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ON THE UNIQUENESS OF THE FOLIATION OF SPHERES OF CONSTANT MEAN CURVATURE IN ASYMPTOTICALLY FLAT 3-MANIFOLDS

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1. Introduction

In the description of the isolated gravitational system in general relativity a space-like time-slice has the structure of a complete Riemannian 3-manifold with an asymptotically flat end. Such an asymptotically flat end is diffeomorphic to $R^3 \setminus B_1(0)$, and the metric on it asymptotically approaches the Euclidean metric near infinity:

$$g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + O(r^{-2}),$$

where r is the Euclidean distance in R^3. The constant m can be interpreted as the total mass of the isolated system and is referred to as an ADM mass in literature [ADM]. It has also been established in [B] that with reasonable conditions ADM mass can be geometrically defined independent of the choices of a coordinate system at infinity.

Often it is better to consider an asymptotically flat end as a perturbation of the static time-slice of the Schwarzchild space-time. Let us start with a precise definition of asymptotically flat 3-manifolds adopted from [HY] for our discussions in this note as follows:

Definition 1.1. A complete Riemannian 3-manifold (M, g) is said to be an asymptotically flat 3-manifold with mass m if there is a compact domain K of M such that $M \setminus K$ is diffeomorphic to $R^3 \setminus B_1(0)$ and the metric g in this coordinate system is given as

$$g_{ij}(x) = (1 + \frac{m}{2|x|})^4 \delta_{ij} + T_{ij}(x),$$

for all $x \in R^3 \setminus B_1(0)$ with a constant C such that

$$|\partial^l T_{ij}|(x) \le C|x|^{-2-l}, \quad 1 \le l \le 4, \tag{1.1}$$

where ∂ denotes partial derivatives with respect to the Euclidean coordinates.

The existence of a unique foliation of spheres of constant mean curvature near the end in an asymptotically flat manifold is a very important question. Among many applications, the unique foliation of spheres of constant mean curvature can

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be used to construct a geometrically canonical coordinate system at infinity of an asymptotically flat end. It can also be used to define a geometric center of mass for an isolated gravitational system (cf. [HY]). It was observed that the study of the Hawking mass is related to stable constant mean curvature 2-spheres in an earlier paper of Christodoulou and Yau [CY]. Indeed the uniqueness of the foliation of spheres of constant mean curvature at the asymptotically flat end is helpful to the study of Penrose inequality regarding the mass (cf. [Br]).

In this note we show that outside a given compact subset in an asymptotically flat 3-manifold with positive mass there is a unique foliation of stable spheres of constant mean curvature. Our main theorem[1] is

Theorem 1.1. *Suppose (M, g) is an asymptotically flat 3-manifold with positive mass. Then there exists a compact domain K such that stable spheres of given constant mean curvature which separates infinity from the compact domain K are unique. Hence the foliation of stable spheres of constant mean curvature outside the compact domain K in M is unique.*

The existence of a foliation of stable spheres of constant mean curvature near asymptotically flat ends was established by Huisken and Yau in [HY] (also see [Ye]). Some uniqueness results with additional assumptions were also proven in [Br], [HY], [Ye]. The major difficulty of establishing the uniqueness of spheres of given constant mean curvature is that possible drifting of the spheres of constant mean curvature presents a hurdle to any useful global a priori estimates on the curvature. As a matter of fact, the uniqueness is known if one assumes no drifting (cf. [HY], [Ye]). Moreover, it was proven in [HY] that, if the drifting was somehow mild, then the uniqueness holds (cf. Theorem 5.1 on page 301 in [HY]).

Our main technical contributions can be summarized as follows: First, as a sharp contrast to the Euclidean space, similar to (5.13) in [HY], we find the following scale invariant integral which detects the nonzero mass. Suppose that N is a surface of constant mean curvature in an asymptotically flat end $(R^3 \setminus B_1(0), g)$ with positive mass m. Then

$$\frac{1}{8\pi}\int_N \frac{H}{|x|}\nu \cdot b d\sigma + \frac{1}{4\pi}\int_N \frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3} d\sigma \leq Cm^{-1}r_0^{-1}, \tag{1.2}$$

where $d\sigma$ is induced from the Euclidean metric, $C > 0$ is some constant, b is any vector in R^3, ν is the unit out-going normal vector of N in R^3 with respect to the Euclidean metric, and

$$r_0 = \min\{|x| : x \in N \subset R^3 \setminus B_1(0)\}, \tag{1.3}$$

provided that

$$\int_N H^2 d\mu < \infty, \tag{1.4}$$

[1]The uniqueness problem addressed here was referred to as the global uniqueness of stable CMC surfaces in [HY]. Their result on this global uniqueness was stated in Theorem 5.1 in [HY]. They proved that for $q > \frac{1}{2}$, if H is sufficiently small, there is a unique stable constant mean curvature surface of mean curvature H outside $B_{H^{-q}}(0)$. It has been a long-standing question whether stable constant mean curvature surfaces are unique outside a fixed compact subset. In the paragraph after Theorem 5.1 on page 301, Huisken and Yau stated: "*it is an open question whether stable constant mean curvature surfaces are actually completely unique outside a fixed compact subset.*" Our main theorem gives an affirmative answer to this question.

where $d\mu$ is induced from g. Secondly[2], we are able to obtain estimates (cf. Corollary 4.4 and Corollary 4.5 in Section 4), which are beyond one individual scale in the blow-down analysis, via an asymptotic analysis used in an early work of ours [QT]. The blow-down for a surface N of constant mean curvature H with the scale H is defined as

$$\tilde{N} = \{\frac{1}{2}Hx : x \in N \subset R^3 \setminus B_1(0)\} \subset R^3. \tag{1.5}$$

The use of the asymptotic analysis introduced in Section 4 is the key which allows us to obtain some finer estimates and untangle the problem that uniform roundness and nondrifting of spheres of constant mean curvature hinge on each other. More precisely, to eliminate the possible drifting, one carefully calculates the two integrals in left-hand side of (1.2) for $\tilde{N}$,

$$\frac{1}{4\pi}\int_{\tilde{N}} \frac{1}{|x|}\nu \cdot b d\sigma + \frac{1}{4\pi}\int_{\tilde{N}} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma, \tag{1.6}$$

with some particular choice of b. If drifting happened, then the rescaled surface $\tilde{N}$ would approach the origin. Then one evaluates the integrals over three different regions: 1) the part of $\tilde{N}$ that is any fixed distance away from the origin; 2) the part of $\tilde{N}$ that is near the origin in the scale of Hr_0; 3) the transition between the above two. We will employ Corollary 4.5 in Section 4 to show that the integrals on the third region contribute something negligible. Consequently we are able to prove that the drifting of stable spheres of constant mean curvature does not happen at all in an asymptotically flat 3-manifold with positive mass. Then using the early uniqueness results in [HY] and [Ye], for instance, Theorem 5.1 in [HY], we may conclude our main theorem.

It is worthwhile to note that the uniqueness of spheres of a given constant mean curvature outside the horizon in the Schwarzchild space is an interesting open problem. In his thesis [Br], Bray proved that the coordinate spheres are the unique minimizing surfaces of given constant mean curvature outside the horizon in Schwarzchild space, in an attempt to prove the Penrose inequality regarding the mass by the foliation of constant mean curvature surfaces. Theorem 1.1 above particularly implies that the coordinate spheres are the only stable sphere of constant mean curvature near infinity of the Schwarzchild space which separates infinity from the horizon.

The paper is organized as follows: In Section 2 we will obtain the curvature estimates based on the Simons' identity and the smallness of the integral of the traceless part of the second fundamental form. In Section 3 we introduce the blow-down analysis in all scales. In Section 4 we present the asymptotic analysis in [QT] and prove a technical proposition. Finally in Section 5 we introduce a sense of the center of mass and prove our main theorem.

[2]In [HY], a global estimate was sought after (cf. Lemma 5.6 in [HY]), with a compromise to assume that the inner radius is not smaller than H^{-q} for $q > \frac{1}{2}$. They stated in the paragraph after Theorem 5.1 (page 21, [HY]) that their assumption on inner radius "*seems to be optimal from a technical point of view*". While in this paper we do different estimates in three different scales. Particularly we establish some decay estimate for the intermediate scales by using an asymptotic analysis developed in [QT].

2. Curvature estimates

First let us recall the Simons' identity [SSY], [Sj] for a hypersurface N in a Riemannian manifold (M, g) (cf. Lemma 1.3 in [HY]):

$$\begin{aligned}\Delta h_{ij} &= \nabla_i\nabla_j H + H h_{ik}h_{jk} - |A|^2 h_{ij} + HR_{3i3j} - h_{ij}R_{3k3k} \\ &\quad + h_{jk}R_{klil} + h_{ik}R_{kljl} - 2h_{lk}R_{iljk} + \nabla_j R_{3kik} + \nabla_k R_{3ijk}\end{aligned} \tag{2.1}$$

where $A = (h_{ij})$ is the second fundamental form for N in M, $H = \mathrm{Tr}A$ is a mean curvature, and R_{ijkl} and ∇R_{ijkl} are curvature and covariant derivatives of curvature for (M, g). When N is a constant mean curvature hypersurface, we rather like to rewrite it as an equation for the traceless part $\mathring{A}$ of A, i.e. $\mathring{A} = A - \frac{1}{2}H$:

$$\begin{aligned}\Delta \mathring{A}_{ij} &= H\mathring{A}_{ik}\mathring{A}_{jk} - \frac{1}{2}H|\mathring{A}|^2\delta_{ij} - (|\mathring{A}|^2 + \frac{1}{2}H^2)\mathring{A}_{ij} \\ &\quad + HR_{3i3j} - \frac{1}{2}HR_{3k3k}\delta_{ij} - \mathring{A}_{ij}R_{3k3k} \\ &\quad + \mathring{A}_{jk}R_{klil} + \mathring{A}_{ik}R_{kljl} - 2\mathring{A}_{lk}R_{likj} \\ &\quad + \nabla_j R_{3kik} - \nabla_k R_{3ikj}.\end{aligned} \tag{2.2}$$

Lemma 2.1. *Suppose that N is a constant mean curvature surface in an asymptotically flat end $(R^3 \setminus B_1(0), g)$. Then*

$$\begin{aligned}-|\mathring{A}|\Delta|\mathring{A}| &\le |\mathring{A}|^4 + CH|\mathring{A}|^3 + CH^2|\mathring{A}|^2 \\ &\quad + C|\mathring{A}|^2|x|^{-3} + CH|\mathring{A}||x|^{-3} + C|\mathring{A}||x|^{-4}.\end{aligned} \tag{2.3}$$

Note that, in an asymptotically flat end (cf. Definition 1.1 in Section 1),

$$|R_{ijkl}| \le C|x|^{-3}, \quad |\nabla R_{ijkl}| \le C|x|^{-4}. \tag{2.4}$$

We refer readers to [HY] for the calculations of curvature of the Schwarzchild space and asymptotically flat ends.

Lemma 2.2. *Suppose that N is a constant mean curvature surface in an asymptotically flat end $(R^3 \setminus B_1(0), g)$. Then $\int_N H_e^2 d\sigma$ is bounded if and only if $\int_N H^2 d\mu$ is bounded, provided that r_0 is sufficiently large.*

Proof. First one may calculate

$$H_e = (1 + \frac{m}{2r})^2 H + 2(1 + \frac{m}{2r})^{-1}\frac{m}{r^3}x\cdot\nu + O(r^{-3}), \tag{2.5}$$

where H_e is the mean curvature of $N \subset R^3$ with respect to the Euclidean metric (cf. Lemma 1.4 in [HY]). Hence

$$H_e^2 = H^2 + O(r^{-1})H^2 + O(r^{-2})H + O(r^{-3}).$$

Following Lemma 5.2 in [HY] and the fact that g is quasi-isometric to the Euclidean metric $|dx|^2$, we have

$$\begin{aligned}\int_N H_e^2 d\sigma &\le C\int_N H_e^2 d\mu \le C\int_N H^2 d\mu + C(\int_N H^2 d\mu)^{\frac{1}{2}}(\int_N r^{-4}d\mu)^{\frac{1}{2}} + C\int_N r^{-3}d\mu \\ &\le C\int_N H^2 d\mu\end{aligned}$$

and

$$(1 - Cr_0^{-1}) \int_N H^2 d\mu \leq C \int_N H_e^2 d\sigma.$$

Thus the lemma is proved. □

Therefore, following Lemma 1 in [Si], we have

Lemma 2.3. *Suppose that N is a constant mean curvature surface in an asymptotically flat end $(R^3 \setminus B_1(0), g)$ with $r_0(N)$ sufficiently large, and that*

$$\int_N H^2 d\mu \leq C.$$

Then

$$C_1 H^{-1} \leq diam(N) \leq C_2 H^{-1}.$$

We would like to point out that, if the surface N separates infinity from the compact part, i.e. the origin is inside $N \subset R^3$, then the above lemma implies

$$C_1 H^{-1} \leq r_1(N) \leq C_2 H^{-1}, \tag{2.6}$$

where the outer radius $r_1(N)$ is defined as

$$r_1(N) = \max\{|x| : x \in N \subset R^3 \setminus B_1(0)\}.$$

Based on Michael and Simon [MS], one has the following Sobolev inequality (cf. Lemma 5.6 in [HY]).

Lemma 2.4. *Suppose that N is a constant mean curvature surface in an asymptotically flat end $(R^3 \setminus B_1(0), g)$ with $r_0(N)$ sufficiently large, and that*

$$\int_N H^2 d\mu \leq C.$$

Then

$$\left(\int_N f^2 d\mu\right)^{\frac{1}{2}} \leq C\left(\int_N |\nabla f| d\mu + \int_N H|f| d\mu\right). \tag{2.7}$$

Now we are ready to state and prove the main curvature estimates:

Theorem 2.5. *Suppose that $(R^3 \setminus B_1(0), g)$ is an asymptotically flat end. Then there exist positive numbers σ_0, ϵ_0 and δ_0 such that for any constant mean curvature surface in the end, which separates infinity from the compact part, we have*

$$|\mathring{A}|^2(x) \leq C|x|^{-2} \int_{B_{\delta_0 |x|}(x)} |\mathring{A}|^2 d\mu + C|x|^{-4}, \tag{2.8}$$

provided that

$$\int_N |\mathring{A}|^2 d\mu \leq \epsilon_0$$

and $r_0(N) \geq \sigma_0$. Also, the corresponding a priori estimates for all covariant derivatives of curvature hold consequently.

Proof. Recall that

$$\begin{aligned} -|\mathring{A}|\Delta|\mathring{A}| \leq\ & |\mathring{A}|^4 + CH|\mathring{A}|^3 + CH^2|\mathring{A}|^2 \\ & + C(|\mathring{A}|^2|x|^{-3} + CH|\mathring{A}||x|^{-3} + C|\mathring{A}||x|^{-4}). \end{aligned}$$

Multiply the two sides with ϕ^3, where ϕ is an appropriate cutoff function of small support, and integrate

$$\int_N -\phi^3|\mathring{A}|\Delta|\mathring{A}|d\mu \leq \int_N \phi^3|\mathring{A}|^4 d\mu + C\int_N H\phi^3|\mathring{A}|^3 d\mu + C\int_N H^2\phi^3|\mathring{A}|^2 d\mu$$
$$+ Cr_0^{-1}\int_N \phi^3(|\mathring{A}|^2|x|^{-2} + CH|\mathring{A}||x|^{-2} + C|\mathring{A}||x|^{-3})d\mu$$

where

$$\begin{aligned}\int_N -\phi^3|\mathring{A}|\Delta|\mathring{A}|d\mu &= \int_N \nabla(\phi^3|\mathring{A}|)\nabla|\mathring{A}|d\mu\\
&= \int_N \phi^2\nabla(\phi|\mathring{A}|)\nabla|\mathring{A}| + \int_N 2\phi^2|\mathring{A}|\nabla\phi\nabla|\mathring{A}|d\mu\\
&= \int_N \phi|\nabla(\phi|\mathring{A}|)|^2 d\mu + \int_N \phi|\mathring{A}|\nabla(\phi|\mathring{A}|)\nabla\phi d\mu + \int_N 2\phi^2|\mathring{A}|\nabla\phi\nabla|\mathring{A}|d\mu\\
&\geq \frac{3}{4}\int_N \phi|\nabla(\phi|\mathring{A}|)|^2 d\mu - C\int_N \phi|\mathring{A}|^2|\nabla\phi|^2 d\mu + \int_N 2\phi^2|\mathring{A}|\nabla\phi\nabla|\mathring{A}|d\mu\\
&\geq \frac{1}{2}\int_N \phi|\nabla(\phi|\mathring{A}|)|^2 d\mu - C\int_N \phi|\mathring{A}|^2|\nabla\phi|^2 d\mu\end{aligned}$$

$$\int_N \phi^3|\mathring{A}|^4 d\mu \leq (\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu)^{\frac{1}{2}}(\int_N (\phi|\mathring{A}|)^6 d\mu)^{\frac{1}{2}}$$

and

$$\int_N H\phi^3|\mathring{A}|^3 d\mu \leq (\int_{\mathrm{supp}(\phi)} H^2 d\mu)^{\frac{1}{2}}(\int_N (\phi|\mathring{A}|)^6 d\mu)^{\frac{1}{2}}.$$

For other terms

$$\int_N H^2\phi^3|\mathring{A}|^2 d\mu \leq Cr_M^{-2}\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu \leq C|x_0|^{-2}\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu,$$

$$\int_N \phi^3|x|^{-2}|\mathring{A}|^2 d\mu \leq C|x_0|^{-2}\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu,$$

$$\int_N \phi^3 H|x|^{-2}|\mathring{A}|d\mu \leq C|x_0|^{-2}(\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu)^{\frac{1}{2}},$$

and

$$\int_N \phi^2|x|^{-3}|\mathring{A}|d\mu \leq C|x_0|^{-2}(\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu)^{\frac{1}{2}}.$$

Note that, for a given point x_0, we may choose the cutoff function ϕ so that it has the suppose of a disk of radius, say, $\delta_0|x_0|$ (δ_0 to be determined). Now, combining all terms, we have

$$\int_N \phi|\nabla(\phi|\mathring{A}|)|^2 d\mu \leq 2(\int_N |\mathring{A}|^2 d\mu)^{\frac{1}{2}}(\int_N (\phi|\mathring{A}|)^6 d\mu)^{\frac{1}{2}}$$
$$+ C(\int_{\mathrm{supp}(\phi)} H^2 d\mu)^{\frac{1}{2}}(\int_N (\phi|\mathring{A}|)^6 d\mu)^{\frac{1}{2}} + C|x_0|^{-2}(\int_{\mathrm{supp}(\phi)} |\mathring{A}|^2 d\mu)^{\frac{1}{2}}.$$

Applying the Sobolev inequality with $f=\phi^3g^3$ where $g=|\mathring{A}|$, we have

$$\begin{aligned}(\int_N(\phi g)^6d\mu)^{\frac12} &\le C(3\int_N(\phi g)^2|\nabla(\phi g)|d\mu+\int_N H(\phi g)^3)\\ &\le C(\int_N \phi^3g^4d\mu)^{\frac12}(\int_N|\nabla(\phi g)|^2\phi d\mu)^{\frac12}+(\int_{\mathrm{supp}(\phi)}H^2d\mu)^{\frac12}(\int_N(\phi g)^6d\mu)^{\frac12}\\ &\le C(\int_N g^2d\mu)^{\frac12}(\int_N(\phi g)^6d\mu)^{\frac12}+C(\int_{\mathrm{supp}(\phi)}H^2d\mu)^{\frac12}(\int_N(\phi g)^6d\mu)^{\frac12}\\ &\quad +C\int_N|\nabla(\phi g)|^2\phi d\mu.\end{aligned}$$

Thus

$$(\int_N(\phi|\mathring{A}|)^6d\mu)^{\frac12}\le C|x_0|^{-2}(\int_{\mathrm{supp}(\phi)}|\mathring{A}|^2d\mu)^{\frac12}, \tag{2.9}$$

which implies

$$\int_N(\phi|\mathring{A}|)^4d\mu\le C|x_0|^{-2}\int_{\mathrm{supp}(\phi)}|\mathring{A}|^2d\mu. \tag{2.10}$$

Note that we have chosen δ_0 small enough so that

$$C\int_{\mathrm{supp}(\phi)}H^2d\mu\le\frac18$$

and $\int_N|\mathring{A}|^2d\mu\le\epsilon_0$, where ϵ_0 is small enough so that

$$C\int_N|\mathring{A}|^2d\mu\le\frac18.$$

Now we proceed to get the point-wise estimates. First, if we take $f=u^2$ in the Sobolev inequality, then

$$\begin{aligned}(\int_N u^4d\mu)^{\frac12} &\le C(2\int_N|u||\nabla u|d\mu+\int_N Hu^2d\mu)\\ &\le C(\int_N u^2d\mu)^{\frac12}(\int_N|\nabla u|^2d\mu)^{\frac12}+C(\int_{\mathrm{supp}(u)}H^2d\mu)^{\frac12}(\int_N u^4d\mu)^{\frac12}.\end{aligned}$$

When u has the support as the cutoff function ϕ, we have

$$(\int_N u^4d\mu)^{\frac12}\le C(\int_N u^2d\mu)^{\frac12}(\int_N|\nabla u|^2d\mu)^{\frac12}. \tag{2.11}$$

To finish the point-wise estimates we use the following rather standard estimate:

Lemma 2.6. *Suppose that a nonnegative function v in L^2 solves*

$$-\Delta v\le fv+h \tag{2.12}$$

on $B_{2R}(x_0)$, where

$$\int_{B_{2R}(x_0)}f^2d\mu\le CR^{-2}$$

and $h\in L^2(B_{2R}(x_0))$. Also, suppose that

$$(\int_N u^4d\mu)^{\frac12}\le C(\int_N u^2d\mu)^{\frac12}(\int_N|\nabla u|^2d\mu)^{\frac12}$$

holds for all u with support inside $B_{2R}(x_0)$. Then

$$\sup_{B_R(x_0)} v \leq CR^{-1}\|v\|_{L^2(B_{2R}(x_0))} + CR\|h\|_{L^2(B_{2R}(x_0))}.$$

Proof. We will simply use the Moser iteration method. For convenience, we may rescale so that we are working on B_2. The correct scales would be

$$v_R(x) = v(Rx), f_R(x) = R^2 f(Rx), \text{ and } h_R = R^2 h(Rx).$$

Let $k = \|h\|_{L^2(B_2)}$ and $\bar{v} = v + k$. Multiply the equation with $\phi^2\bar{v}^{p-1}$ on both sides:

$$\int |\nabla(\phi\bar{v}^{\frac{p}{2}})|^2 \leq \frac{2}{p}\int f\phi^2\bar{v}^p + \int h\phi^2\bar{v}^{p-1} + C\int |\nabla\phi|^2\bar{v}^p.$$

Set $\bar{f} = \frac{h}{k} + f$; we have

$$\int |\nabla(\phi\bar{v}^{\frac{p}{2}})|^2 \leq \frac{2}{p}\int \bar{f}\phi^2\bar{v}^p + C\int |\nabla\phi|^2\bar{v}^p.$$

Note that $\|\bar{f}\|_{L^2(B_2)} \leq 1 + \|f\|_{L^2(B_2)}$. By the assumed Sobolev inequality, we have

$$(\int(\phi\bar{v}^{\frac{p}{2}})^4)^{\frac{1}{2}} \leq C(p\int |\bar{f}|\phi^2\bar{v}^p)^{\frac{1}{2}}(\int \phi^2\bar{v}^p)^{\frac{1}{2}} + C(\int |\nabla\phi|^2 v^p)^{\frac{1}{2}}(\int \phi^2 v^p)^{\frac{1}{2}}.$$

To handle the first term, we apply the Hölder inequality

$$\begin{aligned}(p\int |\bar{f}|\phi^2\bar{v}^p)^{\frac{1}{2}}(\int \phi^2\bar{v}^p)^{\frac{1}{2}} &\leq p^{\frac{1}{2}}(\int |\bar{f}|^2)^{\frac{1}{4}}(\int \phi^2\bar{v}^p)^{\frac{1}{2}}(\int(\phi\bar{v}^{\frac{p}{2}})^4)^{\frac{1}{4}} \\ &\leq \frac{1}{2C}(\int(\phi\bar{v}^{\frac{p}{2}})^4)^{\frac{1}{2}} + Cp(\int |\bar{f}|^2)^{\frac{1}{2}}\int \phi^2\bar{v}^p.\end{aligned}$$

Hence

$$(\int(\phi\bar{v}^{\frac{p}{2}})^4)^{\frac{1}{2}} \leq C(p\|\bar{f}\|_{L^2}\int \phi^2\bar{v}^p + \int |\nabla\phi|^2\bar{v}^p).$$

Now, for $i = 1, 2, \ldots$, let $p = 2^i$ and

$$\phi = \begin{cases} 1 & \forall x \in B_{1+2^{-i}}, \\ 0 & \forall x \notin B_{1+2^{-i+1}}. \end{cases}$$

Then

$$(\int_{B_{1+2^{-i}}} \bar{v}^{2^{i+1}})^{2^{-i-1}} \leq C^{2^{-i}} 2^{i2^{-i}}(\int_{B_{1+2^{-i+1}}} \bar{v}^{2^i})^{2^{-i}}.$$

Thus

$$\sup_{B_1} v \leq \sup_{B_1} \bar{v} \leq C^{\sum_{i=1} 2^{-i}} 2^{\sum_{i=1} i2^{-i}}(\int_{B_2} \bar{v}^2)^{\frac{1}{2}} \leq C(\|v\|_{L^2(B_2)} + \|h\|_{L^2(B_2)}),$$

whose scaled version gives the lemma. □

To get curvature estimates, we write the equation in such way as (2.12) that we may apply the above lemma for

$$f = C(|\mathring{A}|^2 + H|\mathring{A}| + H^2 + r^{-3}) \text{ and } h = C(Hr^{-3} + r^{-4}),$$

in light of (2.9) and (2.10). □

3. Blow-down analysis

In order to understand a surface of constant mean curvature N in an asymptotically flat end $(R^3 \setminus B_1(0), g)$, we will need to blow down the surface in different scales. We first consider, the blow-down by the scale H,

$$\tilde{N} = \frac{1}{2}HN = \{\frac{1}{2}Hx : x \in N\}. \tag{3.1}$$

Suppose that there is a sequence of constant mean curvature surfaces $\{N_i\}$ such that

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi. \tag{3.2}$$

Then, by an argument similar to the proof of Lemma 2.2 in the previous section, we have

$$\lim_{i\to\infty} \int_{N_i} H_e^2 d\sigma = 16\pi. \tag{3.3}$$

Hence, by the curvature estimates established in the previous section combining the proof of Theorem 1 in [Si], we have

Lemma 3.1. *Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi.$$

Also, suppose that N_i separates infinity from the compact part. Then, there is a subsequence of $\{\tilde{N}_i\}$ which converges in Gromov-Hausdorff distance to a round sphere $S_1^2(a)$ of radius 1 and centered at $a \in R^3$. Moreover, the convergence is in C^∞ sense away from the origin.

From the above lemma, the difficulty will be to study the possibility of having the origin lying on the sphere $S^2(a)$, that is,

$$\lim_{i\to\infty} r_0(N_i) = \infty, \text{ and } \lim_{i\to\infty} r_0(N_i)H(N_i) = 0. \tag{3.5}$$

Then, in light of the curvature estimates we obtained in the previous section, we may use the smaller scale $r_0(N_i)$ to blow down the surface

$$\hat{N} = r_0(N)^{-1}N = \{r_0^{-1}x : x \in N\}. \tag{3.6}$$

Lemma 3.2. *Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi.$$

Also, suppose that

$$\lim_{i\to\infty} r_0(N_i)H(N_i) = 0.$$

Then there is a subsequence of $\{\hat{N}_i\}$ that converges to a 2-plane at distance 1 from the origin. Moreover the convergence is in C^∞ in any compact set of R^3.

As one would expect, the real difficulty is to understand the behavior of the surfaces N_i in the scales between $r_0(N_i)$ and $H^{-1}(N_i)$. To start we consider the intermediate scales r_i such that

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0 \text{ and } \lim_{i\to\infty} r_i H(N_i) = 0 \tag{3.7}$$

and blow down the surfaces

$$\bar{N}_i = r_i^{-1} N = \{r_i^{-1} x : x \in N\}. \tag{3.8}$$

Lemma 3.3. *Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi.$$

Also, suppose that $\{r_i\}$ are such that

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0 \text{ and } \lim_{i\to\infty} r_i H(N_i) = 0.$$

Then there is a subsequence of $\{\bar{N}_i\}$ converging to a 2-plane at the origin in Gromov-Hausdorff distance. Moreover the convergence is C^∞ in any compact subset away from the origin.

4. Asymptotic analysis

In this section we would like to apply the asymptotic analysis used in [QT] to obtain some estimate that holds over the whole transition region between the scales $r_0(N_i)$ and $r_1(N_i)$. But first, let us revise Proposition 2.1 in [QT] as follows. Let us denote

$$\|u\|_i^2 = \int_{[(i-1)L, iL]\times S^1} |u|^2 dt d\theta.$$

Lemma 4.1. *Suppose $u \in W^{1,2}(\Sigma, R^k)$ satisfies*

$$\Delta u + A\cdot\nabla u + B\cdot u = h \quad \text{in } \Sigma,$$

where $\Sigma = [0, 3L]\times S^1$. Also, suppose that L is given and large. Then there exists a positive number δ_0 such that, if

$$\|h\|_{L^2(\Sigma)} \le \delta_0 \max_{1\le i\le 3}\{\|u\|_i + \|\nabla u\|_i\}$$

and

$$\|A\|_{L^\infty(\Sigma)} \le \delta_0, \qquad \|B\|_{L^\infty(\Sigma)} \le \delta_0,$$

then

(a) $\|u\|_3 \le e^{-\frac{1}{2}L}\|u\|_2$ implies $\|u\|_2 < e^{-\frac{1}{2}L}\|u\|_1$,
(b) $\|u\|_1 \le e^{-\frac{1}{2}L}\|u\|_2$ implies $\|u\|_2 < e^{-\frac{1}{2}L}\|u\|_3$, and
(c) If both $\int_{L\times S^1} u d\theta$ and $\int_{2L\times S^1} u d\theta \le \delta_0 \max_{1\le i\le 3}\{\|u\|_i\}$, then either $\|u\|_2 < e^{-\frac{1}{2}L}\|u\|_1$ or $\|u\|_2 < e^{-\frac{1}{2}L}\|u\|_3$.

Proof. For the convenience of readers we would like to present a proof here. The first step is to establish the conclusions (a), (b) and

$$\text{(c}'\text{)} \quad \text{If} \int_{L\times S^1} u d\theta = \int_{2L\times S^1} u d\theta = 0, \text{then} \|u\|_2 < e^{-\frac{1}{2}L}\|u\|_1 \text{or} \|u\|_2 < e^{-\frac{1}{2}L}\|u\|_3$$

for harmonic functions. By the separation of variables, we may write a harmonic function as

$$v = a_0 + b_0 t + \sum_{n=1}^{\infty}\{(a_n \cos n\theta + b_n \sin n\theta)e^{nt} + (a_{-n}\cos n\theta + b_{-n}\sin n\theta)e^{-nt}\}.$$

Then it follows that

$$\begin{aligned}\|v\|_i^2 &= 2\pi(a_0^2 L + a_0 b_0 L^2(2i-1) + \frac{1}{3}b_0^2 L^3(3i^2 - 3i + 1)) \\ &\quad + \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{e^{2nL}-1}{n}\{(a_n^2 + b_n^2)e^{2n(i-1)L} + (a_{-n}^2 + b_{-n}^2)e^{-2(i-1)nL}\}.\end{aligned}$$

We then claim

$$\|v\|_2^2 < \frac{1}{2}(e^L\|v\|_3^2 + e^{-L}\|v\|_1^2)$$

provided that $e^L > 4$, which can easily be verified. Hence (a) and (b) can easily be verified for the harmonic function v. To show (c$'$) holds for v, we notice that

$$\|v\|_2^2 < \frac{1}{2}e^{-L}(\|v\|_3^2 + \|v\|_1^2)$$

provided that $e^L > 4$, since $a_0 = b_0 = 0$ under the assumptions of (c$'$). The second step of the proof is to pass limits. For example, let us prove the conclusions (a) and (b). Otherwise suppose that there are a sequence of functions $\|h_k\|_{L^2(\Sigma)} \to 0$ and a sequence of solutions u_k satisfying the elliptic equations such that

$$\frac{1}{2}(e^L\|u_k\|_3^2 + e^{-L}\|u_k\|_1^2) \le \|u_k\|_2^2$$

when either (a) or (b) fails to hold. One may assume

$$1 \le \|u_k\|_1 + \|u_k\|_2 + \|u_k\|_3 \le 3$$

since one may normalize them if necessary. Then the standard elliptic estimates implies that u_k converges, at least for a subsequence, to a harmonic function u strongly on each compact subset of Σ. To see that u is not identically zero we realize that $\|u_k\|_2$ has to be bounded from below by a positive number and u_k strongly converges to u on the middle part $[L, 2L] \times S^1$. Notice that we may use a diagonal method to have a subsequence u_k converge to the harmonic function u almost everywhere over Σ. Then

$$\begin{aligned}\frac{1}{2}(e^L\|u\|_3^2 + e^{-L}\|u\|_1^2) &\le \varliminf_{k\to\infty}\frac{1}{2}(e^L\|u_k\|_3^2 + e^{-L}\|u_k\|_1^2) \\ &\le \lim_{k\to\infty}\|u_k\|_2^2 = \|u\|_2^2,\end{aligned} \tag{4.1}$$

by the Fatou lemma, which contradicts with the conclusion of the first step. (c) can be proven similarly. This finishes the proof of the lemma. □

We would like to point out that Proposition 2.1 in [QT] is overstated since it is not correct for $l > 3$. But, in the proof of Proposition 3.1 in [QT], where Corollary 2.2 is used, one may replace the shifting cylinder with length $3L$ instead of $5L$. The proof still works the same, which is, one push in the direction of growth of the cylinder of length $3L$ when Corollary 2.2 in [QT] applies and it gives the estimates regardless of where one stops applying Corollary 2.2.

Given a surface N in R^3, recall from, for example, (8.5) in [Ka], that

$$\Delta\nu + |\nabla\nu|^2\nu = \nabla H_e \tag{4.2}$$

where ν is the Gauss map from $N \longrightarrow S^2$. For the constant mean curvature surfaces in the asymptotically flat end $(R^3 \setminus B_1(0), g)$,

$$|\nabla H_e|(x) \leq C|x|^{-3}. \tag{4.3}$$

Therefore we consider that the Gauss map of the constant mean curvature surfaces in the asymptotically flat end $(R^3 \setminus B_1(0), g)$ is an almost harmonic map. Hence we are in a situation which is very similar to that in [QT]. We will refer readers to [QT] for rather elementary yet involved analysis since the proof we present here is some modifications from the proof in [QT]. But, for the convenience of the readers, we will in the following present the argument to make this paper self-contained as much as possible. We will not carry the indices for the surfaces N_i if it does not cause any confusion. Set

$$A_{r_1,r_2} = \{x \in N : r_1 \leq |x| \leq r_2\}. \tag{4.4}$$

$A^0_{r_1,r_2}$ stands for the standard annulus in R^2. We are concerned with the behavior of ν on the part $A_{Kr_0(N),sH^{-1}(N)}$ of N where K will be fixed large and s will be fixed small. The first difference from [QT] is that, while we had a fixed domain in [QT], we need the following lemma in order to be in the position to use Lemma 4.1 in the above.

Lemma 4.2. *Suppose that N is a constant mean curvature surface in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$. Then, for any $\epsilon > 0$ and Λ fixed, there are ϵ_0, s and K such that, if*

$$\int_N |\mathring{A}|^2 d\sigma \leq \epsilon_0$$

and $Kr_0(N) < r < sH^{-1}(N)$, then $(r^{-1}A_{r,e^{4\Lambda}r}, r^{-2}g_e)$ may be represented as $(A^0_{1,e^{4\Lambda}}, \bar{g})$ where

$$\|\bar{g} - |dx|^2\|_{C^1(A^0_{1,e^{4\Lambda}})} \leq \epsilon. \tag{4.5}$$

In other words, in the cylindrical coordinates $(S^1 \times [\log r, 4\Lambda + \log r], \bar{g}_c)$,

$$\|\bar{g}_c - (dt^2 + d\theta^2)\|_{C^1(S^1\times[\log r, 4\Lambda+\log r])} \leq \epsilon. \tag{4.6}$$

This is a consequence of Lemma 3.3 in the previous section. Hence we may introduce the following coordinates:

$$I_j = S^1 \times [\log(Kr_0(N)) + jL, \log(Kr_0(N)) + (j+1)L],$$

where

$$j \in [0, n] \text{ and } \log(Kr_0(N)) + (n+1)L = \log(sH(N)^{-1}),$$

on the annulus $A_{Kr_0(N),sH(N)^{-1}}$. The first thing we want is to show (3.8) in [QT], that is, for each $j \in [2, n-1]$, there is a geodesic γ such that

$$\begin{aligned}&\int_{I_j} |\nabla(\nu - \gamma)|^2 dt d\theta \\ &\leq C\,(e^{-\frac{i}{2}L} + e^{-\frac{l_n - i}{2}L})(\|\nabla\nu\|^2_{L^2(A_{Kr_0(N),sH(N)^{-1}})} + r_0^{-2}).\end{aligned} \tag{4.7}$$

To apply Lemma 4.1 to prove (4.7) we choose two points $P, Q \in S^2$ such that

$$|P - \frac{1}{2\pi}\int_{(j-1)L\times S^1} \nu d\theta| \le C \max_{(j-1)L\times S^1} |P-\nu|^2,$$
$$|Q - \frac{1}{2\pi}\int_{jL\times S^1} \nu d\theta| \le C \max_{jL\times S^1} |Q-\nu|^2.$$

P and Q always exist because S^2 is a smooth manifold. We construct a geodesic γ connecting P and Q in S^2. Since P and Q are very close, such a geodesic γ is unique. We then consider the map

$$\nu - \gamma : \Sigma_j = \bigcup_{k=j-1}^{k=j+1} I_k \to S^2$$

and apply Lemma 4.1. The only reason that Lemma 4.1 (c) is not applicable to $\nu - \gamma$ over Σ_j is

$$\int_{I_j} |\nabla(\nu-\gamma)|^2 dt d\theta \le \frac{1}{\delta_0^2}\|\nabla H\|^2_{L^2(\Sigma_j)} \le C r_0^{-2}.$$

The last inequality is because of the size of Σ_j and (4.3). Otherwise, Lemma 4.1 (c) is applicable; then, we have

$$\text{either (A)}\quad \|\nu-\gamma\|_j < e^{-\frac{1}{2}L}\|\nu-\gamma\|_{j-1}$$
$$\text{or (B)}\quad \|\nu-\gamma\|_j < e^{-\frac{1}{2}L}\|\nu-\gamma\|_{j+1}.$$

Suppose that (A) holds. Then we shift to consider the map

$$\nu - \gamma : \Sigma_{j-1} = \bigcup_{k=j-2}^{k=j} \to S^2$$

and to conclude

$$\|\nu-\gamma\|_{j-1} < e^{-\frac{1}{2}L}\|\nu-\gamma\|_{j-2}$$

if Lemma 4.1 (a) is applicable to $\nu-\gamma$ over Σ_{j-1}. If the shifting tube Σ reaches I_1, then by the local gradient estimate (cf. the proof of Lemma 3.2 in [QT]) we have, for $2 \le k \le j$,

$$\|\nu-\gamma\|_k \le e^{-\frac{k}{2}L}(\|\nabla\nu\|^2_{L^2(A_{Kr_0(N), sH(N)^{-1}})} + r_0^{-2}).$$

If, otherwise, the shifting tube stops at I_l, then Lemma 4.1 (a) implies that

$$\|\nu-\gamma\|_l \le C r_0^{-2}$$

as before. Hence, for $l \le k \le j$,

$$\|\nu-\gamma\|_k \le e^{-\frac{k}{2}L}(\|\nabla\nu\|^2_{L^2(A_{Kr_0(N), sH(N)^{-1}})} + r_0^{-2}). \tag{4.8}$$

Suppose that (B) holds instead. Similarly, in light of Lemma 4.1 (b), if the shifting tube Σ stops at I_r, then we have, for $j \le k \le r$,

$$\|\nu-\gamma\|_k \le e^{-\frac{n-k}{2}L}(\|\nabla\nu\|^2_{L^2(A_{Kr_0(N), sH(N)^{-1}})} + r_0^{-2}). \tag{4.9}$$

We would like to point out that at this point our focus is to show (4.7) for I_j. We do not really care about how far left l or how far right r would be. We will need some integral estimates to get (4.7) from the above L^2 estimate (4.8) and (4.9) using the equations over the domain

$$\Sigma_j' = S^1 \times [(j - \frac{1}{2})L, (j + \frac{1}{2})L].$$

Notice that this modification of the argument in [QT] is because the shifting tube should be of length 3 instead of 5 in [QT]. The same broken geodesic argument in [QT] now works here with no more modification and yields (4.7) for I_j with any fixed $j \in [2, n-1]$.

Another difference from [QT] is that we are considering maps with tension fields possibly blowing up at a point but with no energy concentration, while in [QT] we were considering almost harmonic maps with concentration of energy but tension fields uniformly bounded in L^2. In cylindrical coordinates, the tension fields are

$$|\tau(\nu)| = r^2 |\nabla H_e| \leq C r^{-1} = C e^{-t}$$

for $t \in [\log(K r_0), \log(s H^{-1})]$. Thus,

$$\int_{S^1 \times [t, t+L]} |\tau(\nu)|^2 dt d\theta \leq C e^{-2t}$$

which decays nicely, but has no global L^2 bounds exact because of the possibility that the inner radius $r_0(N)$ and the outer $r_1(N)$ may not be in the same scale. To get the growth (or decay) of the energy along the cylinder we need to use the Hopf differential

$$\Phi = |\partial_t \nu|^2 - |\partial_\theta \nu|^2 - 2\sqrt{-1} \partial_t \nu \cdot \partial_\theta \nu$$

and the stationary property, in complex variable $z = t + \sqrt{-1}\theta$,

$$\bar{\partial} \Phi = \partial \nu \cdot \tau(\nu)$$

to bound $\int |\partial_t \nu|^2$ by $\int |\partial_\theta \nu|^2$ (cf. [QT]) as follows:

$$\int_{S^1 \times [t, t+L]} |\partial_t \nu|^2 dt d\theta \leq \int_{S^1 \times [t, t+L]} |\Phi| dt d\theta + \int_{S^1 \times [t, t+L]} |\partial_\theta \nu|^2 dt d\theta.$$

By the elliptic estimates, we have

$$\int_{S^1 \times [t, t+L]} |\Phi| dt d\theta \leq \int_{N \cap B_r^c} |\Phi| dt d\theta \leq C (\int_{N \cap B_r^c} |\nabla \nu|^2 d\sigma)^{\frac{1}{2}} (\int_{N \cap B_r^c} |\tau(\nu)|^2 d\sigma)^{\frac{1}{2}},$$

where $N \cap B_r^c$ is the part of N which is outside of B_r and is a disk since N is a sphere topologically (there is a very good reason not to use $N \bigcap B_r$ in the above estimate). This is the only place one has to use the global L^2 bound of the tension field $\tau(\nu)$. Fortunately the topology of N is a sphere that turns out to be very crucial in carrying the above argument from [QT]. Hence, we have

$$\int_{S^1 \times [t, t+L]} |\Phi| dt d\theta \leq C (\int_{N \cap B_r^c} |\tau(\nu)|^2 d\sigma)^{\frac{1}{2}} \leq C e^{-t}. \tag{4.10}$$

Notice that in [QT] we instead used the fact that the tension fields are uniformly bounded in L^2 inside B_δ (cf. the lines between (3.8) and (3.9) in [QT]). The rest of the proof of Proposition 3.1 in [QT] works with no more modification. Thus we have

Proposition 4.3. *Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi.$$

Also, suppose that

$$\lim_{i\to\infty} r_0(N_i) H(N_i) = 0.$$

Then there exist a large number K, a small number s and i_0 such that, when $i \geq i_0$,

$$\max_{I_j} |\nabla \nu| \leq C\Big(\int_{B_{sH^{-1}(N_i)}\cap N_i} |\nabla\nu|^2 d\sigma + r_0^{-1}\Big)\big(e^{-\frac{1}{4}jL} + e^{-\frac{1}{4}(n_i - j)L}\big), \tag{4.11}$$

where

$$I_j = S^1 \times [\log(Kr_0(N_i)) + jL, \log(Kr_0(N_i)) + (j+1)L]$$

and

$$j \in [0, n_i] \text{ and } \log(Kr_0(N_i)) + (n_i + 1)L = \log(sH(N_i)^{-1}).$$

This finer analysis improves our understanding of the blow-downs that we discussed in the previous section. Namely,

Corollary 4.4. *Assume the same conditions as Proposition 4.3. Then the limit plane in Lemma 3.2 and the limit plane in Lemma 3.3 are all orthogonal to the vector a. In fact, we may choose s small and i large enough so that,*

$$|\nu(x) + a| \leq \epsilon$$

for all $x \in N_i$ and $|x| \leq sH^{-1}(N_i)$.

Also, we have

Corollary 4.5. *Asume the same condition as Proposition 4.3. Let $\nu_i = \nu(p_i)$ for some $p_i \in I_{\frac{n_i}{2}}$. Then*

$$\begin{aligned} &\max_{I_j} |\nu - \nu_i| \\ &\leq C\frac{1}{1 - e^{-\frac{1}{4}L}}\Big(\int_{B_{sH^{-1}(N_i)}\bigcap N_i} |\nabla\nu|^2 d\sigma + r_0^{-1}\Big)\big(e^{-\frac{1}{4}jL} + e^{-\frac{1}{8}n_i L}\big) \end{aligned} \tag{4.12}$$

for $j \in [0, \frac{1}{2}n_i]$ and

$$\begin{aligned} &\max_{I_j} |\nu - \nu_i| \\ &\leq C\frac{1}{1 - e^{-\frac{1}{4}L}}\Big(\int_{B_{sH^{-1}(N_i)}\bigcap N_i} |\nabla\nu|^2 d\sigma + r_0^{-1}\Big)\big(e^{-\frac{1}{8}n_i L} + e^{-\frac{1}{4}(n_i - j)L}\big) \end{aligned} \tag{4.13}$$

for $j \in [\frac{1}{2}n_i, n_i]$.

The two corollaries above will be the key for us to calculate the integrals in the next section to prove our main theorem.

5. Center of mass

First let us recall that, for any embedded surface N in R^3 and any given vector $b \in R^3$,

$$\int_N H_e \nu \cdot b d\sigma = 0. \tag{5.1}$$

One may consider this as the first variation of the area of surface $N_t = N + tb \subset R^3$. On the other hand, if N is a constant mean curvature surface in the asymptotically flat end $(R^3 \setminus B_1(0), g)$, then

$$\int_N H\nu \cdot b d\sigma = H \int_N \nu \cdot b d\sigma = 0, \tag{5.2}$$

since the flux is zero across any surface for a given constant velocity b. Thus, for any constant mean curvature surface in the asymptotically flat end,

$$\int_N (H_e - H)\nu \cdot b d\sigma = 0. \tag{5.3}$$

One may calculate and find

$$H_e - H = m(\frac{H}{|x|} + 2\frac{\nu \cdot x}{|x|^3}) + O(|x|^{-2})H + O(|x|^{-3}). \tag{5.4}$$

Lemma 5.1. *Suppose N is a surface of constant mean curvature in the asymptotically flat end with positive mass $m \neq 0$. Also, suppose that*

$$\int_N H^2 d\mu < \infty$$

and $r_0(N)$ is sufficiently large. Then for any given b and for some $C > 0$,

$$\frac{1}{8\pi}\int_N \frac{H}{|x|}\nu \cdot b d\sigma + \frac{1}{4\pi}\int_N \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma \leq C m^{-1} r_0^{-1}. \tag{5.5}$$

Proof. Simply multiply both sides of (5.4) by $\nu \cdot b$ and integrate over the surface N. Then we have

$$\frac{1}{8\pi}\int_N (H_e - H)\nu \cdot b d\sigma = \frac{m}{8\pi}\int_N \frac{H}{|x|}\nu \cdot b d\sigma + \frac{m}{4\pi}\int_N \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma + O(r_0^{-1}).$$

Here we used the Lemma 5.2 in [HY]. Then the lemma is proved due to (5.3). □

Now, we are ready to state and prove our main theorem in this note as follows:

Theorem 5.2. *Suppose that $\{N_i\}$ is a sequence of spheres of constant mean curvature in a given asymptotically flat end with positive mass $m \neq 0$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty \text{ and } \lim_{i\to\infty}\int_{N_i} H^2 d\sigma = 16\pi.$$

Also, suppose that N_i separates the infinity from the compact part. Then

$$\lim_{i\to\infty}\frac{r_0(N_i)}{r_1(N_i)} = 1. \tag{5.6}$$

Proof. We may apply Lemma 3.1 for the blow-down

$$\tilde{N} = \frac{1}{2}HN = \{\frac{1}{2}Hx : x \in N\}.$$

If the surfaces $\tilde{N}_i$ stay away from the origin, i.e.

$$0 < C \leq H_i^{-1} r_0(N_i)$$

for some positive constants C, then a subsequence of $\tilde{N}_i$ converges to a sphere $S^2(a)$ of radius 1 and centered at $a \in R^3$ in C^∞ by the curvature estimates Theorem 2.5 in Section 2. Also notice that (2.6) implies that the blow-down surfaces $\tilde{N}_i$ always stay within a bounded region in R^3. On one hand, by (5.5) in Lemma 5.1, we have

$$\frac{1}{4\pi}\int_{S^2(a)} \frac{\nu \cdot b}{|x|} d\sigma + \frac{1}{4\pi}\int_{S^2(a)} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma = 0, \tag{5.7}$$

for any b. On the other hand, if $b = -\frac{a}{|a|}$, then

$$\frac{1}{4\pi}\int_{S^2(a)} \frac{\nu \cdot b}{|x|} d\sigma + \frac{1}{4\pi}\int_{S^2(a)} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma = |a| \tag{5.8}$$

due to an explicit calculation when the origin is inside. Therefore

$$a = 0 \text{ and } \lim_{i\to\infty} \frac{1}{2} r_0(N_i) H(N_i) = \lim_{i\to\infty} \frac{r_0(N_i)}{r_1(N_i)} = 1.$$

To conclude that this is all that can happen we need only to exclude the case when

$$\lim_{i\to\infty} H_i^{-1} r_0(N_i) = 0. \tag{5.9}$$

Assume otherwise. According to Lemma 3.1, the blow-down sequence $\tilde{N}_i$ converges to a unit round sphere $S^2(a)$ centered at $a \in R^3$ with $|a| = 1$ in Hausdorff topology. We will take $b = -\frac{a}{|a|}$. From Lemma 5.1, we know

$$\lim_{i\to\infty} (\int_{\tilde{N}_i} \frac{\nu \cdot b}{|x|} d\sigma + \int_{\tilde{N}_i} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma) = 0. \tag{5.10}$$

But, we claim, on the other hand,

$$\lim_{i\to\infty} (\int_{\tilde{N}_i} \frac{\nu \cdot b}{|x|} d\sigma + \int_{\tilde{N}_i} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma) = 4\pi \tag{5.11}$$

which gives us the contradiction. First, we have from explicit calculations

$$\int_{S^2(a)} \frac{\nu \cdot b}{|x|} d\sigma = \frac{4}{3}\pi, \ \int_{S^2(a)} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma = \frac{2}{3}\pi. \tag{5.12}$$

The first term in (5.11) is an easy term because of the uniform integrability

$$\lim_{i\to\infty} \int_{\tilde{N}_i} \frac{\nu \cdot b}{|x|} d\sigma = \int_{S^2(a)} \frac{\nu \cdot b}{|x|} d\sigma = \frac{4}{3}\pi. \tag{5.13}$$

To deal with the second term in (5.11), we break up the integral into three parts. For any fixed small number $s > 0$ and large number $K > 0$,

$$\begin{aligned} \int_{\tilde{N}_i} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma &= \int_{\tilde{N}_i \bigcap B_s^c(0)} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma \\ + \int_{\tilde{N}_i \bigcap (B_s(0) \setminus B_{KHr_0}(0))} &\frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma + \int_{\tilde{N}_i \bigcap B_{KHr_0}(0)} \frac{(\nu \cdot x)(\nu \cdot b)}{|x|^3} d\sigma. \end{aligned} \tag{5.14}$$

Then

$$\lim_{i\to\infty}\int_{\tilde{N}_i\bigcap B_s^c(0)}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma=\int_{S^2(a)\bigcap B_s^c}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma \tag{5.15}$$

and

$$\lim_{i\to\infty}\int_{\tilde{N}_i\bigcap B_{KH(N_i)r_0(N_i)}(0)}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma=\int_{P\bigcap B_K(0)}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma,$$

where P is the limit plane in Lemma 3.2. By Corollary 4.4, we know

$$\int_P\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma=2\pi,$$

due to a simple calculation. Notice that

$$\int_{\tilde{N}_i}\frac{\nu\cdot x}{|x|^3}d\sigma=4\pi \tag{5.16}$$

for any i and

$$\int_{S^2(a)}\frac{\nu\cdot x}{|x|^3}d\sigma=2\pi \tag{5.17}$$

because the origin is on the sphere $S^2(a)$. Since

$$\lim_{i\to\infty}\int_{\tilde{N}_i\bigcap B_s^c(0)}\frac{\nu\cdot x}{|x|^3}d\sigma=\int_{S^2(a)\bigcap B_s^c(0)}\frac{\nu\cdot x}{|x|^3}d\sigma, \tag{5.18}$$

$$\lim_{i\to\infty}\int_{\tilde{N}_i\bigcap B_{KHr_0}(0)}\frac{\nu\cdot x}{|x|^3}d\sigma=\int_{P\bigcap B_K(0)}\frac{\nu\cdot x}{|x|^3}d\sigma \tag{5.19}$$

and

$$\int_P\frac{\nu\cdot x}{|x|^3}d\sigma=2\pi, \tag{5.20}$$

we know

$$\lim_{i\to\infty,s\to 0,K\to\infty}\int_{\tilde{N}_i\bigcap(B_s^c(0)\setminus B_{KHr_0}(0))}\frac{\nu\cdot x}{|x|^3}d\sigma=0. \tag{5.21}$$

Now we are ready to handle the difficult term: the integral over the transition region in (5.14). Our goal is to show that

$$\lim_{i\to\infty,s\to 0,K\to\infty}\int_{\tilde{N}_i\bigcap(B_s(0)\setminus B_{KHr_0}(0))}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}d\sigma=0. \tag{5.22}$$

The key point is to use Corollary 4.5 to prove (5.22) from (5.21). Let ν_i be chosen as in Corollary 4.5. Then

$$\begin{aligned}&\int_{\tilde{N}_i\bigcap(B_s\setminus B_{KHr_0})}\frac{(\nu\cdot x)(\nu\cdot b)}{|x|^3}\\&=(\nu_i\cdot b)\int_{\tilde{N}_i\bigcap(B_s\setminus B_{KHr_0})}\frac{\nu\cdot x}{|x|^3}+\int_{\tilde{N}_i\bigcap(B_s\setminus B_{KHr_0})}\frac{(\nu\cdot x)((\nu-\nu_i)\cdot b)}{|x|^3}.\end{aligned} \tag{5.23}$$

Hence we only need to deal with the second term on the right side of the above (5.23). It is better to use the cylindrical coordinates used in Section 4.

$$\begin{aligned}
&\int_{\tilde{N}_i \bigcap (B_s(0)\setminus B_{KHr_0}(0))} \frac{(\nu\cdot x)((\nu-\nu_i)\cdot b)}{|x|^3} d\sigma \\
&= \sum_{j=1}^{n_i} \int_{I_j} \frac{(\nu\cdot x)((\nu-\nu_i)\cdot b)}{|x|^3} A(t) d\theta dt \\
&\le C \sum_{j=1}^{n_i} L \max_{I_j} |\nu-\nu_i| \\
&= C \sum_{j=1}^{n_i/2} L \max_{I_j} |\nu-\nu_i| + C \sum_{j=n_i/2+1}^{n_i/2} L \max_{I_j} |\nu-\nu_i|.
\end{aligned} \tag{5.24}$$

From (4.12) and (4.13), we have

$$\begin{aligned}
&\int_{\tilde{N}_i \bigcap (B_s(0)\setminus B_{KHr_0}(0))} \frac{(\nu\cdot x)((\nu-\nu_i)\cdot b)}{|x|^3} d\sigma \\
&\le C\eta\Big(\sum_{j=1}^{n_i/2} (e^{-\frac{1}{4}Lj} + e^{-\frac{1}{8}Ln_i}) + \sum_{j=1}^{n_i/2} (e^{-\frac{1}{8}Ln_i} + e^{-\frac{1}{4}(n_i-j)L})\Big) \\
&\le C\eta(n_i e^{-\frac{1}{8}n_i L} + 2),
\end{aligned} \tag{5.25}$$

where

$$\eta = \int_{B_{sH^{-1}(N_i)} \cap N_i} |\nabla \nu|^2 d\sigma + r_0^{-1}$$

and η can be arbitrarily small as long as $s \to 0$ and $r_0(N_i) \to \infty$. Thus (5.22) is proved and the proof of the theorem is completed. □

Corollary 5.3. *Suppose $(R^3 \setminus B_1(0), g)$ is an asymptotically flat end with positive mass. Then there exist a large number $K > 0$ and a small number $\epsilon > 0$ such that, for any $H < \epsilon$, there exists a unique stable sphere N of constant mean curvature H with $N \subset R^3 \setminus B_K(0)$ and which separates infinity from the compact part. Hence there exists a unique foliation of stable spheres of constant mean curvature near infinity.*

Proof. In light of Proposition 5.3 in [HY] we know

$$\lim_{i\to\infty} \int_{N_i} H^2 d\mu = 16\pi, \tag{5.26}$$

provided each N_i is a stable sphere of constant mean curvature. Thus Corollary 5.3 follows from Theorem 5.2 above, Theorem 4.1 in [HY] and the proof of Theorem 5.1 in [HY]. □

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