

PUZZLE GEOMETRY AND RIGIDITY: THE FIBONACCI CYCLE IS HYPERBOLIC

DANIEL SMANIA

1. INTRODUCTION AND RESULTS

1.1. Complex dynamics results. Rigidity is a striking phenomenon in 1-dimensional dynamics which occurs when a weak equivalence between certain dynamical systems implies a stronger one, for instance, a combinatorial equivalence of the orbit structure implying topological equivalence and topological equivalence implying quasiconformal conjugacy. This phenomenon occurs quite often and has important consequences, as in the proof of the real Fatou conjecture for the quadratic family (Lyubich [Lyu97] and Graczyk and Świątek [GS97]) and the hyperbolicity of the renormalization horseshoe (Lyubich [Lyu99]).

Usually the hard part is to conclude that two complex dynamical systems are quasiconformally conjugate, once they are topologically conjugate. The first step is to replace the topological conjugacy by another one which is quasiconformal outside the ‘complicated’ part of the dynamics (the Julia set). This is usually easy. Our first dynamical result is a method to carry out the next step.

Definition 1.1. We say that $f: \bigcup_{i \leq n} U_i \rightarrow V$ is a **generalized polynomial-like map** if

- the sets U_i , $i \leq n$, and V are simply connected domains in the complex plane, U_i is compactly contained in V and $m(\partial V) = m(\partial U_i) = 0$, where m is the 2-dimensional Lebesgue measure;
- f has an analytic extension to a neighborhood of $\bigcup_i \overline{U}_i$;
- $f: U_0 \rightarrow V$ is a ramified covering with a unique critical point;
- for each $i \neq 0$, f is a univalent map from U_i onto V .

The simply connected components of the domain of f^n , for each $n > 0$, will be called **puzzle pieces at level n** associated to f . If the critical point does not **escape** ($f^n(0)$ is well defined, for every $n \geq 0$), there exists a puzzle piece at level n which contains the critical point. This piece is called the **critical piece at level n** and is denoted by C_n .

Received by the editors April 19, 2005.

2000 *Mathematics Subject Classification.* Primary 37F30, 37C15, 30C62, 30C65, 37F25, 37F45, 37E20.

Key words and phrases. Fibonacci combinatorics, generalized renormalization, puzzle, rigidity, quasiconformal conjugacy, removability, hyperbolicity.

This work was supported by grants CNPq-200764/01-2, CNPq-472316/2003-6 and FAPESP-03/03107-9. Visits to Warwick University and ICTP-Trieste were supported by CNPq-Brazil grant 460110/00-4, Warwick University and ICTP-Trieste.

©2007 American Mathematical Society
Reverts to public domain 28 years from publication

Theorem 1 (Rigidity Theorem). *Let $f_j: \bigcup_i U_i^j \rightarrow V^j$, with $j = 1, 2$, be generalized polynomial-like maps whose critical point 0 does not escape and such that the following hold.*

- *There exists $\lambda < 1$ and an increasing sequence n_k so that, if C_n^j denotes the critical puzzle piece at level n of f_j , then*
 - *the diameters of the critical puzzle pieces of f_1 and f_2 decrease to zero;*
 - **puzzle geometry control:** *for any $j = 1, 2$, $k > 0$, there exists x so that*

$$B(x, \lambda \cdot \text{diam } C_{n_k}^j) \subset C_{n_k}^j;$$

- **complex bounds:** *for any $j = 1, 2$, $k > 0$, there exists an annulus $A_k^j \subset V^j$ such that*
 - (1) *mod $A_k^j \geq K_2$;*
 - (2) *the internal boundary of A_k^j is $\partial C_{n_k}^j$;*
 - (3) *for any x which is not contained in $C_{n_k}^j$ and $m > 0$ minimal so that $f_j^m(x) \in C_{n_k}^j$, there exists a neighborhood $V_j(x, m)$ of x which is mapped univalently by f_j^m onto $C_{n_k}^j \cup A_k^j$.*
- *Let $h: \mathbb{C} \rightarrow \mathbb{C}$ be a homeomorphism such that*
 - *h is quasiconformal in $\mathbb{C} - J(f_1)$;*
 - *$h(\partial U_i^1) = \partial U_i^2$ and $h(\partial V^1) = \partial V^2$;*
 - *h is a conjugacy between f_1 and f_2 .*

Then $h: \mathbb{C} \rightarrow \mathbb{C}$ is quasiconformal.

Remark 1.1. We can use the weaker assumption that the conjugacy h is defined only in $\mathbb{C} - J(f_1)$, because the “complex bounds” condition on f_1 and f_2 implies that h will have a continuous extension to $K(f_1) = J(f_1)$, which is a conjugacy on $J(f_1)$.

Remark 1.2. It is easy to state a similar theorem with many critical points involved.

Remark 1.3. The third property of the annulus A_k^j follows for instance from the **unbranched complex bounds** condition

$$A_k^j \cap P(f) = \phi, \quad \text{where } P(f) := \overline{\{f^i(0): i > 0\}}.$$

The puzzle geometry control for quadratic-like maps with Fibonacci combinatorics is a consequence from Lyubich [Lyu93], and for Fibonacci maps with higher criticalities from Buff [Bu]. Lyubich’s result was generalized for an uncountable number of combinatorial types of quadratic polynomials by Perez [P].

Two metric properties are now classic tools to prove rigidity: the “bounded geometry of the postcritical set” and the “linear decay of moduli”. The first one was used to prove the rigidity of the infinitely renormalizable real polynomials (with unique critical point) with bounded combinatorics (see Sullivan [Sul92] and de Melo and van Strien [dMvS]). The second one was used in the proof of the real Fatou conjecture for quadratic polynomials (Lyubich [Lyu97] and Graczyk and Świątek [GS97]). It is quite remarkable that both behaviors can coexist with the puzzle geometry control: the Fibonacci combinatorics is the simplest example. For criticality two, Fibonacci maps have linear decay of modulus (Lyubich and Milnor [LM]); for larger even criticalities, the Fibonacci maps have a postcritical set with bounded geometry (see Bruin, Keller, Nowicki and van Strien [BKNS] for

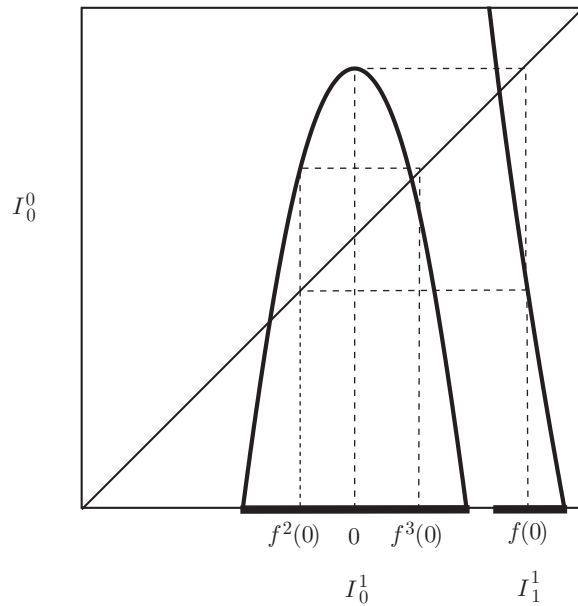


FIGURE 1. A Fibonacci renormalizable map.

high criticalities and Keller and Nowicki [KN]). So the “puzzle geometry control” property seems to be more “primitive” than the two classical properties above.

However, the linear decay of geometry loses its universality for larger criticalities, or even for multimodal maps with quadratic critical points (Swiatek and Vargas [SV]). An even worse fact is the possibility of the existence of real non-renormalizable combinatorics for higher criticalities and/or many critical points, having neither decay of moduli (at any rate) nor bounded geometry (or even essentially bounded geometry). Our hope is that the puzzle geometry control and complex bounds hold for some of these combinatorial types, whose rigidity is seemingly impossible to prove using previously available methods (indeed, there are recent developments in this direction, as noted in Section 15).

In the second part of this work we will use Theorem 1 to prove that the Fibonacci renormalization operator has a hyperbolic cycle. We introduce the Fibonacci renormalization operator before stating the results.

Maps with Fibonacci combinatorics have already attracted a lot of attention: Lyubich and Milnor [LM]; Lyubich [Lyu93]; Bruin, Keller, Nowicki and van Strien [BKNS]; Keller and Nowicki [KN]; van Strien and Nowicki [vSN94]; Buff [Bu]. The main points are that

- the Fibonacci combinatorics is the simplest one which admits wild attractors for high criticalities;
- for critical order larger than two, the Fibonacci combinatorics have bounded geometry (no decay of geometry);
- the Fibonacci combinatorics is the simplest combinatorics which is infinitely renormalizable in the generalized sense.

Let d be an even natural number larger than two (d will be fixed for the rest of this work). If α is a complex number and A a subset of the complex plane, denote $\alpha A := \{\alpha \cdot x : x \in A\}$.

Let f be a real analytic map defined in two intervals, a symmetric interval I_0^1 and I_1^1 . Denote $I_0^0 := f(I_1^1)$. Assume the following.

- $f: I_0^1 \rightarrow I_0^0$ has zero as its unique critical point, which is a maximum. Moreover, the degree of zero is d and $f(\partial I_0^1) \subset \partial I_0^0$.
- $f: I_1^1 \rightarrow I_0^0$ does not have critical points and $f(\partial I_1^1) \subset \partial I_0^0$.

We will say that f is **Fibonacci renormalizable** if

- $f(0) \in I_1^1$;
- $f^2(0) \in I_0^1$;
- $f^3(0) \in I_0^1$;
- the connected component of the domain of f^2 which contains the critical point also contains a fixed point, denoted β_f , so that $Df^2(\beta_f) < 0$.

Denote by I_0^2 the connected component of the domain of the first return map to I_0^1 which contains 0 and by I_1^2 the connected component of the domain of the first return map to I_0^1 which contains $f^2(0)$. Note that $I_0^2 \cap I_1^2 = \emptyset$. The **Fibonacci renormalization** of f , denoted $\mathcal{R}f$, is defined as the map

$$\mathcal{R}f: \frac{1}{\beta_f}I_0^2 \cup \frac{1}{\beta_f}I_1^2 \rightarrow \frac{1}{\beta_f}I_0^1,$$

where $\mathcal{R}f(x)$ is equal to

$$\frac{1}{\beta_f}f^2(\beta_f \cdot x)$$

on $1/\beta_f I_0^2$ and

$$\frac{1}{\beta_f}f(\beta_f \cdot x)$$

on $1/\beta_f I_1^2$. If $\mathcal{R}^n f$ is defined for every $n \geq 0$, we say that f is **infinitely renormalizable** in the Fibonacci sense.

In van Strien and Nowicki [vSN94], it was proved that this operator has a unique periodic orbit of period two, the **Fibonacci cycle**. Indeed, if $\tilde{f}_1: I \cup J \rightarrow T$ is one of the maps in the periodic orbit, then the map \tilde{f}_2 defined as \tilde{f}_1 on I and $-\tilde{f}_1$ on J is exactly $\mathcal{R}\tilde{f}_1$ (see Buff [Bu]). Making a slight modification of the Fibonacci renormalization operator, we can consider these maps as fixed points of the modified operator (see Section 14). The following remark is very important: Since these maps are almost the same, a distinction between the two maps is not important, so sometimes we will denote both of them by \tilde{f} . For instance $|g - \tilde{f}|$ denotes $\min\{|g - \tilde{f}_1|, |g - \tilde{f}_2|\}$.

The map \tilde{f} has a generalized polynomial-like extension $\tilde{f}: \overline{U_0} \cup \overline{U_1} \rightarrow \overline{V}$, where $0 \in U_0$. Let $\mathcal{B}_{nor}(U_0 \cup U_1)$ be the affine space of complex analytic functions on $U_0 \cup U_1$ with continuous extension to $\overline{U_0} \cup \overline{U_1}$ and an appropriate normalization, endowed with the sup norm. Selecting U_0 and U_1 in an appropriate way, we will define an operator $\mathcal{R}: B_{\epsilon_0}(\tilde{f}) \rightarrow \mathcal{B}_{nor}(U_0 \cup U_1)$, where $B_{\epsilon_0}(\tilde{f}) \subset \mathcal{B}_{nor}(U_0 \cup U_1)$ is the ϵ_0 -ball around \tilde{f} , which coincides with the Fibonacci renormalization operator above defined in the real trace of the real functions in $B_{\epsilon_0}(\tilde{f})$, so that we have the following.

Theorem 2 (Hyperbolicity). *The Fibonacci renormalization operator \mathcal{R} , defined in a small neighborhood of \tilde{f} in $\mathcal{B}_{nor}(U_1 \cup U_2)$, is a compact operator with a hyperbolic cycle $\{\tilde{f}_1, \tilde{f}_2\}$ with codimension one stable manifold $W^s(\tilde{f})$. Furthermore, if f is a real analytic map that is infinitely renormalizable in the Fibonacci sense, then there exists $N = N(f)$ so that $\mathcal{R}^N f$ has a complex analytic extension in $W^s(\tilde{f})$.*

Let \mathcal{D} be the set of complex analytic families $f: \overline{\mathbb{D}} \times \overline{U} \rightarrow \mathbb{C}$, with $f_\lambda \in \mathcal{B}_{nor}(U_1 \cup U_2)$, so that for each $\lambda \in [-1, 1]$ the map f_λ is even and preserves the real line. The hyperbolicity of the Fibonacci renormalization operator implies, after a short argument in Section 14, the following universality result:

Theorem 3 (Universality). *Let d be an even natural number larger than two. Then there exists $\gamma = \gamma_d > 1$ so that for an open and dense set of families $f_\lambda \in \mathcal{D}$ the following holds: if f_{λ_∞} , with $\lambda_\infty \in [-\frac{1}{2}, \frac{1}{2}]$, has a polynomial-like restriction which is quasiconformally conjugate with either \tilde{f}_1 or \tilde{f}_2 , then there exist $\delta > 0$, $N \in \mathbb{N}$, and a sequence $\lambda_n \in \mathbb{R}$ so that*

- the sequence λ_n converges to λ_∞ ;
- for $n \geq N$, λ_n is the unique parameter such that $|\lambda_n - \lambda_\infty| \leq \delta$ and whose map f_{λ_n} has a superattractor with period S_n (the Fibonacci sequence) and whose closest returns of the critical point are exactly S_i , $i < n$;
- we have

$$\frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1} - \lambda_{n+2}} \rightarrow \gamma.$$

A similar statement holds for the family $x^d + c$ (perturbations are not necessary in this case).

1.2. Removability result. Removability-type results try to find useful properties for a homeomorphism on \mathbb{C} that is quasiconformal outside a set K that imply quasiconformality everywhere. We will state a new abstract removability-type result: this result is a major step in the proof of Theorem 1. Before stating it, we need to introduce some notation. We will consider only measurable functions.

In this section, we will reserve the word **family** to denote families of domains: a domain is an open and connected set in \mathbb{C} . Moreover, we will assume that all these domains have boundaries with zero 2-dimensional Lebesgue measure. Sometimes we will deal with families of **centered** domains: this is a family of pairs (A, x) , where $x \in A$. We will say that x is a center of A . It may occur in a given family that A has many centers and that a point x is the center of many domains. Recall that $m(A)$ denotes the 2-dimensional Lebesgue measure of A .

Definition 1.2. A family \mathcal{F} covers a set $X \subset \mathbb{C}$ in **arbitrary small scales** if for any $x \in X$ and $\epsilon > 0$ there exists $A \in \mathcal{F}$ such that $x \in A$, $\text{diam } A \leq \epsilon$ and $\partial A \cap X = \emptyset$ for $A \in \mathcal{F}$.

Definition 1.3. A centered family \mathcal{F} covers a set $X \subset \mathbb{C}$ in **all (small) scales** if for any $x \in X$ there exists $C = C(x) > 1$ and $\epsilon_0 = \epsilon_0(x)$ so that for any $\epsilon \leq \epsilon_0$ the point x is the center of one domain $A \in \mathcal{F}$ satisfying

$$\frac{1}{C} \epsilon \leq \text{diam } A \leq C\epsilon.$$

Moreover $X \cap \partial A = \emptyset$ for $A \in \mathcal{F}$. If C and ϵ_0 do not depend on x , we say that \mathcal{F} covers a set X in all scales in a (C, ϵ_0) -uniform way.

Definition 1.4. We say that a family \mathcal{F} has **uniform bounded geometry** if there exists $\lambda < 1$ so that for every $A \in \mathcal{F}$ there exists $x \in A$ such that

$$B(x, \lambda \cdot \text{diam } A) \subset A.$$

Definition 1.5. We say that a centered family \mathcal{F} has **pointwise bounded geometry** if for any $x \in \mathbb{C}$ there exists $C = C(x) < 1$ so that if $(A, x) \in \mathcal{F}$, then $B(x, C \cdot \text{diam } A) \subset A$. The centered family \mathcal{F} has **C -uniform bounded geometry** if C does not depend on x .

Definition 1.6. We say that a family of domains \mathcal{F} is a **Markov family** if we have the following.

- For A_1 and A_2 in \mathcal{F} , one of the following statements holds:
 - $A_1 \cap A_2 = \emptyset$;
 - $A_1 \subset A_2$;
 - $A_2 \subset A_1$.
- Each subfamily $\mathcal{F}' \subset \mathcal{F}$ is finite if it satisfies the following properties:
 - The inclusion is a total order in \mathcal{F}' : if $A_1, A_2 \in \mathcal{F}'$, then either $A_1 \subset A_2$ or $A_2 \subset A_1$.
 - There exists $A_0 \in \mathcal{F}$ such that $A_0 \subset A$ for any $A \in \mathcal{F}'$.

Markov families arise in a very natural way in complex dynamics: the family of the puzzle pieces of any level is a Markov family. Note that any subfamily of a Markov family is a Markov family itself. The main property of Markov families is the following: if a Markov family \mathcal{F} covers a set X , then there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$ which covers each point of X once and each point of \mathbb{C} at most once.

Let h be a homeomorphism in the complex plane and let \mathcal{F} be a (centered) family. Define

$$h(\mathcal{F}) = \{h(A) : A \in \mathcal{F}\},$$

if \mathcal{F} is a family, or define

$$h(\mathcal{F}) = \{(h(A), h(x)) : (A, x) \in \mathcal{F}\},$$

if \mathcal{F} is a centered family.

Theorem 4 (Removability Theorem). *Let $h : \mathbb{C} \rightarrow \mathbb{C}$ be a homeomorphism and let $X \subset \mathbb{C}$ be a compact set. Suppose that h is quasiconformal in $\mathbb{C} - X$. Assume that there exist a decomposition $X = X_1 \cup X_2$, a family \mathcal{F}_1 and a centered family \mathcal{F}_2 such that*

- \mathcal{F}_1 is a Markov family,
- \mathcal{F}_1 covers X_1 in arbitrary small scales,
- \mathcal{F}_1 and $h(\mathcal{F}_1)$ have uniform bounded geometry,

and

- X_2 has zero 2-dimensional Lebesgue measure,
- \mathcal{F}_2 covers X_2 in all scales,
- \mathcal{F}_2 and $h(\mathcal{F}_2)$ have pointwise bounded geometry.

Then h is quasiconformal in \mathbb{C} .

Remark 1.4. The same conclusion holds if we add another exceptional set X_3 with σ -finite 1-dimensional Hausdorff measure.

Remark 1.5. The same proof works replacing \mathbb{C} by \mathbb{R}^n , $n \geq 2$, and quasiconformal by quasisymmetric, assuming that X_2 has zero n -dimensional Lebesgue measure and X_3 has σ -finite $(n - 1)$ -dimensional Hausdorff measure.

Theorem 4 is an almost straightforward generalization of the main result in Kallunki and Koskela [KK00]. In their proof they use the so-called Besicovitch Theorem, which is replaced here by the Markov property. For completeness we present an almost self-contained proof in Section 3. Apart from the above definitions about families, Section 3 is independent of the rest of this work.

In the application of Theorem 4 to prove Theorem 1 (see Section 4), X will be the filled-in Julia set (with empty interior) of a generalized polynomial-like map, X_1 will be the points in the Julia set whose forward orbit accumulates on the critical point and X_2 will be the set of points whose forward orbit is far from the critical point. The family \mathcal{F}_1 will be defined as the family of univalent pullbacks of critical puzzle pieces with bounded geometry. The complex bounds and the Koebe distortion lemma will provide the uniform bounded geometry. For each point $x \in X_2$, there exists a finite number of puzzle pieces (depending on x) so that each domain in \mathcal{F}_2 which covers x is a univalent pullback (with bounded distortion) of one of these domains. This will give us the pointwise bounded geometry.

2. OUTLINE OF THE WORK

Part 1. In the first part of this work (Sections 3–4) we prove Theorem 4 and Theorem 1. We prove Theorem 4 in Section 3 and then deduce Theorem 1 in Section 4.

The naive argument of the proof of Theorem 1 is the following: since we know that the shapes of certain combinatorially defined critical puzzle pieces of *both* maps f_1 and f_2 are under control, we have, by assumption, that the conjugacy h does not distort certain critical puzzle pieces. Using the dynamics, we transport this control of puzzle distortion to any point whose forward orbit accumulates on the critical point (here some “complex bounds” are needed). We can deal with the points whose orbits do not accumulate on the critical point in another way (see below). So we conclude that the conjugacy does not distort a lot of puzzle pieces, around a lot of points in the Julia set (this is done in Section 4): we can expect that this conjugacy is very regular.

To make this argument rigorous, we need Theorem 4. Indeed, the recent proof by Przytycki and Rohde [PR99] that the Julia sets of CE holomorphic repellers are dynamically removable draws our attention to a result by Heinonen and Koskela [HK95], which claims that, to prove that a homeomorphism is quasiconformal, it is sufficient to find, for each point in the domain, a sequence of circles centered on that point, whose diameter goes to zero, so that the geometry of the image of these circles by the homeomorphism is under control. Since we will not use circles, but puzzle pieces (which are hardly circles), a new kind of “removability” result is necessary: certain coverings of sets by discs have very special properties, as in the Besicovitch Theorem. But the advantage of the puzzle pieces is that they satisfy a “Markov property” (see Definition 1.6), which substitutes the Besicovitch Theorem, and moreover they are dynamically defined, which is very unusual for circles!

Unfortunately, we cannot transport the information about the geometry of the critical puzzle pieces in the principal nest and their images to all points in the Julia set, since there are points whose orbits do not accumulate the critical point. The

solution to this problem is that the set of these problematic points is small (zero Lebesgue measure: see Corollary 4.2), so we can be (in some sense) less exigent with them: we need only a pointwise control of distortion, which is obtained using the expansion of the dynamics far from the critical point (Corollary 4.1). Here an argument developed by Kallunki and Koskela [KK00] will be a useful tool. So a new removability-type theorem arises (Theorem 4).

Part 2. The goal of the second part (Sections 5–9) is to show that the local stable manifold of the Fibonacci cycle

$$W_\delta^s(\tilde{f}) := \{f: |\mathcal{R}^i f - \tilde{f}| < \delta, \text{ for all } i \in \mathbb{N}\}$$

is contained in the hybrid class of \tilde{f} (see Theorem 5). Indeed in Section 12 (Part 3) we will improve this result, proving that in the hybrid class of \tilde{f} the Fibonacci renormalization operator is contracting, so the renormalizations of maps in the hybrid class converges to the Fibonacci cycle exponentially fast. These two results give a characterization of the stable manifold: this result will be essential in the proof of the hyperbolicity of the Fibonacci cycle in the third part.

In Section 6 we study the maps in the Fibonacci cycle, using results from Buff [Bu]. We are especially interested in the shape of the puzzle pieces in the principal nest of these maps: after a proper normalization of the maps and domains in the principal nest of Fibonacci maps, we obtain a (fixed) hyperbolic polynomial-like map. As a consequence, the shapes of the puzzle pieces in the principal nest are very close to the shape of the Julia set of such hyperbolic map [Bu].

A complex analytic version of the Fibonacci renormalization operator is defined in Section 7. In Section 8, we prove that the geometry of critical puzzle pieces in the principal nest of maps whose renormalizations are very close to the Fibonacci cycle is under control. Indeed their shapes are close to the shape of the Julia set of the hyperbolic polynomial-like map cited above (Proposition 8.1). Roughly speaking, this follows from the stability of the Julia sets of hyperbolic maps.

In Section 9 we prove that, if the orbit of a map f under the Fibonacci renormalization operator is always very close to the Fibonacci cycle \tilde{f} , then there exists a hybrid conjugacy between that map and one of the maps in the cycle. The first step is to construct a topological conjugacy h that is quasiconformal outside their Julia sets (Proposition 9.1).

The next step is to prove that the conjugacy is indeed quasiconformal in the Julia set. Unfortunately, all the previous methods to prove this kind of result for non-renormalizable (in the classical sense) and non-necessarily real maps are based on the “modulus divergence property”, “linear decay of moduli property” or that the dynamics has some “expansivity” (Collet-Eckmann, summability or Rivera-Letelier’s decay of geometry). None of these properties hold for Fibonacci maps with criticalities larger than two. So we need a new ingredient, i.e., the control of the shape of the puzzle pieces, and Theorem 1 implies that the conjugacy is quasiconformal everywhere.

Part 3. In the third part (Sections 10–15) we prove the hyperbolicity of the Fibonacci cycle and the universality result.

In Section 10 we give an introduction to the theory of infinitesimal perturbations of maps developed in Lyubich [Lyu99] and Avila, Lyubich and de Melo [ALdM], with minor modifications, and in Section 11 we introduce some basic facts about

induced transformations: the Fibonacci renormalization operator is an example of an induced transformation. In Section 12, the exponential contraction of the Fibonacci renormalization operator in the hybrid class of the maps in the Fibonacci cycle is proved (see Corollary 12.2), using the non-existence of invariant line fields in the Fibonacci tower [vSN94] and a simple argument introduced in [Sm02]. Finally, in Section 13 we prove that the Fibonacci cycle is hyperbolic (Theorem 2). The proof of the hyperbolicity of the Fibonacci cycle goes like this: Using the Small Orbits Theorem [Lyu99] and the exponential contraction on the hybrid classes, to prove that the Fibonacci cycle is hyperbolic, it is sufficient to prove that maps whose Fibonacci renormalizations are all very close to the Fibonacci cycle are indeed quasiconformally conjugate to one of the maps in the Fibonacci cycle. So the hyperbolicity is reduced to the rigidity problem solved in the second part.

In Section 14 we derive the universality result (Theorem 3). Future developments and open questions are described in Section 15.

Part 1. Puzzle Geometry and Rigidity

3. REMOVABILITY THEOREM

We are going to prove Theorem 4. The following proof is almost self-contained. We follow closely the ideas of S. Kallunki and P. Koskela [KK00] and J. Heinonen and P. Koskela [HK95]. Denote $B(z_0, r) = \{z: |z - z_0| \leq r\}$, $S(z_0, r) = \{z: |z - z_0| = r\}$ and define $\pi_{z_0}: \mathbb{C} - \{0\} \rightarrow S(z_0, 1)$ by $\pi_{z_0}(z) = \frac{z-z_0}{|z-z_0|} + z_0$. Denote by 1_A the characteristic function of the set A . Here m is the 2-dimensional Hausdorff measure and m_1 is the 1-dimensional Hausdorff measure.

Proof of Theorem 4. Let x_0 be an arbitrary point and $r_0 > 0$. Define

$$l_h(x_0, r_0) := \inf\{|h(x) - h(x_0)|: |x - x_0| \geq r_0\},$$

$$L_h(x_0, r_0) := \sup\{|h(x) - h(x_0)|: |x - x_0| \leq r_0\}.$$

We need to prove that

$$\frac{L_h(x_0, r_0)}{l_h(x_0, r_0)} \leq H$$

for some universal H . Denote $L = L_h(x_0, r_0)$ and $l = l_h(x_0, r_0)$. Define $A := h^{-1}(\{y: l \leq |y - h(x_0)| \leq L\}) - X_2$. Since h is quasiconformal outside X , there exists $\lambda < 1$ such that, for every x outside X , we can find $r_0(x) > 0$ so that

$$B(h(x), \lambda \text{diam } h(B(x, r))) \subset h(B(x, r)),$$

for $r < r_0(x)$. Let j_0 be minimal such that $L2^{-j_0} < l$. We will define a family of closed discs \mathcal{B}_j and a Markov family \mathcal{M}_j in the following way:

- A disc $B(x, R)$ belongs to \mathcal{B}_j , with $j \leq j_0 + 1$, if
 - $x \in A - X$,
 - $r < r_0(x)$ and
 - $h(B(x, r))$ is contained in $R_j := \{y: L2^{-(j+1)} < |y - h(x_0)| < L(2^{-j} + 2^{-(j_0+1)})\}$ and $\text{diam } h(B(x, r)) \leq L2^{-100j_0}$.
- A piece $M \in \mathcal{F}_1$ belongs to \mathcal{M}_j , with $j \leq j_0 + 1$, if $h(M)$ is contained in R_j and $\text{diam } h(M) \leq L2^{-100j_0}$.

Note that $\bigcup_j \mathcal{B}_j$ covers $A - X$ and $\bigcup_j \mathcal{M}_j$ covers $A \cap X_1$. Applying the Besicovitch Covering Theorem to $\bigcup_j \mathcal{B}_j$, we can

- replace \mathcal{B}_j by an enumerable subfamily, denoted by the same symbol \mathcal{B}_j , such that

$$1_{A-X} \leq \sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} 1_B \leq C;$$

- moreover, each family \mathcal{B}_j can be decomposed into Q subfamilies \mathcal{B}_j^i , $i = 1, \dots, Q$, where Q does not depend on anything, so that

if $B_1 \in \mathcal{B}_{j_1}^i$ and $B_2 \in \mathcal{B}_{j_2}^i$, then either $B_1 \cap B_2 = \phi$ or $j_1 = j_2$ and $B_1 = B_2$.

Similarly, since \mathcal{F}_1 is a Markov family, we can replace \mathcal{M}_j by a subfamily, if necessary, so that

$$\sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} 1_M \leq 1$$

almost everywhere in \mathbb{C} and

$$\sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} 1_M = 1$$

in $A \cap X_1$. Note that there exists a universal constant $C > 1$ such that, for any $M \in \bigcup_j \mathcal{M}_j$, there exists a disc B^M which satisfies

- $diam B^M \sim diam M$,
- $diam C B^M > 2 diam M$,
- $B^M \subset M \subset C B^M$.

Define

$$\begin{aligned} \rho := & (\log \frac{L}{l})^{-1} \sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} \frac{diam h(B)}{dist(h(B), h(x_0))} \frac{1}{diam B} 1_{2B} \\ & + (\log \frac{L}{l})^{-1} \sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} \frac{diam h(M)}{dist(h(M), h(x_0))} \frac{1}{diam M} 1_{CB^M}. \end{aligned}$$

We claim that

$$\int \rho^2 dm \leq (\log \frac{L}{l})^{-1}.$$

Denote

$$\begin{aligned} a_B := & \frac{diam h(B)}{dist(h(B), h(x_0))} \frac{1}{diam B}, \\ a_M := & \frac{diam h(M)}{dist(h(M), h(x_0))} \frac{1}{diam M}. \end{aligned}$$

By Lemma 4.2 in [B],

$$\begin{aligned} \int \rho^2 dm &= (\log \frac{L}{l})^{-2} \left(\int \left(\sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} a_B 1_{2B} + \sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} a_M 1_{CB^M} \right)^2 dm \right) \\ &\leq C (\log \frac{L}{l})^{-2} \left(\int \left(\sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} a_B 1_B \right)^2 dm + \int \left(\sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} a_M 1_{B^M} \right)^2 dm \right) \\ &\leq C (\log \frac{L}{l})^{-2} \left(\sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} a_B^2 m(B) + \sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} a_M^2 m(M) \right). \end{aligned}$$

By assumption

$$\frac{m(h(B))}{(\text{diam } h(B))^2}, \frac{m(h(M))}{(\text{diam } h(M))^2}, \text{ and } \frac{m(M)}{(\text{diam } M)^2}$$

are at a definite distance from zero and infinity. We obtain

$$\begin{aligned} \int \rho^2 dm &\leq C \left(\log \frac{L}{l}\right)^{-2} \left(\sum_{j \leq j_0} \sum_{B \in \mathcal{B}_j} \frac{m(h(B))}{\text{dist}(h(B), h(x_0))^2} + \sum_{j \leq j_0} \sum_{M \in \mathcal{M}_j} \frac{m(h(M))}{\text{dist}(h(M), h(x_0))^2} \right) \\ &\leq C \left(\log \frac{L}{l}\right)^{-2} \left(\sum_{j \leq j_0} (2^{-j}L)^{-2} \sum_{B \in \mathcal{B}_j} m(h(B)) + \sum_{j \leq j_0} (2^{-j}L)^{-2} \sum_{M \in \mathcal{M}_j} m(h(M)) \right) \\ &\leq C \left(\log \frac{L}{l}\right)^{-1}. \end{aligned}$$

The last inequality is a consequence of the following:

- $h(\mathcal{B}_j)$ covers each point in R_j at most a universal number of times.
- $h(\mathcal{M}_j)$ covers almost each point in R_j at most once.
- The measure of R_j is commensurable with $(2^{-j}L)^2$.
- Assume that $j_0 > 10$; otherwise there is nothing to prove. Then j_0 is smaller than $C \log \frac{L}{l}$.

Without loss of generality (see [KK00]), we can assume that there are subcontinua $F_1 \subset h^{-1}\{y: |y - h(x_0)| = l\}$ and $F_2 \subset h^{-1}\{y: |y - h(x_0)| = L\}$ such that $\text{dist}(F_1, F_2) = 2 \text{diam } F_1 = 2 \text{diam } F_2 = \frac{1}{2}$. Select $x_i \in F_i$ so that $|x_1 - x_2| = \text{dist}(F_1, F_2)$ and denote by S the unique segment of length one whose mid-point is the intersection point of S and the segment between x_1 and x_2 and, moreover, S cuts this segment perpendicularly, exactly through the middle. Using Lemma 2.1 in [KK00], to get a lower bound for $\int \rho^2 dm$, it is sufficient to prove that, for fixed $x \in F_1$ and $y \in F_2$,

$$\int_{\gamma(x,y,z)} \rho dm \geq \delta > 0$$

for almost every $z \in S$. Here $\gamma(x, y, z)$ is the union of the segments between x and z and y and z . Indeed,

$$\begin{aligned} \int_{\gamma(x,y,z)} \rho dm &\geq C \left(\log \frac{L}{l}\right)^{-1} \sum_{j \leq j_0} (2^{-j}L)^{-1} \sum_{B \in \mathcal{B}_j, B \cap \gamma(x,y,z) \neq \emptyset} \text{diam } h(B) \\ &\quad + C \left(\log \frac{L}{l}\right)^{-1} \sum_{j \leq j_0} (2^{-j}L)^{-1} \sum_{M \in \mathcal{M}_j, M \cap \gamma(x,y,z) \neq \emptyset} \text{diam } h(M). \end{aligned}$$

Assume for a moment that $m_1(h(\gamma(x, y, z)) \cap h(X_2)) = 0$. Then

$$\sum_{B \in \mathcal{B}_j, B \cap \gamma(x,y,z) \neq \emptyset} \text{diam } h(B) + \sum_{M \in \mathcal{M}_j, M \cap \gamma(x,y,z) \neq \emptyset} \text{diam } h(M) \geq C 2^{-j}L.$$

Since $j_0 \sim \log \frac{L}{l}$, we have obtained the lower estimate for $\int_{\gamma(x,y,z)} \rho dm$.

So we need to prove that for x and y fixed, $m_1(h(\gamma(x, y, z)) \cap h(X_2)) = 0$ for almost every $z \in S$. Indeed, fix $z_0 \in \mathbb{C}$ and denote $E_\sigma = \{z_0 + \lambda\sigma: 1 \leq \lambda \leq 2\} \cap X_2$. We claim that $m_1(h(E_\sigma)) = 0$, for almost every $\sigma \in \mathbb{S}^1$. Since $m(X_2) = 0$, we have that, for almost every $\sigma \in \mathbb{S}^1$, $m_1(E_\sigma) = 0$. We can decompose E_σ into a countable number of parts (denote each part by E_σ , again), so that for each part there exist constants $C_1 > 1$, $C_2, C_3 < 1$ and $r_0 > 0$ so that for any $x \in E_\sigma$ and $r < r_0$, there

exists $(A, x) \in \mathcal{F}_2$ satisfying

- $\frac{1}{C_1} r \leq \text{diam } A \leq C_1 r$,
- $B(x, C_2 \text{diam } A) \subset A$,
- $B(h(x), C_3 \text{diam } h(A)) \subset h(A)$.

By a simple argument as in the proof of Lemma 2.4 in [KK00], it is sufficient to prove the following: there exists $C = C(\sigma, C_1, C_2, C_3) > 0$ such that if F is a compact set in $z_0 + \sigma\mathbb{R}$, then $m_1(h(F \cap E_\sigma)) \leq C(m_1(F))^{1/2}$. To prove this, find intervals of disjoint interior $I_i \subset z_0 + \sigma\mathbb{R}$, $i = 1, \dots, p$, with length $r \leq C_2/C_1 r_0$ so that $F \subset \bigcup_i I_i$ and $pr \leq 2 m_1(F)$. For each i so that $E_\sigma \cap I_i \neq \emptyset$, select $x_i \in I_i \cap E$. Then there exists $(A_i, x_i) \in \mathcal{F}_2$ satisfying

- $I_i \subset A_i$,
- $\frac{1}{C} r \leq \text{diam } A_i \leq Cr$.

So $h(F) \subset \bigcup_i h(A_i)$. Note that the family $\{h(A_i)\}_i$ covers each point in \mathbb{C} at most $N(C_1, C_2)$ times. Then

$$\begin{aligned} \left(\sum_i \text{diam } h(A_i)\right)^2 &\leq Cp \sum_i (\text{diam } h(A_i))^2 \\ &\leq Cp \sum_i m(B(h(x_i), C_2 \text{diam } h(A_i))) \\ &\leq C p m(h(\pi_{z_0}^{-1}(B(\sigma, Cr) \cap S(z_0, 1)))). \end{aligned}$$

For Lebesgue almost every $\sigma \in \mathbb{S}^1$ there exists $C = C(\sigma)$ so that

$$\frac{m(h(\pi_{z_0}^{-1}(B(\sigma, R) \cap S(z_0, 1)))}{R} \leq C,$$

provided that R is small. Then

$$\sum_i \text{diam } h(A_i) \leq C pr \leq C (m_1(F))^{1/2},$$

which finishes the proof of the theorem. □

4. RIGIDITY THEOREM

Let f be a generalized polynomial-like map such that $\text{diam } C_n \rightarrow_{n \rightarrow \infty} 0$. Let $V_i, i \leq N = N(n)$, be the puzzle pieces at level n and let $U_j, j \leq N(n+1) - 1$, be the puzzle pieces at level $n+1$ which do not contain the critical point. Define the map $g_n: \bigcup_i U_i \rightarrow \bigcup_j V_j$, as the map f restricted to $\bigcup_i U_i$. Let $K(g_n)$ be the filled-in Julia set of g_n . We can provide each connected component of the Riemann surface $\mathcal{V} = \bigcup_j V_j$ with its hyperbolic metric. The following result is trivial:

Proposition 4.1. *There exist $\alpha_2 > \alpha_1 > 1$, with $\alpha_j = \alpha_j(n)$ such that, for any $x \in K(g_n)$, we have the following.*

- $\alpha_1 \leq \|Df(x)\|_{\mathcal{V}} \leq \alpha_2$.
- For any $m > 0$, let j be so that $f^m(x) \in V_j$. Then there is a domain $D(x, m)$ which contains x so that f^m is univalent in $D(x, m)$ and $f^m(D(x, m)) = V_j$.

The proof of the following corollary is straightforward:

Corollary 4.1. *For $n > 0$, there exists a centered Markov family \mathcal{G}_n so that*

- \mathcal{G}_n covers $K(g_n)$ in all scales in a (C_1, ϵ_0) -uniform way, where (C_1, ϵ_0) depends on f and n ;

- \mathcal{G}_n has $C_2(f, n)$ -uniform bounded geometry.

Furthermore, this centered Markov family is topologically defined in the following sense: Let $f_j: \bigcup_i U_i^j \rightarrow V^j$, $j = 1, 2$, be two generalized polynomial-like maps and let $h: \mathbb{C} \rightarrow \mathbb{C}$ be a homeomorphism so that

- $h(U_i^1) = U_i^2$;
- $h(V^1) = V^2$;
- h is a conjugacy between f_1 and f_2 .

Then $h(\mathcal{G}_n^1) = \mathcal{G}_n^2$.

Proof. Let $x \in K(g_n)$. Then $(A, x) \in \mathcal{G}_n$ if and only if there exists $m \geq 0$ so that $g_n^m(x) \in U_i$ and $A = D(x, m) \cap g_n^{-m}(U_i)$. \square

Corollary 4.2. *The set $\bigcup_n K(g_n)$ has zero 2-dimensional Lebesgue measure.*

Corollary 4.3. *$K(f)$ has empty interior.*

Proof. Suppose $K(f)$ has non-empty interior. By the non-existence of wandering components (Sullivan) and the classification of the periodic components, the critical set must be in the interior of $K(f)$, which is a contradiction, since $\text{diam } C_n \rightarrow 0$. \square

The following is an argument of Levin and van Strien (Proposition 3.1 in [LvS]):

Proposition 4.2. *Let $f: \bigcup_i U_i \rightarrow V$ be a generalized polynomial-like map so that*

- $\text{diam } C_n \rightarrow_n 0$;
- *there exist $M > 0$ and a subsequence of critical pieces C_{n_k} which admit annuli A_k so that*
 - *the internal boundary of A_k is ∂C_{n_k} ;*
 - *$\text{mod } A_k \geq M$;*
 - *for any x which is not contained in C_{n_k} and $m > 0$ minimal so that $f^m(x) \in C_{n_k}$, there exists a neighborhood $V(x, m)$ of x so that*
 - * $f^m(V(x, m)) = C_{n_k} \cup A_k$;
 - * f^m is univalent in $V(x, m)$.

Then any point $x \in K(f)$ admits a sequence of puzzle pieces which shrinks to x . In particular, $K(f)$ is a Cantor set.

Proof. By Proposition 4.1, we can assume that the forward orbit of x accumulates on the critical point. Let m_k be minimal so that $f^{m_k}(x) \in C_{n_k}$. Since m_k is the first entry time, there exists a domain $V(x, k)$ containing x which is mapped univalently by f^{m_k} to $C_{n_k} \cup A_k$. Denote by $P(x, k) \subset V(x, k)$ the piece such that $f^{m_k}P(x, k) = C_{n_k}$. We claim that $\text{diam } P(x, k)$ goes to zero. Indeed, consider the core curve γ_k of the annulus A_k . Since $\text{mod } A_k > M$ and by the Koebe distortion lemma, we have that $\tilde{\gamma}_k = f^{-m_k}(\gamma_k) \cap V(x, k)$ is a curve which contains a disc whose diameter is comparable to $\text{diam } \tilde{\gamma}_k$ and whose center is x . So, if $\text{diam } \tilde{\gamma}_k$ does not converge to zero, the diameter of these balls also does not converge to 0, and we can find a subsequence converging to a ball which is contained in $K(f)$, since f^m , $m > 0$, is a normal family in this ball. This is impossible, since $\text{int } K(f)$ is empty. \square

Proof of Theorem 1. Since the argument is quite symmetric with respect to f_1 and f_2 , we will denote both by f . Decompose $K(f)$ into two disjoint sets $K(f) = X_1 \cup X_2$, where $X_2 = \bigcup_n K(g_n)$. Denote $E_n = K(g_n) - K(g_{n-1})$, with $K(g_0) = \phi$. Then $X_2 = \bigcup_n E_n$. Define the centered Markov family \mathcal{F}_2 in the following way: if

$x \in E_n$, then the domains in \mathcal{F}_2 with center x are exactly the domains with center x belonging to \mathcal{G}_n . It is easy to see that \mathcal{F}_2 is topologically defined, covers X_2 in all scales and has bounded geometry.

Now we will deal with X_1 : this is the set of points in $K(f)$ whose positive orbit accumulates on the critical point. Define the Markov family \mathcal{F}_1 in the following way: for $x \in K_1$, let $m_k \geq 0$ be minimal so that $f^{m_k}(x) \in C_{n_k}$. By assumption, there is a domain which contains x such that f^{m_k} maps it univalently to $C_{n_k} \cup A_k$. So there exists a puzzle piece $V(x, n)$ containing x so that

- f^{m_k} is univalent in $V(x, n)$;
- $f^{m_k}(V(x, n)) = C_{n_k}$;
- moreover, since $\text{mod } A_k > M$ and by the Koebe distortion lemma, there exists $\lambda < 1$ so that for any (x, n) there is $y \in V(x, n)$ satisfying

$$B(y, \lambda V(x, n)) \subset V(x, n)$$

where $\lambda = \lambda(M, K_1, K_2)$.

Note that, for a fixed x , $\text{diam } V(x, n) \rightarrow_n 0$. Define

$$\mathcal{F}_1 = \{(V(x, n), x) : x \in X_1 \text{ and } n > 0\}.$$

So \mathcal{F}_1 covers X_1 in arbitrary small scales, has bounded geometry and, moreover, is topologically defined. By the Removability Theorem (Theorem 4), h is quasiconformal in \mathbb{C} . \square

Part 2. Characterization of the stable manifold of the Fibonacci cycle

5. PRELIMINARIES

5.1. Quasiconformal theory. We will assume that the reader is familiar with the general properties of quasiconformal maps (Lehto and Virtanen [LV] is a good reference) and quasiconformal vector fields (see [McM]). The following property will be very useful:

Proposition 5.1 ([McM]). *Let z_0, z_1 and z_2 be distinct points on the Riemann sphere and let $K > 0$. Then the set of K -quasiconformal vector fields on the Riemann sphere which vanishes at these points is a compact set in the space of continuous vector fields on the Riemann sphere with the sup norm induced by the spherical metric.*

5.2. Banach spaces. We will denote domains (pre-compact open sets in the complex plane) with symbols like $U, V, \tilde{V}, \hat{U}, \dots$. If we denote a domain as, for instance, $U = U_1 \cup \dots \cup U_n$, the reader must assume that U_i are the (simply) connected components of U . If $\alpha \in \mathbb{C}$, then $\alpha U := \{\alpha \cdot x : x \in U\}$. The δ -neighborhood of a set X , denoted $\delta - X$, is the set $\{x : \text{dist}(x, X) \leq \delta \text{ diam } X\}$. Let U be a domain so that \bar{U} does not disconnect the plane. Let $d > 1$. In the Banach space $\mathcal{B}(U)$ of holomorphic functions on U with continuous extension to \bar{U} , equipped with the sup norm, consider the closed affine space $\mathcal{B}_{\text{nor}}(U)$ of functions f such that

- the domain U contains 1 and 0 and furthermore $f(1) = 1$ and $f^{(i)}(0) = 0$, for $1 \leq i < d$.

We will call $\mathcal{B}_{nor}(U)$ a **Banach slice**. The tangent space of \mathcal{B}_{nor} , denoted by $T\mathcal{B}_{nor}(U)$, is the subspace of the vectors $v \in \mathcal{B}(U)$ such that

- $v(1) = 0$,
- $v^{(i)}(0) = 0$, for $1 \leq i < d$.

Let $f: U_0 \cup \dots \cup U_n \rightarrow V$, where U_i are simply connected, be a generalized polynomial-like map and let $\tilde{U} = \tilde{U}_0 \cup \dots \cup \tilde{U}_n$, where $U_i \Subset \tilde{U}_i$ are simply connected. Then we say that the Banach space $\mathcal{B}_{nor}(\tilde{U})$ is **compatible** with f . By a result of Levin and van Strien (Proposition 2.2 in [LvS]), if $g: \hat{U}_1 \cup \dots \cup \hat{U}_n \rightarrow \hat{V}$ is a generalized polynomial-like restriction of f , with $\hat{U}_i \subset \tilde{U}_i$, then $K(g) = K(f)$. We will say that g is a **representation** of f subordinated to \tilde{U} . So all these representations have the same filled-in Julia set and, if the critical points of f do not escape, they have the same postcritical set.

5.3. The principal nest. Let $f: \bigcup_i U_i \rightarrow V$ be a generalized polynomial-like map whose critical point does not escape. Denote $V_0^0 := V$, $V_0^1 := U_0$ and $V_1^1 := U_k$, where the critical point is in U_0 and the critical value is in U_k . Denote $f_0 := f$. The map f restricted to V_0^1 will be denoted f_0^0 and f restricted to V_1^1 by f_0^1 . Denote $\mathcal{E}_0 := \phi$ and $\mathcal{D}_0 := \{V_0^1, V_1^1\}$. Suppose, by induction, that we have defined V_0^n , with $0 \in V_0^n$ and the following property:

$$f^i(\partial V_0^n) \cap V_0^n = \phi,$$

for every $i > 0$ so that f^i is defined on V_0^n . Denote by \mathcal{E}_n the family of (simply) connected components of

$$\{x \in \bigcup_i U_i : \text{there exists } m \geq 0 \text{ so that } f^m(x) \in V_0^n\}.$$

The **first entry map** to V_0^n of a point $x \in \bigcup \mathcal{E}_n$, denoted by $\pi_n(x)$, is defined as $f^m(x)$, where $m \geq 0$ is minimal so that $f^m(x) \in V_0^n$. The **first return map** to V_0^n , denoted by f_n , is defined as $\pi_n \circ f$ in $\bigcup \mathcal{D}_n$, where \mathcal{D}_n is the family of (simply) connected components of $f^{-1}E$, where $E \in \mathcal{E}_n$. Note that $\mathcal{E}_n - \{V_0^n\} \subset \mathcal{D}_n$. Denote by $V_0^{n+1} \in \mathcal{D}_n$ the domain which contains the critical point and by $V_1^{n+1} \in \mathcal{D}_n$ the domain which contains $f_n(0)$. Note that V_0^n is a nested sequence of critical puzzle pieces, called the **principal nest**.

6. ON THE MAPS IN THE FIBONACCI CYCLE

The most important property of the sequence of critical puzzle pieces in the principal nest of a map in the the Fibonacci cycle is that their shape is under control, as we will describe below. It is important to note that the figures in this work are only illustrative: the actual shape of the puzzles and Julia sets depends on the criticality d and they are much more complicated. Taking the figures too literally, the reader might deduce that f_n^0 has degree two, which is not the case. The actual domain D_0 represented in Figure 4, for instance, has more holes (which do not intersect the real line). Realistic pictures of puzzle pieces of Fibonacci maps can be found in Buff [Bu].

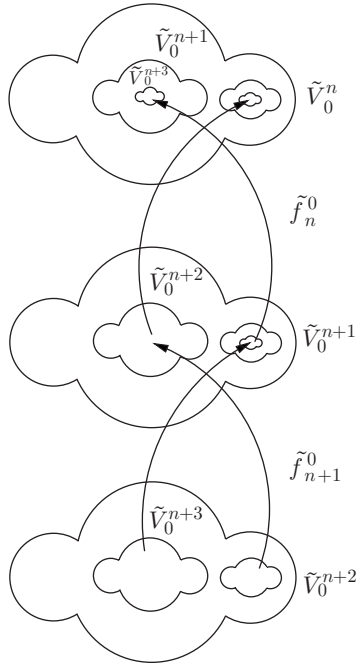


FIGURE 2.

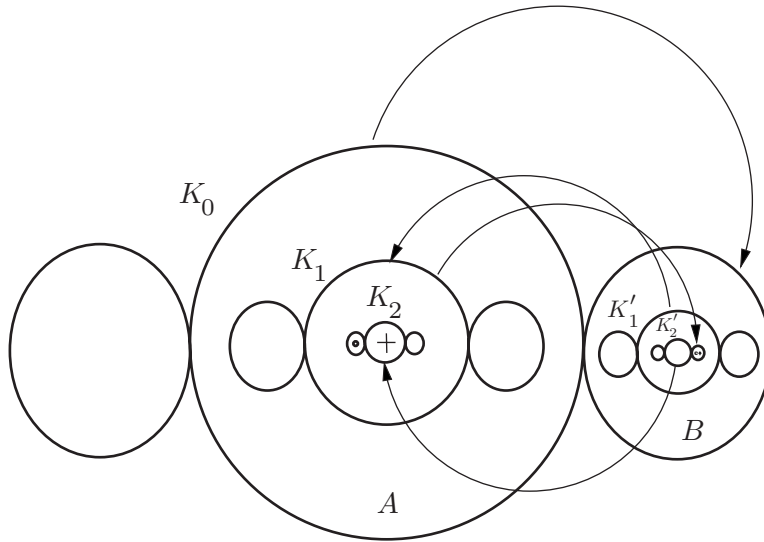


FIGURE 3.

Figure 2 describes how certain puzzles pieces in real Fibonacci maps are permuted by the action of the generalized renormalizations. Buff [Bu] described precisely the shape of V_0^n . Denote $d_H(T_1, T_2) := \max\{d_h(\bar{T}_1, \bar{T}_2), d_h(\partial T_1, \partial T_2)\}$, where $d_h(\cdot, \cdot)$ represents the Hausdorff metric on the set of compact sets.

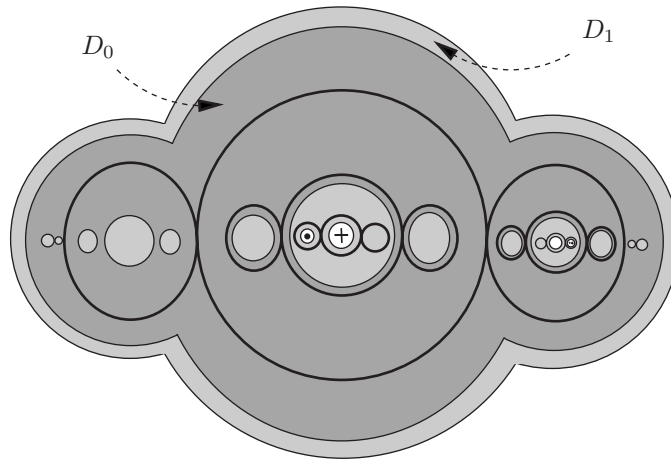


FIGURE 4.

Proposition 6.1 ([Bu]). *Let $\beta \in (0, 1)$ be the closest point with period two to the critical point of \tilde{f} . Then there exists a polynomial-like map g , with a unique critical point and a hyperbolic attractor of period two so that*

$$-1/\beta^2 g^2(\beta z) = g(z),$$

$$\tilde{f}_0^0(z) = g\left(\frac{1}{\beta} \cdot z\right),$$

and

$$\tilde{f}_0^1(z) = \mp \frac{1}{\beta} g(z).$$

In particular $g(z) = 1/\beta^n \tilde{f}_n^0(\beta^{n+1} \cdot z)$. Furthermore

$$\text{dist}_H\left(\frac{1}{\beta^n} \tilde{V}_0^n, K(g)\right) \rightarrow_n 0.$$

Indeed, Buff claimed that $1/\beta^n \overline{\tilde{V}_0^n}$ converges to $K(g)$ in the Hausdorff metric on compact sets, but the slight modification above can be proved in an analogous way (indeed, we will do something similar later, in the proof of Proposition 8.1). We will normalize \tilde{f} so that 0 is a maximum and 1 is the fixed point of \tilde{f} in the central domain so that $D\tilde{f}(1) < 0$. Then the corresponding fixed point of $\mathcal{R}\tilde{f}$ is β (or $-\beta$).

Denote by A the connected component of the interior of $K_0 := K(g)$ containing the critical point and by B the connected component that contains the critical value. We will say that a set O_1 is compactly contained in O_2 , denoted $O_1 \Subset O_2$, if $\overline{O_1} \subset \text{int } O_2$.

The following result will be very important:

Proposition 6.2 ([Bu]). *Denote $K_1 := \beta K_0$, $K_2 := \beta^2 K_0$, $K'_i := g^{-1} K_i \cap B$. Then*

- $-1/\beta^2 g^2(\beta z) = g(z)$,
- $K_1 \subset A$,
- $K_2 \Subset K_1$,
- $g(K_1) = K'_2$.

In particular:

- $\tilde{f}^0(K_1) = K_0,$
- $\tilde{f}^1(K'_1) = K_0,$
- $\tilde{f}^1(K'_2) = K_1.$

Proof. These items are included in, or are an easy consequence of, Buff [Bu]. □

Figure 3 describes how g acts on K_0, K_1, K'_0 and K'_1 : these sets are not represented in the figure, but only the (boundaries of the) connected components (in the figure, circles) of their interior which correspond to the topological discs A, B and $-B$ contained in $K(g)$.

Note that $1/\beta^n \partial \tilde{V}_0^n$ converges to $\partial K_0,$ $1/\beta^n \partial \tilde{V}_0^{n+1}$ converges to ∂K_1 and $1/\beta^n \partial \tilde{V}_1^{n+1}$ converges to $\partial K'_1.$ Since $P(\tilde{f}) \cap (\tilde{V}_0^{n+1} - (\tilde{V}_0^{n+1} \cup \tilde{V}_1^n)) = \phi,$ we conclude that \tilde{f} satisfies all the conditions imposed on f_1 in Theorem 1, with $C_{i_n} := V_0^n.$

The following corollary will be a useful tool in Section 9:

Corollary 6.1. *There exists an open set D_1 which contains the Julia set $J(g)$ in its interior such that, if we denote $D_0 := g^{-2}D_1,$ then*

- $g^2: D_0 \rightarrow D_1$ is an unramified covering of degree $(deg g)^2,$
- $D_0 \Subset D_1,$
- $\beta D_0 \Subset D_1,$
- $g^{-1}(\beta D_0) \cap B \Subset D_1.$

Proof. Let $A_1 \Subset A$ and $B_1 \Subset B$ be domains whose boundaries are Jordan curves very close to $J(g)$ and such that

- $B_1 \Subset g^{-1}A_1,$
- $A_1 \Subset g^{-1}B_1.$

Let U be a small neighborhood of $K(f)$ so that $g^{-1}U \Subset U.$ We define D_1 as (see Figure 4)

$$D_1 := U \setminus (\beta^2 \bar{A}_1 \cup \beta^2 \bar{B}_1 \cup (g^{-1}(\beta^2 \bar{B}_1) \cap B) \cup (g^{-1}(\beta^2 \bar{A}_1) \cap B)).$$

Note that

$$D_0 := g^{-2}D_1 \Subset g^{-1}U \setminus (\beta g^{-1} \bar{A}_1 \cup \beta g^{-1} \bar{B}_1 \cup g^{-1}(\beta g^{-1} \bar{A}_1 \cup \beta g^{-1} \bar{B}_1)) \subset D_1.$$

The other properties of D_1 are immediate consequence of the previous proposition (provided we have selected $\partial A_1, \partial B_1$ and ∂U close to $J(g)$). □

7. THE FIBONACCI RENORMALIZATION OPERATOR: COMPLEX ANALYTIC VERSION

We are going to define the complex analytic version of the Fibonacci operator, defined in a convenient Banach slice. Let \tilde{f} be one of the maps in the Fibonacci cycle. By Proposition 6.2, the δ -neighborhoods of $K_2, \tilde{f}(K_2)$ and $\beta K'_1$ are contained in the interior of $K_1 \cup K'_1,$ for some $\delta > 0.$ Recall that we normalized \tilde{f} so that β (or $-\beta$) is a fixed point of $\tilde{f}^2.$ Taking ϵ_0 small enough, we can assume that β (or $-\beta$) has an analytic continuation, denoted $\beta_f,$ for every f so that $|f - \tilde{f}|_{\mathcal{B}(K_1 \cup K'_1)} \leq \epsilon_0.$ The above considerations imply that there exist $\epsilon_0 > 0$ and $C < 1$ so that if $f \in \mathcal{B}_{nor}(K_1 \cup K'_1)$ and $|f - \tilde{f}|_{\mathcal{B}(K_1 \cup K'_1)} \leq \epsilon_0,$ then f^2 is well defined in a $C\delta$ -neighborhood of $\beta_f K_1$ and f is defined in a $C\delta$ -neighborhood of $\beta_f K'_1.$ We will select a domain which contains $K_1 \cup K'_1$ and whose boundary is more regular:

Let U_0 and U_1 be simply connected domains so that $K_1 \Subset U_0 \subset C\delta/2-K_1$ and $K'_1 \Subset U_1 \subset C\delta/2-K'_1$ and whose boundaries are analytic curves. So for every $f \in \mathcal{B}_{nor}(U_0 \cup U_1)$ satisfying $|f - \tilde{f}|_{\mathcal{B}(U_0 \cup U_1)} \leq \epsilon_0$, f^2 is defined in a $2C\delta/3$ -neighborhood of $\beta_f \overline{U}_0$ and f is defined in a $2C\delta/3$ -neighborhood of $\beta_f \overline{U}_1$. Define the map $\hat{\mathcal{R}}f \in \mathcal{B}_{nor}(2C\delta/3-\overline{U}_0 \cup 2C\delta/3-\overline{U}_1)$ as

$$\frac{1}{\beta_f} f^2(\beta_f \cdot z)$$

on $2C\delta/3-\overline{U}_0$ and

$$\frac{1}{\beta_f} f(\beta_f \cdot z)$$

on $2C\delta/3-\overline{U}_1$. The **Fibonacci renormalization operator** is defined as $\mathcal{R} := i \circ \hat{\mathcal{R}}$, where $i: \mathcal{B}_{nor}(2C\delta/3-\overline{U}_0 \cup 2C\delta/3-\overline{U}_1) \rightarrow \mathcal{B}_{nor}(U_0 \cup U_1)$ is the inclusion $i(f) = f$. This operator is an example of an induced transformation, whose properties will be described in Section 11. In the rest of this work, denote $U := U_1 \cup U_2$. Of course, using the complex bounds result for real Fibonacci maps [vSN94], for a Banach slice $\mathcal{B}_{nor}(\hat{U})$ we can define a complex analytic version of iteration of the Fibonacci renormalization operator in an easier way, without using the special features of \tilde{f} , and carry out a proof of the hyperbolicity in this Banach space as well. The problem is that the exposition would be a little more complicated, especially because we need to deal with an iterate of the Fibonacci operator (not with the operator itself) and there is no guarantee in this approach that $K_1 \cup K'_1 \subset \hat{U}$: so there is no advantage in using this approach in the case of the Fibonacci renormalization (but this alternative method is, possibly, the unique way to define the complex extension in the case of generalizations of this work for other combinatorics, such as non-periodic ones).

8. CONTROL OF THE GEOMETRY IN THE PRINCIPAL NEST

Let f be a function in $\mathcal{B}_{nor}(U)$ such that $\mathcal{R}^n f$ exists and

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \delta,$$

for all $n \geq 0$. We will consider a generalized polynomial-like restriction of f and prove that the shape of the puzzle pieces in its principal nest is under control, provided that δ is small. Indeed we prove that the shapes of the critical puzzle pieces are close to the shape of the Julia set of the polynomial-like map g defined in Section 6. The fact that g is a hyperbolic polynomial-like map is crucial in this result.

Let $D \Subset g(D) \subset V$ be a simply connected domain, which contains $K(g)$ so that $g: D \rightarrow g(D)$ is a ramified covering map and let $D_a \subset A$, $D_b \subset B$, with $\partial D_a \cup \partial D_b$ very close to $J(g)$, be simply connected domains so that $g(D_a) \Subset D_b$ and $g(D_b) \Subset D_a$. Define $E_0 := D \setminus \overline{D}_a \cup \overline{D}_b$ and $E_n := g^{-n} E_0$. Then

$$g: E_{n+1} \rightarrow E_n$$

is a covering map and, because g is a hyperbolic map, if we take a Jordan curve $\gamma \subset E_0$ that is the boundary of a bounded domain containing the postcritical set (and the attractor) of g , then $g^{-n}\gamma$ is also a Jordan curve bounding a domain which contains the postcritical set of g and $d_H(g^{-n}\gamma, J(g)) \leq C\lambda^n$, for some $\lambda < 1$. (Recall that $d_H(T_1, T_2)$ denotes $\max\{d_h(\overline{T}_1, \overline{T}_2), d_h(\partial T_1, \partial T_2)\}$, where $d_h(\cdot, \cdot)$

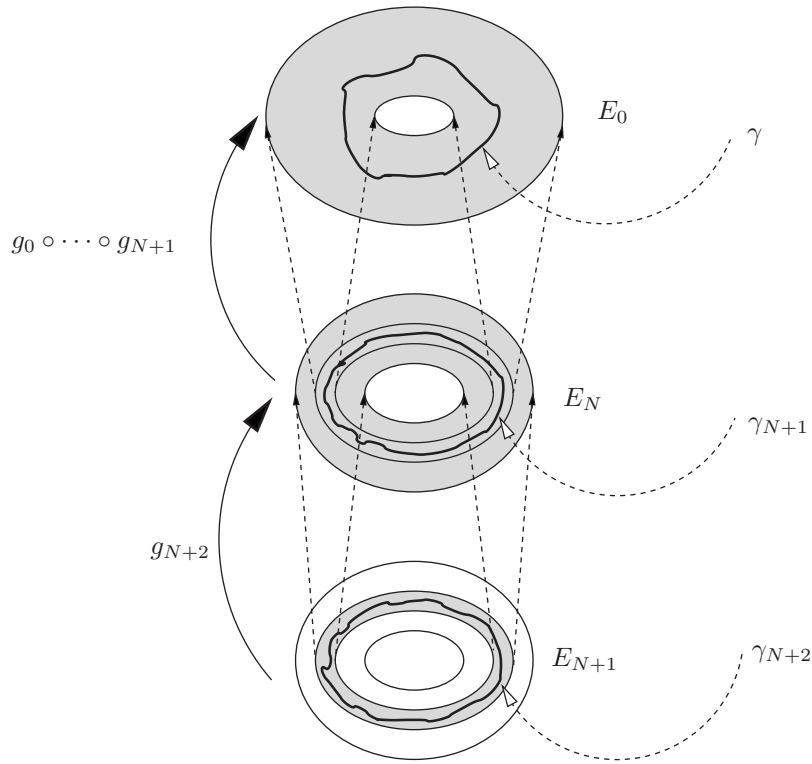


FIGURE 5.

represents the Hausdorff metric under the set of compact sets.) Consider an analytic curve $\gamma \subset E_0$, as above, and denote by V_0^0 the simply connected bounded domain whose boundary is γ . Denote $\tilde{\tau}_n = \beta^n$ and $\beta_i := \beta_{\mathcal{R}^i f}$, and $\tau_n := \beta_{n-1} \cdots \beta_0$. This notation will be used in the rest of this work.

Proposition 8.1. *For every $\epsilon > 0$ there exist $\delta > 0$ and $N = N(\epsilon)$ with the following properties: If $f \in \mathcal{B}_{nor}(U)$ satisfies*

$$|\mathcal{R}^n f - \tilde{f}| \leq \delta,$$

for $n \geq 0$, and V_0^1 and V_1^1 are the connected components of $f^{-1}V_0^0$ which contain the critical point and $f(0)$, then $f: V_0^1 \cup V_1^1 \rightarrow V_0^0$ is a generalized polynomial-like map so that

- (1) the critical point does not escape;
- (2) using the notation defined in Section 5.3, $V_0^{n+1} \neq V_1^{n+1}$ and $P(f) \cap V_0^n \subset V_0^{n+1} \cup V_1^{n+1}$;
- (3) $f_n^0: V_0^{n+1} \rightarrow V_0^n$ is a restriction of f^{S_n} to $\tau_n U_0$ and $f_n^1: V_1^{n+1} \rightarrow V_0^n$ is a restriction of $f^{S_{n-1}}$ to $\tau_n U_1$;

(4) we have

$$\begin{aligned} \text{dist}_H\left(\frac{1}{\tau_n}V_0^n, K_0\right) &\leq \epsilon, \\ \text{dist}_H\left(\frac{1}{\tau_n}V_0^{n+1}, K_1\right) &\leq \epsilon, \\ \text{dist}_H\left(\frac{1}{\tau_n}V_1^{n+1}, K'_1\right) &\leq \epsilon, \\ \text{dist}_H\left(\frac{1}{\tau_n}(f_n^1)^{-1}(V_0^{n+1}), K'_2\right) &\leq \epsilon, \end{aligned}$$

for $n \geq N$.

Furthermore, there exists $C > 0$ such that

$$(1) \quad \text{dist}\left(\frac{1}{\tau_n}f_n^0(0), \frac{1}{\tau_n}\partial(f_n^1)^{-1}(V_0^{n+1})\right) \geq C.$$

In particular, all the assumptions about the generalized polynomial-like maps in Theorem 1 are satisfied by f and, if ϵ is small enough (see Figure 6),

- $V_0^{n-2} \setminus V_0^{n-1} \cup V_1^{n-1} \Subset \tau_{n-2}D_1$,
- $D_0^{n-2} \Subset \tau_{n-2}D_1$,
- $(f_{n-2}^1)^{-1}(\tau_{n-2}D_0^{n-2}) \Subset \tau_{n-2}D_1$,
- $\tau_{n-1}D_0^{n-1} \Subset \tau_{n-2}D_1$,

where $D_0^n := \tau_{n+1}^{-1}(f_n^0)^{-1}\tau_n D_1$.

Proof. Note that by definition $\mathcal{R}^n f$ is an appropriate normalization of $(\mathcal{R}^{n-1} f)^2$ on U_0 and it is a normalized restriction of $\mathcal{R}^{n-1} f$ on U_1 . So it is easy to conclude inductively that

$$(2) \quad f^{S_n}(z) = \tau_n \mathcal{R}^n f(\tau_n^{-1}z) \text{ on } \tau_n U_0$$

and

$$(3) \quad f^{S_{n-1}}(z) = \tau_n \mathcal{R}^n f(\tau_n^{-1}z) \text{ on } \tau_n U_1.$$

Recall that $\tilde{f}(\beta z) = g(z)$ on U_0 . Since the renormalizations of f are very close to \tilde{f} , $g_n(z) := \mathcal{R}^n f(\beta_n z)$ are very close to $g(z)$ on U_0 . Consider N so that $\text{dist}_H(E_N, J(g)) \leq \epsilon/2$. Choosing δ small enough, we can assume that

$$(g_{N+1} \circ \cdots \circ g_0)^{-1}E_0 \Subset E_{N+1},$$

and, for $n \geq 0$,

$$(g_n)^{-1}E_N \Subset E_N.$$

Figure 5 illustrates this, but the topology and shape of the sets E_i are not correct in the figure: E_0 has two holes, and the number of holes of E_i grows exponentially fast with i . Since $\gamma := \partial V_0^0 \subset E_0$, by equation (1) and the above properties, we can inductively define Jordan curves γ_n in the following way: $\gamma_0 := \gamma$, and γ_n is the Jordan curve contained in the set $f^{-S_{n-1}}\gamma_{n-1}$ such that the bounded domain with boundary γ_n , denoted \hat{V}_0^n , contains the critical point. These domains have the following properties:

- $\hat{V}_0^n \Subset \hat{V}_0^{n-1} \subset \tau_{n-1}U_0$;
- $f^{S_n} : \hat{V}_0^{n+1} \rightarrow \hat{V}_0^n$ is a ramified covering with a unique critical point;

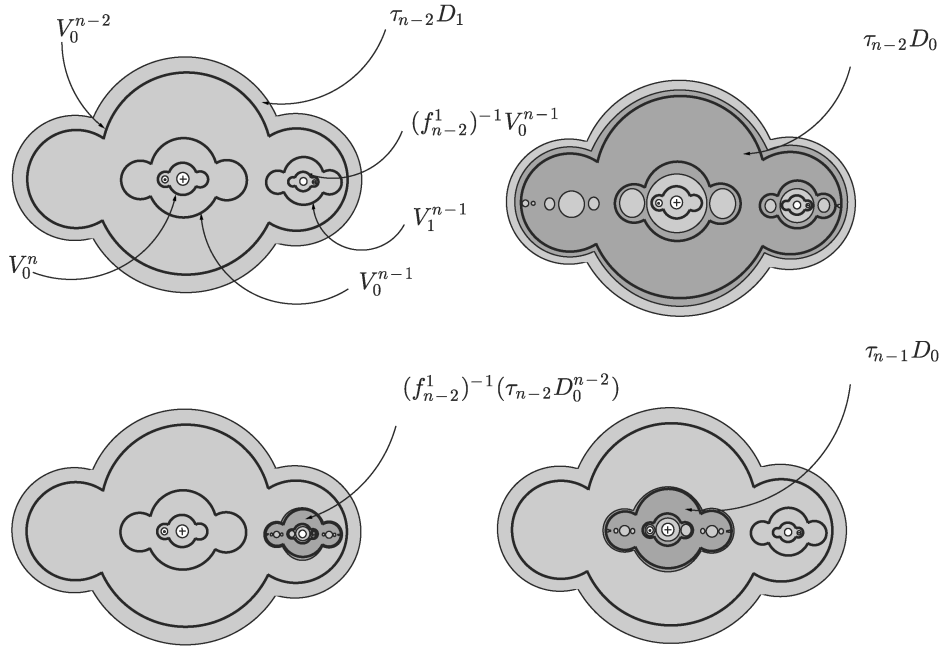


FIGURE 6.

- we have

$$\text{dist}_H(\tau_n^{-1}\hat{V}_0^n, K(g)) \leq \frac{\epsilon}{2},$$

for $n \geq N$.

In particular, the critical point does not escape. We claim that $\hat{V}_0^n = V_0^n$, where V_0^n were defined in Section 5.3. Suppose, by induction, that we proved the claim for V_0^i , $i < n$. In particular, the time of the first return of the critical point to V_0^{n-3} and V_0^{n-2} is S_{n-3} and S_{n-2} . By definition, V_0^n is the domain of the first return map to V_0^{n-1} which contains the critical point. So it is sufficient to prove that the time of the first return m of the critical point to V_0^{n-1} is S_{n-1} . Clearly $m \leq S_{n-1}$. But note that $f^{S_{n-2}}(0) \in V_0^{n-2} \setminus V_0^{n-1}$, because

$$(4) \quad f^{S_{n-2}}(0) = \tau_{n-2}\mathcal{R}^{n-2}f(0) \in \tau_{n-2}U_1,$$

and

$$V_0^{n-1} \subset \tau_{n-1}U_0 \subset \tau_{n-2}U_0.$$

So the first chance for the forward orbit of $f^{S_{n-2}}(0)$ to enter V_0^{n-1} is its iteration by the time of the first return of V_0^{n-2} to V_0^{n-3} ; in other words, $f^{S_{n-2}+S_{n-3}}(0) = f^{S_{n-1}}(0)$. This finishes the proof of the claim. Note that, because $f^{S_{n-1}}(0) \in V_0^{n-1} \setminus V_0^n$, we have that $V_1^n \neq V_0^n$, and since $f^{S_{n-1}+S_{n-2}}(0) \in V_0^n$, by equation (4) we have $V_1^n = (f_n^1)^{-1}V_0^{n-1}$, where f_n^1 , as defined in Section 5.3, is a restriction of $f^{S_{n-2}}$ on $\tau_{n-1}U_1$. By equation (3), and because $\mathcal{R}^n f$ is very close to \tilde{f} and $\tau_n^{-1}V_0^n$ is ϵ -close to K_0 , we have

$$\text{dist}_H(\tau_n^{-1}V_1^{n+1}, K'_1) \leq O(\epsilon).$$

Fix $n \geq 0$. Note that the map f_i on $V_0^{i+1} \cup V_1^{i+1}$, for $i \geq n$, is a composition of iterations of f_n on $V_0^{n+1} \cup V_1^{n+1}$. This is enough to conclude that the postcritical set of f does not intersect $V_0^i \setminus (V_0^{i+1} \cup V_1^{i+1})$. \square

9. QUASICONFORMAL RIGIDITY

A map $f \in \mathcal{B}_{nor}(U)$ satisfying the assumptions in Proposition 8.1 has a dynamical behavior quite similar to the maps in the Fibonacci cycle: the main result in this section is that there exists a hybrid conjugacy between f and either \tilde{f}_1 or \tilde{f}_2 .

If A is a connected domain omitting at least two points of the complex plane, we can provide it with its hyperbolic metric, denoted $dist(\cdot, \cdot)_D$. Consider a map $f \in \mathcal{B}_{nor}(U)$ which is very close to \tilde{f} . Let $\tilde{V}_0 = V_0^0$ be the simply connected domain whose boundary is γ (see the previous section). Define $V_0^1 := (f_0^0)^{-1}V_0$ and $V_1^1 := (f_0^1)^{-1}V_0$ and similarly \tilde{V}_0^1 and \tilde{V}_1^1 . In this section, we will use the notation introduced in Section 5.3. The sets and induced maps defined there for f can also be defined for \tilde{f} , and they will be indicated by a tilde (for instance, $\tilde{f}_n, \tilde{V}_0^n, \dots$). We have

Theorem 5. *There exists $\delta_0 > 0$ with the following property: Assume that $f \in \mathcal{B}_{nor}(U)$ satisfies*

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \delta \leq \delta_0$$

for $n \geq 0$. Then there exists an $O(\delta)$ -quasiconformal map $\phi: \mathbb{C} \rightarrow \mathbb{C}$ which is a hybrid conjugacy between $f: V_0^1 \cup V_1^1 \rightarrow V_0$ and $\tilde{f}: \tilde{V}_0^1 \cup \tilde{V}_1^1 \rightarrow \tilde{V}_0$, with $\phi(V_0) = \tilde{V}_0$.

As the first step to prove the main result of this section, we will prove

Proposition 9.1. *There exists $\delta_0 > 0$ with the following property: Assume that $f \in \mathcal{B}_{nor}(U)$ satisfies*

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \delta \leq \delta_0$$

for $n \geq 0$. Then there exists a topological conjugacy $\phi: \mathbb{C} \rightarrow \mathbb{C}$ between

$$f: V_0^1 \cup V_1^1 \rightarrow V_0 \text{ and } \tilde{f}: \tilde{V}_0^1 \cup \tilde{V}_1^1 \rightarrow \tilde{V}_0$$

such that

- $\phi(V_0) = \tilde{V}_0$,
- ϕ is $O(\delta)$ -quasiconformal outside $K(f)$.

Lemma 9.1. *There exist $\delta_0 > 0, \epsilon_0 > 0$ and $C > 0$ with the following property: If*

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \delta \leq \delta_0$$

for $n \geq 0$ and $\phi_0: A_0 \rightarrow \tilde{A}_0$, where $A_0 = \overline{V_0^0} \setminus (V_0^1 \cup V_1^1)$, is an injective continuous map such that

- $|\phi_0(z) - z| \leq \epsilon$, for $x \in A_0$, for some ϵ so that $\epsilon_0 > \epsilon > C\delta$,
- $\phi_0 \circ f = \tilde{f} \circ \phi_0$ on $\partial V_0^1 \cup \partial V_1^1$,

then there exist a sequence of domains A_n , the corresponding tilded domains \tilde{A}_n , and maps $\phi_n: A_n \rightarrow \tilde{A}_n, \psi_n: A_{n-1} \cup (f_{n-1}^1)^{-1}A_{n-1} \rightarrow \tilde{A}_{n-1} \cup (\tilde{f}_{n-1}^1)^{-1}\tilde{A}_{n-1}$ such that the following hold.

- (1) $A_n := (f_{n-1}^0)^{-1}(A_{n-1} \cup (f_{n-1}^1)^{-1}(A_{n-1}))$.
- (2) $A_n \subset \overline{V_0^n} \setminus (V_0^{n+1} \cup V_1^{n+1})$ and $\partial V_0^n, \partial V_0^{n+1}, \partial V_1^{n+1}$ are connected components of ∂A_n .

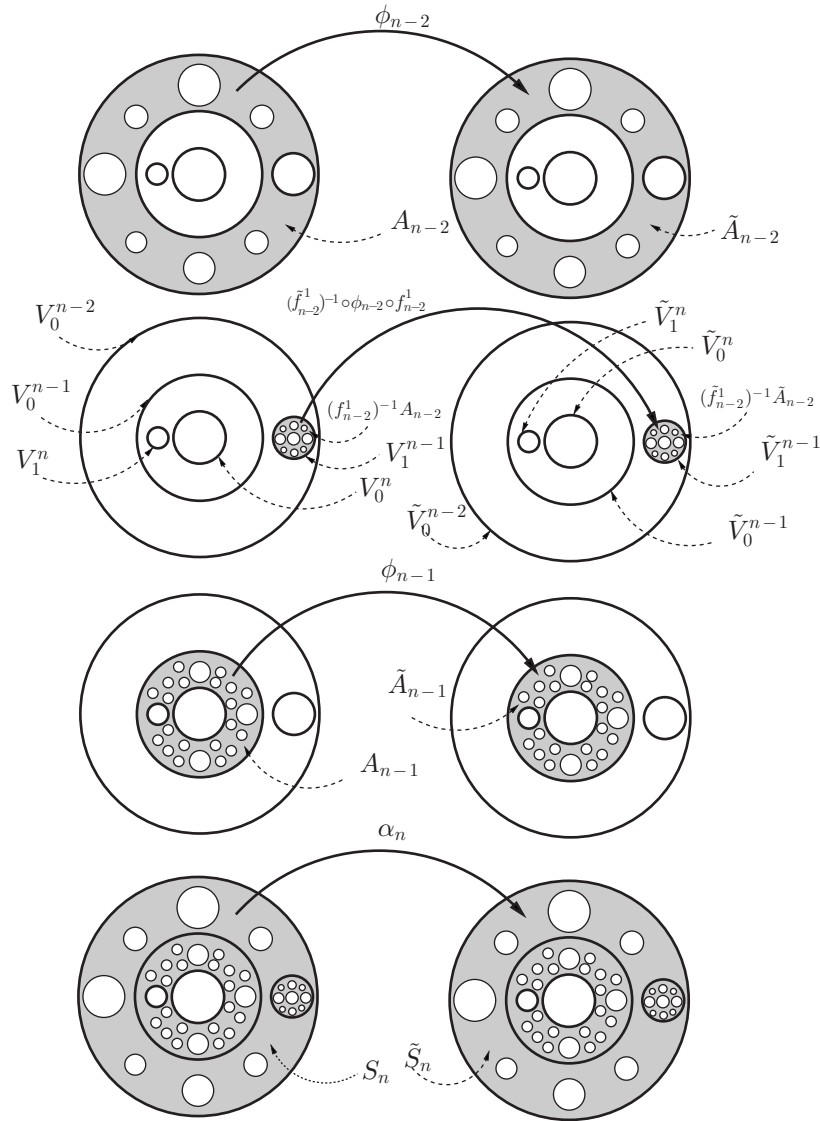


FIGURE 7.

- (3) ψ_n is equal to ϕ_{n-1} on A_{n-1} and it coincides with $\tilde{f}_{n-1}^{-1} \circ \phi_{n-1} \circ f_{n-1}^1$ on $(f_{n-1}^1)^{-1}A_{n-1}$.
- (4) $\psi_n \circ f_n^0 = \tilde{f}_n^0 \circ \phi_n$ on A_n and the maps ϕ_{n-1} , ψ_n and ϕ_n coincide on ∂V_0^n .
- (5) For $z \in \tau_n^{-1}A_n$,

$$\text{dist}(z, \tilde{\tau}_n^{-1} \phi_n(\tau_n \cdot z))_{D_0} \leq \epsilon.$$

Here D_0 is as in Corollary 6.1.

- (6) For $z \in \tau_{n-1}^{-1}(A_{n-1} \cup (f_{n-1}^1)^{-1}A_{n-1})$ we have

$$\text{dist}(z, \tilde{\tau}_{n-1}^{-1} \psi_{n-1}(\tau_{n-1} \cdot z))_{D_0} \leq C\epsilon.$$

Proof. Recall that we assume that all renormalizations of f and \tilde{f} are very close. Define ψ_1 equal to ϕ_0 on A_0 and equal to $(\tilde{f}_0^1)^{-1} \circ \phi_0 \circ f_0^1$ on $(f_0^1)^{-1}(A_0)$. Define ϕ_1 as the unique map such that

- $\phi_1 = \phi_0$ in ∂V_1^0 and
- $\psi_1 \circ f_1^0 = \tilde{f}_1^0 \circ \phi_1$ on A_1 .

ϕ_1 is well defined if $\epsilon_0 + \delta_0$ is small enough. Since the first renormalizations of f and \tilde{f} are very close, we can conclude that ϕ_1 and the extension of ϕ_0 are $O(\delta_0 + \epsilon_0)$ -close to the identity. Assume by induction that we have defined ϕ_i , $i < n$, with the required properties. Define ψ_{n-1} equal to ϕ_{n-2} on A_{n-2} and as $(\tilde{f}_{n-2}^1)^{-1} \circ \phi_{n-2} \circ f_{n-2}^1$ on $(f_{n-2}^1)^{-1}A_{n-2}$. Denote $S_n := A_{n-2} \cup A_{n-1} \cup (f_{n-2}^1)^{-1}A_{n-2}$. Let $\alpha_n: S_n \rightarrow \tilde{S}_n$ be the map which is equal to ψ_{n-1} on $A_{n-2} \cup (f_{n-2}^1)^{-1}A_{n-2}$ and equal to ϕ_{n-1} on A_{n-1} (Figure 7 describes how α_n is defined on each part of S_n). Then define $\phi_n: A_n \rightarrow \tilde{A}_n$ as the unique map (see Figure 8) such that

- $\alpha_n \circ f_{n-2}^0 \circ f_{n-1}^0 = \tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0 \circ \phi_n$ on A_n and
- $\phi_n = \phi_{n-1}$ on ∂V_n^n .

Note that ϕ_n exists because $\tau_i^{-1}f_i^0(\tau_i z)$ is δ -close to $\tilde{f}_i^0(z) = \tilde{\tau}_i^{-1}\tilde{f}_i^0(\tilde{\tau}_i z)$ and $\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)$ is $O(\epsilon + \delta)$ -close to the identity. Recall that

$$g(z) := \tilde{\tau}_i^{-1}\tilde{f}_i^0(\tilde{\tau}_{i+1}z)$$

is a polynomial-like map with a period two attractor, which does not depend on i . In particular, $\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(z) = \tilde{\tau}_{n-2}g^2(\tilde{\tau}_n^{-1}z)$. Denote

$$g_i(z) := \tau_i^{-1}f_i^0(\tau_{i+1}z).$$

Since all renormalizations of f are δ -close to \tilde{f} , Corollary 6.1 remains valid replacing g^2 by $g_{n-2} \circ g_{n-1} = \tau_{n-2}^{-1}f_{n-2}^0 \circ f_{n-1}^0(\tau_n z)$. We claim that

$$\text{dist}(z, \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z))_{D_1} \leq \frac{\epsilon}{\lambda} + C\delta, \text{ on } \tau_{n-2}^{-1}S_n,$$

where $\lambda > 1$ does not depend on n . We will divide S_n into three parts, A_{n-2} , A_{n-1} and $(f_{n-2}^1)^{-1}A_{n-2}$, and prove the claim in each one of them.

The first part is A_{n-2} . Since $D_0 \Subset D_1$, there exists $\lambda_1 > 1$ so that

$$\text{dist}(x, y)_{D_1} \leq \frac{1}{\lambda_1} \text{dist}(x, y)_{D_0}, \text{ if } x, y \in D_0.$$

Because α_n is equal to ϕ_{n-2} on A_{n-2} , then by the inductive hypothesis, we have

$$\text{dist}(z, \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z))_{D_1} \leq \epsilon/\lambda_1$$

on $\tau_{n-2}^{-1}A_{n-2}$.

The second part is A_{n-1} . Recall that $\beta D_0 \Subset D_1$, which implies that there exists $\lambda_2 > 1$ such that

$$\text{dist}(x, y)_{D_1} \leq \frac{1}{\lambda_2} \text{dist}(x, y)_{\beta D_0}, \text{ if } x, y \in \beta D_0.$$

By the inductive hypothesis, we have

$$\text{dist}\left(\frac{z}{\beta_{n-1}}, \tilde{\tau}_{n-1}^{-1}\phi_{n-1}\left(\tau_{n-1}\frac{z}{\beta_{n-1}}\right)\right)_{D_0} \leq \epsilon$$

on $\tau_{n-2}^{-1}A_{n-1}$. Moreover, since $\beta_n - \beta = O(\delta)$, $A_{n-1} \subset \overline{V}_0^{n-1} \setminus (V_0^n \cup V_1^n)$ and

$$\tau_{n-1}^{-1}(\overline{V}_0^{n-1} \setminus (V_0^n \cup V_1^n)) \sim K_0 \setminus (\text{int } K_1 \cup \text{int } K_1') \Subset D_0,$$

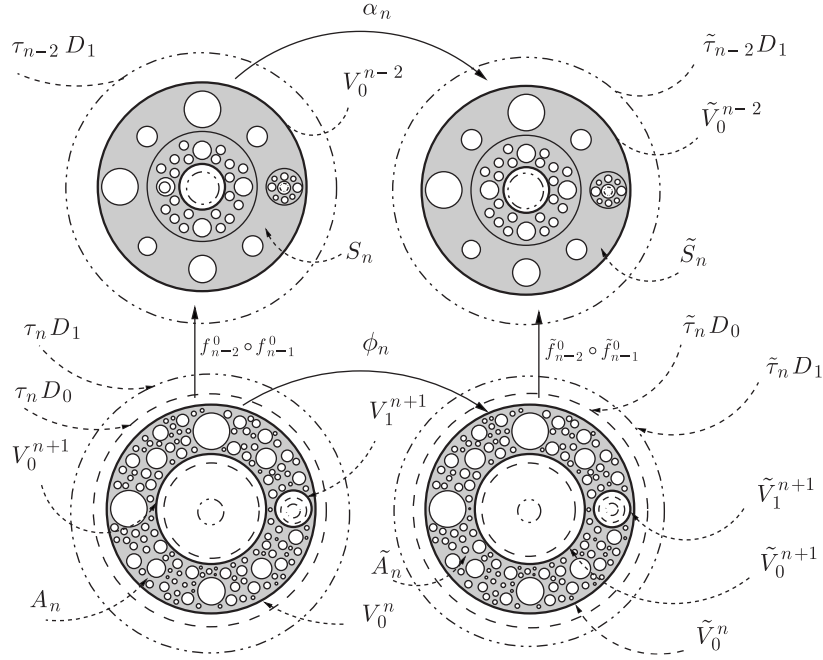


FIGURE 8.

there exists $C > 0$, which does not depend on n , such that, for $z \in \tau_{n-2}^{-1}A_{n-1}$, we have

$$\text{dist}\left(\frac{z}{\beta}, \frac{z}{\beta_{n-1}}\right)_{D_0} \leq C\delta.$$

So we have

$$\begin{aligned} & \text{dist}(z, \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z))_{D_1} \\ &= \text{dist}\left(\beta\frac{z}{\beta}, \beta\tilde{\tau}_{n-1}^{-1}\phi_{n-1}\left(\tau_{n-1}\frac{z}{\beta_{n-1}}\right)\right)_{D_1} \\ &\leq \frac{1}{\lambda_2}\text{dist}\left(\beta\frac{z}{\beta}, \beta\tilde{\tau}_{n-1}^{-1}\phi_{n-1}\left(\tau_{n-1}\frac{z}{\beta_{n-1}}\right)\right)_{\beta D_0} \\ &= \frac{1}{\lambda_2}\text{dist}\left(\frac{z}{\beta}, \tilde{\tau}_{n-1}^{-1}\phi_{n-1}\left(\tau_{n-1}\frac{z}{\beta_{n-1}}\right)\right)_{D_0} \\ &\leq \frac{1}{\lambda_2}\text{dist}\left(\frac{\beta_{n-1}}{\beta}\frac{z}{\beta_{n-1}}, \frac{z}{\beta_{n-1}}\right)_{D_0} + \frac{1}{\lambda_2}\text{dist}\left(\frac{z}{\beta_{n-1}}, \tilde{\tau}_{n-1}^{-1}\phi_{n-1}\left(\tau_{n-1}\frac{z}{\beta_{n-1}}\right)\right)_{D_0} \\ &\leq \frac{\epsilon}{\lambda_2} + C\delta \end{aligned}$$

on $\tau_{n-2}A_{n-1}$.

The last part is $(f_{n-2}^1)^{-1}A_{n-2}$. Because $\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z)$ is very close to $\tilde{f}^1 = \tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^1(\tilde{\tau}_{n-2}z)$, we have $\tau_{n-2}^{-1}(f_{n-2}^1)^{-1}\tau_{n-2}D_0 \sim (\tilde{f}_0^1)^{-1}D_0 \in D_1$, so there exists $\lambda_3 > 1$ such that

$$\text{dist}(x, y)_{D_1} \leq \frac{1}{\lambda_3}\text{dist}(x, y)_{\tau_{n-2}^{-1}(f_{n-2}^1)^{-1}\tau_{n-2}D_0},$$

for $x, y \in \tau_{n-2}^{-1}(f_{n-2}^1)^{-1}\tau_{n-2}D_0$. Here λ_3 does not depend on n . Since

$$\tilde{f}_{n-2}^1 \circ \alpha_n(z) := \phi_{n-2} \circ f_{n-2}^1(z)$$

on $(f_{n-2}^1)^{-1}A_{n-2}$, we get

$$\begin{aligned} & \text{dist}(\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z), \tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)))_{D_0} \\ & \leq \text{dist}(\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z), \tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^1(\tilde{\tau}_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)))_{D_0} \\ & \quad + \text{dist}(\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^1(\tilde{\tau}_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)), \tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)))_{D_0} \\ & \leq \text{dist}(\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z), \tilde{\tau}_{n-2}^{-1}\phi_{n-2}(\tau_{n-2}\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z)))_{D_0} \\ & \quad + \text{dist}(\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^1(\tilde{\tau}_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)), \tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)))_{D_0} \\ & \leq \epsilon + C\delta \end{aligned}$$

on $\tau_{n-2}^{-1}(f_{n-2}^1)^{-1}A_{n-2}$. Because the map

$$\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}\cdot): \tau_{n-2}^{-1}(f_{n-2}^1)^{-1}\tau_{n-2}D_0 \rightarrow D_0$$

is univalent, it preserves the hyperbolic metric, so we get

$$\begin{aligned} & \text{dist}(z, \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z))_{D_1} \\ & \leq \frac{1}{\lambda_3} \text{dist}(z, \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z))_{\tau_{n-2}^{-1}(f_{n-2}^1)^{-1}\tau_{n-2}D_0} \\ & = \frac{1}{\lambda_3} \text{dist}(\tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}z), \tau_{n-2}^{-1}f_{n-2}^1(\tau_{n-2}\tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}z)))_{D_0} \\ & \leq \frac{\epsilon}{\lambda_3} + C\delta, \end{aligned}$$

on $\tau_{n-2}^{-1}(f_{n-2}^1)^{-1}A_{n-2}$. This finishes the proof of the claim (take $\lambda := \min\{\lambda_1, \lambda_2, \lambda_3\}$). Note that

$$\text{dist}(\tau_{n-2}^{-1}f_{n-2}^0 \circ f_{n-1}^0(\tau_n z), \tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n^{-1}z))_{D_1} \leq C\delta$$

on an open set containing $\tau_n^{-1}A_n$ and $\tilde{\tau}_n^{-1}\tilde{A}_n$. So

$$\begin{aligned} & \text{dist}(\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n z), \tilde{\tau}_{n-2}^{-1}\alpha_n(f_{n-2}^0 \circ f_{n-1}^0(\tau_n z)))_{D_0} \\ & \leq \text{dist}(\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n z), \tau_{n-2}^{-1}f_{n-2}^0 \circ f_{n-1}^0(\tau_n z))_{D_0} \\ & \quad + \text{dist}(\tau_{n-2}^{-1}f_{n-2}^0 \circ f_{n-1}^0(\tau_n z), \tilde{\tau}_{n-2}^{-1}\alpha_n(\tau_{n-2}\tau_{n-2}^{-1}f_{n-2}^0 \circ f_{n-1}^0(\tau_n z)))_{D_0} \\ & \leq \frac{\epsilon}{\lambda} + C\delta \end{aligned}$$

on $\tau_n^{-1}A_n$. Thus, since $g^2 = \tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n \cdot): D_0 \rightarrow D_1$ is a covering map (hence it preserves the Riemannian hyperbolic metric), for each $z \in \tau_n^{-1}A_n$, there exists

$$z_0 \in (\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n \cdot))^{-1}(\tilde{\tau}_{n-2}^{-1}\alpha_n \circ f_{n-2}^0 \circ f_{n-1}^0(\tau_n z))$$

so that $\text{dist}(z_0, z)_{D_0} < \epsilon/\lambda + C\delta$. Moreover, since $g^2: D_0 \rightarrow D_1$ is a covering map, there exists $C_1 > 0$ so that

$$\min\{\text{dist}(z_1, z_2)_{D_0} : z_1, z_2 \in g^{-2}(z), z_1 \neq z_2, z \in \tilde{\tau}_{n-2}^{-1}\tilde{S}_n\} > C_1,$$

where C_1 does not depend on n . Now we choose ϵ so that $\epsilon/\lambda + C\delta < \epsilon \ll C_1/3$ (in particular $\epsilon \geq C(1 - 1/\lambda)\delta$), and we conclude that for

$$z_1 \in (\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n \cdot))^{-1}(\tilde{\tau}_{n-2}^{-1}\alpha_n \circ f_{n-2}^0 \circ f_{n-1}^0(\tau_n z)),$$

with $z_1 \neq z_0$, we have $dist(z_1, z)_{D_0} \geq 2C_1/3$. Since

$$\tilde{\tau}_n^{-1}\phi_n(\tau_n z) \in (\tilde{\tau}_{n-2}^{-1}\tilde{f}_{n-2}^0 \circ \tilde{f}_{n-1}^0(\tilde{\tau}_n \cdot))^{-1}(\tilde{\tau}_{n-2}^{-1}\alpha_n \circ f_{n-2}^0 \circ f_{n-1}^0(\tau_n z))$$

and A_n is connected then, either

$$dist(z, \tilde{\tau}_n^{-1}\phi_n(\tau_n z))_{D_0} \leq \epsilon \text{ on } \tau_n^{-1}A_n$$

or

$$dist(z, \tilde{\tau}_n^{-1}\phi_n(\tau_n z))_{D_0} \geq 2C_1/3 \text{ on } \tau_n^{-1}A_n.$$

Since ϕ_n coincides with ϕ_{n-1} on ∂V_n^n , we have $dist(z, \tilde{\tau}_n\phi_n(\tau_n z))_{D_0} < O(\epsilon) \ll 2C_1/3$, for $z \in \tau_n^{-1}\partial V_n^0$, by the inductive hypothesis. This completes the proof. \square

If \mathcal{F} is a family of domains (such as \mathcal{D}_n and \mathcal{E}_n), we denote $\bigcup \mathcal{F} := \bigcup_{D \in \mathcal{F}} D$, $\mathcal{F} \cap D_0 := \{D \in \mathcal{F} : D \subset D_0\}$. The following lemma is easy:

Lemma 9.2. *Let $f: \bigcup_i V_i^1 \rightarrow V_0^0$ be a generalized polynomial-like map whose critical point does not escape and which has a sequence of critical puzzle pieces whose diameters converge to zero. Then there exists a sequence $c_n > 0$ so that, if $D_0 \in \mathcal{D}_n \cap V_0^n$, then there exists an annulus A such that*

- the internal boundary of A is ∂D_0 ;
- $A \subset V_0^n \setminus \overline{\bigcup \mathcal{D}_n}$;
- $mod A \geq c_n$.

Proof of Proposition 9.1. We again use tilde to denote the corresponding objects for f . Using holomorphic motions, it is easy to construct a K -quasiconformal map ϕ_0 in the complex plane, with $K = O(\delta)$, such that $\phi_0(V_0^0) = \tilde{V}_0^0$ and $\phi_0 \circ f = \tilde{f} \circ \phi_0$ on $\partial V_0^1 \cup \partial V_1^1$. Apply Lemma 9.1 to ϕ_0 . We will prove by induction that there exist $K > 0$ and a sequence of injective continuous maps $\phi_n: V_0^0 \rightarrow \tilde{V}_0^0$ such that the following hold.

- ϕ_n coincides on A_n with the map ϕ_n given by Lemma 9.1.
- ϕ_n maps the set of points outside the domain of the first return map of V_0^n to the corresponding set for \tilde{f} . Furthermore

$$\phi_n \circ f = \tilde{f} \circ \phi_n$$

in this set.

- ϕ_n is K -quasiconformal in a neighborhood of every point x satisfying both of the following properties:
 - x does not belong to the domain of the first return map of V_0^n ;
 - x is not in the Julia set.

Note that the points in the boundaries of the domains in \mathcal{D}_n satisfy these properties.

- ϕ_n is equal to ϕ_{n-1} outside the domains of the first return map of V_0^{n-1} .

The pointwise limit of ϕ_n has a continuous extension to the Julia set of f . Moreover, it is a conjugacy between f and \tilde{f} which is K -quasiconformal outside $K(f)$.

Assume by induction that we have constructed ϕ_{n-1} . Let $\{D_j\}$, with $0 \in D_0$, be an enumeration of the subset of domains in \mathcal{D}_{n-1} which are contained in V_0^{n-1} . Enumerate the corresponding family for \tilde{f} in the following way: $\tilde{D}_j := \phi_{n-1}D_j$. Define inductively the family of domains

$$\{D_{i_1, \dots, i_k} : (i_1, \dots, i_k) \in \mathbb{N}_*^k, \text{ for some } k\}$$

($\mathbb{N}_* := \mathbb{N} - \{0\}$) in the following way:

$$D_{i_1, \dots, i_k} := f_{n-1}^{-k}(D_{i_k}) \cap D_{i_1, \dots, i_{k-1}}.$$

Note that

$$(5) \quad \text{mod } D_{i_1, \dots, i_{k-1}} \setminus D_{i_1, \dots, i_{k-1}, i_k} \geq c_{n-1},$$

where c_{n-1} is given in Lemma 9.2. Define $\pi_{i_1, \dots, i_k} : D_{i_1, \dots, i_k} \rightarrow V_0^{n-1}$ by f_{n-1}^k . We are going to define inductively a sequence of continuous injective maps $\psi_n^k : V_0^{n-1} \rightarrow \tilde{V}_0^{n-1}$ in the following way: $\psi_n^0 := \phi_{n-1}$. We define ψ_n^k as equal to ψ_n^{k-1} outside $\bigcup_{(i_1, \dots, i_k) \in \mathbb{N}_*^k} D_{i_1, \dots, i_k}$ and as equal to

$$\tilde{\pi}_{i_1, \dots, i_k}^{-1} \circ \psi_n \circ \pi_{i_1, \dots, i_k}$$

inside D_{i_1, \dots, i_k} . Note that the ψ_n^k are continuous and injective. We claim that the ψ_n^k converge to some injective continuous map $\psi_n^\infty : V_0^{n-1} \rightarrow \tilde{V}_0^{n-1}$. To prove this, we will decompose V_0^{n-1} into three parts, $V_0^{n-1} = P_1 \cup P_2 \cup P_3$.

The set P_1 consists of the points in V_0^{n-1} whose forward orbit enters V_0^n at least once. In other words, $P_1 = \bigcup (\mathcal{E}_n \cap V_0^{n-1})$. The sequence ψ_n^k is eventually locally constant in P_1 (eventually locally constant at a point z means that there exists an open set B , containing z , such that $\psi_n^i = \psi_n^{i+1}$ on B , for large i).

The set P_2 consists of the points in V_0^{n-1} whose forward orbit returns to V_0^{n-1} only a finite number of times and never enters V_0^n . Note that this set includes the boundaries of domains in \mathcal{E}_n which are contained in V_0^{n-1} . In P_2 the sequence ψ_n^k is also eventually locally constant. Moreover, if a point z belongs to $P_2 \setminus K(f)$, then we can conclude that the maps ψ_n^k are K -quasiconformal in a neighborhood of this point, since the map ψ_n^k is defined as a lift of the map ϕ_n by univalent maps, and, by the inductive hypothesis, ϕ_n is K -quasiconformal in a neighborhood of a large iterate of z .

The last set is P_3 . This is the set of points in V_0^{n-1} whose forward orbit returns to V_0^{n-1} infinitely many times but never enters V_0^n . Note that $P_3 \subset K(f)$. We will give a description of P_3 : define $p_{i_1, i_2, \dots}$, with $(i_1, \dots) \in \mathbb{N}_*^{\mathbb{N}}$, as the unique point in

$$\bigcap_k D_{i_1, \dots, i_k}.$$

This set contains a unique point due to equation (5). Then $P_3 = \{p_\sigma : \sigma \in \mathbb{N}_*^{\mathbb{N}}\}$. Note that

$$(6) \quad \psi_n^\ell(D_{i_1, \dots, i_k}) \subset \tilde{D}_{i_1, \dots, i_k}, \text{ for } \ell \text{ large enough.}$$

So, again from equation (5), we have that $\psi_n^k(p_{i_1, i_2, \dots}) \rightarrow_k \tilde{p}_{i_1, i_2, \dots}$.

Thus the pointwise limit of ψ_n^k , denoted by ψ_n^∞ , is well defined at every point in V_0^{n-1} . The continuity at the points in $P_1 \cup P_2$ is obvious. The continuity at the points in P_3 is a consequence of equation (5) and equation (6). We extend the map ψ_n^∞ outside V_0^{n-1} in the following way: if $D \in \mathcal{E}_{n-1}$, $D \neq V_0^{n-1}$, define

$$\psi_{n/D}^\infty := \tilde{\pi}_{n-1/\tilde{D}}^{-1} \circ \psi_n^\infty \circ \pi_{n-1}.$$

Outside \mathcal{E}_{n-1} define $\psi_n^\infty := \phi_{n-1}$. Note that ψ_n^∞ is continuous and injective and satisfies the following properties:

- ψ_n^∞ maps the set of points outside the domain of the first entry map to V_0^n to the corresponding set for \tilde{f} and

$$\psi_n^\infty \circ f = \tilde{f} \circ \psi_n^\infty$$

in this set.

- It is K -quasiconformal in a neighborhood of every point z which satisfies both of the properties
 - z does not belong to the domain of the first entry map to V_0^n ,
 - z does not belong to $K(f)$.

Suppose that $f_{n-1}^0(0) \in D \in \mathcal{E}_n$. Since the maps f_{n-1}^0 and f_{n-1}^1 are very close to \tilde{f}_{n-1}^0 and \tilde{f}_{n-1}^1 and since V_0^{n-1} , V_1^{n-1} and their boundaries are very close of \tilde{V}_0^{n-1} , \tilde{V}_1^{n-1} and their boundaries, the first entry of $f_{n-1}^0(0)$ to V_0^n occurs after applying f_{n-1}^1 once. In particular, D is $(f_{n-1}^0)^{-1}V_0^n$, which is very close to $(\tilde{f}_{n-1}^0)^{-1}\tilde{V}_0^n$. Note that ψ_n^∞ is equal to ψ_n on $\partial(f_{n-1}^0)^{-1}V_0^n$, where ψ_n is the map defined in Lemma 9.1. In particular

$$\text{dist}(z, \tilde{\tau}_{n-1}^{-1}\psi_n^\infty(\tau_{n-1}z)) \leq O(\delta),$$

for $z \in \partial(f_{n-1}^0)^{-1}V_0^n$, which implies that $\tilde{f}_n^0(0) \in \psi_n^\infty((f_{n-1}^0)^{-1}V_0^n)$ (use equation (1) in Proposition 8.1). Since $\tilde{f}_n^0(0) \in (\tilde{f}_{n-1}^0)^{-1}\tilde{V}_0^n$, we have $\psi_n^\infty((f_{n-1}^0)^{-1}V_0^n) = (\tilde{f}_{n-1}^0)^{-1}\tilde{V}_0^n$. Modify ψ_n^∞ only inside V_1^{n+1} to get a new map ψ_n so that $\psi_n^\infty(f_{n-1}^0(0)) = \tilde{f}_{n-1}^0(0)$, but keeping the K -quasiconformality of the map ψ_n in a neighborhood of ∂D . Denote by $\{D_j\} \subset \mathcal{D}_n$, with $0 \in D_0$, the family of connected components of the preimages of the elements of $\mathcal{E}_n \cap V_0^{n-1}$ under the map $f_n^0: V_0^n \rightarrow V_0^{n-1}$. Define $\phi_n: V_0^n \rightarrow \tilde{V}_0^n$ as the unique map such that

- ϕ_n is equal to ψ_n in ∂V_0^n ,
- $\phi_n \circ f_n = \tilde{f}_n \circ \psi_n$ on $V_0^n - \bigcup_i D_i$.

Define $\phi_n := \psi_n$ outside V_0^n . The sequence of maps ϕ_n is eventually locally constant outside $K(f)$. So we obtain a limit map $\phi: \mathbb{C} \setminus K(f) \rightarrow \mathbb{C} \setminus K(\tilde{f})$, with $\phi(\partial V_0^0) = \partial \tilde{V}_0^0$, which is a K -quasiconformal conjugacy. By Proposition 4.2, ϕ has a continuous extension to $K(f)$. □

Proof of Theorem 5. By Proposition 9.1, there exists a map $\phi: \mathbb{C} \rightarrow \mathbb{C}$ which is a conjugacy between f and \tilde{f} , with $\phi(V_0^0) = \tilde{V}_0^0$. Moreover, ϕ is $O(\delta)$ -quasiconformal outside $K(f)$. By Proposition 8.1, the critical pieces in the principal nest of f and \tilde{f} have bounded geometry and satisfy the complex bounds property (see Theorem 1), so by Theorem 1, ϕ is quasiconformal in the complex plane. Because \tilde{f} does not support invariant line fields in $K(\tilde{f})$, $\bar{\partial}\phi = 0$ on $K(\tilde{f})$, and hence ϕ is $O(\delta)$ -quasiconformal in the complex plane. □

Part 3. Hyperbolicity of the Fibonacci cycle

10. INFINITESIMAL THEORY

One of the most elegant and useful tools in Lyubich [Lyu99] and Avila, Lyubich and de Melo [ALdM] is the theory of infinitesimal perturbations of analytic

maps. In this section we give an introduction to this theory, including some minor simplifications and straightforward generalizations that are appropriate in our case.

Let f be a generalized polynomial-like map and \tilde{U} a domain compatible with f . The **horizontal subspace** E_f^h comprises the vectors $v \in T\mathcal{B}_{nor}(\tilde{U})$ (E_f^h depends on \tilde{U} , but we will omit this in the notation) so that there exists a quasiconformal vector field $\alpha: \mathbb{C} \rightarrow \mathbb{C}$, with $\bar{\partial}\alpha = 0$ on $K(f)$, with

$$v = \alpha \circ f - Df \cdot \alpha$$

in a neighborhood of $K(f)$. The horizontal vectors come from quasiconformal deformations of generalized polynomial-like maps. See [Lyu97] and also [Sm02] for details.

The **vertical subspace** E_f^v comprises the vectors $v \in T\mathcal{B}_{nor}(\tilde{U})$ so that there exists a holomorphic vector field α defined in $\bar{\mathbb{C}} - K(f)$, with $\alpha(\infty) = 0$, satisfying

$$(7) \quad v = \alpha \circ f - Df \cdot \alpha$$

in some neighborhood of $K(f)$ in $\bar{\mathbb{C}} - K(f)$. The horizontal and vertical spaces were introduced by M. Lyubich.

If $|\cdot|_{\bar{\mathbb{C}}}$ denotes the Riemannian spherical metric and β is a vector field on $\bar{U} \subset \bar{\mathbb{C}}$, define $|\beta|_{sph} \bar{U} := \sup\{|\beta(x)|_{\bar{\mathbb{C}}}: x \in \bar{U}\}$.

Proposition 10.1 ([Lyu99]). *Let $f \in \mathcal{B}_{nor}(U)$ such that f has a generalized polynomial-like extension compatible with U . Let $V \Subset U$ be a neighborhood of $K(f)$ so that $f^{-1}(V) \Subset V$. Then there exists C such that, for any function g close to f so that 0 does not escape from V and for $v = \alpha \circ g - Dg \cdot \alpha \in E_g^v$, we have*

$$|\alpha|_{sph} \bar{\mathbb{C}} \setminus V \leq C|v|_{\mathcal{B}(V)}.$$

Corollary 10.1. *The vertical space E_f^v is finite-dimensional in any Banach slice compatible with f . Furthermore $\dim E_f^v$ does not depend on the Banach slice.*

Proof. It is sufficient to prove that the unit ball on E_f^v is compact. Indeed, let $v_n = \alpha_n \circ f - Df \cdot \alpha_n \in E_f^v$, with $|v|_{\mathcal{B}(U)} \leq 1$. Let V be as in Proposition 10.1 and define $V_i := f^{-i}V$. Then there exists a sequence C_i so that

$$|\alpha_n|_{sph} \bar{\mathbb{C}} \setminus V_i \leq C_i.$$

In particular, there is a subsequence n_k and a conformal vector field α_∞ on $\bar{\mathbb{C}} \setminus K(f)$ so that $\alpha_i \rightarrow \alpha_\infty$, where the convergence is uniform on compact sets in $\bar{\mathbb{C}} \setminus K(f)$. Furthermore, we can assume that v_{n_k} converges uniformly to a conformal vector field v_∞ in a neighborhood of $K(f)$ (for instance, in V_1). So we have

$$v_\infty = \alpha_\infty \circ f - Df \cdot \alpha_\infty$$

on $V_1 \setminus K(f)$. Consider \tilde{U} , $U \Subset \tilde{U}$, so that f has an analytical extension to \tilde{U} . We use equation (7) to extend v_i and v_∞ to \tilde{U} . By the maximum principle and Proposition 10.1,

$$|v_{n_k} - v_\infty|_{\mathcal{B}(\tilde{U})} \leq C_1 |\alpha_{n_k} - \alpha_\infty|_{sph} \bar{\mathbb{C}} \setminus \tilde{U} \leq C_1 |\alpha_{n_k} - \alpha_\infty|_{sph} \bar{\mathbb{C}} \setminus V_1 \leq C_2 |v_{n_k} - v_\infty|_{\mathcal{B}(V_1)},$$

which finishes the proof: the invariance of the dimension is an immediate consequence of equation (7), since any vertical vector defined in a domain U has an analytical extension to a larger domain \tilde{U} . \square

Let f be a generalized polynomial-like map and let v be a domain compatible with f . Denote $P(f) := \overline{\{f^i(0) : i > 0\}}$. If $v \in T\mathcal{B}(U)$, we say that a quasiconformal vector field α is (f, v) -**equivariant** on $P(f)$ if $v(0) = \alpha(f(0))$ and

$$\alpha(x_0) = \lim_{x \rightarrow x_0} \frac{v(x) - \alpha(f(x))}{Df(x)},$$

for $x_0 \in P(f)$. The concept of equivariance was introduced in Avila, Lyubich and de Melo [ALdM]. The right side of the equation is well defined at the critical point (the unique problematic point) because $v(0) = \alpha(f(0))$ and α is quasiconformal: see [ALdM]. Indeed, we always have $\lim_{x \rightarrow 0} (v(x) - \alpha(f(x)))/Df(x) = 0$. In particular, if $0 \in P(f)$ and α is (f, v) -equivariant on $P(f)$, then $\alpha(0) = 0$.

The Sullivan pullback argument has an infinitesimal counterpart:

Proposition 10.2 (Infinitesimal pullback argument [ALdM]). *Let f be a generalized polynomial-like map compatible with the domain U , whose critical point does not escape. Let $V \Subset U$ be such that $f : f^{-1}V \rightarrow V$ is a representation of f . Then there exist $\epsilon > 0$ and $C > 0$ so that, if $|g - f|_{\mathcal{B}(U)} \leq \epsilon$ and $g : g^{-1}V \rightarrow V$ is a generalized polynomial-like map whose critical point does not escape, then, for any vector $v \in T\mathcal{B}_{nor}$ that admits a (g, v) -equivariant quasiconformal vector field in $P(f)$, there exists a quasiconformal vector field $\alpha : \overline{\mathcal{C}} \rightarrow \overline{\mathcal{C}}$, $\alpha(\infty) = \alpha(1) = 0$, so that $v = \alpha \circ g - Dg \cdot \alpha$ on \overline{V} and*

$$|\overline{\partial}\alpha|_{V \setminus K(g)} \leq C|v|_{\mathcal{B}(U)}.$$

Corollary 10.2 (Semicontinuity of the horizontal subspaces [ALdM]). *Let f, f_n be generalized polynomial-like maps which are compatible with the Banach slice $\mathcal{B}_{nor}(U)$ and so that $f_n \rightarrow f$ in $\mathcal{B}_{nor}(U)$. Assume that f does not support invariant line fields in $K(g)$. If $v_n \in E_{f_n}^h \subset T\mathcal{B}(U)$ and $v_n \rightarrow v$ in $T\mathcal{B}_{nor}(U)$, then $v \in E_f^h$. In particular E_f^h is closed.*

The infinitesimal version of the Douady and Hubbard decomposition of the dynamics of a polynomial-like map in the hybrid class and its external map is the following:

Proposition 10.3 (Lyubich Decomposition [Lyu99]). *In a Banach slice $\mathcal{B}_{nor}(U)$ compatible with f , $T\mathcal{B}_{nor}(U) = E_f^h \oplus E_f^v$.*

M. Lyubich stated the above decomposition for quadratic polynomial-like maps (connected domain), but the proof can be carried out in our case as well.

The proof of Corollary 10.1 does not tell us anything about the finite dimension of E_f^v . The following result will be very useful to solve this problem. The proof of it is a variation of a result in [Lyu99].

Proposition 10.4. *Let f be a generalized polynomial with non-escaping critical points. Assume there are no invariant line fields on $K(f)$ and let U be a domain compatible with f . Then there exists $\epsilon > 0$ so that, for any generalized polynomial-like map g , with non-escaping critical points and ϵ -close to f on $\mathcal{B}_{nor}(U)$, we have $\dim E_f^v = \dim E_g^v$.*

Proof. Firstly, suppose that there exists $f_n \rightarrow f$ on $\mathcal{B}_{nor}(\tilde{U})$, with $\dim E_f^v > \dim E_{f_n}^v$. Then, for every n we can find $v_n \in T\mathcal{B}_{nor}(U)$ so that $v_n \in E_f^v \cap E_{f_n}^h$ and $|v_n|_{\mathcal{B}(U)} = 1$. Since E_f^v is finite dimensional, we can assume that $v_n \rightarrow v \in E_f^v$,

$v \neq 0$. But, by the semi-continuity of the horizontal spaces, $v \in E_f^h$, which is a contradiction. To finish the proof, it is sufficient to prove that, if there exists a sequence of maps $f_n \rightarrow f$ so that $\dim E_{f_n}^v \geq N$, then $\dim E_f^v \geq N$. Indeed, by a classical lemma by Riesz, there exist “almost orthonormal” families of vectors $\{v_n^1, \dots, v_n^N\} \subset E_{f_n}^v$. In other words, $|v_n^i|_{\mathcal{B}(U)} = 1$ and, if we denote

$$\text{span}\{u_1, \dots, u_N\} := \left\{ \sum c_i u_i : c_i \in \mathbb{C} \right\},$$

we have

$$\text{dist}(v_n^i, \text{span}\{v_n^1, \dots, v_n^{i-1}\})_{\mathcal{B}(U)} \geq \frac{1}{2},$$

for $1 < i \leq N$. Let $\hat{U} \Subset U$. Since $|v_n^i|_{\mathcal{B}(U)} = 1$, we can assume that $v_n^i \rightarrow v_\infty^i$ on $T\mathcal{B}_{\text{nor}}(\hat{u})$, for some vectors v_∞^i . By Proposition 10.1,

- $v_\infty^i \in E_f^v \setminus \{0\}$ and
- $\text{dist}(v_\infty^i, \text{span}\{v_\infty^1, \dots, v_\infty^{i-1}\})_{\mathcal{B}(U)} \geq C > 0$, for $1 < i \leq N$.

So $\dim E_f^v \geq N$. □

Denote $\mathcal{O}(x) := \{f^i(x) : i \geq 0\}$. The following is a slight variation of a similar result for maps with strictly preperiodic critical point in [ALdM]:

Proposition 10.5. *Let f be a generalized polynomial-like map with a unique critical point, which is N -periodic. Then, if U is compatible with f , then $v \in E_f^h$ if and only if there is a vector $\alpha : \mathcal{O}(0) \rightarrow \mathbb{C}$ so that $v = \alpha \circ f - Df \cdot \alpha$, with $\alpha(0) = 0$. In particular, $\text{codim } E_f^h = 1$.*

Proof. It is obvious that the assumptions under α are necessary. We will prove that they are sufficient. Replace α by a quasiconformal vector field on $\overline{\mathbb{C}}$ that satisfies the same hypothesis. Then α is equivariant with respect to (f, v) on $P(f)$. We can also assume that α is analytic in a neighborhood of $\mathcal{O}(0)$. It is easy to see in the proof of the infinitesimal pullback argument [ALdM] that there exists a quasiconformal vector field $\alpha : \mathbb{C} \rightarrow \mathbb{C}$, analytic in the interior of $K(f)$, so that $v = \alpha \circ f - Df \cdot \alpha$ on a neighborhood of $K(f)$. Since $m(J(f)) = 0$, $v \in E_f^h$. To prove that $\text{codim } E_f^h = 1$, denote $v_i = v(f^i(0))$ and $\alpha_i = \alpha(f^i(0))$. Then

$$v_i = \alpha_{i+1 \bmod n} - Df(f^i(0)) \cdot \alpha_i.$$

Since the determinant of this system of equations is 1, for each N -tuple (v_0, \dots, v_{N-1}) there exists a unique solution $\alpha_j(v_0, \dots, v_{N-1})$. By the above characterization of E_f^h , the horizontal space is the kernel of the non-zero functional

$$v \rightarrow \alpha_0(v(0), \dots, v(f^{N-1}(0))).$$

□

Corollary 10.3. *Let f be a generalized polynomial-like map with a unique critical point; assume that there are no invariant line fields on $K(f)$ and let U be a domain compatible with f . Assume that there exists a sequence of generalized polynomial-like maps with representations in $\mathcal{B}_{\text{nor}}(U)$ so that f_n has a superattractor and $f_n \rightarrow f$ on $\mathcal{B}_{\text{nor}}(U)$. Then $\dim E_f^v = 1$.*

Proof. Propositions 10.4 and 10.5. □

A generalized polynomial-like map $f_1: \bigcup_i U_1 \rightarrow V_1$ is **hybrid conjugate** to another $f_2: \bigcup_i U_2 \rightarrow V_2$ if there exists a quasiconformal map $h: \mathbb{C} \rightarrow \mathbb{C}$ so that

- $h(V_1) = V_2$,
- $h \circ f_1 = f_2 \circ h$ on V_1 ,
- $\bar{\partial}h$ vanishes at $K(f_1)$.

The **hybrid class** of f_1 is the set of generalized polynomial-like maps which are hybrid conjugate to f_1 .

Corollary 10.4. *Suppose that f_1 has a unique critical point, which does not escape, and $K(f_1)$ does not support invariant line fields. Assume that f_2 is a generalized polynomial-like map which is hybrid conjugate to f_1 . Then the codimension of $E_{f_2}^h$ in an arbitrary Banach slice compatible with f_2 is equal to the codimension of $E_{f_1}^h$.*

Proof. Indeed, let h be a quasiconformal homeomorphism between two generalized polynomial-like maps f_a and f_b in the same hybrid class. So h induces a Beltrami path $h_\lambda \circ f_a \circ h_\lambda^{-1}$ ($h_0 = Id$ and $h_1 = h$). Given two Banach slices $\mathcal{B}(U_a)$ and $\mathcal{B}(U_b)$ compatible with f_a and f_b , we can find, using the compactness of $[0, 1]$ and Proposition 10.4, a sequence of Banach slices $U_a = \hat{U}_0, \dots, \hat{U}_n = U_b$ and closed intervals $I_i \subset [0, 1]$, with $0 \in I_0$ and $1 \in I_n$, $I_i \cap I_{i+1} \neq \emptyset$, so that

- for $\lambda \in I_i$, f_λ is compatible with $\mathcal{B}_{nor}(\hat{U}_{i-1})$, $\mathcal{B}_{nor}(\hat{U}_i)$ and $\mathcal{B}_{nor}(\hat{U}_{i+1})$;
- $codim E_{f_{\lambda_0}}^h = codim E_{f_{\lambda_1}}^h$ on $\mathcal{B}_{nor}(\hat{U}_i)$, for $\lambda_0, \lambda_1 \in I_i$.

This completes the proof. □

11. INDUCED MAPS

In this section we study induced transformations in quite a general setting: induced transformations include the classical renormalization operator and the Fibonacci renormalization operator.

Let f be a generalized polynomial-like map, let $\mathcal{B}(U)$ be a Banach slice compatible with f and let β_g be a non-zero analytical function defined in a neighborhood of f in $\mathcal{B}(U)$ (β_g will be called the **normalization function** or simply ‘normalization’). Consider simply connected domains U_1, \dots, U_n and $j_1, \dots, j_n \in \mathbb{N}$ so that

- the closures of U_i are disjoint and
- the sets $\bar{U}_i, \dots, f^{j_i}(\bar{U}_i)$ are contained in U .

Define $\hat{U}_i = 1/\beta_f U_i$. Then we define a map on a neighborhood of f in $\mathcal{B}(U)$ in the Banach slice $\mathcal{B}(\hat{U}_1 \cup \dots \cup \hat{U}_n)$ in the following way: for each g in a small neighborhood of f in $\mathcal{B}(U)$ define $\mathcal{R}g \in \mathcal{B}(\hat{U}_1 \cup \dots \cup \hat{U}_n)$ as

$$\mathcal{R}g(x) = \frac{1}{\beta_g} g^{j_i}(\beta_g x), \text{ for } x \in \hat{U}_i.$$

The map \mathcal{R} is called the **(normalized) induced transformation** associated with the $2n$ -tuple $(U_1, j_1, \dots, U_n, j_n)$ and the normalization β . The set $U_1 \cup \dots \cup U_n$ is the **support** of the induced transformation.

Let $v \in \mathcal{B}(U)$ be a vector. Denote by a_n the following functions:

$$a_1(x) := v(x),$$

$$a_n(x) := v(f^{n-1}(x)) + Df(f^{n-1}(x))a_{n-1}(x).$$

Then (replacing β_f by β to simplify the notation)

$$\begin{aligned} (D\mathcal{R}_f \cdot v)(x) &= \frac{\partial}{\partial t}_{t=0} \frac{1}{\beta_{f+tv}} (f + tv)^n (\beta_{f+tv} x) \\ &= \frac{-D\beta \cdot v}{\beta} \frac{1}{\beta} f^n(\beta x) + \frac{1}{\beta} (a_n(\beta x) + Df^n(\beta x)(D\beta \cdot v)x). \end{aligned}$$

This is an ugly expression, but in the horizontal space we have a much nicer description of the action of $D\mathcal{R}$:

Proposition 11.1. *Let \mathcal{R} be a normalized induced transformation with normalization β_f . If $v = \alpha \circ f - Df \cdot \alpha \in E_f^h$, then*

$$D\mathcal{R}_f \cdot v = r(\alpha) \circ \mathcal{R}f - D(\mathcal{R}f) \cdot r(\alpha),$$

where

$$r(\alpha) := \frac{1}{\beta_f} \alpha(\beta_f x) - \frac{1}{\beta_f} (D\beta_f \cdot v) \cdot x.$$

Furthermore, if β_f is the analytic continuation of a non-parabolic period point, then

$$r(\alpha) := \frac{1}{\beta_f} \alpha(\beta_f x) - \frac{1}{\beta_f} \alpha(\beta_f) \cdot x.$$

Proof. Assume $v = \alpha \circ f - Df \cdot \alpha$. Substituting this expression in the definition of a_n , we obtain by induction that, for $n > 1$,

$$a_n = \alpha \circ f^n - Df^n \cdot \alpha.$$

It follows that $D\mathcal{R} \cdot v$ can be rewritten as $D\mathcal{R}_f \cdot v = r(\alpha) \circ \mathcal{R}f - D(\mathcal{R}f) \cdot r(\alpha)$, where

$$r(\alpha)(x) := \frac{1}{\beta} \alpha(\beta x) - \frac{1}{\beta} (D\beta \cdot v)x.$$

If we choose β_f to be the analytic continuation of a non-parabolic periodic point of f_0 with period i , then we have $f^i(\beta_f) = \beta_f$, so

$$D\beta_f \cdot v = a_i(\beta) + Df^i(\beta)D\beta_f \cdot v = \alpha(\beta) - Df^i(\beta)\alpha(\beta) + Df^i(\beta)D\beta_f \cdot v;$$

hence $D\beta_f \cdot v = \alpha(\beta)$ and

$$r(\alpha)(x) := \frac{1}{\beta} \alpha(\beta x) - \frac{1}{\beta} \alpha(\beta)x.$$

In particular, $v(1) = r(1) = 0$. □

All items in the proposition below, except the last one, are obvious generalizations of similar statements for the classic renormalization operator in de Melo and van Strien’s book [dMvS], Lyubich [Lyu99] and Avila, Lyubich and de Melo [ALdM] (the second statement in Proposition 11.2 appears in Remark 7.2 in [ALdM]). The third statement in Proposition 11.2 is a consequence of Proposition 11.1. The statement about the independence of induced transformations will be useful only when we are dealing with generalized polynomial-like maps with many critical points, and it will be not used here. Its proof is a straightforward generalization of the proof of the density of the image of the derivative of the classic renormalization operator [ALdM].

Proposition 11.2. *Let \mathcal{R} be a normalized induced transformation associated with the $2n$ -tuple $(U_1, j_1, \dots, U_n, j_n)$ defined in a neighborhood of f in a compatible Banach slice $\mathcal{B}_{nor}(U)$ in $\mathcal{B}_{nor}(U_1 \cup \dots \cup U_n)$. Then we have the following:*

- (1) **Compactness.** *The induced transformation is a compact transformation.*
- (2) **Derivative with dense image.** *The derivative DR is a compact linear transformation with dense image.*
- (3) **Invariance of horizontal subspaces.** *Assume that f and $\mathcal{R}f$ have generalized polynomial-like extensions so that $\mathcal{B}_{nor}(U)$ and $\mathcal{B}_{nor}(U_1 \cup \dots \cup U_n)$ are compatible with them. Then $DR_f(E_f^h) \subset E_g^h$.*
- (4) **Non-singularity in the vertical directions.** *Under the conditions of the previous item, assume that $DR^{-1}E_g^h \subset E_f^h$. Then*

$$\tilde{D}\mathcal{R}_f: T\mathcal{B}_{nor}(U)/E_f^h \rightarrow T\mathcal{B}_{nor}(U_1 \cup \dots \cup U_n)/E^h \tilde{g}$$

is invertible.

Moreover we have the following:

- **Independence between induced transformations.** *If the \mathcal{R}_i , $i \leq i_0$, are induced transformations associated with the $2n$ -tuples*

$$(U_1^i, J_1^i, \dots, U_n^i, J_n^i),$$

where $U_j^i \subset U$ and if the \mathcal{R}_i have two-by-two disjoint supports, then

$$\{(DR_1(f) \cdot v, \dots, DR_i(f) \cdot v, \dots, DR_{i_0}(f) \cdot v) : v \in \mathcal{B}(U)\}$$

is dense in $T\mathcal{B}_{nor}(U_1^1 \cup \dots \cup U_{n_1}^1) \oplus \dots \oplus T\mathcal{B}_{nor}(U_1^i \cup \dots \cup U_{n_i}^i) \oplus \dots \oplus T\mathcal{B}_{nor}(U_1^{i_0} \cup \dots \cup U_{n_{i_0}}^{i_0})$.

12. CONTRACTION ON THE HYBRID CLASS

Let $\tilde{f}_i: \tilde{V}_0^1 \cup \tilde{V}_1^1 \rightarrow \tilde{V}_0^0$, $i = 1, 2$, be the generalized polynomial-like extensions of \tilde{f}_i , as previously defined. Denote by \mathcal{T}_{Fib} , called the **Fibonacci tower** [vSN94], the indexed family of generalized polynomial-like maps $\mathcal{R}^i \tilde{f}_1: \beta^i \tilde{V}_0^1 \cup \beta^i \tilde{V}_1^1 \rightarrow \beta^i \tilde{V}_0^0$, with $i \in \mathbb{Z}$, where either $\mathcal{R}^i \tilde{f}_1(z) := 1/\beta^i \tilde{f}_1(\beta^i \cdot z)$, if i is even, or $\mathcal{R}^i \tilde{f}_1(z) := 1/\beta^i \tilde{f}_2(\beta^i \cdot z)$, if i is odd. This notation is consistent with the previous definitions of the Fibonacci operator \mathcal{R} , since this operator is injective. Let μ be a Beltrami field in \mathbb{C} . We say that μ is invariant by the tower \mathcal{T} if

$$(\mathcal{R}^i \tilde{f}_1)_* \mu = \mu,$$

for $i \in \mathbb{Z}$. We say that the Fibonacci tower \mathcal{T}_{Fib} does not support invariant line fields if every Beltrami field which is invariant by \mathcal{T}_{Fib} vanishes almost everywhere in \mathbb{C} .

Proposition 12.1 ([vSN94]). *The Fibonacci tower \mathcal{T}_{Fib} does not support invariant line fields.*

The following result can be proved using arguments as in van Strien and Nowicki [vSN94]:

Proposition 12.2 (Topological convergence [vSN94]). *Let $f: V_0^1 \cup V_1^1 \rightarrow V_0^0$ be a generalized polynomial-like map which admits a K -quasiconformal map $h: \mathbb{C} \rightarrow \mathbb{C}$ which is a conjugacy between f and \tilde{f} , with $h(V_0^0) = \tilde{V}_0^0$. Then there exists $N = N(K)$ so that the restriction $f_n: V_0^{n+1} \cup V_1^{n+1} \rightarrow V_0^n$ of the first return map to V_0^n (recall the notation in Section 5.3) satisfies the following: if $\tau_n := h^{-1}(\beta^n)$, then*

$$\frac{1}{\tau_n} f_n(\tau_n \cdot z)$$

has an analytical extension to $\mathcal{B}_{nor}(U)$, provided that $n \geq N(f)$, and

$$|\tau_n^{-1}f_n(\tau_n \cdot z) - \tilde{f}|_{\mathcal{B}(U)} \rightarrow 0.$$

The main result in this section is that the rate of convergence is indeed exponential. We will use the method we introduced in [Sm02]. Let $\mathcal{A}(K, V, \epsilon)$ be the set of generalized polynomial-like maps f in $\mathcal{B}_{nor}(U)$ so that

- $\mathcal{R}^n f$ is defined for $n \geq 0$ and

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \epsilon;$$

- there exists a K -quasiconformal map $\phi: \mathbb{C} \rightarrow \mathbb{C}$ so that $\phi(V) \subset \bar{U}$ and $\phi \circ \tilde{f} = f \circ \phi$ on V .

Note that the (closed) set $\mathcal{A}(K, V_0^0, \epsilon)$ is invariant by the action of \mathcal{R} , if ϵ is small enough. Moreover $\overline{\mathcal{R}\mathcal{A}(K, V_0^0, \epsilon)}$ is a compact set.

Corollary 12.1. *Given ϵ small enough, there exists $K_\epsilon = O(\epsilon)$ so that if $f: V_0^1 \cup V_1^1 \rightarrow V_0^0$ is a generalized polynomial-like map which admits a \tilde{K} -quasiconformal map $h: \mathbb{C} \rightarrow \mathbb{C}$ which is a conjugacy between f and \tilde{f} , with $h(V_0^0) = \tilde{V}_0^0$, then, for $n \geq N(\tilde{K})$, $\mathcal{R}^n f \in \mathcal{B}_{nor}(U)$ and*

$$\mathcal{R}^n f \in \mathcal{A}(K_\epsilon, V_0^0, \epsilon).$$

Proof. Given ϵ , by Theorem 5, there exists $K = O(\epsilon)$ so that if f satisfies

$$|\mathcal{R}^n f - \tilde{f}| \leq \epsilon$$

for $n \geq 0$, then there exists a hybrid conjugacy between $f: V_0^1 \cup V_1^1 \rightarrow V_0^0$ and $\tilde{f}: \tilde{V}_0^1 \cup \tilde{V}_1^1 \rightarrow \tilde{V}_0^0$ which is K -quasiconformal in the complex plane. Proposition 12.2 completes the proof. \square

Remark 12.1. The proof of a result similar to Corollary 12.1 is easier in the classic renormalization theory for quadratic polynomials: we do not know if it is possible to prove it without Proposition 9.1. The problem is that there is not a canonical straightening for Fibonacci generalized polynomial-like maps.

Theorem 6 (Contraction in the horizontal direction). *Given $\mathcal{A}(K, V_0^0, \epsilon)$, with ϵ small enough, there exist $\lambda < 1$ and $N \in \mathbb{N}$ so that if $v \in E_f^h$ and $f \in \mathcal{A}(K, V, \epsilon)$, then*

$$|D\mathcal{R}_f^N \cdot v|_{\mathcal{B}(U)} \leq \lambda|v|_{\mathcal{B}(U)}.$$

Proof. The proof is similar to the proof of Proposition 3.9 in [Sm02]. Consider $f \in \mathcal{A} := \mathcal{A}(K, V_0^0, \epsilon)$ and $v \in E_f^h$, $|v|_{\mathcal{B}(U)} \leq 1$. Then, by the infinitesimal pullback argument, there exists a quasiconformal vector field $\alpha: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ so that $v = \alpha \circ f - Df \cdot \alpha$ on $\bar{V}_0^1 \cup \bar{V}_1^1$, with $|\bar{\partial}\alpha|_\infty \leq C$. This implies, by the definition of the Fibonacci renormalization operator, Proposition 10.2 and Proposition 11.1, that, for $n \geq 1$,

$$(8) \quad D\mathcal{R}_f^n \cdot v = \alpha_n \circ \mathcal{R}^n f - D(\mathcal{R}^n f) \cdot \alpha_n,$$

on $U = U_1 \cup U_2$, where

$$\alpha_n(z) := \frac{1}{\beta_{\mathcal{R}^{n-1}f} \cdots \beta_f} \alpha(\beta_{\mathcal{R}^{n-1}f} \cdots \beta_f \cdot z) - \frac{1}{\beta_{\mathcal{R}^{n-1}f} \cdots \beta_f} \alpha(\beta_{\mathcal{R}^{n-1}f} \cdots \beta_f) \cdot z.$$

Note that $|\bar{\partial}\alpha_n|_\infty = |\bar{\partial}\alpha|_\infty \leq C$ on $\bar{\mathbb{C}}$, for $n \geq 1$. Recall that C does not depend on f or v . Since the set of C -quasiconformal vectors which vanish at three points (in our case, 0, 1 and ∞) in the Riemann sphere is a compact set relative to the sup

norm induced by the spherical metric in the Riemann sphere, there exists $C > 0$ so that

$$|D\mathcal{R}_f^n \cdot v|_{\mathcal{B}(U)} \leq C$$

for every $f \in \mathcal{A}(K, V, \epsilon)$, $v \in E_f^h$, $|v| \leq 1$. Since \mathcal{R} is a compact operator, we have that

$$S := \overline{\{(\mathcal{R}f, D\mathcal{R}_f \cdot v) : f \in \mathcal{A}, v \in E_f^h, |v|_{\mathcal{B}(U)} \leq 1\}}$$

is a compact set. Furthermore, since the horizontal subspaces are semi-continuous (Corollary 10.2), we get $S \subset \{(f, v) : f \in \mathcal{A}, v \in E_f^h\}$.

We claim that, for any $v \in E_f^h$, $f \in \mathcal{A}$, we have

$$|D\mathcal{R}_f^n \cdot v|_{\mathcal{B}(U)} \rightarrow 0.$$

It is sufficient to prove that $\alpha_n \rightarrow 0$. Take any convergent subsequence $\alpha_{n_k} \rightarrow \alpha_\infty$. In particular, $\bar{\partial}\alpha_{n_k}$ converges to $\bar{\partial}\alpha_\infty$ in the distributional sense. Denote $\beta_i := \beta_{\mathcal{R}^i f}$. Note that the Beltrami field $\bar{\partial}\alpha_n$ is invariant by the maps

$$\mathcal{R}^n f, \frac{1}{\beta_{n-1}} \mathcal{R}^{n-1} f(\beta_{n-1} \cdot), \dots, \frac{1}{\beta_{n-1} \cdots \beta_0} f(\beta_{n-1} \cdots \beta_0 \cdot).$$

Since $\mathcal{R}^i f \rightarrow_i \tilde{f}$, this sequence of maps converges to the Fibonacci tower, which implies that $\bar{\partial}\alpha_\infty$ is invariant by the Fibonacci tower. Since the Fibonacci tower does not admit invariant line fields, α_∞ is a conformal vector field in the Riemann sphere. So α_∞ is zero (because α_∞ is zero at three points of the Riemann sphere: 0, 1 and ∞). Fix δ small. Then, for every $(f, v) \in S$, there exists $N_{(f,v)}$ so that

$$|D\mathcal{R}_f^n \cdot v|_{\mathcal{B}(U)} \leq \delta,$$

for $n \geq N_{(f,v)}$. Since $|D\mathcal{R}_f^n| \leq C$, we can find a neighborhood of $B_{(f,v)}$ of (f, v) in S so that

$$|D\mathcal{R}_g^n \cdot u|_{\mathcal{B}_{nor}(U)} \leq 2C\delta,$$

for $(g, u) \in B_{(f,v)}$ and $n \geq N_{(f,v)}$. Since S is compact, this finishes the proof. \square

Corollary 12.2 (Contraction on the hybrid class). *There exists $\lambda < 1$ such that, if f is a generalized polynomial-like map which is quasiconformally conjugate to one of the maps in the Fibonacci cycle, then, for $n > N = N(f)$, the n th generalized renormalization of f has a representation in $\mathcal{B}_{nor}(U)$ and, furthermore,*

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \lambda^n.$$

Proof. Let $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be a quasiconformal map which is a conjugacy between f and \tilde{f} in a neighborhood of $J(\tilde{f})$. Consider the Beltrami path induced by ϕ : in other words, $f_t := \phi_t \circ \tilde{f} \circ \phi_t^{-1}$, $|t| \leq 1$, where

$$\frac{\bar{\partial}}{\partial} \phi_t = t \cdot \frac{\bar{\partial}}{\partial} \phi.$$

By the topological convergence of $\mathcal{R}^n f$ to \tilde{f} (see [vSN94]), given ϵ , there exists $N = N(f, \epsilon)$ so that $\mathcal{R}^n f_t \in \mathcal{B}_{nor}(U)$ and

$$|\mathcal{R}^n f_t - \tilde{f}|_{\mathcal{B}(U)} \leq \epsilon,$$

for $|t| \leq 1$. In particular, by Proposition 9.1 and Theorem 1, there exists $K = K(\epsilon)$ and a definitive neighborhood V on $J(\tilde{f})$ so that $\mathcal{R}^n f_t \in \mathcal{A}(K_\epsilon, V_0^0, \epsilon)$, for $n > N_f$

and $|t| \leq 1$. Since

$$\frac{d\mathcal{R}^n f_t}{dt} \in E_{\mathcal{R}^n f_t}^h,$$

we can use Theorem 6 to finish the proof. □

The above argument also works for the classic renormalization operator [Lyu99] (even for non-periodic combinatorics); the multimodal renormalization operator [Sm01]; and possibly in any other situation in complex dynamics where the McMullen arguments with towers can be applied, such as the renormalization theory for critical homeomorphisms or coverings of the circle.

13. HYPERBOLICITY

Proposition 13.1. *There exists a sequence of generalized polynomial-like maps $f_{n,i}$, $i = 1, 2$, real in the real line, such that*

- *the map $f_{n,i}$ has a superattractor with period S_n (the Fibonacci sequence);*
- *the sequence of closest returns of the critical point is exactly $f_{n,i}^{S_i}(0)$, $i < n$;*
- *the maps $f_{n,i}$ are compatible with U and $f_{n,i} \rightarrow \tilde{f}_i$.*

In particular, $\text{codim } E_{\tilde{f}}^h = 1$.

Proof. Consider the Fibonacci parameter c_∞ in the family $x^d + c$. Select a generalized polynomial-like extension $f_{n_0}^{c_\infty} : \hat{U}_1 \cup \hat{U}_2 \rightarrow \hat{V}$ of a generalized renormalization of $x^d + c_\infty$, which has the Fibonacci combinatorics. There exists a unique sequence of parameters c_n such that

- $x^d + c_n$ has a critical point with period S_n (the Fibonacci sequence);
- the sequence of the closest return times of the critical point is S_i , $i < n$.

Because any accumulation point of the sequence c_n is a parameter with Fibonacci combinatorics, we have $c_n \rightarrow c_\infty$, since, by the bounded geometry of the postcritical set, the non-existence of invariant line fields on the Julia set of Fibonacci polynomials and the Sullivan pullback argument, c_∞ is the unique such parameter (see [vSN94]). It is easy to see that $P(x^d + c_n) \rightarrow P(x^d + c_\infty)$. In particular, the n_0 -th generalized renormalization of $x^d + c_n$, denoted $f_{n_0}^{c_n}$, also has a polynomial-like extension with, say, the same image \hat{V} and domains \hat{U}_0^n and \hat{U}_1^n , where these domains are very close to \hat{U}_0 and \hat{U}_1 . This is sufficient to conclude that $\text{codim } E_{\tilde{f}}^h = 1$, since f_{n_0} is in the hybrid class of \tilde{f}_1 (or \tilde{f}_2). Now we could use the dynamics of the Fibonacci renormalization to conclude the proof, but we will use a more general argument: just to fix the notation, suppose that f_{n_0} is hybrid conjugate to \tilde{f}_1 . Then select a generalized polynomial-like extension $f_1 : \tilde{U}_0 \cup \tilde{U}_1 \rightarrow \tilde{V}$ which has a representation in U . Then there exist K -quasiconformal maps, symmetric with respect to the real line,

$$\phi_n : \mathbb{C} - \hat{U}_0^n \cup \hat{U}_1^n \rightarrow \mathbb{C} - \tilde{U}_0 \cup \tilde{U}_1$$

so that

- $\phi_n(\partial\hat{U}_i^n) = \partial\tilde{U}_i^n$,
- $\phi_n \circ f_{n_0}^{c_n} = \tilde{f}_1 \circ \phi_n$ on $\partial\hat{U}_0 \cup \partial\hat{U}_1$.

Let μ_n be the Beltrami field which coincides with $\bar{\partial}\phi_n$ on $\mathbb{C} - \hat{U}_0^n \cup \hat{U}_1^n$. It is invariant under $f_{n_0}^{c_n}$ and $\mu_n = 0$ on $K(f_{n_0}^{c_n})$. Let ψ_n be the K -quasiconformal maps, symmetric with respect to the real line, so that $\bar{\partial}\psi_n = \mu_n$. We claim that the

generalized polynomial-like maps $\hat{f}_n := \psi_n \circ f_{n_0}^{c_n} \circ \psi_n^{-1}$ converge to \tilde{f}_1 . Clearly any convergent subsequence converges to a generalized polynomial-like map of the form $\hat{f} := \psi \circ f_{n_0}^{c_\infty} \circ \psi^{-1}$. Note that the map $\phi_n \circ \psi_n^{-1}: \overline{\mathbb{C}} - \psi_n(\hat{U}_0^n \cup \hat{U}_1^n) \rightarrow \overline{\mathbb{C}} - (\tilde{U}_0 \cap \tilde{U}_1)$ is conformal. Moreover $\phi_n \circ \psi_n^{-1} \circ \hat{f}_n = \tilde{f}_1 \circ \phi_n \circ \psi_n^{-1}$ on $\psi_n(\partial\hat{U}_0 \cup \partial\hat{U}_1)$. Taking the limit, we conclude that there exists $\phi \circ \psi^{-1}$ such that

- $\phi \circ \psi^{-1}$ is conformal outside $\psi(\hat{U}_0^n \cup \hat{U}_1^n)$,
- $\phi \circ \psi^{-1} \circ \hat{f} = \tilde{f}_1 \circ \phi \circ \psi^{-1}$ on $\psi(\partial\hat{U}_0 \cup \partial\hat{U}_1)$.

Construct a real symmetric quasiconformal map which coincides with a quasisymmetric conjugacy between \hat{f} and \tilde{f}_1 , on the real line, and with $\phi \circ \psi^{-1}$ on $\overline{\mathbb{C}} - \psi(\partial\hat{U}_0 \cup \partial\hat{U}_1)$. By the Sullivan pullback argument, we obtain a quasiconformal map $h: \overline{\mathbb{C}} \rightarrow \mathbb{C}$ which is conformal outside $K(\hat{f})$ and is a conjugacy between \hat{f} and \tilde{f}_1 . Since there is no invariant line field on $K(\hat{f})$, h is an affine map, which finishes the proof. □

Proposition 13.2 (No Small Orbits). *There exists $\epsilon > 0$ such that if $f \in \mathcal{B}_{nor}(U)$ satisfies*

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq \epsilon$$

for $n \geq 0$, then f has a generalized polynomial-like restriction which is hybrid conjugate with either \tilde{f}_1 or \tilde{f}_2 and

$$|\mathcal{R}^n f - \tilde{f}|_{\mathcal{B}(U)} \leq C\lambda^n,$$

for some $C > 0$ and $\lambda < 1$.

Proof. This is an obvious consequence of Theorem 5 and Proposition 12.2. □

Theorem 7. *The Fibonacci cycle is hyperbolic with 1-dimensional unstable manifold.*

Proof. Note that, by Propositions 10.3 and 13.1, $\text{codim } E_f^h = 1$. We already proved that $D\mathcal{R}_f^2$ remains invariant and is a contraction on the Banach subspace E_f^h . Note that the spectrum of $D\mathcal{R}_f^2$ cannot be contained inside the open disc, otherwise \tilde{f} is an attracting fixed point, which implies, by Proposition 9.1 and Theorem 1, that any function in $\mathcal{B}_{nor}(U)$ very close to \tilde{f} admits a quasiconformal conjugacy with \tilde{f} , contradicting Proposition 13.1. So, if $D\mathcal{R}_f^2$ is not hyperbolic, then the spectrum is contained in the closed unit disc and intersects the unit circle. By the Small Orbits Theorem [Lyu99], for any small ϵ , there exists f so that $|\mathcal{R}^i f - \tilde{f}| \leq \epsilon$, for $i \geq 0$, but $\mathcal{R}^i f$ does not converge exponentially to \tilde{f} , which contradicts Proposition 13.2. This finishes the proof. □

14. UNIVERSALITY

In this section, we will restrict ourselves to the subspace of even maps (but we will keep the same notation for the Banach slices). This restriction is not really necessary in the universality result, but this assumption will simplify the exposition a little. Recall that the Fibonacci renormalization operator is defined in two balls B_1 and B_2 in the Banach slice $\mathcal{B}_{nor}(U_0 \cup U_1)$: the ball B_i is an ϵ_0 -ball whose center is \tilde{f}_i . Consider the linear transformation $\phi: \mathcal{B}_{nor}(U_0 \cup U_1) \rightarrow \mathcal{B}_{nor}(U_0 \cup U_1)$ defined by $\phi(g) = \hat{g}$, where \hat{g} coincides with g on \overline{U}_0 and with $-g$ on \overline{U}_1 . Then

- ϕ is a linear isometry of $\mathcal{B}_{nor}(U_0 \cup U_1)$,

- $\phi \circ \phi = Id$,
- $\phi \circ \mathcal{R} = \mathcal{R} \circ \phi$.

In particular $(\mathcal{R} \circ \phi)^2 = \mathcal{R}^2$. Since $\phi(\tilde{f}_i) = \tilde{f}_{1-i}$, we have that the \tilde{f}_i are hyperbolic fixed points of the operator $\mathcal{R} \circ \phi$. It is a question of taste to work with either \mathcal{R} or $\mathcal{R} \circ \phi$: we prefer to use the former in most of this work because its definition is more ‘dynamical’, which simplifies the exposition. The following lemma can be proved using the argument applied in [ALdM] to prove that the set of real analytic maps which are topologically conjugate with a given map f is a real analytic manifold (indeed, our situation is simpler, since we will restrict ourselves to maps with a superattractor):

Proposition 14.1. *There exists $C > 0$ so that for each small $\delta > 0$ there exists $\tilde{\delta} < \delta$ satisfying the following: Denote by $H(f_{n,i}) \subset \mathcal{B}_{nor}(U)$ the subset of maps $f \in \mathcal{B}_{nor}(U)$ which are hybrid conjugate to $f_{n,i}$ (defined in Proposition 13.1). Then, for n large enough, each connected component \mathcal{C} of the intersection of $H(f_{n,i})$ with*

$$Q_\delta := \{\tilde{f}_i + v_1 + v_2 : v_1 \in E_{\tilde{f}_i}^h, v_2 \in E_{\tilde{f}_i}^v, |v_1|_{\mathcal{B}(U)} \leq \delta \text{ and } |v_2|_{\mathcal{B}(U)} \leq C\delta\}$$

that satisfies

$$dist_{\mathcal{B}_{nor}(U)}(\tilde{f}_i, \mathcal{C}) < \tilde{\delta}$$

is a connected complex analytic manifold which is the graph of a complex analytic function

$$h_{n,i}: \{v_1 : v_1 \in E_{\tilde{f}_i}^h, |v_1| \leq \delta\} \rightarrow \{v_2 : v_2 \in E_{\tilde{f}_i}^v, |v_2| \leq C\delta\}.$$

Note that $\delta, \tilde{\delta}$ do not depend on n .

The following corollary is a monotonicity result.

Corollary 14.1. *For each small $\epsilon > 0$ there exists $\tilde{\epsilon} < \epsilon$ so that $H(f_{n,i}) \cap Q_\epsilon$, for n large enough, has a unique connected component $H_{\mathbb{R}}(f_{n,i})$ that satisfies*

$$dist_{\mathcal{B}_{nor}(U)}(\tilde{f}_i, H_{\mathbb{R}}(f_{n,i})) < \tilde{\epsilon}$$

and contains real maps.

Proof. Note that, by the Sullivan pullback argument, for each small ϵ there exists a small $\hat{\epsilon}$ so that the following holds: if two real maps belong to $H(f_{n,i}) \cap Q_\epsilon$, then they belong to the same connected component of $H(f_{n,i}) \cap Q_{\hat{\epsilon}}$.

Let $\tilde{\epsilon} = \tilde{\delta}$, where $\tilde{\delta}$ is as in Proposition 14.1, in such a way that it works for $\delta = \epsilon$ and $\delta = \hat{\epsilon}$. If c_∞ is the Fibonacci parameter, then the family $x^d + c$ is transversal with $E_{x^d+c_\infty}^h$ (because this family is tangent to the vertical direction). By the contraction of the Fibonacci renormalization operator, there exists an induced transformation \mathcal{I} from a neighborhood of $x^d + c_\infty$ in some Banach slice $\mathcal{B}_{nor}(V)$ (V is simply connected) to $\mathcal{B}_{nor}(U)$ so that $\mathcal{I}(x^d + c_\infty) \in W^s(\tilde{f}_i)$ and

$$dist_{\mathcal{B}_{nor}(U)}(\tilde{f}_i, \mathcal{I}(x^d + c_\infty)) < \tilde{\delta}.$$

Furthermore, by the infinitesimal pullback argument and the non-existence of invariant line fields on $K(x^d + c_\infty)$, it is easy to see that $D\mathcal{I}^{-1}(E_{\mathcal{I}(x^d+c_\infty)}^h) = E_{x^d+c_\infty}^h$. So the family $\mathcal{I}(x^d + c)$, $|c - c_\infty| < \eta$, with η small, is transversal to $E_{\tilde{f}_i}^h$. By Proposition 14.1 if n is large, the family $\mathcal{I}(x^d + c)$ intersects each connected component \mathcal{C} of $H(f_{n,i}) \cap Q_{\tilde{\epsilon}}$ satisfying

$$(9) \quad dist_{\mathcal{B}_{nor}(U)}(\tilde{f}_i, \mathcal{C}) < \tilde{\delta}$$

at least once. This family cannot hit $H(f_{n,i})$ in two real parameters c_1, c_2 , since $x^d + c_1$ and $x^d + c_2$ will have the same combinatorics and, by a well-known result, for each unimodal combinatorics where the critical point is periodic, there exists only one real parameter c such that $x^d + c$ (recall that d is an even number) has this combinatorics. Let $c_{n,i}$ be such a real parameter. If a real map f belongs to a connected component \mathcal{C} of $H(f_{n,i}) \cap Q_\epsilon$ satisfying equation (9), then f belongs to the connected component of $H(f_{n,i}) \cap Q_\epsilon$ that contains $\mathcal{I}(x^d + c_{n,i})$. By Proposition 14.1, they belong to the same connected component of $H(f_{n,i}) \cap Q_\epsilon$. \square

Proof of Theorem 3. Let $H_{\mathbb{R}}(f_{n,i})$ be as in Corollary 14.1. Reducing ϵ , if necessary, we have

$$\phi(H_{\mathbb{R}}(f_{n,i}) \cap Q_\epsilon) = H_{\mathbb{R}}(f_{n,3-i}) \cap Q_\epsilon$$

and

$$\mathcal{R}^{-1}(H_{\mathbb{R}}(f_{n-1,i}) \cap Q_\epsilon) \cap Q_\epsilon \subset H_{\mathbb{R}}(f_{n,3-i}),$$

so

$$(\mathcal{R} \circ \phi)^{-1}(H_{\mathbb{R}}(f_{n-1,i}) \cap Q_\epsilon) \cap Q_\epsilon \subset H_{\mathbb{R}}(f_{n,i}).$$

Let $f_\lambda \in \mathcal{B}_{nor}(U_0 \cup U_1)$, $\lambda \in \mathbb{C}$, be a complex analytic family so that $f_\lambda(z)$ is real if λ and z are real. Assume that f_0 has the Fibonacci combinatorics. By the topological convergence of the Fibonacci renormalization operator, we can define an induced transformation \mathcal{I} from a neighborhood of f_0 to a neighborhood of a point in the local stable manifold of \tilde{f}_1 .

Since the image of the derivative of any induced transformation is dense, we can find a vector $v \in T\mathcal{B}_{nor}(U_0 \cup U_1)$ so that the family $\mathcal{I}(f_\lambda + \lambda \cdot v)$ is transverse to $W_{loc}^s(\tilde{f}_1)$.

The tangent space of $H_{\mathbb{R}}(f_{n,i})$ at a point g is exactly E_g^h . So $H_{\mathbb{R}}(f_{n,i})$ is transversal to the unstable direction of \tilde{f}_i . Now consider the operator $\mathcal{R} \circ \phi$ and its fixed points \tilde{f}_i and apply a well-known fact about (complex) hyperbolic fixed points with (complex) 1-dimensional complex unstable manifold, which implies that, if f_λ is a curve in $\mathcal{B}_{nor}(U)$ that intersect $W_{loc}^s(\tilde{f}_i)$ in a transversal way at $\lambda = 0$, then there exists, for n large enough, a unique sequence λ_n (the uniqueness follows that of Corollary 14.1) such that $f_{\lambda_n} \in H_{\mathbb{R}}(f_{n,i})$ and

$$\frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1} - \lambda_{n+2}}$$

converges to the unique eigenvalue of $D(\mathcal{R}_{\tilde{f}} \circ \phi)$ larger than one. Because $(\mathcal{R} \circ \phi)^2 = \mathcal{R}^2$, the absolute value of this eigenvalue is exactly the square root of the modulus of the unique eigenvalue larger than one of the operators $D\mathcal{R}_{\tilde{f}_1}^2$ and $D\mathcal{R}_{\tilde{f}_2}^2$ (these two linear operators have the same spectrum, as $\mathcal{R}^2 \circ \phi = \phi \circ \mathcal{R}^2$).

For the family of generalized polynomials induced by the family $x^d + c$, a perturbation is not necessary: consider the induced transformation defined in the proof of Corollary 14.1. We concluded there that the family $\mathcal{I}(x^d + c)$ is transversal to $W_{loc}^s(\tilde{f}_1)$. This finishes the proof. \square

15. FURTHER DEVELOPMENTS AND OPEN QUESTIONS

Quite a lot has happened concerning the rigidity of polynomials and puzzle geometry control since this work came out as a preprint in March 2002, in such a way that it was necessary to modify this section. With the exception of this section, some misprints, reduction of some sections (as suggested by the referee), modifications in

Proposition 14.1, Corollary 14.1 and Theorem 3, and the modification mentioned below, the mathematical content of this article matches exactly with that preprint.

The new development is the work entitled “Rigidity for real polynomials”, by O. Kozlovski, W. Shen, and S. van Strien. They claim quasiconformal rigidity for all real polynomials with only real critical points, which implies the density of hyperbolic polynomials in this class. The methods introduced in this work, in particular the use of puzzle geometry control to obtain rigidity, have a major role in their proof. For instance, they prove that all non-renormalizable real unimodal polynomials (with “non-trivial” combinatorics) have sequences of critical puzzle pieces with bounded geometry. Actually, they show that the puzzle geometry control is a property shared by essentially all real polynomials with real critical points (including, in some sense, infinitely renormalizable polynomials). In view of this, it was necessary to remove the following line from the introduction: “On the other hand, the linear decay of modulus holds for all real non-renormalizable quadratic polynomials (Lyubich [Lyu94]), what is no longer true to the puzzle geometry control property.”

Actually, the puzzle geometry control does not hold (in general) in the critical pieces in the principal nest, but Kosłowski, Shen and van Strien show that it is always possible to find a sequence of critical pieces that does satisfy this condition.

However, the following conjecture and analogous statements for polynomials with many critical points remain open:

Conjecture 1. *The set of parameters $c \in \mathbb{R}$ so that $x^d + c$ has a wild attractor has zero Lebesgue measure.*

We hope to give a first approximation to Conjecture 1, using generalized renormalization to prove that sets of parameters corresponding to large sets of combinatorics which admit wild attractors have zero Lebesgue measure in the parameter space. The idea is to prove the existence of a hyperbolic set for the generalized renormalization operator associated with these combinatorics, so that this set contains all limit behaviors of the action of the generalized renormalization operator on the maps which are infinitely renormalizable with respect to it. In view of the work of Lyubich in the quadratic family [Lyu99], the existence of hyperbolic sets for generalized renormalization operators implies that the set of parameters in the real family $x^d + c$ whose corresponding map is infinitely renormalizable with respect to these generalized renormalization operators has zero Lebesgue measure in the parameter space (in the families $x^d + c$). The methods applied in this work to the Fibonacci renormalization operator can be generalized to a lot of other combinatorics, including non-periodic ones. Note that, to prove the hyperbolicity, one needs more than the puzzle geometry control: after normalizing the domains of the generalized renormalizations in an appropriate way, we must obtain polynomial-like maps which are hyperbolic, in such a way that we can control the critical puzzle geometry of maps whose renormalizations are very close to the original map. We also expect that this kind of argument can be generalized to many other combinatorics, including the non-periodic case.

ACKNOWLEDGMENTS

I would like to express my thanks for the hospitality of IMS-SUNY-Stony Brook, ICTP-Trieste, the Mathematics Institute of Warwick University, and IMPA, where

parts of this work were done. I would like to thank especially M. Lyubich for my exciting period at Stony Brook and S. van Strien for my wonderful stay at Warwick. Thanks to A. Avila for discussions about Avila, Lyubich and de Melo's work and the correction of a mistake in the proof of Corollary 10.1. I am also grateful to Juan Rivera-Letelier, whose comments were useful for improving the presentation of this work, for the fruitful suggestion of an inductive argument with the hyperbolic metric in the proof of Lemma 9.1, and for his great support in the fall of 2001.

REFERENCES

- [AB] L. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math.*, 72:385–404, 1960. MR0115006 (22:5813)
- [ALdM] A. Avila, M. Lyubich and W. de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. *Invent. Math.*, 154:451–550, 2003. MR2018784 (2006i:37083)
- [B] B. Bojarski. Remarks on Sobolev imbedding inequalities. In *Proc. of the Conference on Complex Analysis, Joensuu, 1987*, Lecture Notes in Math., 1351, Springer-Verlag, 1988. MR0982072 (90b:46068)
- [Br] B. Branner. Cubic polynomials: turning around the connectedness locus. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, Publish or Perish, Houston, 1993. MR1215972 (94c:58168)
- [BH98] B. Branner and J. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.*, 160:143–206, 1998. MR0945011 (90d:30073)
- [Bru] H. Bruin. Topological conditions for the existence of absorbing Cantor sets. *Trans. Amer. Math. Soc.*, 350:2229–2263, 1998. MR1458316 (99f:58064)
- [BKNS] H. Bruin, G. Keller, T. Nowicki and S. van Strien. Wild Cantor attractors exist. *Ann. of Math. (2)*, 143:97–130, 1996. MR1370759 (96m:58145)
- [Bu] X. Buff. Fibonacci fixed point of renormalization. *Ergodic Theory and Dynam. Systems*, 20:1287–1317, 2000. MR1786715 (2001m:37086)
- [GS96] J. Graczyk and G. Świątek. Induced expansion for quadratic polynomials. *Ann. Sci. École Norm. Sup. (4)*, 29:399–482, 1996. MR1386222 (98d:58152)
- [GS97] J. Graczyk and G. Świątek. Generic hyperbolicity in the logistic family. *Ann. of Math. (2)*, 146:1–52, 1997. MR1469316 (99b:58079)
- [HK95] J. Heinonen and P. Koskela. Definitions of quasiconformality. *Invent. Math.*, 120:61–79, 1995. MR1323982 (96e:30051)
- [KK00] S. Kallunki and P. Koskela. Exceptional sets for the definition of quasiconformality. *American Journal of Mathematics*, 122:735–743, 2000. MR1771571 (2001h:37095)
- [KN] G. Keller and T. Nowicki. Fibonacci maps re(al)visited. *Ergodic Theory and Dynam. Systems*, 15:99–120, 1995. MR1314971 (95k:58098)
- [LV] O. Lehto and K. Virtanen. Quasiconformal mappings in the plane. Springer-Verlag, New York-Heidelberg, 1973. MR0344463 (49:9202)
- [LvS] G. Levin and S. van Strien. Local connectivity of the Julia set of real polynomials. *Ann. of Math.*, 147:471–541, 1998. MR1637647 (99e:58143)
- [Lyu93] M. Lyubich. Teichmüller space of Fibonacci maps. *Preprint*, ims93-12, IMS-SUNY at Stony Brook, 1993.
- [Lyu97] M. Lyubich. Dynamics of quadratic polynomials. I, II. *Acta Math.*, 178:185–297, 1997. MR1459261 (98e:58145)
- [Lyu94] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. *Ann. of Math. (2)*, 140:347–404, 1994. MR1298717 (95j:58108)
- [Lyu99] M. Lyubich. Feigenbaum-Collet-Tresser universality and Milnor's hairness conjecture. *Ann. of Math.*, 149:319–420, 1999. MR1689333 (2000d:37051)
- [LM] M. Lyubich and J. Milnor. The Fibonacci unimodal map. *J. Am. Math. Soc.*, 6:425–457, 1993. MR1182670 (93h:58080)
- [dMvS] W. de Melo and S. van Strien. One-dimensional dynamics. Springer-Verlag, Berlin, 1993. MR1239171 (95a:58035)
- [McM] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. *Annals of Mathematics Studies*, 142, Princeton University Press, Princeton, 1996. MR1401347 (97f:57022)

- [P] R. A. Perez. On the combinatorics of the principal nest. Ph. D. Thesis, IMS-SUNY at Stony Brook, 2002.
- [PR99] F. Przytycki and S. Rohde. Rigidity of holomorphic Collet-Eckmann repellers. *Ark. Mat.*, 37:357–371, 1999. MR1714763 (2000i:37064)
- [Sh] W. Shen. On the metric property of multimodal maps and C^2 density of Axiom A. *Invent. Math.*, 156, no. 2, 301–403, 2004. MR2052610 (2005d:37078)
- [Sm01] D. Smania. Phase space universality for multimodal maps. *Bull. Braz. Math. Soc, New Series* 36(2), 225–274, 2005. MR2152018 (2006f:37057)
- [Sm02] D. Smania. On the hyperbolicity of the period-doubling fixed point. *Trans. Amer. Math. Soc.*, 358, no. 4, 1827–1846, 2006. MR2186998
- [vSN94] S. van Strien and T. Nowicki. Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps. *Unpublished*, 1994.
- [Sul92] D. Sullivan. Bounds, quadratic differentials, and renormalization conjectures. In *AMS Centennial Publications: Mathematics into the Twenty-first Century*, volume 2. AMS, 1992. MR1184622 (93k:58194)
- [SV] G. Świątek and E. Vargas. Decay of geometry in the cubic family. *Ergodic Theory Dynam. Systems*, 18:1311–1329, 1998. MR1653256 (99h:58161)

INSTITUTE FOR MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT STONY BROOK,
STONY BROOK, NEW YORK 11794-3660

Current address: Departamento de Matemática, ICMC-USP-Campus de São Carlos, Caixa
Postal 668, São Carlos-SP, CEP 13560-970, Brazil

E-mail address: smania@icmc.usp.br

URL: www.icmc.usp.br/~smania