_{U_0}$ and $\mathcal{C}$ is a chain of isogenies

$$\mathcal{C} : \mathcal{G}_A = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_A/\mathcal{G}_A[w]$$

of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\#k(w)$ and with composite equal to the canonical map $\mathcal{G}_A \to \mathcal{G}_A/\mathcal{G}_A[w]$. There is a natural transformation of functors $\mathfrak{X}_U \to \mathfrak{X}_{U_0}$.

Lemma 3.2. *If U^p is sufficiently small, the functor $\mathfrak{X}_U$ is represented by a scheme X_U which is finite over X_{U_0}. The scheme X_U has some irreducible components of dimension n.*

Proof. By denoting the kernel of $\mathcal{G}_0 \to \mathcal{G}_j$ by $\mathcal{K}_j \subset \mathcal{G}[w]$, we can view the above chain as a flag

$$0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{G}[w]$$

of closed finite flat subgroup schemes with $\mathcal{O}_{F,w}$-action, with each $\mathcal{K}_j/\mathcal{K}_{j-1}$ having order $\#k(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over X_{U_0} defining $\mathcal{G}[w]$. Then $\mathfrak{X}_U$ is represented by a closed subscheme X_U of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus X_U is projective over $\mathcal{O}_{F,w}$. At each closed geometric point s of X_{U_0} the number of possible $\mathcal{O}_{F,w}$-submodules of $\mathcal{G}[w]_s \cong \mathcal{G}[w]_s^0 \times \mathcal{G}[w]_s^{\mathrm{et}}$ is finite, so X_U is finite over X_{U_0}. To see that X_U has some components of dimension n it suffices to note that on the generic fibre the map to X_{U_0} is finite étale. □

We say an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\#k(w)$ over a scheme S of characterstic p has *connected kernel* if it induces the zero map on $\mathrm{Lie}\,\mathcal{G}$. We will denote the Frobenius map by $F : \mathcal{G} \to \mathcal{G}^{(p)}$ and let $f = [k(w) : \mathbb{F}_p]$, and then $F^f : \mathcal{G} \to \mathcal{G}^{(\#k(w))}$ is an isogeny of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\#k(w)$ and has connected kernel.

We have the following rigidity lemma.

Lemma 3.3. *Let W denote the ring of integers of the completion of the maximal unramified extension of F_w. Suppose that R is an Artinian local W-algebra with residue field $k(w)^{ac}$. Suppose also that*

$$\mathcal{C} : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[w]$$

is a chain of isogenies of degree $\#k(w)$ of one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F,w}$-modules over R of $\mathcal{O}_{F,w}$-height n with composite equal to multiplication by ϖ_w. If every isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel (for $i = 1, ..., n$), then R is a $k(w)^{ac}$-algebra and $\mathcal{C}$ is the pull-back of a chain of Barsotti-Tate $\mathcal{O}_{F,w}$-modules over $k(w)^{ac}$, with all the isogenies isomorphic to F^f.

Proof. As the composite of the n isogenies induces multiplication by ϖ_w on the tangent space, $\varpi_w = 0$ in R, i.e., R is a $k(w)^{ac}$-algebra. Choose a parameter T_i for $\mathcal{G}_i$ over R. With respect to these choices, let $f_i(T_i) \in R[[T_i]]$ represent $\mathcal{G}_{i-1} \to \mathcal{G}_i$. We can write $f_i(T_i) = g_i(T_i^{p^{h_i}})$ with $h_i \in \mathbb{Z}_{\geq 0}$ and $g_i'(0) \neq 0$. (See [Fr], chapter I, §3, Theorem 2.) As $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel, $f_i'(0) = 0$ and $h_i > 0$. As f_i commutes with the action $[r]$ for all $r \in \mathcal{O}_{F,w}$, we have $\overline{r}^{p^{h_i}} = \overline{r}$ for all $\overline{r} \in k(w)$; hence h_i is a multiple of $f = [k(w) : \mathbb{F}_p]$. Reducing modulo the maximal ideal of R we see that $h_i \leq f$ and so in fact $h_i = f$ and $g_i'(0) \in R^\times$. Thus $\mathcal{G}_i \cong \mathcal{G}_0^{(\#k(w)^i)}$ in such a way that the isogeny $\mathcal{G}_0 \to \mathcal{G}_i$ is identified with F^{fi}. In particular $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^n)}$ and hence $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^{nm})}$ for any $m \in \mathbb{Z}_{\geq 0}$. As R is Artinian some power of the absolute Frobenius on R factors through $k(w)^{ac}$. Thus $\mathcal{G}_0$ is a pull-back from $k(w)^{ac}$ and the lemma follows. □

Now let $\overline{X}_U = X_U \times_{\mathrm{Spec}\,\mathcal{O}_{F,w}} \mathrm{Spec}\,k(w)$ denote the special fibre of X_U, and let $Y_{U,i}$ denote the closed subscheme of $\overline{X}_U$ over which $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel.

The following proposition may be known to experts. The case that $F_w/\mathbb{Q}_p$ is unramified follows from results of [G]; however, it is essential for our purposes to include the (harder) case where $F_w/\mathbb{Q}_p$ is ramified. The proposition could be proved via the usual "reduction to formal models" argument and some linear algebra. However the following argument seems to be easier and identifies formal parameters for the completion of the strict Henselisation of X_U at a geometric point of $X_U^{(0)}$. More precisely, they can be taken to be the scalars giving the linear maps $\mathrm{Lie}\,\mathcal{G}_{i-1} \to \mathrm{Lie}\,\mathcal{G}_i$ with respect to some bases.

Proposition 3.4. (1) *X_U has pure dimension n and semistable reduction over $\mathcal{O}_{F,w}$; that is, for all closed points x of the special fibre $\overline{X}_U$, there exists an* *étale morphism $V \to X_U$ with x contained in the image of V and an étale $\mathcal{O}_{F,w}$-morphism:*

$$V \longrightarrow \operatorname{Spec} \mathcal{O}_{F,w}[T_1, ..., T_n]/(T_1 \cdots T_m - \varpi_w)$$

for some $1 \leq m \leq n$, where ϖ_w is a uniformizer of $\mathcal{O}_{F,w}$.

(2) *X_U is regular and the natural map $X_U \to X_{U_0}$ is finite and flat.*

(3) *Each $Y_{U,i}$ is smooth over $\operatorname{Spec} k(w)$ of pure dimension $n-1$, $\overline{X}_U = \bigcup_{i=1}^n Y_{U,i}$ and, for $i \neq j$ the schemes $Y_{U,i}$ and $Y_{U,j}$ share no common connected component. In particular, X_U has strictly semistable reduction.*

Proof. In this proof we will make repeated use of the following version of Deligne's homogeneity principle ([DR]). Write W for the ring of integers of the completion of the maximal unramified extension of F_w. In what follows, if s is a closed geometric point of an $\mathcal{O}_{F,w}$-scheme X locally of finite type, then we write $\mathcal{O}^\wedge_{X,s}$ for the completion of the strict Henselisation of X at s, i.e., $\mathcal{O}^\wedge_{X \times \operatorname{Spec} W,s}$. Let $\mathbf{P}$ be a property of complete Noetherian local W-algebras such that if X is a $\mathcal{O}_{F,w}$-scheme locally of finite type, then the set of closed geometric points s of X for which $\mathcal{O}^\wedge_{X,s}$ has property $\mathbf{P}$ is Zariski open. Also let $X \to X_{U_0}$ be a finite morphism with the following properties.

(i) If s is a closed geometric point of $\overline{X}^{(h)}_{U_0}$, then, up to isomorphism, $\mathcal{O}^\wedge_{X,s}$ does not depend on s (but only on h).

(ii) There is a unique geometric point of X above any geometric point of $\overline{X}^{(0)}_{U_0}$.

If $\mathcal{O}^\wedge_{X,s}$ has property $\mathbf{P}$ for every geometric point of X over $\overline{X}^{(0)}_{U_0}$, then $\mathcal{O}^\wedge_{X,s}$ has property $\mathbf{P}$ for every closed geometric point of X. Indeed, if we let Z denote the closed subset of X where $\mathbf{P}$ does not hold, then its image in X_{U_0} is closed and either is empty or contains some $\overline{X}^{(h)}_{U_0}$. In the latter case, by Lemma 3.1, it also contains $\overline{X}^{(0)}_{U_0}$, which is impossible. Thus Z must be empty.

Note that both $X = X_U$ and $X = Y_{U,i}$ satisfy the above condition (ii) for the homogeneity principle, by letting $R = k(w)^{ac}$ in Lemma 3.3.

(1): The dimension of $\mathcal{O}^\wedge_{X_U,s}$ as s runs over geometric points of X_U above $\overline{X}^{(0)}_{U_0}$ is constant, say m. Applying the homogeneity principle to $X = X_U$ with $\mathbf{P}$ being 'dimension m', we see that X_U has pure dimension m. By Lemma 3.2 we must have $m = n$ and X_U has pure dimension n.

Now we will apply the above homogeneity principle to $X = X_U$ taking $\mathbf{P}$ to be 'isomorphic to $W[[T_1, ..., T_n]]/(T_1 \cdots T_m - \varpi_w)$ for some $m \leq n$'. By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of X_U have this property $\mathbf{P}$, then X_U is semistable of pure dimension n.

Let s be a geometric point of X_U over a point of $\overline{X}^{(0)}_{U_0}$. Choose a basis e_i of $\operatorname{Lie} \mathcal{G}_i$ over $\mathcal{O}^\wedge_{X_U,s}$ such that e_n maps to e_0 under the isomorphism $\mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[w] \xrightarrow{\sim} \mathcal{G}_0$ induced by ϖ_w. With respect to these bases let $X_i \in \mathcal{O}^\wedge_{X_U,s}$ represent the linear map $\operatorname{Lie} \mathcal{G}_{i-1} \to \operatorname{Lie} \mathcal{G}_i$. Then

$$X_1 \cdots X_n = \varpi_w.$$

Moreover, it follows from Lemma 3.3 that $\mathcal{O}^\wedge_{X_U,s}/(X_1, ..., X_n) = k(w)^{ac}$. (Because, by lemma III.4.1 of [HT], $\mathcal{O}^\wedge_{X_{U_0},s}$ is the universal deformation space of $\mathcal{G}_s$. Hence by Lemma 3.3, $\mathcal{O}^{\wedge}_{X_U,s}$ is the universal deformation space of the chain

$$\mathcal{G}_s \xrightarrow{F^f} \mathcal{G}_s^{(\#k(w))} \xrightarrow{F^f} \cdots \xrightarrow{F^f} \mathcal{G}_s^{(\#k(w)^n)} \cong \mathcal{G}_s/\mathcal{G}_s[\varpi_w].)$$

Thus we get a surjection

$$W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w) \twoheadrightarrow \mathcal{O}^{\wedge}_{X_U,s}$$

and as $\mathcal{O}^{\wedge}_{X_U,s}$ has dimension n this map must be an isomorphism.

(2): We see at once that X_U is regular. Then [AK] V, 3.6 tells us that $X_U \to X_{U_0}$ is flat.

(3): We apply the homogeneity principle to $X = Y_{U,i}$, taking $\mathbf{P}$ to be 'formally smooth of dimension $n-1$'. If s is a geometric point of $Y_{U,i}$ above $\overline{X}_{U_0}^{(0)}$, then we see that $\mathcal{O}^{\wedge}_{Y_{U,i},s}$ is cut out in $\mathcal{O}^{\wedge}_{X_U,s} \cong W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w)$ by the single equation $X_i = 0$. (We are using the parameters X_i defined above.) Thus

$$\mathcal{O}^{\wedge}_{Y_{U,i},s} \cong k(w)^{ac}[[X_1, ..., X_{i-1}, X_{i+1}, ..., X_n]]$$

is formally smooth of dimension $n-1$. We deduce that $Y_{U,i}$ is smooth of pure dimension $n-1$.

As our $\mathcal{G}/\overline{X}_U$ is one-dimensional, over a closed point, at least one of the isogenies $\mathcal{G}_{i-1} \to \mathcal{G}_i$ must have connected kernel, which shows that $\overline{X}_U = \bigcup_i Y_{U,i}$. Suppose $Y_{U,i}$ and $Y_{U,j}$ share a connected component Y for some $i \neq j$. Then Y would be finite flat over $\overline{X}_{U_0}$ and so the image of Y would meet $\overline{X}_{U_0}^{(n-1)}$. This is impossible, because above a closed point of $\overline{X}_{U_0}^{(n-1)}$ only one isogeny among the chain can have connected kernel. Thus, for $i \neq j$ the closed subschemes $Y_{U,i}$ and $Y_{U,j}$ have no connected component in common. □

By the strict semistability, if we write, for $S \subset \{1, ..., n\}$,

$$Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supsetneq S} Y_{U,T},$$

then $Y_{U,S}$ is smooth over $\operatorname{Spec} k(w)$ of pure dimension $n - \#S$ and $Y_{U,S}^0$ are disjoint for different S. With respect to the finite flat map $\overline{X}_U \to \overline{X}_{U_0}$, the inverse image of $\overline{X}_{U_0}^{[h]}$ is exactly the locus where at least $n-h$ of the isogenies have connected kernel, i.e., $\bigcup_{\#S \geq n-h} Y_{U,S}$. Hence the inverse image of $\overline{X}_{U_0}^{(h)}$ is equal to $\bigcup_{\#S = n-h} Y_{U,S}^0$.

The inverse systems $X_U \to X_{U_0}$, as U^p varies, have compatible actions of $G(\mathbb{A}^{\infty,p})$. For any S, the systems of subvarieties $Y_{U,S}$ and $Y_{U,S}^0$ are stable under this action. As in characterstic zero, these actions extend to actions on the universal abelian varieties $\mathcal{A}_{U_0}$ and $\mathcal{A}_U$ over these bases (which we again denote by the same symbols $\mathcal{A}_{U_0}$ and $\mathcal{A}_U$). This action is by prime-to-p quasi-isogenies. (By a prime-to-p quasi-isogeny we mean a quasi-isogeny such that the multiple by some integer prime to p is an isogeny of degree prime to p.)

Let l be a prime and ξ be an irreducible representation of G over $\mathbb{Q}_l^{ac}$. If $l \neq p$, then the sheaf $\mathcal{L}_\xi$ extends to a lisse sheaf on our integral models of X_{U_0} and X_U (using exactly the same construction as in characteristic zero), and $a_\xi = a_{\xi,N} \in \operatorname{End}(\mathcal{A}_U^{m_\xi}/X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$ extends to the $\mathcal{A}_U^{m_\xi}$ over integral models. We take N prime to p, so that a_ξ extend as étale morphisms on $\mathcal{A}_U^{m_\xi}$. Also we will denote $\mathcal{A}_U^{m_\xi} \times_{X_U} Y_{U,S}$ by $\mathcal{A}_{U,S}^{m_\xi}$ for simplicity. We make the following definitions.

- Define the admissible $G(\mathbb{A}^{\infty,p})$-modules with a commuting continuous action of $\mathrm{Gal}\,(F^{ac}/F)$:

$$H^j(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)=\varinjlim_{U^p} H^j(X_U\times_F F^{ac},\mathcal{L}_\xi)=H^j(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)},$$
$$H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathbb{Q}_l^{ac})=\varinjlim_{U^p} H^j(\mathcal{A}^{m_\xi}_U\times_F F^{ac},\mathbb{Q}_l^{ac}).$$

- If $l\neq p$, define admissible $G(\mathbb{A}^{\infty,p})\times \mathrm{Frob}_w^{\mathbb{Z}}$-modules:

$$H^j(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)=\varinjlim_{U^p} H^j(Y_{U,S}\times_{k(w)} k(w)^{ac},\mathcal{L}_\xi),$$
$$H^j_c(Y^0_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)=\varinjlim_{U^p} H^j_c(Y^0_{U,S}\times_{k(w)} k(w)^{ac},\mathcal{L}_\xi),$$
$$H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac})=\varinjlim_{U^p} H^j(\mathcal{A}^{m_\xi}_{U,S}\times_{k(w)} k(w)^{ac},\mathbb{Q}_l^{ac}).$$

- If $l=p$ and $\tau: W_0\hookrightarrow \mathbb{Q}_l^{ac}$ over $\mathbb{Z}_p=\mathbb{Z}_l$, set (let W_0 denote the Witt ring of $k(w)$):

$$H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}=\varinjlim_{U^p} H^j(\mathcal{A}^{m_\xi}_{U,S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac},$$

  an admissible $G(\mathbb{A}^{\infty,p})\times\mathrm{Frob}_w^{\mathbb{Z}}$-module. (Here $H^j(\mathcal{A}^{m_\xi}_{U,S}/W_0)$ denote crystalline cohomology and we let Frob_w act by the $[k(w):\mathbb{F}_p]$-power of the crystalline Frobenius.)

Note that if $l\neq p$, then a_ξ is an idempotent on $H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac})$ and

$$a_\xi H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac})=H^j(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)$$

(for the same reason this is true in characteristic zero). Similarly a_ξ defines an idempotent on each $H^j(\mathcal{A}^{m_\xi}_{U,S}/W_0)$ and hence on $H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}$, by the crystalline analogue of the same argument.

We will call two irreducible admissible representations π and π' of $G(\mathbb{A}^{\infty,p})$ *nearly equivalent* if $\pi_x\cong\pi'_x$ for all but finitely many rational primes $x\not| p$. If M is an admissible $G(\mathbb{A}^{\infty,p})$-module and π is an irreducible admissible representation of $G(\mathbb{A}^{\infty,p})$, then we define the π-*near isotypic* component $M[\pi]$ of M to be the largest $G(\mathbb{A}^{\infty,p})$-submodule of M all of whose irreducible subquotients are nearly equivalent to π. Then

$$M=\bigoplus M[\pi]$$

as π runs over representatives of near equivalence classes of irreducible admissible $G(\mathbb{A}^\infty)$-modules. (This follows from the following fact. Suppose that A is a (commutative) polynomial algebra over $\mathbb{C}$ in countably many variables, and that M is an A-module which is finitely generated over $\mathbb{C}$. Then we can write

$$M=\bigoplus_{\mathfrak{m}} M_{\mathfrak{m}},$$

where $\mathfrak{m}$ runs over maximal ideals of A with residue field $\mathbb{C}$.)

Note that as $\mathcal{A}_U$ is smooth over X_U the varieties $\mathcal{A}^{m_\xi}_U$ are strictly semistable. The special fibre is the union of the smooth subschemes $\mathcal{A}^{m_\xi}_{U,i}=\mathcal{A}^{m_\xi}_U\times_{X_U} Y_{U,i}$ for $1\leq i\leq n$, and for $i\neq j$ the subschemes $\mathcal{A}^{m_\xi}_{U,i}$ and $\mathcal{A}^{m_\xi}_{U,j}$ have nocomponent in common. Now we have:

Proposition 3.5. *Suppose that π is an irrreducible admissible representation of $G(\mathbb{A}^{\infty,p})$. For each rational prime l, there is a spectral sequence*

$$E_1^{i,j}(\mathrm{Iw}(m),\xi)[\pi] \Rightarrow \mathrm{WD}\big(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)}\big)[\pi],$$

where $E_1^{i,j}(\mathrm{Iw}(m),\xi) = \bigoplus_{s\geq \max(0,-i)} \bigoplus_{\#S=i+2s+1} H^j_{S,s}$ *and*

$$H^j_{S,s} = \begin{cases} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac}(t_\xi-s)) = H^{j-2s}(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi(-s)) & (l\neq p),\\ a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}(t_\xi-s) & (l=p).\end{cases}$$

Proof. We will use the functoriality of weight spectral sequences with respect to the pull-back by étale morphisms. This follows immediately from the étale local nature of the construction of weight spectral sequences. For $\ell=p$, this property has been used in [Sa1] and [O]. For $\ell\neq p$ this is proven for general morphisms (not necessarily étale) in [Sa2].

If $l\neq p$, then the Rapoport-Zink weight spectral sequence ([RZ], [Sa2]) is a spectral sequence

$$\begin{aligned} E_1^{i,j} &= \bigoplus_{s\geq\max(0,-i)}\bigoplus_{\#S=i+2s+1} H^{j-2s}(\mathcal{A}^{m_\xi}_{U,S}\times_{k(w)} k(w)^{ac},\mathbb{Q}_l^{ac}(t-s))\\ &\Rightarrow H^{i+j}(\mathcal{A}_U^{m_\xi}\times_F F_w^{ac},\mathbb{Q}_l^{ac}(t)).\end{aligned}$$

Taking $t=t_\xi$, applying a_ξ, replacing j by $j+m_\xi$, and passing to the direct limit over U^p we get a spectral sequence of $G(\mathbb{A}^{\infty,p})$-modules

$$\begin{aligned} E_1^{i,j}(\mathrm{Iw}(m),\xi) &= \bigoplus_{s\geq\max(0,-i)}\bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac}(t_\xi-s))\\ &\Rightarrow a_\xi H^{i+j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathbb{Q}_l^{ac}(t_\xi)) = H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi),\end{aligned}$$

or equivalently

$$\begin{aligned} E_1^{i,j}(\mathrm{Iw}(m),\xi) &= \bigoplus_{s\geq\max(0,-i)}\bigoplus_{\#S=i+2s+1} H^{j-2s}(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi(-s))\\ &\Rightarrow H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi).\end{aligned}$$

Hence we get the desired spectral sequence after passing to π-near isotypic components and identifying $a_\xi H^{i+j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathbb{Q}_l^{ac}(t_\xi))$ (resp. $H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)$) with their associated Weil-Deligne representations. Note that I_{F_w} acts trivially on these spaces, the spectral sequence is equivariant for the action of $\mathrm{Frob}_w^{\mathbb{Z}}$ and the endomorphism N is induced by the identity map

$$\begin{aligned} N\colon &\bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac}(t_\xi-s))\\ &\xrightarrow{\sim} \bigoplus_{\#S=(i+2)+2(s-1)+1} a_\xi H^{(j-2)+m_\xi-2(s-1)}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S},\mathbb{Q}_l^{ac}(t_\xi-(s-1)))\end{aligned}$$

(resp.

$$N\colon \bigoplus_{\#S=i+2s+1} H^{j-2s}(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi(-s)) \xrightarrow{\sim} \bigoplus_{\#S=(i+2)+2(s-1)+1} H^{(j-2)-2(s-1)}(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi(1-s))).$$

If $l=p$, then the Mokrane [M] weight spectral sequence is a spectral sequence

$$E_1^{i,j} = \bigoplus_{s\geq\max(0,-i)} \bigoplus_{\#S=i+2s+1} H^{j-2s}(\mathcal{A}^{m_\xi}_{U,S}/W_0)(-s) \Rightarrow H^{i+j}((\mathcal{A}^{m_\xi}_U)^\times/W_0^\times),$$

computing the log-crystalline cohomology of $\mathcal{A}^{m_\xi}_U$ in terms of the crystalline cohomology of the $\mathcal{A}^{m_\xi}_{U,S}$. Combining this with Tsuji's comparison theorem [Ts] we get, for any choice of an embedding $\tau : W_0 \hookrightarrow \mathbb{Q}_l^{ac}$ over $\mathbb{Z}_p=\mathbb{Z}_l$, a spectral sequence

$$E_1^{i,j} = \bigoplus_{s\geq\max(0,-i)} \bigoplus_{\#S=i+2s+1} H^{j-2s}(\mathcal{A}^{m_\xi}_{U,S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}(t-s) \Rightarrow \mathrm{WD}\big(H^{i+j}(\mathcal{A}^{m_\xi}_U\times_F F_w^{ac},\mathbb{Q}_l^{ac}(t))\big).$$

Taking $t=t_\xi$, applying a_ξ (which is a linear combination of étale morphisms by our choice of N), replacing j by $j+m_\xi$, and passing to the direct limit over U^p we get a spectral sequence of $G(\mathbb{A}^{\infty,p})$-modules

$$E_1^{i,j}(\mathrm{Iw}(m),\xi) = \bigoplus_{s\geq\max(0,-i)} \bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}(t_\xi-s) \Rightarrow \mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)}).$$

Hence we get the desired spectral sequence after passing to π-near isotypic components (because $G(\mathbb{A}^{\infty,p})$ acts by étale morphisms). On

$$\mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)}),$$

the inertia group I_{F_w} acts trivially, the action of Frob_w is compatible with the action of the crystalline Frobenius on $a_\xi H^{j-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)(t_\xi-s)$, and the endomorphism N of $\mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})$ is induced by the identity maps

$$N\colon \bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)(t_\xi-s)\otimes_{W_0,\tau}\mathbb{Q}_l^{ac} \xrightarrow{\sim} \bigoplus_{\#S=(i+2)+2(s-1)+1} a_\xi H^{(j-2)+m_\xi-2(s-1)}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0)(t_\xi-(s-1))\otimes_{W_0,\tau}\mathbb{Q}_l^{ac}.$$

□

4. Computing the cohomology of $Y_{U,S}$

In this section we use the results of [HT] to compute $H^j(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)$. As a result we can show that a large part of the above spectral sequences degenerates at E_1 and from this we deduce our main theorems.

We will keep the notation of the last section.

First we will relate the open strata $Y^0_{U,S}$ to the Igusa varieties of the first kind defined in [HT]. For $0\leq h\leq n-1$, $m_1\in\mathbb{Z}_{\geq 0}$ and $m\in\mathbb{Z}^{r-1}_{\geq 0}$, we write $I^{(h)}_{U^p,(m_1,m)}$

for the Igusa varieties of the first kind defined on page 121 of [HT]. We also define an *Iwahori-Igusa variety of the first kind*

$$I_U^{(h)}/\overline{X}_{U_0}^{(h)}$$

as the moduli space of chains of isogenies

$$\mathcal{G}^{\mathrm{et}} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_h = \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$$

of étale Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each isogeny having degree $\#k(w)$ and with composite equal to the natural map $\mathcal{G}^{\mathrm{et}} \to \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$. Then $I_U^{(h)}$ is finite étale over $\overline{X}_{U_0}^{(h)}$, and as the Igusa variety $I_{U^p,(1,m)}^{(h)}$ classifies the isomorphisms

$$\alpha_1^{\mathrm{et}} : (w^{-1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^h_{\overline{X}_{U_0}^{(h)}} \longrightarrow \mathcal{G}^{\mathrm{et}}[w],$$

the natural map

$$I_{U^p,(1,m)}^{(h)} \longrightarrow I_U^{(h)}$$

is finite étale and Galois with Galois group $B_h(k(w))$. Hence we can identify $I_U^{(h)}$ with $I_{U^p,(1,m)}^{(h)}/B_h(k(w))$. Note that the system $I_U^{(h)}$ for varying U^p naturally inherits the action of $G(\mathbb{A}^{\infty,p})$.

Lemma 4.1. *For $S \subset \{1, ..., n\}$ with $\#S = n - h$, there exists a finite map of $\overline{X}_{U_0}^{(h)}$-schemes*

$$\varphi : Y_{U,S}^0 \longrightarrow I_U^{(h)}$$

which is bijective on the geometric points.

Proof. The map is defined in a natural way from the chain of isogenies $\mathcal{C}$ by passing to the étale quotient $\mathcal{G}^{\mathrm{et}}$, and it is finite as $Y_{U,S}^0$ (resp. $I_U^{(h)}$) is finite (resp. finite étale) over $\overline{X}_{U_0}^{(h)}$. Let s be a closed geometric point of $I_U^{(h)}$ with a chain of isogenies

$$\mathcal{G}_s^{\mathrm{et}} = \mathcal{G}_0^{\mathrm{et}} \to \cdots \to \mathcal{G}_h^{\mathrm{et}} = \mathcal{G}_s^{\mathrm{et}}/\mathcal{G}_s^{\mathrm{et}}[w].$$

For $1 \le i \le n$ let $j(i)$ denote the number of elements of S which are less than or equal to i. Set $\mathcal{G}_i = (\mathcal{G}_s^0)^{(\#k(w)^{j(i)})} \times \mathcal{G}_{i-j(i)}^{\mathrm{et}}$. If $i \notin S$, define an isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ to be the identity times the given isogeny $\mathcal{G}_{i-1}^{\mathrm{et}} \to \mathcal{G}_i^{\mathrm{et}}$. If $i \in S$, define an isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ to be F^f times the identity. Then

$$\mathcal{G}_0 \to \cdots \to \mathcal{G}_n$$

defines the unique geometric point of $Y_{U,S}^0$ above s. □

Corollary 4.2. *Suppose that $l \neq p$. For every $S \subset \{1, ..., n\}$ with $\#S = n - h$ and every $i \in \mathbb{Z}_{\ge 0}$, we have isomorphisms*

$$\begin{aligned} H_c^i(Y_{U,S}^0 \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) &\xrightarrow{\sim} H_c^i(I_U^{(h)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) \\ &\xrightarrow{\sim} H_c^i(I_{U^p,(1,m)}^{(h)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi)^{B_h(k(w))} \end{aligned}$$

that are compatible with the actions of $G(\mathbb{A}^{\infty,p})$ when we vary U^p.

Proof. By Lemma 4.1, for any lisse $\mathbb{Q}_l^{ac}$-sheaf $\mathcal{F}$ on $I_U^{(h)}$, we have $\mathcal{F} \cong \varphi_*\varphi^*\mathcal{F}$ by looking at the stalks at all geometric points. As φ is finite the first isomorphism follows. The second isomorphism follows easily as $I_{U^p,(1,m)}^{(h)} \to I_U^{(h)}$ is finite étale and Galois with Galois group $B_h(k(w))$. □

If $l \neq p$, we define

$$H^j_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^j_c(I^{(h)}_U \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) = H^j_c(I^{(h)}, \mathcal{L}_\xi)^{U^w_p \times \mathrm{Iw}_{h,w}}$$

in the notation of page 136 of [HT]. It is an admissible $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}}$-module. In the notation of page 136 of [HT], Frob_w acts as

$$(1, p^{-[k(w):\mathbb{F}_p]}, -1, 1, 1) \in G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times/\mathbb{Z}_p^\times) \times \mathbb{Z} \times GL_h(F_w) \times \left(\prod_{i=2}^r (B^{\mathrm{op}}_{w_i})^\times\right),$$

where we have identified $D^\times_{F_w,n-h}/\mathcal{O}^\times_{D_{F_w,n-h}}$ with $\mathbb{Z}$ via $w \circ \det$. We also define elements of $\mathrm{Groth}\,(G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}})$ (we write $\mathrm{Groth}\,(G)$ for the Grothendieck group of admissible G-modules) as follows:

$$\begin{aligned}
\big[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)\big] &= \sum_i (-1)^{n-\#S-i} H^i(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi),\\
\big[H_c(Y^0_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)\big] &= \sum_i (-1)^{n-\#S-i} H^i_c(Y^0_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi),\\
\big[H_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)\big] &= \sum_i (-1)^{h-i} H^i_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi).
\end{aligned}$$

Finally set

$$[H(X, \mathcal{L}_\xi)] = \sum_j (-1)^{n-1-j} H^j(X, \mathcal{L}_\xi) \in \mathrm{Groth}\,(G(\mathbb{A}^\infty)).$$

Theorem V.5.4 of [HT] tells us that (for $l \neq p$)

$$\begin{aligned}
n\big[H_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)\big] &= n\big[H_c(I^{(h)}, \mathcal{L}_\xi)\big]^{U^w_p(m) \times \mathrm{Iw}_{h,w}}\\
&= \sum_i (-1)^{n-1-i} \mathrm{Red}^{(h)}\big[H^i(X, \mathcal{L}_\xi)^{U^w_p(m)}\big]
\end{aligned}$$

in $\mathrm{Groth}\,(G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}})$, where

$$\mathrm{Red}^{(h)} : \mathrm{Groth}\,(GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \mathrm{Groth}\,(\mathrm{Frob}_w^{\mathbb{Z}})$$

is the composite of the normalised Jacquet functor

$$J_{N_h^{\mathrm{op}}} : \mathrm{Groth}\,(GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \mathrm{Groth}\,(GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times)$$

with the functor

$$\mathrm{Groth}\,(GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \mathrm{Groth}\,(\mathrm{Frob}_w^{\mathbb{Z}})$$

which sends $[\alpha \otimes \beta \otimes \gamma]$ to

$$\begin{aligned}
&\sum_\phi \mathrm{vol}\,(D^\times_{F_w,n-h}/F_w^\times)^{-1}\,\mathrm{tr}\,\alpha(\varphi_{\mathrm{Sp}_{n-h}(\phi)})(\dim \beta^{\mathrm{Iw}_{h,w}})\\
&\quad \times \Big[\mathrm{rec}\big(\phi^{-1}|\ |_w^{\frac{1-n}{2}}(\gamma^{\mathbb{Z}_p^\times} \circ \mathbf{N}_{F_w/E_u})^{-1}\big)\Big],
\end{aligned}$$

where the sum is over characters ϕ of $F_w^\times/\mathcal{O}^\times_{F,w}$. (We just took the $\mathrm{Iw}_{h,w}$-invariant part of $\mathrm{Red}^{(h)}_1$, which is defined on p. 182 of [HT].)

Then, because

$$Y_{U,S} = \bigcup_{T \supset S} Y^0_{U,T}$$

for each $U = U^p \times \mathrm{Iw}(m)$, we have equalities

$$\begin{aligned}\big[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)\big] &= \sum_{T\supset S} (-1)^{(n-\#S)-(n-\#T)} \big[H_c(Y^0_{\mathrm{Iw}(m),T}, \mathcal{L}_\xi)\big] \\ &= \sum_{T\supset S} (-1)^{(n-\#S)-(n-\#T)} \big[H_c(I^{(n-\#T)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)\big].\end{aligned}$$

As there are $\binom{n-\#S}{h}$ subsets T with $\#T = n-h$ and $T \supset S$, we have proved the following lemma.

Lemma 4.3. *If $l \neq p$, then for every $S \subset \{1, ..., n\}$ we have an equality*

$$\big[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)\big] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \mathrm{Red}^{(h)} \big[H^i(X, \mathcal{L}_\xi)\big]^{U^w_p(m)}$$

in the Grothendieck group of admissible $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}}$-modules over $\mathbb{Q}_l^{ac}$.

The main innovation of our work is the following proposition.

Proposition 4.4. *Suppose that $l \neq p$ and $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Suppose also that π is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ such that*

- $\pi_p^{\mathrm{Iw}(m)} \neq (0)$,
- $R^j_{\xi,l}(\pi) \neq (0)$ *for some j,*
- $\mathrm{BC}_\imath(\pi)$ *is cuspidal.*

Then

$$H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)[\pi^p] = (0)$$

for $j \neq n - \#S$.

Proof. Let $\mathrm{BC}_\imath(\pi) = (\psi, \Pi)$. Then $H^j(X, \mathcal{L}_\xi)^{U^w_p(m)}[\pi^p] = (0)$ if $j \neq n-1$, while the space $H^{n-1}(X, \mathcal{L}_\xi)^{U^w_p(m)}[\pi^p]^{U^p}$ is $\imath^{-1}(\Pi_w \times \psi_u)$-isotypic for any open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$. (See Lemmas 2.1 and 2.2.) Moreover Π_w has an Iwahori fixed vector and ψ_u is unramified, so that

$$\begin{aligned}&(\dim \Pi_w^{\mathrm{Iw}_{n,w}}) \big[H^{n-1}(X, \mathcal{L}_\xi)^{U^w_p(m)}[\pi^p]^{U^p}\big] \\ &\quad= (\dim H^{n-1}(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p}) \big[\imath^{-1}(\Pi_w \otimes \psi_u)\big],\end{aligned}$$

and

$$\begin{aligned}&n(\dim \Pi_w^{\mathrm{Iw}_{n,w}}) \big[H_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)[\pi^p]^{U^p}\big] \\ &\quad= (\dim H^{n-1}(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p})\, \imath^{-1}\mathrm{Red}^{(h)} \big[\Pi_w \otimes \psi_u\big].\end{aligned}$$

Combining this with Lemma 4.3, we get

$$\begin{aligned}&n(\dim \Pi_w^{\mathrm{Iw}_{n,w}}) \big[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)[\pi^p]^{U^p}\big] \\ &= (\dim H^{n-1}(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p}) \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \imath^{-1}\mathrm{Red}^{(h)} \big[\Pi_w \otimes \psi_u\big].\end{aligned}$$

As Π_w is tempered, it is a full normalised induction of the form

$$\text{n-Ind}_{P(F_w)}^{GL_n(F_w)} (\mathrm{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \mathrm{Sp}_{s_t}(\pi_t)),$$

where π_i is an irreducible cuspidal representation of $GL_{g_i}(F_w)$ and P is a parabolic subgroup of GL_n with Levi component $GL_{s_1 g_1} \times \cdots \times GL_{s_t g_t}$. As Π_w has an Iwahori fixed vector, we must have $g_i = 1$ and π_i unramified for all i. (See [Ca].) Note that, for this type of representation (full induced from square integrables $\mathrm{Sp}_{s_i}(\pi_i)$ with π_i an unramified character of $F_w^\times$),

$$\begin{aligned}&\dim\big(\text{n-Ind}\,{}_{P(F_w)}^{GL_n(F_w)}(\mathrm{Sp}_{s_1}(\pi_1)\otimes\cdots\otimes\mathrm{Sp}_{s_t}(\pi_t))\big)^{\mathrm{Iw}_{n,w}}\\ &= \#P(k(w))\backslash GL_n(k(w))/B_n(k(w)) = \frac{n!}{\prod_j s_j!}.\end{aligned}$$

We can compute $\mathrm{Red}^{(h)}\big[\Pi_w\otimes\psi_u\big]$ using lemma I.3.9 of [HT] (but note the typo there — "positive integers $h_1,...,h_t$" should read "nonnegative integers $h_1,...,h_t$"). Putting $V_i = \mathrm{rec}\big(\pi_i^{-1}|\ |_w^{\frac{1-n}{2}}(\psi_u\circ\mathbf{N}_{F_w/E_u})^{-1}\big)$, we see that

$$\begin{aligned}&\mathrm{Red}^{(h)}\big[\Pi_w\otimes\psi_u\big]\\ &\quad=\sum_i \dim\big(\text{n-Ind}\,{}_{P'(F_w)}^{GL_h(F_w)}(\mathrm{Sp}_{s_i+h-n}(\pi_i|\ |^{n-h})\otimes\bigotimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi_j))\big)^{\mathrm{Iw}_{h,w}}\big[V_i\big]\\ &\quad=\sum_i\frac{h!}{(s_i+h-n)!\prod_{j\neq i}s_j!}\big[V_i\big],\end{aligned}$$

where the sum runs only over those i for which $s_i\geq n-h$, and $P'\subset GL_h$ is a parabolic subgroup. Thus

$$\begin{aligned}&n\frac{n!}{\prod_j s_j!}\big[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}\big]\\ &= D\sum_{h=0}^{n-\#S}(-1)^{n-\#S-h}\binom{n-\#S}{h}\sum_{i:\, s_i\geq n-h}\frac{h!}{(s_i+h-n)!\prod_{j\neq i}s_j!}\big[V_i\big]\\ &= D\sum_{i=1}^{t}\frac{(n-\#S)!}{(s_i-\#S)!\prod_{j\neq i}s_j!}\sum_{h=n-s_i}^{n-\#S}(-1)^{n-\#S-h}\binom{s_i-\#S}{h+s_i-n}\big[V_i\big]\\ &= D\sum_{i:\, s_i=\#S}\frac{(n-\#S)!}{\prod_{j\neq i}s_j!}\big[V_i\big],\end{aligned}$$

where $D=\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p}$, and so

$$n\binom{n}{\#S}\big[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}\big] = (\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p})\sum_{s_i=\#S}\big[V_i\big].$$

As Π_w is tempered, $\mathrm{rec}\big(\Pi_w^\vee\otimes(\psi_u^\vee\circ\mathbf{N}_{F_w/E_u})|\det|^{\frac{1-n}{2}}\big)$ is pure of weight $m_\xi - 2t_\xi+(n-1)$. Hence

$$V_i=\mathrm{rec}\big(\pi_i^{-1}|\ |_w^{\frac{1-\#S}{2}}(\psi_u\circ\mathbf{N}_{F_w/E_u})^{-1}|\ |_w^{\frac{\#S-n}{2}}\big)$$

is strictly pure of weight $m_\xi-2t_\xi+(n-\#S)$. The Weil conjectures then tell us that

$$H^j(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}=(0)$$

for $j\neq n-\#S$ and the proposition follows. □

Corollary 4.5. *Suppose that $l=p$, that $\tau: W_0\hookrightarrow\mathbb{Q}_l^{ac}$ over $\mathbb{Z}_p=\mathbb{Z}_l$ and that $\imath:\mathbb{Q}_l^{ac}\xrightarrow{\sim}\mathbb{C}$. Suppose also that π is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ such that*

- $\pi_p^{\mathrm{Iw}(m)}\neq(0)$,

- $R^j_{\xi,p}(\pi) \neq (0)$ *for some* j,
- $\mathrm{BC}_\imath(\pi)$ *is cuspidal.*

Then

$$a_\xi(H^{j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\tau} \mathbb{Q}_l^{ac})[\pi^p] = (0)$$

for $j \neq n - \#S$.

Proof. Choose a prime $l' \neq l$ and an isomorphism $\imath' : \mathbb{Q}_{l'}^{ac} \xrightarrow{\sim} \mathbb{C}$. Set $\xi' = (\imath')^{-1}\imath\xi$ and $\pi' = (\imath')^{-1}\imath\pi$. For any open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$ we have

$$\begin{aligned}
&\dim_{\mathbb{Q}_l^{ac}} a_\xi(H^{j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\tau} \mathbb{Q}_l^{ac})[\pi^p]^{U^p} \\
=&\dim_{\mathbb{Q}_{l'}^{ac}} a_{\xi'} H^{j+m_{\xi'}}(\mathcal{A}^{m_{\xi'}}_{\mathrm{Iw}(m),S}, \mathbb{Q}_{l'}^{ac})[(\pi')^p]^{U^p} \\
=&\dim_{\mathbb{Q}_{l'}^{ac}} H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_{\xi'})[(\pi')^p]^{U^p}.
\end{aligned}$$

(Use the main theorems of [KMe] and [GM].) The corollary now follows from the proposition. □

Corollary 4.6. *Suppose that* $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. *Suppose also that* π *is an irreducible admissible representation of* $G(\mathbb{A}^\infty)$ *such that*

- $\pi_p^{\mathrm{Iw}(m)} \neq (0)$,
- $R^j_{\xi,p}(\pi) \neq (0)$ *for some* j,
- $\mathrm{BC}_\imath(\pi)$ *is cuspidal.*

Then $\mathrm{WD}(R^{n-1}_{\xi,l}(\pi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})$ *is pure of weight* $m_\xi - 2t_\xi + n - 1$.

Proof. The spectral sequence of Proposition 3.5,

$$E_1^{i,j}(\mathrm{Iw}(m), \xi)[\pi^p] \Rightarrow \mathrm{WD}\big(H^{i+j}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)}\big)[\pi^p]$$

degenerates at E_1, as $E_1^{i,j}(\mathrm{Iw}(m), \xi)[\pi^p] = (0)$ unless $i+j = n-1$ by Proposition 4.4 and the previous corollary. Therefore the abutment is pure of the desired weight. (See the proof of Proposition 3.5 for a description of the action of the monodromy operator N.) The description of $H^{i+j}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)$ in section 2 then tells us that

$$\begin{aligned}
&\mathrm{WD}\big(H^{i+j}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{\mathrm{Gal}\,(F_w^{ac}/F_w)}\big)[\pi^p] \\
&\quad = \bigoplus_{\pi' \sim \pi} (\pi')^{\mathrm{Iw}(m)} \otimes \mathrm{WD}(R^{n-1}_{\xi,l}(\pi')|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})
\end{aligned}$$

(where the sum is over irreducible admissible π' with $\pi'_x \cong \pi_x$ for all but finitely many finite places x of $\mathbb{Q}$), hence the corollary. □

We now conclude the proof of Theorem 1.5 and hence of Theorem 1.2. We return to the notation of Theorem 1.5. Recall that L is an imaginary CM field and that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- Π_∞ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from L to $\mathbb{Q}$ of GL_n;
- for some finite place x of L the representation Π_x is square integrable.

Recall also that v is a place of L above a rational prime p, that l is a second rational prime and that $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the l-adic representation associated to Π.

Choose a quadratic CM extension L'/L in which v and x split. Choose places $v'|v$ and $x'|x$ of L' with $v' \not| x'(x')^c$. Also choose an imaginary quadratic field E and a totally real field F^+ such that

- $[F^+ : \mathbb{Q}]$ is even;
- $F = EF^+$ is soluble and Galois over L';
- p splits as $p = uu^c$ in E;
- if we denote by Π_F the base change of Π to $GL_n(\mathbb{A}_F)$, there is a place w of F above u and v' such that $\Pi_{F,w}$ has an Iwahori fixed vector;
- x lies above a rational prime which splits in E and x' splits in F.

Note that the component of Π_F at a place above x' is square integrable and hence Π_F is cuspidal.

Choose a place z of F above x' and a division algebra B with centre F as in section 2 and satisfying

- B_y is split for all places $y \neq z, z^c$ of F.

Also choose $*$, β and G as in section 2. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

- a character $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$,
- an irreducible algebraic representation ξ of G over $\mathbb{Q}_l^{ac}$,
- an irreducible admissible representation π of $G(\mathbb{A}^\infty)$,

such that

- $R^{n-1}_{\xi,l}(\pi) \neq (0)$,
- ψ is unramified above p,
- $\psi^c|_{E_\infty^\times}$ is the inverse of the restriction of $\imath\xi$ to $E_\infty^\times \subset G(\mathbb{R})$,
- ψ^c/ψ is the restriction of the central character of Π_F to $\mathbb{A}_E^\times$,
- $\mathrm{BC}_\imath(\pi) = (\psi, \Pi_F)$,
- $\pi_w = \Pi_{F,w}$.

By the previous corollary $\mathrm{WD}((R^{n-1}_{\xi,l}(\pi) \otimes R_{l,\imath}(\psi)^{-1})|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})$ is pure. Moreover

$$R^{n-1}_{\xi,l}(\pi)^{\mathrm{ss}} \otimes R_{l,\imath}(\psi)^{-1} \cong R_{l,\imath}(\Pi)|^a_{\mathrm{Gal}\,(F^{ac}/F)}$$

for some $a \in \mathbb{Z}_{>0}$. Hence Theorem 1.5 follows from Lemma 1.7.

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