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AN ALGEBRO-GEOMETRIC PROOF OF WITTEN'S CONJECTURE

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1. Introduction

Let $\overline{\mathcal{M}}_{g;n}$ denote the Deligne–Mumford moduli space of genus g stable complex curves with n marked points [3]. For each $i \in \{1, \dots, n\}$, consider the line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g;n}$ whose fiber over a point $(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g;n}$ is the cotangent line to the curve C at the marked point x_i. Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g;n})$ denote the first Chern class of this line bundle, $\psi_i = c_1(\mathcal{L}_i)$. Consider the generating function

$$F(t_0, t_1, \dots) = \sum \langle \tau_{d_1} \dots \tau_{d_n} \rangle \frac{t_{d_1} \dots t_{d_n}}{|\mathrm{Aut}(d_1, \dots, d_n)|} \tag{1.1}$$

for the intersection numbers of these classes,

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{d_1} \dots \psi_n^{d_n},$$

where the genus g is uniquely determined from the identity

$$d_1 + \dots + d_n = \dim\ \overline{\mathcal{M}}_{g;n} = 3g - 3 + n.$$

The first few terms of this function are

$$\begin{aligned} F \quad = \quad & \frac{1}{24}t_1 + \frac{1}{6}t_0^3 + \frac{1}{48}t_1^2 + \frac{1}{24}t_0t_2 + \frac{1}{6}t_0^3t_1 + \frac{1}{1152}t_4 + \frac{1}{72}t_1^3 + \frac{1}{12}t_0t_1t_2 + \frac{1}{48}t_0^2t_3 \\ & + \frac{1}{6}t_0^3t_1^2 + \frac{1}{24}t_0^4t_2 + \frac{29}{5760}t_2t_3 + \frac{1}{384}t_1t_4 + \frac{1}{1152}t_0t_5 + \dots . \end{aligned}$$

The celebrated Witten conjecture asserts that *the second derivative* $U = \partial^2 F/\partial t_0^2$ *of the generating function* F *satisfies the KdV equation*

$$\frac{\partial U}{\partial t_1} = U\frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}. \tag{1.2}$$

The motivation for this conjecture can be found in [18], and for a detailed exposition suitable for a mathematically-minded reader see, e.g., [10]. Note that the KdV equation can be interpreted as a recurrence formula allowing one to calculate all the

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intersection indices provided "initial conditions" are given. Witten has shown that the function F satisifes the so-called string and dilaton equations, which together with the KdV equation generate the whole KdV hierarchy and provide necessary initial conditions.

A number of proofs of the Witten conjecture are known, but all of them exploit techniques that do not seem to be intrinsically related to the initial problem: Kontsevich's proof [9] makes use of Jenkins–Strebel differentials and matrix integrals; the proof due to Okounkov and Pandharipande [13], which starts with Hurwitz numbers, also involves matrix integrals and graphs on surfaces study, as well as asymptotic analysis; and, finally, Mirzakhani's proof [11] is based on the Riemannian geometry properties of moduli spaces. The goal of this paper is to present a new proof using purely algebro-geometric techniques. Similarly to the proof due to Okounkov and Pandharipande, we start with Hurwitz numbers, but then we follow a different line.

Hurwitz numbers enumerate ramified coverings of the 2-sphere with prescribed ramification points and ramification types over these points. We deal only with ramified coverings whose ramification type is simple over each ramification point but one. Our proof is based on the following, now well-known, properties of these numbers:

- the ELSV formula [4, 5] relating Hurwitz numbers to the intersection theory on moduli spaces,
- the relationship between Hurwitz numbers and integrable hierarchies conjectured by Pandharipande [14] and proved, in a stronger form, by Okounkov [12].

Using the ELSV formula, we express the intersection indices of the ψ-classes in terms of Hurwitz numbers. The partial differential equations governing the generating series for Hurwitz numbers then lead to the KdV equation for the intersection indices. Note that the existence of such a proof has been predicted in [7][1]. One of the main features of the proof consists in the fact that known effective algorithms for computing the Hurwitz numbers, which are relatively simple combinatorial objects, lead to an independent tool for computing the intersection indices. We describe the properties of the Hurwitz numbers in detail and deduce Witten's conjecture from them in Section 2. Section 3 is devoted to a discussion of the proof.

2. Proof

2.1. **Hurwitz numbers.** Fix a sequence $b_1, \dots, b_n$ of positive integers. Consider ramified coverings of the sphere S^2 by compact oriented two-dimensional surfaces of genus g with ramification type $(b_1, \dots, b_n)$ over one point and the simplest possible ramification type $(2, 1, 1, \dots, 1)$ over all other points of ramification. According to the Riemann–Hurwitz formula, the total number m of these points of simple

[1]After the present paper was submitted, one more proof of Witten's conjecture appeared in L. Chen, Y. Li, K. Liu, *Localization, Hurwitz numbers, and the Witten conjecture*, math.AG/0609263. It also starts with the cut-and-join equation and the ELSV formula, but it uses a different way to express the intersection indices of the ψ-classes in terms of the Hurwitz numbers. This way is based on the approach used in I. P. Goulden, D. M. Jackson, and R. Vakil, *A short proof of λ_g-conjecture without Gromov–Witten theory: Hurwitz theory and the moduli of curves*, math.AG/0604297, to prove the so-called λ_g-conjecture.

ramification is

$$m = 2g - 2 + n + B, \tag{2.1}$$

where $B = b_1 + \cdots + b_n$ is the degree of the covering. If we fix the ramification points in the target sphere, then the number of topologically distinct ramified coverings of this type becomes finite, and we denote by $h_{g;b_1,\dots,b_n}$ the number of these coverings, with marked preimages of the point of degenerate ramification, counted with the weight inverse to the order of the automorphism group of the covering. These numbers are called *Hurwitz numbers*.

The ELSV formula [4, 5] expresses the Hurwitz numbers in terms of Hodge integrals over the moduli spaces of stable complex curves:

$$h_{g;b_1,\dots,b_n} = m! \prod_{i=1}^{n} \frac{b_i^{b_i}}{b_i!} \int_{\overline{\mathcal{M}}_{g;n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{(1 - b_1\psi_1)\dots(1 - b_n\psi_n)} \tag{2.2}$$

for $g > 0, n \geq 1$ or $g = 0, n \geq 3$. The numerator of the integrand is the total Chern class of the vector bundle over $\overline{\mathcal{M}}_{g;n}$ dual to the Hodge bundle (whose fiber is the g-dimensional vector space of holomorphic differentials on the curve), $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{g;n})$. The integral in (2.2) is understood as the result of expanding the fraction as a power series, with further selection of monomials whose degree coincides with the dimension of the base (there are finitely many of them) and integration of each of these monomials.

The integral (2.2) is a sum of intersection indices of both ψ- and λ-classes. In [13], the λ-classes are excluded by considering asymptotics of integrals of this kind. In contrast, in the present paper, the exclusion of the λ-classes is based on simple combinatorial considerations originating in [17, 19], see Section 2.2 below.

Now consider the following exponential generating function for the Hurwitz numbers:

$$H(\beta; p_1, p_2, \dots) = \sum h_{g;b_1,\dots,b_n} \frac{p_{b_1} \dots p_{b_n}}{n!} \frac{\beta^m}{m!},$$

where the summation is taken over all finite sequences $b_1, \dots, b_n$ of positive integers and all nonnegative values of g, with m given by (2.1). According to Okounkov [12], the exponent e^H of this generating function is a τ-function for the KP-hierarchy. In fact, Okounkov proved a much stronger theorem stating that the generating function for double Hurwitz numbers (those having degenerate ramification over two points rather than one) satisfies the Toda lattice equations. We do not need this statement in such generality, and in Section 3 we discuss a simple proof of the fact we really need. (Various kinds of Hurwitz numbers have been long since known to lead to solutions of integrable hierarchies, but we were unable to trace exact statements and origins.) The function e^H being a τ-function for the KP hierarchy means, in particular, that H satisfies the *KP-equation*

$$\frac{\partial^2 H}{\partial p_2^2} = \frac{\partial^2 H}{\partial p_3 \partial p_1} - \frac{1}{2}\left(\frac{\partial^2 H}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 H}{\partial p_1^4}. \tag{2.3}$$

We use this equation below to deduce from it the KdV equation for the function U.

2.2. **Expressing intersection indices of ψ-classes via Hurwitz numbers.** Obviously, for each nonnegative integer d there exist constants c_b^d, $b = 1, \dots, d+1$, such that

$$\sum_{b=1}^{d+1} \frac{c_b^d}{1-b\psi} = \psi^d + O(\psi^{d+1}), \tag{2.4}$$

and these constants are uniquely determined by this requirement. They are given by the formula

$$c_b^d = \frac{(-1)^{d-b+1}}{(d-b+1)!(b-1)!}.$$

Indeed, we need to prove that the first $d-1$ derivatives in ψ of the linear combination

$$\sum_{b=1}^{d+1} \frac{c_b^d}{1-b\psi}$$

vanish at 0, while the dth derivative is $d!$. The ith derivative of this linear combination evaluated at $\psi = 0$ is

$$(-1)^{i+1}\frac{1}{i!}\left(\binom{d}{0}1^i - \binom{d}{1}2^i + \binom{d}{2}3^i - \dots \pm \binom{d}{d}d^i\right).$$

The sum in parentheses coincides with the result of applying the ith iteration of the operator xd/dx to the polynomial $(1-x)^d$ and evaluating at $x = 1$, which is 0 for $0 \le i < d$ and $(-1)^d d!$ for $i = d$, as desired.

Multiplying identities (2.4) for different d, we obtain the following equality:

$$\sum_{b_1=1}^{d_1+1} \dots \sum_{b_n=1}^{d_n+1} \frac{c_{b_1}^{d_1} \dots c_{b_n}^{d_n}}{(1-b_1\,\psi_1)\dots(1-b_n\,\psi_n)} = \prod_{i=1}^{n} \psi_i^{d_i} + \dots,$$

where the dots on the right-hand side denote cohomology classes of degree greater than $d_1 + \dots + d_n$. This means, in particular, that for $d_1 + \dots + d_n = 3g-3+n$ the linear combination

$$\sum_{b_1=1}^{d_1+1} \dots \sum_{b_n=1}^{d_n+1} c_{b_1}^{d_1} \dots c_{b_n}^{d_n} \int_{\overline{\mathcal{M}}_{g;n}} \frac{1-\lambda_1+\dots\pm\lambda_g}{(1-b_1\psi_1)\dots(1-b_n\psi_n)}$$

is simply $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$, because the integral of the terms of higher degree vanishes. Taking into account the coefficient of the integral in (2.2), we obtain the following explicit identity.

Theorem 2.1. *For any sequence of nonnegative integers $d_1, \dots, d_n$, we have*

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \sum_{b_1=1}^{d_1+1} \dots \sum_{b_n=1}^{d_n+1} \left(\frac{1}{m!} \prod_{i=1}^{n} \frac{(-1)^{d_i+1-b_i}}{(d_i+1-b_i)!\,b_i^{b_i-1}} \right) h_{g;b_1,\dots,b_n},$$

where g is determined by the left-hand side, $\sum d_i = 3g-3+n$, and $m = 2g-2+B+n$.

It is convenient to reformulate the statement of the theorem in terms of generating functions. Decompose the generating function H into the sum

$$H = H_{0;1} + H_{0;2} + H_{\mathrm{st}}, \tag{2.5}$$

where the stable part H_{st} contains all the monomials whose coefficients are given by the ELSV formula (2.2), and $H_{0;1}$ and $H_{0;2}$ are the generating functions for the numbers of ramified coverings of the sphere by the sphere with 1 ("polynomial") and 2 ("trigonometric polynomial") preimages over the distinguished ramification point, respectively. The latter generating functions are known since Hurwitz; see also [1]:

$$\begin{aligned} H_{0;1} &= \sum_{b=1}^{\infty} h_{0;b} p_b \frac{\beta^{b-1}}{(b-1)!} = \sum_{b=1}^{\infty} \frac{b^{b-2}}{b!} p_b \beta^{b-1}, \\ H_{0;2} &= \frac{1}{2} \sum_{b_1,b_2=1}^{\infty} h_{0;b_1,b_2} p_{b_1} p_{b_2} \frac{\beta^{b_1+b_2}}{(b_1+b_2)!} = \frac{1}{2} \sum_{b_1,b_2=1}^{\infty} \frac{b_1^{b_1} b_2^{b_2}}{(b_1+b_2) b_1! b_2!} p_{b_1} p_{b_2} \beta^{b_1+b_2}. \end{aligned}$$

In fact, we are going to use below not the precise formulas for $H_{0;1}, H_{0;2}$, but the fact that they contain only terms of degree at most 2 in p_i, which yields

$$\begin{aligned} \frac{\partial^2}{\partial p_1^2}(H_{0;1} + H_{0;2}) &= \frac{1}{2}\beta^2, \\ \frac{\partial^2}{\partial p_2^2}(H_{0;1} + H_{0;2}) &= \beta^4, \\ \frac{\partial^2}{\partial p_1 \partial p_3}(H_{0;1} + H_{0;2}) &= \frac{9}{8}\beta^4. \end{aligned}$$

Denote by $G_{\mathrm{st}} = G_{\mathrm{st}}(\beta; t_0, t_1, \dots)$ the result of the following change of variables in the series H_{st}:

$$p_b = \beta^{-b-\frac{1}{3}} \frac{b!}{b^b} \sum_{d=b-1}^{\infty} c_b^d \, \beta^{-\frac{2}{3}d} t_d = \sum_{d=b-1}^{\infty} \frac{(-1)^{d-b+1}}{(d-b+1)! b^{b-1}} \beta^{-b-\frac{2d+1}{3}} t_d. \tag{2.6}$$

The result of this substitution is a series in $t_0, t_1, \dots$ whose coefficients are formal Laurent expansions in $\beta^{2/3}$. Indeed, the powers of β in the contribution of a monomial $p_{b_1} \dots p_{b_n}$ to the expansion have the form

$$(b_1 + \dots + b_n + 2g - 2 + n) + \sum_{i=1}^{n} \left(-b_i - \frac{2d_i + 1}{3} \right) = \frac{2}{3} \left(3g - 3 + n - \sum_{i=1}^{n} d_i \right);$$

hence they become even integers when multiplied by 3. On the other hand, the powers of β at each t_d are bounded from below, because t_d enters only the expansions for $p_1, \dots, p_{d+1}$.

Theorem 2.2. (1) *The series G_{st} contains no terms with negative powers of β.*

(2) *The free term in β, $G_{\mathrm{st}}|_{\beta=0}$ (which is correctly defined due to the first statement), coincides with the generating function F for the intersection numbers given by* (1.1),

$$F(t_0, t_1, \dots) = G_{\mathrm{st}}(0; t_0, t_1, \dots).$$

Proof. Let us represent the function $H_{\mathrm{st}} = H - H_{0,1} - H_{0,2}$ in the form $H_{\mathrm{st}} = \sum_{n=1}^{\infty} \frac{1}{n!} H_n$, where we collect in H_n the terms corresponding to the Hurwitz numbers enumerating coverings with exactly n preimages of infinity. Then, since the number of simple critical points is equal to

$$m = 2g - 2 + n + \sum b_i = \tfrac{2}{3}(3g - 3 + n) + \sum (b_i + \tfrac{1}{3}) = \tfrac{2}{3} \dim \overline{\mathcal{M}}_{g,n} + \sum (b_i + \tfrac{1}{3}),$$

we get from the ELSV formula

$$\begin{aligned} H_n &= \sum_{g,b_1,\dots,b_n} h_{g,b_1,\dots,b_n} \frac{\beta^m}{m!} p_{b_1}\dots p_{b_n} \\ &= \sum_{g,b_1,\dots,b_n} \prod_{i=1}^n \frac{b_i^{b_i}\,\beta^{b_i+\frac13}}{b_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\beta^{\frac23}\lambda_1+\beta^{\frac43}\lambda_2-\dots}{\prod_{i=1}^n(1-b_i\beta^{\frac23}\psi_i)} p_{b_1}\dots p_{b_n} \\ &= \left\langle (1-\beta^{\frac23}\lambda_1+\beta^{\frac43}\lambda_2-\dots)\prod_{i=1}^n \sum_{b\geq 1} \frac{b^b\beta^{b+\frac13}}{b!}\frac{p_b}{(1-b\beta^{\frac23}\psi_i)}\right\rangle, \end{aligned}$$

where we understand the fraction as a formal power series and the angle brackets mean integration of each monomial over the space $\overline{\mathcal{M}}_{g;n}$ whose dimension coincides with the degree of the monomial. Applying our change of variables (2.6), we get

$$\begin{aligned} \sum_{b\geq 1} \frac{b^b\beta^{b+\frac13}}{b!}\frac{p_b}{(1-b\beta^{\frac23}\psi_i)} &= \sum_{d\geq b-1\geq 0} \frac{c_b^d}{(1-b\beta^{\frac23}\psi_i)}\beta^{-\frac23 d}t_d \\ &= \sum_{d\geq 0}\big((\beta^{\frac23}\psi_i)^d+\dots\big)\beta^{-\frac23 d}t_d = \sum_{d\geq 0}\big(\psi_i^d+\dots\big)\,t_d, \qquad i=1,2,\dots,n, \end{aligned}$$

where the dots denote terms of higher degree in β. Therefore, the corresponding summand of $G_{\rm st}$ is equal to

$$(G_{\rm st})_n = \sum_{d_1,\dots,d_n} \langle \prod_{i=1}^n \psi_i^{d_i}+\dots\rangle t_{d_1}\dots t_{d_n}.$$

Theorem 2.2 is proved. $\square$

2.3. **Reduction to the KdV equation.** By Theorem 2.2, $G_{\rm st}$ is a power series in $\beta^{2/3}, t_0, t_1, \dots$ whose coefficient $G|_{\beta=0}$ of β^0 coincides with the function F. By definition, $G_{\rm st}$ is the result of the substitution (2.6) to the series $H_{\rm st} = H - H_{0;1} - H_{0;2}$. Equation (2.3) together with the explicit formulas for $H_{0;1}$ and $H_{0;2}$ imply that $H_{\rm st}$ satisfies the equation

$$\frac{\partial^2 H_{\rm st}}{\partial p_2^2} = \frac{\partial^2 H_{\rm st}}{\partial p_3\partial p_1} - \frac12\left(\frac{\partial^2 H_{\rm st}}{\partial p_1^2}\right)^2 - \frac12\beta^2\frac{\partial^2 H_{\rm st}}{\partial p_1^2} - \frac{1}{12}\frac{\partial^4 H_{\rm st}}{\partial p_1^4}.$$

The change of variables (2.6) results in the following change of partial derivatives:

$$\begin{aligned} \frac{\partial}{\partial p_1} &= \beta^{4/3}\frac{\partial}{\partial t_0}, \\ \frac{\partial}{\partial p_2} &= 2\beta^{9/3}\frac{\partial}{\partial t_1} + 2\beta^{7/3}\frac{\partial}{\partial t_0}, \\ \frac{\partial}{\partial p_3} &= 9\beta^{14/3}\frac{\partial}{\partial t_2} + 9\beta^{12/3}\frac{\partial}{\partial t_1} + \frac92\beta^{10/3}\frac{\partial}{\partial t_0}. \end{aligned}$$

After substituting this into the equation above and dividing the result by $\beta^{16/3}$, we rewrite it as

$$\left(\frac{\partial^2 G_{\rm st}}{\partial t_1\partial t_0} - \frac12\left(\frac{\partial^2 G_{\rm st}}{\partial t_0^2}\right)^2 - \frac{1}{12}\frac{\partial^4 G_{\rm st}}{\partial t_0^4}\right) + \beta^{2/3}\left(9\frac{\partial^2 G_{\rm st}}{\partial t_0\partial t_2} - 4\frac{\partial^2 G_{\rm st}}{\partial t_0^4}\right) = 0. \tag{2.7}$$

The coefficient of β^0 in (2.7) has the form

$$\frac{\partial^2 F}{\partial t_1 \partial t_0} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 - \frac{1}{12}\frac{\partial^4 F}{\partial t_0^4} = 0.$$

Differentiating the last equation over t_0, we obtain the desired KdV equation (1.2) for the function $U = \partial^2 F/\partial^2 t_0^2$. This completes the proof of Witten's conjecture.

3. Odds and ends

Trying to make the present paper more self-contained, we discuss in this section the Hirota bilinear equations and the KP hierarchy, as well as an explicit presentation of the function e^H as a τ-function for the KP hierarchy. Although specialists in integrable hierarchies are well aware of these facts, it is not an easy task to find their compact and readable exposition. We refer the reader to [15], [2], and [16] for a description of the relationship between integrable hierarchies and the geometry of semi-infinite Grassmannians. The KP hierarchy is a system of partial differential equations for an unknown function H. The KP equation (2.3) is the first equation in this system. Similarly to the case of the KdV equation, the expansion of a solution to the KP equation can be reconstructed from "initial conditions". The exponent $\tau = e^H$ of a solution H to the KP hierarchy is called a *τ-function* of the hierarchy. The equations of the KP hierarchy rewritten for τ-functions are also partial differential equations; they are called the Hirota equations. They possess the nice property of being quadratic with respect to τ.

3.1. **Semi-infinite Grassmannian, Hirota–Plücker bilinear equations, and integrable hierarchies.** Define the *charge zero Fock space* $\mathcal{F}$ as the completion of the infinite dimensional coordinate vector space over $\mathbb{C}$ whose basic elements s_λ are labeled by partitions,

$$\mathcal{F} = \overline{\bigoplus \mathbb{C} s_\lambda}.$$

Recall that a *partition* is a nonincreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, having finitely many nonzero terms. Elements of $\mathcal{F}$ are infinite formal linear combinations of the vectors s_λ. We shall use the following two realizations of the Fock space.

(1) The space $\mathcal{F}$ can be identified with the space $\mathcal{F} = \mathbb{C}[[p_1, p_2, \dots]]$ of formal power series in infinitely many variables $p_1, p_2, \dots$ by setting s_λ to be the corresponding Schur function. The *Schur function* corresponding to a one-part partition is defined by the expansion

$$s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4 + \cdots = e^{p_1 z + p_2 \frac{z^2}{2} + p_3 \frac{z^3}{3} + \cdots},$$

and for a general partition λ it is given by the determinant

$$s_\lambda = \det\ ||s_{\lambda_j - j + i}||. \tag{3.1}$$

The indices i, j here run over the set $\{1, 2, \dots, n\}$ for n large enough, and since $\lambda_i = 0$ for i sufficiently large, the determinant, whence s_λ, is independent of n. Here are a few first Schur polynomials:

$$s_0 = 1, \qquad s_1 = p_1, \qquad s_2 = \frac{1}{2}(p_1^2 + p_2), \qquad s_3 = \frac{1}{6}(p_1^3 + 3p_1p_2 + 2p_3),$$

$$s_{1,1} = \frac{1}{2}(p_1^2 - p_2), \qquad s_{2,1} = \frac{1}{3}(p_1^3 - p_3), \qquad s_{1,1,1} = \frac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3).$$

(2) Let $V = \mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials in z, which we treat as the vector space with the basis z^i, $i \in \mathbb{Z}$. Identify $\mathcal{F}$ with the *semi-infinite wedge space* $\mathcal{F} = \Lambda^{\infty/2}V$ freely spanned by the formal infinite wedge products of the form

$$s_\lambda = z^{k_1} \wedge z^{k_2} \wedge z^{k_3} \wedge \dots, \qquad k_i = i - \lambda_i,$$

for all partions λ. Sequences k_i appearing on the right-hand side can be characterized as arbitrary strictly increasing sequences of integers satisfying $k_i = i$ for i large enough.

The theory of the KP hierarchy can be summarized as follows.

The projectivization $P\mathcal{F} = P\Lambda^{\infty/2}V$ is the ambient space of the standard Plücker embedding $Gr \hookrightarrow P\mathcal{F}$, where $Gr = Gr_{\infty/2}(V)$ is the Grassmannian of "half-infinite dimensional subspaces", often referred to as the Sato Grassmannian. By definition, the elements of Gr are subspaces spanned by linearly independent vectors $\varphi_1, \varphi_2, \varphi_3, \dots$ in (the formal completion of) V such that for i large enough we have $\varphi_i = z^i + \dots$, where the dots denote terms of lower order in z. Such a vector space can be interpreted as the wedge product

$$\tau = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \dots. \tag{3.2}$$

Indeed, another choice of a basis does not affect this wedge product, up to a scalar factor. If all the functions φ_i are Laurent polynomials and φ_i is simply the monomial z^i for all i sufficiently large, then the wedge product τ can be represented, after expanding the brackets, as a *finite* linear combination $\tau = \sum_\lambda c_\lambda s_\lambda$, $c_\lambda \in \mathbb{C}$. If the functions φ_i contain infinitely many terms, then the function τ is a formal linear combination of s_λ and can be obtained in the following way: *when expanding the brackets in the infinite wedge product* (3.2) *pick one monomial summand in each* φ_i *in such a way that this summand is* z^i *for all but finitely many indices* i *and do this in all possible ways.*

More explicitly, if $\varphi_i = \sum_{j \in \mathbb{Z}} a_{i,j} z^j$, then

$$\tau = \sum_\lambda \det ||a_{i,j-\lambda_i}||_{i,j\geq 1} s_\lambda = \det || \sum_{k \in \mathbb{Z}} a_{i,k} s_{j-k} ||_{i,j \geq 1}. \tag{3.3}$$

Theorem 3.1 ([15, 2, 16]). *A* (*nonzero*) *function* τ *is a* τ*-function for the KP hierarchy iff the corresponding point* $[\tau] \in P\mathcal{F}$ *belongs to the Grassmannian* $Gr \subset P\mathcal{F}$.

In particular, each Schur polynomial s_λ and any linear combination of the Schur functions s_i corresponding to one-part partitions produces a solution to the KP equation (2.3).

The image of the Plücker embedding of the Grassmannian is known to be given by a system of quadratic equations. In our case, these algebraic equations for the Taylor coefficients of the function τ take the form of partial differential equations for this function, known as the *bilinear Hirota equations*.

3.2. **Formulas for the generating function for Hurwitz numbers.** The exponent e^H of the generating function for the Hurwitz numbers is nothing but the generating function for the numbers of ramified coverings of the 2-sphere by *all*, not necessarily connected, compact oriented surfaces of Euler characteristic $2-2g$. Take such a covering and let a point of simple ramification in the target sphere tend to the point of degenerate ramification. Then one can express the number of such coverings as a linear combination of the numbers of similar coverings with fewer points of simple ramification. This recurrence relation (the "cut-and-join equation" of [6]) expressed in terms of generating functions reads as follows:

$$\frac{\partial e^H}{\partial \beta} = \frac{1}{2}\sum_{i,j=1}^{\infty}\left((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right)e^H = Ae^H, \tag{3.4}$$

where we denote by A the differential operator on the right-hand side. For an algebro-geometric interpretation of the cut-and-join equation, see [17].

Equation (3.4) can be solved explicitly. The operator A acts linearly on the space of weighted homogeneous polynomials in the variables $p_1, p_2, \dots$, with the weight of the variable p_i equal to i. Moreover, it preserves the weighted degree of the polynomials, whence it can be split into a direct sum of finite dimensional linear operators. The Schur functions s_λ are eigenvectors of A, and they form a complete set of eigenvectors.

Denote by $f(\lambda)$ the eigenvalue of the eignevector s_λ. It can be checked that

$$f(\lambda) = \frac{1}{2}\sum_{i=1}^{\infty}\left[(i-\tfrac{1}{2}-\lambda_i)^2 - (i-\tfrac{1}{2})^2\right].$$

It follows that any solution of (3.4) can be represented as a sum

$$\sum_{\lambda} c_\lambda s_\lambda e^{f(\lambda)\beta},$$

over all partitions λ, for some coefficients c_λ. For the solution e^H, these coefficients can be computed from the initial value $H|_{\beta=0} = p_1$, which yields

$$e^H = \sum_{\lambda} s_\lambda(1,0,0,\dots)s_\lambda e^{f(\lambda)\beta}. \tag{3.5}$$

In [12], this form of the function e^H was deduced from the representation theory of symmetric groups. Note that $c_\lambda = s_\lambda(1,0,0,\dots) = (\dim\ R_\lambda)/|\lambda|!$, where $|\lambda| = \lambda_1 + \lambda_2 + \dots$ and R_λ is the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to the partition λ.

Taking the logarithm, we obtain the few first terms in the expansion of H:

$$\begin{aligned} H = {} & p_1 + \frac{1}{4}(e^\beta - 2 + e^{-\beta})p_1^2 + \frac{1}{4}(e^\beta - e^{-\beta})p_2 + \frac{1}{36}(e^{3\beta} - 9e^\beta + 16 - 9e^{-\beta} + e^{-3\beta})p_1^3 \\ & + \frac{1}{12}(e^{3\beta} - 3e^\beta + 3e^{-\beta} - e^{-3\beta})p_1p_2 + \frac{1}{18}(e^{3\beta} - 2 + e^{-3\beta})p_3 + \dots. \end{aligned}$$

3.3. **The exponent of the generating function for Hurwitz numbers as an element of the semi-infinite Grassmannian.** In Section 2.2, the fact that the function e^H is a τ-function for the KP hierarchy was established by a reference to Okounkov's paper [12]. Here we present a more direct argument.

Theorem 3.2. *The function $e^H \in \mathcal{F}$ can be represented by the infinite wedge product $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \dots$ with*

$$\varphi_i(z) = z^i \sum_{\ell=0}^{\infty} e^{\frac{\beta}{2}((i-\frac{1}{2}-\ell)^2 - (i-\frac{1}{2})^2)}\frac{z^{-\ell}}{\ell!}, \qquad i = 1,2,3,\dots.$$

To prove this statement, we remark that the exponents on the right-hand side of the formula for φ_i contribute to the following coefficient of $s_\lambda = \bigwedge_{i=1}^{\infty} z^{i-\lambda_i}$:

$$\prod_{i=1}^{\infty} e^{\frac{\beta}{2}((i-\frac{1}{2}-\lambda_i)^2-(i-\frac{1}{2})^2)} = e^{f(\lambda)\beta}.$$

This agrees with (3.5); that is, it suffices to prove the assertion of the theorem for $\beta = 0$.

For $\beta = 0$, the wedge product reduces to $e^{z^{-1}} z \wedge e^{z^{-1}} z^2 \wedge e^{z^{-1}} z^3 \wedge \dots$. Here z^{-1} denotes the operator of multiplication by z^{-1} belonging to the special linear Lie algebra acting on the Fock space. Its exponent $e^{z^{-1}}$ is an element of the special linear Lie group. The fact that the wedge product for $\beta = 0$ corresponds to $e^{p_1} = H|_{\beta=0}$ follows from (3.3) by elementary considerations. Alternatively, one can observe that the operator of multiplication by z^{-1} acts as the multiplication by p_1 on the space of power series in variables p_i. Therefore,

$$e^{z^{-1}} z \wedge e^{z^{-1}} z^2 \wedge e^{z^{-1}} z^3 \wedge \dots = e^{z^{-1}} (z \wedge z^2 \wedge z^3 \wedge \dots) = e^{p_1} s_\varnothing = e^{p_1}.$$

This yields an independent proof of the following

Corollary 3.3. *The function H satisfies the KP hierarchy, in particular, it is a solution to the KP equation* (2.3).

The KP hierarchy degenerates into the KdV hierarchy if we are looking for solutions independent of variables with even indices (that is, the derivatives with respect to these variables vanish identically). In the normalization of the present paper, the variables t_i are related to the variables p_j by $t_i = \frac{1}{(2i-1)!!} p_{2i+1}$ which, in particular, makes F independent of $p_2, p_4, \dots$. (Recall that $(2i-1)!!$ denotes the product of odd numbers from 1 to $2i-1$, and $(-1)!! = 1$.) Since the function F given by (1.1) satisfies the KdV hierarchy, its exponent e^F is a τ-function for the KdV hierarchy. Its explicit presentation similar to that of e^H is given in [8].

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References

[1] V. I. Arnold *Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges*, Funct. Anal. and its Appl., **30**, no. 1 (1996), 1–14. MR1387484 (97d:32053)

[2] E. Date, M. Kashivara, M. Jimbo, T. Miwa, *Transformation groups for soliton equations*, Proc. of RIMS Symposium on Non-Linear Integrable Systems, Singapore, World Science Publ. Co., 1983, pp. 39–119. MR725700 (86a:58093)

[3] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109. MR0262240 (41:6850)

[4] T. Ekedahl, S. K. Lando, M. Shapiro, A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris Sér I Math., **328** (1999), 1175–1180. MR1701381 (2001b:14083)

[5] T. Ekedahl, S. K. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math., **146** (2001), 297–327. MR1864018 (2002j:14034)

[6] I. P. Goulden, D. M. Jackson, *Transitive factorisation into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc., **125**, no. 1 (1997), 51–60. MR1396978 (97j:05007)

[7] I. P. Goulden, D. M. Jackson, R. Vakil, *The Gromov–Witten potential of a point, Hurwitz numbers, and Hodge integrals*, Proc. London Math. Soc. (3), **83** (2001), 563–581. MR1851082 (2002h:14043)

[8] V. Kac, A. Schwarz, *Geometric interpretation for the partition function of 2D gravity*, Phys. Lett. B, **257**, no. 3–4 (1991), 329–334. MR1100639 (93b:58066)

[9] M. Kontsevich, *Intersection theory on the moduli space of curves and the Airy function*, Comm. Math. Phys., **147** (1992), 1–23. MR1171758 (93e:32027)

[10] S. K. Lando, A. K. Zvonkin, *Graphs on surfaces and their applications*, Springer, Berlin, 2004. MR2036721 (2005b:14068)

[11] M. Mirzakhani, *Weil–Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc., **20**, no. 1 (2007), 1–23. MR2257394

[12] A. Okounkov, *Toda equations for Hurtwitz numbers*, Math. Res. Lett. **7**, no. 4 (2000), 447–453. MR1783622 (2001i:14047)

[13] A. Okounkov, R. Pandharipande, *Gromov–Witten theory, Hurwitz numbers, and matrix models I*, math.AG/0101147 (2001).

[14] R. Pandharipande, *The Toda equations and the Gromov–Witten theory of the Riemann sphere*, Lett. Math. Phys., **53**, no. 1 (2000), 59–74. MR1799843 (2001j:14074)

[15] M. Sato, Y. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, Nonlinear partial differential equations in applied science, North-Holland, Amsterdam, 1983, pp. 259–271. MR730247 (86m:58072)

[16] G. Segal, G. Wilson, *Loop groups and equations of the KdV type*, Inst. Hautes Études Sci. Publ. Math., **61** (1985), 5–65. MR783348 (87b:58039)

[17] S. V. Shadrin, *Geometry of meromorphic functions and intersections on moduli spaces of curves*, Int. Math. Res. Notes, **38** (2003), 2051–2094. MR1994776 (2004h:14032)

[18] E. Witten, *Two-dimensional gravity and intersection theory on moduli spaces*, Surveys in Differential Geometry, **1** (1991), 243–269. MR1144529 (93e:32028)

[19] D. Zvonkine, *Enumeration of ramified coverings of the sphere and 2-dimensional gravity*, math.AG/0506248 (2005).

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