JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, January 2008, Pages 283–304
S 0894-0347(07)00567-X
Article electronically published on June 5, 2007

FORMAL DEGREES AND ADJOINT γ-FACTORS

KAORU HIRAGA, ATSUSHI ICHINO, AND TAMOTSU IKEDA

Dedicated to Professor Hiroshi Saito on the occasion of his sixtieth birthday

Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over a local field F and let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$ over F. Set $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$. Let π be an irreducible unitary representation of G and let V_π be the space of π. For $v \in V_\pi$, we will consider the integral

$$\int_H (\pi(h)v, v)\, dh. \tag{0.1}$$

We can regard this integral as an analogue of (the square of the absolute value of) a period integral of an automorphic form and expect that it is related to L and ϵ-factors. For example, let $\mathbf{G} = \mathrm{SO}(n+1) \times \mathrm{SO}(n)$ and $\mathbf{H} = \mathrm{SO}(n)$. Let $\pi = \pi_1 \otimes \pi_0$, where π_1 (resp. π_0) is an irreducible unramified tempered representation of $\mathrm{SO}(n+1, F)$ (resp. $\mathrm{SO}(n, F)$). Then (0.1) can be expressed in terms of

$$\frac{L(\frac{1}{2}, \pi_1 \times \pi_0)}{L(1, \pi_1, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})}$$

if v is unramified (cf. [20]). Now let $\mathbf{G} = \mathbf{H} \times \mathbf{H}$, where $\mathbf{H}$ is a connected reductive algebraic group over F. For simplicity, we assume that the connected center of $\mathbf{H}$ is anisotropic. Let $\pi = \pi_H \otimes \check{\pi}_H$, where π_H is a discrete series representation of H and $\check{\pi}_H$ is the contragredient representation of π_H. Then (0.1) can be expressed in terms of the formal degree $d(\pi_H)$ of π_H. In this paper, we give a conjectural formula for $d(\pi_H)$ in terms of the adjoint γ-factor

$$\gamma(s, \pi_H, \mathrm{Ad}, \psi) = \epsilon(s, \pi_H, \mathrm{Ad}, \psi) \cdot \frac{L(1-s, \check{\pi}_H, \mathrm{Ad})}{L(s, \pi_H, \mathrm{Ad})}$$

(cf. Conjecture 1.4). Here Ad is the adjoint representation of the L-group ${}^L H$ of $\mathbf{H}$ on the Lie algebra $\mathrm{Lie}(\hat{H})$ of the dual group of $\mathbf{H}$ and ψ is a non-trivial additive character of F.

Our conjecture is supported by various examples. For example, we assume that $F = \mathbb{R}$ and $\mathbf{H}$ is anisotropic. We take the Haar measure dh on H determined by a Chevalley basis of $\mathrm{Lie}(H) \otimes \mathbb{C}$. Let π_H be an irreducible finite dimensional representation of H. Then the conjecture for π_H asserts that

$$\frac{\dim \pi_H}{\mathrm{vol}(H)} = \frac{1}{2^l} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|$$

Received by the editors July 5, 2006.
2000 *Mathematics Subject Classification.* Primary 22E50.

and it is compatible with the Weyl dimension formula. Here l is the rank of $\mathbf{H}$ and $\psi(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{R}$. Also, if F is non-archimedean, then the conjecture for $\mathrm{GL}(n)$ is compatible with the result of Silberger and Zink [35], [37].

Moreover, we provide some evidence in the case of the quasi-split unitary group in three variables. To be precise, let F be a non-archimedean local field of characteristic zero. Let E be a quadratic extension of F and let σ be the non-trivial automorphism of E over F. Put

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let

$$\mathbf{H} = \mathrm{U}(3) = \{h \in \mathrm{Res}_{E/F}\,\mathrm{GL}(3) \mid \theta(h) = h\},$$

where $\theta(h) = \mathrm{Ad}(J_3)(\sigma({}^t h^{-1}))$. Following Gross [11], we choose a Haar measure dh on H. Let π_H be a stable discrete series representation of H. We will verify the conjecture for π_H, i.e.,

$$d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)| \tag{0.2}$$

(cf. Theorem 8.6).

To prove (0.2), we use twisted endoscopy. Let $J(\pi_H) = \mathrm{trace}\,\pi_H$ be the character of π_H and let $c_0(\pi_H)$ be the coefficient associated to the trivial orbit in the local character expansion of $J(\pi_H)$ (cf. [15]). Recall that

$$c_0(\pi_H) \doteq d(\pi_H). \tag{0.3}$$

Here the notation $\doteq$ indicates equality up to constants which do not depend on the representations. Let π be the base change of π_H to $\mathrm{GL}(3, E)$. Then π is square integrable since π_H is stable. Also, π is isomorphic to $\pi \circ \theta$. We fix an isomorphism $\pi(\theta) : \pi \to \pi \circ \theta$ such that $\pi(\theta)^2 = \mathrm{id}$. Let $J^\theta(\pi) = \mathrm{trace}\,\pi \circ \pi(\theta)$ be the twisted character of π and let $c_{0,\theta}(\pi)$ be the coefficient associated to the trivial orbit in the local character expansion of $J^\theta(\pi)$ (cf. [5]). The character identity between $J^\theta(\pi)$ and $J(\pi_H)$ was proved by Rogawski [31] and implies that

$$|c_{0,\theta}(\pi)| \doteq |c_0(\pi_H)|.$$

We also have an analogue

$$c_{0,\theta}(\pi) \cdot (v, \pi(\theta)v') \doteq d(\pi) \cdot J^\theta(1, f)$$

of (0.3). Here f is a matrix coefficient of π given by $f(g) = (\pi(g)v, v')$ and $J^\theta(1, f)$ is the twisted orbital integral of f at the identity element. By the result of Silberger and Zink [35], [37], we have

$$d(\pi) \doteq |\lim_{s\to 0} s^{-1}\gamma(s, \pi \times \check{\pi}, \psi)|.$$

By the results of Shahidi [32] and Goldberg [10], we have

$$|J^\theta(1, f)| \doteq |\lim_{s\to 0} s^{-1}\gamma(s, \pi, r, \psi)|^{-1} \cdot |(v, \pi(\theta)v')|,$$

where r is the Asai representation. Thus we obtain (0.2).

This paper is organized as follows. In §1, we formulate a conjecture on formal degrees and relate it to the Plancherel formula. In §2, we verify the conjecture in the archimedean case. In §3, we present various examples in the non-archimedean case. For example, the conjecture for $\mathrm{GL}(n)$ is compatible with the result of Silberger and Zink [35], [37]. Using the results of Shahidi [32], [33], [34], we give a new proof of their result in §4. In §5, we give a description of the coefficient associated to the trivial orbit in the local character expansion of a certain twisted character. After recalling some facts about twisted orbital integrals in §6, we prove this description in §7. In §8, we verify the conjecture for a stable discrete series representation of U(3).

1. Conjectures

In this section, we formulate a conjecture on formal degrees (cf. Conjecture 1.4).

Let F be a local field of characteristic zero and let ψ be a non-trivial additive character of F. Let $|\cdot|_F$ denote the absolute value on F. If F is non-archimedean, let $\mathfrak{o}_F$ be the maximal compact subring of F, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, and $q = q_F$ the cardinality of $\mathfrak{o}_F/\mathfrak{p}_F$. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ denote the absolute Galois group of F, W_F the Weil group of F, W'_F the Weil-Deligne group of F, and L_F the Langlands group of F given by

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times \mathrm{SL}(2,\mathbb{C}) & \text{if } F \text{ is non-archimedean.} \end{cases}$$

Let $\mathbf{G}$ be a connected reductive algebraic group over F. Set $G = \mathbf{G}(F)$. Let $\mathbf{G}^*$ be the quasi-split inner form of $\mathbf{G}$ and choose an inner twist $\eta : \mathbf{G} \to \mathbf{G}^*$. Let $\hat{G}$ denote the dual group of $\mathbf{G}$ and ${}^LG = \hat{G} \rtimes W_F$ the L-group of $\mathbf{G}$. We fix an F-splitting $(\mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\})$ of $\mathbf{G}^*$ and a Γ-splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ of $\hat{G}$.

Let π be a discrete series representation of G and let V_π be the space of π. We fix an invariant hermitian inner product $(\cdot,\cdot)$ on V_π. Let $d(\pi) \in \mathbb{R}_{>0}$ denote the formal degree of π. By definition, we have

$$\int_{G/A} (\pi(g)u, u')\overline{(\pi(g)v, v')}\, dg = d(\pi)^{-1}(u,v)\overline{(u',v')}$$

for $u, u', v, v' \in V_\pi$, where $\mathbf{A}$ is the split component of the center of $\mathbf{G}$ and $A = \mathbf{A}(F)$. We remark that $d(\pi) = d(\pi, dg)$ depends on the choice of dg. Following Gross [11], we take a Haar measure $\mu_{G/A,\psi}$ on G/A defined as follows. (This should not be confused with the Euler-Poincaré measure μ_G on G in the notation of [11].) We may assume that $\mathbf{A} = \{1\}$. Moreover, we may assume that $\mathbf{G}$ has an anisotropic inner form if F is archimedean. Let $\omega_{\mathbf{G}}$ be a differential form of top degree on $\mathbf{G}$ over F as in Sections 4 and 7 of [11]. Let $\mu_{G,\psi}$ denote the Haar measure on G determined by $\omega_{\mathbf{G}}$ and the self-dual measure on F with respect to ψ. Then

$$\mu_{G,\psi_a} = |a|_F^{\dim \mathbf{G}/2} \cdot \mu_{G,\psi}, \tag{1.1}$$

where $a \in F^\times$ and $\psi_a(x) = \psi(ax)$ for $x \in F$. If F is non-archimedean, ψ is of order zero, and $\mathbf{G}$ is unramified, then

$$\mu_{G,\psi}(\mathbf{G}(\mathfrak{o}_F)) = q^{-\dim \mathbf{G}}|\mathbf{G}(\mathbb{F}_q)|.$$

Here we extend $\mathbf{G}$ to a smooth group scheme over $\mathfrak{o}_F$ associated to a hyperspecial maximal compact subgroup of G.

Lemma 1.1. *Let π be a discrete series representation of G. Let $a \in F^\times$. We define a non-trivial additive character ψ_a of F by $\psi_a(x) = \psi(ax)$ for $x \in F$. Then*

$$d(\pi, \mu_{G/A,\psi_a}) = |a|_F^{-n/2} \cdot d(\pi, \mu_{G/A,\psi}),$$

where $n = \dim \mathbf{G}/\mathbf{A}$.

Proof. The lemma follows from (1.1). □

Let $\phi : L_F \to {}^L G$ be a Langlands parameter. We say that ϕ is tempered if $\phi(W_F)$ is bounded and that ϕ is elliptic if $\phi(L_F)$ is not contained in any proper parabolic subgroup of ${}^L G$. For each finite dimensional representation r of ${}^L G$, put

$$\gamma(s, r \circ \phi, \psi) = \epsilon(s, r \circ \phi, \psi) \cdot \frac{L(1 - s, \check{r} \circ \phi)}{L(s, r \circ \phi)},$$

where $\check{r}$ is the contragredient representation of r. Let Ad denote the adjoint representation of ${}^L G$ on $\mathrm{Lie}(\hat{G})/\mathrm{Lie}(Z(\hat{G})^\Gamma)$. Note that Ad is self-dual.

Lemma 1.2. *Let $\phi : L_F \to {}^L G$ be an elliptic Langlands parameter. Then at $s = 0$, $\gamma(s, \mathrm{Ad} \circ \phi, \psi)$ is holomorphic and non-zero.*

Proof. Since ϕ is elliptic, $\mathrm{Ad} \circ \phi$ does not contain the trivial representation of L_F (cf. Lemma 10.3.1 of [22]). Hence the lemma follows from the multiplicativity of γ-factors. □

Lemma 1.3. *Let $\phi : L_F \to {}^L G$ be an elliptic Langlands parameter. Let $a \in F^\times$. We define a non-trivial additive character ψ_a of F by $\psi_a(x) = \psi(ax)$ for $x \in F$. Then*

$$|\gamma(0, \mathrm{Ad} \circ \phi, \psi_a)| = |a|_F^{-n/2} \cdot |\gamma(0, \mathrm{Ad} \circ \phi, \psi)|,$$

where $n = \dim \mathbf{G}/\mathbf{A}$.

Proof. Note that $n = \dim \mathrm{Lie}(\hat{G})/\mathrm{Lie}(Z(\hat{G})^\Gamma)$. By definition, we have

$$|\epsilon(s, \mathrm{Ad} \circ \phi, \psi_a)| = |a|_F^{n(s-1/2)} \cdot |\epsilon(s, \mathrm{Ad} \circ \phi, \psi)|.$$

This yields the lemma. □

Let $\Pi(G)$ denote the set of equivalence classes of irreducible admissible representations of G. The local Langlands conjecture asserts that there exists a partition

$$\coprod_\phi \Pi_\phi(G)$$

of $\Pi(G)$ into finite subsets, where ϕ runs over equivalence classes of Langlands parameters $\phi : L_F \to {}^L G$. Let $\pi \in \Pi_\phi(G)$. If ϕ is tempered (resp. elliptic), then π is expected to be tempered (resp. essentially square integrable). For each finite dimensional representation r of ${}^L G$, put

$$L(s, \pi, r) = L(s, r \circ \phi),$$
$$\epsilon(s, \pi, r, \psi) = \epsilon(s, r \circ \phi, \psi),$$

and

$$\gamma(s, \pi, r, \psi) = \epsilon(s, \pi, r, \psi) \cdot \frac{L(1 - s, \check{\pi}, r)}{L(s, \pi, r)},$$

where $\check{\pi}$ is the contragredient representation of π.

Let $\phi : L_F \to {}^L G$ be a tempered Langlands parameter. Following [19, §1], set

$$S_\phi = \{s \in \hat{G}_{\mathrm{sc}} \mid \operatorname{Int} s \circ \phi = \phi \bmod \mathrm{B}^1(W_F, Z(\hat{G}))\}, \qquad \mathcal{S}_\phi = \pi_0(S_\phi),$$
$$S_\phi^\natural = \{s \in \hat{G}^\natural \mid \operatorname{Int} s \circ \phi = \phi\}, \qquad \mathcal{S}_\phi^\natural = \pi_0(S_\phi^\natural),$$

where $\hat{G}_{\mathrm{sc}}$ is the simply connected cover of the derived group of $\hat{G}$ and $\hat{G}^\natural$ is the dual group of $\mathbf{G}/\mathbf{A}$. Let $\mathcal{Z}_\phi$ be the image of $Z(\hat{G}_{\mathrm{sc}})$ in $\mathcal{S}_\phi$. Let $\chi_{\mathbf{G}}$ be the character of $Z(\hat{G}_{\mathrm{sc}})^\Gamma$ associated to $\mathbf{G}$ by the map

$$\mathrm{H}^1(F, \mathbf{G}^*_{\mathrm{ad}}) \longrightarrow \pi_0(Z(\hat{G}_{\mathrm{sc}})^\Gamma)^D$$

defined by Kottwitz [22], [23]. Here $\mathbf{G}^*_{\mathrm{ad}}$ is the adjoint group of $\mathbf{G}^*$. By Lemma 9.1 of [19], we can regard $\chi_{\mathbf{G}}$ as a character of the image of $Z(\hat{G}_{\mathrm{sc}})^\Gamma$ in $\mathcal{S}_\phi$. We extend $\chi_{\mathbf{G}}$ to a character of $\mathcal{Z}_\phi$. Let $\Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ denote the set of equivalence classes of irreducible representations of $\mathcal{S}_\phi$ such that $\mathcal{Z}_\phi$ acts via $\chi_{\mathbf{G}}$. It is believed that there exists a map

$$\Pi_\phi(G) \longrightarrow \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$$

which satisfies certain conditions on characters (cf. [2]). For example,

$$\sum_{\pi \in \Pi_\phi(G)} \langle 1, \pi \rangle \operatorname{trace} \pi$$

is required to be the unique (up to a scalar) stable distribution in the space of virtual characters generated by $\Pi_\phi(G)$, where

$$\langle 1, \pi \rangle = \dim \rho_\pi$$

if $\rho_\pi \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ is associated to $\pi \in \Pi_\phi(G)$. Moreover, the quantity $\langle 1, \pi \rangle$ is expected to be canonically determined by π.

Conjecture 1.4. *Let $\phi : L_F \to {}^L G$ be an elliptic tempered Langlands parameter. Then*

$$d(\pi) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_\phi^\natural|} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|$$

for $\pi \in \Pi_\phi(G)$.

We will relate Conjecture 1.4 to the Plancherel formula. We fix a non-trivial additive character ψ of F. Let Θ be the set of pairs $(\mathfrak{O}, \mathbf{P} = \mathbf{MN})$, where $\mathbf{P}$ is a semi-standard parabolic subgroup of $\mathbf{G}$, $\mathbf{M}$ is the Levi subgroup of $\mathbf{P}$, $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$, and $\mathfrak{O}$ is an orbit in the set of equivalence classes of discrete series representations of M under the action of the group of unramified unitary characters of M. For $(\mathfrak{O}, \mathbf{P} = \mathbf{MN}) \in \Theta$ and $\pi \in \mathfrak{O}$, put

$$d\nu(\pi) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_{\phi_M}^\natural|} \cdot |\gamma(0, \pi, r_M, \psi)| \cdot d\pi.$$

Here $\phi_M : L_F \to {}^L M$ is the (conjectural) Langlands parameter associated to π, r_M is the adjoint representation of ${}^L M$ on $\operatorname{Lie}(\hat{G})/\operatorname{Lie}(Z(\hat{M})^\Gamma)$, and $d\pi$ is the Lebesgue measure on $\mathfrak{O}$ (cf. [36, pp. 239 and 302]). Then the Plancherel formula (cf. Theorem 27.3 of [14] and Théorème VIII.1.1 of [36]), Langlands' conjecture on Plancherel measures (cf. Appendix II of [26]), and Conjecture 1.4 suggest that the following conjecture holds.

Conjecture 1.5. *There exist explicit constants $c_M \in \mathbb{R}_{>0}$ which do not depend on $\mathfrak{O}$ such that*

$$f(1) = \sum_{(\mathfrak{O}, \mathbf{P}=\mathbf{MN}) \in \Theta} c_M \int_{\mathfrak{O}} \operatorname{trace} \operatorname{Ind}_P^G(\pi)(f)\, d\nu(\pi)$$

for $f \in C_c^\infty(G)$.

2. Examples: The archimedean case

In this section, we verify Conjecture 1.4 in the archimedean case.

Let $F = \mathbb{R}$. By Lemmas 1.1 and 1.3, we may assume that $\psi(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{R}$. Let $\mathbf{G}$ be a connected reductive algebraic group of rank l over $\mathbb{R}$. For simplicity, we assume that the connected center of $\mathbf{G}$ is anisotropic. We may assume that $\mathbf{G}$ has an anisotropic inner form $\mathbf{G}_{\mathrm{an}}$.

Proposition 2.1. *Let π be a discrete series representation of G. Then*

$$d(\pi) = \frac{1}{2^l} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|.$$

In particular, Conjecture 1.4 holds for π.

The rest of this section is devoted to the proof of Proposition 2.1. Let $\check{\Sigma}$ denote the set of roots of $\mathcal{T}$ in $\hat{G}$ and $\check{\Sigma}^+$ the subset of positive roots determined by $\mathcal{B}$. Let N be the number of positive roots. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $X_*(\mathcal{T}) \otimes \mathbb{Q}$ and $X^*(\mathcal{T}) \otimes \mathbb{Q}$.

Lemma 2.2. *Let π_λ be a discrete series representation of G with Harish-Chandra parameter λ. Then*

$$|\gamma(0, \pi_\lambda, \mathrm{Ad}, \psi)| = \pi^{-l} \cdot (2\pi)^{-N} \prod_{\check{\alpha} \in \check{\Sigma}^+} |\langle \lambda, \check{\alpha} \rangle|.$$

Proof. Let $\phi : W_\mathbb{R} \to {}^L G$ be the Langlands parameter associated to π_λ. Then $\phi(z) = z^\lambda \bar{z}^{-\lambda}$ for $z \in W_\mathbb{C}$. The action $\mathrm{Ad} \circ \phi$ of $W_\mathbb{R}$ on $\mathrm{Lie}(\mathcal{T})$ (resp. $\mathbb{C}\mathcal{X}_{\check{\alpha}} \oplus \mathbb{C}\mathcal{X}_{-\check{\alpha}}$) is given by the sign character (resp. $\operatorname{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}(\phi_{\check{\alpha}})$). Here $\check{\alpha} \in \check{\Sigma}^+$ and $\phi_{\check{\alpha}}(z) = z^{\langle \lambda, \check{\alpha} \rangle} \bar{z}^{-\langle \lambda, \check{\alpha} \rangle}$ for $z \in W_\mathbb{C}$. Hence we have

$$L(s, \pi_\lambda, \mathrm{Ad}) = \Gamma_\mathbb{R}(s+1)^l \prod_{\check{\alpha} \in \check{\Sigma}^+} \Gamma_\mathbb{C}(s + |\langle \lambda, \check{\alpha} \rangle|),$$

where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. By definition, $\epsilon(s, \pi_\lambda, \mathrm{Ad}, \psi)$ is a power of $\sqrt{-1}$. This yields the lemma. □

Set $\mathfrak{g} = \mathrm{Lie}(G)$. Let θ be a Cartan involution of $\mathfrak{g}$ and let B be a symmetric bilinear form on $\mathfrak{g}$ over $\mathbb{R}$ which satisfies the conditions of Lemma 3.2 of [13]. Then the quadratic form

$$\|X\|^2 = -B(X, \theta(X))$$

for $X \in \mathfrak{g}$ is positive definite. This norm $\|\cdot\|$ on $\mathfrak{g}$ defines a Lebesgue measure on $\mathfrak{g}$ and hence a Haar measure dG on G via the exponential map. Let $\mathbf{T}$ be an anisotropic maximal torus of $\mathbf{G}$ such that $\mathrm{Lie}(T)$ is θ-invariant. Similarly, we can define a Haar measure dT on T.

Set $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes \mathbb{C}$ and $\mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}(T) \otimes \mathbb{C}$. Let Σ denote the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ and Σ^+ the subset of positive roots. We extend B to a symmetric bilinear form on $\mathfrak{g}_{\mathbb{C}}$ over $\mathbb{C}$. For $\alpha \in \Sigma$, we define $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ by

$$B(H, H_\alpha) = \alpha(H)$$

for $H \in \mathfrak{t}_{\mathbb{C}}$. Put

$$\varpi = \prod_{\alpha \in \Sigma^+} H_\alpha.$$

Lemma 2.3. *Let π_λ be a discrete series representation of G with Harish-Chandra parameter λ. Then*

$$d(\pi_\lambda, dG) = (2\pi)^{-N} \cdot |\varpi(\lambda)| \cdot \mathrm{vol}(T, dT)^{-1}.$$

Proof. Let K be a maximal compact subgroup of G such that $\mathrm{Lie}(K)$ is θ-invariant and let dK be the Haar measure on K determined by $\|\cdot\|$. Let dx be the standard measure on G as in [13, §7]. By Lemma 37.2 of [13], we have

$$dx = 2^{\nu/2} \cdot \mathrm{vol}(K, dK)^{-1} \cdot dG,$$

where $\nu = \dim G/K - \mathrm{rank}\, G/K$. By Corollary of Lemma 23.1 of [14], we have

$$d(\pi_\lambda, dx) = c_G^{-1} \cdot |W| \cdot |\varpi(\lambda)|.$$

Here W is the Weyl group of T in G and

$$c_G = 2^{\nu/2} \cdot (2\pi)^N \cdot |W| \cdot \frac{\mathrm{vol}(T, dT)}{\mathrm{vol}(K, dK)}$$

(cf. Lemma 37.3 of [13]). This completes the proof. $\square$

Lemma 2.4. *Let π be a discrete series representation of G and let $\phi : W_{\mathbb{R}} \to {}^L G$ be the Langlands parameter associated to π. Let π_{an} be the irreducible finite dimensional representation of G_{an} associated to ϕ by the local Langlands correspondence. Then*

$$d(\pi) = d(\pi_{\mathrm{an}}).$$

Proof. We extend θ to an anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ over $\mathbb{C}$. Set $\mathfrak{g}_{\mathrm{an}} = \mathrm{Lie}(G_{\mathrm{an}})$. We may identify $\mathfrak{g}_{\mathrm{an}}$ with $\mathfrak{g}_{\mathbb{C}}^{\theta}$. Then the restrictions of θ and B to $\mathfrak{g}_{\mathrm{an}}$ define a norm $\|\cdot\|_{\mathrm{an}}$ on $\mathfrak{g}_{\mathrm{an}}$. Let dG_{an} be the Haar measure on G_{an} determined by $\|\cdot\|_{\mathrm{an}}$. Then dG and dG_{an} are compatible. By Lemma 2.3, we have

$$d(\pi, dG) = d(\pi_{\mathrm{an}}, dG_{\mathrm{an}}).$$

By definition, $\mu_{G,\psi}$ and $\mu_{G_{\mathrm{an}},\psi}$ are also compatible. This yields the lemma. $\square$

By Lemma 2.4, to prove Proposition 2.1, we may assume that $\mathbf{G}$ is anisotropic. Let π be an irreducible finite dimensional representation of G. By Lemmas 2.2 and 2.3, there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on π such that

$$d(\pi) = c|\gamma(0, \pi, \mathrm{Ad}, \psi)|.$$

By [27], [11, §7], we have

$$\mathrm{vol}(G) = 2^N \prod_{i=1}^{l} \frac{2\pi^{m_i+1}}{m_i!} = (2\pi)^{l+N} \prod_{\check{\alpha} \in \check{\Sigma}^+} \langle \rho, \check{\alpha} \rangle^{-1}.$$

Here $m_1, \ldots, m_l$ are the exponents of $\mathbf{G}$ and ρ is half the sum of positive roots. Note that

$$\sum_{i=1}^{l} m_i = N,$$
$$\prod_{i=1}^{l} m_i! = \prod_{\check{\alpha} \in \check{\Sigma}^+} \langle \rho, \check{\alpha} \rangle.$$

By Lemma 2.2, we have $\mathrm{vol}(G) = 2^l |\gamma(0, \pi_\rho, \mathrm{Ad}, \psi)|^{-1}$, where π_ρ is the trivial representation of G. Hence we have $c = 2^{-l}$. This completes the proof of Proposition 2.1.

3. Examples: The non-archimedean case

Let F be a non-archimedean local field of characteristic zero. By Lemmas 1.1 and 1.3, we may assume that ψ is of order zero.

3.1. **Inner forms of** $\mathrm{GL}(n)$**.** We first recall the following result of Silberger and Zink [35], [37].

Theorem 3.1. *Let π be a discrete series representation of* $\mathrm{GL}(n, F)$*. Then*

$$d(\pi) = \frac{1}{n} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|.$$

In particular, Conjecture 1.4 *holds for π.*

To be precise, let π be the unique irreducible subrepresentation of an induced representation

$$\sigma |\det|_F^{(e-1)/2} \times \sigma |\det|_F^{(e-3)/2} \times \cdots \times \sigma |\det|_F^{-(e-1)/2},$$

where σ is an irreducible unitary supercuspidal representation of $\mathrm{GL}(m, F)$ with $n = em$. Using the theory of types, Silberger and Zink showed that $d(\pi)$ is equal to

$$r \cdot \frac{q^{em} - 1}{q^{er} - 1} \cdot q^{e(r-m)/2 + e^2(f+r-m^2)/2} \cdot \frac{1}{n} \prod_{i=1}^{n-1} (q^i - 1) \cdot \mathrm{vol}(\mathrm{GL}(n, \mathfrak{o}_F)/\mathfrak{o}_F^\times)^{-1}$$

(cf. Theorems 6.5 and 6.9 of [3]). Here r is the torsion number of σ and f is the conductor of $\sigma \times \check{\sigma}$. It is easy to check that this quantity coincides with $n^{-1} |\gamma(0, \pi, \mathrm{Ad}, \psi)|$. In §4, we will give a new proof of Theorem 3.1 which does not rely on the theory of types.

Let $\mathbf{G}$ be an inner form of $\mathrm{GL}(n)$ over F. Then $G = \mathrm{GL}(n', D)$ with $n = dn'$, where D is a division algebra of dimension d^2 over F. Let π be a discrete series representation of G. By Theorem 7.2 of [3], we have

$$d(\pi) = \prod_{\substack{1 \le i \le n \\ i \not\equiv 0 \bmod d}} (q^i - 1)^{-1} \cdot d(\pi^*) \cdot \frac{\mathrm{vol}(\mathrm{GL}(n, \mathfrak{o}_F)/\mathfrak{o}_F^\times)}{\mathrm{vol}(\mathrm{GL}(n', \mathfrak{o}_D)/\mathfrak{o}_F^\times)},$$

where π^* is the discrete series representation of $\mathrm{GL}(n, F)$ associated to π by the Deligne-Kazhdan-Vignéras correspondence [9]. Since

$$\mathrm{vol}(\mathrm{GL}(n', \mathfrak{o}_D)) = q^{-(d-1)dn'^2/2} \prod_{i=1}^{n'} (1 - q^{-di}),$$

we obtain

$$d(\pi) = d(\pi^*).$$

3.2. **Inner forms of** $\mathrm{SL}(n)$**.** Let $\tilde{\mathbf{G}}$ be an inner form of $\mathrm{GL}(n)$ over F and let $\mathbf{G}$ be the derived group of $\tilde{\mathbf{G}}$. Then $\mathbf{G}$ is an inner form of $\mathrm{SL}(n)$ over F. Let $\mathbf{G}_{\mathrm{ad}}$ be the adjoint group of $\mathbf{G}$. Set $C = \mathrm{cok}(G \to G_{\mathrm{ad}})$.

Let $\phi : L_F \to {}^L G$ be an elliptic Langlands parameter. Then there exists an elliptic tempered Langlands parameter $\tilde{\phi} : L_F \to {}^L\tilde{G}$ such that $\phi = \mathrm{pr} \circ \tilde{\phi}$, where $\mathrm{pr} : {}^L\tilde{G} \to {}^L G$ is the projection. Let $\tilde{\pi}$ be the discrete series representation of $\tilde{G}$ associated to $\tilde{\phi}$ by the local Langlands correspondence [16], [17] and let $V_{\tilde{\pi}}$ be the space of $\tilde{\pi}$. Let $\Pi_\phi(G)$ denote the set of equivalence classes of irreducible constituents of the restriction of $\tilde{\pi}$ to G. Note that $\Pi_\phi(G)$ does not depend on the choice of $\tilde{\phi}$. Put

$$X(\tilde{\pi}) = \{\omega \in C^D \mid \tilde{\pi} \otimes \omega \simeq \tilde{\pi}\},$$

where C^D is the Pontrjagin dual of C and ω is regarded as a character of G_{ad}. For $s \in S_\phi$, we have

$$\mathrm{Int}\, s \circ \tilde{\phi} = a_s \cdot \tilde{\phi},$$

where a_s is a 1-cocycle of W_F in $Z(\hat{G}_{\mathrm{sc}})$. Let ω_s be the character of C determined by a_s. Then the map $s \mapsto \omega_s$ induces an exact sequence

$$1 \longrightarrow \mathcal{Z}_\phi \longrightarrow \mathcal{S}_\phi \longrightarrow X(\tilde{\pi}) \longrightarrow 1.$$

By Theorem 1.4 of [19], there exists an action of $\mathcal{S}_\phi$ on $V_{\tilde{\pi}}$ such that $\mathcal{Z}_\phi$ acts via $\chi_{\mathbf{G}}$ and such that

$$\tilde{\pi} \circ s = s \circ (\tilde{\pi} \otimes \omega_s)$$

for $s \in \mathcal{S}_\phi$. Moreover, if we write a decomposition of $V_{\tilde{\pi}}$ as a representation of $\mathcal{S}_\phi \times G$ in the form

$$\bigoplus_{\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})} \rho \otimes \pi_\rho,$$

then the map $\rho \mapsto \pi_\rho$ defines a bijection between $\Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ and $\Pi_\phi(G)$ (cf. Theorem 1.1 of [19]).

Lemma 3.2. *For $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$, we have*

$$d(\pi_\rho) = n^2 \cdot \frac{\dim \rho}{|\mathcal{S}_\phi|} \cdot d(\tilde{\pi}).$$

Proof. We fix an invariant hermitian inner product $(\cdot,\cdot)$ on $V_{\tilde{\pi}}$. Then $(\cdot,\cdot)$ is $\mathcal{S}_\phi$-invariant. Let v be an element in the π_ρ-isotypic subspace of $V_{\tilde{\pi}}$. Recall that the sequence

$$1 \longrightarrow G/\mu_n(F) \longrightarrow G_{\mathrm{ad}} \longrightarrow C \longrightarrow 1$$

is exact. Here μ_n is the group of n-th roots of unity. Since the pullback of $\omega_{\mathbf{G}_{\mathrm{ad}}}$ to $\mathbf{G}$ is $n\omega_{\mathbf{G}}$ and $|C|^{-1}\sum_{\omega \in C^D} \omega$ is the characteristic function of $G/\mu_n(F)$, we have

$$d(\pi_\rho)^{-1}(v,v)\overline{(v,v)} = \frac{|\mu_n(F)|}{|n|_F \cdot |C|} \sum_{\omega \in C^D} \int_{G_{\mathrm{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)\overline{(\tilde{\pi}(g)v, v)}\, dg.$$

By the Schur orthogonality relations, we have

$$\int_{G_{\mathrm{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)\overline{(\tilde{\pi}(g)v, v)}\, dg = 0$$

unless $\omega \in X(\tilde{\pi})$. Moreover, we have

$$\int_{G_{\mathrm{ad}}} ((\tilde{\pi} \otimes \omega_s)(g)v, v)\overline{(\tilde{\pi}(g)v, v)}\, dg = \int_{G_{\mathrm{ad}}} (\tilde{\pi}(g)sv, sv)\overline{(\tilde{\pi}(g)v, v)}\, dg$$
$$= d(\tilde{\pi})^{-1}(sv, v)\overline{(sv, v)}$$

for $s \in \mathcal{S}_\phi$. Thus we obtain

$$d(\pi_\rho)^{-1}(v, v)\overline{(v, v)} = \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \sum_{s \in \mathcal{S}_\phi} d(\tilde{\pi})^{-1}(sv, v)\overline{(sv, v)}$$
$$= \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \cdot \frac{|\mathcal{S}_\phi|}{\dim \rho} \cdot d(\tilde{\pi})^{-1}(v, v)\overline{(v, v)}.$$

Note that

$$|n|_F = \frac{|\mathrm{H}^0(F, \mu_n)| \cdot |\mathrm{H}^2(F, \mu_n)|}{|\mathrm{H}^1(F, \mu_n)|} = \frac{|\mu_n(F)| \cdot n}{|C|}.$$

This yields the lemma. □

By Theorem 3.1 and Lemma 3.2, we have

$$d(\pi_\rho) = n \cdot \frac{\dim \rho}{|\mathcal{S}_\phi|} \cdot |\gamma(0, \pi_\rho, \mathrm{Ad}, \psi)|$$

for $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$.

3.3. **Steinberg representations.** Let $\mathbf{G}$ be a connected reductive algebraic group over F. For simplicity, we assume that the connected center of $\mathbf{G}$ is anisotropic. Let π_0 be the Steinberg representation of G. Note that the formal degree of π_0 was computed by Borel [4]. Using the results of Kottwitz [24] and Gross [11], [12], we will verify Conjecture 1.4 for π_0. In particular, if $\mathbf{G}$ is an anisotropic torus, then Conjecture 1.4 holds.

Let $\mu_{G,\mathrm{EP}}$ denote the Euler-Poincaré measure on G and let $f_{\mathrm{EP}} \in C_c^\infty(G)$ denote the Euler-Poincaré function with respect to $\mu_{G,\mathrm{EP}}$. By Theorems 2 and 2′ of [24] and the Plancherel formula (cf. Théorème VIII.1.1 of [36]), we have $f_{\mathrm{EP}}(1) = 1$ and

$$|d(\pi_0, \mu_{G,\mathrm{EP}})| = 1.$$

By Theorem 5.5 of [11], we have

$$e(\mathbf{G}) \cdot |\mathrm{H}^1(F, \mathbf{G})| \cdot L(M) \cdot \mu_{G,\mathrm{EP}} = L(M^\vee(1)) \cdot \mu_{G,\psi}.$$

Here $e(\mathbf{G}) = \pm 1$ is the Kottwitz sign, M is the motive of $\mathbf{G}$ as in [11], and $M^\vee(1) = M^\vee \otimes \mathbb{Q}(1)$ is the Tate twist of the dual motive of M. Hence we have

$$d(\pi_0) = |\mathrm{H}^1(F, \mathbf{G})|^{-1} \cdot \frac{|L(M^\vee(1))|}{|L(M)|}.$$

Let $\phi_0 : L_F \to {}^L G$ be the Langlands parameter associated to π_0. Then ϕ_0 is trivial on W_F and the restriction of ϕ_0 to $\mathrm{SL}(2, \mathbb{C})$ corresponds to the regular unipotent orbit in $\hat{G}$. Hence the centralizer of $\phi_0(L_F)$ in $\hat{G}$ is $Z(\hat{G})^\Gamma$ and $|\mathcal{S}_{\phi_0}^\natural| = |\mathrm{H}^1(F, \mathbf{G})|$.

Lemma 3.3.

$$|\gamma(0, \pi_0, \mathrm{Ad}, \psi)| = \frac{|L(M^\vee(1))|}{|L(M)|}.$$

Proof. By Corollary 6.5 of [12], we have $L(M^\vee(1)) = L(1, \pi_0, \mathrm{Ad})$. Thus it suffices to show that

$$|L(M)|^{-1} = |\epsilon(0, \pi_0, \mathrm{Ad}, \psi)| \cdot |L(0, \pi_0, \mathrm{Ad})|^{-1}.$$

Recall that $M = \bigoplus_{d \geq 1} V_d(1-d)$, where V_d is the Artin motive as in [11, §1]. By Proposition 6.4 of [12], we have

$$L(s, \pi_0, \mathrm{Ad})^{-1} = \prod_{d \geq 1} \det(1 - q^{-s-d+1} \cdot \mathrm{Frob}\,;\, V_d^{I_F}),$$

where I_F is the inertia group of F. Since V_d is self-dual, we have

$$\begin{aligned} |L(M)|^{-1} &= \prod_{d \geq 1} |\det(1 - q^{d-1} \cdot \mathrm{Frob}\,;\, V_d^{I_F})| \\ &= \prod_{d \geq 1} q^{(d-1)\dim V_d^{I_F}} \cdot |L(0, \pi_0, \mathrm{Ad})|^{-1}. \end{aligned}$$

Set $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$. Let (ρ, N) be the representation of W_F' on $\hat{\mathfrak{g}}$ associated to $\mathrm{Ad} \circ \phi_0$. We can regard N as a regular nilpotent element in $\hat{\mathfrak{g}}$. By definition, we have

$$|\epsilon(s, \pi_0, \mathrm{Ad}, \psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)},$$

where $a(\hat{\mathfrak{g}}) = \dim \hat{\mathfrak{g}}^{I_F} - \dim \hat{\mathfrak{g}}_N^{I_F}$. By Proposition 5.2 of [12], we have $\hat{\mathfrak{g}} = \bigoplus_{d \geq 1} V_d \otimes \rho_{2d-2}$ as a representation of $\Gamma \times \mathrm{SL}(2, \mathbb{C})$. Here ρ_k is the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ of dimension $k+1$. Hence we have

$$a(\hat{\mathfrak{g}}) = 2 \sum_{d \geq 1} (d-1) \dim V_d^{I_F}.$$

This completes the proof. □

Thus we obtain

$$d(\pi_0) = \frac{1}{|\mathcal{S}_{\phi_0}^\natural|} \cdot |\gamma(0, \pi_0, \mathrm{Ad}, \psi)|.$$

3.4. **Unipotent discrete series representations.** Let $\mathbf{G}$ be a connected adjoint split exceptional group of rank l over F. Let $\phi : L_F \to {}^L G$ be an elliptic Langlands parameter. We assume that ϕ is trivial on the inertia group I_F of F. Put

$$t = \phi\left(\mathrm{Frob} \times \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}\right).$$

We may assume that $t \in \mathcal{T}$. Let $\check{\Sigma}$ denote the set of roots of $\mathcal{T}$ in $\hat{G}$. For $i \in \mathbb{Z}$, put

$$\check{\Sigma}(i) = \{\check{\alpha} \in \check{\Sigma} \,|\, \check{\alpha}(t) = q^{-i/2}\}.$$

For each $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$, Reeder defined a discrete series representation π_ρ of G and showed that

$$d(\pi_\rho) = \frac{q^N \dim \rho}{|\mathcal{S}_\phi^\natural|} \cdot \frac{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(0)} (\check{\alpha}(t) - 1)}{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(2)} (q\check{\alpha}(t) - 1)} \cdot \mathrm{vol}(I)^{-1}$$

(cf. [29, (0.3)]). Here N is the number of positive roots and I is an Iwahori subgroup of G. Note that $\mathrm{vol}(I) = q^{-N}(1 - q^{-1})^l$.

Lemma 3.4.

$$|\gamma(0,\pi_\rho,\mathrm{Ad},\psi)| = q^{2N}(1-q^{-1})^{-l}\cdot \frac{\prod_{\check\alpha\in\check\Sigma-\check\Sigma(0)}|\check\alpha(t)-1|}{\prod_{\check\alpha\in\check\Sigma-\check\Sigma(2)}|q\check\alpha(t)-1|}.$$

Proof. It is easy to check that

$$\gamma(s,\pi_\rho,\mathrm{Ad},\psi) = \left(\frac{1-q^{-s}}{1-q^{-1+s}}\right)^l \prod_{\check\alpha\in\check\Sigma}\frac{1-\check\alpha(t)^{-1}q^{-s}}{1-\check\alpha(t)^{-1}q^{-1+s}}.$$

By [29, (7.2a)], we have $|\check\Sigma(2)| = |\check\Sigma(0)| + l$. This yields the lemma. □

Thus we obtain

$$d(\pi_\rho) = \frac{\dim\rho}{|\mathcal{S}_\phi^\natural|}\cdot|\gamma(0,\pi_\rho,\mathrm{Ad},\psi)|$$

for $\rho\in\Pi(\mathcal{S}_\phi,\chi_{\mathbf{G}})$.

3.5. **Depth-zero supercuspidal representations.** Let $\mathbf{G}$ be a connected reductive algebraic group of rank l over F. We assume that $\mathbf{G}^*$ is unramified and $\mathbf{G}$ is a pure inner form of $\mathbf{G}^*$. For simplicity, we assume that the connected center of $\mathbf{G}$ is anisotropic. Let $\phi: L_F\to {}^LG$ be an elliptic Langlands parameter. We assume that ϕ is trivial on I_F^+ and that the centralizer of $\phi(I_F)$ in $\hat G$ is $\mathcal{T}$. Here I_F is the inertia group of F and I_F^+ is the wild inertia subgroup of I_F. Note that ϕ is trivial on $\mathrm{SL}(2,\mathbb{C})$. Put $\sigma=\phi(\mathrm{Frob})$. Then σ normalizes $\mathcal{T}$ and $\mathcal{S}_\phi^\natural$ is isomorphic to $\mathcal{T}^\sigma$, where $\mathcal{T}^\sigma$ is the centralizer of σ in $\mathcal{T}$.

Let $\mathbf{T}$ be an unramified maximal torus of $\mathbf{G}$ determined by σ. Then $\mathbf{T}$ is anisotropic. Let $\mathbf{T}_0$ be an unramified maximal torus of $\mathbf{G}$ which is maximally split. We extend $\mathbf{T}$ and $\mathbf{T}_0$ to smooth group schemes over $\mathfrak{o}_F$. For each $\rho\in\Pi(\mathcal{S}_\phi,\chi_{\mathbf{G}})$, DeBacker and Reeder defined an irreducible supercuspidal representation π_ρ of G and showed that

$$d(\pi_\rho) = q^{l/2}|\mathbf{T}(\mathbb{F}_q)|^{-1}\cdot q^{-l/2}|\mathbf{T}_0(\mathbb{F}_q)|\cdot\mathrm{vol}(I)^{-1}$$

(cf. [8, §5.3]). Here I is an Iwahori subgroup of G. By [11, (4.11)], we have $\mathrm{vol}(I)=q^{-l-N}|\mathbf{T}_0(\mathbb{F}_q)|$, where $N=(\dim\mathbf{G}-l)/2$. Hence we have

$$d(\pi_\rho) = q^{l+N}|\mathbf{T}(\mathbb{F}_q)|^{-1}.$$

Lemma 3.5.

$$|\gamma(0,\pi_\rho,\mathrm{Ad},\psi)| = q^{l+N}|\mathbf{T}(\mathbb{F}_q)|^{-1}\cdot|\mathcal{T}^\sigma|.$$

Proof. Set $\hat{\mathfrak{g}}=\mathrm{Lie}(\hat G)$. Then $\hat{\mathfrak{g}}^{I_F}=\mathrm{Lie}(\mathcal{T})$. Here the action of W_F' on $\hat{\mathfrak{g}}$ is associated to $\mathrm{Ad}\circ\phi$. By definition, we have

$$L(s,\pi_\rho,\mathrm{Ad}) = \det(1-q^{-s}\cdot\sigma\,;\,\mathrm{Lie}(\mathcal{T}))^{-1}.$$

Hence we have

$$L(1,\pi_\rho,\mathrm{Ad}) = q^l|\mathbf{T}(\mathbb{F}_q)|^{-1}.$$

Since $\mathcal{T}^\sigma$ is isomorphic to $\{x\in X_*(\mathcal{T})\otimes\mathbb{C}\,|\,(1-\sigma)x\in X_*(\mathcal{T})\}/X_*(\mathcal{T})$, we have

$$|L(0,\pi_\rho,\mathrm{Ad})|^{-1} = |\det(1-\sigma\,;\,\mathrm{Lie}(\mathcal{T}))| = |\mathcal{T}^\sigma|.$$

By definition, we have

$$|\epsilon(s,\pi_\rho,\mathrm{Ad},\psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)},$$

where $a(\hat{\mathfrak{g}})=\dim\hat{\mathfrak{g}}/\hat{\mathfrak{g}}^{I_F}=2N$. This completes the proof. □

Thus we obtain

$$d(\pi_\rho) = \frac{1}{|\mathcal{S}_\phi^\natural|} \cdot |\gamma(0, \pi_\rho, \mathrm{Ad}, \psi)|$$

for $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$.

4. Proof of Theorem 3.1

In this section, we give a new proof of the result of Silberger and Zink [35], [37].

Let π be a discrete series representation of $\mathrm{GL}(n, F)$ and let V_π be the space of π. We fix an invariant hermitian inner product $(\cdot, \cdot)$ on V_π and equip $V_\pi \otimes V_\pi$ with the invariant hermitian inner product such that $(u \otimes v, u' \otimes v') = (u, u')(v, v')$.

Let $G^\sharp = \mathrm{GL}(2n, F)$. Let $P^\sharp = M^\sharp N^\sharp$ be a parabolic subgroup of $G^\sharp$ given by

$$M^\sharp = \left\{ \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} \,\middle|\, a, a' \in \mathrm{GL}(n, F) \right\},$$
$$N^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \,\middle|\, x \in \mathrm{Mat}_{n \times n}(F) \right\}.$$

We consider an induced representation

$$I(s, \pi \otimes \pi) = \mathrm{Ind}_{P^\sharp}^{G^\sharp}(\pi |\det|_F^{s/2} \otimes \pi |\det|_F^{-s/2})$$

for $s \in \mathbb{C}$. Put

$$w = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \in G^\sharp.$$

For $\phi \in I(s, \pi \otimes \pi)$ and $g \in G^\sharp$, the integral

$$M(s, w, \pi \otimes \pi)\phi(g) = \int_{\mathrm{Mat}_{n \times n}(F)} \phi \left(w^{-1} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} g \right) dx$$

is absolutely convergent for $\mathrm{Re}(s) > 0$, has a meromorphic continuation to the whole s-plane, and defines an intertwining operator

$$M(s, w, \pi \otimes \pi) : I(s, \pi \otimes \pi) \longrightarrow I(-s, w(\pi \otimes \pi)).$$

Here dx is the Haar measure on $\mathrm{Mat}_{n \times n}(F)$ with $\mathrm{vol}(\mathrm{Mat}_{n \times n}(\mathfrak{o}_F), dx) = 1$.

Lemma 4.1. *There exists a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that*

$$\begin{aligned}&(\operatorname*{Res}_{s=0} M(s, w, \pi \otimes \pi)\phi(1), u' \otimes v') \\ &= \alpha (\log q)^{-1}(1 - q^{-1})\gamma(0, \pi, \mathrm{Ad}, \psi)^{-1}(\phi(1), v' \otimes u')\end{aligned}$$

for $\phi \in I(s, \pi \otimes \pi)$ and $u', v' \in V_\pi$. Here ψ is a non-trivial additive character of F of order zero.

Proof. Set $I(\pi \otimes \pi) = I(0, \pi \otimes \pi)$. Let $\mathrm{sw} : V_\pi \otimes V_\pi \to V_\pi \otimes V_\pi$ be an isomorphism given by $\mathrm{sw}(u \otimes v) = v \otimes u$. Then sw induces an isomorphism $\mathrm{sw} : I(w(\pi \otimes \pi)) \to I(\pi \otimes \pi)$. We define a normalized intertwining operator

$$N(w, \pi \otimes \pi) : I(\pi \otimes \pi) \longrightarrow I(\pi \otimes \pi)$$

by

$$N(w, \pi \otimes \pi) = \mathrm{sw} \lim_{s \to 0} \gamma(s, \pi \times \check{\pi}, \psi) M(s, w, \pi \otimes \pi).$$

By Theorem 7.9 of [32], $N(w, \pi \otimes \pi)$ is unitary. Since $I(\pi \otimes \pi)$ is irreducible, there exists a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$N(w, \pi \otimes \pi) = \alpha \, \mathrm{id},$$

i.e.,

$$\mathrm{sw}\operatorname*{Res}_{s=0} M(s,w,\pi\otimes\pi)\phi(g) = \alpha \operatorname*{Res}_{s=0}\gamma(s,\pi\times\check{\pi},\psi)^{-1}\phi(g)$$

for $\phi \in I(s,\pi\otimes\pi)$ and $g \in G^\sharp$. □

Let

$$\bar{N}^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \middle| \, x \in \mathrm{Mat}_{n\times n}(F) \right\}$$

and $L = \mathrm{Mat}_{n\times n}(\mathfrak{o}_F)$. Let $\mathbf{1}_L$ denote the characteristic function of L.

Lemma 4.2. *Let $u, v \in V_\pi$. We define $\phi \in I(s,\pi\otimes\pi)$ which has compact support in $P^\sharp\bar{N}^\sharp$ modulo $P^\sharp$ by*

$$\phi\left(\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix}\right) = \begin{cases} u\otimes v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

Then

$$\begin{aligned}&(\operatorname*{Res}_{s=0} M(s,w,\pi\otimes\pi)\phi(1), u'\otimes v') \\ &= (n\log q)^{-1}(1-q^{-1})d(\pi)^{-1}(\phi(1), v'\otimes u')\end{aligned}$$

for $u', v' \in V_\pi$.

Proof. The lemma follows from Proposition 5.1 of [34]. We include the proof for the sake of completeness.

As in [33], [34], we have

$$\begin{aligned}&(M(s,w,\pi\otimes\pi)\phi(1), u'\otimes v') \\ &= \int_{\mathrm{GL}(n,F)} \mathbf{1}_L(x^{-1})|\det(x)|_F^{-s-n}(\pi(x^{-1})u\otimes\pi(x)v, u'\otimes v')\,dx \\ &= \int_{\mathrm{GL}(n,F)} \mathbf{1}_L(x)|\det(x)|_F^{s}(\pi(x)u,u')\overline{(\pi(x)v',v)}\,d^\times x \\ &= \int_{\mathrm{GL}(n,F)/F^\times} \varphi_s(x)(\pi(x)u,u')\overline{(\pi(x)v',v)}\,d^\times x,\end{aligned}$$

where $d^\times x = |\det(x)|_F^{-n}\,dx$ and

$$\varphi_s(x) = |\det(x)|_F^s \int_{F^\times} \mathbf{1}_L(zx)|z|_F^{ns}\,d^\times z.$$

For $x = (x_{ij}) \in \mathrm{GL}(n,F)$, we have

$$\int_{F^\times} \mathbf{1}_L(zx)|z|_F^{ns}\,d^\times z = \int_{\mathfrak{p}_F^{-m}} |z|_F^{ns}\,d^\times z = q^{mns}(1-q^{-ns})^{-1}(1-q^{-1}),$$

where $m = \min(\mathrm{ord}_F(x_{ij}))$. Note that this integral is absolutely convergent for $\mathrm{Re}(s) > 0$. Hence we have

$$\begin{aligned}&(\operatorname*{Res}_{s=0} M(s,w,\pi\otimes\pi)\phi(1), u'\otimes v') \\ &= (n\log q)^{-1}(1-q^{-1})\int_{\mathrm{GL}(n,F)/F^\times} (\pi(x)u,u')\overline{(\pi(x)v',v)}\,d^\times x \\ &= (n\log q)^{-1}(1-q^{-1})d(\pi)^{-1}(u,v')(v,u').\end{aligned}$$

This calculation is justified since

$$\varphi_s(x) \leq (1-q^{-ns})^{-1}(1-q^{-1})$$

for $s \in \mathbb{R}_{>0}$. $\square$

By Lemmas 4.1 and 4.2, we have $d(\pi) = \alpha^{-1}n^{-1}\gamma(0,\pi,\mathrm{Ad},\psi)$. This completes the proof of Theorem 3.1.

5. Twisted characters

Let F be a non-archimedean local field of characteristic zero and let ψ be a non-trivial additive character of F of order zero. Let $\mathbf{G} = \mathrm{Res}_{E/F}\,\mathrm{GL}(n)$, where E is a quadratic extension of F and n is odd. Let σ be the non-trivial automorphism of E over F and let $\omega_{E/F}$ be the quadratic character of $F^\times$ associated to E/F by class field theory. Put $\theta(g) = \mathrm{Ad}(J_n)(\sigma({}^t g^{-1}))$ for $g \in \mathbf{G}$, where

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}(n)$$

and σ is regarded as an automorphism of $\mathbf{G}$ over F.

Let π be a discrete series representation of G such that $\pi \simeq \pi\circ\theta$. We fix an isomorphism $\pi(\theta) : \pi \to \pi\circ\theta$ such that $\pi(\theta)^2 = \mathrm{id}$ and we define a distribution $J^\theta(\pi)$ by

$$J^\theta(\pi)(f) = J^\theta(\pi, f) = \mathrm{trace}(\pi(f)\pi(\theta))$$

for $f \in C_c^\infty(G)$. By Theorem 1 of [5], $J^\theta(\pi)$ is a locally integrable function on G which is locally constant on $G_{\theta\text{-reg}}$. Here $G_{\theta\text{-reg}}$ is the set of θ-regular and θ-semisimple elements in G. Let G_θ denote the identity component of $\{g \in G \mid \theta(g) = g\}$ and $\mathfrak{g}_\theta$ the Lie algebra of G_θ. By Theorem 3 of [5], we have the expansion

$$J^\theta(\pi, \exp(X)) = \sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi)\hat{\mu}_{\mathcal{O}}(X) \tag{5.1}$$

for $X \in \mathfrak{g}_\theta$ sufficiently near zero, where $\mathcal{O}$ runs over nilpotent G_θ-orbits in $\mathfrak{g}_\theta$ and where $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of the invariant measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$.

Theorem 5.1. *Let π be a discrete series representation of G such that $\pi \simeq \pi\circ\theta$ and such that $L(s,\pi,r)$ has a pole at $s=0$, where r is the Asai representation of ${}^L G$. Then there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on π such that*

$$|c_{0,\theta}(\pi)| = c|\gamma(0,\pi,r',\psi)|,$$

where $r' = r \otimes \omega_{E/F}$.

Remark 5.2. By the result of Henniart [18], we have

$$L_{\mathrm{LS}}(s,\pi,r) = L(s,\pi,r),$$
$$\gamma_{\mathrm{LS}}(s,\pi,r',\psi) = \alpha\gamma(s,\pi,r',\psi),$$

where the subscript LS indicates local factors defined by the Langlands-Shahidi method and $\alpha \in \mathbb{C}$ is a root of unity.

6. Twisted orbital integrals

Let F be a non-archimedean local field of characteristic zero. Let $\mathbf{G}$ be a connected reductive algebraic group over F and let θ be a quasi-semisimple automorphism of $\mathbf{G}$ over F. Let $\mathbf{Z}$ be the center of $\mathbf{G}$. For $\gamma \in G$ and $f \in C_c^\infty(G/(1-\theta)Z)$, put

$$J^\theta(\gamma, f) = \int_{ZG_{\gamma\theta}\backslash G} f(g^{-1}\gamma\theta(g))\, dg.$$

Here $G_{\gamma\theta}$ is the identity component of $\{g \in G \,|\, g^{-1}\gamma\theta(g) = \gamma\}$. By [28] and Lemma 2.1 of [1], this integral is absolutely convergent. By Proposition 7.1 of [33], we have the expansion

$$J^\theta(\exp(X), f) = \sum_u \Gamma_{u,\theta}(X) J^\theta(u, f) \tag{6.1}$$

for θ-regular and θ-semisimple elements X in $\mathfrak{g}_\theta$ sufficiently near zero. Here u runs over representatives for unipotent orbits in G_θ.

Let $\mathbf{G}$ and θ be as in §5. By [6], $J^\theta(\gamma, f)$ is absolutely convergent and (6.1) holds even if f is a Schwartz function.

7. Proof of Theorem 5.1

Throughout this section, we ignore the normalization of measures since it does not affect the proof. Let $\mathbf{G}$ and θ be as in §5. Let π be a discrete series representation of G such that $\pi \simeq \pi \circ \theta$ and such that $L(s, \pi, r)$ has a pole at $s = 0$, where r is the Asai representation of ${}^L G$. Let V_π be the space of π. We fix an invariant hermitian inner product $(\cdot, \cdot)$ on V_π and an isomorphism $\pi(\theta) : V_\pi \to V_\pi$ such that $\pi(\theta)\pi(g) = \pi(\theta(g))\pi(\theta)$ for $g \in G$ and such that $\pi(\theta)^2 = \mathrm{id}$. Then $(\cdot, \cdot)$ is $\pi(\theta)$-invariant.

Let

$$G^\sharp = \mathrm{U}(2n, F) = \{g \in \mathrm{GL}(2n, E) \,|\, gQ_n\sigma({}^t g) = Q_n\},$$

where

$$Q_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Let $P^\sharp = M^\sharp N^\sharp$ be a parabolic subgroup of $G^\sharp$ given by

$$M^\sharp = \left\{ \begin{pmatrix} a & 0 \\ 0 & \theta(a) \end{pmatrix} \,\middle|\, a \in \mathrm{GL}(n, E) \right\},$$
$$N^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \,\middle|\, x \in X \right\},$$

where $X = \{x \in \mathrm{Mat}_{n\times n}(E) \,|\, \mathrm{Ad}(J_n)(\sigma({}^t x)) = x\}$. As in §4, we consider an induced representation

$$I(s, \pi) = \mathrm{Ind}_{P^\sharp}^{G^\sharp}(\pi |\det|_E^{s/2})$$

for $s \in \mathbb{C}$. Put

$$w = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \in G^\sharp.$$

For $\phi \in I(s, \pi)$ and $g \in G^\sharp$, the integral

$$M(s, w, \pi)\phi(g) = \int_X \phi\left(w^{-1} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} g \right) dx$$

is absolutely convergent for $\mathrm{Re}(s) > 0$, has a meromorphic continuation to the whole s-plane, and defines an intertwining operator

$$M(s, w, \pi) : I(s, \pi) \longrightarrow I(-s, w(\pi)).$$

Lemma 7.1. *There exists a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that*

$$(\operatorname*{Res}_{s=0} M(s, w, \pi)\phi(1), v') = \alpha \operatorname*{Res}_{s=0} \gamma(s, \pi, r, \psi)^{-1}(\phi(1), \pi(\theta)v')$$

for $\phi \in I(s, \pi)$ and $v' \in V_\pi$.

Proof. We remark that $I(0, \pi)$ is irreducible since $L(s, \pi, r)$ has a pole at $s = 0$. As in the proof of Lemma 4.1, the lemma follows from the result of Shahidi [32]. □

Let

$$\bar{N}^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \,\middle|\, x \in X \right\}$$

and let $L = X \cap \mathrm{Mat}_{n\times n}(\mathfrak{o}_E)$. Let $\mathbf{1}_L$ denote the characteristic function of L.

Lemma 7.2. *Let $v, v' \in V_\pi$. Let f be a matrix coefficient of π given by $f(g) = (\pi(g)v, v')$ for $g \in G$. We define $\phi \in I(s, \pi)$ which has compact support in $P^\sharp \bar{N}^\sharp$ modulo $P^\sharp$ by*

$$\phi\left(\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix}\right) = \begin{cases} v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

Then there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on π such that

$$(\operatorname*{Res}_{s=0} M(s, w, \pi)\phi(1), v') = cJ^\theta(1, f).$$

Proof. The lemma follows from the result of Goldberg [10]. We include the proof for the sake of completeness.

We fix $\delta \in F^\times - \mathrm{N}_{E/F}(E^\times)$. Set $X' = \{x \in X \mid \det(x) \neq 0\}$. Then G acts on X' by $x \mapsto g^{-1}x\theta(g)$. Let $G\backslash X'$ denote the set of G-orbits in X'. Note that $\{1, \delta\}$ is a set of representatives for $G\backslash X'$. We define a G-invariant measure $d^\times x$ on X' by $d^\times x = |\det(x)|_E^{-n/2}\, dx$.

As in [10, §2], we have

$$\begin{aligned}
&(M(s, w, \pi)\phi(1), v') \\
&= \int_{X'} \mathbf{1}_L(x^{-1}) |\det(x)|_E^{-s/2-n/2} (\pi(x^{-1})v, v')\, dx \\
&= \int_{X'} \mathbf{1}_L(x) |\det(x)|_E^{s/2} f(x)\, d^\times x \\
&= \sum_{\gamma \in G\backslash X'} \int_{G_{\gamma\theta}\backslash G} \mathbf{1}_L(g^{-1}\gamma\theta(g)) |\det(g^{-1}\gamma\theta(g))|_E^{s/2} f(g^{-1}\gamma\theta(g))\, dg \\
&= \sum_{\gamma \in G\backslash X'} \int_{ZG_{\gamma\theta}\backslash G} \varphi_s(g^{-1}\gamma\theta(g)) f(g^{-1}\gamma\theta(g))\, dg,
\end{aligned}$$

where

$$\varphi_s(x) = |\det(x)|_E^{s/2} \int_{E^\times} \mathbf{1}_L(z\sigma(z)x) |z|_E^{ns}\, d^\times z.$$

For $x = (x_{ij}) \in X'$, we have

$$\int_{E^\times} \mathbf{1}_L(z\sigma(z)x) |z|_E^{ns}\, d^\times z = q_E^{[m/2]ns}(1 - q_E^{-ns})^{-1}(1 - q_E^{-1}),$$

where $m = \min(\operatorname{ord}_E(x_{ij}))$. Note that this integral is absolutely convergent for $\operatorname{Re}(s) > 0$. Hence we have

$$(\operatorname*{Res}_{s=0} M(s,w,\pi)\phi(1), v') = (n \log q_E)^{-1}(1 - q_E^{-1}) \sum_{\gamma \in G\backslash X'} J^\theta(\gamma, f).$$

This calculation is justified since

$$\varphi_s(x) \leq (1 - q_E^{-ns})^{-1}(1 - q_E^{-1})$$

for $s \in \mathbb{R}_{>0}$. As in [10, §2], the central character of π is trivial on $F^\times$. Hence we have

$$J^\theta(\delta, f) = J^\theta(1, f).$$

This completes the proof. □

Lemma 7.3. *Let $v, v' \in V_\pi$. Let f be a matrix coefficient of π given by $f(g) = (\pi(g)v, v')$ for $g \in G$. Then*

$$J^\theta(\gamma, f) = d(\pi)^{-1}(v, \pi(\theta)v')J^\theta(\pi, \gamma)$$

for θ-regular and θ-elliptic elements γ in G.

Proof. We proceed as in the proof of Proposition 5 of [7]. Let γ be a θ-regular and θ-elliptic element in G. Let $\varphi \in C_c^\infty(G)$. We assume that the support of φ is contained in the set of θ-regular and θ-elliptic elements in G. By the Schur orthogonality relations, we have

$$\int_{Z\backslash G} (\pi(g^{-1})\pi(\varphi)\pi(\theta(g))v, v')\, dg = d(\pi)^{-1}(v, \pi(\theta)v')J^\theta(\pi, \varphi).$$

The left-hand side is equal to

$$\int_{Z\backslash G}\int_G \varphi(h) f(g^{-1}h\theta(g))\, dh\, dg = \int_G \varphi(h) \int_{Z\backslash G} f(g^{-1}h\theta(g))\, dg\, dh.$$

Let φ tend to the Dirac measure at γ. This yields the lemma. □

Let $v, v' \in V_\pi$. Let f be a matrix coefficient of π given by $f(g) = (\pi(g)v, v')$ for $g \in G$. By (5.1), (6.1), and Lemma 7.3, we have

$$(v, \pi(\theta)v') \sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi)\hat{\mu}_{\mathcal{O}}(X) = d(\pi) \sum_u \Gamma_{u,\theta}(X) J^\theta(u, f)$$

for θ-regular and θ-elliptic elements X in $\mathfrak{g}_\theta$ sufficiently near zero. For $t \in F^\times$, we have

$$\hat{\mu}_{\mathcal{O}}(t^2X) = |t|_F^{-\dim \mathcal{O}}\hat{\mu}_{\mathcal{O}}(X),$$
$$\Gamma_{u,\theta}(t^2X) = |t|_F^{-\dim \operatorname{Ad}(G_\theta)(u)}\Gamma_{u,\theta}(X).$$

Note that $\hat{\mu}_0 = \Gamma_{1,\theta} = 1$ if measures are suitably normalized. By homogeneity, we obtain

$$(v, \pi(\theta)v')c_{0,\theta}(\pi) = d(\pi)J^\theta(1, f).$$

By Theorem 3.1, $d(\pi)$ is equal to

$$|\lim_{s\to 0} s^{-1}\gamma(s, \pi \times \check{\pi}, \psi)| = |\lim_{s\to 0} s^{-1}\gamma(s, \pi, r, \psi)| \cdot |\gamma(0, \pi, r', \psi)|$$

up to a constant which does not depend on π. Thus Theorem 5.1 follows from Lemmas 7.1 and 7.2.

Remark 7.4. Let $\mathbf{G} = \mathrm{GL}(n)$, where n is even. Put $\theta(g) = \mathrm{Ad}(J_n)({}^t g^{-1})$ for $g \in \mathbf{G}$. Let r (resp. r') be the exterior (resp. symmetric) square representation of ${}^L G$. Using the result of Shahidi [33], one can prove an analogue of Theorem 5.1 for an irreducible unitary supercuspidal representation π of G such that $\pi \simeq \pi \circ \theta$ and such that $L(s, \pi, r)$ has a pole at $s = 0$. To drop the assumption that π is supercuspidal, one has to show that twisted orbital integrals of Schwartz functions are absolutely convergent.

8. Twisted endoscopy

Let F be a non-archimedean local field of characteristic zero. Let $\mathbf{G}$ and θ be as in §5. We consider a set of endoscopic data $(\mathbf{H}, {}^L H, 1, \xi)$ for $(\mathbf{G}, \theta, 1)$ defined as follows. Recall that $\mathbf{G} = \mathrm{Res}_{E/F} \mathrm{GL}(n)$, where E is a quadratic extension of F and where n is odd. We have ${}^L G = \hat{G} \rtimes W_F$, where $\hat{G} = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ and the action of $w \in W_F$ is given by

$$(g_1, g_2) \longmapsto \begin{cases} (g_1, g_2) & \text{if } w \in W_E, \\ (g_2, g_1) & \text{if } w \notin W_E. \end{cases}$$

Let $\mathbf{H} = \mathrm{U}(n)$ be the quasi-split unitary group in n variables. Then ${}^L H = \hat{H} \rtimes W_F$, where $\hat{H} = \mathrm{GL}(n, \mathbb{C})$ and the action of $w \in W_F$ is given by

$$h \longmapsto \begin{cases} h & \text{if } w \in W_E, \\ \mathrm{Ad}(J_n)({}^t h^{-1}) & \text{if } w \notin W_E. \end{cases}$$

We define $\xi : {}^L H \to {}^L G$ by $\xi(h \times w) = (h, \mathrm{Ad}(J_n)({}^t h^{-1})) \times w$.

Lemma 8.1. *Let r be the Asai representation of ${}^L G$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. Then the adjoint representation* Ad *of ${}^L H$ on $\mathrm{Lie}(\hat{H})$ is isomorphic to $r' \circ \xi$, where $r' = r \otimes \omega_{E/F}$.*

Proof. Recall that r is defined by

$$r((g_1, g_2) \times 1)(x \otimes y) = g_1 x \otimes g_2 y,$$
$$r((1,1) \times w)(x \otimes y) = \begin{cases} x \otimes y & \text{if } w \in W_E, \\ y \otimes x & \text{if } w \notin W_E. \end{cases}$$

It is easy to check that

$$\mathrm{Ad}(h \times 1)(X) = \mathrm{Ad}(h)(X),$$
$$\mathrm{Ad}(1 \times w)(X) = \begin{cases} X & \text{if } w \in W_E, \\ -\mathrm{Ad}(J_n)({}^t X) & \text{if } w \notin W_E. \end{cases}$$

Hence the isomorphism $\mathbb{C}^n \otimes \mathbb{C}^n \to \mathrm{Lie}(\hat{H})$ given by $x \otimes y \mapsto x\,{}^t y J_n$ is intertwining. $\square$

It is believed that the following conjectures hold (cf. [2]).

Conjecture 8.2. *For $f \in C_c^\infty(G)$, there exists $f^H \in C_c^\infty(H)$ such that f and f^H have matching orbital integrals (cf.* [25, §5.5]*).*

Conjecture 8.3. *Let $\phi_H : L_F \to {}^L H$ be a tempered Langlands parameter. We define a tempered Langlands parameter $\phi : L_F \to {}^L G$ by $\phi = \xi \circ \phi_H$. Let π be the irreducible tempered representation of G associated to ϕ by the local Langlands* *correspondence* [16], [17]*. Then there exists a constant $c \in \mathbb{C}^\times$ such that $|c|$ does not depend on ϕ_H and such that*

$$J^\theta(\pi, f) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle J(\pi_H, f^H).$$

Here f and f^H have matching orbital integrals.

For $f \in C_c^\infty(G)$, let $f^{G_\theta} \in C_c^\infty(G_\theta)$ be a decent of f, where G_θ is the identity component of $\{g \in G \,|\, \theta(g) = g\}$. Let $f^H \in C_c^\infty(H)$. Assume that the supports of f and f^H are sufficiently small. By [15], [5], we have

$$J^\theta(\pi, f) = \sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi) \hat{\mu}_{\mathcal{O}}(f^{G_\theta} \circ \exp),$$
$$J(\pi_H, f^H) = \sum_{\mathcal{O}_H} c_{\mathcal{O}_H}(\pi_H) \hat{\mu}_{\mathcal{O}_H}(f^H \circ \exp).$$

Here $\mathcal{O}$ (resp. $\mathcal{O}_H$) runs over nilpotent G_θ-orbits (resp. H-orbits) in $\mathfrak{g}_\theta = \mathrm{Lie}(G_\theta)$ (resp. $\mathfrak{h} = \mathrm{Lie}(H)$).

Lemma 8.4. *Assume that Conjectures* 8.2 *and* 8.3 *hold. Let $\phi_H : L_F \to {}^L H$ be a tempered Langlands parameter and let π be the irreducible tempered representation of G as in Conjecture* 8.3*. Then there exists a constant $c \in \mathbb{C}$ such that $|c|$ does not depend on ϕ_H and such that*

$$c_{0,\theta}(\pi) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H).$$

Proof. We proceed as in [32, §9], [21, §8]. Assume that the supports of f and f^H are sufficiently small. If $t \in F^\times$ is sufficiently small, then we can define $f_t \in C_c^\infty(G)$ by $f_t(\exp(X)) = f(\exp(t^{-1}X))$. Similarly, we can define $f_t^H \in C_c^\infty(H)$. Then

$$\hat{\mu}_{\mathcal{O}}(f_{t^2}^{G_\theta} \circ \exp) = |t|_F^{2\dim \mathfrak{g}_\theta - \dim \mathcal{O}} \hat{\mu}_{\mathcal{O}}(f^{G_\theta} \circ \exp),$$
$$\hat{\mu}_{\mathcal{O}_H}(f_{t^2}^{H} \circ \exp) = |t|_F^{2\dim \mathfrak{h} - \dim \mathcal{O}_H} \hat{\mu}_{\mathcal{O}_H}(f^{H} \circ \exp).$$

Note that $\dim \mathfrak{g}_\theta = \dim \mathfrak{h}$.

Assume that f and f^H have matching orbital integrals. By Lemma 8.5 of [21], f_t and f_t^H have matching orbital integrals. Hence we have

$$\sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi) \hat{\mu}_{\mathcal{O}}(f_{t^2}^{G_\theta} \circ \exp) = c \sum_{\mathcal{O}_H} \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_{\mathcal{O}_H}(\pi_H) \hat{\mu}_{\mathcal{O}_H}(f_{t^2}^H \circ \exp).$$

By homogeneity, we obtain

$$c_{0,\theta}(\pi) \hat{\mu}_0(f^{G_\theta} \circ \exp) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H) \hat{\mu}_0(f^H \circ \exp).$$

□

Let π_H be a discrete series representation of H and let $\phi_H : L_F \to {}^L H$ be the (conjectural) Langlands parameter associated to π_H. Let π be the irreducible tempered representation of G as in Conjecture 8.3. If π_H is stable, then π is expected to be square integrable.

Proposition 8.5. *Assume that Conjectures* 8.2 *and* 8.3 *hold. Let* π_H *be a stable discrete series representation of* H*. Then*

$$d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|.$$

Proof. By [15], [30], we have

$$c_0(\pi_H) = (-1)^{l_0} d(\pi_{H,0})^{-1} \cdot d(\pi_H),$$

where l_0 is the semisimple F-rank of $\mathbf{H}$ and $\pi_{H,0}$ is the Steinberg representation of H. By Theorem 5.1 and Lemmas 8.1 and 8.4, there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on π_H such that

$$d(\pi_H) = c|\gamma(0, \pi_H, \mathrm{Ad}, \psi)|.$$

Since $\pi_{H,0}$ is stable and $d(\pi_{H,0}) = 2^{-1}|\gamma(0, \pi_{H,0}, \mathrm{Ad}, \psi)|$, we have $c = 2^{-1}$. □

For $n = 3$, Conjectures 8.2 and 8.3 were proved by Rogawski [31]. Thus we obtain the following theorem.

Theorem 8.6. *Let* $\mathbf{H} = \mathrm{U}(3)$ *be the quasi-split unitary group in three variables. Let* π_H *be a stable discrete series representation of* H*. Then*

$$d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|.$$

In particular, Conjecture 1.4 *holds for* π_H*.*

References

[1] J. Arthur, *The local behaviour of weighted orbital integrals*, Duke Math. J. **56** (1988), 223–293. MR932848 (89h:22036)

[2] ———, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71. MR1021499 (91f:22030)

[3] A.-M. Aubert and R. Plymen, *Plancherel measure for* $\mathrm{GL}(n,F)$ *and* $\mathrm{GL}(m,D)$*: explicit formulas and Bernstein decomposition*, J. Number Theory **112** (2005), 26–66. MR2131140 (2006d:22018)

[4] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259. MR0444849 (56:3196)

[5] L. Clozel, *Characters of nonconnected, reductive p-adic groups*, Canad. J. Math. **39** (1987), 149–167. MR889110 (88i:22039)

[6] ———, *The fundamental lemma for stable base change*, Duke Math. J. **61** (1990), 255–302. MR1068388 (91h:22036)

[7] ———, *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*, Harmonic analysis on reductive groups, Progr. Math. **101**, Birkhäuser Boston, 1991, pp. 101–121. MR1168480 (93h:22020)

[8] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, preprint.

[9] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, 1984, pp. 33–117. MR771672 (86h:11044)

[10] D. Goldberg, *Some results on reducibility for unitary groups and local Asai L-functions*, J. Reine Angew. Math. **448** (1994), 65–95. MR1266747 (95g:22031)

[11] B. H. Gross, *On the motive of a reductive group*, Invent. Math. **130** (1997), 287–313. MR1474159 (98m:20060)

[12] ———, *On the motive of G and the principal homomorphism* $\mathrm{SL}_2 \to \hat{G}$, Asian J. Math. **1** (1997), 208–213. MR1480995 (99d:20077)

[13] Harish-Chandra, *Harmonic analysis on real reductive groups. I. The theory of the constant term*, J. Functional Analysis **19** (1975), 104–204. MR0399356 (53:3201)

[14] ———, *Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula*, Ann. of Math. **104** (1976), 117–201. MR0439994 (55:12875)

[15] ______, *Admissible invariant distributions on reductive p-adic groups*, University Lecture Series **16**, American Mathematical Society, 1999. MR1702257 (2001b:22015)
[16] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies **151**, Princeton University Press, 2001. MR1876802 (2002m:11050)
[17] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), 439–455. MR1738446 (2001e:11052)
[18] ______, *Correspondance de Langlands et fonctions L des carrés extérieur et symétrique*, preprint.
[19] K. Hiraga and H. Saito, *On L-packets for inner forms of* SL_n, preprint.
[20] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, preprint.
[21] T. Konno, *Twisted endoscopy and the generic packet conjecture*, Israel J. Math. **129** (2002), 253–289. MR1910946 (2004d:22008)
[22] R. E. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), 611–650. MR757954 (85m:11080)
[23] ______, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399. MR858284 (88d:22027)
[24] ______, *Tamagawa numbers*, Ann. of Math. **127** (1988), 629–646. MR942522 (90e:11075)
[25] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255** (1999). MR1687096 (2000k:22024)
[26] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics **544**, Springer-Verlag, 1976. MR0579181 (58:28319)
[27] I. G. Macdonald, *The volume of a compact Lie group*, Invent. Math. **56** (1980), 93–95. MR558859 (81h:22018)
[28] R. Ranga Rao, *Orbital integrals in reductive groups*, Ann. of Math. **96** (1972), 505–510. MR0320232 (47:8771)
[29] M. Reeder, *Formal degrees and L-packets of unipotent discrete series representations of exceptional p-adic groups*, J. Reine Angew. Math. **520** (2000), 37–93. MR1748271 (2001k:22039)
[30] J. D. Rogawski, *An application of the building to orbital integrals*, Compositio Math. **42** (1980/81), 417–423. MR607380 (83g:22011)
[31] ______, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies **123**, Princeton University Press, 1990. MR1081540 (91k:22037)
[32] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. **132** (1990), 273–330. MR1070599 (91m:11095)
[33] ______, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. **66** (1992), 1–41. MR1159430 (93b:22034)
[34] ______, *Poles of intertwining operators via endoscopy: the connection with prehomogeneous vector spaces*, Compositio Math. **120** (2000), 291–325. MR1748913 (2001i:11060)
[35] A. J. Silberger and E.-W. Zink, *The formal degree of discrete series representations of central simple algebras over p-adic fields*, Max-Planck-Institut für Mathematik, 1996.
[36] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), 235–333. MR1989693 (2004d:22009)
[37] E.-W. Zink, *Comparison of* GL_N *and division algebra representations II*, Max-Planck-Institut für Mathematik, 1993.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: hiraga@math.kyoto-u.ac.jp

Department of Mathematics, Graduate School of Science, Osaka City University, Osaka 558-8585, Japan
E-mail address: ichino@sci.osaka-cu.ac.jp

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: ikeda@math.kyoto-u.ac.jp