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UPPER BOUNDS IN QUANTUM DYNAMICS

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1. Introduction

If H is a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, the time-dependent Schrödinger equation of quantum mechanics, $i\partial_t\psi = H\psi$, leads to a unitary dynamical evolution in $\mathcal{H}$,

$$\psi(t) = e^{-itH}\psi(0). \tag{1}$$

Of course, the case of main interest is where $\mathcal{H}$ is given by $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, H is a Schrödinger operator of the form $-\Delta + V$, and $\psi(0)$ is a localized wavepacket.

Under the time evolution (1), the wavepacket $\psi(t)$ will in general spread out with time. Finding quantitative bounds for this spreading in concrete cases and developing methods for proving such bounds has been the objective of a great number of papers in recent years. One is often interested in the growth behavior of the moments of the position operator, that is, the function $t \mapsto \||X|^{p/2}\psi(t)\|^2$.

These questions are particularly complicated and interesting in situations where singular continuous spectra occur. The discovery of the occurrence of such spectra for many models of physical relevance has triggered some activity on the quantum dynamical side. For models with absolutely continuous spectra, on the other hand, quantum dynamical questions have been investigated much earlier in the context of scattering theory and have been answered to a great degree of satisfaction. Models with point spectrum, or more restrictively, models exhibiting Anderson localization, have also been investigated from a quantum dynamical point of view. While it is not true in general that spectral localization implies dynamical localization [26, 73], it has been shown that the three main approaches to spectral localization, the fractional moment method [3, 4, 5], multi-scale analysis [28, 27, 75], and the Bourgain-Goldstein-Schlag method [12, 37], yield dynamical localization as a by-product [2, 13, 22, 31, 34].

Quantum dynamics for Schrödinger operators with singular continuous spectra is much less understood. The two primary examples motivated by physics are the Harper model, also known as the almost Mathieu operator at critical coupling, which describes a Bloch electron in a constant magnetic field with irrational flux through a unit cell, and the Fibonacci model, which describes a standard one-dimensional quasicrystal. Both models have been studied heavily in both the physics and mathematics communities.

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The standard approach to dynamical questions is via the spectral theorem. Recall that each initial vector $\psi(0) = \psi$ has a spectral measure, defined as the unique Borel measure obeying

$$\langle \psi, f(H)\psi \rangle = \int_{\sigma(H)} f(E)\, d\mu_\psi(E)$$

for every measurable function f. Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathcal{H}$. A very important discovery of Guarneri [39, 40], which has been extended by other authors [14, 41, 62, 58], was that suitable continuity properties of the spectral measure $d\mu_\psi$ imply power-law lower bounds for the time-averaged moments of the position operator. Such continuity properties follow from upper bounds on the measure of intervals, $\mu_\psi([E-\varepsilon, E+\varepsilon]),\ E \in \sigma(H),\ \varepsilon \to 0$. Later on, extensions and refinements of Guarneri's method were developed [8, 42, 71], which allowed the authors of those papers to obtain more general dynamical lower bounds that take into account the whole statistics of $\mu_\psi([E-\varepsilon, E+\varepsilon]),\ E \in \mathbb{R}$. In particular, lower bounds on $\mu_\psi([E-\varepsilon, E+\varepsilon])$ for E from some set of positive Lebesgue measure imply dynamical lower bounds in the case of both singular continuous and pure point spectrum [33]. Better lower bounds can be obtained if one has information about both the measure of intervals and the growth of the generalized eigenfunctions $u_\psi(n, E)$ [58, 71].

In the case of Schrödinger operators in one space dimension, the required information on the spectral measure (and on generalized eigenfunctions) is intimately related to properties of solutions to the differential/difference equation $Hu = Eu$ [20, 25, 33, 51, 52, 74]. An analysis of these solutions has been carried out in a number of concrete cases [18, 20, 33, 51, 52, 58], giving rise to explicit lower bounds on spreading rates for (generalizations of) the Fibonacci model, sparse potentials, and (random) decaying potentials. Thus, in one dimension, there is a three-step procedure for obtaining lower bounds on spreading rates for a given model: properties of the model $\rightsquigarrow$ solution estimates $\rightsquigarrow$ bounds on the measure of intervals $\mu_\psi([E-\varepsilon, E+\varepsilon]) \rightsquigarrow$ dynamical lower bounds.

There is a second approach to dynamical lower bounds in one dimension, which is based on the Parseval formula,[1]

$$(2) \qquad 2\pi \int_0^\infty e^{-2t/T} |\langle e^{-itH}\delta_1, \delta_n \rangle|^2\, dt = \int_{-\infty}^\infty \left| \langle (H - E - \tfrac{i}{T})^{-1} \delta_1, \delta_n \rangle \right|^2\, dE.$$

This method was developed in [23, 24, 72]. It has the advantage that it gives dynamical bounds rather directly without any knowledge of the properties of spectral measure (although additional information on spectral measure allows one to improve the results [23, 72]). The input required for this method is based on upper bounds for solutions corresponding to some set of energies, which can be very small (nonempty is sufficient [23, 24]). This method is the basis for the results in [21, 53]. Moreover, a combination of this approach with the traditional one (based on the spectral theorem) leads to optimal dynamical bounds for growing sparse potentials [72] and the best dynamical lower bounds for the Fibonacci operator known to this date [25].

To a certain extent, there is a fairly good understanding of how to prove dynamical lower bounds, particularly in one space dimension, as a consequence of

[1] The formula (2) was used earlier by Kiselev, Killip, and Last in their study of dynamical upper bounds on the slow part of the wavepacket [57], which will be discussed momentarily.

the results described above. The problem of proving upper bounds, on the other hand, is much harder and has essentially eluded the researchers in this field up to this point. The known lower bounds for models with singular continuous spectra do not exclude ballistic motion and hence are not able to distinguish them from models with absolutely continuous spectra. However, ballistic motion is not expected to occur for the two main models of interest, the Harper model and the Fibonacci model. While we explain the terminology below, let us mention at this point that the motion is expected to be almost-diffusive in the Harper model, while it is expected to be anomalous (i.e., neither localized nor diffusive nor ballistic) in the Fibonacci model. There is a lot of numerical and heuristic evidence for these claims in the physics literature. To prove such a statement, one needs two-sided estimates for the dynamical quantities associated with these models.

Proving upper bounds is so hard because one needs to control the entire wavepacket. In fact, the dynamical lower bounds that are typically established only bound some (fast) part of the wavepacket from below and this is sufficient for the desired growth of the standard dynamical quantities. It is of course much easier to prove upper bounds only for a portion of the wavepacket and Killip, Kiselev, and Last succeeded in doing so [57]. Namely, their work provides explicit criteria for upper bounds on the slow part of the wavepacket in terms of lower bounds on solutions. They applied their general method to the Fibonacci operator. Their result further supports the conjecture that this model exhibits anomalous transport.

It is the purpose of this paper to develop a general method connecting solution properties to dynamical upper bounds and to use this method to prove dynamical upper bounds for the Fibonacci operator and the almost Mathieu operator.

The general method is based on the Parseval formula (2), which has already proved its utility when studying dynamics. We are able to bound the entire wavepacket from above, provided that suitable lower bounds for solution (or rather transfer matrix) growth at complex energies are available.

We apply this method to the Fibonacci operator and establish anomalous transport at sufficiently large coupling. In combination with the known lower bounds, our result shows that the large coupling asymptotics of the dynamics follows a law predicted by Abe and Hiramoto in 1987.

Finally, we prove upper bounds for the almost Mathieu operator and extensions thereof at sufficiently large coupling. It is shown that positive Lyapunov exponents imply a weak form of dynamical localization in a sense that will be made precise below. This result complements a stunning observation by Last who proved almost ballistic transport on a sequence of time scales for the same model in the case of some Liouville frequencies. Our result shows that on other sequences of time scales, one has essentially no transport. Thus, the almost Mathieu operator with a Liouville frequency exhibits quite interesting dynamics. We also show that the phenomenon discovered by Last is impossible if the frequency satisfies a weak Brjuno-type condition.

2. Description of the results and overview

Consider a discrete one-dimensional Schrödinger operator,

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{3}$$

on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, \ldots\}$. In the case of $\ell^2(\mathbb{Z}_+)$, we will work with a Dirichlet boundary condition, $\psi(0) = 0$, but our results easily extend to all other self-adjoint boundary conditions.

A number of recent papers (e.g., [23, 24, 25, 51, 52, 53, 57, 71, 72]) were devoted to proving lower bounds on the spreading of an initially localized wavepacket, say $\psi = \delta_1$, under the dynamics governed by H, typically in situations where the spectral measure of δ_1 with respect to H is purely singular and sometimes even pure point.

A standard quantity that is considered to measure the spreading of the wavepacket is the following: For $p > 0$, define

$$\langle |X|^p_{\delta_1} \rangle(T) = \sum_n |n|^p a(n,T), \tag{4}$$

where

$$a(n,T) = \frac{2}{T} \int_0^\infty e^{-2t/T} |\langle e^{-itH}\delta_1, \delta_n\rangle|^2 \, dt. \tag{5}$$

Clearly, the faster $\langle |X|^p_{\delta_1} \rangle(T)$ grows, the faster $e^{-itH}\delta_1$ spreads out, at least averaged in time. One typically wants to prove power-law bounds on $\langle |X|^p_{\delta_1} \rangle(T)$ and hence it is natural to define the following quantity: For $p > 0$, define the *lower transport exponent* $\beta^-_{\delta_1}(p)$ by

$$\beta^-_{\delta_1}(p) = \liminf_{T\to\infty} \frac{\log \langle |X|^p_{\delta_1} \rangle(T)}{p \log T}$$

and the *upper transport exponent* $\beta^+_{\delta_1}(p)$ by

$$\beta^+_{\delta_1}(p) = \limsup_{T\to\infty} \frac{\log \langle |X|^p_{\delta_1} \rangle(T)}{p \log T}.$$

Both functions $\beta^\pm_{\delta_1}(p)$ are nondecreasing.

Another way to describe the spreading of the wave-function is in terms of probabilities. We define time-averaged *outside probabilities* by

$$P(N,T) = \sum_{|n|>N} a(n,T).$$

Following [33], for any $\alpha \in [0, +\infty]$ define[2]

$$S^-(\alpha) = -\liminf_{T\to\infty} \frac{\log P(T^\alpha - 2, T)}{\log T} \tag{6}$$

and

$$S^+(\alpha) = -\limsup_{T\to\infty} \frac{\log P(T^\alpha - 2, T)}{\log T}. \tag{7}$$

For every α, $0 \le S^+(\alpha) \le S^-(\alpha) \le \infty$.

These numbers control the power decaying tails of the wavepacket. In particular, the following critical exponents are of interest:

$$\alpha_l^\pm = \sup\{\alpha \ge 0 : S^\pm(\alpha) = 0\}, \tag{8}$$

$$\alpha_u^\pm = \sup\{\alpha \ge 0 : S^\pm(\alpha) < \infty\}. \tag{9}$$

[2] We take $T^\alpha - 2$ so that $P(T^0 - 2, T) = 1$ for all T.

We have that $0 \le \alpha_l^- \le \alpha_u^- \le 1$, $0 \le \alpha_l^+ \le \alpha_u^+ \le 1$, and also that $\alpha_l^- \le \alpha_l^+$, $\alpha_u^- \le \alpha_u^+$. One can interpret $\alpha_l^\pm$ as the (lower and upper) rates of propagation of the essential part of the wavepacket and $\alpha_u^\pm$ as the rates of propagation of the fastest (polynomially small) part of the wavepacket; compare [33]. In particular, if $\alpha > \alpha_u^+$, then $P(T^\alpha, T)$ goes to 0 faster than any inverse power of T. Since a ballistic upper bound holds in our case (for any potential V), Theorem 4.1 in [33] yields

$$\lim_{p\to 0} \beta_{\delta_1}^\pm(p) = \alpha_l^\pm$$

and

$$\lim_{p\to \infty} \beta_{\delta_1}^\pm(p) = \alpha_u^\pm.$$

In particular, since the $\beta_{\delta_1}^\pm(p)$ are nondecreasing, we have that

$$\beta_{\delta_1}^\pm(p) \le \alpha_u^\pm \quad \text{ for every } p > 0. \tag{10}$$

When one wants to bound all these dynamical quantities for specific models, it is useful to connect them to the qualitative behavior of the solutions of the difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n) \tag{11}$$

since there are effective methods for studying these solutions. Presently, the known general results are limited to one-sided estimates of the transport exponents. Namely, as already alluded to in the introduction, a number of approaches to lower bounds on $\beta_{\delta_1}^\pm(p)$ have been found in recent years.

It should be stressed that there were no general methods known to bound $\alpha_l^\pm, \alpha_u^\pm$, or $\beta_{\delta_1}^\pm(p)$ nontrivially from above. In the present paper we propose a first general approach to proving upper bounds on $\alpha_u^\pm$ (which in turn bound $\alpha_l^\pm$ and $\beta^\pm(p)$ for all $p > 0$ from above, as well). This approach relates the dynamical quantities introduced above to the behavior of the solutions to the difference equation (11) for complex energies z. To state this result, let us recall the reformulation of (11) in terms of transfer matrices. These matrices are uniquely determined by the requirement that

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \Phi(n,z) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

for every solution u of (11). Consequently,

$$\Phi(n,z) = \begin{cases} T(n,z)\cdots T(1,z) & n \ge 1, \\ \mathrm{Id} & n = 0, \\ [T(n+1,z)]^{-1}\cdots[T(0,z)]^{-1} & n \le -1, \end{cases} \tag{12}$$

where

$$T(m,z) = \begin{pmatrix} z - V(m) & -1 \\ 1 & 0 \end{pmatrix}.$$

We have the following result:

Theorem 1. *Suppose H is given by* (3)*, where V is a bounded real-valued function, and $K \ge 4$ is such that $\sigma(H) \subseteq [-K+1, K-1]$. Suppose that, for some $C \in (0,\infty)$ and $\alpha \in (0,1)$, we have*

$$\int_{-K}^{K} \left(\max_{1\le n \le CT^\alpha} \left\| \Phi\left(n, E + \tfrac{i}{T}\right)\right\|^2 \right)^{-1} dE = O(T^{-m}) \tag{13}$$

and

$$\int_{-K}^{K} \left(\max_{1 \le -n \le CT^\alpha} \left\| \Phi \left(n, E + \tfrac{i}{T} \right) \right\|^2 \right)^{-1} dE = O(T^{-m}) \tag{14}$$

for every $m \ge 1$*. Then*

$$\alpha_u^+ \le \alpha. \tag{15}$$

In particular,

$$\beta^+(p) \le \alpha \quad \textit{for every } p > 0. \tag{16}$$

Remarks. (a) If the conditions of the theorem are fulfilled for some sequence of times $T_k \to \infty$, we get an upper bound for α_u^-.

(b) The statement of the theorem follows from upper bounds for outside probabilities described in Theorem 7 below.

The next issue we want to address is the stability of the transport exponents $\alpha_u^\pm$ with respect to suitable perturbations of the potential. For example, it is reasonable to expect that they are invariant with respect to finitely supported perturbations. However, such a property has not been established yet. Moreover, the approaches to lower bounds for transport exponents that are based on dimensional properties of spectral measures are not suited to prove this invariance. It is well known that these dimensions are rather sensitive with respect to finite rank perturbations. As pointed out in [23, 24], the approach developed in those papers is stable in the sense that if it can be applied to a certain model, it can also be applied (and yields the same bounds) to finitely supported (or even suitable power-decaying) perturbations.

Of course, it is even more desirable to prove that the transport exponents (as opposed to one-sided bounds) are invariants. The following result establishes this fact:

Theorem 2. *Let* H_1, H_2 *be two operators of the form* (3) *with bounded potentials and let* $K \ge 4$ *be such that* $\sigma(H_{1,2}) \subseteq [-K+1, K-1]$*. Denote the corresponding transfer matrices by* Φ_1, Φ_2 *and the corresponding transport exponents by* $\alpha_{1,u}^\pm$, $\alpha_{2,u}^\pm$*. Assume there exists* $A > 0$ *such that for all* $E \in [-K, K]$, $0 < \varepsilon \le 1$, $|n| \le 1/\varepsilon$,

$$\varepsilon^A \|\Phi_1(n, E + i\varepsilon)\| \lesssim \|\Phi_2(n, E + i\varepsilon)\| \lesssim \varepsilon^{-A} \|\Phi_1(n, E + i\varepsilon)\|. \tag{17}$$

Then, $\alpha_{1,u}^\pm = \alpha_{2,u}^\pm$*.*

Here and in what follows, $f \lesssim g$ means that $f \le Cg$ for some positive constant C that we leave implicit.

The bounds (17) clearly hold if the potentials differ only on a finite set. Thus, Theorem 2 yields the invariance mentioned above. For example, assume that dynamical localization holds for the operator H_1. That means, in particular, that $\beta_{1,\delta_1}^+(p) = 0$ for every $p > 0$ and thus $\alpha_{1,u}^+ = 0$. Theorem 2 yields $\alpha_{2,u}^+ = 0$, so that $\beta_{2,\delta_1}^+(p) = 0$ — some weak form of dynamical localization for H_2. This result should be compared with that of [26] about semi-stability of dynamical localization.

In most cases where nontrivial dynamical lower bounds have been proven, the results obtained imply $\alpha_l^\pm > 0$ and $\alpha_u^\pm = 1$. This is the case for sparse potentials [33, 72], random decaying potentials [33], the Thue-Morse Hamiltonian [24], and random polymer models [53]. Although it has not yet been proven, it is expected that $\alpha_l^\pm < 1$ for some of these models (for sparse potentials, however, it was shown in [15] that $\alpha_l^+ = 1$).

On the other hand, there are models for which it is expected that $\alpha_u^\pm$ may take values other than zero (this is the case when one has dynamical localization) or one (ballistic transport at least for $p \gg 1$). Such a situation is called *anomalous transport* by some authors. More restrictively, $\alpha_u^\pm = 1/2$ is sometimes called *diffusive transport* and anomalous transport refers to a situation where one does not have ballistic transport, diffusive transport, or dynamical localization.

Two very prominent examples are given by the Fibonacci operator and the almost Mathieu operator. We will apply our general result, Theorem 1, to both of them.

The Fibonacci operator has been studied for decades by mathematicians and physicists. The potential is given by

$$V(n) = \lambda \chi_{[1-\theta,1)}(n\theta \mod 1), \quad \theta = \frac{\sqrt{5}-1}{2}. \tag{18}$$

This potential belongs to the more general class of Sturmian potentials, given by

$$V(n) = \lambda \chi_{[1-\theta,1)}(n\theta + \omega \mod 1) \tag{19}$$

with general irrational $\theta \in (0,1)$ and arbitrary $\omega \in [0,1)$. These sequences provide standard models for one-dimensional quasicrystals.[3] Early numerical and heuristic studies of the spectral properties of the Fibonacci model were performed by Kohmoto, Kadanoff, and Tang [59] and Ostlund, Pandit, Rand, Schellnhuber, and Siggia [64]. It was suggested that the spectrum is always purely singular continuous. This was rigorously established by Sütő for the Fibonacci case [69, 70] and by Bellissard, Iochum, Scoppola, and Testard [11] and Damanik, Killip, and Lenz [20] in the general Sturmian case. Abe and Hiramoto studied the transport exponents for the Fibonacci model numerically [1, 44]. They found that they are decreasing in λ and behave like

$$\alpha_l^\pm,\, \alpha_u^\pm \asymp \frac{1}{\log \lambda} \tag{20}$$

as $\lambda \to \infty$. (Here and in the following, we write $f \asymp g$ if $C^{-1}g \le f \le Cg$ for some $C \ge 1$.) See also [29, 55] for other numerical studies of Fibonacci quantum dynamics.

The general approaches to lower bounds for the transport exponent described above have all been applied to the Fibonacci Hamiltonian (and some Sturmian models). The best lower bound for α_l^- was obtained by Killip, Kiselev, and Last in [57]. It reads[4]

$$\alpha_l^- \ge \frac{2\kappa}{\zeta(\lambda) + \kappa + 1/2},$$

where

$$\kappa = \frac{\log \frac{\sqrt{17}}{4}}{5 \log\left(\frac{\sqrt{5}+1}{2}\right)} \approx 0.0126$$

and $\zeta(\lambda)$, chosen so that one can prove a result like

$$\sum_{n=1}^{L} \|\Phi(n,E)\|^2 \le CL^{2\zeta(\lambda)+1}$$

[3]See [66] for the discovery of quasicrystals and [6] for surveys on the mathematical theory of quasicrystals in general and the role of the Fibonacci operator in particular.

[4]The expression for $\zeta(\lambda)$ is given in terms of its large λ asymptotics which is of main interest here. See [24, 57] for its values at small λ.

for energies in the spectrum of H (our definition differs from that of [57]), obeys

$$\zeta(\lambda) = \frac{3\log\sqrt{5}}{\log\left(\frac{\sqrt{5}+1}{2}\right)}\left(\log\lambda + O(1)\right).$$

This shows in particular that α_l^- admits a lower bound of the type (20).

The best lower bound for α_u^- was found in [24], where it was shown that

$$\alpha_u^- \geq \frac{1}{\zeta(\lambda)+1}.$$

In terms of the exponents $\beta_{\delta_1}^-(p)$, the best known lower bounds are (see [25])

$$\beta_{\delta_1}^-(p) \geq \begin{cases} \frac{p+2\kappa}{(p+1)(\zeta(\lambda)+\kappa+1/2)}, & p \leq 2\zeta(\lambda)+1, \\ \frac{1}{\zeta(\lambda)+1}, & p > 2\zeta(\lambda)+1. \end{cases}$$

We also want to mention work on upper bounds for the slow part of the wavepacket by Killip, Kiselev, and Last [57]. More precisely, they showed that there exists a $\delta \in (0,1)$ such that for λ large enough (so that $p(\lambda)$ defined in (21) below is less than one),

$$P(C_2 T^{p(\lambda)}, T) \leq 1 - \delta.$$

Here,

$$p(\lambda) = \frac{6\log\frac{\sqrt{5}+1}{2}}{\log\xi(\lambda)} \tag{21}$$

and

$$\xi(\lambda) = \frac{\lambda - 4 + \sqrt{(\lambda-4)^2 - 12}}{2}. \tag{22}$$

See [57, Theorem 1.6.(i)]. However, this result does not say anything about the fast part of the wavepacket, and in particular, no statement for any of the transport exponents can be derived.

With the help of Theorem 1 we can prove upper bounds for α_u^+ for the Fibonacci model at sufficiently large coupling. These upper bounds show that (20) is indeed true.

The precise result is as follows:

Theorem 3. *Consider the Fibonacci Hamiltonian, that is, the operator* (3) *with potential* (18). *Assume that* $\lambda \geq 8$ *and let*

$$\alpha(\lambda) = \frac{2\log\frac{\sqrt{5}+1}{2}}{\log\xi(\lambda)}$$

with $\xi(\lambda)$ *as in* (22). *Then,*

$$\alpha_u^+ \leq \alpha(\lambda), \tag{23}$$

and hence

$$\beta^+(p) \leq \alpha(\lambda) \quad \text{for every } p > 0. \tag{24}$$

One can observe that $\alpha(\lambda) < p(\lambda)$, where $p(\lambda)$ is the number given in (21)—the power for which [57] proved upper bounds for the slow part. Note that $\xi(\lambda) = \lambda + O(1)$ as $\lambda \to \infty$. Moreover,

$$\alpha(8) = \frac{2 \log \frac{\sqrt{5}+1}{2}}{\log 3} \approx 0.876$$

and $\alpha(\lambda)$ is a decreasing function of λ for[5] $\lambda \geq 8$. Thus, we establish anomalous transport for the Fibonacci Hamiltonian with coupling $\lambda \geq 8$ and confirm the asymptotic dependence of the transport exponents $\alpha_u^\pm$ on the coupling constant λ that was predicted by Abe and Hiramoto. We emphasize again that this is the first model for which anomalous transport, in the sense that $0 < \alpha_l^- \leq \alpha_u^+ < 1$, can be shown rigorously.

While we only deal with the Fibonacci case explicitly, let us remark that by combining the ideas of the present paper with those from [19], it is possible to perform a similar quantum dynamical analysis for general Sturmian potentials given by (19).

The almost Mathieu operator is given by (3) with potential

$$V(n) = \lambda \cos(2\pi(n\theta + \omega)) \tag{25}$$

with θ irrational and $\omega \in [0, 1)$. The almost Mathieu operator in the critical case $\lambda = 2$ is also called the Harper model,

$$V(n) = 2 \cos(2\pi(n\theta + \omega)). \tag{26}$$

Operators with potential (25) have been studied extensively by mathematicians and physicists for decades. The spectral theory is almost completely understood. For almost every frequency θ and phase ω, one has purely absolutely continuous spectrum for $0 < \lambda < 2$, purely singular continuous spectrum for $\lambda = 2$, and pure point spectrum with exponentially decaying eigenfunctions (*Anderson localization*) for $\lambda > 2$.

On the other hand, for θ's that are very well approximated by rational numbers, one can use Gordon's Lemma [38] to prove absence of eigenvalues so that localization fails for these θ's and all ω's when $\lambda > 2$. But even for a Diophantine θ, absence of eigenvalues holds generically, so that for a dense G_δ set of ω's, one has again no eigenvalues [54]. To a certain extent, the situation was mollified by Jitomirskaya and Last [52], who proved that for $\lambda > 2$, all irrational frequencies and all phases, the spectral measures are zero-dimensional.[6]

On the dynamical side, the result of Last that for $0 < \lambda < 2$, every irrational θ, and every[7] ω, there exists some absolutely continuous spectrum [61], implies ballistic motion in this coupling regime. In the regime $\lambda > 2$, it is of course desirable to establish dynamical localization whenever spectral localization holds. For results in this direction, see [30, 32, 50]. As a result, dynamical localization holds for $\lambda > 2$, Diophantine θ, and almost every ω.

On the other hand, it was not clear what should be expected for the dynamical quantities $\beta_{\delta_1}^-(p) = \beta_{\delta_1}^-(p, \lambda, \theta, \omega)$ in cases of exceptional frequencies and/or phases

[5] The minimum value, λ_0, can be chosen slightly smaller than 8. By monotonicity, all we need to require from λ_0 is that $\xi(\lambda_0) > 1$ and $\alpha(\lambda_0) < 1$.

[6] This was proved earlier in the special case of Liouville frequencies by Last [62].

[7] Last proves this for almost every ω; it holds for every ω by a result of Last and Simon [63].

at coupling $\lambda > 2$. Since the Guarneri-Combes-Last bound is one-sided, zero-dimensionality of spectral measures has no dynamical implications. In fact, Last proved the following result in [62]: $\beta^+_{\delta_1}(2, \lambda, \theta, \omega) = 1$ (and hence $\alpha^+_u(\lambda, \theta, \omega) = 1$) for every λ, ω, and a suitable (λ, ω)-dependent Liouville frequency θ. Del Rio, Jitomirskaya, Last, and Simon constructed θ such that $\beta^+_{\delta_1}(p, 3, \theta, \omega) = 1$ for every $p > 0, \omega \in [0, 1)$ [26] (their proof given for $p = 2$ can be easily generalized to any $p > 0$). Thus, for such a value of θ, $\alpha^+_l(3, \theta, \omega) = 1$. Later on, it was shown in [33] that given $\lambda > 2$, one has $\alpha^+_l(\lambda, \theta, \omega) = 1$ for all θ's from some dense G_δ set.

Thus, in these exceptional cases, one can only hope to bound $\alpha^-_u(\lambda, \theta, \omega)$ from above nontrivially.

Recall the continued fraction expansion of θ,

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

with uniquely determined integers $a_0 \in \mathbb{Z}$ and $a_n \geq 1$, $n \geq 1$; see [56] for background information. Truncation of this expansion gives rise to best rational approximants p_k/q_k of θ,

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_k}}}.$$

The following condition on θ will be useful below:

$$\lim_{k \to \infty} \frac{\log q_{k+1}}{q_k} = 0. \tag{27}$$

This condition is satisfied for Lebesgue almost every θ; see, for example, [56, Theorem 31]. It is related to, but weaker than, the Brjuno condition, which states that $(\log q_{k+1})/q_k \in \ell^1$. The Brjuno condition is crucial, for example, in Yoccoz's work on analytic linearization of circle diffeomorphisms [77]. Every Diophantine number satisfies the Brjuno condition and the inclusion is strict.

We have the following dynamical results for the almost Mathieu operator at super-critical coupling:

Theorem 4. *For the almost Mathieu operator with coupling $\lambda > 2$, any irrational θ, and any $\omega \in [0, 1)$, we have $\alpha^-_u(\lambda, \theta, \omega) = 0$ (and hence $\beta^-_{\delta_1}(p, \lambda, \theta, \omega) = 0$ for every $p > 0$). If in addition θ obeys* (27)*, then $\alpha^+_u(\lambda, \theta, \omega) = 0$ for every $\omega \in [0, 1)$ (and hence $\beta^+_{\delta_1}(p, \lambda, \theta, \omega) = 0$ for every $p > 0$).*

Remarks. (i) In particular, for any given coupling $\lambda > 2$, any phase ω, and an associated Liouville frequency θ, there is a sequence of time scales, for which one has almost ballistic transport, but there is essentially no transport along certain other sequences of time scales. Thus, the Liouville case may exhibit interesting dynamical phenomena.

(ii) Under the rather weak condition (27) we get a certain form of dynamical localization. While this result is slightly weaker than the one established by Germinet and Jitomirskaya under a Diophantine assumption on θ [32], it has several advantages:

- Our result holds for all phases ω, while Germinet and Jitomirskaya consider a phase average and, in particular, lose control over zero measure sets. As mentioned above, even under a Diophantine condition, there are exceptional values of ω for which one has no eigenvalues. Germinet and Jitomirskaya cannot make any statement for these ω's, while we can handle every ω. In a way, our result bears some similarity with a dynamical upper bound established by del Rio, Jitomirskaya, Last, and Simon in [26, Theorem 8.1] who considered exceptional couplings in a rank-one perturbation situation: While there are exceptional phases, for which spectral localization fails, the failure of (the strong form of) dynamical localization cannot be too severe.
- Our proof is much shorter. It only relies on positive Lyapunov exponents and does not need to establish spectral localization first. Note that [32] relies on [12, 37, 48], while our proof is self-contained up to a proof of positive Lyapunov exponents, which can be proved in a few lines [43]; see also [17, pp. 199–200].

We will prove Theorem 4 in a more general form. Let us consider an operator $H = H_{\theta,\omega}$ of the form (3) whose potential is given by

$$V(n) = V_{\theta,\omega}(n) = f(n\theta + \omega),$$

where f is a 1-periodic trigonometric polynomial, θ is irrational, and $\omega \in [0, 1)$.

The Lyapunov exponent, whose existence follows from Kingman's subadditive ergodic theorem, is given by

$$\gamma_\theta(z) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \log \|\Phi_n(z, \theta, \omega)\| \, d\omega = \inf_{n\geq 1} \frac{1}{n} \int_0^1 \log \|\Phi_n(z, \theta, \omega)\| \, d\omega. \tag{28}$$

One gets the same quantity if one considers $n \leq -1$.

Theorem 5. *Let $H_{\theta,\omega}$ be as just described. Assume that there is a number $\Gamma > 0$ such that $\gamma_\theta(z) \geq \Gamma$ for every $z \in \mathbb{C}$. Then $\alpha_u^-(\theta, \omega) = 0$ for every $\omega \in [0, 1)$. If in addition θ obeys* (27)*, then $\alpha_u^+(\theta, \omega) = 0$ for every $\omega \in [0, 1)$.*

A classical result of Herman [43] states that if $f(\omega) = \lambda \cos(2\pi\omega)$, then

$$\gamma_\theta(z) \geq \log \frac{|\lambda|}{2} \quad \text{for every } \theta \in \mathbb{R} \setminus \mathbb{Q}, \ z \in \mathbb{C}.$$

Thus, Theorem 5 implies Theorem 4.

For a general nonconstant trigonometric polynomial f, Herman's argument shows that the assumption of Theorem 5 holds for sufficiently large leading coefficient. Thus, we may state the following

Theorem 6. *Suppose g is a nonconstant trigonometric polynomial and $f = \lambda g$. Then, there exists λ_0 such that for every $\lambda > \lambda_0$, every irrational θ, and every $\omega \in [0, 1)$, we have $\alpha_u^-(\lambda, \theta, \omega) = 0$. Moreover, if θ obeys* (27)*, then $\alpha_u^+(\lambda, \theta, \omega) = 0$ for every $\lambda > \lambda_0$ and $\omega \in [0, 1)$.*

Note that Theorems 4 and 6, combined with the Guarneri-Combes-Last bound, provide a new proof of the zero-dimensionality of spectral measures since we do not use the Jitomirskaya-Last inequality from [51] on which the proof in [52] was based.

We believe that one can extend Theorem 5, and hence Theorem 6, to the case of $f = \lambda g$ with g analytic and λ sufficiently large. Positivity of the Lyapunov exponent was shown in this case by Sorets and Spencer [68]; see also [12, 41]. Zero-dimensionality of spectral measures for analytic potentials and large coupling is a result of Jitomirskaya and Landrigan [49].

Given Germinet and Jitomirskaya [32], Last [62], and Theorem 4 above, we have a good understanding of the dynamical picture in the super-critical case, $\lambda > 2$. Due to the existence of absolutely continuous spectrum in the sub-critical case [61], $0 < \lambda < 2$, this leaves the intriguing problem of studying the critical case, $\lambda = 2$, that is, when the potential is given by (26). There are no explicit rigorous results to this date. Based on numerics [29, 45, 55, 76], it is expected that one has almost diffusive transport in the sense that the transport exponents are close, but not equal, to 1/2. For Diophantine frequencies, Bellissard, Guarneri, and Schulz-Baldes have established a lower bound on phase-averaged transport exponents in terms of the multifractal dimensions of the density of states [10]. Since there are no known lower bounds for these dimensions, however, this result does not yield any explicit estimates for the dynamical quantities at this point in time.

The organization is as follows. In Section 3 we prove upper bounds for outside probabilities in terms of the norms of transfer matrices at complex energies; see Theorem 7. These bounds immediately imply Theorem 1 above. We also express the numbers $\alpha_u^\pm$ in terms of transfer matrix norms (see Theorem 8) and prove Theorem 2. In Section 4 we study the Fibonacci Hamiltonian. Suitable bounds on transfer matrix norms are obtained by studying the traces of these matrices. Since they obey recursive relations, we have to investigate a dynamical system, the so-called *trace map*. As opposed to earlier studies of the trace map, we need to regard this as a complex dynamical system and use ideas from conformal mapping theory, the Koebe Distortion Theorem in particular, to prove the required estimates. Once these estimates have been established, Theorem 3 follows from Theorem 1. Theorem 5 is proved in Section 5.

3. Upper bounds on the tail of the wavepacket

In this section we prove an upper bound on the tail of the (time-averaged) wavepacket in terms of transfer matrix norms for complex energies. It is shown how this leads to upper bounds on transport exponents in suitable cases.

Observe that, by unitarity, $\sum_n a(n,T) = 1$ for any T. For an integer $N \geq 1$, define

$$P_r(N,T) = \sum_{n>N} a(n,T) \quad \text{and} \quad P_l(N,T) = \sum_{n<-N} a(n,T)$$

so that $P(N,T) = P_l(N,T) + P_r(N,T)$. We call $P_r(N,T)$ (resp., $P_l(N,T)$) the right (resp., left) outside probability. Our first goal in this section is to find upper bounds for $P_r(N,T)$ and $P_l(N,T)$. We will give the details for $P_r(N,T)$. Analogous bounds for $P_l(N,T)$ can be obtained in the exact same way.

To be precise, we want to find $N(T) \to \infty$ such that

$$\lim_{T\to\infty} P_r(N(T),T) = 0. \tag{29}$$

The intuitive meaning of (29) is that for T large, the wavepacket $\psi(t) = e^{-itH}\delta_1$, $0 \leq t \leq T$, has essentially no weight in the region $(N(T), \infty)$ for times up to T, at least in a time-averaged sense. A ballistic upper bound shows that (29) always holds if we let $N(T) = T^{1+\nu}$ for some $\nu > 0$. In the case of a dynamically localized system (e.g., when the potential is given by independent identically distributed random variables), (29) trivially holds for any $N(T)$ such that $N(T) \to \infty$. The most interesting case is the intermediate one, where (29) holds for $N(T) = T^\alpha$ with some $\alpha \in (0,1)$ (and does not hold for $N(T) = T^\gamma$ with some $0 < \gamma < \alpha$).

When proving (29) and the analogous bound for $P_l(N(T),T)$, two kinds of results are of interest:

(i) Find $N(T)$ such that $P_r(N(T),T)$ and $P_l(N(T),T)$ go to zero fast, for example, faster than any inverse power of T. In particular, if $N(T) = T^\alpha$ with $\alpha \in (0,1)$, we get

$$\beta^+_{\delta_1}(p) \le \alpha^+_u \le \alpha, \; p > 0. \tag{30}$$

(ii) Find smaller values of $N(T)$ such that $P_r(N(T),T)$ and $P_l(N(T),T)$ go to zero according to some inverse power of T. This allows one to prove upper bounds for $\alpha^\pm_l$ and upper bounds for $\beta^\pm(p)$ that depend on p and are better than (30).

Here we are mainly interested in the first problem, which deals with the sharp front of the wavepacket.

The following result gives upper bounds on outside probabilities in terms of quantities involving the norms of transfer matrices at complex energies.

Theorem 7. *Suppose H is given by* (3)*, where V is a bounded real-valued function, and $K \ge 4$ is such that $\sigma(H) \subseteq [-K+1, K-1]$. Then, the outside probabilities can be bounded from above in terms of transfer matrix norms as follows:*

$$P_r(N,T) \lesssim \exp(-cN) + T^3 \int_{-K}^{K} \left(\max_{1 \le n \le N} \left\| \Phi\left(n, E + \tfrac{i}{T}\right) \right\|^2 \right)^{-1} dE,$$
$$P_l(N,T) \lesssim \exp(-cN) + T^3 \int_{-K}^{K} \left(\max_{-N \le n \le -1} \left\| \Phi\left(n, E + \tfrac{i}{T}\right) \right\|^2 \right)^{-1} dE.$$

The implicit constants depend only on K and c is a universal positive constant.

The starting point is the Parseval formula [24, 57],

$$a(n,T) = \frac{1}{T\pi} \int_{-\infty}^{\infty} \left| \langle (H - E - \tfrac{i}{T})^{-1} \delta_1, \delta_n \rangle \right|^2 dE. \tag{31}$$

Let us write $\varepsilon = 1/T$ and $R(z) = (H - zI)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. We assume $T \ge 1$, so that $0 < \varepsilon \le 1$. Thus, we have

$$P_r(N,T) = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} M_r(N, E + i\varepsilon)\, dE, \tag{32}$$

where

$$M_r(N,z) = \sum_{n > N} |\langle R(z)\delta_1, \delta_n \rangle|^2 = \|\chi_N R(z) \delta_1\|^2,$$

and $\chi_N(n) = 0,\ n \le N,\ \chi_N(n) = 1,\ n \ge N+1$. The key problem is to bound $M_r(N,z)$ from above. If the energy E is outside the spectrum of the operator H, this is a trivial problem. Assume that $E \in (-\infty, -K) \cup (K, +\infty)$, so that $\eta = \operatorname{dist}(E + i\varepsilon, \sigma(H)) \ge 1$. In this case the well-known Combes-Thomas estimate yields

$$|\langle R(z)\delta_1, \delta_n \rangle| \le \frac{2}{\eta} \exp(-c \min\{c\eta, 1\} |n-1|) \tag{33}$$

with some universal positive c; compare, for example, [33, (A.11) on p. 825]. Using (33) one can easily see that

$$\int_{E:|E|\geq K} M_r(N, E+i\varepsilon)\, dE \leq C(K)\exp(-cN),\ c>0,$$

and thus goes fast to 0 for any $N(T)=T^\alpha,\ \alpha>0$. The main problem is to estimate

$$L_r(N,\varepsilon)=\int_{-K}^{K} M_r(N,E+i\varepsilon)\, dE,$$

where $E+i\varepsilon$ may be close to the spectrum of H. We will link $M_r(N,z)$ to the complex solutions to the stationary equation $Hu=zu$.

Define two operators with truncated potential:

$$H_N^{\pm}\psi(n)=\psi(n-1)+\psi(n+1)+V_N^{\pm}(n)\psi(n),$$

where

$$V_N^{\pm}(n)=V(n),\ n\leq N,\quad V_N^{\pm}(n)=\pm 2K,\ n\geq N+1.$$

Such operators already appeared (in a different context) in [33]. We denote by $R_N^{\pm}(z)$ their resolvents and define

$$S^{\pm}(N,z)=\|\chi_N R_N^{\pm}(z)\delta_1\|^2.$$

Lemma 1. *For any $E\in[-K,K], 0<\varepsilon<1$, we have*

$$\varepsilon^2 S^{\pm}(N,E+i\varepsilon)\lesssim M_r(N,E+i\varepsilon)\lesssim \varepsilon^{-2}S^{\pm}(N,E+i\varepsilon),$$

where the implicit constants depend only on K.

Proof. By the resolvent identity, it follows that

$$R(z)\delta_1-R_N^{\pm}(z)\delta_1=R(z)(V-V_N^{\pm})R_N^{\pm}(z)\delta_1=R(z)(\chi_N(V(n)\mp 2K))R_N^{\pm}(z)\delta_1.$$

Since $\|R(z)\|\leq\varepsilon^{-1},\ \varepsilon<1$, and V is bounded, we get

$$\begin{aligned} M(N,z) &\leq 2S^{\pm}(N,z)+2\|R(z)(V-V_N^{\pm})R_N^{\pm}(z)\delta_1\|^2 \\ &\leq 2S^{\pm}(N,z)+2\varepsilon^{-2}\|\chi_N(V\mp 2K)R_N^{\pm}(z)\delta_1\|^2 \\ &\leq c_2(K)\varepsilon^{-2}S^{\pm}(N,z). \end{aligned}$$

The other bound can be proved in a similar way. □

The next step is to link the quantities $S^{\pm}(N,z)$ with the solutions to the stationary equation. For any complex z, define $u_0(n,z)$ as a solution to $Hu_0=zu_0,\ u_0(0,z)=0,\ u_0(1,z)=1$ and define $u_1(n,z)$ as a solution to $Hu_1=zu_1,\ u_1(0,z)=1,\ u_1(1,z)=0$. An important observation is the following: since $V(n)=V_N^{\pm}(n),\ n\leq N$, the solutions u_0,u_1 for H and for $H_N^{\pm}$ coincide for all $n\leq N+1$. We will consider u_0,u_1 only for such n; thus we use the same notation for u_0,u_1 in the case of H and $H_N^{\pm}$.

For any $E\in[-K,K],\ 0<\varepsilon<1,\ z=E+i\varepsilon$, we define

$$\lambda_{1,2}^{\pm}(z)=\frac{1}{2}(z\mp 2K+\sqrt{(z\mp 2K)^2-4}).$$

Observe that $|z\mp 2K|\geq K\geq 4$. Thus, one of the branches of the square root yields $|\lambda_1^{\pm}(z)|<1$ and the other $|\lambda_2^{\pm}(z)|>1$. For the values of z under consideration, we clearly have

$$|\lambda_1^{\pm}(z)|\leq b(K)<1,\quad |\lambda_1^{+}(z)-\lambda_1^{-}(z)|\geq c(K)>0$$

with uniform constants $b(K), c(K)$. The numbers $\lambda^{\pm}_{1,2}(z)$ in fact are the eigenvalues of the free equation

$$u(n-1) + u(n+1) = (z \mp 2K)u(n), \tag{34}$$

whose general solution is

$$u(n) = C_1\lambda_1^n + C_2\lambda_2^n.$$

Lemma 2. *Let $z = E + i\varepsilon$, where $E \in [-K, K]$, $0 < \varepsilon < 1$. Then, for $n \geq N \geq 3$, we have*

$$|\langle R^{\pm}_N(z)\delta_1, \delta_n\rangle| \leq 2\varepsilon^{-1}\frac{|\lambda^{\pm}_1(z)|^{n-N}}{|\lambda^{\pm}_1(z)u_0(N,z) - u_0(N+1,z)|} \tag{35}$$

and

$$|\langle R^{\pm}_N(z)\delta_1, \delta_n\rangle| \leq \varepsilon^{-1}\frac{|\lambda^{\pm}_1(z)|^{n-N}}{|\lambda^{\pm}_1(z)u_1(N,z) - u_1(N+1,z)|}. \tag{36}$$

Proof. The following formula for the resolvent of any operator of the form (3) holds (see, e.g., [57]):

$$\langle R(z)\delta_1, \delta_n\rangle = d(z)u_0(n,z) + b(z)u_1(n,z),\ n \geq 1, \tag{37}$$

$$\langle R(z)\delta_1, \delta_n\rangle = d(z)u_0(n,z) + c(z)u_1(n,z),\ n < 1, \tag{38}$$

where

$$d(z) = \frac{-m_+(z)m_-(z)}{m_+(z) + m_-(z)},$$

$$b(z) = \frac{m_-(z)}{m_+(z) + m_-(z)},$$

$$c(z) = \frac{-m_+(z)}{m_+(z) + m_-(z)},$$

with some complex functions $m_+(z), m_-(z)$ (which depend on the potential). Since $\|R(z)\delta_1\| \leq \varepsilon^{-1}$ and

$$d(z) = \langle R(z)\delta_1, \delta_1\rangle,\ c(z) = \langle R(z)\delta_1, \delta_0\rangle,$$

we get $|d(z)| \leq \varepsilon^{-1}$, $|c(z)| \leq \varepsilon^{-1}$. Since $b(z) = 1 + c(z)$ and $\varepsilon \leq 1$, we also get $|b(z)| \leq 2\varepsilon^{-1}$. Summarizing,

$$|d(z)| \leq \varepsilon^{-1}, |c(z)| \leq \varepsilon^{-1}, |b(z)| \leq 2\varepsilon^{-1}. \tag{39}$$

One should stress that the bounds (39) hold for operators of the form (3) with any potential, and the constants are universal.

Consider the operators $H^{\pm}_N$ and define $\phi^{\pm} = R^{\pm}_N(z)\delta_1$ (the vectors $\phi^{\pm}$ depend on N, of course). Since

$$(H^{\pm}_N - z)\phi^{\pm} = \delta_1,$$

the function $\phi^{\pm}(n) = \langle\phi^{\pm}, \delta_n\rangle$ obeys the equation $H^{\pm}_N\phi(n) = z\phi(n), n \geq 2$. Since $V_N(n) = \pm 2K,\ n \geq N+1$, $\phi(n)$ obeys the free equation (34) for $n \geq N+1$. Hence,

$$(\phi^{\pm}(N+k+1), \phi^{\pm}(N+k))^T = C^{\pm}_1\lambda^{\pm}_1(z)^k e_1 + C^{\pm}_2\lambda^{\pm}_2(z)^k e_2,\ k \geq 0. \tag{40}$$

Here $e^{\pm}_{1,2} = (\lambda^{\pm}_{1,2}(z), 1)^T$ are two eigenvectors of the matrix corresponding to the equation (34), and the constants $C^{\pm}_1, C^{\pm}_2$ are defined by

$$(\phi^{\pm}(N+1), \phi^{\pm}(N))^T = C^{\pm}_1 e_1 + C^{\pm}_2 e_2.$$

Since $|\lambda_1^\pm(z)| < 1$, $|\lambda_2^\pm(z)| > 1$ and $\phi^\pm \in \ell^2(\mathbb{Z})$, the identity (40) implies that $C_2^\pm = 0$, and thus

$$(\phi^\pm(N+1), \phi^\pm(N))^T = D_N^\pm(z)(\lambda_1^\pm(z), 1)^T,\ D_N^\pm(z) \neq 0. \tag{41}$$

On the other hand, (37) implies

$$\phi^\pm(N+1) = d_N^\pm(z)u_0(N+1,z) + b_N^\pm(z)u_1(N+1,z), \tag{42}$$

$$\phi^\pm(N) = d_N^\pm(z)u_0(N,z) + b_N^\pm(z)u_1(N,z). \tag{43}$$

Note that $d_N^\pm(z) \neq 0$, since it is a Borel transform of the spectral measure corresponding to the vector δ_1 [57]. The same is true for $b_N^\pm(z), c_N^\pm(z)$ since $m_+(z), m_-(z)$ are functions with positive imaginary part (it also follows from $d_N^\pm(z) \neq 0$ and the expressions of b, c). Using the fact that $u_0(n+1,z)u_1(n,z) - u_0(n,z)u_1(n+1,z) = 1$ for any n (in particular, for $n = N$), and (41)–(43), it is easy to calculate $D_N^\pm(z)$:

$$D_N^\pm(z) = \frac{d_N^\pm(z)}{\lambda_1^\pm(z)u_1(N,z) - u_1(N+1,z)} = \frac{b_N^\pm(z)}{\lambda_1^\pm(z)u_0(N,z) - u_0(N+1,z)}. \tag{44}$$

As observed above, the solutions u_0, u_1 are the same for $H, H_N^\pm$ if $n \leq N+1$. It follows from (40), where $C_1 = D^\pm$ and $C_2 = 0$, that

$$\langle R_N^\pm(z)\delta_1, \delta_n\rangle = D_N^\pm(z)\lambda_1^\pm(z)^{n-N},\ n \geq N.$$

The result of the lemma follows now directly from (39) and (44). □

Lemma 3. *For $z = E + i\varepsilon$, with $E \in [-K, K]$ and $0 < \varepsilon \leq 1$, and $N \geq 3$, we have*

$$M_r(N,z) \leq C(K)\varepsilon^{-4}\left(\max_{3\leq n\leq N} \|\Phi(n,z)\|^2\right)^{-1}.$$

Proof. The second bound of Lemma 1 and the bound (35) of Lemma 2 yield

$$M_r(N,z) \leq A(K)\varepsilon^{-4}|\lambda_1^\pm(z)u_0(N,z) - u_0(N+1,z)|^{-2}\sum_{k=0}^{\infty}|\lambda_1^\pm(z)|^{2k} \tag{45}$$

$$\leq B(K)\varepsilon^{-4}|\lambda_1^\pm(z)u_0(N,z) - u_0(N+1,z)|^{-2} \tag{46}$$

with uniform $B(K)$, since $|\lambda_1^\pm(z)| \leq b(K) < 1$.

Denote $U^\pm = \lambda_1^\pm(z)u_0(N,z) - u_0(N+1,z)$. It is not hard to see that

$$|U^-| + |U^+| \geq |\lambda_1^-(z) - \lambda_1^+(z)||u_0(N,z)| \tag{47}$$

and

$$|U^-| + |U^+| \geq |\lambda_1^-(z)||U^-| + |\lambda_1^+(z)||U^+| \geq |\lambda_1^-(z) - \lambda_1^+(z)||u_0(N+1,z)|. \tag{48}$$

Since $|\lambda_1^-(z) - \lambda_1^+(z)| \geq c(K) > 0$, (45)–(48) imply

$$M_r(N,z) \leq C(K)\varepsilon^{-4}\left(|u_0(N,z)|^2 + |u_0(N+1,z)|^2\right)^{-1}.$$

Using (36), we can prove a similar bound with u_0 replaced by u_1. Thus,

$$M_r(N,z) \leq C(K)\varepsilon^{-4}\|\Phi(N,z)\|^{-2}.$$

Since $M_r(n,z)$ is decreasing in n, the asserted bound follows. □

Proof of Theorem 7. The assertion is an immediate consequence of (32) and Lemma 3 (and the analogous results on the left half-line). Note that we can replace $[3,N]$ by $[1,N]$ since this modification only changes the K-dependent constant. □

Proof of Theorem 1. Let us choose $N(T) = \lfloor CT^\alpha \rfloor$, where $C \in (0,\infty)$ and $\alpha \in (0,1)$ are chosen such that (13) and (14) hold. Observe that

$$P(N(T),T) = P(\lfloor N(T) \rfloor, T).$$

Then Theorem 7 shows that $P_r(N(T),T)$ and $P_l(N(T),T)$ go to 0 faster than any inverse power of T. By definition of $S^+(\alpha)$ and α_u^+ (cf. (7) and (9)), it follows that $\alpha_u^+ \le \alpha$, which is (15). Finally, (16) follows from (10). $\square$

Let us now show that the transport exponents $\alpha_u^\pm$ can be expressed in terms of the integrals

$$I(N,T) = \int_{-K}^{K} \left(\left\| \Phi\left(N, E + \tfrac{i}{T}\right) \right\|^{-2} + \left\| \Phi\left(-N, E + \tfrac{i}{T}\right) \right\|^{-2} \right) dE.$$

It follows from Theorem 7 that

$$P(N,T) \lesssim \exp(-cN) + T^3 I(N,T). \tag{49}$$

Define

$$W^-(\alpha) = -\liminf_{T\to\infty} \frac{\log I(\lfloor T^\alpha - 2 \rfloor, T)}{\log T},$$
$$W^+(\alpha) = -\limsup_{T\to\infty} \frac{\log I(\lfloor T^\alpha - 2 \rfloor, T)}{\log T},$$
$$\gamma^\pm = \sup\{\alpha \ge 0 : W^\pm(\alpha) < \infty\}.$$

Theorem 8. *We have that*

$$\alpha_u^\pm = \gamma^\pm.$$

Proof. We first prove a lower bound for $a(n+1,T)+a(n,T)$. Let $\phi(n) = \langle R(z)\delta_1, \delta_n \rangle$, where $z = E + i/T$. The identities (37), (38) imply

$$(\phi(n+1), \phi(n))^T = \Phi(n,z)(d(z), b(z))^T, \ n \ge 1,$$
$$(\phi(n+1), \phi(n))^T = \Phi(n,z)(d(z), c(z))^T, \ n < 1.$$

Since $\det \Phi(n,z) = 1$, we obtain (see, e.g., [24] for details)

$$|\phi(n+1)|^2 + |\phi(n)|^2 \ge \|\Phi(n,z)\|^{-2} |d(z)|^2 \tag{50}$$

for any n. Since $d(z) = \langle R(z)\delta_1, \delta_1 \rangle$ is the Borel transform of a compactly supported measure, it is easy to see that $|d(E + \frac{i}{T})| \ge \operatorname{Im} d(E + \frac{i}{T}) \ge C/T$ with a constant that is uniform in $E \in [-K,K], T \ge 1$. Therefore, (31) and (50) yield

$$a(n+1,T) + a(n,T) \ge \frac{C}{T^3} \int_{-K}^{K} \left\| \Phi\left(n, E + \tfrac{i}{T}\right) \right\|^{-2} dE$$

and

$$P(N,T) \ge \frac{C}{T^3} \int_{-K}^{K} \left(\left\| \Phi\left(N+1, E + \tfrac{i}{T}\right) \right\|^{-2} + \left\| \Phi\left(-N-2, E + \tfrac{i}{T}\right) \right\|^{-2} \right) dE. \tag{51}$$

Since the potential V is bounded and $E \in [-K,K], \ T \ge 1$, we have that

$$\|\Phi(N+1, E + \tfrac{i}{T})\| \asymp \|\Phi(N, E + \tfrac{i}{T})\|,$$
$$\|\Phi(-N-2, E + \tfrac{i}{T})\| \asymp \|\Phi(-N, E + \tfrac{i}{T})\|,$$

with constants depending only on K. Therefore, (51) implies

$$P(N,T) \geq \frac{C}{T^3} I(N,T). \tag{52}$$

Let us show that $\alpha_u^+ = \gamma^+$. Assume first that $\gamma^+ < \alpha_u^+$. Then there exists α such that $\gamma^+ < \alpha < \alpha_u^+$. The definition of γ^+ implies $W^+(\alpha) = \infty$ and thus

$$I(\lfloor T^\alpha - 2 \rfloor, T) \leq \frac{C_M}{T^M} \tag{53}$$

for any $M > 0$ uniformly in T. It follows from (49) that

$$P(T^\alpha - 2, T) = P(\lfloor T^\alpha - 2 \rfloor, T)$$

also obeys a bound like (53), and thus $S^+(\alpha) = \infty$. However, this is impossible since $\alpha < \alpha_u^+$. Therefore, $\gamma^+ \geq \alpha_u^+$. Similarly, $\gamma^+ \leq \alpha_u^+$ follows from the bound (52). The identity $\alpha_u^- = \gamma^-$ can be proven in the same way (with inequalities like (53) valid for some sequence of times $T_k \to \infty$). □

Proof of Theorem 2. It follows directly from (17) and the definition of $W^\pm(\alpha)$ that $|W_1^\pm(\alpha) - W_2^\pm(\alpha)| \leq 2A$ for any α, and hence $\gamma_1^\pm = \gamma_2^\pm$. The result now follows from Theorem 8. (Since $\alpha_u^\pm \leq 1$, it is sufficient that (17) holds for $|n| \leq 1/\varepsilon$.) □

4. The Fibonacci operator

In this section we prove upper bounds for the transport exponents $\beta^+(p)$ in the case of the Fibonacci potential given by (18) for λ sufficiently large.

In order to apply Theorem 7, we need to find lower bounds for the norms of transfer matrices at complex energies. Rather than norms, we will study traces of transfer matrices. This will be sufficient because lower bounds for traces give rise to lower bounds for norms in a trivial way. Moreover, the study of traces is natural in the case of the Fibonacci model since it displays a well-known hierarchical structure on the level of transfer matrices that induces recursion relations for the traces of transfer matrices over intervals whose length is given by a Fibonacci number. This in turn is most conveniently described by a dynamical system, the so-called trace map. We remark that in the existing proofs of lower bounds for transport exponents, it was required to find upper bounds for transfer matrix norms. Interestingly enough, it was sufficient also in that case to study traces. This less trivial transition from traces to norms can be achieved in the situation at hand using an observation of Iochum and Testard [47]; see also [46].

While the trace map has been studied by many authors (e.g., [11, 47, 57, 59, 64, 65, 69]), it has been analyzed so far only for real energies. Thus, the trace map dynamical system is usually considered as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Theorem 7 suggests studying complex energies. Thus, we are led to study the trace map as a complex dynamical system, that is, as a map from $\mathbb{C}^3$ to $\mathbb{C}^3$. The initial element of $\mathbb{C}^3$ will be an energy-dependent vector. We investigate the stable set, that is, the set of complex energies such that the associated vector in $\mathbb{C}^3$ has a bounded trace map orbit. This set of energies is, in fact, a subset of $\mathbb{R}$, but it is important to study the canonical approximants to this set as subsets of $\mathbb{C}$. Their complements are escaping regions in the sense that once an orbit enters such a set, it escapes to infinity at a super-exponential rate. Since all nonreal energies give rise to escaping trace map orbits, we can study in this way the maximum number of iterates it takes, for a given imaginary part ε of the energy z, to enter an escaping region.

From this point on, we can control the increase of the trace and this will prove sufficient for us to obtain the lower bounds we need.

Our goal is to derive Theorem 3 from Theorem 1. Since $V(-n) = V(n-1)$ for $n \geq 2$ (see [69]), we can restrict our attention to one half-line. Thus, we will give details only for (13) with suitable α. The proof of (14) is completely analogous.

For $z \in \mathbb{C}$, define the matrices $M_k(z)$ by

$$\Phi(F_k, z) = M_k(z), \quad k \geq 1, \tag{54}$$

where F_k is the k-th Fibonacci number, that is, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$. It is well known that

$$M_{k+1}(z) = M_{k-1}(z) M_k(z), \quad k \geq 2.$$

For the variables $x_k(z) = (\mathrm{Tr}\ M_k(z))/2$, $k \geq 1$, we have the recursion

$$x_{k+1}(z) = 2x_k(z)x_{k-1}(z) - x_{k-2}(z) \tag{55}$$

and the invariant

$$x_{k+1}(z)^2 + x_k(z)^2 + x_{k-1}(z)^2 - 2x_{k+1}(z)x_k(z)x_{k-1}(z) - 1 \equiv \frac{\lambda^2}{4}. \tag{56}$$

Letting $x_{-1}(z) = 1$ and $x_0(z) = z/2$, the recursion (55) holds for all $k \geq 0$. See, for example, [59, 64, 69], for these results.

For $\delta \geq 0$, consider the sets

$$\sigma_k^\delta = \{z \in \mathbb{C} : |x_k(z)| \leq 1 + \delta\}.$$

For $S \subseteq \mathbb{C}$, we let $S^{\mathbb{R}} = S \cap \mathbb{R}$. The set σ_k^0 is the spectrum of the k-th periodic approximant to the Fibonacci operator. In other words, x_k is the discriminant of this periodic operator and hence the set $(\sigma_k^0)^{\mathbb{R}}$ consist of F_k closed bands that, in general, have disjoint interiors. However, Raymond has shown that in these particular cases, even the closed bands are mutually disjoint [65]. Thus, there are exactly F_k disjoint bands making up the set $(\sigma_k^0)^{\mathbb{R}}$, and each one of them contains exactly one zero of x_k and each one of the $F_k - 1$ bounded gaps contains exactly one critical point of x_k.

We have

$$\sigma_k^\delta \cup \sigma_{k-1}^\delta \supseteq \sigma_{k+1}^\delta \cup \sigma_k^\delta \to \sigma,$$

the latter set being the spectrum of the Fibonacci Hamiltonian. Outside of these sets, the traces grow super-exponentially. More precisely, a modification of Sütő's proof shows the following:

Lemma 4. *A necessary and sufficient condition that $x_k(z)$ be unbounded is that*

$$|x_{N-1}(z)| \leq 1 + \delta, \quad |x_N(z)| > 1 + \delta, \quad |x_{N+1}(z)| > 1 + \delta \tag{57}$$

for some $N \geq 0$. This N is unique. Moreover, in this case we have

$$|x_{n+2}(z)| > |x_{n+1}(z)x_n(z)| \text{ for } n \geq N \tag{58}$$

and

$$|x_{N+n}(z)| \geq (1+\delta)^{F_n} \text{ for } n \geq 0. \tag{59}$$

Proof. Suppose that (57) holds true for some $N \geq 0$. Then

$$\begin{aligned} |x_{N+2}(z)| &\geq |x_{N+1}(z)x_N(z)| + (|x_{N+1}(z)x_N(z)| - |x_{N-1}(z)|) \\ &\geq |x_{N+1}(z)x_N(z)| + \delta(1+\delta). \end{aligned}$$

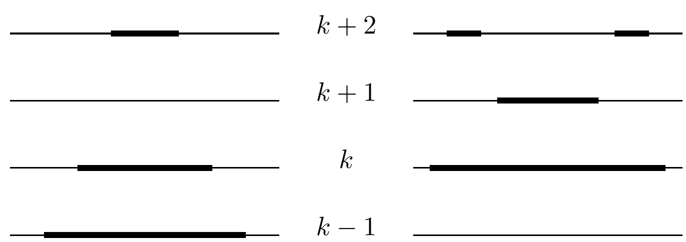

FIGURE 1. The structure of the sets $(\sigma_k^\delta)^{\mathbb{R}}$ when (61) holds.

This also implies $|x_{N+2}(z)x_{N+1}(z)| > |x_N(z)|$. Thus, (58) follows by induction. This in turn shows both (59) and that there is at most one $N \geq 0$ with (57).

Now suppose that (57) holds for no value of $N \geq 0$. Since $x_{-1}(z) = 1$, this implies that for every n with $|x_n(z)| > 1 + \delta$, we must have both $|x_{n-1}(z)| \leq 1 + \delta$ and $|x_{n+1}(z)| \leq 1 + \delta$. The invariant (56) therefore shows that the sequence $x_n(z)$ must be bounded. □

Moreover, assuming $\lambda > \lambda_0(\delta)$, where

$$\lambda_0(\delta) = [12(1+\delta)^2 + 8(1+\delta)^3 + 4]^{1/2}, \tag{60}$$

the invariant (56) implies

$$\sigma_k^\delta \cap \sigma_{k+1}^\delta \cap \sigma_{k+2}^\delta = \emptyset. \tag{61}$$

Lemma 5. *Fix $\delta \geq 0$ and $\lambda > \lambda_0(\delta)$. Then the set $(\sigma_k^\delta)^{\mathbb{R}}$ consists of F_k closed bands that are mutually disjoint. Each one of them contains exactly one zero of x_k and each one of the $F_k - 1$ bounded gaps contains exactly one critical point of x_k.*

Proof. For $\delta = 0$, these statements are known and were recalled above. If $\delta > 0$, let t run from 0 to δ. The bands of $(\sigma_k^t)^{\mathbb{R}}$ fatten up and what needs to be avoided is that two consecutive bands touch. By the assumption $\lambda > \lambda_0(\delta)$ this is impossible, however, as can be seen from (61) and the structure of the sets $(\sigma_k^\delta)^{\mathbb{R}}$ induced by this property (compare Figure 1 and the discussion of the sets $(\sigma_k^0)^{\mathbb{R}}$ in [57, Section 5] and [65, Section 6]). □

Lemma 6. *Fix $\delta \geq 0$ and $\lambda > \lambda_0(\delta)$. Then the set σ_k^δ has exactly F_k connected components. Each of them is a topological disk that is symmetric about the real axis.*

Proof. Since x_k has degree F_k, the set σ_k^δ has at most F_k connected components. Since the coefficients of x_k are real, it follows also that each such component must be symmetric about the real axis. On the other hand, Lemma 5 shows that the set $(\sigma_k^\delta)^{\mathbb{R}}$ has exactly F_k connected components. Thus, the maximum modulus principle shows that each component of σ_k^δ is a topological disk and there are exactly F_k of them. □

The final ingredients we need are the following bounds on $|x_k'(z)|$ for $z \in (\sigma_k^0)^{\mathbb{R}}$ (cf. [57, Proposition 5.2] and [24, equation (57)]):

Proposition 1. *If $\lambda > 8$, then*

$$|x_k'(z)| \geq \xi(\lambda)^{k/2} \tag{62}$$

for all $k \geq 3$ *and* $z \in (\sigma_k^0)^{\mathbb{R}}$*. Here,*

$$\xi(\lambda) = \frac{\lambda - 4 + \sqrt{(\lambda-4)^2 - 12}}{2}$$

and hence $\xi(\lambda) = \lambda + O(1)$ *as* $\lambda \to \infty$*.*

Proposition 2. *If* $\lambda > 4$*, then*

$$|x_k'(z)| \leq C(2\lambda + 22)^k \tag{63}$$

for all $k \geq 1$ *and* $z \in (\sigma_k^0)^{\mathbb{R}}$*.*

Now we can prove our main result on the structure of the set σ_k^δ. Denote $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$.

Proposition 3. *Fix* $k \geq 3$*,* $\delta > 0$*, and* $\lambda > \lambda_0(2\delta)$*. Then, there are constants* $c_\delta, d_\delta > 0$ *such that*

$$\bigcup_{j=1}^{F_k} B\left(z_k^{(j)}, r_k\right) \subseteq \sigma_k^\delta \subseteq \bigcup_{j=1}^{F_k} B\left(z_k^{(j)}, R_k\right), \tag{64}$$

where $\{z_k^{(j)}\}_{1 \leq j \leq F_k}$ *are the zeros of* x_k*,* $r_k = c_\delta(2\lambda + 22)^{-k}$*, and* $R_k = d_\delta \xi(\lambda)^{-k/2}$*.*

Remark. For our purpose here, the second inclusion in (64) is sufficient. Note, however, that it is possible to improve the known dynamical lower bounds for the Fibonacci Hamiltonian by employing the first inclusion in (64) together with the main result of [24].

Proof. Consider a connected component C_j of $\sigma_k^{2\delta}$. By the assumption $\lambda > \lambda_0(2\delta)$, C_j contains exactly one of the F_k real zeros, $z_k^{(j)}$. Moreover, it contains exactly one connected component of σ_k^δ, which we denote by $\tilde{C}_j$. Our goal is to show that with the radii r_k, R_k described in the proposition,

$$B\left(z_k^{(j)}, r_k\right) \subseteq \tilde{C}_j \subseteq B\left(z_k^{(j)}, R_k\right). \tag{65}$$

Clearly, (64) follows from (65).

Since C_j contains exactly one zero of x_k, it follows from the Maximum Modulus Theorem and Rouché's Theorem (e.g., [7, Theorems 6.13 and 10.10]) that

$$x_k : \mathrm{int}(C_j) \to B(0, 1 + 2\delta)$$

is univalent, and hence

$$x_k^{-1} : B(0, 1 + 2\delta) \to \mathrm{int}(C_j)$$

is well defined and univalent as well. Consequently, the following mapping is a Schlicht function:

$$F : B(0,1) \to \mathbb{C}, \quad F(z) = \frac{x_k^{-1}((1+2\delta)z) - z_k^{(j)}}{(1+2\delta)[(x_k^{-1})'(0)]}.$$

That is, F is a univalent function on $B(0,1)$ with $F(0) = 0$ and $F'(0) = 1$.

The Koebe Distortion Theorem (see [16, Theorem 7.9]) implies that

$$\frac{|z|}{(1+|z|)^2} \leq |F(z)| \leq \frac{|z|}{(1-|z|)^2} \text{ for } |z| \leq 1. \tag{66}$$

Evaluate the bound (66) on the circle $|z| = \frac{1+\delta}{1+2\delta}$. For such z, we obtain

$$\frac{(1+\delta)(1+2\delta)}{(2+3\delta)^2} \le |F(z)| \le \frac{(1+\delta)(1+2\delta)}{\delta^2}.$$

By definition of F this means that

$$|x_k^{-1}((1+2\delta)z) - z_k^{(j)}| \le \frac{(1+\delta)(1+2\delta)}{\delta^2}(1+2\delta)|(x_k^{-1})'(0)|$$

and

$$|x_k^{-1}((1+2\delta)z) - z_k^{(j)}| \ge \frac{(1+\delta)(1+2\delta)}{(2+3\delta)^2}(1+2\delta)|(x_k^{-1})'(0)|$$

for all z with $|z| = \frac{1+\delta}{1+2\delta}$. In other words, if $|z| = 1+\delta$, then

$$|x_k^{-1}(z) - z_k^{(j)}| \le \frac{(1+\delta)(1+2\delta)^2}{\delta^2}|(x_k^{-1})'(0)| \tag{67}$$

and

$$|x_k^{-1}(z) - z_k^{(j)}| \ge \frac{(1+\delta)(1+2\delta)^2}{(2+3\delta)^2}|(x_k^{-1})'(0)|. \tag{68}$$

Since $|(x_k^{-1})'(0)| = |x_k'(z_k^{(j)})|^{-1}$, we obtain from Proposition 1 and (67)

$$|x_k^{-1}(z) - z_k^{(j)}| < \left(\frac{(1+\delta)(1+2\delta)}{\delta}\right)^2 \xi(\lambda)^{-k/2} \tag{69}$$

for all z of magnitude $1+\delta$. Similarly, Proposition 2 and (68) give

$$|x_k^{-1}(z) - z_k^{(j)}| \ge \frac{(1+\delta)(1+2\delta)^2}{(2+3\delta)^2}\frac{1}{C}(2\lambda+22)^{-k} \tag{70}$$

for these values of z. Note that as z runs through the circle of radius $1+\delta$ around zero, the point $x_k^{-1}(z)$ runs through the entire boundary of $\tilde{C}_j$. Thus, (65) follows from (69) and (70). □

Proof of Theorem 3. By the assumption $\lambda > 8$, it is possible to choose $\delta > 0$ such that $\lambda > \lambda_0(2\delta)$; compare (60). Since the $z_k^{(j)}$ are real, the last proposition gives in particular that

$$\sigma_k^\delta \subseteq \{z \in \mathbb{C} : |\mathrm{Im}\ z| < d_\delta \xi(\lambda)^{-k/2}\} \subseteq \{z \in \mathbb{C} : |\mathrm{Im}\ z| < d_\delta F_k^{-\gamma(\lambda)}\}$$

for a suitable $\gamma(\lambda)$. This implies that

$$\sigma_k^\delta \cup \sigma_{k+1}^\delta \subseteq \{z \in \mathbb{C} : |\mathrm{Im}\ z| < d_\delta F_k^{-\gamma(\lambda)}\}. \tag{71}$$

If $\eta = (\sqrt{5}+1)/2$ (so that F_k behaves like η^k for k large enough), then the constant $\gamma(\lambda)$ may be chosen according to

$$\gamma(\lambda) = \frac{\log \xi(\lambda)}{2(1+\nu)\log \eta}$$

for some $\nu > 0$. Since we are interested in large T behavior, we can work with any positive ν and then have the inclusion above for $k \ge k_0(\nu)$.

In other words, for each $\varepsilon = \mathrm{Im}\ z > 0$, one obtains lower bounds on $|x_n(E+i\varepsilon)|$ which are uniform for $E \in [-K, K] \subseteq \mathbb{R}$. Namely, given $\varepsilon > 0$, choose k minimal with the property $d_\delta F_k^{-\gamma(\lambda)} < \varepsilon$. By (71), we infer that $|x_k(E+i\varepsilon)| > 1+\delta$ and

$|x_{k+1}(E+i\varepsilon)| > 1+\delta$. Since $|x_{-1}(E+i\varepsilon)| = 1 \le 1+\delta$, we must have the situation of Lemma 4 for some $N \le k$. In particular, for $n > k$, (59) shows that

$$|x_n(E+i\varepsilon)| \ge (1+\delta)^{F_{n-k}}.$$

This motivates the following definitions. Fix some small $\delta > 0$. For $T > 1$, denote by $k(T)$ the unique integer with

$$\frac{F_{k(T)-1}^{\gamma(\lambda)}}{d_\delta} \le T < \frac{F_{k(T)}^{\gamma(\lambda)}}{d_\delta}$$

and let

$$N(T) = F_{k(T)+\lfloor\sqrt{k(T)}\rfloor}.$$

Thus, for every $\tilde{\nu} > 0$, there is a constant $C_{\tilde{\nu}} > 0$ such that

$$N(T) \le C_{\tilde{\nu}} T^{\frac{1}{\gamma(\lambda)}} T^{\tilde{\nu}}. \tag{72}$$

It follows from Theorem 7 and the argument above that

$$\begin{aligned} P_r(N(T),T) &\lesssim \exp(-cN(T)) + T^3 \int_{-K}^{K} \left(\max_{3\le n\le N(T)} \left\|\Phi\left(n,0;E+\tfrac{i}{T}\right)\right\|^2 \right)^{-1} dE \\ &\lesssim \exp(-cN(T)) + T^3(1+\delta)^{-2F_{\lfloor\sqrt{k(T)}\rfloor}}. \end{aligned}$$

From this bound, we see that $P_r(N(T),T)$ goes to zero faster than any inverse power. We note again that we can obtain the same result on the left half-line due to the symmetry of the potential. Therefore, by (72), we can apply Theorem 1 with

$$\alpha = \frac{1}{\gamma(\lambda)} + \tilde{\nu} = \frac{2(1+\nu)\log\eta}{\log\xi(\lambda)} + \tilde{\nu},$$

and hence (23) and (24) follow from (15) and (16), respectively, since we can take ν and $\tilde{\nu}$ arbitrarily small. □

5. The almost Mathieu operator

Suppose we are given an operator family $\{H_{\theta,\omega}\}$ satisfying the assumptions of Theorem 5.

Let $K = \|f\|_\infty + 3$. Then,

$$\sigma(H_{\theta,\omega}) \subseteq [-K+1, K-1] \quad \text{for all } \omega \in [0,1). \tag{73}$$

We will show the following

Proposition 4. *Assume that $H_{\theta,\omega}$ obeys the assumptions of Theorem 5. Denote*

$$\begin{aligned} I_r(N,T) &= \int_{-K}^{K} \left(\max_{1\le n\le N} \left\|\Phi\left(n,E+\tfrac{i}{T},\theta,\omega\right)\right\|^2 \right)^{-1} dE, \\ I_l(N,T) &= \int_{-K}^{K} \left(\max_{1\le -n\le N} \left\|\Phi\left(n,E+\tfrac{i}{T},\theta,\omega\right)\right\|^2 \right)^{-1} dE. \end{aligned}$$

(a) *For every $\alpha > 0$, there exists a sequence $T_k \to \infty$ such that*

$$I_r(T_k^\alpha, T_k) = O(T_k^{-m}) \tag{74}$$

and

$$I_l(T_k^\alpha, T_k) = O(T_k^{-m}) \tag{75}$$

for every $m \geq 1$.

(b) *Assume moreover that* θ *obeys* (27). *Then for any* $\alpha > 0$,

$$I_r(T^\alpha, T) = O(T^{-m}),\ I_l(T^\alpha, T) = O(T^{-m}) \tag{76}$$

for every $m \geq 1$.

Remarks. (a) The α-dependent sequence $\{T_k\}$ will be related to the denominators q_k of the continued fraction approximants p_k/q_k of θ (cf. [56]).

(b) Our proof of Proposition 4 will have some elements in common with the arguments given by Jitomirskaya and Last in [52] in their proof of zero-dimensionality of spectral measures.

Proof of Theorem 5. The assertion is an immediate consequence of Proposition 4 together with Theorem 1 and the first remark after that theorem. $\square$

Proof of Proposition 4. Denote

$$\mathcal{R} = \{z \in \mathbb{C} : -K \leq \operatorname{Re} z \leq K,\ 0 \leq \operatorname{Im} z \leq 1\}.$$

It is easy to see that there exists a constant Γ' such that

$$\|\Phi(n, z, \theta, \omega)\| \leq e^{|n|\Gamma'} \text{ for all } \omega \in [0, 1),\ n \in \mathbb{Z},\ z \in \mathcal{R}. \tag{77}$$

Combining this with the assumption of Theorem 5, we obtain

$$0 < \Gamma \leq \gamma_\theta(z) \leq \Gamma' < \infty \text{ for all } z \in \mathcal{R}. \tag{78}$$

For $n \geq 1$, let

$$A_n = \left\{\omega \in [0, 1) : \|\Phi(n, z, \theta, \omega)\| > \exp\left(\frac{n\gamma_\theta(z)}{2}\right)\right\}.$$

It follows from (28) and (77) that

$$\begin{aligned} n\gamma_\theta(z) &\leq \int_0^1 \log \|\Phi(n, z, \theta, \omega)\|\, d\omega \\ &= \int_{A_n} \log \|\Phi(n, z, \theta, \omega)\|\, d\omega + \int_{[0,1)\setminus A_n} \log \|\Phi(n, z, \theta, \omega)\|\, d\omega \\ &\leq |A_n| n\Gamma' + (1 - |A_n|) n \frac{\gamma_\theta(z)}{2}, \end{aligned}$$

where $|\cdot|$ denotes Lebesgue measure.

Therefore,

$$|A_n| \geq \frac{\gamma_\theta(z)}{2\Gamma' - \gamma_\theta(z)} \geq \frac{\Gamma}{2\Gamma' - \Gamma} \equiv c.$$

Let d be the degree of the trigonometric polynomial f.[8] Then each entry of $\Phi(n, z, \theta, \omega)^* \Phi(n, z, \theta, \omega)$ is a trigonometric polynomial of degree at most $2nd$, and

[8] Note that we have $d \geq 1$ since the assumptions of the theorem preclude the case $d = 0$.

the set A_n consists of no more than $4nd$ intervals. Therefore, there exists an interval $I_n \subseteq A_n$ with[9]

$$|I_n| \geq \frac{c}{4nd}.$$

Let

$$n_k = \left\lfloor \frac{cq_k}{4d} \right\rfloor + 1.$$

By a standard lemma in continued fraction approximation (see, e.g., [52, Lemma 9] for a formulation suitable to our purpose), for every ω, there exists a $j \in \{0, 1, \dots, q_k + q_{k-1} - 1\}$ such that $j\theta + \omega \mod 1$ belongs to I_{n_k} and hence to A_{n_k}. That is, $\|\Phi(n_k, z, \theta, j\theta + \omega)\| > \exp\left(\frac{n_k \gamma_\theta(z)}{2}\right)$. Since

$$\Phi(j + n_k, z, \theta, \omega) = \Phi(n_k, z, \theta, j\theta + \omega)\Phi(j, z, \theta, \omega)$$

and each Φ is unimodular, we see that either $\|\Phi(j + n_k, z, \theta, \omega)\|$ or $\|\Phi(j, z, \theta, \omega)\|$ is greater than $\exp\left(\frac{n_k \gamma_\theta(z)}{4}\right)$.

Let

$$j_k = \min \left\{ j \in \{0, \dots, q_k + q_{k-1} - 1 + n_k\} : \|\Phi(j, z, \theta, \omega)\| > \exp\left(\frac{n_k \gamma_\theta(z)}{4}\right) \right\}.$$

By definition of j_k, (77), and (78), we have $\frac{n_k \Gamma}{4\Gamma'} \leq j_k$ and hence there are z-independent constants C_1, C_2 such that

$$C_1 q_k \leq j_k \leq C_2 q_k. \tag{79}$$

Moreover, there is a z-independent constant C_3 such that

$$\|\Phi(j_k, z, \theta, \omega)\| > \exp\left(C_3 q_k\right). \tag{80}$$

Thus, the definition of $I_r(N, T)$ and (80) imply

$$I_r(C_2 q_k, T) \leq 2K \exp(-2C_3 q_k) \tag{81}$$

for any $T \geq 1, k \in \mathbb{N}$. To prove part (a) of the proposition, consider, for a given $\alpha > 0$, the sequence $T_k = [C_2 q_k]^{1/\alpha}$. The bound (74) follows from (81). Since

$$\Phi(n, z, \theta, \omega) = \Phi(-n, z, -\theta, \theta + \omega) \tag{82}$$

and the q_k's are the same for θ and $-\theta$, we see that we can prove (75) with the same sequence $\{T_k\}$.

Let us now prove part (b) of the proposition. Choose some $\alpha > 0$. For any given $T \geq 1$, one can find some $k = k(T)$ such that

$$(C_2 q_k)^{1/\alpha} \leq T \leq (C_2 q_{k+1})^{1/\alpha}. \tag{83}$$

Let

$$\gamma_k = \frac{\log q_{k+1}}{q_k}. \tag{84}$$

It follows from the definition of I_r, (81) and (83)–(84) that

$$I_r(T^\alpha, T) \leq I_r(C_2 q_k, T) \leq 2K \exp(-2C_3 q_k) = 2K q_{k+1}^{-2C_3/\gamma_k} \leq C T^{-2\alpha C_3/\gamma_k}.$$

By assumption we have $\lim_{k\to\infty} \gamma_k = 0$. Since $k(T) \to \infty$ as $T \to \infty$, we get $I_r(T^\alpha, T) = O(T^{-m})$ for every $m \geq 1$. Due to the observation (82), the same is true for $I_l(T^\alpha, T)$. This gives (76) and concludes the proof. $\square$

[9] If one considers the Hilbert-Schmidt norm $\|\Phi\|_2^2 = \mathrm{Tr}(\Phi^*\Phi)$, which is of course equivalent to the operator norm, this claim is obvious.

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