FORMAL DEGREES AND ADJOINT $\gamma$-FACTORS

KAORU HIRAGA, ATSUSHI ICHINO, AND TAMOTSU IKEDA

Dedicated to Professor Hiroshi Saito on the occasion of his sixtieth birthday

INTRODUCTION

Let $G$ be a connected reductive algebraic group over a local field $F$ and let $H$ be a closed subgroup of $G$ over $F$. Set $G = G(F)$ and $H = H(F)$. Let $\pi$ be an irreducible unitary representation of $G$ and let $V_\pi$ be the space of $\pi$. For $v \in V_\pi$, we will consider the integral

\[(0.1) \int_H (\pi(h)v, v) \, dh.\]

We can regard this integral as an analogue of (the square of the absolute value of) a period integral of an automorphic form and expect that it is related to $L$ and $\epsilon$-factors. For example, let $G = SO(n + 1) \times SO(n)$ and $H = SO(n)$. Let $\pi = \pi_1 \otimes \pi_0$, where $\pi_1$ (resp. $\pi_0$) is an irreducible unramified tempered representation of $SO(n + 1, F)$ (resp. $SO(n, F)$). Then (0.1) can be expressed in terms of

\[
\frac{L(\frac{1}{2}, \pi_1 \times \pi_0)}{L(1, \pi_1, \text{Ad})L(1, \pi_0, \text{Ad})}
\]

if $v$ is unramified (cf. [20]). Now let $G = H \times H$, where $H$ is a connected reductive algebraic group over $F$. For simplicity, we assume that the connected center of $H$ is anisotropic. Let $\pi = \pi_H \otimes \tilde{\pi}_H$, where $\pi_H$ is a discrete series representation of $H$ and $\tilde{\pi}_H$ is the contragredient representation of $\pi_H$. Then (0.1) can be expressed in terms of the formal degree $d(\pi_H)$ of $\pi_H$. In this paper, we give a conjectural formula for $d(\pi_H)$ in terms of the adjoint $\gamma$-factor

\[
\gamma(s, \pi_H, \text{Ad}, \psi) = \epsilon(s, \pi_H, \text{Ad}, \psi) \cdot \frac{L(1 - s, \tilde{\pi}_H, \text{Ad})}{L(s, \pi_H, \text{Ad})}
\]

(cf. Conjecture [1,4].) Here $\text{Ad}$ is the adjoint representation of the $L$-group $^L H$ of $H$ on the Lie algebra $\text{Lie}(H)$ of the dual group of $H$ and $\psi$ is a non-trivial additive character of $F$.

Our conjecture is supported by various examples. For example, we assume that $F = \mathbb{R}$ and $H$ is anisotropic. We take the Haar measure $dh$ on $H$ determined by a Chevalley basis of $\text{Lie}(H) \otimes \mathbb{C}$. Let $\pi_H$ be an irreducible finite dimensional representation of $H$. Then the conjecture for $\pi_H$ asserts that

\[
\frac{\dim \pi_H}{\text{vol}(H)} = \frac{1}{2^n} \cdot |\gamma(0, \pi_H, \text{Ad}, \psi)|
\]
and it is compatible with the Weyl dimension formula. Here \( l \) is the rank of \( H \) and \( \psi(x) = \exp(2\pi \sqrt{-1}x) \) for \( x \in \mathbb{R} \). Also, if \( F \) is non-archimedean, then the conjecture for \( \text{GL}(n) \) is compatible with the result of Silberger and Zink \[35, 37\].

Moreover, we provide some evidence in the case of the quasi-split unitary group in three variables. To be precise, let \( F \) be a non-archimedean local field of characteristic zero. Let \( E \) be a quadratic extension of \( F \) and let \( \sigma \) be the non-trivial automorphism of \( E \) over \( F \). Put

\[
J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Let

\[
H = U(3) = \{ h \in \text{Res}_{E/F} \text{GL}(3) \mid \theta(h) = h \},
\]

where \( \theta(h) = \text{Ad}(J_3)(\sigma(h^{-1})) \). Following Gross \[11\], we choose a Haar measure \( dh \) on \( H \). Let \( \pi_H \) be a stable discrete series representation of \( H \). We will verify the conjecture for \( \pi_H \), i.e.,

\[
d(\pi_H) = \frac{1}{2} |\gamma(0, \pi_H, \text{Ad}, \psi)|
\]

(cf. Theorem 8.1).

To prove (0.2), we use twisted endoscopy. Let \( J(\pi_H) = \text{trace} \pi_H \) be the character of \( \pi_H \) and let \( c_0(\pi_H) \) be the coefficient associated to the trivial orbit in the local character expansion of \( J(\pi_H) \) (cf. [15]). Recall that

\[
c_0(\pi_H) \doteq d(\pi_H).
\]

Here the notation \( \doteq \) indicates equality up to constants which do not depend on the representations. Let \( \pi \) be the base change of \( \pi_H \) to \( \text{GL}(3, E) \). Then \( \pi \) is square integrable since \( \pi_H \) is stable. Also, \( \pi \) is isomorphic to \( \pi \circ \theta \). We fix an isomorphism \( \pi(\theta) : \pi \to \pi \circ \theta \) such that \( \pi(\theta)^2 = \text{id} \). Let \( J^\theta(\pi) = \text{trace} \pi \circ \pi(\theta) \) be the twisted character of \( \pi \) and let \( c_{0, \theta}(\pi) \) be the coefficient associated to the trivial orbit in the local character expansion of \( J^\theta(\pi) \) (cf. [5]). The character identity between \( J^\theta(\pi) \) and \( J(\pi_H) \) was proved by Rogawski \[31\] and implies that

\[
|c_{0, \theta}(\pi)| \doteq |c_0(\pi_H)|.
\]

We also have an analogue

\[
c_{0, \theta}(\pi) \cdot (v, \pi(\theta)v') \doteq d(\pi) \cdot J^\theta(1, f)
\]

of (0.3). Here \( f \) is a matrix coefficient of \( \pi \) given by \( f(g) = (\pi(g)v, v') \) and \( J^\theta(1, f) \) is the twisted orbital integral of \( f \) at the identity element. By the result of Silberger and Zink \[35, 37\], we have

\[
d(\pi) = |\lim_{s \to 0} s^{-1} \gamma(s, \pi \times \tilde{\pi}, \psi)|.
\]

By the results of Shahidi \[32\] and Goldberg \[10\], we have

\[
|J^\theta(1, f)| \doteq |\lim_{s \to 0} s^{-1} \gamma(s, \pi, r, \psi)|^{-1} \cdot |(v, \pi(\theta)v')|,
\]

where \( r \) is the Asai representation. Thus we obtain (0.2).

This paper is organized as follows. In [11] we formulate a conjecture on formal degrees and relate it to the Plancherel formula. In [32] we verify the conjecture in the archimedean case. In [33] we present various examples in the non-archimedean case. For example, the conjecture for \( \text{GL}(n) \) is compatible with the result of Silberger.
and Zink [35, 37]. Using the results of Shahidi [32, 33, 34], we give a new proof of their result in [4]. In [5] we give a description of the coefficient associated to the trivial orbit in the local character expansion of a certain twisted character. After recalling some facts about twisted orbital integrals in [6] we prove this description in [7]. In [8] we verify the conjecture for a stable discrete series representation of $U(3)$.

1. Conjectures

In this section, we formulate a conjecture on formal degrees (cf. Conjecture 1.4).

Let $F$ be a local field of characteristic zero and let $\psi$ be a non-trivial additive character of $F$. Let $| \cdot |_F$ denote the absolute value on $F$. If $F$ is non-archimedean, let $\mathfrak{o}_F$ be the maximal compact subring of $F$, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, and $q = q_F$ the cardinality of $\mathfrak{o}_F/\mathfrak{p}_F$. Let $\Gamma = \text{Gal}(\bar{F}/F)$ denote the absolute Galois group of $F$, $W_F$ the Weil group of $F$, $W'_F$ the Weil-Deligne group of $F$, and $L_F$ the Langlands group of $F$ given by

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean}, \\ W_F \times \text{SL}(2, \mathbb{C}) & \text{if } F \text{ is non-archimedean}. \end{cases}$$

Let $G$ be a connected reductive algebraic group over $F$. Set $G = G(F)$. Let $G^*$ be the quasi-split inner form of $G$ and choose an inner twist $\eta : G \to G^*$. Let $\hat{G}$ denote the dual group of $G$ and $L^*G = G \times W_F$ the $L$-group of $G$. We fix an $F$-splitting $(B^*, T^*, \{X_\alpha\})$ of $G^*$ and a $\Gamma$-splitting $(B, T, \{X_\alpha\})$ of $G$.

Let $\pi$ be a discrete series representation of $G$ and let $V_{\pi}$ be the space of $\pi$. Let $d(\pi) \in \mathbb{R}_{>0}$ denote the formal degree of $\pi$. By definition, we have

$$\int_{G/A} (\pi(g)u, u')|(\pi(g)v, v')| dg = d(\pi)^{-1}(u, v)(u', v')$$

for $u, u', v, v' \in V_{\pi}$, where $A$ is the split component of the center of $G$ and $A = A(F)$. We remark that $d(\pi) = d(\pi, dg)$ depends on the choice of $dg$. Following Gross [11], we take a Haar measure $\mu_{G/A, \psi}$ on $G/A$ defined as follows. (This should not be confused with the Euler-Poincaré measure $\mu_G$ on $G$ in the notation of [11].) We may assume that $A = \{1\}$. Moreover, we may assume that $G$ has an anisotropic inner form if $F$ is archimedean. Let $\omega_G$ be a differential form of top degree on $G$ over $F$ as in Sections 4 and 7 of [11]. Let $\mu_{G, \psi}$ denote the Haar measure on $G$ determined by $\omega_G$ and the self-dual measure on $F$ with respect to $\psi$. Then

$$\mu_{G, \psi}(a) = |\mathfrak{o}_F|^{\dim G/2} \cdot \mu_{G, \psi},$$

where $a \in F^\times$ and $\psi_a(x) = \psi(ax)$ for $x \in F$. If $F$ is non-archimedean, $\psi$ is of order zero, and $G$ is unramified, then

$$\mu_{G, \psi}(G(\mathfrak{o}_F)) = q^{-\dim G}|G(\mathbb{F}_q)|.$$

Here we extend $G$ to a smooth group scheme over $\mathfrak{o}_F$ associated to a hyperspecial maximal compact subgroup of $G$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 1.1. Let $\pi$ be a discrete series representation of $G$. Let $a \in F^\times$. We define a non-trivial additive character $\psi_a$ of $F$ by $\psi_a(x) = \psi(ax)$ for $x \in F$. Then

$$d(\pi, \mu_{G/A, \psi_a}) = |a|_F^{-n/2} \cdot d(\pi, \mu_{G/A, \psi}),$$

where $n = \dim G/A$.

Proof. The lemma follows from (1.1).

Let $\phi : L_F \to LG$ be a Langlands parameter. We say that $\phi$ is tempered if $\phi(W_F)$ is bounded and that $\phi$ is elliptic if $\phi(L_F)$ is not contained in any proper parabolic subgroup of $L G$. For each finite dimensional representation $r$ of $L G$, put

$$\gamma(s, r \circ \phi, \psi) = \epsilon(s, r \circ \phi, \psi) \cdot \frac{L(1 - s, \check{r} \circ \phi)}{L(s, r \circ \phi)},$$

where $\check{r}$ is the contragredient representation of $r$. Let $\text{Ad}$ denote the adjoint representation of $L G$ on $\text{Lie}(\hat{G})/\text{Lie}(Z(\hat{G})^F)$. Note that $\text{Ad}$ is self-dual.

Lemma 1.2. Let $\phi : L_F \to L G$ be an elliptic Langlands parameter. Then at $s = 0$,

$$\gamma(s, \text{Ad} \circ \phi, \psi) = \epsilon(s, \text{Ad} \circ \phi, \psi) \text{ holomorphic and non-zero.}$$

Proof. Since $\phi$ is elliptic, $\text{Ad} \circ \phi$ does not contain the trivial representation of $L_F$ (cf. Lemma 10.3.1 of [22]). Hence the lemma follows from the multiplicativity of $\gamma$-factors.

Lemma 1.3. Let $\phi : L_F \to L G$ be an elliptic Langlands parameter. Let $a \in F^\times$. We define a non-trivial additive character $\psi_a$ of $F$ by $\psi_a(x) = \psi(ax)$ for $x \in F$. Then

$$|\gamma(0, \text{Ad} \circ \phi, \psi_a)| = |a|_F^{-n/2} \cdot |\gamma(0, \text{Ad} \circ \phi, \psi)|,$$

where $n = \dim G/A$.

Proof. Note that $n = \dim \text{Lie}(\hat{G})/\text{Lie}(Z(\hat{G})^F)$. By definition, we have

$$|\epsilon(s, \text{Ad} \circ \phi, \psi_a)| = |a|_F^{n(s-1/2)} \cdot |\epsilon(s, \text{Ad} \circ \phi, \psi)|.$$}

This yields the lemma.

Let $\Pi(G)$ denote the set of equivalence classes of irreducible admissible representations of $G$. The local Langlands conjecture asserts that there exists a partition

$$\prod_{\phi} \Pi_{\phi}(G)$$

of $\Pi(G)$ into finite subsets, where $\phi$ runs over equivalence classes of Langlands parameters $\phi : L_F \to L G$. Let $\pi \in \Pi_{\phi}(G)$. If $\phi$ is tempered (resp. elliptic), then $\pi$ is expected to be tempered (resp. essentially square integrable). For each finite dimensional representation $r$ of $L G$, put

$$L(s, \pi, r) = L(s, r \circ \phi),$$

$$\epsilon(s, \pi, r, \psi) = \epsilon(s, r \circ \phi, \psi),$$

and

$$\gamma(s, \pi, r, \psi) = \epsilon(s, \pi, r, \psi) \cdot \frac{L(1 - s, \check{\pi}, r)}{L(s, \pi, r)},$$

where $\check{\pi}$ is the contragredient representation of $\pi$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let \( \phi : L_F \to L^G \) be a tempered Langlands parameter. Following [19, §1], set

\[
S_\phi = \{ s \in \hat{G}_{sc} | \text{Int} \circ \phi = \phi \mod B^1(W_F, Z(\hat{G})) \}, \quad S_\phi = \pi_0(S_\phi), \\
S_\phi^2 = \{ s \in \hat{G}^2 | \text{Int} \circ \phi = \phi \}, \quad S_\phi^2 = \pi_0(S_\phi^2),
\]

where \( \hat{G}_{sc} \) is the simply connected cover of the derived group of \( \hat{G} \) and \( \hat{G}^2 \) is the dual group of \( G/A \). Let \( Z_\phi \) be the image of \( Z(\hat{G}_{sc}) \) in \( S_\phi \). Let \( \chi_G \) be the character of \( Z(\hat{G}_{sc})^F \) associated to \( G \) by the map

\[
H^1(F, G^{ad}_*) \longrightarrow \pi_0(Z(\hat{G}_{sc})^F)^D
\]
defined by Kottwitz [22, 23]. Here \( G^{ad}_* \) is the adjoint group of \( G^* \). By Lemma 9.1 of [19], we can regard \( \chi_G \) as a character of the image of \( Z(\hat{G}_{sc})^F \) in \( S_\phi \). We extend \( \chi_G \) to a character of \( Z_\phi \). Let \( \Pi(S_\phi, \chi_G) \) denote the set of equivalence classes of irreducible representations of \( S_\phi \) such that \( Z_\phi \) acts via \( \chi_G \). It is believed that there exists a map

\[
\Pi_\phi(G) \longrightarrow \Pi(S_\phi, \chi_G)
\]
which satisfies certain conditions on characters (cf. [2]). For example,

\[
\sum_{\pi \in \Pi_\phi(G)} \langle 1, \pi \rangle \text{trace} \pi
\]
is required to be the unique (up to a scalar) stable distribution in the space of virtual characters generated by \( \Pi_\phi(G) \), where

\[
\langle 1, \pi \rangle = \dim \rho_\pi
\]
if \( \rho_\pi \in \Pi(S_\phi, \chi_G) \) is associated to \( \pi \in \Pi_\phi(G) \). Moreover, the quantity \( \langle 1, \pi \rangle \) is expected to be canonically determined by \( \pi \).

**Conjecture 1.4.** Let \( \phi : L_F \to L^G \) be an elliptic tempered Langlands parameter. Then

\[
d(\pi) = \frac{\langle 1, \pi \rangle}{|S_\phi^2|} \cdot |\gamma(0, \pi, \text{Ad}, \psi)|
\]

for \( \pi \in \Pi_\phi(G) \).

We will relate Conjecture 1.4 to the Plancherel formula. We fix a non-trivial additive character \( \psi \) of \( F \). Let \( \Theta \) be the set of pairs \( (\mathfrak{D}, P = MN) \), where \( P \) is a semi-standard parabolic subgroup of \( G \), \( M \) is the Levi subgroup of \( P \), \( N \) is the unipotent radical of \( P \), and \( \mathfrak{D} \) is an orbit in the set of equivalence classes of discrete series representations of \( M \) under the action of the group of unramified unitary characters of \( M \). For \( (\mathfrak{D}, P = MN) \in \Theta \) and \( \pi \in \mathfrak{D} \), put

\[
d\nu(\pi) = \frac{\langle 1, \pi \rangle}{|S_{\phi_M}^2|} \cdot |\gamma(0, \pi, r_M, \psi)| \cdot d\pi.
\]

Here \( \phi_M : L_F \to L^M \) is the (conjectural) Langlands parameter associated to \( \pi \), \( r_M \) is the adjoint representation of \( L^M \) on \( \text{Lie}(\hat{G})/\text{Lie}(Z(\hat{M}))^F \), and \( d\pi \) is the Lebesgue measure on \( \mathfrak{D} \) (cf. [36, pp. 239 and 302]). Then the Plancherel formula (cf. Theorem 27.3 of [14] and Théorème VIII.1.1 of [36]), Langlands' conjecture on Plancherel measures (cf. Appendix II of [26]), and Conjecture 1.4 suggest that the following conjecture holds.
Conjecture 1.5. There exist explicit constants $c_M \in \mathbb{R}_{>0}$ which do not depend on $\mathcal{D}$ such that

$$f(1) = \sum_{(\mathcal{D}, P=MN) \in \Theta} c_M \int_{\mathcal{D}} \text{trace} \text{Ind}_{\pi}^{G}(f) \, d\nu(\pi)$$

for $f \in C_c^{\infty}(G)$.

2. Examples: The archimedean case

In this section, we verify Conjecture 1.4 in the archimedean case.

Let $F = \mathbb{R}$. By Lemmas 1.1 and 1.3 we may assume that $\psi(x) = \exp(2\pi \sqrt{-1}x)$ for $x \in \mathbb{R}$. Let $G$ be a connected reductive algebraic group of rank $l$ over $\mathbb{R}$. For simplicity, we assume that the connected center of $G$ is anisotropic. We may assume that $G$ has an anisotropic inner form $G_{\text{an}}$.

Lemma 2.2. Let $\pi$ be a discrete series representation of $G$. Then

$$d(\pi) = \frac{1}{2^l} |\gamma(0, \pi, \text{Ad}, \psi)|.$$

In particular, Conjecture 1.4 holds for $\pi$.

The rest of this section is devoted to the proof of Proposition 2.1. Let $\hat{\Sigma}$ denote the set of roots of $T$ in $\hat{G}$ and $\hat{\Sigma}^+$ the subset of positive roots determined by $B$. Let $N$ be the number of positive roots. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $X_+(T) \otimes \mathbb{Q}$ and $X^+(T) \otimes \mathbb{Q}$.

Lemma 2.2. Let $\pi_\lambda$ be a discrete series representation of $G$ with Harish-Chandra parameter $\lambda$. Then

$$|\gamma(0, \pi_\lambda, \text{Ad}, \psi)| = \pi^{-l} \cdot (2\pi)^{-N} \prod_{\alpha \in \hat{\Sigma}^+} |\langle \lambda, \hat{\alpha} \rangle|.$$

Proof. Let $\phi : W_\mathbb{R} \to L^G$ be the Langlands parameter associated to $\pi_\lambda$. Then $\phi(z) = z^\lambda \overline{z}^{-\lambda}$ for $z \in W_\mathbb{C}$. The action $\text{Ad} \circ \phi$ of $W_\mathbb{R}$ on $\text{Lie}(T)$ (resp. $\mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$) is given by the sign character (resp. $\text{Ind}_{W_\mathbb{C}}^{G}(\phi_\alpha)$). Here $\hat{\alpha} \in \hat{\Sigma}^+$ and $\phi_\alpha(z) = z^{\langle \lambda, \hat{\alpha} \rangle} \overline{z}^{-\langle \lambda, \hat{\alpha} \rangle}$ for $z \in W_\mathbb{C}$. Hence we have

$$L(s, \pi_\lambda, \text{Ad}) = \Gamma_{\mathbb{R}}(s+1) \prod_{\alpha \in \hat{\Sigma}^+} \Gamma_{\mathbb{C}}(s + |\langle \lambda, \hat{\alpha} \rangle|),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$. By definition, $\epsilon(s, \pi_\lambda, \text{Ad}, \psi)$ is a power of $\sqrt{-1}$. This yields the lemma.

Set $g = \text{Lie}(G)$. Let $\theta$ be a Cartan involution of $g$ and let $B$ be a symmetric bilinear form on $g$ over $\mathbb{R}$ which satisfies the conditions of Lemma 3.2 of [13]. Then the quadratic form

$$\|X\|^2 = -B(X, \theta(X))$$

for $X \in g$ is positive definite. This norm $\| \cdot \|$ on $g$ defines a Lebesgue measure on $g$ and hence a Haar measure $dG$ on $G$ via the exponential map. Let $T$ be an anisotropic maximal torus of $G$ such that $\text{Lie}(T)$ is $\theta$-invariant. Similarly, we can define a Haar measure $dT$ on $T$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Set $g_C = \text{Lie}(G) \otimes \mathbb{C}$ and $t_C = \text{Lie}(T) \otimes \mathbb{C}$. Let $\Sigma$ denote the set of roots of $t_C$ in $g_C$ and $\Sigma^+$ the subset of positive roots. We extend $B$ to a symmetric bilinear form on $g_C$ over $\mathbb{C}$. For $\alpha \in \Sigma$, we define $H_\alpha \in t_C$ by

$$B(H, H_\alpha) = \alpha(H)$$

for $H \in t_C$. Put

$$\varpi = \prod_{\alpha \in \Sigma^+} H_\alpha.$$

**Lemma 2.3.** Let $\pi_\lambda$ be a discrete series representation of $G$ with Harish-Chandra parameter $\lambda$. Then

$$d(\pi_\lambda, dG) = (2\pi)^{-N} \cdot |\varpi(\lambda)| \cdot \text{vol}(T, dT)^{-1}.$$

*Proof.* Let $K$ be a maximal compact subgroup of $G$ such that $\text{Lie}(K)$ is $\theta$-invariant and let $dK$ be the Haar measure on $K$ determined by $\| \cdot \|$. Let $dx$ be the standard measure on $G$ as in [13, §7]. By Lemma 37.2 of [13], we have

$$dx = 2^{\nu/2} \cdot \text{vol}(K, dK)^{-1} \cdot dG,$$

where $\nu = \dim G/K - \text{rank } G/K$. By Corollary of Lemma 23.1 of [14], we have

$$d(\pi_\lambda, dx) = c_G^{-1} \cdot |W| \cdot |\varpi(\lambda)|.$$

Here $W$ is the Weyl group of $T$ in $G$ and

$$c_G = 2^{\nu/2} \cdot (2\pi)^N \cdot |W| \cdot \frac{\text{vol}(T, dT)}{\text{vol}(K, dK)}$$

(cf. Lemma 37.3 of [13]). This completes the proof. \hfill $\Box$

**Lemma 2.4.** Let $\pi$ be a discrete series representation of $G$ and let $\phi : W_R \to \mathbb{C}$ be the Langlands parameter associated to $\pi$. Let $\pi_{\text{an}}$ be the irreducible finite dimensional representation of $G_{\text{an}}$ associated to $\phi$ by the local Langlands correspondence. Then

$$d(\pi) = d(\pi_{\text{an}}).$$

*Proof.* We extend $\theta$ to an anti-linear involution of $g_C$ over $\mathbb{C}$. Set $g_{\text{an}} = \text{Lie}(G_{\text{an}})$. We may identify $g_{\text{an}}$ with $g^\theta_C$. Then the restrictions of $\theta$ and $B$ to $g_{\text{an}}$ define a norm $\| \cdot \|_{\text{an}}$ on $g_{\text{an}}$. Let $dG_{\text{an}}$ be the Haar measure on $G_{\text{an}}$ determined by $\| \cdot \|_{\text{an}}$. Then $dG$ and $dG_{\text{an}}$ are compatible. By Lemma 2.3, we have

$$d(\pi, dG) = d(\pi_{\text{an}}, dG_{\text{an}}).$$

By definition, $\mu_G, \psi$ and $\mu_{G_{\text{an}}, \psi}$ are also compatible. This yields the lemma. \hfill $\Box$

By Lemma 2.4 to prove Proposition 2.1, we may assume that $G$ is anisotropic. Let $\pi$ be an irreducible finite dimensional representation of $G$. By Lemmas 2.2 and 2.3 there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on $\pi$ such that

$$d(\pi) = c |\gamma(0, \pi, \text{Ad}, \psi)|.$$

By [27, 11, §7], we have

$$\text{vol}(G) = 2^N \prod_{i=1}^l \frac{2\pi^{m_i+1}}{m_i!} = (2\pi)^{l+N} \prod_{\alpha \in \Sigma^+} \langle \rho, \alpha \rangle^{-1}.$$
Here \( m_1, \ldots, m_l \) are the exponents of \( G \) and \( \rho \) is half the sum of positive roots. Note that
\[
\sum_{i=1}^{l} m_i = N,
\]
\[
\prod_{i=1}^{l} m_i! = \prod_{\alpha \in \Sigma^+} \langle \rho, \alpha \rangle.
\]
By Lemma 2.2, we have \( \text{vol}(G) = 2^{l} |\gamma(0, \pi, \text{Ad}, \psi)|^{-1} \), where \( \pi \) is the trivial representation of \( G \). Hence we have \( c = 2^{-l} \). This completes the proof of Proposition 2.1.

3. Examples: The non-archimedean case

Let \( F \) be a non-archimedean local field of characteristic zero. By Lemmas 1.1 and 1.3, we may assume that \( \psi \) is of order zero.

3.1. Inner forms of \( \text{GL}(n) \). We first recall the following result of Silberger and Zink [35], [37].

**Theorem 3.1.** Let \( \pi \) be a discrete series representation of \( \text{GL}(n, F) \). Then
\[
d(\pi) = \frac{1}{n} |\gamma(0, \pi, \text{Ad}, \psi)|.
\]
In particular, Conjecture 1.4 holds for \( \pi \).

To be precise, let \( \pi \) be the unique irreducible subrepresentation of an induced representation
\[
\sigma| \det(F^{(e-1)/2} \times \sigma) | \det(F^{(e-3)/2} \times \cdots \times \sigma) | \det(F^{-(e-1)/2}),
\]
where \( \sigma \) is an irreducible unitary supercuspidal representation of \( \text{GL}(m, F) \) with \( n = em \). Using the theory of types, Silberger and Zink showed that \( d(\pi) \) is equal to
\[
r \cdot q^{em} - 1 \cdot q^{er} - 1 \cdot q^{e(r-m)/2 + e^2(f+r-m^2)/2} \cdot \frac{1}{n} \prod_{i=1}^{n-1} (q^i - 1) \cdot \text{vol}(\text{GL}(n, \mathcal{O}_F)/\mathcal{O}_F^\times)^{-1}
\]
(cf. Theorems 6.5 and 6.9 of [3]). Here \( r \) is the torsion number of \( \sigma \times \sigma \). It is easy to check that this quantity coincides with \( n^{-1} |\gamma(0, \pi, \text{Ad}, \psi)| \). In [4] we will give a new proof of Theorem 3.1 which does not rely on the theory of types.

Let \( G \) be an inner form of \( \text{GL}(n) \) over \( F \). Then \( G = \text{GL}(n', D) \) with \( n = dn' \), where \( D \) is a division algebra of dimension \( d^2 \) over \( F \). Let \( \pi \) be a discrete series representation of \( G \). By Theorem 7.2 of [3], we have
\[
d(\pi) = \prod_{\substack{1 \leq i \leq n \\ i \neq 0 \mod d}} (q^i - 1)^{-1} \cdot d(\pi^*) \cdot \frac{\text{vol}(\text{GL}(n, \mathcal{O}_F)/\mathcal{O}_F^\times)}{\text{vol}(\text{GL}(n', \mathcal{O}_D)/\mathcal{O}_D^\times)},
\]
where \( \pi^* \) is the discrete series representation of \( \text{GL}(n, F) \) associated to \( \pi \) by the Deligne-Kazhdan-Vignéras correspondence [9]. Since
\[
\text{vol}(\text{GL}(n', \mathcal{O}_D)) = q^{-(d-1)dn'^2/2} \prod_{i=1}^{n'} (1 - q^{-di}),
\]
we obtain
\[ d(\pi) = d(\pi^*). \]

3.2. Inner forms of SL(n). Let \( \tilde{G} \) be an inner form of GL(n) over \( F \) and let \( G \) be the derived group of \( \tilde{G} \). Then \( G \) is an inner form of SL(n) over \( F \). Let \( G_{\text{ad}} \) be the adjoint group of \( G \). Set \( C = \text{cok}(G \to G_{\text{ad}}) \).

Let \( \phi : L_F \to L_G \) be an elliptic Langlands parameter. Then there exists an elliptic tempered Langlands parameter \( \tilde{\phi} : L_F \to \tilde{L}_G \) such that \( \phi = \text{pr} \circ \tilde{\phi} \), where \( \text{pr} : L_G \to L_G \) is the projection. Let \( \tilde{\pi} \) be the discrete series representation of \( \tilde{G} \) associated to \( \tilde{\phi} \) by the local Langlands correspondence \([10], [17]\) and let \( V_{\tilde{\pi}} \) be the space of \( \tilde{\pi} \). Let \( \Pi_{\phi}(G) \) denote the set of equivalence classes of irreducible constituents of the restriction of \( \tilde{\pi} \) to \( G \). Note that \( \Pi_{\phi}(G) \) does not depend on the choice of \( \tilde{\phi} \). Put
\[ X(\tilde{\pi}) = \{ \omega \in C^D \mid \tilde{\pi} \otimes \omega \simeq \hat{\pi} \}, \]
where \( C^D \) is the Pontrjagin dual of \( C \) and \( \omega \) is regarded as a character of \( G_{\text{ad}} \). For \( s \in S_{\phi} \), we have
\[ \text{Int} s \circ \tilde{\phi} = a_s \cdot \tilde{\phi}, \]
where \( a_s \) is a 1-cocycle of \( W_F \) in \( Z(\tilde{G}_{\text{sc}}) \). Let \( \omega_s \) be the character of \( C \) determined by \( a_s \). Then the map \( s \mapsto \omega_s \) induces an exact sequence
\[ 1 \longrightarrow Z_{\phi} \longrightarrow S_{\phi} \longrightarrow X(\tilde{\pi}) \longrightarrow 1. \]
By Theorem 1.4 of \([19]\), there exists an action of \( S_{\phi} \) on \( V_{\tilde{\pi}} \) such that \( Z_{\phi} \) acts via \( \chi_G \) and such that
\[ \tilde{\pi} \circ s = s \circ (\tilde{\pi} \otimes \omega_s) \]
for \( s \in S_{\phi} \). Moreover, if we write a decomposition of \( V_{\tilde{\pi}} \) as a representation of \( S_{\phi} \times G \) in the form
\[ \bigoplus_{\rho \in \Pi(S_{\phi}, \chi_G)} \rho \otimes \pi_{\rho}, \]
then the map \( \rho \mapsto \pi_{\rho} \) defines a bijection between \( \Pi(S_{\phi}, \chi_G) \) and \( \Pi_{\phi}(G) \) (cf. Theorem 1.1 of \([19]\)).

Lemma 3.2. For \( \rho \in \Pi(S_{\phi}, \chi_G) \), we have
\[ d(\pi_{\rho}) = n^2 \cdot \dim \rho \cdot d(\tilde{\pi}). \]

Proof. We fix an invariant hermitian inner product \( (\cdot, \cdot) \) on \( V_{\tilde{\pi}} \). Then \( (\cdot, \cdot) \) is \( S_{\phi} \)-invariant. Let \( v \) be an element in the \( \pi_{\rho} \)-isotypic subspace of \( V_{\tilde{\pi}} \). Recall that the sequence
\[ 1 \longrightarrow G/\mu_n(F) \longrightarrow G_{\text{ad}} \longrightarrow C \longrightarrow 1 \]
is exact. Here \( \mu_n \) is the group of \( n \)-th roots of unity. Since the pullback of \( \omega_{G_{\text{ad}}} \) to \( G \) is \( n \omega_G \) and \( |C|^{-1} \sum_{\omega \in C^D} \omega \) is the characteristic function of \( G/\mu_n(F) \), we have
\[ d(\pi_{\rho})^{-1}(v, v)(v, v) = \frac{|\mu_n(F)|}{|n|} \cdot \frac{1}{|C|} \sum_{\omega \in C^D} \int_{G_{\text{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)(\tilde{\pi}(g)v, v) \, dg. \]
By the Schur orthogonality relations, we have
\[ \int_{G_{\text{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)(\overline{\tilde{\pi}(g)v, v}) \, dg = 0 \]
unless \( \omega \in X(\hat{\pi}) \). Moreover, we have
\[
\int_{G_{ad}} ((\tilde{\pi} \otimes \omega_n)(g)v, v)(\tilde{\pi}(g)v, v) dg = \int_{G_{ad}} (\tilde{\pi}(g)sv, sv)(\tilde{\pi}(g)v, v) dg = d(\tilde{\pi})^{-1}(sv, v)(sv, v)
\]
for \( s \in S_\phi \). Thus we obtain
\[
d(\pi_\rho)^{-1}(v, v)(\bar{v}, \bar{v}) = \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \sum_{s \in S_\phi} d(\tilde{\pi})^{-1}(sv, v)(sv, v)
\]
\[= \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \cdot |S_\phi| \cdot d(\tilde{\pi})^{-1}(v, v)(\bar{v}, \bar{v}).
\]
Note that
\[|n|_F = \frac{|H^0(F, \mu_n)| \cdot |H^2(F, \mu_n)|}{|H^1(F, \mu_n)|} = \frac{|\mu_n(F)| \cdot n}{|C|}.
\]
This yields the lemma. \( \square \)

By Theorem \([5.1]\) and Lemma \([5.2]\) we have
\[d(\pi_\rho) = n \cdot \dim \rho \cdot |S_\phi| \cdot |\gamma(0, \pi_\rho, \Ad, \psi)|
\]
for \( \rho \in \Pi(S_\phi, \chi G) \).

### 3.3. Steinberg representations

Let \( G \) be a connected reductive algebraic group over \( F \). For simplicity, we assume that the connected center of \( G \) is anisotropic. Let \( \pi_0 \) be the Steinberg representation of \( G \). Note that the formal degree of \( \pi_0 \) was computed by Borel \([4]\). Using the results of Kottwitz \([24]\) and Gross \([11], [12]\), we will verify Conjecture \([1.4]\) for \( \pi_0 \). In particular, if \( G \) is an anisotropic torus, then Conjecture \([1.4]\) holds.

Let \( \mu_{G,EP} \) denote the Euler-Poincaré measure on \( G \) and let \( f_{EP} \in C_c^\infty(G) \) denote the Euler-Poincaré function with respect to \( \mu_{G,EP} \). By Theorems 2 and 2′ of \([24]\) and the Plancherel formula (cf. Théorème VIII.1.1 of \([39]\)), we have \( f_{EP}(1) = 1 \) and
\[|d(\pi_0, \mu_{G,EP})| = 1.
\]
By Theorem 5.5 of \([11]\), we have
\[e(G) \cdot |H^1(F, G)| \cdot L(M) \cdot \mu_{G,EP} = L(M^\vee(1)) \cdot \mu_G, \psi.
\]
Here \( e(G) = \pm 1 \) is the Kottwitz sign, \( M \) is the motive of \( G \) as in \([11]\), and \( M^\vee(1) = M^\vee \otimes \mathbb{Q}(1) \) is the Tate twist of the dual motive of \( M \). Hence we have
\[d(\pi_0) = |H^1(F, G)|^{-1} \cdot \frac{|L(M^\vee(1))|}{|L(M)|}.
\]

Let \( \phi_0 : L_F \rightarrow \mathbb{L} G \) be the Langlands parameter associated to \( \pi_0 \). Then \( \phi_0 \) is trivial on \( W_F \) and the restriction of \( \phi_0 \) to \( \SL(2, \mathbb{C}) \) corresponds to the regular unipotent orbit in \( \tilde{G} \). Hence the centralizer of \( \phi_0(L_F) \) in \( \tilde{G} \) is \( Z(\tilde{G})^F \) and \( |S_{\phi_0}| = |H^1(F, G)| \).

**Lemma 3.3.**
\[ |\gamma(0, \pi_0, \Ad, \psi)| = \frac{|L(M^\vee(1))|}{|L(M)|}.
\]
Proof. By Corollary 6.5 of [12], we have $L(M^V(1)) = L(1, \pi_0, \text{Ad})$. Thus it suffices to show that
\[
|L(M)|^{-1} = |\epsilon(0, \pi_0, \text{Ad}, \psi)| \cdot |L(0, \pi_0, \text{Ad})|^{-1}.
\]
Recall that $M = \bigoplus_{d \geq 1} V_d(1-d)$, where $V_d$ is the Artin motive as in [11, §1]. By Proposition 6.4 of [12], we have
\[
L(s, \pi_0, \text{Ad})^{-1} = \prod_{d \geq 1} \det(1 - q^{-s-d+1} \cdot \text{Frob}; V_d^f).
\]
where $I_F$ is the inertia group of $F$. Since $V_d$ is self-dual, we have
\[
|L(M)|^{-1} = \prod_{d \geq 1} |\det(1 - q^{d-1} \cdot \text{Frob}; V_d^f)|
\]
\[
= \prod_{d \geq 1} q^{(d-1) \dim V_d^f} \cdot |L(0, \pi_0, \text{Ad})|^{-1}.
\]
Set $\hat{g} = \text{Lie}(\hat{G})$. Let $(\rho, N)$ be the representation of $W_F^e$ on $\hat{g}$ associated to $\text{Ad} \circ \phi_0$. We can regard $N$ as a regular nilpotent element in $\hat{g}$. By definition, we have
\[
|\epsilon(s, \pi_0, \text{Ad}, \psi)| = q^{-a(\hat{g})(s-1/2)},
\]
where $a(\hat{g}) = \dim \hat{g}^f - \dim \hat{g}_N^f$. By Proposition 5.2 of [12], we have $\hat{g} = \bigoplus_{d \geq 1} V_d \otimes \rho_{2d-2}$ as a representation of $\Gamma \times \text{SL}(2, \mathbb{C})$. Here $\rho_k$ is the irreducible representation of $\text{SL}(2, \mathbb{C})$ of dimension $k+1$. Hence we have
\[
a(\hat{g}) = 2 \sum_{d \geq 1} (d-1) \dim V_d^f.
\]
This completes the proof.

Thus we obtain
\[
d(\pi_0) = \frac{1}{|S_{\phi_0}^s|} \cdot |\gamma(0, \pi_0, \text{Ad}, \psi)|.
\]

3.4. Unipotent discrete series representations. Let $G$ be a connected adjoint split exceptional group of rank $l$ over $F$. Let $\phi: L_F \to {}^L G$ be an elliptic Langlands parameter. We assume that $\phi$ is trivial on the inertia group $I_F$ of $F$. Put
\[
t = \phi \left( \text{Frob} \times \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \right).
\]
We may assume that $t \in T$. Let $\hat{\Sigma}$ denote the set of roots of $T$ in $\hat{G}$. For $i \in \mathbb{Z}$, put
\[
\hat{\Sigma}(i) = \{ \hat{\alpha} \in \hat{\Sigma} \mid \hat{\alpha}(t) = q^{-i/2} \}.
\]
For each $\rho \in \Pi(S_{\phi}, \chi_G)$, Reeder defined a discrete series representation $\pi_\rho$ of $G$ and showed that
\[
d(\pi_\rho) = \frac{q^N \dim \rho}{|S_{\phi}^s|} \cdot \prod_{\hat{\alpha} \in \hat{\Sigma} - \Sigma(0)} (\hat{\alpha}(t) - 1) \cdot \frac{\text{vol}(I)}{\prod_{\hat{\alpha} \in \hat{\Sigma} - \Sigma(2)} (q\hat{\alpha}(t) - 1) \cdot \text{vol}(I)^{-1}}
\]
(cf. [29] (0.3)). Here $N$ is the number of positive roots and $I$ is an Iwahori subgroup of $G$. Note that $\text{vol}(I) = q^{-N}(1 - q^{-1})^l$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 3.4.

$$|\gamma(0, \pi_\rho, \Ad, \psi)| = q^{2N}(1 - q^{-1})^{-l} \cdot \prod_{\alpha \in \Sigma - \Sigma(0)} |\hat{\alpha}(t) - 1| \prod_{\alpha \in \Sigma - \Sigma(2)} |q\hat{\alpha}(t) - 1|. $$

Proof. It is easy to check that

$$\gamma(s, \pi_\rho, \Ad, \psi) = \left(\frac{1 - q^{-s}}{1 - q^{-1+s}}\right) \prod_{\alpha \in \Sigma} \frac{1 - \hat{\alpha}(t)^{-1}q^{-s}}{1 - \hat{\alpha}(t)^{-1}q^{-1+s}}. $$

By [29 (7.2a)], we have $|\Sigma(2)| = |\Sigma(0)| + l$. This yields the lemma. \qed

Thus we obtain

$$d(\pi_\rho) = \frac{\dim \rho}{|S^\Sigma_\phi|} \cdot |\gamma(0, \pi_\rho, \Ad, \psi)|$$

for $\rho \in \Pi(S_\phi, \chi_G)$.

3.5. Depth-zero supercuspidal representations. Let $G$ be a connected reductive algebraic group of rank $l$ over $F$. We assume that $G^*$ is unramified and $G$ is a pure inner form of $G^*$. For simplicity, we assume that the connected center of $G$ is anisotropic. Let $\phi : L_F \to LG$ be an elliptic Langlands parameter. We assume that $\phi$ is trivial on $I_F^+$ and that the centralizer of $\phi(I_F)$ in $G$ is $T$. Here $I_F$ is the inertia group of $F$ and $I_F^+$ is the wild inertia subgroup of $I_F$. Note that $\phi$ is trivial on $\SL(2, \mathbb{C})$. Put $\sigma = \phi(Frob)$. Then $\sigma$ normalizes $T$ and $S^\Sigma_\phi$ is isomorphic to $T^\sigma$, where $T^\sigma$ is the centralizer of $\sigma$ in $T$.

Let $T$ be an unramified maximal torus of $G$ determined by $\sigma$. Then $T$ is anisotropic. Let $T_0$ be an unramified maximal torus of $G$ which is maximally split. We extend $T$ and $T_0$ to smooth group schemes over $\sigma_F$. For each $\rho \in \Pi(S_\phi, \chi_G)$, DeBacker and Reeder defined an irreducible supercuspidal representation $\pi_\rho$ of $G$ and showed that

$$d(\pi_\rho) = q^{l/2}|T(\mathbb{F}_q)|^{-1} \cdot q^{-l/2}|T_0(\mathbb{F}_q)| \cdot \vol(I)^{-1}$$

(cf. [8 §5.3]). Here $I$ is an Iwahori subgroup of $G$. By [11 (4.11)], we have $\vol(I) = q^{-l-N}|T_0(\mathbb{F}_q)|$, where $N = (\dim G - l)/2$. Hence we have

$$d(\pi_\rho) = q^{l+N}|T(\mathbb{F}_q)|^{-1}. $$

Lemma 3.5.

$$|\gamma(0, \pi_\rho, \Ad, \psi)| = q^{l+N}|T(\mathbb{F}_q)|^{-1} \cdot |T^\sigma|. $$

Proof. Set $\hat{\mathfrak{g}} = \Lie(\hat{G})$. Then $\hat{\mathfrak{g}}^T = \Lie(T)$. Here the action of $W_F$ on $\hat{\mathfrak{g}}$ is associated to $\Ad \circ \phi$. By definition, we have

$$L(s, \pi_\rho, \Ad) = \det(1 - q^{-s} \cdot \sigma; \Lie(T))^{-1}. $$

Hence we have

$$L(1, \pi_\rho, \Ad) = q^k|T(\mathbb{F}_q)|^{-1}. $$

Since $T^\sigma$ is isomorphic to $\{x \in X_*(T) \otimes \mathbb{C} | (1 - \sigma)x \in X_*(T)\}/X_*(T)$, we have

$$|L(0, \pi_\rho, \Ad)|^{-1} = |\det(1 - \sigma; \Lie(T))| = |T^\sigma|. $$

By definition, we have

$$|\epsilon(s, \pi_\rho, \Ad, \psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)},$$

where $a(\hat{\mathfrak{g}}) = \dim \hat{\mathfrak{g}}/\hat{\mathfrak{g}}^T = 2N$. This completes the proof. \qed
Thus we obtain
\[ d(\pi_\rho) = \frac{1}{|\mathcal{S}_\phi^2|} \cdot |\gamma(0, \pi_\rho, \Ad, \psi)| \]
for \( \rho \in \Pi(S_\phi, \chi_G) \).

4. Proof of Theorem 3.1

In this section, we give a new proof of the result of Silberger and Zink [35], [37].

Let \( \pi \) be a discrete series representation of \( GL(n, F) \) and let \( V_\pi \) be the space of \( \pi \). We fix an invariant hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( V_\pi \) and equip \( V_\pi \otimes V_\pi \) with the invariant hermitian inner product such that \( \langle u \otimes v, u' \otimes v' \rangle = \langle u, u' \rangle \langle v, v' \rangle \).

Let \( G^2 = GL(2n, F) \). Let \( F^2 = M^2 \sigma^2 \) be a parabolic subgroup of \( G^2 \) given by
\[
M^2 = \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & a' \end{array} \right) \mid a, a' \in GL(n, F) \right\},
\]
\[
N^2 = \left\{ \left( \begin{array}{cc} 1_n & x \\ 0 & 1_n \end{array} \right) \mid x \in \text{Mat}_{n \times n}(F) \right\}.
\]

We consider an induced representation
\[
I(s, \pi \otimes \pi) = \text{Ind}_{F^2}^{G^2}(\pi | \det |_{F^2}^{s/2} \otimes \pi | \det |_{F^2}^{-s/2})
\]
for \( s \in \mathbb{C} \). Put
\[
w = \left( \begin{array}{cc} 0 & 1_n \\ -1_n & 0 \end{array} \right) \in G^2.
\]
For \( \phi \in I(s, \pi \otimes \pi) \) and \( g \in G^2 \), the integral
\[
M(s, w, \pi \otimes \pi)\phi(g) = \int_{\text{Mat}_{n \times n}(F)} \phi \left( w^{-1} \left( \begin{array}{cc} 1_n & x \\ 0 & 1_n \end{array} \right) g \right) \, dx
\]
is absolutely convergent for \( \text{Re}(s) > 0 \), has a meromorphic continuation to the whole \( s \)-plane, and defines an intertwining operator
\[
M(s, w, \pi \otimes \pi) : I(s, \pi \otimes \pi) \longrightarrow I(-s, w(\pi \otimes \pi)).
\]
Here \( dx \) is the Haar measure on \( \text{Mat}_{n \times n}(F) \) with \( \text{vol}(\text{Mat}_{n \times n}(F), \, dx) = 1 \).

**Lemma 4.1.** There exists a constant \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \) such that
\[
(\text{Res} \ M(s, w, \pi \otimes \pi)\phi(1), u' \otimes v')
\]
\[= \alpha (\log q)^{-1} (1 - q^{-1}) \gamma(0, \pi, \Ad, \psi)^{-1} (\phi(1), v' \otimes u') \]
for \( \phi \in I(s, \pi \otimes \pi) \) and \( u', v' \in V_\pi \). Here \( \psi \) is a non-trivial additive character of \( F \) of order zero.

**Proof.** Set \( I(\pi \otimes \pi) = I(0, \pi \otimes \pi) \). Let \( \text{sw} : V_\pi \otimes V_\pi \to V_\pi \otimes V_\pi \) be an isomorphism given by \( \text{sw}(u \otimes v) = v \otimes u \). Then \( \text{sw} \) induces an isomorphism \( \text{sw} : I(w(\pi \otimes \pi)) \to I(\pi \otimes \pi) \). We define a normalized intertwining operator
\[
N(w, \pi \otimes \pi) : I(\pi \otimes \pi) \longrightarrow I(\pi \otimes \pi)
\]
by
\[N(w, \pi \otimes \pi) = \text{sw} \lim_{s \to 0} \gamma(s, \pi \times \pi, \psi) M(s, w, \pi \otimes \pi).
\]
By Theorem 7.9 of [32], \( N(w, \pi \otimes \pi) \) is unitary. Since \( I(\pi \otimes \pi) \) is irreducible, there exists a constant \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \) such that
\[N(w, \pi \otimes \pi) = \alpha \text{id},\]
Let \( \text{Lemma 4.2.} \) in \( P \), i.e.,

\[
\text{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(g) = \alpha \text{Res}_{s=0} \gamma(s, \pi \times \pi, \psi)^{-1} \phi(g)
\]

for \( \phi \in I(s, \pi \otimes \pi) \) and \( g \in G^s \).

Let

\[
\tilde{N}^s = \left\{ \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \middle| x \in \text{Mat}_{n \times n}(F) \right\}
\]

and \( L = \text{Mat}_{n \times n}(a_F) \). Let \( 1_L \) denote the characteristic function of \( L \).

**Lemma 4.2.** Let \( u, v \in V_\pi \). We define \( \phi \in I(s, \pi \otimes \pi) \) which has compact support in \( P^s \tilde{N}^s \) modulo \( P^s \) by

\[
\phi \left( \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \right) = \begin{cases} u \otimes v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}
\]

Then

\[
\begin{align*}
\text{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v' \\
= (n \log q)^{-1} (1 - q^{-1}) d(\pi)^{-1} (\phi(1), v' \otimes u')
\end{align*}
\]

for \( u', v' \in V_\pi \).

**Proof.** The lemma follows from Proposition 5.1 of [34]. We include the proof for the sake of completeness.

As in [33], [34], we have

\[
\begin{align*}
(M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v') \\
= \int_{\text{GL}(n, F)} 1_L(x^{-1}) |\det(x)|^{-s-n} (\pi(x^{-1})u \otimes \pi(x)v, u' \otimes v') \, dx \\
= \int_{\text{GL}(n, F)} 1_L(x) |\det(x)|^{-s-n} (\pi(x)u, u' \otimes v') (\pi(x)v', v) \, d^x x \\
= \int_{\text{GL}(n, F)/F^x} \varphi_s(x)(\pi(x)u, u' \otimes v') (\pi(x)v', v) \, d^x x,
\end{align*}
\]

where \( d^x x = |\det(x)|^{-n} \, dx \) and

\[
\varphi_s(x) = |\det(x)|^{-n} \int_{F^x} 1_L(zx) |z|_{F}^{-ns} \, d^x z.
\]

For \( x = (x_{ij}) \in \text{GL}(n, F) \), we have

\[
\int_{F^x} 1_L(zx) |z|_{F}^{-ns} \, d^x z = \int_{F^{nm}} |z|_{F}^{-ns} \, d^x z = q^{mn s} (1 - q^{-ns})^{-1} (1 - q^{-1}),
\]

where \( m = \min(\text{ord}_F(x_{ij})) \). Note that this integral is absolutely convergent for \( \text{Re}(s) > 0 \). Hence we have

\[
\begin{align*}
\text{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v' \\
= (n \log q)^{-1} (1 - q^{-1}) \int_{\text{GL}(n, F)/F^x} (\pi(x)u, u' \otimes v') (\pi(x)v', v) \, d^x x \\
= (n \log q)^{-1} (1 - q^{-1}) d(\pi)^{-1} (u, v')(v, u').
\end{align*}
\]
This calculation is justified since
\[ \varphi_s(x) \leq (1 - q^{-ns})^{-1}(1 - q^{-1}) \]
for \( s \in \mathbb{R}_{>0} \). □

By Lemmas 4.1 and 4.2 we have \( d(\pi) = \alpha_n^{-1}n^{-1}\gamma(0, \pi, \text{Ad}, \psi) \). This completes the proof of Theorem 3.1.

5. Twisted characters

Let \( F \) be a non-archimedean local field of characteristic zero and let \( \psi \) be a non-trivial additive character of \( F \) of order zero. Let \( G = \text{Res}_{E/F} \text{GL}(n) \), where \( E \) is a quadratic extension of \( F \) and \( n \) is odd. Let \( \sigma \) be the non-trivial automorphism of \( E \) over \( F \) and let \( \omega_{E/F} \) be the quadratic character of \( F^\times \) associated to \( E/F \) by class field theory. Put \( \theta(g) = \text{Ad}(J_n)(\sigma(1g^{-1})) \) for \( g \in G \), where
\[
J_n = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & -1 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
(-1)^{n-1} & \cdots & 0 & 0
\end{pmatrix} \in \text{GL}(n)
\]
and \( \sigma \) is regarded as an automorphism of \( G \) over \( F \).

Let \( \pi \) be a discrete series representation of \( G \) such that \( \pi \simeq \pi \circ \theta \). We fix an isomorphism \( \pi(\theta) : \pi \rightarrow \pi \circ \theta \) such that \( \pi(\theta)^2 = \text{id} \) and we define a distribution \( J^\theta(\pi) \) by
\[
J^\theta(\pi)(f) = J^0(\pi, f) = \text{trace}(\pi(f)\pi(\theta))
\]
for \( f \in C_0^\infty(G) \). By Theorem 1 of [5], \( J^0(\pi) \) is a locally integrable function on \( G \) which is locally constant on \( G_{\theta, \text{reg}} \). Here \( G_{\theta, \text{reg}} \) is the set of \( \theta \)-regular and \( \theta \)-semisimple elements in \( G \). Let \( G_\theta \) denote the identity component of \( \{ g \in G | \theta(g) = g \} \) and \( g_\theta \) the Lie algebra of \( G_\theta \). By Theorem 3 of [5], we have the expansion
\[ J^0(\pi, \exp(X)) = \sum_{\mathcal{O}} c_{\mathcal{O}, \theta}(\pi) \hat{\mu}_\mathcal{O}(X) \]
for \( X \in g_\theta \) sufficiently near zero, where \( \mathcal{O} \) runs over nilpotent \( G_\theta \)-orbits in \( g_\theta \) and where \( \hat{\mu}_\mathcal{O} \) is the Fourier transform of the invariant measure \( \mu_\mathcal{O} \) on \( \mathcal{O} \).

**Theorem 5.1.** Let \( \pi \) be a discrete series representation of \( G \) such that \( \pi \simeq \pi \circ \theta \) and such that \( L(s, \pi, r) \) has a pole at \( s = 0 \), where \( r \) is the Asai representation of \( LG \). Then there exists a constant \( c \in \mathbb{R}_{>0} \) which does not depend on \( \pi \) such that
\[ |c_{\mathcal{O}, \theta}(\pi)| = c|\gamma(0, \pi, r, \psi)|, \]
where \( r' = r \otimes \omega_{E/F} \).

**Remark 5.2.** By the result of Henniart [18], we have
\[
L_{LS}(s, \pi, r) = L(s, \pi, r),
\]
\[
\gamma_{LS}(s, \pi, r', \psi) = \alpha \gamma(s, \pi, r', \psi),
\]
where the subscript \( \text{LS} \) indicates local factors defined by the Langlands-Shahidi method and \( \alpha \in \mathbb{C} \) is a root of unity.
6. Twisted orbital integrals

Let \( F \) be a non-archimedean local field of characteristic zero. Let \( G \) be a connected reductive algebraic group over \( F \) and let \( \theta \) be a quasi-semisimple automorphism of \( G \) over \( F \). Let \( Z \) be the center of \( G \). For \( \gamma \in G \) and \( f \in C_c^\infty(G/(1-\theta)Z) \), put

\[
J^\theta(\gamma, f) = \int_{ZG_{\gamma}\theta G} f(g^{-1}\gamma\theta(g)) \, dg.
\]

Here \( G_{\gamma\theta} \) is the identity component of \( \{g \in G \mid g^{-1}\gamma\theta(g) = \gamma\} \). By [28] and Lemma 2.1 of [1], this integral is absolutely convergent. By Proposition 7.1 of [33], we have the expansion

\[
J^\theta(\exp(X), f) = \sum_u \Gamma_{u,\theta}(X) J^\theta(u, f)
\]

for \( \theta \)-regular and \( \theta \)-semisimple elements \( X \) in \( G_{\theta} \) sufficiently near zero. Here \( u \) runs over representatives for unipotent orbits in \( G_{\theta} \).

Let \( G \) and \( \theta \) be as in [1]. By [6], \( J^\theta(\gamma, f) \) is absolutely convergent and (6.1) holds even if \( f \) is a Schwartz function.

7. Proof of Theorem 5.1

Throughout this section, we ignore the normalization of measures since it does not affect the proof. Let \( G \) and \( \theta \) be as in [1]. Let \( \pi \) be a discrete series representation of \( G \) such that \( \pi \simeq \pi \circ \theta \) and such that \( L(s, \pi, r) \) has a pole at \( s = 0 \), where \( r \) is the Asai representation of \( L^G \). Let \( V_\pi \) be the space of \( \pi \). We fix an invariant hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( V_\pi \) and an isomorphism \( \pi(\theta) : V_\pi \rightarrow V_\pi \) such that \( \pi(\theta)\pi(g) = \pi(\theta(g))\pi(\theta) \) for \( g \in G \) and such that \( \pi(\theta)^2 = 1 \). Then \( \langle \cdot, \cdot \rangle \) is \( \pi(\theta) \)-invariant.

Let \( G^\sharp = U(2n, F) = \{g \in GL(2n, E) \mid gQ_n\sigma(g) = Q_n\} \), where

\[
Q_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.
\]

Let \( P^\sharp = M^\sharp N^\sharp \) be a parabolic subgroup of \( G^\sharp \) given by

\[
M^\sharp = \left\{ \begin{pmatrix} a & 0 \\ 0 & \theta(a) \end{pmatrix} \mid a \in GL(n, E) \right\},
\]

\[
N^\sharp = \left\{ \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \mid x \in X \right\},
\]

where \( X = \{x \in \text{Mat}_{n \times n}(E) \mid \text{Ad}(J_n)(\sigma(t)x) = x\} \). As in [24] we consider an induced representation

\[
I(s, \pi) = \text{Ind}_{P^\sharp E}^{G^\sharp}(\pi | \det |_{E}^{s/2})
\]

for \( s \in \mathbb{C} \). Put

\[
w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in G^\sharp.
\]

For \( \phi \in I(s, \pi) \) and \( g \in G^\sharp \), the integral

\[
M(s, w, \pi)\phi(g) = \int_X \phi \left( w^{-1} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} g \right) \, dx
\]
Then there exists a constant \( c \in \mathbb{C} \) with \(|c| = 1\) such that

\[
\left( \Res_{s=0} M(s, w, \pi) \phi(1), v' \right) = c \Res_{s=0} \gamma(s, \pi, r, \psi)^{-1}(\phi(1), \pi(\theta)v')
\]

for \( \phi \in I(s, \pi) \) and \( v' \in V_\pi \).

**Proof.** We remark that \( I(0, \pi) \) is irreducible since \( L(s, \pi, r) \) has a pole at \( s = 0 \). As in the proof of Lemma 4.1, the lemma follows from the result of Shahidi 32. \( \square \)

Let

\[
\tilde{N}^a = \left\{ \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \mid x \in X \right\}
\]

and let \( L = X \cap \text{Mat}_{n \times n}(\mathbb{O}_E) \). Let \( 1_L \) denote the characteristic function of \( L \).

**Lemma 7.2.** Let \( v, v' \in V_\pi \). Let \( f \) be a matrix coefficient of \( \pi \) given by \( f(g) = (\pi(g)v, v') \) for \( g \in G \). We define \( \phi \in I(s, \pi) \) which has compact support in \( P^1 \tilde{N}^a \) modulo \( P^1 \pi \) by

\[
\phi \left( \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \right) = \begin{cases} v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}
\]

Then there exists a constant \( c \in \mathbb{R}_{>0} \) which does not depend on \( \pi \) such that

\[
\left( \Res_{s=0} M(s, w, \pi) \phi(1), v' \right) = cJ^0(1, f).
\]

**Proof.** The lemma follows from the result of Goldberg 10. We include the proof for the sake of completeness.

We fix \( \delta \in F^\times - N_{E/F}(E^\times) \). Set \( X' = \{ x \in X \mid \det(x) \neq 0 \} \). Then \( G \) acts on \( X' \) by \( x \mapsto g^{-1}x\theta(g) \). Let \( G' \backslash X' \) denote the set of \( G \)-orbits in \( X' \). Note that \( \{1, \delta\} \) is a set of representatives for \( G \backslash X' \). We define a \( G \)-invariant measure \( d^x x \) on \( X' \) by \( d^x x = |\det(x)|_E^{-n/2} dx \).

As in 10 2, we have

\[
(M(s, w, \pi)\phi(1), v')
\]

\[
= \int_{X'} 1_L(x^{-1}) |\det(x)|_E^{-s/2-n/2}(\pi(x^{-1})v, v') dx
\]

\[
= \int_{X'} 1_L(x) |\det(x)|_E^{s/2} f(x) d^x x
\]

\[
= \sum_{\gamma \in G \backslash X'} \int_{G \gamma, \theta \backslash G} 1_L(g^{-1} \gamma \theta(g)) |\det(g^{-1} \gamma \theta(g))|_E^{s/2} f(g^{-1} \gamma \theta(g)) dg
\]

\[
= \sum_{\gamma \in G \backslash X'} \int_{ZG \gamma, \theta \backslash G} \varphi_s(g^{-1} \gamma \theta(g)) f(g^{-1} \gamma \theta(g)) dg,
\]

where

\[
\varphi_s(x) = |\det(x)|_E^{s/2} \int_{E^\times} 1_L(z \sigma(z)x)|z|_E^{ns} d^z z.
\]

For \( x = (x_{ij}) \in X' \), we have

\[
\int_{E^\times} 1_L(z \sigma(z)x)|z|_E^{ns} d^z z = q_E^{(m/2)ns}(1-q_E^{-ns})^{-1}(1-q_E^{-1}),
\]

is absolutely convergent for \( \text{Re}(s) > 0 \), has a meromorphic continuation to the whole \( s \)-plane, and defines an intertwining operator

\[
M(s, w, \pi) : I(s, \pi) \rightarrow I(-s, w(\pi)).
\]
where \( m = \min(\text{ord}_E(x_{ij})) \). Note that this integral is absolutely convergent for \( \Re(s) > 0 \). Hence we have
\[
(\text{Res}_{s=0} M(s, w, \pi)\phi(1), v') = (n \log q_E)^{-1}(1 - q_E^{-1}) \sum_{\gamma \in G \setminus X'} J^0(\gamma, f).
\]

This calculation is justified since
\[
\phi_s(x) \leq (1 - q_E^{-ns})^{-1}(1 - q_E^{-1})
\]
for \( s \in \mathbb{R}_{>0} \). As in [10, §2], the central character of \( \pi \) is trivial on \( F^\times \). Hence we have
\[
J^0(\delta, f) = J^0(1, f).
\]

This completes the proof. \( \square \)

**Lemma 7.3.** Let \( v, v' \in V_\pi \). Let \( f \) be a matrix coefficient of \( \pi \) given by \( f(g) = (\pi(g)v, v') \) for \( g \in G \). Then
\[
J^0(\gamma, f) = d(\pi)^{-1}(v, \pi(\theta)v')\mu(\gamma)
\]
for \( \theta \)-regular and \( \theta \)-elliptic elements \( \gamma \) in \( G \).

**Proof.** We proceed as in the proof of Proposition 5 of [7]. Let \( \gamma \) be a \( \theta \)-regular and \( \theta \)-elliptic element in \( G \). Let \( \varphi \in C_c^\infty(G) \). We assume that the support of \( \varphi \) is contained in the set of \( \theta \)-regular and \( \theta \)-elliptic elements in \( G \). By the Schur orthogonality relations, we have
\[
\int_{Z \setminus G} (\pi(g)^{-1}\pi(\varphi)\pi(\theta(g))v, v') dg = d(\pi)^{-1}(v, \pi(\theta)v')\mu(\gamma).
\]
The left-hand side is equal to
\[
\int_{Z \setminus G} \int_G \varphi(h)f(g^{-1}h\theta(g)) dh dg = \int_G \varphi(h)\int_{Z \setminus G} f(g^{-1}h\theta(g)) dg dh.
\]
Let \( \varphi \) tend to the Dirac measure at \( \gamma \). This yields the lemma. \( \square \)

Let \( v, v' \in V_\pi \). Let \( f \) be a matrix coefficient of \( \pi \) given by \( f(g) = (\pi(g)v, v') \) for \( g \in G \). By [5.1], [6.1], and Lemma 7.3 we have
\[
(v, \pi(\theta)v') \sum_\mathcal{O} c_{\mathcal{O}, \theta}(\pi)\mu_\mathcal{O}(X) = d(\pi)\sum_u \Gamma_{u, \theta}(X)J^0(u, f)
\]
for \( \theta \)-regular and \( \theta \)-elliptic elements \( X \) in \( g_\theta \) sufficiently near zero. For \( t \in F^\times \), we have
\[
\mu_\mathcal{O}(t^2X) = |t|_F^{-\text{dim} \mathcal{O}}\mu_\mathcal{O}(X),
\]
\[
\Gamma_{u, \theta}(t^2X) = |t|_F^{-\text{dim} \text{Ad}(G_\theta)(u)}\Gamma_{u, \theta}(X).
\]
Note that \( \mu_0 = \Gamma_{1, \theta} = 1 \) if measures are suitably normalized. By homogeneity, we obtain
\[
(v, \pi(\theta)v')c_{0, \theta}(\pi) = d(\pi)J^0(1, f).
\]
By Theorem 5.1 \( d(\pi) \) is equal to
\[
|\lim_{s \to 0} s^{-1}\gamma(s, \pi \times \hat{\pi}, \psi)| = |\lim_{s \to 0} s^{-1}\gamma(s, \pi, r, \psi)| \cdot |\gamma(0, \pi, r', \psi)|
\]
up to a constant which does not depend on \( \pi \). Thus Theorem 5.1 follows from Lemmas 7.1 and 7.2.
Lemma 8.1. Let $\phi$ be a tempered Langlands parameter. We define a tempered Langlands parameter $\phi : L_F \to L^\infty G$ by $\phi = \xi \circ \phi_H$. Let $\pi$ be the irreducible tempered representation of $G$ associated to $\phi$ by the local Langlands conjectures hold (cf. [2]).

8. Twisted endoscopy

Let $F$ be a non-archimedean local field of characteristic zero. Let $G$ and $\theta$ be as in §5. We consider a set of endoscopic data $(H, H, 1, \xi)$ for $(G, \theta, 1)$ defined as follows. Recall that $G = \text{Res}_{E/F} \text{GL}(n)$, where $E$ is a quadratic extension of $F$ and where $n$ is odd. We have $L^\infty G = G \times W_F$, where $G = \text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})$ and the action of $w \in W_F$ is given by

$$ (g_1, g_2) \mapsto \begin{cases} (g_1, g_2) & \text{if } w \in W_E, \\ (g_2, g_1) & \text{if } w \notin W_E. \end{cases} $$

Let $H = \text{U}(n)$ be the quasi-split unitary group in $n$ variables. Then $L^\infty H = H \times W_F$, where $H = \text{GL}(n, \mathbb{C})$ and the action of $w \in W_F$ is given by

$$ h \mapsto \begin{cases} h & \text{if } w \in W_E, \\ \text{Ad}(J_n)(h^{-1}) & \text{if } w \notin W_E. \end{cases} $$

We define $\xi : L^\infty H \to L^\infty G$ by $\xi(h \times w) = (h, \text{Ad}(J_n)(h^{-1})) \times w$.

Lemma 8.1. Let $r$ be the Asai representation of $L^\infty G$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. Then the adjoint representation $\text{Ad}$ of $L^\infty H$ on $\text{Lie}(H)$ is isomorphic to $r' \circ \xi$, where $r' = r \otimes \omega_{E/F}$.

Proof. Recall that $r$ is defined by

$$ r((g_1, g_2) \times 1)(x \otimes y) = g_1 x \otimes g_2 y, $$

$$ r((1, 1) \times w)(x \otimes y) = \begin{cases} x \otimes y & \text{if } w \in W_E, \\ y \otimes x & \text{if } w \notin W_E. \end{cases} $$

It is easy to check that

$$ \text{Ad}(h \times 1)(X) = \text{Ad}(h)(X), $$

$$ \text{Ad}(1 \times w)(X) = \begin{cases} X & \text{if } w \in W_E, \\ -\text{Ad}(J_n)(X) & \text{if } w \notin W_E. \end{cases} $$

Hence the isomorphism $\mathbb{C}^n \otimes \mathbb{C}^n \to \text{Lie}(H)$ given by $x \otimes y \mapsto x'yJ_n$ is intertwining.

It is believed that the following conjectures hold (cf. [2]).

Conjecture 8.2. For $f \in C_c^\infty(G)$, there exists $f^H \in C_c^\infty(H)$ such that $f$ and $f^H$ have matching orbital integrals (cf. [25, §5.5]).

Conjecture 8.3. Let $\phi_H : L_F \to L^\infty H$ be a tempered Langlands parameter. We define a tempered Langlands parameter $\phi : L_F \to L^\infty G$ by $\phi = \xi \circ \phi_H$. Let $\pi$ be the irreducible tempered representation of $G$ associated to $\phi$ by the local Langlands conjectures hold (cf. [2]).
correspondence [16], [17]. Then there exists a constant $c \in \mathbb{C}$ such that $|c|$ does not depend on $\phi_H$ and such that

$$J^\theta(\pi, f) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle J(\pi_H, f^H).$$

Here $f$ and $f^H$ have matching orbital integrals.

For $f \in C_c^\infty(G)$, let $f^{G_\theta} \in C_c^\infty(G_\theta)$ be a decent of $f$, where $G_\theta$ is the identity component of $\{g \in G | \theta(g) = g\}$. Let $f^H \in C_c^\infty(H)$. Assume that the supports of $f$ and $f^H$ are sufficiently small. By [15], [5], we have

$$J^\theta(\pi, f) = \sum_{\mathcal{O}} c_{\mathcal{O}, \theta}(\pi) \mu_{\mathcal{O}}(f^{G_\theta} \circ \exp),$$

$$J(\pi_H, f^H) = \sum_{\mathcal{O}_H} c_{\mathcal{O}_H}(\pi_H) \mu_{\mathcal{O}_H}(f^H \circ \exp).$$

Here $\mathcal{O}$ (resp. $\mathcal{O}_H$) runs over nilpotent $G_\theta$-orbits (resp. $H$-orbits) in $g_\theta = \text{Lie}(G_\theta)$ (resp. $\mathfrak{h} = \text{Lie}(H)$).

**Lemma 8.4.** Assume that Conjectures 8.2 and 8.3 hold. Let $\phi_H : L_F \to L^H$ be a tempered Langlands parameter and let $\pi$ be the irreducible tempered representation of $G$ as in Conjecture 8.3. Then there exists a constant $c \in \mathbb{C}$ such that $|c|$ does not depend on $\phi_H$ and such that

$$c_{0, \theta}(\pi) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H).$$

**Proof.** We proceed as in [22 §9], [21 §8]. Assume that the supports of $f$ and $f^H$ are sufficiently small. If $t \in F^\infty$ is sufficiently small, then we can define $f_t \in C_c^\infty(G)$ by $f_t(\exp(X)) = f(\exp(t^{-1}X))$. Similarly, we can define $f_t^H \in C_c^\infty(H)$. Then

$$\hat{\mu}_{\mathcal{O}}(f_{t_2}^{G_\theta} \circ \exp) = |t|_F^{2 \dim g_\theta - \dim \mathcal{O}} \mu_{\mathcal{O}}(f^{G_\theta} \circ \exp),$$

$$\hat{\mu}_{\mathcal{O}_H}(f_{t_2}^H \circ \exp) = |t|_F^{2 \dim \mathfrak{h} - \dim \mathcal{O}_H} \mu_{\mathcal{O}_H}(f^H \circ \exp).$$

Note that $\dim g_\theta = \dim \mathfrak{h}$.

Assume that $f$ and $f^H$ have matching orbital integrals. By Lemma 8.5 of [21], $f_t$ and $f_t^H$ have matching orbital integrals. Hence we have

$$\sum_{\mathcal{O}} c_{\mathcal{O}, \theta}(\pi) \hat{\mu}_{\mathcal{O}}(f_{t_2}^{G_\theta} \circ \exp) = c \sum_{\mathcal{O}_H} \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_{\mathcal{O}_H}(\pi_H) \hat{\mu}_{\mathcal{O}_H}(f_{t_2}^H \circ \exp).$$

By homogeneity, we obtain

$$c_{0, \theta}(\pi) \hat{\mu}_0(\pi) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H) \hat{\mu}_0(f^H \circ \exp).$$

□

Let $\pi_H$ be a discrete series representation of $H$ and let $\phi_H : L_F \to L^H$ be the (conjectural) Langlands parameter associated to $\pi_H$. Let $\pi$ be the irreducible tempered representation of $G$ as in Conjecture 8.3. If $\pi_H$ is stable, then $\pi$ is expected to be square integrable.
Proposition 8.5. Assume that Conjectures 8.2 and 8.3 hold. Let $\pi_H$ be a stable discrete series representation of $H$. Then
\[ d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \text{Ad}, \psi)|. \]

Proof. By [15], [30], we have
\[ c_0(\pi_H) = (-1)^l_0 d(\pi_{H,0})^{-1} \cdot d(\pi_H), \]
where $l_0$ is the semisimple $F$-rank of $H$ and $\pi_{H,0}$ is the Steinberg representation of $H$. By Theorems 8.1 and Lemmas 8.1 and 8.4, there exists a constant $c \in \mathbb{R}_{>0}$ which does not depend on $\pi_H$ such that
\[ d(\pi_H) = c |\gamma(0, \pi_H, \text{Ad}, \psi)|. \]
Since $\pi_{H,0}$ is stable and $d(\pi_{H,0}) = 2^{-1} |\gamma(0, \pi_{H,0}, \text{Ad}, \psi)|$, we have $c = 2^{-1}$. \hfill \Box

For $n = 3$, Conjectures 8.2 and 8.3 were proved by Rogawski [31]. Thus we obtain the following theorem.

Theorem 8.6. Let $H = U(3)$ be the quasi-split unitary group in three variables. Let $\pi_H$ be a stable discrete series representation of $H$. Then
\[ d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \text{Ad}, \psi)|. \]
In particular, Conjecture 1.4 holds for $\pi_H$.

References

Acknowledgments

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: hiraga@math.kyoto-u.ac.jp

Department of Mathematics, Graduate School of Science, Osaka City University, Osaka 558-8585, Japan
E-mail address: ichino@sci.osaka-cu.ac.jp

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: ikeda@math.kyoto-u.ac.jp