

## FORMAL DEGREES AND ADJOINT $\gamma$ -FACTORS

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*Dedicated to Professor Hiroshi Saito on the occasion of his sixtieth birthday*

### INTRODUCTION

Let  $\mathbf{G}$  be a connected reductive algebraic group over a local field  $F$  and let  $\mathbf{H}$  be a closed subgroup of  $\mathbf{G}$  over  $F$ . Set  $G = \mathbf{G}(F)$  and  $H = \mathbf{H}(F)$ . Let  $\pi$  be an irreducible unitary representation of  $G$  and let  $V_\pi$  be the space of  $\pi$ . For  $v \in V_\pi$ , we will consider the integral

$$(0.1) \quad \int_H (\pi(h)v, v) dh.$$

We can regard this integral as an analogue of (the square of the absolute value of) a period integral of an automorphic form and expect that it is related to  $L$  and  $\epsilon$ -factors. For example, let  $\mathbf{G} = \mathrm{SO}(n+1) \times \mathrm{SO}(n)$  and  $\mathbf{H} = \mathrm{SO}(n)$ . Let  $\pi = \pi_1 \otimes \pi_0$ , where  $\pi_1$  (resp.  $\pi_0$ ) is an irreducible unramified tempered representation of  $\mathrm{SO}(n+1, F)$  (resp.  $\mathrm{SO}(n, F)$ ). Then (0.1) can be expressed in terms of

$$\frac{L(\frac{1}{2}, \pi_1 \times \pi_0)}{L(1, \pi_1, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})}$$

if  $v$  is unramified (cf. [20]). Now let  $\mathbf{G} = \mathbf{H} \times \mathbf{H}$ , where  $\mathbf{H}$  is a connected reductive algebraic group over  $F$ . For simplicity, we assume that the connected center of  $\mathbf{H}$  is anisotropic. Let  $\pi = \pi_H \otimes \tilde{\pi}_H$ , where  $\pi_H$  is a discrete series representation of  $H$  and  $\tilde{\pi}_H$  is the contragredient representation of  $\pi_H$ . Then (0.1) can be expressed in terms of the formal degree  $d(\pi_H)$  of  $\pi_H$ . In this paper, we give a conjectural formula for  $d(\pi_H)$  in terms of the adjoint  $\gamma$ -factor

$$\gamma(s, \pi_H, \mathrm{Ad}, \psi) = \epsilon(s, \pi_H, \mathrm{Ad}, \psi) \cdot \frac{L(1-s, \tilde{\pi}_H, \mathrm{Ad})}{L(s, \pi_H, \mathrm{Ad})}$$

(cf. Conjecture 1.4). Here  $\mathrm{Ad}$  is the adjoint representation of the  $L$ -group  ${}^L H$  of  $\mathbf{H}$  on the Lie algebra  $\mathrm{Lie}(\hat{H})$  of the dual group of  $\mathbf{H}$  and  $\psi$  is a non-trivial additive character of  $F$ .

Our conjecture is supported by various examples. For example, we assume that  $F = \mathbb{R}$  and  $\mathbf{H}$  is anisotropic. We take the Haar measure  $dh$  on  $H$  determined by a Chevalley basis of  $\mathrm{Lie}(H) \otimes \mathbb{C}$ . Let  $\pi_H$  be an irreducible finite dimensional representation of  $H$ . Then the conjecture for  $\pi_H$  asserts that

$$\frac{\dim \pi_H}{\mathrm{vol}(H)} = \frac{1}{2^l} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|$$

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and it is compatible with the Weyl dimension formula. Here  $l$  is the rank of  $\mathbf{H}$  and  $\psi(x) = \exp(2\pi\sqrt{-1}x)$  for  $x \in \mathbb{R}$ . Also, if  $F$  is non-archimedean, then the conjecture for  $\mathrm{GL}(n)$  is compatible with the result of Silberger and Zink [35], [37].

Moreover, we provide some evidence in the case of the quasi-split unitary group in three variables. To be precise, let  $F$  be a non-archimedean local field of characteristic zero. Let  $E$  be a quadratic extension of  $F$  and let  $\sigma$  be the non-trivial automorphism of  $E$  over  $F$ . Put

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let

$$\mathbf{H} = \mathrm{U}(3) = \{h \in \mathrm{Res}_{E/F} \mathrm{GL}(3) \mid \theta(h) = h\},$$

where  $\theta(h) = \mathrm{Ad}(J_3)(\sigma({}^t h^{-1}))$ . Following Gross [11], we choose a Haar measure  $dh$  on  $H$ . Let  $\pi_H$  be a stable discrete series representation of  $H$ . We will verify the conjecture for  $\pi_H$ , i.e.,

$$(0.2) \quad d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|$$

(cf. Theorem 8.6).

To prove (0.2), we use twisted endoscopy. Let  $J(\pi_H) = \mathrm{trace} \pi_H$  be the character of  $\pi_H$  and let  $c_0(\pi_H)$  be the coefficient associated to the trivial orbit in the local character expansion of  $J(\pi_H)$  (cf. [15]). Recall that

$$(0.3) \quad c_0(\pi_H) \doteq d(\pi_H).$$

Here the notation  $\doteq$  indicates equality up to constants which do not depend on the representations. Let  $\pi$  be the base change of  $\pi_H$  to  $\mathrm{GL}(3, E)$ . Then  $\pi$  is square integrable since  $\pi_H$  is stable. Also,  $\pi$  is isomorphic to  $\pi \circ \theta$ . We fix an isomorphism  $\pi(\theta) : \pi \rightarrow \pi \circ \theta$  such that  $\pi(\theta)^2 = \mathrm{id}$ . Let  $J^\theta(\pi) = \mathrm{trace} \pi \circ \pi(\theta)$  be the twisted character of  $\pi$  and let  $c_{0,\theta}(\pi)$  be the coefficient associated to the trivial orbit in the local character expansion of  $J^\theta(\pi)$  (cf. [5]). The character identity between  $J^\theta(\pi)$  and  $J(\pi_H)$  was proved by Rogawski [31] and implies that

$$|c_{0,\theta}(\pi)| \doteq |c_0(\pi_H)|.$$

We also have an analogue

$$c_{0,\theta}(\pi) \cdot (v, \pi(\theta)v') \doteq d(\pi) \cdot J^\theta(1, f)$$

of (0.3). Here  $f$  is a matrix coefficient of  $\pi$  given by  $f(g) = (\pi(g)v, v')$  and  $J^\theta(1, f)$  is the twisted orbital integral of  $f$  at the identity element. By the result of Silberger and Zink [35], [37], we have

$$d(\pi) \doteq \left| \lim_{s \rightarrow 0} s^{-1} \gamma(s, \pi \times \tilde{\pi}, \psi) \right|.$$

By the results of Shahidi [32] and Goldberg [10], we have

$$|J^\theta(1, f)| \doteq \left| \lim_{s \rightarrow 0} s^{-1} \gamma(s, \pi, r, \psi) \right|^{-1} \cdot |(v, \pi(\theta)v')|,$$

where  $r$  is the Asai representation. Thus we obtain (0.2).

This paper is organized as follows. In §1, we formulate a conjecture on formal degrees and relate it to the Plancherel formula. In §2, we verify the conjecture in the archimedean case. In §3, we present various examples in the non-archimedean case. For example, the conjecture for  $\mathrm{GL}(n)$  is compatible with the result of Silberger

and Zink [35], [37]. Using the results of Shahidi [32], [33], [34], we give a new proof of their result in §4. In §5, we give a description of the coefficient associated to the trivial orbit in the local character expansion of a certain twisted character. After recalling some facts about twisted orbital integrals in §6, we prove this description in §7. In §8, we verify the conjecture for a stable discrete series representation of  $U(3)$ .

1. CONJECTURES

In this section, we formulate a conjecture on formal degrees (cf. Conjecture 1.4).

Let  $F$  be a local field of characteristic zero and let  $\psi$  be a non-trivial additive character of  $F$ . Let  $|\cdot|_F$  denote the absolute value on  $F$ . If  $F$  is non-archimedean, let  $\mathfrak{o}_F$  be the maximal compact subring of  $F$ ,  $\mathfrak{p}_F$  the maximal ideal of  $\mathfrak{o}_F$ , and  $q = q_F$  the cardinality of  $\mathfrak{o}_F/\mathfrak{p}_F$ . Let  $\Gamma = \text{Gal}(\bar{F}/F)$  denote the absolute Galois group of  $F$ ,  $W_F$  the Weil group of  $F$ ,  $W'_F$  the Weil-Deligne group of  $F$ , and  $L_F$  the Langlands group of  $F$  given by

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times \text{SL}(2, \mathbb{C}) & \text{if } F \text{ is non-archimedean.} \end{cases}$$

Let  $\mathbf{G}$  be a connected reductive algebraic group over  $F$ . Set  $G = \mathbf{G}(F)$ . Let  $\mathbf{G}^*$  be the quasi-split inner form of  $\mathbf{G}$  and choose an inner twist  $\eta : \mathbf{G} \rightarrow \mathbf{G}^*$ . Let  $\hat{G}$  denote the dual group of  $\mathbf{G}$  and  ${}^L G = \hat{G} \rtimes W_F$  the  $L$ -group of  $\mathbf{G}$ . We fix an  $F$ -splitting  $(\mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\})$  of  $\mathbf{G}^*$  and a  $\Gamma$ -splitting  $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$  of  $\hat{G}$ .

Let  $\pi$  be a discrete series representation of  $G$  and let  $V_\pi$  be the space of  $\pi$ . We fix an invariant hermitian inner product  $(\cdot, \cdot)$  on  $V_\pi$ . Let  $d(\pi) \in \mathbb{R}_{>0}$  denote the formal degree of  $\pi$ . By definition, we have

$$\int_{G/A} (\pi(g)u, u') \overline{(\pi(g)v, v')} dg = d(\pi)^{-1} (u, v) \overline{(u', v')}$$

for  $u, u', v, v' \in V_\pi$ , where  $\mathbf{A}$  is the split component of the center of  $\mathbf{G}$  and  $A = \mathbf{A}(F)$ . We remark that  $d(\pi) = d(\pi, dg)$  depends on the choice of  $dg$ . Following Gross [11], we take a Haar measure  $\mu_{G/A, \psi}$  on  $G/A$  defined as follows. (This should not be confused with the Euler-Poincaré measure  $\mu_G$  on  $G$  in the notation of [11].) We may assume that  $\mathbf{A} = \{1\}$ . Moreover, we may assume that  $\mathbf{G}$  has an anisotropic inner form if  $F$  is archimedean. Let  $\omega_{\mathbf{G}}$  be a differential form of top degree on  $\mathbf{G}$  over  $F$  as in Sections 4 and 7 of [11]. Let  $\mu_{G, \psi}$  denote the Haar measure on  $G$  determined by  $\omega_{\mathbf{G}}$  and the self-dual measure on  $F$  with respect to  $\psi$ . Then

$$(1.1) \quad \mu_{G, \psi_a} = |a|_F^{\dim \mathbf{G}/2} \cdot \mu_{G, \psi},$$

where  $a \in F^\times$  and  $\psi_a(x) = \psi(ax)$  for  $x \in F$ . If  $F$  is non-archimedean,  $\psi$  is of order zero, and  $\mathbf{G}$  is unramified, then

$$\mu_{G, \psi}(\mathbf{G}(\mathfrak{o}_F)) = q^{-\dim \mathbf{G}} |\mathbf{G}(\mathbb{F}_q)|.$$

Here we extend  $\mathbf{G}$  to a smooth group scheme over  $\mathfrak{o}_F$  associated to a hyperspecial maximal compact subgroup of  $G$ .

**Lemma 1.1.** *Let  $\pi$  be a discrete series representation of  $G$ . Let  $a \in F^\times$ . We define a non-trivial additive character  $\psi_a$  of  $F$  by  $\psi_a(x) = \psi(ax)$  for  $x \in F$ . Then*

$$d(\pi, \mu_{G/A, \psi_a}) = |a|_F^{-n/2} \cdot d(\pi, \mu_{G/A, \psi}),$$

where  $n = \dim \mathbf{G}/\mathbf{A}$ .

*Proof.* The lemma follows from (1.1). □

Let  $\phi : L_F \rightarrow {}^L G$  be a Langlands parameter. We say that  $\phi$  is tempered if  $\phi(W_F)$  is bounded and that  $\phi$  is elliptic if  $\phi(L_F)$  is not contained in any proper parabolic subgroup of  ${}^L G$ . For each finite dimensional representation  $r$  of  ${}^L G$ , put

$$\gamma(s, r \circ \phi, \psi) = \epsilon(s, r \circ \phi, \psi) \cdot \frac{L(1-s, \check{r} \circ \phi)}{L(s, r \circ \phi)},$$

where  $\check{r}$  is the contragredient representation of  $r$ . Let  $\text{Ad}$  denote the adjoint representation of  ${}^L G$  on  $\text{Lie}(\hat{G})/\text{Lie}(Z(\hat{G})^\Gamma)$ . Note that  $\text{Ad}$  is self-dual.

**Lemma 1.2.** *Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic Langlands parameter. Then at  $s = 0$ ,  $\gamma(s, \text{Ad} \circ \phi, \psi)$  is holomorphic and non-zero.*

*Proof.* Since  $\phi$  is elliptic,  $\text{Ad} \circ \phi$  does not contain the trivial representation of  $L_F$  (cf. Lemma 10.3.1 of [22]). Hence the lemma follows from the multiplicativity of  $\gamma$ -factors. □

**Lemma 1.3.** *Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic Langlands parameter. Let  $a \in F^\times$ . We define a non-trivial additive character  $\psi_a$  of  $F$  by  $\psi_a(x) = \psi(ax)$  for  $x \in F$ . Then*

$$|\gamma(0, \text{Ad} \circ \phi, \psi_a)| = |a|_F^{-n/2} \cdot |\gamma(0, \text{Ad} \circ \phi, \psi)|,$$

where  $n = \dim \mathbf{G}/\mathbf{A}$ .

*Proof.* Note that  $n = \dim \text{Lie}(\hat{G})/\text{Lie}(Z(\hat{G})^\Gamma)$ . By definition, we have

$$|\epsilon(s, \text{Ad} \circ \phi, \psi_a)| = |a|_F^{n(s-1/2)} \cdot |\epsilon(s, \text{Ad} \circ \phi, \psi)|.$$

This yields the lemma. □

Let  $\Pi(G)$  denote the set of equivalence classes of irreducible admissible representations of  $G$ . The local Langlands conjecture asserts that there exists a partition

$$\coprod_{\phi} \Pi_{\phi}(G)$$

of  $\Pi(G)$  into finite subsets, where  $\phi$  runs over equivalence classes of Langlands parameters  $\phi : L_F \rightarrow {}^L G$ . Let  $\pi \in \Pi_{\phi}(G)$ . If  $\phi$  is tempered (resp. elliptic), then  $\pi$  is expected to be tempered (resp. essentially square integrable). For each finite dimensional representation  $r$  of  ${}^L G$ , put

$$\begin{aligned} L(s, \pi, r) &= L(s, r \circ \phi), \\ \epsilon(s, \pi, r, \psi) &= \epsilon(s, r \circ \phi, \psi), \end{aligned}$$

and

$$\gamma(s, \pi, r, \psi) = \epsilon(s, \pi, r, \psi) \cdot \frac{L(1-s, \check{\pi}, r)}{L(s, \pi, r)},$$

where  $\check{\pi}$  is the contragredient representation of  $\pi$ .

Let  $\phi : L_F \rightarrow {}^L G$  be a tempered Langlands parameter. Following [19, §1], set

$$\begin{aligned} S_\phi &= \{s \in \hat{G}_{\text{sc}} \mid \text{Int } s \circ \phi = \phi \bmod B^1(W_F, Z(\hat{G}))\}, & \mathcal{S}_\phi &= \pi_0(S_\phi), \\ S_\phi^\natural &= \{s \in \hat{G}^\natural \mid \text{Int } s \circ \phi = \phi\}, & \mathcal{S}_\phi^\natural &= \pi_0(S_\phi^\natural), \end{aligned}$$

where  $\hat{G}_{\text{sc}}$  is the simply connected cover of the derived group of  $\hat{G}$  and  $\hat{G}^\natural$  is the dual group of  $\mathbf{G}/\mathbf{A}$ . Let  $\mathcal{Z}_\phi$  be the image of  $Z(\hat{G}_{\text{sc}})$  in  $\mathcal{S}_\phi$ . Let  $\chi_{\mathbf{G}}$  be the character of  $Z(\hat{G}_{\text{sc}})^\Gamma$  associated to  $\mathbf{G}$  by the map

$$H^1(F, \mathbf{G}_{\text{ad}}^*) \longrightarrow \pi_0(Z(\hat{G}_{\text{sc}})^\Gamma)^D$$

defined by Kottwitz [22], [23]. Here  $\mathbf{G}_{\text{ad}}^*$  is the adjoint group of  $\mathbf{G}^*$ . By Lemma 9.1 of [19], we can regard  $\chi_{\mathbf{G}}$  as a character of the image of  $Z(\hat{G}_{\text{sc}})^\Gamma$  in  $\mathcal{S}_\phi$ . We extend  $\chi_{\mathbf{G}}$  to a character of  $\mathcal{Z}_\phi$ . Let  $\Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$  denote the set of equivalence classes of irreducible representations of  $\mathcal{S}_\phi$  such that  $\mathcal{Z}_\phi$  acts via  $\chi_{\mathbf{G}}$ . It is believed that there exists a map

$$\Pi_\phi(G) \longrightarrow \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$$

which satisfies certain conditions on characters (cf. [2]). For example,

$$\sum_{\pi \in \Pi_\phi(G)} \langle 1, \pi \rangle \text{trace } \pi$$

is required to be the unique (up to a scalar) stable distribution in the space of virtual characters generated by  $\Pi_\phi(G)$ , where

$$\langle 1, \pi \rangle = \dim \rho_\pi$$

if  $\rho_\pi \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$  is associated to  $\pi \in \Pi_\phi(G)$ . Moreover, the quantity  $\langle 1, \pi \rangle$  is expected to be canonically determined by  $\pi$ .

**Conjecture 1.4.** *Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic tempered Langlands parameter. Then*

$$d(\pi) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_\phi^\natural|} \cdot |\gamma(0, \pi, \text{Ad}, \psi)|$$

for  $\pi \in \Pi_\phi(G)$ .

We will relate Conjecture 1.4 to the Plancherel formula. We fix a non-trivial additive character  $\psi$  of  $F$ . Let  $\Theta$  be the set of pairs  $(\mathfrak{D}, \mathbf{P} = \mathbf{M}\mathbf{N})$ , where  $\mathbf{P}$  is a semi-standard parabolic subgroup of  $\mathbf{G}$ ,  $\mathbf{M}$  is the Levi subgroup of  $\mathbf{P}$ ,  $\mathbf{N}$  is the unipotent radical of  $\mathbf{P}$ , and  $\mathfrak{D}$  is an orbit in the set of equivalence classes of discrete series representations of  $M$  under the action of the group of unramified unitary characters of  $M$ . For  $(\mathfrak{D}, \mathbf{P} = \mathbf{M}\mathbf{N}) \in \Theta$  and  $\pi \in \mathfrak{D}$ , put

$$d\nu(\pi) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_{\phi_M}^\natural|} \cdot |\gamma(0, \pi, r_M, \psi)| \cdot d\pi.$$

Here  $\phi_M : L_F \rightarrow {}^L M$  is the (conjectural) Langlands parameter associated to  $\pi$ ,  $r_M$  is the adjoint representation of  ${}^L M$  on  $\text{Lie}(\hat{G})/\text{Lie}(Z(\hat{M})^\Gamma)$ , and  $d\pi$  is the Lebesgue measure on  $\mathfrak{D}$  (cf. [36, pp. 239 and 302]). Then the Plancherel formula (cf. Theorem 27.3 of [14] and Théorème VIII.1.1 of [36]), Langlands' conjecture on Plancherel measures (cf. Appendix II of [26]), and Conjecture 1.4 suggest that the following conjecture holds.

**Conjecture 1.5.** *There exist explicit constants  $c_M \in \mathbb{R}_{>0}$  which do not depend on  $\mathfrak{D}$  such that*

$$f(1) = \sum_{(\mathfrak{D}, \mathbf{P}=\mathbf{MN}) \in \Theta} c_M \int_{\mathfrak{D}} \text{trace Ind}_{\mathbf{P}}^G(\pi)(f) d\nu(\pi)$$

for  $f \in C_c^\infty(G)$ .

2. EXAMPLES: THE ARCHIMEDEAN CASE

In this section, we verify Conjecture 1.4 in the archimedean case.

Let  $F = \mathbb{R}$ . By Lemmas 1.1 and 1.3, we may assume that  $\psi(x) = \exp(2\pi\sqrt{-1}x)$  for  $x \in \mathbb{R}$ . Let  $\mathbf{G}$  be a connected reductive algebraic group of rank  $l$  over  $\mathbb{R}$ . For simplicity, we assume that the connected center of  $\mathbf{G}$  is anisotropic. We may assume that  $\mathbf{G}$  has an anisotropic inner form  $\mathbf{G}_{\text{an}}$ .

**Proposition 2.1.** *Let  $\pi$  be a discrete series representation of  $G$ . Then*

$$d(\pi) = \frac{1}{2^l} \cdot |\gamma(0, \pi, \text{Ad}, \psi)|.$$

In particular, Conjecture 1.4 holds for  $\pi$ .

The rest of this section is devoted to the proof of Proposition 2.1. Let  $\check{\Sigma}$  denote the set of roots of  $\mathcal{T}$  in  $\hat{G}$  and  $\check{\Sigma}^+$  the subset of positive roots determined by  $\mathcal{B}$ . Let  $N$  be the number of positive roots. Let  $\langle \cdot, \cdot \rangle$  denote the pairing between  $X_*(\mathcal{T}) \otimes \mathbb{Q}$  and  $X^*(\mathcal{T}) \otimes \mathbb{Q}$ .

**Lemma 2.2.** *Let  $\pi_\lambda$  be a discrete series representation of  $G$  with Harish-Chandra parameter  $\lambda$ . Then*

$$|\gamma(0, \pi_\lambda, \text{Ad}, \psi)| = \pi^{-l} \cdot (2\pi)^{-N} \prod_{\check{\alpha} \in \check{\Sigma}^+} |\langle \lambda, \check{\alpha} \rangle|.$$

*Proof.* Let  $\phi : W_{\mathbb{R}} \rightarrow {}^L G$  be the Langlands parameter associated to  $\pi_\lambda$ . Then  $\phi(z) = z^\lambda \bar{z}^{-\lambda}$  for  $z \in W_{\mathbb{C}}$ . The action  $\text{Ad} \circ \phi$  of  $W_{\mathbb{R}}$  on  $\text{Lie}(\mathcal{T})$  (resp.  $\mathbb{C}\mathcal{X}_{\check{\alpha}} \oplus \mathbb{C}\mathcal{X}_{-\check{\alpha}}$ ) is given by the sign character (resp.  $\text{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(\phi_{\check{\alpha}})$ ). Here  $\check{\alpha} \in \check{\Sigma}^+$  and  $\phi_{\check{\alpha}}(z) = z^{\langle \lambda, \check{\alpha} \rangle} \bar{z}^{-\langle \lambda, \check{\alpha} \rangle}$  for  $z \in W_{\mathbb{C}}$ . Hence we have

$$L(s, \pi_\lambda, \text{Ad}) = \Gamma_{\mathbb{R}}(s+1)^l \prod_{\check{\alpha} \in \check{\Sigma}^+} \Gamma_{\mathbb{C}}(s + |\langle \lambda, \check{\alpha} \rangle|),$$

where  $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$  and  $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$ . By definition,  $\epsilon(s, \pi_\lambda, \text{Ad}, \psi)$  is a power of  $\sqrt{-1}$ . This yields the lemma.  $\square$

Set  $\mathfrak{g} = \text{Lie}(G)$ . Let  $\theta$  be a Cartan involution of  $\mathfrak{g}$  and let  $B$  be a symmetric bilinear form on  $\mathfrak{g}$  over  $\mathbb{R}$  which satisfies the conditions of Lemma 3.2 of [13]. Then the quadratic form

$$\|X\|^2 = -B(X, \theta(X))$$

for  $X \in \mathfrak{g}$  is positive definite. This norm  $\|\cdot\|$  on  $\mathfrak{g}$  defines a Lebesgue measure on  $\mathfrak{g}$  and hence a Haar measure  $dG$  on  $G$  via the exponential map. Let  $\mathbf{T}$  be an anisotropic maximal torus of  $\mathbf{G}$  such that  $\text{Lie}(\mathcal{T})$  is  $\theta$ -invariant. Similarly, we can define a Haar measure  $dT$  on  $T$ .

Set  $\mathfrak{g}_{\mathbb{C}} = \text{Lie}(G) \otimes \mathbb{C}$  and  $\mathfrak{t}_{\mathbb{C}} = \text{Lie}(T) \otimes \mathbb{C}$ . Let  $\Sigma$  denote the set of roots of  $\mathfrak{t}_{\mathbb{C}}$  in  $\mathfrak{g}_{\mathbb{C}}$  and  $\Sigma^+$  the subset of positive roots. We extend  $B$  to a symmetric bilinear form on  $\mathfrak{g}_{\mathbb{C}}$  over  $\mathbb{C}$ . For  $\alpha \in \Sigma$ , we define  $H_{\alpha} \in \mathfrak{t}_{\mathbb{C}}$  by

$$B(H, H_{\alpha}) = \alpha(H)$$

for  $H \in \mathfrak{t}_{\mathbb{C}}$ . Put

$$\varpi = \prod_{\alpha \in \Sigma^+} H_{\alpha}.$$

**Lemma 2.3.** *Let  $\pi_{\lambda}$  be a discrete series representation of  $G$  with Harish-Chandra parameter  $\lambda$ . Then*

$$d(\pi_{\lambda}, dG) = (2\pi)^{-N} \cdot |\varpi(\lambda)| \cdot \text{vol}(T, dT)^{-1}.$$

*Proof.* Let  $K$  be a maximal compact subgroup of  $G$  such that  $\text{Lie}(K)$  is  $\theta$ -invariant and let  $dK$  be the Haar measure on  $K$  determined by  $\|\cdot\|$ . Let  $dx$  be the standard measure on  $G$  as in [13, §7]. By Lemma 37.2 of [13], we have

$$dx = 2^{\nu/2} \cdot \text{vol}(K, dK)^{-1} \cdot dG,$$

where  $\nu = \dim G/K - \text{rank } G/K$ . By Corollary of Lemma 23.1 of [14], we have

$$d(\pi_{\lambda}, dx) = c_G^{-1} \cdot |W| \cdot |\varpi(\lambda)|.$$

Here  $W$  is the Weyl group of  $T$  in  $G$  and

$$c_G = 2^{\nu/2} \cdot (2\pi)^N \cdot |W| \cdot \frac{\text{vol}(T, dT)}{\text{vol}(K, dK)}$$

(cf. Lemma 37.3 of [13]). This completes the proof. □

**Lemma 2.4.** *Let  $\pi$  be a discrete series representation of  $G$  and let  $\phi : W_{\mathbb{R}} \rightarrow {}^L G$  be the Langlands parameter associated to  $\pi$ . Let  $\pi_{\text{an}}$  be the irreducible finite dimensional representation of  $G_{\text{an}}$  associated to  $\phi$  by the local Langlands correspondence. Then*

$$d(\pi) = d(\pi_{\text{an}}).$$

*Proof.* We extend  $\theta$  to an anti-linear involution of  $\mathfrak{g}_{\mathbb{C}}$  over  $\mathbb{C}$ . Set  $\mathfrak{g}_{\text{an}} = \text{Lie}(G_{\text{an}})$ . We may identify  $\mathfrak{g}_{\text{an}}$  with  $\mathfrak{g}_{\mathbb{C}}^{\theta}$ . Then the restrictions of  $\theta$  and  $B$  to  $\mathfrak{g}_{\text{an}}$  define a norm  $\|\cdot\|_{\text{an}}$  on  $\mathfrak{g}_{\text{an}}$ . Let  $dG_{\text{an}}$  be the Haar measure on  $G_{\text{an}}$  determined by  $\|\cdot\|_{\text{an}}$ . Then  $dG$  and  $dG_{\text{an}}$  are compatible. By Lemma 2.3, we have

$$d(\pi, dG) = d(\pi_{\text{an}}, dG_{\text{an}}).$$

By definition,  $\mu_{G, \psi}$  and  $\mu_{G_{\text{an}}, \psi}$  are also compatible. This yields the lemma. □

By Lemma 2.4, to prove Proposition 2.1, we may assume that  $\mathbf{G}$  is anisotropic. Let  $\pi$  be an irreducible finite dimensional representation of  $G$ . By Lemmas 2.2 and 2.3, there exists a constant  $c \in \mathbb{R}_{>0}$  which does not depend on  $\pi$  such that

$$d(\pi) = c|\gamma(0, \pi, \text{Ad}, \psi)|.$$

By [27], [11, §7], we have

$$\text{vol}(G) = 2^N \prod_{i=1}^l \frac{2\pi^{m_i+1}}{m_i!} = (2\pi)^{l+N} \prod_{\check{\alpha} \in \check{\Sigma}^+} \langle \rho, \check{\alpha} \rangle^{-1}.$$

Here  $m_1, \dots, m_l$  are the exponents of  $\mathbf{G}$  and  $\rho$  is half the sum of positive roots. Note that

$$\sum_{i=1}^l m_i = N,$$

$$\prod_{i=1}^l m_i! = \prod_{\check{\alpha} \in \check{\Sigma}^+} \langle \rho, \check{\alpha} \rangle.$$

By Lemma 2.2, we have  $\text{vol}(G) = 2^l |\gamma(0, \pi_\rho, \text{Ad}, \psi)|^{-1}$ , where  $\pi_\rho$  is the trivial representation of  $G$ . Hence we have  $c = 2^{-l}$ . This completes the proof of Proposition 2.1.

### 3. EXAMPLES: THE NON-ARCHIMEDEAN CASE

Let  $F$  be a non-archimedean local field of characteristic zero. By Lemmas 1.1 and 1.3, we may assume that  $\psi$  is of order zero.

**3.1. Inner forms of  $\text{GL}(n)$ .** We first recall the following result of Silberger and Zink [35], [37].

**Theorem 3.1.** *Let  $\pi$  be a discrete series representation of  $\text{GL}(n, F)$ . Then*

$$d(\pi) = \frac{1}{n} \cdot |\gamma(0, \pi, \text{Ad}, \psi)|.$$

*In particular, Conjecture 1.4 holds for  $\pi$ .*

To be precise, let  $\pi$  be the unique irreducible subrepresentation of an induced representation

$$\sigma | \det|_F^{(e-1)/2} \times \sigma | \det|_F^{(e-3)/2} \times \dots \times \sigma | \det|_F^{-(e-1)/2},$$

where  $\sigma$  is an irreducible unitary supercuspidal representation of  $\text{GL}(m, F)$  with  $n = em$ . Using the theory of types, Silberger and Zink showed that  $d(\pi)$  is equal to

$$r \cdot \frac{q^{em} - 1}{q^{er} - 1} \cdot q^{e(r-m)/2 + e^2(f+r-m^2)/2} \cdot \frac{1}{n} \prod_{i=1}^{n-1} (q^i - 1) \cdot \text{vol}(\text{GL}(n, \mathfrak{o}_F) / \mathfrak{o}_F^\times)^{-1}$$

(cf. Theorems 6.5 and 6.9 of [3]). Here  $r$  is the torsion number of  $\sigma$  and  $f$  is the conductor of  $\sigma \times \check{\sigma}$ . It is easy to check that this quantity coincides with  $n^{-1} |\gamma(0, \pi, \text{Ad}, \psi)|$ . In §4, we will give a new proof of Theorem 3.1 which does not rely on the theory of types.

Let  $\mathbf{G}$  be an inner form of  $\text{GL}(n)$  over  $F$ . Then  $G = \text{GL}(n', D)$  with  $n = dn'$ , where  $D$  is a division algebra of dimension  $d^2$  over  $F$ . Let  $\pi$  be a discrete series representation of  $G$ . By Theorem 7.2 of [3], we have

$$d(\pi) = \prod_{\substack{1 \leq i \leq n \\ i \not\equiv 0 \pmod{d}}} (q^i - 1)^{-1} \cdot d(\pi^*) \cdot \frac{\text{vol}(\text{GL}(n, \mathfrak{o}_F) / \mathfrak{o}_F^\times)}{\text{vol}(\text{GL}(n', \mathfrak{o}_D) / \mathfrak{o}_F^\times)},$$

where  $\pi^*$  is the discrete series representation of  $\text{GL}(n, F)$  associated to  $\pi$  by the Deligne-Kazhdan-Vignéras correspondence [9]. Since

$$\text{vol}(\text{GL}(n', \mathfrak{o}_D)) = q^{-(d-1)dn'^2/2} \prod_{i=1}^{n'} (1 - q^{-di}),$$



we obtain

$$d(\pi) = d(\pi^*).$$

**3.2. Inner forms of  $SL(n)$ .** Let  $\tilde{\mathbf{G}}$  be an inner form of  $GL(n)$  over  $F$  and let  $\mathbf{G}$  be the derived group of  $\tilde{\mathbf{G}}$ . Then  $\mathbf{G}$  is an inner form of  $SL(n)$  over  $F$ . Let  $\mathbf{G}_{\text{ad}}$  be the adjoint group of  $\mathbf{G}$ . Set  $C = \text{cok}(G \rightarrow G_{\text{ad}})$ .

Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic Langlands parameter. Then there exists an elliptic tempered Langlands parameter  $\tilde{\phi} : L_F \rightarrow {}^L \tilde{G}$  such that  $\phi = \text{pr} \circ \tilde{\phi}$ , where  $\text{pr} : {}^L \tilde{G} \rightarrow {}^L G$  is the projection. Let  $\tilde{\pi}$  be the discrete series representation of  $\tilde{G}$  associated to  $\tilde{\phi}$  by the local Langlands correspondence [16], [17] and let  $V_{\tilde{\pi}}$  be the space of  $\tilde{\pi}$ . Let  $\Pi_{\phi}(G)$  denote the set of equivalence classes of irreducible constituents of the restriction of  $\tilde{\pi}$  to  $G$ . Note that  $\Pi_{\phi}(G)$  does not depend on the choice of  $\tilde{\phi}$ . Put

$$X(\tilde{\pi}) = \{\omega \in C^D \mid \tilde{\pi} \otimes \omega \simeq \tilde{\pi}\},$$

where  $C^D$  is the Pontrjagin dual of  $C$  and  $\omega$  is regarded as a character of  $G_{\text{ad}}$ . For  $s \in S_{\phi}$ , we have

$$\text{Int } s \circ \tilde{\phi} = a_s \cdot \tilde{\phi},$$

where  $a_s$  is a 1-cocycle of  $W_F$  in  $Z(\hat{G}_{\text{sc}})$ . Let  $\omega_s$  be the character of  $C$  determined by  $a_s$ . Then the map  $s \mapsto \omega_s$  induces an exact sequence

$$1 \longrightarrow \mathcal{Z}_{\phi} \longrightarrow \mathcal{S}_{\phi} \longrightarrow X(\tilde{\pi}) \longrightarrow 1.$$

By Theorem 1.4 of [19], there exists an action of  $\mathcal{S}_{\phi}$  on  $V_{\tilde{\pi}}$  such that  $\mathcal{Z}_{\phi}$  acts via  $\chi_{\mathbf{G}}$  and such that

$$\tilde{\pi} \circ s = s \circ (\tilde{\pi} \otimes \omega_s)$$

for  $s \in \mathcal{S}_{\phi}$ . Moreover, if we write a decomposition of  $V_{\tilde{\pi}}$  as a representation of  $\mathcal{S}_{\phi} \times G$  in the form

$$\bigoplus_{\rho \in \Pi(\mathcal{S}_{\phi}, \chi_{\mathbf{G}})} \rho \otimes \pi_{\rho},$$

then the map  $\rho \mapsto \pi_{\rho}$  defines a bijection between  $\Pi(\mathcal{S}_{\phi}, \chi_{\mathbf{G}})$  and  $\Pi_{\phi}(G)$  (cf. Theorem 1.1 of [19]).

**Lemma 3.2.** *For  $\rho \in \Pi(\mathcal{S}_{\phi}, \chi_{\mathbf{G}})$ , we have*

$$d(\pi_{\rho}) = n^2 \cdot \frac{\dim \rho}{|\mathcal{S}_{\phi}|} \cdot d(\tilde{\pi}).$$

*Proof.* We fix an invariant hermitian inner product  $(\cdot, \cdot)$  on  $V_{\tilde{\pi}}$ . Then  $(\cdot, \cdot)$  is  $\mathcal{S}_{\phi}$ -invariant. Let  $v$  be an element in the  $\pi_{\rho}$ -isotypic subspace of  $V_{\tilde{\pi}}$ . Recall that the sequence

$$1 \longrightarrow G/\mu_n(F) \longrightarrow G_{\text{ad}} \longrightarrow C \longrightarrow 1$$

is exact. Here  $\mu_n$  is the group of  $n$ -th roots of unity. Since the pullback of  $\omega_{\mathbf{G}_{\text{ad}}}$  to  $\mathbf{G}$  is  $n\omega_{\mathbf{G}}$  and  $|C|^{-1} \sum_{\omega \in C^D} \omega$  is the characteristic function of  $G/\mu_n(F)$ , we have

$$d(\pi_{\rho})^{-1}(v, v)\overline{(v, v)} = \frac{|\mu_n(F)|}{|n|_F \cdot |C|} \sum_{\omega \in C^D} \int_{G_{\text{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)\overline{(\tilde{\pi}(g)v, v)} dg.$$

By the Schur orthogonality relations, we have

$$\int_{G_{\text{ad}}} ((\tilde{\pi} \otimes \omega)(g)v, v)\overline{(\tilde{\pi}(g)v, v)} dg = 0$$

unless  $\omega \in X(\tilde{\pi})$ . Moreover, we have

$$\begin{aligned} \int_{G_{\text{ad}}} ((\tilde{\pi} \otimes \omega_s)(g)v, v) \overline{(\tilde{\pi}(g)v, v)} dg &= \int_{G_{\text{ad}}} (\tilde{\pi}(g)sv, sv) \overline{(\tilde{\pi}(g)v, v)} dg \\ &= d(\tilde{\pi})^{-1}(sv, v) \overline{(sv, v)} \end{aligned}$$

for  $s \in \mathcal{S}_\phi$ . Thus we obtain

$$\begin{aligned} d(\pi_\rho)^{-1}(v, v) \overline{(v, v)} &= \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \sum_{s \in \mathcal{S}_\phi} d(\tilde{\pi})^{-1}(sv, v) \overline{(sv, v)} \\ &= \frac{|\mu_n(F)|}{|n|_F \cdot |C| \cdot n} \cdot \frac{|\mathcal{S}_\phi|}{\dim \rho} \cdot d(\tilde{\pi})^{-1}(v, v) \overline{(v, v)}. \end{aligned}$$

Note that

$$|n|_F = \frac{|\mathbf{H}^0(F, \mu_n)| \cdot |\mathbf{H}^2(F, \mu_n)|}{|\mathbf{H}^1(F, \mu_n)|} = \frac{|\mu_n(F)| \cdot n}{|C|}.$$

This yields the lemma. □

By Theorem 3.1 and Lemma 3.2, we have

$$d(\pi_\rho) = n \cdot \frac{\dim \rho}{|\mathcal{S}_\phi|} \cdot |\gamma(0, \pi_\rho, \text{Ad}, \psi)|$$

for  $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ .

**3.3. Steinberg representations.** Let  $\mathbf{G}$  be a connected reductive algebraic group over  $F$ . For simplicity, we assume that the connected center of  $\mathbf{G}$  is anisotropic. Let  $\pi_0$  be the Steinberg representation of  $G$ . Note that the formal degree of  $\pi_0$  was computed by Borel [4]. Using the results of Kottwitz [24] and Gross [11], [12], we will verify Conjecture 1.4 for  $\pi_0$ . In particular, if  $\mathbf{G}$  is an anisotropic torus, then Conjecture 1.4 holds.

Let  $\mu_{G, \text{EP}}$  denote the Euler-Poincaré measure on  $G$  and let  $f_{\text{EP}} \in C_c^\infty(G)$  denote the Euler-Poincaré function with respect to  $\mu_{G, \text{EP}}$ . By Theorems 2 and 2' of [24] and the Plancherel formula (cf. Théorème VIII.1.1 of [36]), we have  $f_{\text{EP}}(1) = 1$  and

$$|d(\pi_0, \mu_{G, \text{EP}})| = 1.$$

By Theorem 5.5 of [11], we have

$$e(\mathbf{G}) \cdot |\mathbf{H}^1(F, \mathbf{G})| \cdot L(M) \cdot \mu_{G, \text{EP}} = L(M^\vee(1)) \cdot \mu_{G, \psi}.$$

Here  $e(\mathbf{G}) = \pm 1$  is the Kottwitz sign,  $M$  is the motive of  $\mathbf{G}$  as in [11], and  $M^\vee(1) = M^\vee \otimes \mathbb{Q}(1)$  is the Tate twist of the dual motive of  $M$ . Hence we have

$$d(\pi_0) = |\mathbf{H}^1(F, \mathbf{G})|^{-1} \cdot \frac{|L(M^\vee(1))|}{|L(M)|}.$$

Let  $\phi_0 : L_F \rightarrow {}^L G$  be the Langlands parameter associated to  $\pi_0$ . Then  $\phi_0$  is trivial on  $W_F$  and the restriction of  $\phi_0$  to  $\text{SL}(2, \mathbb{C})$  corresponds to the regular unipotent orbit in  $\hat{G}$ . Hence the centralizer of  $\phi_0(L_F)$  in  $\hat{G}$  is  $Z(\hat{G})^\Gamma$  and  $|\mathcal{S}_{\phi_0}^\natural| = |\mathbf{H}^1(F, \mathbf{G})|$ .

**Lemma 3.3.**

$$|\gamma(0, \pi_0, \text{Ad}, \psi)| = \frac{|L(M^\vee(1))|}{|L(M)|}.$$

*Proof.* By Corollary 6.5 of [12], we have  $L(M^\vee(1)) = L(1, \pi_0, \text{Ad})$ . Thus it suffices to show that

$$|L(M)|^{-1} = |\epsilon(0, \pi_0, \text{Ad}, \psi)| \cdot |L(0, \pi_0, \text{Ad})|^{-1}.$$

Recall that  $M = \bigoplus_{d \geq 1} V_d(1-d)$ , where  $V_d$  is the Artin motive as in [11, §1]. By Proposition 6.4 of [12], we have

$$L(s, \pi_0, \text{Ad})^{-1} = \prod_{d \geq 1} \det(1 - q^{-s-d+1} \cdot \text{Frob}; V_d^{I_F}),$$

where  $I_F$  is the inertia group of  $F$ . Since  $V_d$  is self-dual, we have

$$\begin{aligned} |L(M)|^{-1} &= \prod_{d \geq 1} |\det(1 - q^{d-1} \cdot \text{Frob}; V_d^{I_F})| \\ &= \prod_{d \geq 1} q^{(d-1) \dim V_d^{I_F}} \cdot |L(0, \pi_0, \text{Ad})|^{-1}. \end{aligned}$$

Set  $\hat{\mathfrak{g}} = \text{Lie}(\hat{G})$ . Let  $(\rho, N)$  be the representation of  $W'_F$  on  $\hat{\mathfrak{g}}$  associated to  $\text{Ad} \circ \phi_0$ . We can regard  $N$  as a regular nilpotent element in  $\hat{\mathfrak{g}}$ . By definition, we have

$$|\epsilon(s, \pi_0, \text{Ad}, \psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)},$$

where  $a(\hat{\mathfrak{g}}) = \dim \hat{\mathfrak{g}}^{I_F} - \dim \hat{\mathfrak{g}}_N^{I_F}$ . By Proposition 5.2 of [12], we have  $\hat{\mathfrak{g}} = \bigoplus_{d \geq 1} V_d \otimes \rho_{2d-2}$  as a representation of  $\Gamma \times \text{SL}(2, \mathbb{C})$ . Here  $\rho_k$  is the irreducible representation of  $\text{SL}(2, \mathbb{C})$  of dimension  $k+1$ . Hence we have

$$a(\hat{\mathfrak{g}}) = 2 \sum_{d \geq 1} (d-1) \dim V_d^{I_F}.$$

This completes the proof. □

Thus we obtain

$$d(\pi_0) = \frac{1}{|\mathcal{S}_{\phi_0}^{\natural}|} \cdot |\gamma(0, \pi_0, \text{Ad}, \psi)|.$$

**3.4. Unipotent discrete series representations.** Let  $\mathbf{G}$  be a connected adjoint split exceptional group of rank  $l$  over  $F$ . Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic Langlands parameter. We assume that  $\phi$  is trivial on the inertia group  $I_F$  of  $F$ . Put

$$t = \phi \left( \text{Frob} \times \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \right).$$

We may assume that  $t \in \mathcal{T}$ . Let  $\check{\Sigma}$  denote the set of roots of  $\mathcal{T}$  in  $\hat{G}$ . For  $i \in \mathbb{Z}$ , put

$$\check{\Sigma}(i) = \{ \check{\alpha} \in \check{\Sigma} \mid \check{\alpha}(t) = q^{-i/2} \}.$$

For each  $\rho \in \Pi(\mathcal{S}_{\phi}, \chi_{\mathbf{G}})$ , Reeder defined a discrete series representation  $\pi_{\rho}$  of  $G$  and showed that

$$d(\pi_{\rho}) = \frac{q^N \dim \rho}{|\mathcal{S}_{\phi}^{\natural}|} \cdot \frac{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(0)} (\check{\alpha}(t) - 1)}{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(2)} (q\check{\alpha}(t) - 1)} \cdot \text{vol}(I)^{-1}$$

(cf. [29, (0.3)]). Here  $N$  is the number of positive roots and  $I$  is an Iwahori subgroup of  $G$ . Note that  $\text{vol}(I) = q^{-N}(1 - q^{-1})^l$ .

**Lemma 3.4.**

$$|\gamma(0, \pi_\rho, \text{Ad}, \psi)| = q^{2N} (1 - q^{-1})^{-l} \cdot \frac{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(0)} |\check{\alpha}(t) - 1|}{\prod_{\check{\alpha} \in \check{\Sigma} - \check{\Sigma}(2)} |q\check{\alpha}(t) - 1|}.$$

*Proof.* It is easy to check that

$$\gamma(s, \pi_\rho, \text{Ad}, \psi) = \left( \frac{1 - q^{-s}}{1 - q^{-1+s}} \right)^l \prod_{\check{\alpha} \in \check{\Sigma}} \frac{1 - \check{\alpha}(t)^{-1} q^{-s}}{1 - \check{\alpha}(t)^{-1} q^{-1+s}}.$$

By [29, (7.2a)], we have  $|\check{\Sigma}(2)| = |\check{\Sigma}(0)| + l$ . This yields the lemma. □

Thus we obtain

$$d(\pi_\rho) = \frac{\dim \rho}{|\mathcal{S}_\phi^\natural|} \cdot |\gamma(0, \pi_\rho, \text{Ad}, \psi)|$$

for  $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ .

**3.5. Depth-zero supercuspidal representations.** Let  $\mathbf{G}$  be a connected reductive algebraic group of rank  $l$  over  $F$ . We assume that  $\mathbf{G}^*$  is unramified and  $\mathbf{G}$  is a pure inner form of  $\mathbf{G}^*$ . For simplicity, we assume that the connected center of  $\mathbf{G}$  is anisotropic. Let  $\phi : L_F \rightarrow {}^L G$  be an elliptic Langlands parameter. We assume that  $\phi$  is trivial on  $I_F^+$  and that the centralizer of  $\phi(I_F)$  in  $\hat{G}$  is  $\mathcal{T}$ . Here  $I_F$  is the inertia group of  $F$  and  $I_F^+$  is the wild inertia subgroup of  $I_F$ . Note that  $\phi$  is trivial on  $\text{SL}(2, \mathbb{C})$ . Put  $\sigma = \phi(\text{Frob})$ . Then  $\sigma$  normalizes  $\mathcal{T}$  and  $\mathcal{S}_\phi^\natural$  is isomorphic to  $\mathcal{T}^\sigma$ , where  $\mathcal{T}^\sigma$  is the centralizer of  $\sigma$  in  $\mathcal{T}$ .

Let  $\mathbf{T}$  be an unramified maximal torus of  $\mathbf{G}$  determined by  $\sigma$ . Then  $\mathbf{T}$  is anisotropic. Let  $\mathbf{T}_0$  be an unramified maximal torus of  $\mathbf{G}$  which is maximally split. We extend  $\mathbf{T}$  and  $\mathbf{T}_0$  to smooth group schemes over  $\mathfrak{o}_F$ . For each  $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ , DeBacker and Reeder defined an irreducible supercuspidal representation  $\pi_\rho$  of  $G$  and showed that

$$d(\pi_\rho) = q^{l/2} |\mathbf{T}(\mathbb{F}_q)|^{-1} \cdot q^{-l/2} |\mathbf{T}_0(\mathbb{F}_q)| \cdot \text{vol}(I)^{-1}$$

(cf. [8, §5.3]). Here  $I$  is an Iwahori subgroup of  $G$ . By [11, (4.11)], we have  $\text{vol}(I) = q^{-l-N} |\mathbf{T}_0(\mathbb{F}_q)|$ , where  $N = (\dim \mathbf{G} - l)/2$ . Hence we have

$$d(\pi_\rho) = q^{l+N} |\mathbf{T}(\mathbb{F}_q)|^{-1}.$$

**Lemma 3.5.**

$$|\gamma(0, \pi_\rho, \text{Ad}, \psi)| = q^{l+N} |\mathbf{T}(\mathbb{F}_q)|^{-1} \cdot |\mathcal{T}^\sigma|.$$

*Proof.* Set  $\hat{\mathfrak{g}} = \text{Lie}(\hat{G})$ . Then  $\hat{\mathfrak{g}}^{I_F} = \text{Lie}(\mathcal{T})$ . Here the action of  $W'_F$  on  $\hat{\mathfrak{g}}$  is associated to  $\text{Ad} \circ \phi$ . By definition, we have

$$L(s, \pi_\rho, \text{Ad}) = \det(1 - q^{-s} \cdot \sigma; \text{Lie}(\mathcal{T}))^{-1}.$$

Hence we have

$$L(1, \pi_\rho, \text{Ad}) = q^l |\mathbf{T}(\mathbb{F}_q)|^{-1}.$$

Since  $\mathcal{T}^\sigma$  is isomorphic to  $\{x \in X_*(\mathcal{T}) \otimes \mathbb{C} \mid (1 - \sigma)x \in X_*(\mathcal{T})\} / X_*(\mathcal{T})$ , we have

$$|L(0, \pi_\rho, \text{Ad})|^{-1} = |\det(1 - \sigma; \text{Lie}(\mathcal{T}))| = |\mathcal{T}^\sigma|.$$

By definition, we have

$$|\epsilon(s, \pi_\rho, \text{Ad}, \psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)},$$

where  $a(\hat{\mathfrak{g}}) = \dim \hat{\mathfrak{g}} / \hat{\mathfrak{g}}^{I_F} = 2N$ . This completes the proof. □

Thus we obtain

$$d(\pi_\rho) = \frac{1}{|\mathcal{S}_\phi^\sharp|} \cdot |\gamma(0, \pi_\rho, \text{Ad}, \psi)|$$

for  $\rho \in \Pi(\mathcal{S}_\phi, \chi_{\mathbf{G}})$ .

#### 4. PROOF OF THEOREM 3.1

In this section, we give a new proof of the result of Silberger and Zink [35], [37].

Let  $\pi$  be a discrete series representation of  $\text{GL}(n, F)$  and let  $V_\pi$  be the space of  $\pi$ . We fix an invariant hermitian inner product  $(\cdot, \cdot)$  on  $V_\pi$  and equip  $V_\pi \otimes V_\pi$  with the invariant hermitian inner product such that  $(u \otimes v, u' \otimes v') = (u, u')(v, v')$ .

Let  $G^\sharp = \text{GL}(2n, F)$ . Let  $P^\sharp = M^\sharp N^\sharp$  be a parabolic subgroup of  $G^\sharp$  given by

$$M^\sharp = \left\{ \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} \mid a, a' \in \text{GL}(n, F) \right\},$$

$$N^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \mid x \in \text{Mat}_{n \times n}(F) \right\}.$$

We consider an induced representation

$$I(s, \pi \otimes \pi) = \text{Ind}_{P^\sharp}^{G^\sharp} (\pi | \det|_F^{s/2} \otimes \pi | \det|_F^{-s/2})$$

for  $s \in \mathbb{C}$ . Put

$$w = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \in G^\sharp.$$

For  $\phi \in I(s, \pi \otimes \pi)$  and  $g \in G^\sharp$ , the integral

$$M(s, w, \pi \otimes \pi)\phi(g) = \int_{\text{Mat}_{n \times n}(F)} \phi \left( w^{-1} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} g \right) dx$$

is absolutely convergent for  $\text{Re}(s) > 0$ , has a meromorphic continuation to the whole  $s$ -plane, and defines an intertwining operator

$$M(s, w, \pi \otimes \pi) : I(s, \pi \otimes \pi) \longrightarrow I(-s, w(\pi \otimes \pi)).$$

Here  $dx$  is the Haar measure on  $\text{Mat}_{n \times n}(F)$  with  $\text{vol}(\text{Mat}_{n \times n}(\mathfrak{o}_F), dx) = 1$ .

**Lemma 4.1.** *There exists a constant  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$  such that*

$$\begin{aligned} & (\text{Res}_{s=0} M(s, w, \pi \otimes \pi)\phi(1), u' \otimes v') \\ &= \alpha (\log q)^{-1} (1 - q^{-1}) \gamma(0, \pi, \text{Ad}, \psi)^{-1} (\phi(1), v' \otimes u') \end{aligned}$$

for  $\phi \in I(s, \pi \otimes \pi)$  and  $u', v' \in V_\pi$ . Here  $\psi$  is a non-trivial additive character of  $F$  of order zero.

*Proof.* Set  $I(\pi \otimes \pi) = I(0, \pi \otimes \pi)$ . Let  $\text{sw} : V_\pi \otimes V_\pi \rightarrow V_\pi \otimes V_\pi$  be an isomorphism given by  $\text{sw}(u \otimes v) = v \otimes u$ . Then  $\text{sw}$  induces an isomorphism  $\text{sw} : I(w(\pi \otimes \pi)) \rightarrow I(\pi \otimes \pi)$ . We define a normalized intertwining operator

$$N(w, \pi \otimes \pi) : I(\pi \otimes \pi) \longrightarrow I(\pi \otimes \pi)$$

by

$$N(w, \pi \otimes \pi) = \text{sw} \lim_{s \rightarrow 0} \gamma(s, \pi \times \check{\pi}, \psi) M(s, w, \pi \otimes \pi).$$

By Theorem 7.9 of [32],  $N(w, \pi \otimes \pi)$  is unitary. Since  $I(\pi \otimes \pi)$  is irreducible, there exists a constant  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$  such that

$$N(w, \pi \otimes \pi) = \alpha \text{id},$$

i.e.,

$$\operatorname{sw} \operatorname{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(g) = \alpha \operatorname{Res}_{s=0} \gamma(s, \pi \times \tilde{\pi}, \psi)^{-1} \phi(g)$$

for  $\phi \in I(s, \pi \otimes \pi)$  and  $g \in G^\sharp$ . □

Let

$$\bar{N}^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \mid x \in \operatorname{Mat}_{n \times n}(F) \right\}$$

and  $L = \operatorname{Mat}_{n \times n}(\mathfrak{o}_F)$ . Let  $\mathbf{1}_L$  denote the characteristic function of  $L$ .

**Lemma 4.2.** *Let  $u, v \in V_\pi$ . We define  $\phi \in I(s, \pi \otimes \pi)$  which has compact support in  $P^\sharp \bar{N}^\sharp$  modulo  $P^\sharp$  by*

$$\phi \left( \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \right) = \begin{cases} u \otimes v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

Then

$$\begin{aligned} & (\operatorname{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v') \\ &= (n \log q)^{-1} (1 - q^{-1}) d(\pi)^{-1} (\phi(1), v' \otimes u') \end{aligned}$$

for  $u', v' \in V_\pi$ .

*Proof.* The lemma follows from Proposition 5.1 of [34]. We include the proof for the sake of completeness.

As in [33], [34], we have

$$\begin{aligned} & (M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v') \\ &= \int_{\operatorname{GL}(n, F)} \mathbf{1}_L(x^{-1}) |\det(x)|_F^{-s-n} (\pi(x^{-1})u \otimes \pi(x)v, u' \otimes v') dx \\ &= \int_{\operatorname{GL}(n, F)} \mathbf{1}_L(x) |\det(x)|_F^s (\pi(x)u, u') \overline{(\pi(x)v', v)} d^\times x \\ &= \int_{\operatorname{GL}(n, F)/F^\times} \varphi_s(x) (\pi(x)u, u') \overline{(\pi(x)v', v)} d^\times x, \end{aligned}$$

where  $d^\times x = |\det(x)|_F^{-n} dx$  and

$$\varphi_s(x) = |\det(x)|_F^s \int_{F^\times} \mathbf{1}_L(zx) |z|_F^{ns} d^\times z.$$

For  $x = (x_{ij}) \in \operatorname{GL}(n, F)$ , we have

$$\int_{F^\times} \mathbf{1}_L(zx) |z|_F^{ns} d^\times z = \int_{\mathfrak{p}_F^{-m}} |z|_F^{ns} d^\times z = q^{mns} (1 - q^{-ns})^{-1} (1 - q^{-1}),$$

where  $m = \min(\operatorname{ord}_F(x_{ij}))$ . Note that this integral is absolutely convergent for  $\operatorname{Re}(s) > 0$ . Hence we have

$$\begin{aligned} & (\operatorname{Res}_{s=0} M(s, w, \pi \otimes \pi) \phi(1), u' \otimes v') \\ &= (n \log q)^{-1} (1 - q^{-1}) \int_{\operatorname{GL}(n, F)/F^\times} (\pi(x)u, u') \overline{(\pi(x)v', v)} d^\times x \\ &= (n \log q)^{-1} (1 - q^{-1}) d(\pi)^{-1} (u, v')(v, u'). \end{aligned}$$

This calculation is justified since

$$\varphi_s(x) \leq (1 - q^{-ns})^{-1}(1 - q^{-1})$$

for  $s \in \mathbb{R}_{>0}$ . □

By Lemmas 4.1 and 4.2, we have  $d(\pi) = \alpha^{-1}n^{-1}\gamma(0, \pi, \text{Ad}, \psi)$ . This completes the proof of Theorem 3.1.

### 5. TWISTED CHARACTERS

Let  $F$  be a non-archimedean local field of characteristic zero and let  $\psi$  be a non-trivial additive character of  $F$  of order zero. Let  $\mathbf{G} = \text{Res}_{E/F} \text{GL}(n)$ , where  $E$  is a quadratic extension of  $F$  and  $n$  is odd. Let  $\sigma$  be the non-trivial automorphism of  $E$  over  $F$  and let  $\omega_{E/F}$  be the quadratic character of  $F^\times$  associated to  $E/F$  by class field theory. Put  $\theta(g) = \text{Ad}(J_n)(\sigma({}^t g^{-1}))$  for  $g \in \mathbf{G}$ , where

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix} \in \text{GL}(n)$$

and  $\sigma$  is regarded as an automorphism of  $\mathbf{G}$  over  $F$ .

Let  $\pi$  be a discrete series representation of  $G$  such that  $\pi \simeq \pi \circ \theta$ . We fix an isomorphism  $\pi(\theta) : \pi \rightarrow \pi \circ \theta$  such that  $\pi(\theta)^2 = \text{id}$  and we define a distribution  $J^\theta(\pi)$  by

$$J^\theta(\pi)(f) = J^\theta(\pi, f) = \text{trace}(\pi(f)\pi(\theta))$$

for  $f \in C_c^\infty(G)$ . By Theorem 1 of [5],  $J^\theta(\pi)$  is a locally integrable function on  $G$  which is locally constant on  $G_{\theta\text{-reg}}$ . Here  $G_{\theta\text{-reg}}$  is the set of  $\theta$ -regular and  $\theta$ -semisimple elements in  $G$ . Let  $G_\theta$  denote the identity component of  $\{g \in G \mid \theta(g) = g\}$  and  $\mathfrak{g}_\theta$  the Lie algebra of  $G_\theta$ . By Theorem 3 of [5], we have the expansion

$$(5.1) \quad J^\theta(\pi, \exp(X)) = \sum_{\mathcal{O}} c_{\mathcal{O}, \theta}(\pi) \hat{\mu}_{\mathcal{O}}(X)$$

for  $X \in \mathfrak{g}_\theta$  sufficiently near zero, where  $\mathcal{O}$  runs over nilpotent  $G_\theta$ -orbits in  $\mathfrak{g}_\theta$  and where  $\hat{\mu}_{\mathcal{O}}$  is the Fourier transform of the invariant measure  $\mu_{\mathcal{O}}$  on  $\mathcal{O}$ .

**Theorem 5.1.** *Let  $\pi$  be a discrete series representation of  $G$  such that  $\pi \simeq \pi \circ \theta$  and such that  $L(s, \pi, r)$  has a pole at  $s = 0$ , where  $r$  is the Asai representation of  ${}^L G$ . Then there exists a constant  $c \in \mathbb{R}_{>0}$  which does not depend on  $\pi$  such that*

$$|c_{0, \theta}(\pi)| = c |\gamma(0, \pi, r', \psi)|,$$

where  $r' = r \otimes \omega_{E/F}$ .

*Remark 5.2.* By the result of Henniart [18], we have

$$\begin{aligned} L_{\text{LS}}(s, \pi, r) &= L(s, \pi, r), \\ \gamma_{\text{LS}}(s, \pi, r', \psi) &= \alpha \gamma(s, \pi, r', \psi), \end{aligned}$$

where the subscript LS indicates local factors defined by the Langlands-Shahidi method and  $\alpha \in \mathbb{C}$  is a root of unity.

6. TWISTED ORBITAL INTEGRALS

Let  $F$  be a non-archimedean local field of characteristic zero. Let  $\mathbf{G}$  be a connected reductive algebraic group over  $F$  and let  $\theta$  be a quasi-semisimple automorphism of  $\mathbf{G}$  over  $F$ . Let  $\mathbf{Z}$  be the center of  $\mathbf{G}$ . For  $\gamma \in G$  and  $f \in C_c^\infty(G/(1-\theta)Z)$ , put

$$J^\theta(\gamma, f) = \int_{ZG_{\gamma\theta}\backslash G} f(g^{-1}\gamma\theta(g)) dg.$$

Here  $G_{\gamma\theta}$  is the identity component of  $\{g \in G \mid g^{-1}\gamma\theta(g) = \gamma\}$ . By [28] and Lemma 2.1 of [1], this integral is absolutely convergent. By Proposition 7.1 of [33], we have the expansion

$$(6.1) \quad J^\theta(\exp(X), f) = \sum_u \Gamma_{u,\theta}(X) J^\theta(u, f)$$

for  $\theta$ -regular and  $\theta$ -semisimple elements  $X$  in  $\mathfrak{g}_\theta$  sufficiently near zero. Here  $u$  runs over representatives for unipotent orbits in  $G_\theta$ .

Let  $\mathbf{G}$  and  $\theta$  be as in §5. By [6],  $J^\theta(\gamma, f)$  is absolutely convergent and (6.1) holds even if  $f$  is a Schwartz function.

7. PROOF OF THEOREM 5.1

Throughout this section, we ignore the normalization of measures since it does not affect the proof. Let  $\mathbf{G}$  and  $\theta$  be as in §5. Let  $\pi$  be a discrete series representation of  $G$  such that  $\pi \simeq \pi \circ \theta$  and such that  $L(s, \pi, r)$  has a pole at  $s = 0$ , where  $r$  is the Asai representation of  ${}^L G$ . Let  $V_\pi$  be the space of  $\pi$ . We fix an invariant hermitian inner product  $(\cdot, \cdot)$  on  $V_\pi$  and an isomorphism  $\pi(\theta) : V_\pi \rightarrow V_\pi$  such that  $\pi(\theta)\pi(g) = \pi(\theta(g))\pi(\theta)$  for  $g \in G$  and such that  $\pi(\theta)^2 = \text{id}$ . Then  $(\cdot, \cdot)$  is  $\pi(\theta)$ -invariant.

Let

$$G^\sharp = \text{U}(2n, F) = \{g \in \text{GL}(2n, E) \mid gQ_n\sigma({}^t g) = Q_n\},$$

where

$$Q_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Let  $P^\sharp = M^\sharp N^\sharp$  be a parabolic subgroup of  $G^\sharp$  given by

$$M^\sharp = \left\{ \begin{pmatrix} a & 0 \\ 0 & \theta(a) \end{pmatrix} \mid a \in \text{GL}(n, E) \right\},$$

$$N^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \mid x \in X \right\},$$

where  $X = \{x \in \text{Mat}_{n \times n}(E) \mid \text{Ad}(J_n)(\sigma({}^t x)) = x\}$ . As in §4, we consider an induced representation

$$I(s, \pi) = \text{Ind}_{P^\sharp}^{G^\sharp}(\pi \mid \det|_E^{s/2})$$

for  $s \in \mathbb{C}$ . Put

$$w = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \in G^\sharp.$$

For  $\phi \in I(s, \pi)$  and  $g \in G^\sharp$ , the integral

$$M(s, w, \pi)\phi(g) = \int_X \phi\left(w^{-1} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} g\right) dx$$



is absolutely convergent for  $\text{Re}(s) > 0$ , has a meromorphic continuation to the whole  $s$ -plane, and defines an intertwining operator

$$M(s, w, \pi) : I(s, \pi) \longrightarrow I(-s, w(\pi)).$$

**Lemma 7.1.** *There exists a constant  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$  such that*

$$(\text{Res}_{s=0} M(s, w, \pi)\phi(1), v') = \alpha \text{Res}_{s=0} \gamma(s, \pi, r, \psi)^{-1}(\phi(1), \pi(\theta)v')$$

for  $\phi \in I(s, \pi)$  and  $v' \in V_\pi$ .

*Proof.* We remark that  $I(0, \pi)$  is irreducible since  $L(s, \pi, r)$  has a pole at  $s = 0$ . As in the proof of Lemma 4.1, the lemma follows from the result of Shahidi [32].  $\square$

Let

$$\bar{N}^\sharp = \left\{ \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \mid x \in X \right\}$$

and let  $L = X \cap \text{Mat}_{n \times n}(\mathfrak{o}_E)$ . Let  $\mathbf{1}_L$  denote the characteristic function of  $L$ .

**Lemma 7.2.** *Let  $v, v' \in V_\pi$ . Let  $f$  be a matrix coefficient of  $\pi$  given by  $f(g) = (\pi(g)v, v')$  for  $g \in G$ . We define  $\phi \in I(s, \pi)$  which has compact support in  $P^\sharp \bar{N}^\sharp$  modulo  $P^\sharp$  by*

$$\phi \left( \begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} \right) = \begin{cases} v & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

Then there exists a constant  $c \in \mathbb{R}_{>0}$  which does not depend on  $\pi$  such that

$$(\text{Res}_{s=0} M(s, w, \pi)\phi(1), v') = cJ^\theta(1, f).$$

*Proof.* The lemma follows from the result of Goldberg [10]. We include the proof for the sake of completeness.

We fix  $\delta \in F^\times - N_{E/F}(E^\times)$ . Set  $X' = \{x \in X \mid \det(x) \neq 0\}$ . Then  $G$  acts on  $X'$  by  $x \mapsto g^{-1}x\theta(g)$ . Let  $G \backslash X'$  denote the set of  $G$ -orbits in  $X'$ . Note that  $\{1, \delta\}$  is a set of representatives for  $G \backslash X'$ . We define a  $G$ -invariant measure  $d^\times x$  on  $X'$  by  $d^\times x = |\det(x)|_E^{-n/2} dx$ .

As in [10, §2], we have

$$\begin{aligned} & (M(s, w, \pi)\phi(1), v') \\ &= \int_{X'} \mathbf{1}_L(x^{-1}) |\det(x)|_E^{-s/2-n/2} (\pi(x^{-1})v, v') dx \\ &= \int_{X'} \mathbf{1}_L(x) |\det(x)|_E^{s/2} f(x) d^\times x \\ &= \sum_{\gamma \in G \backslash X'} \int_{G_{\gamma\theta} \backslash G} \mathbf{1}_L(g^{-1}\gamma\theta(g)) |\det(g^{-1}\gamma\theta(g))|_E^{s/2} f(g^{-1}\gamma\theta(g)) dg \\ &= \sum_{\gamma \in G \backslash X'} \int_{ZG_{\gamma\theta} \backslash G} \varphi_s(g^{-1}\gamma\theta(g)) f(g^{-1}\gamma\theta(g)) dg, \end{aligned}$$

where

$$\varphi_s(x) = |\det(x)|_E^{s/2} \int_{E^\times} \mathbf{1}_L(z\sigma(z)x) |z|_E^{ns} d^\times z.$$

For  $x = (x_{ij}) \in X'$ , we have

$$\int_{E^\times} \mathbf{1}_L(z\sigma(z)x) |z|_E^{ns} d^\times z = q_E^{\lfloor m/2 \rfloor ns} (1 - q_E^{-ns})^{-1} (1 - q_E^{-1}),$$

where  $m = \min(\text{ord}_E(x_{ij}))$ . Note that this integral is absolutely convergent for  $\text{Re}(s) > 0$ . Hence we have

$$(\text{Res}_{s=0} M(s, w, \pi)\phi(1), v') = (n \log q_E)^{-1}(1 - q_E^{-1}) \sum_{\gamma \in G \setminus X'} J^\theta(\gamma, f).$$

This calculation is justified since

$$\varphi_s(x) \leq (1 - q_E^{-ns})^{-1}(1 - q_E^{-1})$$

for  $s \in \mathbb{R}_{>0}$ . As in [10, §2], the central character of  $\pi$  is trivial on  $F^\times$ . Hence we have

$$J^\theta(\delta, f) = J^\theta(1, f).$$

This completes the proof. □

**Lemma 7.3.** *Let  $v, v' \in V_\pi$ . Let  $f$  be a matrix coefficient of  $\pi$  given by  $f(g) = (\pi(g)v, v')$  for  $g \in G$ . Then*

$$J^\theta(\gamma, f) = d(\pi)^{-1}(v, \pi(\theta)v')J^\theta(\pi, \gamma)$$

for  $\theta$ -regular and  $\theta$ -elliptic elements  $\gamma$  in  $G$ .

*Proof.* We proceed as in the proof of Proposition 5 of [7]. Let  $\gamma$  be a  $\theta$ -regular and  $\theta$ -elliptic element in  $G$ . Let  $\varphi \in C_c^\infty(G)$ . We assume that the support of  $\varphi$  is contained in the set of  $\theta$ -regular and  $\theta$ -elliptic elements in  $G$ . By the Schur orthogonality relations, we have

$$\int_{Z \setminus G} (\pi(g^{-1})\pi(\varphi)\pi(\theta(g))v, v') dg = d(\pi)^{-1}(v, \pi(\theta)v')J^\theta(\pi, \varphi).$$

The left-hand side is equal to

$$\int_{Z \setminus G} \int_G \varphi(h)f(g^{-1}h\theta(g)) dh dg = \int_G \varphi(h) \int_{Z \setminus G} f(g^{-1}h\theta(g)) dg dh.$$

Let  $\varphi$  tend to the Dirac measure at  $\gamma$ . This yields the lemma. □

Let  $v, v' \in V_\pi$ . Let  $f$  be a matrix coefficient of  $\pi$  given by  $f(g) = (\pi(g)v, v')$  for  $g \in G$ . By (5.1), (6.1), and Lemma 7.3, we have

$$(v, \pi(\theta)v') \sum_{\mathcal{O}} c_{\mathcal{O}, \theta}(\pi)\hat{\mu}_{\mathcal{O}}(X) = d(\pi) \sum_u \Gamma_{u, \theta}(X)J^\theta(u, f)$$

for  $\theta$ -regular and  $\theta$ -elliptic elements  $X$  in  $\mathfrak{g}_\theta$  sufficiently near zero. For  $t \in F^\times$ , we have

$$\begin{aligned} \hat{\mu}_{\mathcal{O}}(t^2X) &= |t|_F^{-\dim \mathcal{O}} \hat{\mu}_{\mathcal{O}}(X), \\ \Gamma_{u, \theta}(t^2X) &= |t|_F^{-\dim \text{Ad}(G_\theta)(u)} \Gamma_{u, \theta}(X). \end{aligned}$$

Note that  $\hat{\mu}_0 = \Gamma_{1, \theta} = 1$  if measures are suitably normalized. By homogeneity, we obtain

$$(v, \pi(\theta)v')c_{0, \theta}(\pi) = d(\pi)J^\theta(1, f).$$

By Theorem 3.1,  $d(\pi)$  is equal to

$$|\lim_{s \rightarrow 0} s^{-1}\gamma(s, \pi \times \tilde{\pi}, \psi)| = |\lim_{s \rightarrow 0} s^{-1}\gamma(s, \pi, r, \psi)| \cdot |\gamma(0, \pi, r', \psi)|$$

up to a constant which does not depend on  $\pi$ . Thus Theorem 5.1 follows from Lemmas 7.1 and 7.2.

*Remark 7.4.* Let  $\mathbf{G} = \mathrm{GL}(n)$ , where  $n$  is even. Put  $\theta(g) = \mathrm{Ad}(J_n)({}^t g^{-1})$  for  $g \in \mathbf{G}$ . Let  $r$  (resp.  $r'$ ) be the exterior (resp. symmetric) square representation of  ${}^L G$ . Using the result of Shahidi [33], one can prove an analogue of Theorem 5.1 for an irreducible unitary supercuspidal representation  $\pi$  of  $G$  such that  $\pi \simeq \pi \circ \theta$  and such that  $L(s, \pi, r)$  has a pole at  $s = 0$ . To drop the assumption that  $\pi$  is supercuspidal, one has to show that twisted orbital integrals of Schwartz functions are absolutely convergent.

8. TWISTED ENDOSCOPY

Let  $F$  be a non-archimedean local field of characteristic zero. Let  $\mathbf{G}$  and  $\theta$  be as in §5. We consider a set of endoscopic data  $(\mathbf{H}, {}^L H, 1, \xi)$  for  $(\mathbf{G}, \theta, 1)$  defined as follows. Recall that  $\mathbf{G} = \mathrm{Res}_{E/F} \mathrm{GL}(n)$ , where  $E$  is a quadratic extension of  $F$  and where  $n$  is odd. We have  ${}^L G = \hat{G} \rtimes W_F$ , where  $\hat{G} = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$  and the action of  $w \in W_F$  is given by

$$(g_1, g_2) \mapsto \begin{cases} (g_1, g_2) & \text{if } w \in W_E, \\ (g_2, g_1) & \text{if } w \notin W_E. \end{cases}$$

Let  $\mathbf{H} = \mathrm{U}(n)$  be the quasi-split unitary group in  $n$  variables. Then  ${}^L H = \hat{H} \rtimes W_F$ , where  $\hat{H} = \mathrm{GL}(n, \mathbb{C})$  and the action of  $w \in W_F$  is given by

$$h \mapsto \begin{cases} h & \text{if } w \in W_E, \\ \mathrm{Ad}(J_n)({}^t h^{-1}) & \text{if } w \notin W_E. \end{cases}$$

We define  $\xi : {}^L H \rightarrow {}^L G$  by  $\xi(h \times w) = (h, \mathrm{Ad}(J_n)({}^t h^{-1})) \times w$ .

**Lemma 8.1.** *Let  $r$  be the Asai representation of  ${}^L G$  on  $\mathbb{C}^n \otimes \mathbb{C}^n$ . Then the adjoint representation  $\mathrm{Ad}$  of  ${}^L H$  on  $\mathrm{Lie}(\hat{H})$  is isomorphic to  $r' \circ \xi$ , where  $r' = r \otimes \omega_{E/F}$ .*

*Proof.* Recall that  $r$  is defined by

$$\begin{aligned} r((g_1, g_2) \times 1)(x \otimes y) &= g_1 x \otimes g_2 y, \\ r((1, 1) \times w)(x \otimes y) &= \begin{cases} x \otimes y & \text{if } w \in W_E, \\ y \otimes x & \text{if } w \notin W_E. \end{cases} \end{aligned}$$

It is easy to check that

$$\begin{aligned} \mathrm{Ad}(h \times 1)(X) &= \mathrm{Ad}(h)(X), \\ \mathrm{Ad}(1 \times w)(X) &= \begin{cases} X & \text{if } w \in W_E, \\ -\mathrm{Ad}(J_n)({}^t X) & \text{if } w \notin W_E. \end{cases} \end{aligned}$$

Hence the isomorphism  $\mathbb{C}^n \otimes \mathbb{C}^n \rightarrow \mathrm{Lie}(\hat{H})$  given by  $x \otimes y \mapsto x {}^t y J_n$  is intertwining. □

It is believed that the following conjectures hold (cf. [2]).

**Conjecture 8.2.** *For  $f \in C_c^\infty(G)$ , there exists  $f^H \in C_c^\infty(H)$  such that  $f$  and  $f^H$  have matching orbital integrals (cf. [25, §5.5]).*

**Conjecture 8.3.** *Let  $\phi_H : L_F \rightarrow {}^L H$  be a tempered Langlands parameter. We define a tempered Langlands parameter  $\phi : L_F \rightarrow {}^L G$  by  $\phi = \xi \circ \phi_H$ . Let  $\pi$  be the irreducible tempered representation of  $G$  associated to  $\phi$  by the local Langlands*

correspondence [16], [17]. Then there exists a constant  $c \in \mathbb{C}^\times$  such that  $|c|$  does not depend on  $\phi_H$  and such that

$$J^\theta(\pi, f) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle J(\pi_H, f^H).$$

Here  $f$  and  $f^H$  have matching orbital integrals.

For  $f \in C_c^\infty(G)$ , let  $f^{G_\theta} \in C_c^\infty(G_\theta)$  be a decent of  $f$ , where  $G_\theta$  is the identity component of  $\{g \in G \mid \theta(g) = g\}$ . Let  $f^H \in C_c^\infty(H)$ . Assume that the supports of  $f$  and  $f^H$  are sufficiently small. By [15], [5], we have

$$\begin{aligned} J^\theta(\pi, f) &= \sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi) \hat{\mu}_{\mathcal{O}}(f^{G_\theta} \circ \exp), \\ J(\pi_H, f^H) &= \sum_{\mathcal{O}_H} c_{\mathcal{O}_H}(\pi_H) \hat{\mu}_{\mathcal{O}_H}(f^H \circ \exp). \end{aligned}$$

Here  $\mathcal{O}$  (resp.  $\mathcal{O}_H$ ) runs over nilpotent  $G_\theta$ -orbits (resp.  $H$ -orbits) in  $\mathfrak{g}_\theta = \text{Lie}(G_\theta)$  (resp.  $\mathfrak{h} = \text{Lie}(H)$ ).

**Lemma 8.4.** *Assume that Conjectures 8.2 and 8.3 hold. Let  $\phi_H : L_F \rightarrow {}^L H$  be a tempered Langlands parameter and let  $\pi$  be the irreducible tempered representation of  $G$  as in Conjecture 8.3. Then there exists a constant  $c \in \mathbb{C}$  such that  $|c|$  does not depend on  $\phi_H$  and such that*

$$c_{0,\theta}(\pi) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H).$$

*Proof.* We proceed as in [32, §9], [21, §8]. Assume that the supports of  $f$  and  $f^H$  are sufficiently small. If  $t \in F^\times$  is sufficiently small, then we can define  $f_t \in C_c^\infty(G)$  by  $f_t(\exp(X)) = f(\exp(t^{-1}X))$ . Similarly, we can define  $f_t^H \in C_c^\infty(H)$ . Then

$$\begin{aligned} \hat{\mu}_{\mathcal{O}}(f_t^{G_\theta} \circ \exp) &= |t|_F^{2 \dim \mathfrak{g}_\theta - \dim \mathcal{O}} \hat{\mu}_{\mathcal{O}}(f^{G_\theta} \circ \exp), \\ \hat{\mu}_{\mathcal{O}_H}(f_t^H \circ \exp) &= |t|_F^{2 \dim \mathfrak{h} - \dim \mathcal{O}_H} \hat{\mu}_{\mathcal{O}_H}(f^H \circ \exp). \end{aligned}$$

Note that  $\dim \mathfrak{g}_\theta = \dim \mathfrak{h}$ .

Assume that  $f$  and  $f^H$  have matching orbital integrals. By Lemma 8.5 of [21],  $f_t$  and  $f_t^H$  have matching orbital integrals. Hence we have

$$\sum_{\mathcal{O}} c_{\mathcal{O},\theta}(\pi) \hat{\mu}_{\mathcal{O}}(f_t^{G_\theta} \circ \exp) = c \sum_{\mathcal{O}_H} \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_{\mathcal{O}_H}(\pi_H) \hat{\mu}_{\mathcal{O}_H}(f_t^H \circ \exp).$$

By homogeneity, we obtain

$$c_{0,\theta}(\pi) \hat{\mu}_0(f^{G_\theta} \circ \exp) = c \sum_{\pi_H \in \Pi_{\phi_H}(H)} \langle 1, \pi_H \rangle c_0(\pi_H) \hat{\mu}_0(f^H \circ \exp).$$

□

Let  $\pi_H$  be a discrete series representation of  $H$  and let  $\phi_H : L_F \rightarrow {}^L H$  be the (conjectural) Langlands parameter associated to  $\pi_H$ . Let  $\pi$  be the irreducible tempered representation of  $G$  as in Conjecture 8.3. If  $\pi_H$  is stable, then  $\pi$  is expected to be square integrable.

**Proposition 8.5.** *Assume that Conjectures 8.2 and 8.3 hold. Let  $\pi_H$  be a stable discrete series representation of  $H$ . Then*

$$d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \text{Ad}, \psi)|.$$

*Proof.* By [15], [30], we have

$$c_0(\pi_H) = (-1)^{l_0} d(\pi_{H,0})^{-1} \cdot d(\pi_H),$$

where  $l_0$  is the semisimple  $F$ -rank of  $\mathbf{H}$  and  $\pi_{H,0}$  is the Steinberg representation of  $H$ . By Theorem 5.1 and Lemmas 8.1 and 8.4, there exists a constant  $c \in \mathbb{R}_{>0}$  which does not depend on  $\pi_H$  such that

$$d(\pi_H) = c|\gamma(0, \pi_H, \text{Ad}, \psi)|.$$

Since  $\pi_{H,0}$  is stable and  $d(\pi_{H,0}) = 2^{-1}|\gamma(0, \pi_{H,0}, \text{Ad}, \psi)|$ , we have  $c = 2^{-1}$ .  $\square$

For  $n = 3$ , Conjectures 8.2 and 8.3 were proved by Rogawski [31]. Thus we obtain the following theorem.

**Theorem 8.6.** *Let  $\mathbf{H} = \text{U}(3)$  be the quasi-split unitary group in three variables. Let  $\pi_H$  be a stable discrete series representation of  $H$ . Then*

$$d(\pi_H) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \text{Ad}, \psi)|.$$

*In particular, Conjecture 1.4 holds for  $\pi_H$ .*

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