

CONVERGENCE OF KÄHLER-RICCI FLOW

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0. INTRODUCTION

In this paper, we prove a theorem on convergence of Kähler-Ricci flow on a compact Kähler manifold M which admits a Kähler-Ricci soliton. A Kähler metric h is called a Kähler-Ricci soliton if its Kähler form ω_h satisfies equation

$$\text{Ric}(\omega_h) - \omega_h = L_X \omega_h,$$

where $\text{Ric}(\omega_h)$ is the Ricci form of h and $L_X \omega_h$ denotes the Lie derivative of ω_h along a holomorphic vector field X on M . As usual, we denote a Kähler-Ricci soliton by a pair (g_{KS}, X) . According to [TZ1], X should lie in the center of a reductive Lie subalgebra $\eta_r(M)$ of $\eta(M)$, which consists of all holomorphic vector fields on M . If $X = 0$, ω_h is just a Kähler-Einstein metric. Since ω_h is d -closed, we may write $L_X \omega_h = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta$ for some real-valued smooth function θ . It follows that the first Chern class $c_1(M)$ is positive and it is represented by ω_h .

The Ricci flow was first introduced by R. Hamilton in [Ha]. If the underlying manifold M is Kähler with positive first Chern class, it is more natural to study the following Kähler-Ricci flow (normalized):

$$(0.1) \quad \begin{cases} \frac{\partial g(t, \cdot)}{\partial t} = -\text{Ric}(g(t, \cdot)) + g(t, \cdot), \\ g(0, \cdot) = g_0, \end{cases}$$

where g_0 is a given metric with its Kähler class representing $c_1(M)$. It can be shown that (0.1) preserves the Kähler class.

Let $\text{Aut}_r(M)$ be the connected Lie subgroup of the automorphism group of M corresponding to $\eta_r(M)$. Let K be a maximal compact subgroup of $\text{Aut}_r(M)$. According to [TZ1], we may assume that a Kähler-Ricci soliton (g_{KS}, X) is K -invariant and the imaginary part $\text{Im}(X)$ of X generates a one-parameter subgroup K_X of K . The following is our main result.

Main Theorem. *Let M be a compact Kähler manifold which admits a Kähler-Ricci soliton (g_{KS}, X) . Then any solution $g(t, \cdot)$ of (0.1) will converge to the g_{KS} in the sense of Cheeger-Gromov if the initial Kähler metric g_0 is K_X -invariant.*

Corollary. *If M admits a Kähler-Einstein metric, then any solution of (0.1) will converge to a Kähler-Einstein metric in the sense of Cheeger-Gromov for any initial Kähler metric g_0 with Kähler class $c_1(M)$.*

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This corollary was first announced by G. Perelman [P2] when he was visiting MIT in the Spring of 2003.

Our proof of the main theorem is to study a certain complex Monge-Ampère flow which arises from (0.1). The flow of this type has been studied before by many people (cf. [Ca], [CT1], [CT2]). Indeed, our proof used a deep estimate of Perelman ([P2]; also see [ST]). We combined Perelman's estimate with estimates on solutions of complex Monge-Ampère equations we have used in our proving the uniqueness of Kähler-Ricci solitons [TZ1].

The organization of this paper is as follows. In Section 1, we describe an unpublished estimate of Perelman on the time derivative of potential functions of evolved Kähler metrics along the Kähler-Ricci flow. In Section 2, we use the relative capacity theory, which was first developed in [Ko] and adapted to the case of Kähler-Ricci solitons in [TZ1], to obtain a C^0 -estimate for a certain Monge-Ampère equation. Then in Section 3, we obtain a global Harnack-type inequality for solutions of complex Monge-Ampère flow associated to the Kähler-Ricci flow (0.1). In Section 4, we derive monotonicity of the generalized K-energy introduced in [TZ2]. A C^0 -estimate on a modified complex Monge-Ampère flow is obtained in Section 5 and the Main Theorem will be proved in Section 6.

1. AN ESTIMATE OF PERELMAN

In this section, we first reduce the Kähler-Ricci flow to a fully nonlinear flow on Kähler potentials. Then we discuss a recent and deep estimate of Perelman.

Let (M, g) be an n -dimensional compact Kähler manifold with its Kähler form ω_g representing the first Chern class $c_1(M) > 0$. In local coordinates z_1, \dots, z_n , we have

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right).$$

Moreover, the Ricci form $\text{Ric}(\omega_g)$ is given by

$$\begin{cases} R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log(\det(g_{k\bar{l}})), \\ \text{Ric}(\omega_g) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n R_{i\bar{j}} dz^i \wedge d\bar{z}^j. \end{cases}$$

Since the Ricci form represents $c_1(M)$, there exists a smooth function h on M such that

$$(1.1) \quad \text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h.$$

An easy computation shows that the flow (0.1) preserves the Kähler class of its solution $g(t)$, so we may write the Kähler form of $g(t)$ at a solvable time t as

$$\omega_\varphi = \omega_{g_0} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$$

for some smooth function $\varphi = \varphi(t, \cdot) = \varphi_t$. This φ is usually called a Kähler potential function associated to the Kähler metric $g(t)$. Using the Maximal Principle, one can show that (0.1) is equivalent to the following complex Monge-Ampère flow

for $\varphi(t, \cdot)$:

$$(1.2) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} + \varphi - h, \\ \varphi(0, \cdot) = 0. \end{cases}$$

Observe that $\frac{\partial \varphi}{\partial t}|_{t=0} = -h$.

Differentiating on both sides of (1.2) on t , we have

$$(1.3) \quad \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial t} = \Delta' \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t},$$

where Δ' denotes the Laplacian operator associated to the metric ω_φ . Then it follows from the standard Maximal Principle that

$$\left| \frac{\partial}{\partial t} \varphi(t, \cdot) \right| \leq C e^t,$$

and consequently,

$$|\varphi(t, \cdot)| \leq C e^t.$$

By using these facts and arguments in deriving the higher order estimates in Yau's solution of the Calabi conjecture [Ya], H.D. Cao showed that (1.2) is solvable for all $t \in (0, +\infty)$ [Ca].

Using his W -functional and arguments in proving noncollapsing of the Ricci flow [P1], recently, G. Perelman proved the following deep estimate [P2].

Lemma 1.1. *Let φ_t be a solution of Monge-Ampère flow (1.2). Choose c_t by the condition $h_t = -\frac{\partial \varphi}{\partial t} + c_t$ such that*

$$\int_M e^{h_t} \omega_{\varphi_t}^n = \int_M \omega_{g_0}^n.$$

Then there is a uniform constant A independent of t such that

$$(1.4) \quad |h_t| \leq A.$$

For the reader's convenience, we will present a proof of Lemma 1.1 taken from [ST] in the appendix. Lemma 1.1 is crucial in proving our main theorem. Recall that h_t is defined by (1.1) with ω_g replaced by ω_{φ_t} and can be different from a constant. In Section 4 below we will further prove that c_t is uniformly bounded and so $\frac{\partial \varphi}{\partial t}$ is.

2. RELATIVE CAPACITY AND C^0 -ESTIMATE

In this section, as in [TZ1], we will use the relative capacity theory for plurisubharmonic functions first developed in [Ko] to derive a C^0 -estimate on a certain Monge-Ampère equation.

First we recall some notation which can be found in [BT]. For any compact subset K of a strictly pseudoconvex domain Ω in \mathbb{C}^n , its relative capacity in Ω is defined as

$$\text{cap}(K, \Omega) = \sup \left\{ \int_K (\sqrt{-1} \partial \bar{\partial} u)^n \mid u \in \text{PSH}(\Omega), -1 \leq u < 0 \right\},$$

where $\text{PSH}(\Omega)$ denotes the space of plurisubharmonic functions (abbreviated as psh) in the weak sense. For any open set $U \subset \Omega$, we have

$$\text{cap}(U, \Omega) = \sup \{ \text{cap}(K, \Omega) \mid \text{for any compact } K \subset U \}.$$

The extremal function of K relative to Ω is defined by

$$u_K(z) = \sup\{u(z) \mid u \in \text{PSH}(\Omega) \cap L^\infty(\Omega), u < 0 \text{ and } u|_K \leq -1\}.$$

One can show that $u_K^*(z) = \overline{\lim}_{z' \rightarrow z} u_K(z')$ is a psh function. It is called the upper semicontinuous regularization of u_K . A compact set K is said to be regular if $u_K^* = u_K$. The following are some properties of u_K^* (cf. [BT], [AT]):

$$\begin{aligned} u_K^* &\in \text{PSH}(\Omega), \quad -1 \leq u_K^* \leq 0, \quad \lim_{z \rightarrow \partial\Omega} u_K^* = 0, \\ (\sqrt{-1}\partial\bar{\partial}u^*)^n &= 0 \quad \text{on } \Omega \setminus K, \\ u_K^* &= -1 \quad \text{on } K, \text{ except on a set of relative capacity zero.} \end{aligned}$$

Moreover, we have

$$(2.1) \quad \text{cap}(K, \Omega) = \int_{\Omega} (\sqrt{-1}\partial\bar{\partial}u_K^*)^n = \int_K (\sqrt{-1}\partial\bar{\partial}u_K^*)^n.$$

Lemma 2.1. *Let Ω be a strictly pseudoconvex domain in C^n and let $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on Ω :*

$$\det(u_{i\bar{j}}) = f.$$

Suppose that u and f satisfy

$$(2.2) \quad \begin{aligned} u(p) &> c \quad (p \in \Omega) \quad \text{and} \\ \int_K f dv &\leq A \text{cap}(K, \Omega) \frac{\text{cap}(K, \Omega)^{\frac{1}{n}}}{1 + \text{cap}(K, \Omega)^{\frac{1}{n}}} \end{aligned}$$

for any compact subset K of Ω . If the sets

$$U(s) = \{z \mid u(z) < s\} \cap \Omega''$$

are nonempty and relatively compact in $\Omega'' \subset \Omega' \subset\subset \Omega$ for any $s \in [S, S + D]$, where S is some number, then there is a uniform constant C , which depends only on $c, D, \delta, \Omega', \Omega$, such that

$$(2.3) \quad -\inf_{\Omega''} u \leq CA^\delta + D.$$

Proof. This lemma is essentially due to [Ko]. For the reader's convenience, we will include a proof using an argument from [TZ1]. Put

$$a(s) = \text{cap}(U(s), \Omega) \quad \text{and} \quad b(s) = \int_{U(s)} (\sqrt{-1}\partial\bar{\partial}u)^n.$$

Then we define an increasing sequence s_0, s_1, \dots, s_N by setting $s_0 = S$ and

$$s_j = \sup\{s \mid a(s) \leq \lim_{t \rightarrow s_{j-1}^+} e a(t)\}$$

for $j = 1, \dots, N$, where N is chosen to be the greatest integer such that $s_N \leq S + D$.

By using an argument in Lemma 4.1 of [TZ1], we can prove

$$(2.4) \quad S + D - s_N \leq (Ae)^{\frac{1}{n}} a(S + D)^{\frac{1}{n\delta}}$$

and

$$(2.5) \quad s_N - S \leq 2(Ae)^{\frac{1}{n}} (1 + n\delta) a(S + D)^{\frac{1}{n\delta}}.$$

However, it was proved in [AT] (or Theorem 1.2.11 in [Ko]) that

$$\text{cap}(\{u < s\} \cap \Omega', \Omega) \leq \frac{c'}{|s|},$$

where c' depends only on c and Ω' . It implies that

$$(2.6) \quad a(S + D) \leq \frac{c'}{-D - S}.$$

Combining (2.4)–(2.6), we get

$$D \leq 2(2 + n\delta)(Ae)^{\frac{1}{n}} \left(\frac{c'}{-D - S}\right)^{\frac{1}{n\delta}}.$$

It follows that

$$-S \leq c' \left(\frac{2(2 + n\delta)}{D}\right)^{n\delta} e^\delta A^\delta + D.$$

Consequently, we have

$$-\inf_{\Omega''} u \leq c' \left(\frac{2(1 + n\delta)}{D}\right)^{n\delta} e^\delta A^\delta + D,$$

so (2.3) is proved. \square

Lemma 2.2. *Let Ω be a strictly pseudoconvex domain in C^n and let $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on Ω :*

$$\det(u_{i\bar{j}}) = f,$$

where $f(\geq 0) \in L^{1+\epsilon_0}(\Omega)$ for some $\epsilon_0 > 0$. Suppose that u satisfies

$$u(p) > c \quad (p \in \Omega).$$

Define $U(s)$ as in last lemma. If the $U(s)$ are nonempty and relatively compact in Ω'' for any $s \in [S, S + D]$ for some S , then for any positive $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$, there is a uniform constant $C = C(c, D, \delta_0, \epsilon_0, \Omega', \Omega)$ such that

$$-\inf_{\Omega''} u \leq C \left(\frac{1}{\delta\epsilon}\right)^{n+\delta} \|f\|_{L^{1+\epsilon}(\Omega)}^\delta + D.$$

Proof. Let u_K be the relative extremal function of a regular set K with respect to Ω and let $v = \text{cap}^{-\frac{1}{n}}(K, \Omega)u_K$. Then v is a psh function and satisfies

$$\int_{\Omega} (\sqrt{-1}\partial\bar{\partial}v)^n = 1 \quad \text{and} \quad \lim_{z \rightarrow \partial\Omega} v = 0.$$

By Lemma 2.5.1 in [Ko], we have

$$\lambda(U'(s)) \leq c' \exp\{-2\pi|s|\}$$

for some uniform constant c' independent of v , where $\lambda(U'(s))$ is the Lebesgue measure of $U'(s) = \{v < s\}$. It follows that for any $q \geq 1$,

$$(2.7) \quad \begin{aligned} \int_{\Omega} |v|^q d\mu &\leq |\Omega| + \sum_{i=1}^{\infty} \int_{-s-1 \leq v \leq -s} |v|^q d\mu \\ &\leq |\Omega| + c' \sum_{i=1}^{\infty} (s+1)^q e^{-2\pi s} \\ &\leq |\Omega| + c' e^{4\pi} \int_2^{+\infty} s^q e^{-2\pi s} ds \\ &\leq C_1 2^{q+2} ([q] + 2)! \leq C_1 2^{q+2} (q+2)^{q+2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & \text{cap}(K, \Omega)^{-1}(1 + \text{cap}^{-1/\delta}(K, \Omega)) \int_K f d\mu \\
 & \leq \int_K |v|^n (1 + |v|^{\frac{n}{\delta}}) f d\mu \\
 (2.8) \quad & \leq \int_{\Omega} (|v|^n + |v|^{n(1+\frac{1}{\delta})}) f d\mu \\
 & \leq [(\int_{\Omega} |v|^{\frac{n(1+\epsilon)}{\epsilon}} d\mu)^{\frac{\epsilon}{1+\epsilon}} + (\int_{\Omega} |v|^{\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon}} d\mu)^{\frac{\epsilon}{1+\epsilon}}] \|f\|_{L^{1+\epsilon}(\Omega)}.
 \end{aligned}$$

Combining (2.7) and (2.8), we get

$$\int_{\Omega} f d\mu \leq A \text{cap}(K, \Omega) \frac{\text{cap}(K, \Omega)^{\frac{1}{\delta}}}{1 + \text{cap}(K, \Omega)^{\frac{1}{\delta}}},$$

where

$$\begin{aligned}
 A &= 2C_1 2^{\frac{n(1+\delta)}{\delta}+2} (\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon} + 2)^{\frac{n(1+\delta)}{\delta}+2} \|f\|_{L^{1+\epsilon}(\Omega)} \\
 &\leq C_2 (\frac{C_0}{\delta\epsilon})^{\frac{n(1+\delta)}{\delta}+2} \|f\|_{L^{1+\epsilon}(\Omega)}.
 \end{aligned}$$

Therefore, it follows from Lemma 2.1 that

$$-\inf_{\Omega''} u \leq C (\frac{C_0}{\delta\epsilon})^{n+3n\delta} \|f\|_{L^{1+\epsilon}(\Omega)}^{\delta} + D.$$

Now the lemma follows from replacing $3n\delta$ by δ . □

Proposition 2.1. *Let (M, g) be a compact Kähler manifold and let φ be a smooth solution of the complex Monge-Ampère equation on M ,*

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) f, \\ \sup_M \varphi = 0. \end{cases}$$

Then, for any positive $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$, there are two uniform constants C, C' which depend only on g, δ_0, ϵ_0 such that

$$-\inf_M \varphi \leq C (\frac{1}{\delta\epsilon})^{n+\delta} \|f\|_{L^{1+\epsilon}(M)}^{\delta} + C'.$$

Proof. This is a direct corollary of Lemma 2.2 (cf. the proof of Proposition 4.1 in [TZ1]). We omit its proof. □

3. A HARNACK-TYPE INEQUALITY

In this section, we use Proposition 2.1 to develop a C^0 -estimate for solutions of the Monge-Ampère flow (1.2).

Proposition 3.1. *Let $\varphi = \varphi_t$ be any solution of (1.2). Then, for any positive $\delta \leq 1$, there are two uniform constants $C = C(g, n, \delta)$ and $C' = C'(g, n, \delta)$ such that*

$$\text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C (\int_M \varphi(\omega_g^n - \omega_{\varphi}^n))^{n+\delta} + C'.$$

Proposition 3.1 is analogous to Proposition 4.2 in [TZ1] for the complex Monge-Ampère equation which arises from the equation for Kähler-Ricci solitons.

As before we assume that (M, g) is an n -dimensional compact Kähler manifold such that ω_g represents the first Chern class $c_1(M) > 0$. We need two lemmas in order to prove Proposition 3.1.

Lemma 3.1 (Poincaré-type inequality). *Let h be the smooth function determined by the relation (1.1). Let $C^\infty(M, \mathbb{C})$ be the space of complex-valued smooth functions. Then for any $\psi \in C^\infty(M, \mathbb{C})$, we have*

$$(3.1) \quad \int_M |\bar{\partial}\tilde{\psi}|^2 e^h \omega_g^n \geq \int_M |\tilde{\psi}|^2 e^h \omega_g^n,$$

where

$$\tilde{\psi} = \psi - \frac{1}{V} \int_M \psi e^h \omega_g^n.$$

In particular, for any $\varphi \in C^\infty(M)$, we have

$$(3.2) \quad \int_M |\bar{\partial}\varphi|^2 e^h \omega_g^n \geq \int_M \varphi^2 e^h \omega_g^n - \frac{1}{V} \left(\int_M \varphi e^h \omega_g^n \right)^2.$$

Proof. Let L be the linear differential operator on $C^\infty(M, \mathbb{C})$ defined by

$$L\psi = \Delta\psi + \langle \bar{\partial}h, \bar{\partial}\psi \rangle, \quad \text{for } \psi \in C^\infty(M, \mathbb{C}),$$

where Δ denotes the Laplacian operator of g . Then L is elliptic and self-adjoint with respect to the following Hermitian inner product:

$$(\psi, \psi')_h = \int_M \psi \bar{\psi}' e^h \omega_g^n, \quad \text{for } \psi, \psi' \in C^\infty(M, \mathbb{C});$$

namely,

$$(L\psi, \psi')_h = (\psi, L\psi')_h.$$

It follows that all eigenvalues of L are real. Denote by $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_i \leq \dots$ the sequence of eigenvalues of L and by ψ_i ($i = 0, 1, 2, \dots$) the corresponding sequence of eigenfunctions with the property $(\psi_i, \psi_j)_h = \delta_{ij}$, for any i, j . Note that ψ_0 is constant. Then $\{\psi_i\}$ is a complete orthonormal basis of the space $W^{1,2}(M, \mathbb{C})$ with respect to the weighted L^2 -norm $(\cdot, \cdot)_h$. On the other hand, using the Bochner's technique, one can prove that $\lambda_1 \geq 1$ ([F2]). So (3.1) holds, so does (3.2). \square

Lemma 3.2. *Let $\varphi = \varphi_t$ be any solution of (1.2). Then there is a uniform constant $c_0 > 0$ independent of t such that*

$$(3.3) \quad \int_M \exp\left\{-\frac{c_0}{I(\varphi)}(\varphi - \sup_M \varphi)\right\} \omega_\varphi^n \leq C,$$

where $I(\varphi) = \frac{1}{V} \int_M \varphi(\omega_g^n - \omega_\varphi^n) > 0$.

Proof. As in [TZ1], we will use an iteration argument to prove this lemma. Without loss of generality, we may assume $I(\varphi) > 1$.

Let $\bar{\varphi} = \varphi - \sup_M \varphi$. Then for any $p > 0$, we have

$$\begin{aligned} \int_M (-\bar{\varphi})^p (\omega_\varphi^n - \omega_\varphi^{n-1} \wedge \omega_g) &= \frac{\sqrt{-1}}{2\pi} \int_M (-\bar{\varphi})^p \partial \bar{\partial}(-\bar{\varphi}) \wedge \omega_\varphi^{n-1} \\ &= p \frac{\sqrt{-1}}{2\pi} \int_M (-\bar{\varphi})^{p-1} (-\partial \bar{\varphi}) \wedge (-\bar{\partial} \bar{\varphi}) \wedge \omega_\varphi^{n-1} \\ &= \frac{4p}{n(p+1)^2} \int_M |\bar{\partial}(-\bar{\varphi})^{\frac{p+1}{2}}|^2 \omega_\varphi^n. \end{aligned}$$

It follows that

$$(3.4) \quad \int_M |\bar{\partial}(-\bar{\varphi})^{\frac{p+1}{2}}|^2 \omega_\varphi^n \leq \frac{n(p+1)^2}{4p} \int_M (-\bar{\varphi})^p \omega_\varphi^n.$$

Applying Lemma 3.1 to function $(-\bar{\varphi})^{\frac{p+1}{2}}$ in the case of the metric ω_φ , we have

$$(3.5) \quad \int_M |\bar{\partial}(-\bar{\varphi})^{\frac{p+1}{2}}|^2 e^{h_t} \omega_{\varphi_t}^n \geq \int_M (-\bar{\varphi})^{p+1} e^{h_t} \omega_{\varphi_t}^n - \frac{1}{V} \left(\int_M (-\bar{\varphi})^{(p+1)/2} e^{h_t} \omega_{\varphi_t}^n \right)^2,$$

where $h_t = -\frac{\partial \varphi}{\partial t} + c_t$ which are uniformly bounded for t (cf. Lemma 1.1). Thus by using the Hölder inequality, we get

$$\int_M (-\bar{\varphi})^{p+1} e^{h_t} \omega_{\varphi_t}^n \leq cp \int_M (-\bar{\varphi})^p e^{h_t} \omega_{\varphi_t}^n + \frac{1}{V} \int_M (-\bar{\varphi})^p e^{h_t} \omega_{\varphi_t}^n \cdot \int_M (-\bar{\varphi}) e^{h_t} \omega_{\varphi_t}^n,$$

and consequently

$$(3.6) \quad \int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \leq c' [p \int_M (-\bar{\varphi})^p \omega_\varphi^n + \frac{1}{V} \int_M (-\bar{\varphi})^p \omega_\varphi^n \cdot \int_M (-\bar{\varphi}) \omega_\varphi^n],$$

where c, c' are uniform constants.

By the mean-value inequality, we have

$$\sup_M \varphi \leq V^{-1} \int_M \varphi \omega_g^n + C.$$

It follows that

$$\begin{aligned} \int_M (-\bar{\varphi}) \omega_\varphi^n &= V \sup_M \varphi + \int_M (-\varphi) \omega_\varphi^n \\ &\leq \int_M \varphi (\omega_g^n - \omega_\varphi^n) + CV \\ &\leq a I(\varphi), \end{aligned}$$

where a is a uniform constant. Thus inserting this inequality into (3.6), we get

$$(3.7) \quad \int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \leq ac'(p + I(\varphi)) \int_M (-\bar{\varphi})^p \omega_\varphi^n.$$

Iterating (3.7), we have

$$\int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \leq 2(ac' I(\varphi))^p (p+1)! \int_M (-\bar{\varphi}) \omega_\varphi^n \leq (c' I(\varphi))^{p+1} (p+1)!.$$

Now choosing $\varepsilon < \frac{1}{ac'I(\varphi)}$, we obtain

$$\begin{aligned} \int_M \exp\{-\varepsilon\bar{\varphi}\}\omega_\varphi^n &= \sum_{p=0}^{+\infty} \frac{\varepsilon^p}{p!} \int_M (-\bar{\varphi})^p \omega_\varphi^n \\ &\leq \sum_{p=0}^{+\infty} (\varepsilon ac'I(\varphi))^p \\ &\leq \frac{1}{1 - ac'I(\varphi)\varepsilon}. \end{aligned}$$

Put $c_0 = \frac{1}{ac'}$. Then (3.3) is proved. \square

Proof of Proposition 3.1. Let c_t and h_t be given as in Lemma 1.1. Let $\bar{\varphi} = \varphi - c_t$ and $\tilde{\varphi} = \varphi - \sup_M \varphi$. Then (1.2) becomes

$$(3.8) \quad \begin{cases} \det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}}) = \det(g_{i\bar{j}})f, \\ \sup_M \tilde{\varphi} = 0, \end{cases}$$

where $f = h + \frac{\partial\varphi}{\partial t} - c_t - \bar{\varphi} = h - h_t - \bar{\varphi}$. Since

$$\int_M e^{h_t} \omega_\varphi^n = V,$$

we have

$$(3.9) \quad 0 < c_1 \leq \int_M e^{-\bar{\varphi}} \omega_g^n \leq c_2$$

for some uniform constants c_1 and c_2 . This implies

$$(3.10) \quad \sup_M \bar{\varphi} \geq -C \text{ and } \inf_M \bar{\varphi} \leq C.$$

By (3.10) and Lemma 3.2, we have

$$\begin{aligned} \int_M \exp\left\{-\left(1 + \frac{c_0}{I(\varphi)}\right)\bar{\varphi}\right\}\omega_g^n &\leq e^{c_0 C} \int_M \exp\left\{-\frac{c_0}{I(\varphi)}(\bar{\varphi} - \sup_M \bar{\varphi}) - \bar{\varphi}\right\}\omega_g^n \\ &= e^{c_0 C} \int_M \exp\left\{-\frac{c_0}{I(\varphi)}(\varphi - \sup_M \varphi) - \bar{\varphi}\right\}\omega_g^n \\ &\leq C_1 \int_M \exp\left\{-\frac{c_0}{I(\varphi)}(\varphi - \sup_M \varphi)\right\}\omega_\varphi^n \leq C_2. \end{aligned}$$

It follows that

$$\|f\|_{L^{1+\frac{c_0}{I(\varphi)}}(M)} \leq C_3.$$

Thus, applying Proposition 2.1 to (3.8), we see that for any $\delta > 0$ there are uniform constants C_4 and C_5 only depending on δ such that

$$\sup_M \varphi - \inf_M \varphi = -\inf_M \tilde{\varphi} \leq C_4 I(\varphi)^{n+\delta} + C_5. \quad \square$$

4. MONOTONICITY OF GENERALIZED K-ENERGY

In this section, we show the monotonicity of the generalized K-energy introduced in [TZ2] along the Kähler-Ricci flow. Since M has positive first Chern class, associated to any holomorphic vector field X on M , there is a unique smooth, complex-valued function $\theta_X = \theta_X(g)$ on M such that

$$(4.1) \quad \begin{cases} i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_X, \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n, \end{cases}$$

where $i_X \omega_g$ is the interior product of X with ω_g , that is, $i_X \omega_g(Y) = \omega_g(X, Y)$ for any vector Y . The first relation above implies

$$L_X \omega_g = d[i_X(\omega_g)] = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X.$$

Thus we see that (g_{KS}, X) is a Kähler-Ricci soliton if and only if $h = \theta_X$ modulo a constant, where h is a smooth function determined by the relation (1.1) associated to the metric g_{KS} . As an obstruction to Kähler-Ricci solitons, the following holomorphic invariant was introduced in [TZ2]:

$$F_X(v) = \int_M v(h - \theta_X) e^{\theta_X} \omega_g^n, \quad v \in \eta(M),$$

where $\eta(M)$ denotes the Lie algebra consisting of all holomorphic vector fields on M . When $X \equiv 0$, the functional $F_X(\cdot)$ is just the Futaki invariant [F1]. It was shown in [TZ2] that $F_X(\cdot)$ is independent of the choice of Kähler metric g and $F_X(\cdot) \equiv 0$ if there exists a Kähler-Ricci soliton (g_{KS}, X) on M .

Let $\eta_r(M)$ be a reductive Lie subalgebra of $\eta(M)$ and let $X \in \eta_r(M)$. Let K_X be the one-parameter group generated by $\text{Im}(X)$ and let g be a K_X -invariant Kähler metric. Set

$$\mathcal{M}_X(\omega_g) = \{ \varphi \in C^\infty(M) \mid \omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi > 0, \text{Im}(X)(\varphi) = 0 \}.$$

Associated to the invariant $F_X(\cdot)$, the following functional on $\mathcal{M}_X(\omega_g)$ was introduced in [TZ2]:

$$\begin{aligned} \tilde{\mu}_{\omega_g}(\varphi) = & -\frac{n}{V} \int_0^1 \int_M \psi [\text{Ric}(\omega_\psi) - \omega_\psi \\ & - \frac{\sqrt{-1}}{2\pi} (\partial \bar{\partial} \theta_X(\omega_\psi) - \partial(h_{\omega_\psi} - \theta_X(\omega_\psi)) \wedge \bar{\partial} \theta_X(\omega_\psi))] \wedge e^{\theta_X(\omega_\psi)} \omega_\psi^{n-1} \wedge dt, \end{aligned}$$

where $\psi = \psi_t$ ($0 \leq t \leq 1$) is a path connecting 0 to φ in $\mathcal{M}_X(\omega_g)$, $\theta_X(\omega_\psi) = \theta_X + X(\psi)$ and $\text{Ric}(\omega_\psi)$ denotes the Ricci form of ω_ψ . If $X = 0$, then $\tilde{\mu}_{\omega_g}(\varphi)$ is just Mabuchi's K -energy [Ma].

Let X be in the center of $\eta_r(M)$ and let $\text{Aut}_r(M)$ be the connected subgroup associated to $\eta_r(M)$. Then

$$\rho^* L_X \omega_\varphi = L_X(\rho^* \omega_\varphi), \quad \forall \rho \in \text{Aut}_r(M).$$

Thus if $\varphi \in \mathcal{M}_X(\omega_g)$, then

$$\varphi_\rho \in \mathcal{M}_X(\omega_g), \quad \forall \rho \in \text{Aut}_r(M),$$

where φ_ρ is the Kähler potential function determined by

$$\rho^* \omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_\rho.$$

Moreover, analogous to the case of the K -energy and the Futaki invariant, we have the following for $\tilde{\mu}_{\omega_g}$:

$$\frac{d\tilde{\mu}_{\omega_g}(\varphi_{\rho_t})}{dt} = \text{Re}(F_X(v)),$$

where $\rho_t = \exp\{tv\}$ is a one-parameter subgroup generated by a holomorphic vector field v on M . This implies that if $F_X(\cdot) \equiv 0$, then

$$(4.2) \quad \tilde{\mu}_{\omega_g}(\varphi_\rho) = \tilde{\mu}_{\omega_g}(\varphi), \quad \forall \rho \in \text{Aut}_r(M).$$

Note that X must lie in the center of $\eta_r(M)$ if $F_X(\cdot) \equiv 0$ [TZ1].

Lemma 4.1. *Let φ be a solution of the flow (1.2). Suppose that the initial metric g is K_X -invariant and that the holomorphic invariant $F_X(\cdot)$ vanishes. Then*

$$(4.3) \quad \tilde{\mu}_{\omega_g}(\varphi) \leq 0.$$

Proof. Let $\sigma_t = \exp\{tX\}$ be a one-parameter subgroup in $\text{Aut}_r(M)$ generated by X and let $\varphi' = \varphi_{\sigma_t}$ be defined as above for the subgroup $\{\sigma_t\}$. Then $\omega_{\varphi'}$ satisfies the modified Kähler-Ricci flow,

$$(4.4) \quad \frac{\partial}{\partial t}\omega_{\varphi'} = -\text{Ric}(\omega_{\varphi'}) + \omega_{\varphi'} + L_X\omega_{\varphi'}.$$

Then, using integration by parts, we have

$$(4.5) \quad \frac{d\tilde{\mu}_{\omega_g}(\varphi')}{dt} = -\frac{1}{V} \int_M \|\bar{\partial}\frac{\partial\varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n \leq 0.$$

It follows that

$$\tilde{\mu}_{\omega_g}(\varphi') \leq 0,$$

and consequently, by (4.2), we get (4.3). □

It was shown in [TZ2] that if M admits a Kähler-Ricci soliton (g_{KS}, X) , then the functional $\tilde{\mu}_{\omega_g}(\cdot)$ is bounded from below on $\mathcal{M}_X(\omega_g)$. So by (4.5), we have

$$\frac{1}{V} \int_0^\infty \int_M \|\bar{\partial}\frac{\partial\varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n \wedge dt \leq C.$$

It follows that

$$(4.6) \quad \frac{1}{V} \int_0^\infty \int_M \|\bar{\partial}\frac{\partial\varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi') - t} (\omega_{\varphi'})^n \wedge dt \leq C.$$

By using the Maximal Principle, we get from (4.4) that

$$(4.7) \quad \frac{\partial\varphi'}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \varphi'_{i\bar{j}})}{\det(g_{i\bar{j}})} + X(\varphi') + \varphi' - h + \theta_X.$$

Differentiating the above equation on t , we obtain

$$(4.8) \quad \frac{\partial}{\partial t} \frac{\partial\varphi'}{\partial t} = \Delta' \frac{\partial\varphi'}{\partial t} + X\left(\frac{\partial\varphi'}{\partial t}\right) + \frac{\partial\varphi'}{\partial t}.$$

Let

$$a_t = \int_M \frac{\partial\varphi'}{\partial t} e^{\theta_X + X(\varphi')} \omega_{\varphi'}^n.$$

Then it is easy to see that

$$(4.9) \quad \frac{da_t}{dt} = a_t - \int_M \|\bar{\partial}\frac{\partial\varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n.$$

Lemma 4.2. *Suppose that M admits a Kähler-Ricci soliton (g_{KS}, X) . Let g be a K_X -invariant Kähler metric on M . We choose h in equation (1.2) by adding a suitable constant so that*

$$(4.10) \quad \frac{1}{V} \int_M (h - \theta_X) \omega_g^n = -\frac{1}{V} \int_0^\infty \int_M \|\bar{\partial} \frac{\partial \varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi') - t} (\omega_{\varphi'})^n \wedge dt.$$

Then the solution $\varphi'(t, \cdot)$ of (4.7) with the initial condition $\varphi'(0, \cdot) = 0$ satisfies

$$\lim_{t \rightarrow \infty} \int_M \frac{\partial \varphi'}{\partial t} e^{\theta_X + X(\varphi')} \omega_{\varphi'}^n = 0.$$

Proof. By (4.9), we have

$$(4.11) \quad \frac{(e^{-t} a_t)}{dt} = - \int_M e^{-t} \|\bar{\partial} \frac{\partial \varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n.$$

Note that

$$h - \theta_X = -\frac{\partial \varphi'}{\partial t} \Big|_{t=0}$$

and

$$a_0 = \int_0^\infty \int_M \|\bar{\partial} \frac{\partial \varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi') - t} (\omega_{\varphi'})^n \wedge dt.$$

Then we have

$$\begin{aligned} a_t &= \int_t^\infty e^{t-s} \|\bar{\partial} \frac{\partial \varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n \wedge ds \\ &\leq \int_t^\infty \|\bar{\partial} \frac{\partial \varphi'}{\partial t}\|^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n \wedge ds \\ &\rightarrow 0, \text{ as } t \rightarrow \infty. \end{aligned}$$

Thus the lemma is true. □

Proposition 4.1. *Suppose that M admits a Kähler-Ricci soliton (g_{KS}, X) . Let φ be a solution of (1.2). Suppose that the initial metric g is K_X -invariant and that h is normalized as in Lemma 4.2. Then the constants c_t in Lemma 1.1 are uniformly bounded. Consequently, the $\frac{\partial \varphi}{\partial t}$ are uniformly bounded.*

Proof. Let $\varphi'(t, \cdot)$ be the solution of equation (4.7). Then

$$\frac{\partial \varphi'}{\partial t} = \frac{\partial \varphi}{\partial t} + \theta_X + X(\varphi').$$

Thus by Lemma 4.2, we get

$$\left| \int_M -\frac{\partial \varphi}{\partial t} e^{\theta_X + X(\varphi')} \omega_{\varphi'}^n \right| \leq C$$

for some uniform constant C , and so

$$V|c_t| \leq C + \left| \int_M h_t e^{\theta_X + X(\varphi')} \omega_{\varphi'}^n \right|.$$

Here we use the fact that the $|X(\psi)|$ are uniformly bounded for any $\psi \in \mathcal{M}_X(\omega_g)$ [Zh]. Therefore by Lemma 1.1, we obtain

$$|c_t| \leq \frac{C}{V} + A$$

and

$$\left| \frac{\partial \varphi}{\partial t} \right| \leq \frac{C}{V} + 2A. \quad \square$$

5. A C^0 -ESTIMATE

In this section, we use the properness and monotonicity of $\tilde{\mu}_{\omega_g}(\cdot)$ to derive the C^0 -estimate for solutions of (1.2). We will assume that M admits a Kähler-Ricci soliton (g_{KS}, X) .

Let us recall two other functionals on $\mathcal{M}_X(\omega_g)$ introduced in [TZ2]:

$$\begin{aligned} \tilde{F}_{\omega_g}(\varphi) &= \tilde{J}(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega_g^n - \log\left(\frac{1}{V} \int_M e^{h-\varphi} \omega_g^n\right) \\ (5.1) \quad &= -\frac{1}{V} \int_0^1 \int_M \frac{\partial \psi}{\partial t} e^{\theta_X + X(\psi)} \omega_\psi^n \wedge dt - \log\left(\frac{1}{V} \int_M e^{h-\varphi} \omega_g^n\right) \end{aligned}$$

and

$$\tilde{J}(\varphi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial \psi}{\partial t} (e^{\theta_X} \omega_g^n - e^{\theta_X + X(\psi)} \omega_\psi^n) \wedge dt,$$

where $\psi = \psi_t$ ($0 \leq t \leq 1$) is any path from 0 to φ and both functionals are independent of the choice of path [Zh]. The following result was proved in [TZ2].

Lemma 5.1. *We have*

$$\tilde{\mu}_{\omega_g}(\varphi) = \tilde{F}_{\omega_g}(\varphi) - \frac{1}{V} \int_M (h_{\omega_\varphi} - \theta_X - X(\varphi)) e^{\theta_X + X(\varphi)} \omega_\varphi^n + \frac{1}{V} \int_M (h - \theta_X) e^{\theta_X} \omega_g^n,$$

where h and h_{ω_φ} are smooth functions determined by the relation (1.1) associated to ω_g and ω_φ , respectively, and normalized by

$$\int_M e^h \omega_g^n = \int_M e^{h_{\omega_\varphi}} \omega_\varphi^n = V.$$

In particular, we have

$$(5.2) \quad \tilde{\mu}_{\omega_g}(\varphi) \geq \tilde{F}_{\omega_g}(\varphi) - C.$$

The following lemma was proved in [CTZ] by using the same arguments in [TZ1].

Lemma 5.2. *Assume that M admits a Kähler-Ricci soliton (g_{KS}, X) , and let $\Lambda_1(\omega_{KS}, X)$ be the space of eigenfunctions of the first nonzero eigenvalue of the second order elliptic operator L defined on $\mathcal{M}_X(\omega_{KS})$, where*

$$L\varphi = \Delta_{g_{KS}}\varphi + X(\varphi).$$

Then for any $\varphi \in \Lambda_1(X, g_{KS})^\perp$, we have

$$(5.3) \quad \tilde{F}_{\omega_{KS}}(\varphi) \geq CI_{\omega_{KS}}(\varphi)^{\frac{1}{4n+5}} - C'$$

for some uniform constants C and C' , where

$$I_{\omega_{KS}}(\varphi) = \frac{1}{V} \int_M \varphi (\omega_{KS}^n - (\omega_{KS} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi)^n).$$

In order to apply the above lemma to obtain a C^0 -estimate for solutions of (1.2), we shall modify the flow. Let $\rho_t \in \text{Aut}_r(M)$ be a family of holomorphic transformations on M . Let

$$(5.4) \quad \omega_{\tilde{\varphi}} = \rho_t^* \omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\tilde{\varphi},$$

where ω_φ is the evolved metrics in the flow (1.2). Then $\omega_{\tilde{\varphi}}$ satisfies

$$(5.5) \quad \frac{\partial}{\partial t} \omega_{\tilde{\varphi}} = -\text{Ric}(\omega_{\tilde{\varphi}}) + \omega_{\tilde{\varphi}} + L_{\text{Re}(\tilde{X})} \omega_{\tilde{\varphi}},$$

where $\tilde{X} = \tilde{X}_t = \rho_t^{-1} \cdot \frac{d\rho_t}{dt} \in \eta(M)$ is a family of holomorphic vector fields on M . Consequently we get a Monge-Ampère flow for $\tilde{\varphi}$ (modulo a constant),

$$(5.6) \quad \frac{\partial \tilde{\varphi}}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}})}{\det(g_{i\bar{j}})} + \operatorname{Re}(\tilde{X}(\tilde{\varphi})) + \tilde{\varphi} - h + \operatorname{Re}(\theta_{\tilde{X}}),$$

where $\theta_{\tilde{X}}$ is the smooth function defined by (4.1) for the holomorphic vector field \tilde{X} . From the proof of the uniqueness of Kähler-Ricci solitons in [TZ1], one can choose a holomorphic transformation ρ_t for each t so that $\tilde{\varphi}(t, \cdot) - \Psi_0 \in \Lambda_1(\omega_{KS}, X)^\perp$, where Ψ_0 is a smooth function satisfying

$$(5.7) \quad \omega_g = \omega_{KS} - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Psi_0.$$

Proposition 5.1. *Suppose that M admits a Kähler-Ricci soliton (g_{KS}, X) . Let $\tilde{\varphi}$ be a solution of (5.6). Suppose that the initial metric g is K_X -invariant and that h is normalized as in Lemma 4.2. Then there is a uniform constant C such that*

$$(5.8) \quad \|\tilde{\varphi}\|_{C^0(M)} \leq C.$$

Proof. Note that the existence of Kähler-Ricci soliton (g_{KS}, X) implies that $F_X(\cdot) \equiv 0$. By (4.3) in Lemma 4.1, we have

$$\tilde{\mu}_{\omega_g}(\tilde{\varphi}) = \tilde{\mu}_{\omega_g}(\varphi) \leq 0.$$

It follows that

$$\begin{aligned} \tilde{\mu}_{\omega_{KS}}(\tilde{\varphi} - \Psi_0) &= \tilde{\mu}_{\omega_{KS}}(-\Psi_0) + \tilde{\mu}_{\omega_g}(\tilde{\varphi}) \\ &\leq \tilde{\mu}_{g_{KS}}(-\Psi_0). \end{aligned}$$

On the other hand, by Lemma 5.2, we have

$$\tilde{F}_{\omega_{KS}}(\tilde{\varphi} - \Psi_0) \geq C I_{\omega_{KS}}(\tilde{\varphi} - \Psi_0)^{\frac{1}{4n+5}} - C'.$$

Thus by Lemma 5.1, we see that there is a uniform constant C_1 such that

$$I_{\omega_{KS}}(\tilde{\varphi} - \Psi_0) \leq C_1.$$

This shows that

$$(5.9) \quad I(\tilde{\varphi}) \leq C_2.$$

It is clear that (5.6) is equal to

$$(5.10) \quad \det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\left\{h + \frac{\partial\varphi}{\partial t} \circ \rho_t - \tilde{\varphi}\right\}.$$

Since $\frac{\partial\varphi}{\partial t} \circ \rho_t$ is uniformly bounded (Proposition 4.1), we can apply Proposition 3.1 to the solution $\tilde{\varphi}$ of (5.10). Thus by (5.9), we have

$$\operatorname{osc}_M \tilde{\varphi} \leq C_3(I(\tilde{\varphi})^{n+1} + 1) \leq C_4.$$

The estimate (5.8) follows from this. □

6. HIGHER ORDER ESTIMATES AND PROOF OF MAIN THEOREM

In order to get higher order estimates of solutions to (5.6), we shall modify the gauge transformations again which appeared in Section 5. The argument used here is similar to one in Sections 6 and 7 in [CT2]. First we observe

Lemma 6.1. *Let ρ_t be a family of holomorphic transformations defined in Section 5. Then for each integer $i = 0, 1, \dots$, we have*

$$(6.1) \quad \|\rho_{i+1}^{-1} \cdot \rho_i - I\| \leq C$$

for some uniform constant C , where the norm $\|\cdot\|$ denotes the distance between two elements in $\text{Aut}(M)$.

Proof. Let $\tilde{\varphi}$ be a solution of (5.6). Then

$$(\rho_i)^* \omega_{\varphi_i} = \omega_{\tilde{\varphi}_i} = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\tilde{\varphi}_i$$

and

$$(\rho_{i+1})^* \omega_{\varphi_{i+1}} = \omega_{\tilde{\varphi}_{i+1}} = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\tilde{\varphi}_{i+1}.$$

It follows that

$$\begin{aligned} (\rho_i)^* \omega_{\varphi_{i+1}} - (\rho_i)^* \omega_{\varphi_i} &= (\rho_{i+1} \cdot (\rho_{i+1}^{-1} \cdot \rho_i))^* \omega_{\varphi_{i+1}} - (\rho_i)^* \omega_{\varphi_i} \\ &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}[\tilde{\varphi}_{i+1} \cdot (\rho_{i+1}^{-1} \rho_i) - \tilde{\varphi}_i] + [(\rho_{i+1}^{-1} \rho_i)^* \omega_g - \omega_g]. \end{aligned}$$

Thus

$$(\rho_{i+1}^{-1} \rho_i)^* \omega_g - \omega_g = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}[\tilde{\varphi}_{i+1} \cdot (\rho_{i+1}^{-1} \rho_i) - \tilde{\varphi}_i] + \left(\int_i^{i+1} \frac{\partial\varphi}{\partial t} dt\right) \cdot \rho_i.$$

Since the smooth function

$$G = \tilde{\varphi}_{i+1} \cdot (\rho_{i+1}^{-1} \rho_i) - \tilde{\varphi}_i + \left(\int_i^{i+1} \frac{\partial\varphi}{\partial t} dt\right) \cdot \rho_i$$

is uniformly bounded by Proposition 4.1 and (5.1), we have

$$\|\rho_{i+1}^{-1} \rho_i - I\| \leq C$$

for some uniform constant C . □

Using Lemma 6.1, one can choose a modified family of holomorphic transformations $\bar{\rho}_t \in \text{Aut}_r(M)$ ($0 < t < \infty$) to replace ρ_t such that for any $t \in (0, \infty)$ (cf. [CT2]),

$$(6.2) \quad \|\rho_t^{-1} \bar{\rho}_t - Id\| \leq C$$

and

$$(6.3) \quad \|(\bar{\rho}_t^{-1})_* \frac{d\bar{\rho}_t}{dt}\|_g \leq C,$$

where the $(\bar{\rho}_t^{-1})_* \frac{d\bar{\rho}_t}{dt} = \bar{X}_t \in \eta_r(M)$ induce a family of holomorphic vector fields on M . Furthermore, for any $k \geq 0$, we may assume that there is a constant C_k such that

$$\left\| \frac{\partial^k \bar{X}_t}{\partial t^k} \right\|_g \leq C_k.$$

Let $\bar{\varphi} = \bar{\varphi}_t$ be a family of smooth functions determined by

$$(\bar{\rho}_t)^*\omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}.$$

Such a family is unique modulo addition by constants. Then by adding appropriate constants if needed, we may assume that $\bar{\varphi}$ satisfies a complex Monge-Ampère equation

$$(6.4) \quad \det(g_{i\bar{j}} + \bar{\varphi}_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\{h + \frac{\partial\varphi}{\partial t} \cdot \bar{\rho} - \bar{\varphi}\}.$$

Since

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi} = (\rho_t^{-1}\bar{\rho}_t - I)^*\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}((\rho_t^{-1}\bar{\rho}_t)^*\bar{\varphi}),$$

Proposition 5.1 implies that

$$(6.5) \quad \|\bar{\varphi}\|_{C^0(M)} \leq C$$

for some uniform C .

To obtain a C^2 -estimate and C^3 -estimate for $\bar{\varphi}$, we need to further modify the Kähler-Ricci flow equation. Let $\sigma_t = \exp\{tX\}$ be an one-parameter subgroup generated by X associated to the Kähler-Ricci soliton ω_{KS} and let $\omega_\rho = (\bar{\rho}_t^{-1} \cdot \sigma_t)^*\omega_{KS}$. Define $f_\rho = f_{\rho_t}$ by

$$(\bar{\rho}_t^{-1} \cdot \sigma_t)^*\omega_{KS} = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_\rho.$$

Since $(\bar{\rho}_t^{-1} \cdot \sigma_t)^*\omega_{KS}$ is also a Kähler-Ricci soliton w.r.t to X , we may assume that f_ρ modulo a constant satisfies the equation

$$(6.6) \quad \det(g_{i\bar{j}} + (f_\rho + \Psi_0)_{i\bar{j}}) = \det(g_{i\bar{j}})e^{h-\theta_X - X(f_\rho + \Psi_0) - f_\rho - \Psi_0},$$

where Ψ_0 is given as in (5.7). Differentiating the equation on t , we have

$$\Delta_\rho \frac{\partial f_\rho}{\partial t} + \frac{\partial f_\rho}{\partial t} = -X\left(\frac{\partial f_\rho}{\partial t}\right),$$

where $\Delta_\rho = \Delta_{\rho_t}$ is the Laplacian operator of the metric ω_ρ . This implies

$$(6.7) \quad \frac{\partial f_\rho}{\partial t} = (\bar{\rho}_0\bar{\rho}_t^{-1} \cdot \sigma_t)^*\frac{\partial f_\rho}{\partial t}|_{t=0}.$$

On the other hand, $\frac{\partial f_\rho}{\partial t}$ satisfies

$$L_{X-\bar{X}}(\omega_{\rho_t}) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\frac{\partial f_\rho}{\partial t}.$$

It follows that modulo constants, we have

$$\frac{\partial f_\rho}{\partial t} = \theta_{X-\bar{X}}(\omega_{\rho_t}),$$

where $\theta_{X-\bar{X}}(\omega_{\rho_t})$ is determined by

$$\begin{cases} i_{X-\bar{X}}(\omega_{\rho_t}) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta_{X-\bar{X}}(\omega_{\rho_t}), \\ \int_M e^{\theta_{X-\bar{X}}(\omega_{\rho_t})}\omega_{\rho_t}^n = V. \end{cases}$$

Observing that

$$\int_M \frac{\partial f_\rho}{\partial t} e^{\theta_{X-\bar{X}}(\omega_{\rho_t})}\omega_{\rho_t}^n = 0,$$

we can deduce

$$|\theta_{X-\bar{X}}(\omega_{\rho_t}) - \frac{\partial f_\rho}{\partial t}| \leq C_0$$

for some uniform constant C_0 . In particular,

$$\|\frac{\partial f_\rho}{\partial t}|_{t=0}\|_{C^0(M)} \leq \|\theta_{X-\bar{X}}(\omega_{\rho_t})|_{t=0}\|_{C^0(M)} + C_0.$$

Combining this with (6.7), we get

$$\|\frac{\partial f_\rho}{\partial t}\|_{C^0(M)} \leq C.$$

Moreover, for any integer $l \geq 0$, one can have

$$(6.8) \quad \|\nabla_\rho^l \frac{\partial f_\rho}{\partial t}\|_{\omega_\rho} \leq C,$$

where ∇_ρ^l denotes the l -th covariant derivative with respect to the metric ω_ρ .

Let $\psi = \psi_t = (\bar{\rho}_t^{-1} \cdot \sigma_t)^*(\bar{\varphi} - \Psi_0)$. Then ψ is uniformly bounded. Note that ψ satisfies

$$\begin{aligned} \omega_{\varphi'} &= (\bar{\rho}^{-1} \sigma)^* \omega_{KS} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi \\ &= \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (\psi + f_\rho + \Psi_0), \end{aligned}$$

where $\omega_{\varphi'}$ is defined as in Section 4. Thus by (4.4) and (6.6), we get a parabolic equation for ψ :

$$\frac{\partial \psi}{\partial t} = \log \frac{\omega_{\varphi'}^n}{\omega_\rho^n} + X(\psi) + \psi - \frac{\partial f_\rho}{\partial t}.$$

This is equivalent to

$$(6.9) \quad \frac{\partial \psi}{\partial t} = \log \frac{(\omega_\rho + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi)^n}{\omega_\rho^n} + X(\psi) + \psi - \frac{\partial f_\rho}{\partial t}.$$

Since

$$\frac{(\omega_\rho + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi)^n}{\omega_\rho^n} = (\bar{\rho}_t^{-1} \cdot \sigma_t)^* \left(\frac{\omega_{\bar{\varphi}}^n}{\omega_{KE}^n} \right)$$

and

$$\begin{aligned} \left| \log \frac{\omega_{\bar{\varphi}}^n}{\omega_{KS}^n} \right| &= \left| \log \frac{\omega_{\bar{\varphi}}^n}{\omega_g^n} + \log \frac{\omega_g^n}{\omega_{KS}^n} \right| \\ &= \left| h + \frac{\partial \varphi}{\partial t} \cdot \bar{\rho} - \bar{\varphi} + \log \frac{\omega_g^n}{\omega_{KS}^n} \right| \leq C, \end{aligned}$$

we have

$$\left| \log \left(\frac{(\omega_\rho + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi)^n}{\omega_\rho^n} \right) \right| \leq C.$$

It follows that

$$\left| \frac{\partial \psi}{\partial t} \right| \leq \left| \log \left(\frac{(\omega_\rho + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi)^n}{\omega_\rho^n} \right) \right| + |X(\psi)| + |\psi| + \left| \frac{\partial f_\rho}{\partial t} \right| \leq C'.$$

Here we have used the fact that the $|X(\psi)|$ are uniformly bounded for all $\psi \in \mathcal{M}_X(\omega_g)$ [Zh].

Lemma 6.2.

$$(6.10) \quad n + \Delta_\rho \psi \leq C.$$

Proof. Let $\Delta' = \Delta'_t$ be the Laplacian operator associated to $\omega_{\varphi'_t}$. Set

$$F = \frac{\partial \psi}{\partial t} - X(\psi) - \psi + \frac{\partial f_\rho}{\partial t}.$$

Then for sufficiently large c , using (6.9) and following arguments in [Ya], we compute

$$(6.11) \quad \begin{aligned} \Delta'(e^{-c\psi}(n + \Delta_\rho \psi)) &\geq e^{-c\psi}[\Delta F - C_1 - cn(n + \Delta_\rho \psi) \\ &\quad + \frac{c}{2}(n + \Delta_\rho \psi) \sum_{i=1}^n \frac{1}{1 + \psi_{i\bar{i}}}], \end{aligned}$$

where we choose a local coordinate system at a given point $p \in M$ so that $\omega_\rho = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$ at p and C_1 depends only on the metric ω_{KS} . Note that

$$\begin{aligned} \Delta_\rho F &= \Delta_\rho \frac{\partial \psi}{\partial t} - (n + \Delta_\rho \psi) + n - \Delta_\rho(X(\psi)) + \Delta_\rho \frac{\partial f_\rho}{\partial t} \\ &\geq \frac{\partial}{\partial t}(n + \Delta_\rho \psi) - (\|\nabla_\rho X\|_{\omega_\rho} + 1)(n + \Delta_\rho \psi) - c(n + \Delta_\rho \psi) \sup_M X(\psi) \\ &\quad - e^{c\psi}(X(e^{-c\psi}(n + \Delta_\rho \psi))) - \sum_{i=1}^n \left(\frac{\partial f_\rho}{\partial t}\right)_{i\bar{i}} \psi_{i\bar{i}} + \Delta_\rho \frac{\partial f_\rho}{\partial t} \end{aligned}$$

and

$$\|\nabla_\rho X\|_{\omega_\rho} = \|\nabla X\|_{\omega_{KS}}.$$

Thus by (6.8), we get

$$(6.12) \quad \begin{aligned} \Delta_\rho F &\geq \frac{\partial}{\partial t}(n + \Delta_\rho \psi) - C_2(n + \Delta_\rho \psi) - C_3 \\ &\quad - e^{c\psi}(X(e^{-c\psi}(n + \Delta_\rho \psi))). \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=1}^n \frac{1}{1 + \psi_{i\bar{i}}} &\geq \prod_{i=1}^n (1 + \psi_{i\bar{i}})^{-\frac{1}{n-1}} (n + \Delta_\rho \psi)^{\frac{1}{n-1}} \\ &= e^{-\frac{F}{n-1}} (n + \Delta_\rho \psi)^{\frac{1}{n-1}} \\ &\geq C_4 e^{\frac{\psi}{n-1}} (n + \Delta_\rho \psi)^{\frac{1}{n-1}} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}(e^{-c\psi}(n + \Delta_\rho \psi)) &= e^{-c\psi} \frac{\partial}{\partial t}(n + \Delta_\rho \psi) - c \frac{\partial \psi}{\partial t}(n + \Delta_\rho \psi) \\ &\leq e^{-c\psi} \frac{\partial}{\partial t}(n + \Delta_\rho \psi) + C_5(n + \Delta_\rho \psi), \end{aligned}$$

we get from (6.11) and (6.12) that

$$(6.13) \quad \begin{aligned} -\frac{\partial}{\partial t}(e^{-c\psi}(n + \Delta_\rho \psi)) + \Delta'(e^{-c\psi}(n + \Delta_\rho \psi)) + X(e^{-c\psi}(n + \Delta_\rho \psi)) \\ \geq e^{-c\psi}(-C_6 - C_7(n + \Delta_\rho \psi) + C_8 e^{\frac{1}{n-1}\psi}(n + \Delta_\rho \psi)^{1 + \frac{1}{n-1}}). \end{aligned}$$

Applying the Maximal Principle to function $e^{-c\psi}(n + \Delta_\rho\psi)$ in (6.13), we obtain

$$n + \Delta_\rho\psi \leq C \exp\{c(\psi - \inf_M \psi)\}(1 + \exp\{-\inf_M \psi\}).$$

Then (6.10) follows from (6.5). □

Proposition 6.1. *Assume that M admits a Kähler-Ricci soliton (g_{KS}, X) . Then*

- (i) *the metric $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}$ is uniformly equivalent to the metric ω_g ;*
- (ii) *we have a uniform estimate*

$$\|\bar{\varphi}\|_{C^3(M)} \leq C.$$

Proof. It follows from (6.9) and Lemma 6.2 that

$$0 < c_1 \leq n + \Delta_\rho\psi \leq c_2.$$

It follows that

$$c_1 \leq n + \Delta_{\omega_{KS}}(\bar{\varphi} - \Psi_0) \leq c_2.$$

Then (i) follows easily from this.

For the third derivative estimate, we consider Calabi's function

$$S = g'^{i\bar{r}}g'^{s\bar{j}}g'^{k\bar{l}}\psi_{i\bar{j}k}\psi_{\bar{r}s\bar{l}},$$

where $(g'^{i\bar{j}})$ is the inverse of the Hermitian matrix function associated to the metric $\omega_\rho + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$. Note that S is equal to the quantity

$$\bar{g}^{i\bar{r}}\bar{g}^{s\bar{j}}\bar{g}^{k\bar{l}}(\bar{\varphi} - \Psi_0)_{i\bar{j}k}(\bar{\varphi} - \Psi_0)_{\bar{r}s\bar{l}},$$

where $(\bar{g})^{i\bar{r}}$ denotes the inverse of the Hermitian matrix function associated to the metric $\omega_{\bar{\varphi}}$.

Following Calabi's computation as in [Ya], by Lemma 6.2, one can get

$$(\Delta_\rho - \frac{\partial}{\partial t})(S + c\Delta_\rho\psi) \geq C_1S - C_2.$$

Here c is sufficiently large, and C_1 and C_2 are uniform positive constants depending only on the metric ω_{KS} . Then by the Maximal Principle, we see that $S \leq C$. Hence by part (i), we obtain

$$\|\bar{\varphi}\|_{C^3(M)} \leq C'. \quad \square$$

Proof of Main Theorem. We shall prove that $\omega_{\bar{\varphi}_t}$ converges to a Kähler-Ricci soliton. First we show that the solution $\bar{\varphi}$ of (6.4) has uniformly bounded C^k -norms. Note that $\bar{\varphi}$ is a solution of the following equation:

$$(6.14) \quad \frac{\partial \bar{\varphi}}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \bar{\varphi}_{i\bar{j}})}{\det(g_{i\bar{j}})} + \text{Re}(\bar{X}(\bar{\varphi})) + \bar{\varphi} - h + \text{Re}(\theta_{\bar{X}}).$$

Differentiating this equation on z_k in a coordinate chart of M with local holomorphic coordinates (z_1, \dots, z_n) , we have

$$(\Delta' - \frac{\partial}{\partial t})(\frac{\partial \bar{\varphi}}{\partial z_k}) = g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z^k} - g'^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z^k} - \bar{\varphi}_k - (h - \text{Re}(\theta_{\bar{X}} - \bar{X}(\bar{\varphi})))_k.$$

Since Δ' is uniformly elliptic and is at least of C^α ($0 < \alpha < 1$), by the standard Schauder estimates, we see that $\frac{\partial \bar{\varphi}}{\partial z_k}$ has a uniform $C^{2,\alpha}$ -norm. Similarly, $\frac{\partial \bar{\varphi}}{\partial \bar{z}_k}$ has a uniform $C^{2,\alpha}$ -norm. Hence, the $C^{3,\alpha}$ -norm of $\bar{\varphi}$ is uniformly bounded. Repeating this process, one can easily show that the C^k -norm of $\bar{\varphi}$ is uniformly bounded for each integer $k \geq 0$.

From the above arguments, we see that for any sequence of functions $\bar{\varphi}_t$, there is a subsequence which converges to a smooth function $\bar{\varphi}_\infty$ on M in the C^k -topology. Next we show that $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}_\infty$ is a Kähler-Ricci soliton. Let $\sigma_t = \exp\{tX\}$ and $\sigma' = \sigma'_t = \sigma_t^{-1} \cdot \bar{\rho}_t$. Since

$$\int_M \|\bar{\partial}((\sigma')^* \frac{\partial\varphi'}{\partial t})\|_{\omega_{\bar{\varphi}}}^2 e^{\theta_X + X(\bar{\varphi})} \omega_{\bar{\varphi}}^n = \int_M \|\bar{\partial} \frac{\partial\varphi'}{\partial t}\|_{\omega_{\varphi'}}^2 e^{\theta_X + X(\varphi')} (\omega_{\varphi'})^n,$$

then by (4.5), we have

$$(6.15) \quad \int_M \|\bar{\partial}((\sigma')^* \frac{\partial\varphi'}{\partial t})\|_{\omega_{\bar{\varphi}}}^2 e^{\theta_X + X(\bar{\varphi})} \omega_{\bar{\varphi}}^n = -\frac{d\tilde{\mu}_{\omega_g}(\varphi')}{dt}.$$

By the lower bound of $\tilde{\mu}_{\omega_g}(\varphi')$ [TZ2], one concludes that there is a sequence of t_i , $i = 1, 2, \dots$, such that $\bar{X}_i \rightarrow X_\infty$ and

$$(6.16) \quad \int_M \|\bar{\partial}((\sigma')^* \frac{\partial\varphi'}{\partial t}|_{t_i})\|_{\omega_{\bar{\varphi}_{t_i}}}^2 \omega_{\bar{\varphi}_{t_i}}^n \rightarrow 0, \text{ as } i \rightarrow \infty.$$

On the other hand, by (4.4), we have

$$(6.17) \quad \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}[(\sigma')^* \frac{\partial\varphi'}{\partial t}] = -\text{Ric}(\omega_{\bar{\varphi}}) + \omega_{\bar{\varphi}} + L_X \omega_{\bar{\varphi}}.$$

Then the C^k -norm of $(\sigma')^* \frac{\partial\varphi'}{\partial t}$ is uniformly bounded, so there exists a convergent subsequence of $(\sigma')^* \frac{\partial\varphi'}{\partial t}|_{t_i}$. Hence by (6.16) and Lemma 4.2, we conclude that $(\sigma')^* \frac{\partial\varphi'}{\partial t}(t_i, \cdot)$ (still denoted by the same indices t_i) converges to zero in the C^k -topology, and consequently, by (6.17), Kähler metrics $\omega_{\bar{\varphi}_{t_i}}$ converge to a Kähler-Ricci soliton associated to a holomorphic vector field X_∞ . By the uniqueness [TZ1] and [TZ2], we see that $X_\infty = X$ and the limit Kähler-Ricci soliton $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}_\infty$ must be $\beta^* \omega_{KS}$ for some element $\beta \in \text{Aut}_r(M)$. For simplicity, we may assume that β is just the identity in $\text{Aut}_r(M)$.

It remains to prove that $\omega_{\bar{\varphi}_t}$ converges to $(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}_\infty, X)$ as t goes to infinity. If this is false, then there is a convergent sequence $\omega_{\bar{\varphi}_{t_i}}$ whose limit is not $(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}_\infty, X)$. But by (6.15) and the lower bound of $\tilde{\mu}_{\omega_g}(\varphi')$, we see that for any fixed number $\epsilon > 0$ there is a sequence $s_i \in (-\epsilon + t_i, \epsilon + t_i)$, $i = 1, 2, \dots$, such that relation (6.16) holds for the sequence $(\sigma')^* \frac{\partial\varphi'}{\partial t}|_{t=s_i}$. It follows from (6.17) that $\omega_{\bar{\varphi}_{s_i}}$ converges to $(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\bar{\varphi}_\infty, X)$ in the C^k sense. On other hand, on each interval $[-\epsilon + t_i, \epsilon + t_i]$, $\bar{\varphi} - \bar{\varphi}_{s_i}$ satisfies the following Monge-Ampère flow:

$$\frac{\partial(\bar{\varphi} - \bar{\varphi}_{s_i})}{\partial t} = \log \frac{(\omega_{\bar{\varphi}_{s_i}} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\bar{\varphi} - \bar{\varphi}_{s_i}))^n}{\omega_{\bar{\varphi}_{s_i}}^n} + \text{Re}(\bar{X}(\bar{\varphi} - \bar{\varphi}_{s_i})) + (\bar{\varphi} - \bar{\varphi}_{s_i}) + G,$$

where

$$H = -\frac{\partial\bar{\varphi}}{\partial t}|_{s_i} + \text{Re}(\theta_{\bar{X}} - \theta_{\bar{X}_{s_i}}) + \text{Re}((\bar{X} - \bar{X}_{s_i})\bar{\varphi}_{s_i}).$$

Note that $\|H\|_{C^k(M)} \leq C\epsilon$, when i is sufficiently large. Thus by using the implicit function theorem, we see that

$$\|\bar{\varphi}_t - \bar{\varphi}_{s_i}\|_{C^k(M)} \leq \delta(\epsilon), \quad \forall t \in [-\epsilon + t_i, \epsilon + t_i],$$

when i is sufficiently large. Here $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since ϵ can be chosen to be small, the sequence $\bar{\varphi}_{t_i}$ will converge to $\bar{\varphi}_\infty$, which contradicts the assumption. The proof is completed. \square

Remark 6.1. To get the C^2 -estimate and C^3 -estimate of $\bar{\varphi}_t$, we used (6.9) instead of (6.14). The reason for this is that in (6.9), the holomorphic vector field X is fixed and $X(\psi_t)$ is uniformly bounded according to a result in [Zh], while in (6.14), we do not know how to bound $\text{Re}(\bar{X}_t(\bar{\varphi}_t))$.

Remark 6.2. Following arguments in Section 10 in [CT1], one can further prove that the evolved metrics $\omega_{\bar{\varphi}_t}$ of (6.17) will exponentially converge to a Kähler-Ricci soliton with respect to X in the sense of Cheeger-Gromov. More precisely, there is a family of automorphisms $\beta_t \in \text{Aut}_r(M)$ with $\|\beta_t - Id\| \leq C$ such that for any integer $k \geq 0$ it holds that

$$\|D^k[\beta_t^*(\text{Ric}(\omega_{\bar{\varphi}_t}) - \omega_{\bar{\varphi}_t} - L_X\omega_{\bar{\varphi}_t})]\|_{g_{KS}} \leq C_k(\epsilon)e^{-2(\lambda_2-1-\epsilon)t} \rightarrow 0, \text{ as } t \rightarrow \infty,$$

where D^k denotes the covariant derivative of degree k with respect to the Kähler-Ricci soliton g_{KS} and $\epsilon > 0$ is a small number and $\lambda_2 > 1$ is the second nonzero eigenvalue of operator $\Delta_{g_{KS}} + X(\cdot)$ on the K_X -invariant functions space.

APPENDIX

In this appendix, we outline a proof of Lemma 1.1 which is taken from [ST]. The proof consists of three main steps.

Step 1. Let $u = -h_t$. Then we show that u is uniformly bounded from below. To prove this, we need to introduce Perelman’s W -functional for a pair (g, f) :

$$(A.1) \quad W(g, f) = \int_M [(R(g) + |Df|^2) + f]e^{-f} dV_g,$$

where g is a Riemannian metric on M and $R(g)$ denotes the scalar curvature of g and where f is a smooth function on M which satisfies a normalization condition

$$(A.2) \quad \int_M e^{-f} dV_g = \int_M dV_g \equiv V.$$

Let

$$\mu(g) = \inf\{W(g, f) \mid f \text{ satisfies (A.2)}\}.$$

It is easy to see that $\mu(g)$ can be attained by some f . In [P1], it was shown that $\mu(g(t))$ is increasing if $g(t)$ is a family of evolved Riemannian metrics along the Ricci flow. Since in our case of Kähler-Ricci flow (0.1),

$$R(g(t)) = 4(-\Delta u + n),$$

we have

$$\begin{aligned} C &\leq \mu(g(t)) \leq W(g(t), h_t) \\ &= \int_M [4(-\Delta u + |\nabla u|^2 + n) + u]e^{-u}\omega_\varphi^n = 4n \int_M \omega_{g_0}^n + \int_M ue^{-u}\omega_\varphi^n, \end{aligned}$$

where Δ and ∇ denote the $\bar{\partial}$ -Laplace operator and the covariant derivative of type (1,0) associated to metric $g(t)$, respectively. Thus

$$(A.3) \quad \int_M ue^{-u}\omega_\varphi^n \geq -C_1.$$

By (1.3), we have

$$(A.4) \quad \frac{du}{dt} = \Delta u + u + a,$$

for some constants $a = a_t$. Using (A.2), we get

$$a = -\frac{1}{V} \int_M u e^{-u} \omega_\varphi^n \leq C_1.$$

On the other hand, by using the Maximal Principle, it is known that scalar curvature $R = R(g(t))$ is uniformly bounded from below. Thus

$$(A.5) \quad \frac{\partial u}{\partial t} = -R + n + u + a \leq C_2 + u,$$

where $R = -\Delta u + n$ is the scalar curvature of $g(t)$ after a scale for the case of Kähler metrics.

Now we assume that $u(t_0)(x_0) < -2C_2 < 0$ for some time t_0 and some $x_0 \in M$. So it holds that

$$u(t_0)|_U \leq -2C_2$$

for some neighborhood U of x_0 . Thus

$$\frac{\partial u}{\partial t}(t_0)|_U \leq C_2 + u < -C_2 < 0.$$

It follows that

$$(A.6) \quad u(t)|_U \leq -C_2 e^{t-t_0} \leq -C_3 e^t,$$

for any $t \geq t_0$.

We choose some constants b_t so that $\tilde{\varphi} = \tilde{\varphi}_t = \varphi_t + b_t$ and $u = \frac{\partial \tilde{\varphi}}{\partial t}$. Then by (A.6), we have

$$\tilde{\varphi}(t)|_U \leq -C'_3 e^t,$$

for any $t \geq t_0$. Since $\tilde{\varphi}(t)$ is an almost subharmonic function, one can apply the Green's formula to obtain

$$(A.7) \quad \max_M \tilde{\varphi}_t \leq \max_U \tilde{\varphi}_t + C(g_0) \leq -C'_3 e^t + C(g_0),$$

for any $t \geq t_0$. On the other hand, by (A.5), one sees

$$\frac{\partial}{\partial t}(u - \tilde{\varphi}) \leq C_2.$$

It follows that

$$(A.8) \quad \max_M \tilde{\varphi}_t \geq -C_2 t - \max_M(u_0 - \tilde{\varphi}_0).$$

(A.8) is a contradiction to (A.7) for sufficiently large t . This implies that $u(t) \geq -2C_2$ for any t and so Step 1 is proved.

Step 2. Choose B so that $u + B > 0$. Then we claim

$$(A.9) \quad |\nabla u|^2 \leq C(u + B)$$

and

$$(A.10) \quad -\Delta u \leq C(u + B).$$

(A.9) and (A.10) can be proved by using the standard Maximal Principle for the parabolic operator $\frac{\partial}{\partial t} - \Delta$ (see [ST] for details). For simplicity, we outline a proof for (A.9).

By (A.4), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Delta u) = -|\nabla \bar{\nabla} u|^2 + \Delta u$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 = -|\nabla \nabla u|^2 - |\nabla \bar{\nabla} u|^2 + |\nabla u|^2.$$

Let $H = \frac{|\nabla u|^2}{u+B}$. Then

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)H &= \frac{|\nabla u|^2(B-a)}{(u+B)^2} - \frac{|\nabla \nabla u|^2 + |\nabla \bar{\nabla} u|^2}{u+B} \\ &+ 2\frac{\langle \nabla u, \nabla |\nabla u|^2 \rangle}{(u+B)^2} - 2\frac{|\nabla u|^4}{(u+B)^3}. \end{aligned} \tag{A.11}$$

After a simple estimate of the last three terms on the right side of (A.11), one shows

$$\left(\frac{\partial}{\partial t} - \Delta\right)H \leq \frac{|\nabla u|^2(B-a)}{(u+B)^2} + (2-\epsilon)\frac{\langle \nabla u, \nabla H \rangle}{u+B} - \frac{\epsilon}{2}\frac{|\nabla u|^4}{(u+B)^3},$$

for some small ϵ . Thus one can use the Maximal Principle to get

$$\max_M H \leq C.$$

This proves (A.9).

By (A.9), we get

$$u(x, t) \leq CDiam(M, g(t))^2 + C'. \tag{A.12}$$

So to complete Lemma 1.1, it suffices to prove that the diameters $Diam(M, g(t))$ of M are uniformly bounded. Note that we can also get from (A.10) and (A.12) that

$$R(g(t)) \leq CDiam(M, g(t))^2 + C' \tag{A.13}$$

and

$$|\nabla u| \leq CDiam(M, g(t)) + C'. \tag{A.14}$$

Step 3. Prove

$$Diam(M, g(t)) \leq C. \tag{A.15}$$

We use an argument by contradiction to prove (A.15). The proof is similar to one of Perelman's noncollapsing theorems ([P1]). Let $x = x_t \in M$ be a minimal point of function $u(t, y)$ and denote $d_t(z) = \text{dist}_t(x, z)$. Let $B_t(k_1, k_2) = \{z \mid 2^{k_1} \leq d_t(z) \leq 2^{k_2}\}$ be an annuli in M . Then by using the noncollapsing result of Ricci flow ([P1], [KL]) together with (A.13) and (A.14), one sees that the following statement is true ([ST]): If $Diam(M, g(t))$ is large enough, then for every $\epsilon > 0$ there exists $B_t(k_1, k_2)$ such that

$$\text{Vol}(B_t(k_1, k_2)) < \epsilon \text{ and } \text{Vol}(B_t(k_1, k_2)) \leq 2^{10n} \text{Vol}(B_t(k_1 + 2, k_2 - 2)). \tag{A.16}$$

Moreover, there exists r_1 and r_2 and a uniform constant C such that $2^{k_1} \leq r_1 \leq 2^{k_1+1}$, $2^{k_2} \leq r_2 \leq 2^{k_2+1}$ and

$$\int_{B_t(r_1, r_2)} R(g(t))\omega_{\varphi_t}^n \leq C\text{Vol}(B_t(k_1, k_2)). \tag{A.17}$$

Now we suppose that there is a sequence t_i such that $Diam(M, g(t_i)) \rightarrow \infty$. Then by the above statement there are sequences $\{k_1^i\}$, $\{k_2^i\}$, $\{r_1^i\}$ and $\{r_2^i\}$ such that (A.16) and (A.17) are satisfied for these sequences. Let f_i be a sequence of smooth cut-off functions on \mathbb{R} such that $f_i \equiv 1$ on $[2^{k_1^i+2}, 2^{k_2^i-2}]$ and $f_i \equiv 0$ on

$(-\infty, r_1^i] \cup [r_2^i, \infty)$. Let $u_i(z) = e^{C_i} f_i(d_t(z))$ be a sequence of functions on M such that $\int_M u_i^2 \omega_{\varphi_{t_i}}^n = \int_M \omega_{g_0}^n$ by choosing suitable constants C_i . Since

$$\text{Vol}(B_{t_i}(k_1^i, k_2^i)) < \epsilon_i \rightarrow 0,$$

it is easy to see that $C_i \rightarrow 0$.

By the monotonicity of $\mu(g(t))$, we have

$$\begin{aligned} (A.18) \quad C &\leq \mu(g(t_i)) \leq W(g(t_i), -2 \ln u_i) \\ &= e^{2C_i} \int_{B_{t_i}(r_1^i, r_2^i)} (4|f_i'(\text{dist}_{t_i}(z))|^2 - 2f_i^2 \ln f_i) \omega_{\varphi_{t_i}}^n \\ &\quad + \int_{B_{t_i}(r_1^i, r_2^i)} R(g(t_i)) u_i^2 \omega_{\varphi_{t_i}}^n - 2C_i. \end{aligned}$$

On the other hand, by (A.16) and (A.17), we estimate

$$\begin{aligned} (A.19) \quad \int_{B_{t_i}(r_1^i, r_2^i)} R(g(t_i)) u_i^2 \omega_{\varphi_{t_i}}^n &\leq e^{2C_i} \int_{B_{t_i}(r_1^i, r_2^i)} R(g(t_i)) \omega_{\varphi_{t_i}}^n \\ &\leq C_1 e^{2C_i} \text{Vol}(B_{t_i}(k_1^i, k_2^i)) \\ &\leq C_1 e^{2C_i} 2^{10n} \text{Vol}(B_{t_i}(k_1^i + 2, k_2^i - 2)) \\ &\leq C_1 2^{10n} \int_M u_i^2 \omega_{\varphi_{t_i}}^n = C_1 2^{10n} \int_M \omega_{g_0}^n. \end{aligned}$$

Similarly by (A.16), we have

$$(A.20) \quad e^{2C_i} \int_{B_{t_i}(r_1^i, r_2^i)} (4|f_i'(\text{dist}_{t_i}(z))|^2 - 2f_i^2 \ln f_i) \omega_{\varphi_{t_i}}^n \leq C_2 2^{10n} \int_M \omega_{g_0}^n.$$

Thus inserting (A.19) and (A.20) into (A.18), we get

$$C \leq C_3 - 2C_i \rightarrow -\infty,$$

which is impossible. Therefore, (A.15) is true.

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