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K-REGULARITY, cdh-FIBRANT HOCHSCHILD HOMOLOGY, AND A CONJECTURE OF VORST

G. CORTIÑAS, C. HAESEMEYER, AND C. WEIBEL

Introduction

It is a well-known fact that algebraic K-theory is homotopy invariant as a functor on regular schemes; if X is a regular scheme, then the natural map $K_n(X) \to K_n(X \times \mathbb{A}^1)$ is an isomorphism for all $n \in \mathbb{Z}$. This is false in general for nonregular schemes and rings.

To express this failure, Bass introduced the terminology that, for any contravariant functor $\mathcal{P}$ defined on schemes, a scheme X is called *$\mathcal{P}$-regular* if the pullback maps $\mathcal{P}(X) \to \mathcal{P}(X \times \mathbb{A}^r)$ are isomorphisms for all $r \geq 0$. If $X = \mathrm{Spec}(R)$, we also say that R is $\mathcal{P}$-regular. Thus regular schemes are K_n-regular for every n. In contrast, it was observed as long ago as in [2] that a nonreduced affine scheme can never be K_1-regular. In particular, if A is an Artinian ring (that is, a 0-dimensional Noetherian ring), then A is regular (that is, reduced) if and only if A is K_1-regular. In [17], Vorst conjectured that for an affine scheme X, of finite type over a field F and of dimension d, regularity and K_{d+1}-regularity are equivalent; Vorst proved this conjecture for $d = 1$ (by proving that K_2-regularity implies normality).

In this paper, we prove Vorst's conjecture in all dimensions provided the characteristic of the ground field F is zero. In fact we prove a stronger statement. We say that X is *regular in codimension* $< n$ if $\mathrm{Sing}(X)$ has codimension $\geq n$ in X. Note that for all $n \in \mathbb{Z}$, if a ring R is K_n-regular, then it is K_{n-1}-regular. This is proved in [17] for $n \geq 1$ and in [6, 4.4] for $n \leq 0$.

Let $\mathcal{F}_K$ denote the presheaf of spectra such that $\mathcal{F}_K(X)$ is the homotopy fiber of the natural map $K(X) \to KH(X)$, where $K(X)$ is the algebraic K-theory spectrum of X and $KH(X)$ is the homotopy K-theory of X defined in [19]. We write $\mathcal{F}_K(R)$ for $\mathcal{F}_K(\mathrm{Spec}(R))$.

Theorem 0.1. *Let R be a commutative ring which is essentially of finite type over a field F of characteristic* 0. *Then the following hold.*

(a) *If $\mathcal{F}_K(R)$ is n-connected, then R is regular in codimension $< n$.*
(b) *If R is K_n-regular, then R is regular in codimension $< n$.*
(c) *(Vorst's conjecture) If R is $K_{1+\dim(R)}$-regular, then R is regular.*

It was observed in [19, Proposition 1.5] that if X is K_n-regular, then $K_i(X) \to KH_i(X)$ is an isomorphism for $i \leq n$ and a surjection for $i = n+1$, so that $\mathcal{F}_K(X)$

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is n-connected. Thus (a) implies (b) in this theorem, and (c) is a special case of (b).

The bounds in (a) and (b) are the best possible, because it follows from Vorst's results ([18, Thm. A], [17, Thm. 3.6]) that for an affine singular seminormal curve X, $\mathcal{F}_K(X)$ is 1-connected, but not 2-connected. The converse of (c) is trivial, but those of (a) and (b) are false. Indeed, affine normal surfaces are regular in codimension 1 but may not be K_{-1}-regular, much less K_2-regular; see [23, Affine Cones 5.8.1].

Finally the analogue of (c) –and thus also of (a) and (b)– for K-theory of general nonaffine schemes is false. Indeed we give the following example of a nonreduced (and in particular nonregular) projective curve which is K_n-regular for all n.

Theorem 0.2. *Let (X, Q) be an elliptic curve over a field of characteristic 0, and let P be a rational point on X such that the line bundle $L = L(P - Q)$ does not have odd order in the Picard group $Pic(X)$. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O} \oplus L$, where L is regarded as a square-zero ideal.*

Then Y is K_n-regular for all n, and $\mathcal{F}_K(Y)$ is contractible.

The idea of the proof of Theorem 0.1 is to use results from [4] to translate K-theoretic statements into statements about Hochschild homology and, ultimately, into statements about differentials. Thus the original K-theoretic problem is reinterpreted as a characterization of regularity in terms of Hochschild homology and differential forms, which we solve.

In more detail, the structure of the proof of Theorem 0.1 is as follows. Using results from [4], we translate the failure of K-theory to be homotopy invariant into a statement about the failure of Hochschild homology taken over $\mathbb{Q}$ to satisfy *cdh*-descent (roughly speaking this means that the Hochschild homology of a singularity cannot be computed from that of a resolution of its singularities; see Section 1). By hypothesis, the K-theory of R is homotopy invariant up to some degree; therefore, its Hochschild homology over $\mathbb{Q}$ satisfies *cdh*-descent up to some degree (Corollary 1.7). Using the transitivity spectral sequence of [11] and its *cdh*-sheafified version (Lemmas 4.1 and 4.2), we conclude that this implies that the Hochschild homology of R taken *over* F satisfies *cdh*-descent up to some degree (Proposition 4.8). Finally, we show that this has implications for the algebra of differentials of R over F (Lemmas 2.3 and 3.2) that imply the regularity of R (Theorems 4.11 and 4.12).

As mentioned above, the proof employs results from our paper with M. Schlichting [4] that allow us to describe $\mathcal{F}_K$ in terms of cyclic homology; the necessary statements will be recalled in Section 1. In Section 2, we study the *cdh*-fibrant version of Hochschild homology and its Hodge decomposition. Section 3 contains a smoothness criterion (Theorem 3.1) using *cdh*-fibrant Hochschild homology, which is of independent interest and is generalized in Theorem 4.11. Section 4 (see Theorem 4.12) contains the discussion of the transitivity spectral sequence and the proof of part (a) of Theorem 0.1. As explained above, parts (b) and (c) follow from this. Finally Section 5 is devoted to the counterexample stated in Theorem 0.2 (and restated as Theorem 5.2).

Notation. All rings considered in this paper are commutative and Noetherian. We shall write Sch/F for the category of schemes essentially of finite type over a field F. Objects of Sch/F shall be called *F-schemes*.

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum* $\mathcal{E}$ is a sequence of simplicial sets $\mathcal{E}_n$ together with bonding maps $b_n : \mathcal{E}_n \to \Omega\mathcal{E}_{n+1}$. We say that $\mathcal{E}$ is an Ω-*spectrum* if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an Ω-spectrum.

We say that a presheaf E of spectra on Sch/F satisfies the *Mayer-Vietoris-property* (or MV-property, for short) for a cartesian square of schemes

$$(\square) \qquad \begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

if applying E to this square results in a homotopy cartesian square of spectra. We say that E satisfies the Mayer-Vietoris property for a class of squares provided it satisfies the MV-property for each square in the class.

We say that E satisfies *Nisnevich descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares in Sch/F; an *elementary Nisnevich square* is a cartesian square of schemes $(\square)$ for which $Y \to X$ is an open embedding, $X' \to X$ is étale and $(X' - Y') \to (X - Y)$ is an isomorphism. It is a result of Nisnevich that this is equivalent to the assertion that $E(X) \to \mathbb{H}_{nis}(X, E)$ is a weak equivalence for each scheme X, where $\mathbb{H}_{nis}(-, E)$ is a fibrant replacement for E in a suitable model structure; see [4, Theorem 3.4] for a proof.

We say that E satisfies *cdh-descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares (Nisnevich descent) and for all abstract blow-up squares in Sch/F; an *abstract blow-up square* is a square $(\square)$ such that $Y \to X$ is a closed embedding, $X' \to X$ is proper and the induced morphism $(X' - Y')_{red} \to (X - Y)_{red}$ is an isomorphism. With M. Schlichting, we showed in Theorem 3.7 of [4] that cdh-descent is equivalent to the assertion that $E(X) \to \mathbb{H}_{cdh}(X, E)$ is a weak equivalence for each scheme X, where $\mathbb{H}_{cdh}(-, E)$ is a fibrant replacement for the presheaf E in a suitable model structure. We abbreviate $\mathbb{H}_{cdh}(-E)$ as $\mathbb{H}(-, E)$ if no confusion can arise, and we write $\mathbb{H}_n(X, E) = \mathbb{H}^{-n}(X, E)$ for $\pi_n\mathbb{H}(X, E)$.

We use cohomological indexing for all chain complexes in this paper; for a complex A, $A[p]^q = A^{p+q}$. It is well-known that there is an Eilenberg-Mac Lane functor $A \mapsto |A|$ from chain complexes of abelian groups to spectra and from presheaves of chain complexes of abelian groups to presheaves of spectra. See, for example, [20, 10.9.19]. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra and satisfies $\pi_n(|A|) = H^{-n}(A)$. In this spirit, we will use descent terminology for presheaves of complexes.

For example, the Hochschild, cyclic, periodic and negative cyclic homology of schemes over a field k (such as F-schemes over a field $F \supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [21] and [4, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively, and regard them as presheaves of either cochain complexes or spectra. When k is omitted, it is understood that $k = \mathbb{Q}$ is intended. Finally, we write $\Omega^i_{/k}$ for the presheaf $X \mapsto \Omega^i_{X/k}$, while $\Omega^\bullet_{/k}$ denotes the presheaf of algebraic de Rham complexes and $\Omega^{\leq i}_{/k}$ denotes its brutal truncation in degree i.

1. *cdh*-DESCENT

In this section we recall the main results from [4] and prove that the failure of K-theory to be homotopy invariant can be measured using cyclic homology. We work on the category Sch/F of F-schemes essentially of finite type over a field F of characteristic 0.

Here are two of the main results of [4]. Recall that infinitesimal K-theory $\mathcal{K}^{inf}(X)$ is the homotopy fiber of the Jones-Goodwillie Chern character $K(X) \to HN(X)$. The first one is Theorem 4.6 of [4]:

Theorem 1.1. *The presheaf of spectra $\mathcal{K}^{inf}$ satisfies cdh-descent.*

The second one is a slight modification of Corollary 3.13 of [4].

Theorem 1.2. *For each subfield $k \subseteq F$, the presheaf $HP(/k)$ satisfies cdh-descent on* Sch/F. *In particular, HP satisfies cdh-descent.*

Proof. For $k = \mathbb{Q}$, this is proven in [4, Corollary 3.13]. As in *loc. cit.,* the result for $k \subset F$ follows from [4, Theorem 3.12], once we verify that the hypotheses hold. But this follows from three observations: (1) the Cuntz-Quillen excision theorem holds over k (see [5, 5.3], noting that the condition that the base field be algebraically closed is not needed; see [5, p. 69]). Cuntz-Quillen is a theorem about bivariant periodic cyclic homology; to obtain the result we need, simply set the first variable equal to k. (2) Goodwillie's theorem that periodic cyclic homology is invariant under infinitesimal extension does not require rings of finite type over k (see [7, II.5.1] or [12, 4.1.15]); and (3) the results of [4, Section 2] hold for (Hochschild, cyclic, periodic, negative) homology over k. □

Example 1.3. Consider the presheaf of spectra KH associated to homotopy K-theory. The main theorem of [9] implies that the natural transformation $K \to KH$ induces an equivalence $\mathbb{H}_{cdh}(-, K) \simeq KH$, because KH satisfies *cdh*-descent. Thus the following definition of $\mathcal{F}_K$ is compatible (up to canonical natural equivalence) with the definition of $\mathcal{F}_K$ given in the introduction.

Definition 1.4. For any presheaf of spectra E, we write $\mathcal{F}_E$ for the homotopy fiber of $E \to \mathbb{H}_{cdh}(-, E)$. If $\mathcal{F}_E(X)$ is n-connected for some scheme X (for all X in some subcategory of Sch/F), we say that E satisfies n-*cdh*-descent on X (resp., on the subcategory). Note that if E satisfies n-*cdh*-descent for all n on a subcategory, then E satisfies *cdh*-descent on that subcategory.

Since the fibrant replacement functor $\mathbb{H}_{cdh}$ preserves (objectwise) fibration sequences, it follows that $\mathcal{F}$ does too. (See the first paragraph of [4, Section 5].) We record this as the following observation.

Lemma 1.5. *Let $E_1 \to E_2 \to E_3$ be a fibration sequence of presheaves of spectra. Then there is a natural induced fibration sequence*

$$\mathcal{F}_{E_1} \to \mathcal{F}_{E_2} \to \mathcal{F}_{E_3}.$$

Theorem 1.6. *For any scheme X, essentially of finite type over a field of characteristic* 0, *the Chern character $K \to HN = HN(/\mathbb{Q})$ induces natural weak equivalences*

$$\mathcal{F}_K(X) \xrightarrow{\simeq} \mathcal{F}_{HN}(X) \xleftarrow{\simeq} \Omega^{-1}\mathcal{F}_{HC}(X).$$

Proof. The first weak equivalence follows from Lemma 1.5 and Theorem 1.1. The second weak equivalence follows from Lemma 1.5, Theorem 1.2 and the SBI fibration sequence $\Omega HP \to \Omega^{-1}HC \to HN \to HP$. □

Corollary 1.7. *Let $X \in \mathrm{Sch}/F$. Then K satisfies $(n+1)$-cdh-descent on X if and only if HH satisfies n-cdh-descent on X.*

Proof. Since $HH(X)$ and $\mathbb{H}(X, HH)$ are n-connected for $n < -\dim(X)$ by [21], this follows from Theorem 1.6 and the SBI sequence $\Omega^{-1}HC \to HH \to HC$. Indeed, by Lemma 1.5 and the theorem, the SBI sequence induces a fibration sequence $\mathcal{F}_K(X) \to \mathcal{F}_{HH}(X) \to \Omega\mathcal{F}_K(X)$ in which all terms have bounded below homotopy. Clearly, if K satisfies $(n+1)$-descent, then HH satisfies n-descent. Conversely, if HH satisfies n-descent, then $\pi_k\mathcal{F}_K(X) \cong \pi_{k-2}\mathcal{F}_K(X)$ for $k \leq n+1$; since $\pi_k\mathcal{F}_K(X) = 0$ for $k << 0$, this completes the proof. □

2. cdh-fibrant Hochschild and cyclic homology

In this section we study the cdh-fibrant version of Hochschild homology and its Hodge decomposition, and we establish some of their basic properties. We will use the terminology of [4, Section 3].

For legibility, we will write a for the natural morphism of sites from the cdh-site to the Zariski site on Sch/F. If A is a Zariski sheaf, its associated cdh-sheaf will be written as A_{cdh} or a^*A. We will simplify notation and write $H^*_{cdh}(X, A)$ for the cohomology of A_{cdh}. In particular this applies to the sheaf $X \mapsto \Omega^i_{X/k}$ of Kähler i-differential forms ($i \geq 0$); $H^*_{cdh}(X, \Omega^i_{/k})$ is the cohomology of $a^*\Omega^i_{/k}$. If A is a complex of presheaves of abelian groups on Sch/F, then we write $\mathbb{H}_{cdh}(A)$ for a cdh-fibrant replacement of A and $\mathbb{H}_{cdh}(X, A)$ for its complex of sections over X; the usual hypercohomology $\mathbb{H}^n_{cdh}(X, A)$ is just $H^n\mathbb{H}_{cdh}(X, A)$. For example, if A is a presheaf, considered as a complex concentrated in degree zero, then $\mathbb{H}_{cdh}(A)$ is just an injective resolution of the cdh-sheafification A_{cdh}, and $\mathbb{H}^n_{cdh}(X, A)$ is the usual cohomology $H^n_{cdh}(X, A_{cdh})$ of A_{cdh}.

When A is an unbounded complex, such as a complex representing Hochschild homology, then $\mathbb{H}_{cdh}(A)$ may be constructed using product total complexes of flasque Cartan-Eilenberg resolutions. This works because the columns of the Cartan-Eilenberg double complex are locally cohomologically bounded by [15].

The cdh-site is Noetherian (every covering has a finite subcovering), so cdh-cohomology commutes with filtered direct limits of sheaves. A typical application of this fact is that if M is a sheaf of F-modules and V is a vector space, then $H^n_{cdh}(X, V \otimes_F M) \cong V \otimes_F H^n_{cdh}(X, M)$. (See [1], Exp. VI, 2.11 and 5.2.)

The Hochschild and cyclic homology of schemes over a field k (such as F-schemes over a field $F \supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of mixed complexes; see [21] and [4, 2.7] for precise definitions. It was observed in [22, 3.0] that, because the mixed complexes already admit a Hodge decomposition, so do the complexes $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$. Taking fibrant replacements in the cdh-topology (or, in fact, in any topology) respects such product decompositions up to canonical global equivalence, as recorded in the following proposition.

Proposition 2.1. *Let X be a scheme over a field k of characteristic 0 (such as an F-scheme for $F \supseteq k$). Then the cdh-fibrant Hochschild, cyclic, negative cyclic and periodic cyclic homology of X over k admit natural Hodge decompositions.*

That is, if H denotes any of $HH(/k)$, $HC(/k)$, $HP(/k)$ or $HN(/k)$, then

$$\mathbb{H}_{cdh}(X,H) \cong \prod \mathbb{H}_{cdh}(X,H^{(i)}).$$

Proof. Let $H = \prod H^{(i)}$ be any of the product decompositions in question. Since the condition of being fibrant is defined by a right lifting property, the product $\prod \mathbb{H}_{cdh}(-,H^{(i)})$ is fibrant. The natural map $\mathbb{H}_{cdh}(-,H) \to \prod \mathbb{H}_{cdh}(-,H^{(i)})$ is a local weak equivalence; since both source and target are fibrant, it is a global weak equivalence, as asserted. □

Moreover, using the computations of these decompositions provided in [22, Theorem 3.3] and the fact that all F-schemes are smooth locally in the *cdh*-topology by resolution of singularities, it is possible to compute the Hodge decomposition explicitly in terms of the *cdh*-hypercohomology of the de Rham complex.

Theorem 2.2. *Let $k \subseteq F$ be a subfield. There are natural isomorphisms for every F-scheme X:*

$$\pi_n \mathbb{H}_{cdh}(X, HH^{(i)}(/k)) \cong H^{i-n}_{cdh}(X, \Omega^i_{/k});$$
$$\pi_n \mathbb{H}_{cdh}(X, HC^{(i)}(/k)) \cong \mathbb{H}^{2i-n}_{cdh}(X, \Omega^{\leq i}_{/k});$$
$$\pi_n \mathbb{H}_{cdh}(X, HN^{(i)}(/k)) \cong \mathbb{H}^{2i-n}_{cdh}(X, \Omega^{\geq i}_{/k});$$
$$\pi_n \mathbb{H}_{cdh}(X, HP^{(i)}(/k)) \cong \mathbb{H}^{2i-n}_{cdh}(X, \Omega^{\bullet}_{/k}).$$

Proof. Let $\mathbf{C}(X)$ denote the mixed complex computing the Hochschild and cyclic homology of X over k. The functor $\mathbf{C} : X \mapsto \mathbf{C}(X)$ is a presheaf of mixed complexes. By [20, 9.8.12], there is a Hodge decomposition $\mathbf{C} \cong \prod_i \mathbf{C}^{(i)}$ and a natural map of mixed complexes $e : \mathbf{C} \to (\Omega^*_{X/k}, 0, d)$ that sends the Hochschild chain complex $HH^{(i)}(X) = (\mathbf{C}^{(i)}(X), b)$ to $\Omega^i_{X/k}[i]$. As observed in [22], the induced map on Connes' double complexes sends $\mathcal{B}^{(i)}_{**}$ to $\Omega^{\leq i}_{X/k}[2i]$. It suffices to prove that these are locally quasi-isomorphisms for the *cdh*-topology. This boils down to showing that e induces a quasi-isomorphism $HH^{(i)}(R/k) \to \Omega^i_{R/k}[i]$ for every regular local F-algebra R. For $k = F$, this is the Hochschild-Kostant-Rosenberg theorem ([20, 9.4.7]). The general case follows from this and the fact that R is the union of smooth k-algebras. (It also follows from the Kassel-Sletsjœ spectral sequence of [11, 4.3a], which we recall in Lemma 4.1 below.) □

Lemma 2.3. *Let R be an F-algebra essentially of finite type, $k \subseteq F$ a subfield. Then for each n, $HH(/k)$ satisfies n-cdh-descent on $X = \mathrm{Spec}(R)$ if and only if the following three conditions hold simultaneously:*

(2.3a) $$HH^{(q)}_m(R/k) = 0 \quad \text{if } 0 \leq q < m \leq n;$$

(2.3b) $$\Omega^q_{R/k} \to H^0_{cdh}(X, \Omega^q_{/k}) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n+1;$$

(2.3c) $$H^p_{cdh}(X, \Omega^q_{/k}) = 0 \quad \text{if } p > 0 \text{ and } 0 \leq q \leq p+n+1.$$

Note that (2.3a) is vacuous if $n \leq 0$, and (2.3b) is vacuous if $n \leq -2$. In particular, $HH(/k)$ satisfies (-2)-cdh-descent just in case $H^p_{cdh}(X, \Omega^q_{/k}) = 0$ for all $p > q \geq 0$.

Proof. This follows easily from the Hodge decomposition and the isomorphisms

$$HH_q^{(q)}(R/k) \cong \Omega^q_{R/k} \quad \text{and} \quad HH_m^{(q)}(R/k) = 0 \text{ for } q > m.$$

In more detail, we see from Proposition 2.1 and Theorem 2.2 that the maps

$$HH_m^{(q)}(R/k) \to H^{q-m}_{cdh}(X, \Omega^q_{/k})$$

must be isomorphisms for $m \le n$ and onto for $m = n+1$. □

On smooth schemes, all our functors are well-behaved. Recall from [15] that the *scdh*-topology on Sm/F is the restriction of the *cdh*-topology on Sch/F. Since every scheme is smooth locally in the *cdh*-topology by resolution of singularities, every *cdh*-cover has a smooth refinement and it follows that $\mathbb{H}_{scdh}(X, A)$ is just $\mathbb{H}_{cdh}(X, A)$ for every presheaf A. (See the argument of the first part of the proof of [4, 3.12].) If A satisfies *scdh*-descent, then $A(X) \cong \mathbb{H}_{cdh}(X, A)$ for all smooth X.

Recall that k is a subfield of F.

Theorem 2.4. *Let H denote any of $HH(/k)$, $HC(/k)$, $HN(/k)$ or $HP(/k)$, and let $H^{(i)}$ denote the ith component in the Hodge decomposition of H.*

Then H and $H^{(i)}$ satisfy scdh-descent on Sm/F. In particular, if X is smooth over F, then $H^{(i)}(X) \cong \mathbb{H}_{cdh}(X, H^{(i)})$.

Proof. By [4, Theorem 3.9 and Theorems 2.9 and 2.10], which hold for homology taken relative to k, Hochschild, cyclic, negative and periodic cyclic homology (relative to k) all satisfy *scdh*-descent on Sm/F.

By Proposition 2.1, the quasi-isomorphisms $H(X) \cong \mathbb{H}_{cdh}(X, H) = \mathbb{H}_{scdh}(X, H)$ induce quasi-isomorphisms $H^{(i)}(X) \cong \mathbb{H}^{(i)}_{cdh}(X, H) = \mathbb{H}^{(i)}_{scdh}(X, H)$ for all i. □

The special case $H^*_{Zar}(X, \mathcal{O}) \cong H^*_{cdh}(X, \mathcal{O})$ (for smooth X) of the following corollary was proven in [4, 6.3].

Corollary 2.5. *If X is smooth over F, then $H^p_{Zar}(X, \Omega^i_{/k}) \cong H^p_{cdh}(X, \Omega^i_{/k})$ for all p and i. In particular, $\Omega^i_{X/k} \cong H^0_{cdh}(X, \Omega^i_{/k})$.*

Proof. Consider the map $e^{(i)} : HH^{(i)} = (\mathbf{C}^{(i)}, b) \to \Omega^i_{/F}[i]$ of complexes of Zariski sheaves. By [22, 3.3], it is a quasi-isomorphism over every smooth scheme X over F, inducing $HH^{(i)}_{i-p}(X) \cong H^{p-i}_{Zar}(X, \mathbf{C}^{(i)}) \cong H^p_{Zar}(X, \Omega^i_{/k})$. The map $e^{(i)}$ remains a quasi-isomorphism after sheafifying for the *cdh*-topology, so that $\mathbb{H}^{p-i}_{cdh}(X, HH^{(i)}) \cong H^{p-i}_{cdh}(X, \mathbf{C}^{(i)}) \cong H^p_{cdh}(X, \Omega^i_{/k})$. By Theorem 2.4, $HH^{(i)}_n(X) \cong \mathbb{H}^{-n}_{cdh}(X, HH^{(i)})$, whence the result. □

The next result is proven by copying the proof of [4, 6.1], replacing $\mathcal{O}$ with $\Omega^i_{/k}$.

Proposition 2.6. *If X is a d-dimensional scheme, essentially of finite type over F, and $k \subseteq F$ is a subfield, then*

$$H^d_{Zar}(X, \Omega^i_{/k}) \to H^d_{cdh}(X, \Omega^i_{/k})$$

is surjective. In particular, if X is affine and $d > 0$, then $H^d_{cdh}(X, \Omega^i_{/k}) = 0$.

The following useful theorem is proven in [15, 12.1].

Theorem 2.7. *For every abstract blow-up square* ($\square$) *and for every complex of sheaves of abelian groups A, there is a long exact Mayer-Vietoris sequence:*

$$\cdots H^n_{cdh}(X,A) \to H^n_{cdh}(X',A) \oplus H^n_{cdh}(Y,A) \to H^n_{cdh}(Y',A) \to H^{n+1}_{cdh}(X,A) \cdots .$$

Consider the change-of-topology morphism $a : (\mathrm{Sch}/F)_{cdh} \to (\mathrm{Sch}/F)_{Zar}$.

Lemma 2.8. *If a Zariski sheaf M on* Sch/F *is a quasi-coherent sheaf (resp., coherent sheaf) on each X_{Zar} and if M satisfies scdh-descent on* Sm/F*, then the cohomology sheaves $R^q a_*(a^*M)$ are also quasi-coherent (resp., coherent) on each X_{Zar}.*

If $X = \mathrm{Spec}(R)$ is affine, then $R^q a_*(a^*M)$ is the quasi-coherent sheaf associated to the R-module $H^q_{cdh}(X,M)$, and the natural map $M(X) \to H^0_{cdh}(X,M)$ is R-linear.

Proof. We proceed by induction on $\dim X$, the case $\dim(X) = 0$ being clear. Using resolution of singularities, pick a smooth abstract blow-up X' of X, and form the abstract blow-up square ($\square$). By Theorem 2.7, we get a triangle on X_{Zar}: $Ra_*(a^*M)|_X \to Ra_*(a^*M)|_{X' \coprod Y} \to Ra_*(a^*M)|_{Y'}$. As the latter two terms have quasi-coherent (resp., coherent) cohomology sheaves, by induction and *scdh*-descent on X', so does the first.

If X is affine, then $H^p_{Zar}(X, R^q a_* M) = 0$ for $p > 0$. Hence the Leray spectral sequence collapses to yield $H^q_{cdh}(X,M) = H^0_{Zar}(X, R^q a_*(a^*M))$. $\square$

Corollary 2.9. *Suppose that $X = \mathrm{Spec}(R)$ is affine. Then $U \mapsto \pi_n \mathcal{F}_{HH(/k)}(U)$ and $U \mapsto \pi_n \mathbb{H}_{cdh}(U, HH(/k))$ are quasi-coherent Zariski sheaves on X for all n.*

3. A criterion for smoothness

In this section we present a local criterion for smoothness of schemes over a field F, in terms of the Hochschild homology and *cdh*-fibrant Hochschild homology of their local rings over F (see Theorem 3.1). As an application we prove Vorst's conjecture for algebras of finite type over $\mathbb{Q}$ and their localizations at maximal ideals (see Theorem 3.3).

A stronger global version of the following result shall be proved in Section 4 below (see Theorem 4.11).

Recall that F is a field of characteristic 0.

Theorem 3.1. *Let R be the local ring of a d-dimensional F-algebra of finite type at a maximal ideal. If $HH(/F)$ satisfies d-cdh-descent on R, then R is smooth over F.*

Proof. Recall that $\Omega^\bullet_{/F}$ denotes the de Rham complex, whose terms are the Zariski sheaves $\Omega^i_{/F}$, while $\Omega^{\leq i}_{/F}$ denotes its brutal truncation in degrees at most i. By Theorems 2.2 and 1.2, we have isomorphisms

$$HP^{(j)}_n(X/F) \xrightarrow{\cong} \mathbb{H}^{2j-n}_{cdh}(X, \Omega^\bullet_{/F})$$

for any $X \in \mathrm{Sch}/F$ and all n and j. Moreover, by the proof of Theorem 2.2, this isomorphism factors through a natural map $e : HP^{(j)}_n(X/F) \to \mathbb{H}^{2j-n}_{Zar}(X, \Omega^\bullet_{/F})$. Now specialize to the case $X = \mathrm{Spec}\, R$, where R is as in the theorem. By [15, Proposition 5.9 and Lemma 12.4] and resolution of singularities, every *cdh*-cover of X has

a refinement consisting of (essentially) smooth affine F-schemes of transcendence degree at most d over F. Hence we have

$$\mathbb{H}^*_{cdh}(X, \Omega^\bullet_{/F}) = \mathbb{H}^*_{cdh}(X, \Omega^{\leq d}_{/F}).$$

Moreover, Lemma 2.3 implies that the hypercohomology spectral sequence for $\mathbb{H}^*_{cdh}$ degenerates to yield an isomorphism

$$H^*(\Omega^{\leq d}_{R/F}, d) \to \mathbb{H}^*_{cdh}(X, \Omega^{\leq d}_{/F}).$$

The canonical map $S : HP_n^{(j)}(R/F) \to HC_{n-2}^{(j-1)}(R/F)$ fits into the commutative diagram

$$\begin{tikzcd}
HP_{d+2}^{(d+1)}(R/F) \arrow[r, "e"'] \arrow[d, "S"'] & H^d_{dR}(R/F) \arrow[r] \arrow[d, hook] & \mathbb{H}^d_{cdh}(X, \Omega^\bullet_{/F}) \arrow[d, "\wr"] \\
HC_d^{(d)}(R/F) \arrow[r, "\cong", "e"'] & \Omega^d_{R/F}/d\Omega^{d-1}_{R/F} \arrow[r, "\cong"] & \mathbb{H}^d_{cdh}(X, \Omega^{\leq d}_{/F}).
\end{tikzcd}$$

We have seen that the top composite, the right vertical and both bottom arrows are isomorphisms. It follows that the middle vertical inclusion is the identity map, i.e., that $d\Omega^d_{R/F} = 0$. On the other hand, $d\Omega^d_{R/F}$ generates $\Omega^{d+1}_{R/F}$ as an R-module; therefore we can infer that $\Omega^{d+1}_{R/F} = 0$. By Lemma 3.2 below, R is regular, and hence smooth over F. □

Lemma 3.2. *Let F be any perfect field. Suppose R is the local ring of a d-dimensional F-algebra of finite type at a maximal ideal. If $\Omega^{d+1}_{R/F} = 0$, then R is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Since $L := R/\mathfrak{m}$ is smooth over F, the Second Fundamental Theorem [20, 9.3.5] shows that there is an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to L \otimes_R \Omega^1_{R/F}$ sending x to dx. Consequently, there is a surjection from $\Omega^{d+1}_{R/F}$ onto $\Lambda^{d+1}_L(\mathfrak{m}/\mathfrak{m}^2)$, which is a nonzero vector space unless R is regular. □

As an application, we can now verify Vorst's conjecture for algebras of finite type over $\mathbb{Q}$ and their localizations at maximal ideals.

Theorem 3.3. *Let R be a d-dimensional $\mathbb{Q}$-algebra which is either of finite type over $\mathbb{Q}$ or a localization of a $\mathbb{Q}$-algebra of finite type at a maximal ideal.*

If R is K_{d+1}-regular, then R is regular.

Proof. First assume R is of finite type over $\mathbb{Q}$ and K_{d+1}-regular. To prove R is regular, we may replace R by its localization at a maximal ideal; these local rings are also K_{d+1}-regular, by Vorst's localization theorem (see [17, 1.9]; also see [16, 9.12] as to why the condition that R be reduced is not needed in the localization theorem). Thus we are reduced to proving the theorem in the local case.

As remarked in the introduction, if R is K_{d+1}-regular, then $\mathcal{F}_K(R)$ is $(d+1)$-connected (see [19]). By Corollary 1.7, $\mathcal{F}_{HH(/\mathbb{Q})}(R)$ is d-connected. Now Theorem 3.1 applies to prove that R is smooth over $\mathbb{Q}$ and hence regular. □

4. Vorst's conjecture

In this section we will prove Theorem 0.1. Throughout, F will be a fixed field of characteristic zero, $k \subseteq F$ a subfield, R an F-algebra essentially of finite type, and $X = \mathrm{Spec}(R)$. Note that we write HH for $HH(/\mathbb{Q})$.

Lemma 4.1 (Kassel-Sletsjœ, [11, 4.3a])**.** *Let $k \subseteq F$ and let $p \geq 1$ be fixed. Then there is a bounded second quadrant homological spectral sequence ($0 \leq i < p$, $j \geq 0$):*

$$ {}_pE^1_{-i,i+j} = \Omega^i_{F/k} \otimes_F HH^{(p-i)}_{p-i+j}(R/F) \Rightarrow HH^{(p)}_{p+j}(R/k). $$

Lemma 4.2. *Let $k \subseteq F$ and let $p \geq 1$ be fixed. Then there is a spectral sequence:*

$$ {}^pE_1^{i,j} = \Omega^i_{F/k} \otimes_F H^{i+j}_{cdh}(X, \Omega^{p-i}_{/F}) \Rightarrow H^{i+j}_{cdh}(X, \Omega^p_{/k}). $$

Proof. Consider the sheaf of ideals $I := \ker(\Omega^*_{/k} \to \Omega^*_{/F})$. The I-adic filtration of $\Omega^*_{/k}$ induces a filtration $\mathcal{G} = \mathcal{G}(p)$ on $\Omega^p_{/k}$. If R is any F-algebra essentially of finite type, we have a natural surjection

$$ \Omega^i_{F/k} \otimes_F \Omega^{p-i}_{R/F} \twoheadrightarrow \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R), \tag{4.3} $$

which is an isomorphism if R is smooth. To see this, consider the fundamental sequence $\Omega^1_{F/k} \otimes R \to \Omega^1_{R/k} \to \Omega^1_{R/F} \to 0$. Note that $\Omega^n_{R/k}$ receives a map from $\Omega^i_{F/k} \otimes \Omega^{p-i}_{R/k}$ and that the image of this map is $\mathcal{G}^i\Omega^n_{R/k}$. The surjection $\Omega^i_{F/k} \otimes \Omega^{n-i}_{R/F} \to \mathcal{G}^i\Omega^n_{R/k}$ is clear from this description. Note also that this implies that $\mathcal{G}^{p+1} = 0$. If R/F is smooth, then the fundamental sequence splits, and a choice of splitting induces a direct sum decomposition which in turn induces $\Omega^n_{R/k} = \bigoplus_i \Omega^i_{F/k} \otimes \Omega^{n-i}_{R/F}$. By construction, this gives a filtered isomorphism between $\Omega^n_{R/k}$ and the graded module associated to the filtration $\mathcal{G}$. This proves that the maps (4.3) are isomorphisms in this case.

Thus the *cdh*-sheafification of (4.3) is an isomorphism. Since $\Omega^i_{F/k}$ is a vector space, the spectral sequence of the lemma is the one associated to the corresponding filtration of the *cdh*-sheaf $a^*\Omega^p_{/k}$. □

Lemma 4.4. *Let $X = \mathrm{Spec}(R)$ be affine, and fix $n \geq 0$. Assume that*

$$ \Omega^q_{R/k} \to H^0_{cdh}(X, \Omega^q_{/k}) \quad \textit{is bijective if } q \leq n \textit{ and onto if } q = n+1, \tag{4.4a} $$

$$ H^1_{cdh}(X, \Omega^q_{/F}) = 0 \textit{ if } q \leq n+1. \tag{4.4b} $$

Then $\Omega^q_{R/F} \to H^0_{cdh}(X, \Omega^q_{/F})$ is bijective if $q \leq n$ and onto if $q = n+1$.

Proof. By induction on q. If $q = 0$, there is nothing to prove. Fix $q > 0$, and consider the filtration $\mathcal{G}^i = \mathcal{G}^i(q)$, $0 \leq i \leq q$, considered in the proof of Lemma 4.2. We have a commutative diagram

$$ \begin{array}{ccc} \Omega^i_{F/k} \otimes_F \Omega^{q-i}_{R/F} & \longrightarrow & \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R) \\ \downarrow & & \downarrow \\ \Omega^i_{F/k} \otimes_F H^0_{cdh}(X, \Omega^{q-i}_{/F}) & \longrightarrow & H^0_{cdh}(X, \mathcal{G}^i/\mathcal{G}^{i+1}). \end{array} \tag{4.5} $$

The top arrow is surjective for all i and an isomorphism for $i = 0$. The bottom arrow is an isomorphism by the proof of Lemma 4.2. By the inductive hypothesis, the left vertical arrow is an isomorphism for $0 < i$. It follows that the top arrow is

an isomorphism for all $0 \le i \le q$ and that the arrow on the right is an isomorphism for $i > 0$. By (4.4b) we have an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0_{cdh}(X, \mathcal{G}^{i+1}) \arrow[r] & H^0_{cdh}(X, \mathcal{G}^i) \arrow[r] & \Omega^i_{F/k} \otimes_F H^0_{cdh}(X, \Omega^{q-i}_{/F}) \arrow[d] \\ & 0 & H^1_{cdh}(X, \mathcal{G}^i) \arrow[l] & H^1_{cdh}(X, \mathcal{G}^{i+1}) \arrow[l] \end{tikzcd}$$

Since $\mathcal{G}^{q+1} = 0$, we deduce, by descending induction on i, that for all $i > 0$,

$$H^1_{cdh}(X, \mathcal{G}^i) = 0. \tag{4.6}$$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^{i+1}(R) \arrow[r] \arrow[d] & \mathcal{G}^i(R) \arrow[r] \arrow[d] & \Omega^i_{F/k} \otimes \Omega^{q-i}_{R/F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0_{cdh}(X, \mathcal{G}^{i+1}) \arrow[r] & H^0_{cdh}(X, \mathcal{G}^i) \arrow[r] & \Omega^i_{F/k} \otimes H^0_{cdh}(X, \Omega^{q-i}_{/F}) \arrow[r] & 0. \end{tikzcd}$$

Using descending induction on i again, we obtain from this diagram that

$$\mathcal{G}^i(R) \cong H^0_{cdh}(X, \mathcal{G}^i) \qquad (i > 0). \tag{4.7}$$

We have a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^1(R) \arrow[r] \arrow[d] & \Omega^q_{R/k} \arrow[r] \arrow[d] & \Omega^q_{R/F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0_{cdh}(X, \mathcal{G}^1) \arrow[r] & H^0_{cdh}(X, \Omega^q_{/k}) \arrow[r] & H^0_{cdh}(X, \Omega^q_{/F}) \arrow[r] & 0. \end{tikzcd}$$

The third map in the bottom row is onto by (4.6). The first vertical map is an isomorphism by (4.7). By (4.4a), the second is an isomorphism if $q \le n$ and onto if $q = n + 1$. It follows that the same is true of the third vertical map. □

Proposition 4.8. *Assume $n \ge 0$. If $HH(/k)$ satisfies n-cdh-descent on R, then so does $HH(/F)$.*

Proof. By Lemma 2.3, the hypothesis is equivalent to saying that the following conditions hold simultaneously:

$$HH^{(q)}_m(R/k) = 0 \quad \text{if } 0 \le q < m \le n, \tag{4.8a}$$

$$\Omega^q_{R/k} \to H^0_{cdh}(X, \Omega^q_{/k}) \quad \text{is bijective if } q \le n \text{ and onto if } q = n + 1, \tag{4.8b}$$

$$H^p_{cdh}(X, \Omega^q_{/k}) = 0 \quad \text{if } p > 0 \text{ and } q \le p + n + 1. \tag{4.8c}$$

We have to prove that the following conditions hold:

$$HH^{(q)}_m(R/F) = 0 \quad \text{if } 0 \le q < m \le n, \tag{4.9a}$$

$$\Omega^q_{R/F} \to H^0_{cdh}(X, \Omega^q_{/F}) \quad \text{is bijective if } q \le n \text{ and onto if } q = n + 1, \tag{4.9b}$$

$$H^p_{cdh}(X, \Omega^q_{/F}) = 0 \quad \text{if } p > 0 \text{ and } q \le p + n + 1. \tag{4.9c}$$

Using (4.8c), the spectral sequence of Lemma 4.2, and induction, we obtain (4.9c). Hence (4.8b) implies (4.9b), by Lemma 4.4. To prove (4.9a), we proceed by induction on q. The case $q = 0$ is just the fact that $HH^{(0)}_m(A/k) = 0$ for any $m > 0$, any

field k and any k-algebra A. Assume $n \geq q \geq 1$ and that we have $HH_m^{(q')}(R/F) = 0$ for all $m \leq n$ and all $q' < \min\{m, q\}$. By (4.9b) and (4.5), the spectral sequence of Lemma 4.1 collapses for $j = 0$ to yield

$$_qE^{\infty}_{i,-i} = {}_qE^{1}_{i,-i}.$$

Given this, (4.9a) follows from (4.8a) by induction. □

Lemma 4.10. *Let F be a field and R a local F-algebra essentially of finite type. Then there exists a field $F \subset E \subset R$ such that R is isomorphic to the localization of a finite type algebra over E at a maximal ideal.*

Proof. The hypothesis on R means that there exist an F-algebra A of finite type and a prime ideal $P \subset A$ such that $R = A_P$. Suppose that $\dim(A) = k + h$ and $ht(P) = h$. By Noether normalization, there is a polynomial subring $S = F[t_1, \ldots, t_k]$ of A meeting P in 0, and the field $R/P = A_P/PA_P$ is a finite extension of $E = F(t_1, \ldots, t_k)$. There is an evident inclusion of $E = S_{P\cap S}$ into $R = A_P$, and R is the localization of the finite type E-algebra $A \otimes_S E$ at a maximal ideal. □

The results of this section, together with Theorem 3.1, allow us to prove the following global regularity criterion.

Theorem 4.11. *Let $k \subseteq F$ be fields of characteristic 0, and let R be an F-algebra essentially of finite type.*

If $HH(/k)$ satisfies h-cdh-descent on R, then R is smooth in codimension h. That is, for every prime ideal P of height h, the local ring R_P is regular.

Proof. Let P be a prime ideal of R of height h. Since $HH(/F)$ satisfies h-cdh-descent on R by Proposition 4.8, it also satisfies h-cdh-descent on the localization R_P, by Corollary 2.9. By Lemma 4.10, there is a field $F \subset E \subset R_P$ such that R_P is the localization at a maximal ideal of an algebra of finite type over E. By Proposition 4.8, $HH(/E)$ satisfies h-cdh-descent on R_P. Because $\dim(R_P) = h$, Theorem 3.1 implies that R_P is smooth over E and hence is regular. □

Once again, let F be a field of characteristic 0.

Theorem 4.12. *Suppose R is an F-algebra essentially of finite type. If R is K_{h+1}-regular for some $h \geq 0$, then R is regular in codimension h. In particular, if R is $K_{\dim(R)+1}$-regular, then R is regular and hence smooth over F.*

Proof. If R is K_{h+1}-regular, then $HH(/\mathbb{Q})$ satisfies h-descent on R, by Corollary 1.7. The assertion now follows from Theorem 4.11. □

5. A nonreduced scheme which is K-regular

This section is devoted to the counterexample stated in Theorem 0.2, which reappears here as Theorem 5.2.

Lemma 5.1. *Let X be a smooth projective elliptic curve over a field F with basepoint Q, and let L be a degree zero line bundle L on X. Assume that L is not an element of odd order in the Picard group. Then $H^*(X, L^{\otimes 2n+1}) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Because $\mathcal{O}_X$ is a dualizing sheaf, we are reduced by Serre duality to proving that $H^0(X, L^{2n+1}) = 0$. Because X is elliptic, there exists a rational point $P \in X$ such that $L := L(P - Q)$. Now if $H^0(X, L^{\otimes 2n+1})$ were nonzero, there would exist

an element f in the function field of X with $\mathrm{div}(f) = (2n+1)(P-Q)$. But because L is not an odd torsion element, there is no such f. □

Theorem 5.2. *Let X be a smooth projective elliptic curve over a field F of characteristic 0, and let L be as in Lemma 5.1. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L$, where L is regarded as a square-zero ideal; that is, the product $(f,s)\cdot(g,t) = (fg, ft+gs)$ for sections f and g of $\mathcal{O}_X$ and s and t of L.*

Then for all n, $K_n(Y) = K_n(X)$ and Y is K_n-regular.

Proof. As X is regular, and hence K_n-regular, it suffices to show that $K(Y\times\mathbb{A}^m) \to K(X\times\mathbb{A}^m) \cong K(X)$ is an equivalence for all $m \geq 0$. We shall prove the equivalent assertion that the relative homotopy groups $K_n(Y\times\mathbb{A}^m, X\times\mathbb{A}^m)$ are zero. By Goodwillie's theorem [8] and Zariski descent, these relative K-groups are isomorphic to the corresponding relative cyclic homology groups over $\mathbb{Q}$. By base-change (see [10, Example (3.3)]) it suffices to show that the relative groups $HH_n^{rel} = HH_n(Y,X)$ and $HC_n(Y,X)$ vanish for all n. Using the SBI sequence for the relative groups and the fact that $HC_n(Y,X)$ vanishes for $n << 0$, we see that it suffices to show that $HH_n^{rel} = 0$ for all n. There is a Zariski descent spectral sequence whose E_2-term consists of the groups $H^0(X, HH_n^{rel})$ and $H^1(X, HH_n^{rel})$ (note that the cohomological dimension of X is 1); therefore, it suffices to prove that these vanish for all n. From Lemma 5.3 below and the fact that $\Omega^1_{X/F} \cong \mathcal{O}_X$, we see that the Zariski sheaves HH_n^{rel} are sums of odd tensor powers of L when F is a number field, and odd tensor powers of L tensored over F with vector spaces $\Omega^i_{F/\mathbb{Q}}$ in general. But the cohomology of such powers vanishes by Lemma 5.1. □

The following lemma is well-known, at least in the case when L is free. We include a proof for the sake of completeness. For simplicity, we write $HH_*(R)$ for $HH_*(R/k)$.

Lemma 5.3. *Let k be a field with $\mathrm{char}(k) \neq 2$, R a commutative k-algebra, and L a projective R-module of rank 1. Put $A = R \oplus L$, with L considered as square-zero ideal. Let M_* denote the graded R-module*

$$M_p = \begin{cases} L^{\otimes_R (p+1)}, & p \geq 0 \text{ even,} \\ L^{\otimes_R p}, & p > 0 \text{ odd.} \end{cases}$$

Then for relative Hochschild homology over k,

$$HH_n(A,L) = \bigoplus_{p+q=n} M_p \otimes_R HH_q(R).$$

Proof. Let $C_*(A,L)$ be the relative Hochschild complex; the subspace $L^{\otimes_k 2m+1}$ of $C_{2m}(A,L)$ consists of cycles and induces a map $M_{2m} = L^{\otimes_R 2m+1} \to HH_{2m}(A,L)$, because for $x_i \in L$ and $r \in R$ we have

$$(-1)^i b(x_0 \otimes \cdots x_i \otimes r \otimes x_{i+1} \cdots) = (x_0 \otimes \cdots x_i r \otimes x_{i+1} \cdots) - (x_0 \otimes \cdots x_i \otimes r x_{i+1} \cdots).$$

Because $Bb + bB = 0$, where $B : C_*(A,L) \to C_{*+1}(A,L)$ is the Connes operator, the subspace $B(L^{\otimes_k 2m+1})$ of $C_{2m+1}(A,L)$ also consists of cycles and induces a map $M_{2m+1} = L^{\otimes_R 2m+1} \to HH_{2m}(A,L)$. Thus we have a graded map $M_* \to HH_{2m}(A,L)$. Because $HH_*(A,L)$ is a graded module over $HH_*(R)$, we get a canonical R-module map from $M_* \otimes_R HH_*(R)$ to $HH_*(A,L)$. To see that it is an isomorphism, we may assume R is local, whence $A = R[x]/\langle x^2\rangle$. By the Künneth

formula, we are reduced to the case $R = k$, which is straightforward (see [12, 1.1.6] for the necessary computation). □

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References

[1] M. Artin, A. Grothendieck, and J. L. Verdier. *Théorie des topos et cohomologie étale des schémas. Tome 2*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.

[2] H. Bass and M. P. Murthy. Grothendieck groups and Picard groups of abelian group rings. *Annals of Math.*, 86:16–73, 1967. MR0219592 (36:2671)

[3] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978. MR513569 (80e:55021)

[4] G. Cortiñas, C. Haesemeyer, M. Schlichting and C. A. Weibel. Cyclic homology, *cdh*-cohomology and negative K-theory. To appear in Annals of Math. *Preprint*. Available at http://www.math.uiuc.edu/K-theory/0722/, 2005.

[5] J. Cuntz, D. Quillen. Excision in bivariant periodic cyclic homology. *Invent. Math.* 127:67-98, 1997. MR1423026 (98g:19003)

[6] B. Dayton and C. Weibel. K-theory of hyperplanes. *Trans. AMS* 257:119-141, 1980. MR549158 (81e:18015)

[7] Thomas G. Goodwillie. Cyclic homology, derivations, and the free loop space. *Topology*, 24(2):187–215, 1985. MR793184 (87c:18009)

[8] Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math. (2)*, 124(2):347–402, 1986. MR855300 (88b:18008)

[9] Christian Haesemeyer. Descent properties of homotopy K-theory. *Duke Math. J.*, 125:589–620, 2004. MR2166754 (2006g:19002)

[10] Christian Kassel. Cyclic homology, comodules and mixed complexes. J. of Algebra, 107:195–216, 1987. MR883882 (88k:18019)

[11] C. Kassel, A. B. Sletsjœ. Base change, transitivity and Künneth formulas for the Quillen decomposition of Hochschild homology. *Math. Scand.*, 70:186–192, 1992. MR1189973 (93i:16017)

[12] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco. MR1217970 (94a:19004)

[13] Carlo Mazza, Vladimir Voevodsky and Charles Weibel. *Lecture notes on motivic cohomology*. Clay Mathematics Monographs, 2. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR2242284 (2007e:14035)

[14] Y. Nisnevich. The completely decomposed topology, pp. 241–341 in NATO ASI 279, Kluwer, 1989. MR1045853 (91c:19004)

[15] Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000. MR1744945 (2001g:14031)

[16] R. W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Volume III*, volume 88 of *Progress in Math.*, pages 247–436. Birkhäuser, Boston, Basél, Berlin, 1990. MR1106918 (92f:19001)

[17] Ton Vorst. Localization of the K-theory of polynomial extensions. *Math. Ann.*, 244:33–54, 1979. MR550060 (80k:18016)

[18] Ton Vorst. Polynomial extensions and excision for K_1 *Math. Ann.*, 244:193–204, 1979. MR553251 (82a:13002)

[19] Charles Weibel. Homotopy algebraic K-theory *AMS Contemp Math.*, 83:461–488, 1989. MR991991 (90d:18006)

[20] Charles Weibel. *An introduction to homological algebra.* Cambridge Univ. Press, 1994. MR1269324 (95f:18001)

[21] Charles Weibel. Cyclic homology for schemes. *Proc. AMS*, 124:1655–1662, 1996 MR1277141 (96h:19003)

[22] Charles Weibel. The Hodge filtration and cyclic homology. *K-theory* 12:145–164, 1997. MR1469140 (98h:19004)

[23] Charles Weibel. The negative K-theory of normal surfaces. *Duke Math. J.*, 108:1–35, 2001. MR1831819 (2002b:14012)

Departamento Matemática, FCEyN-Universidad de Buenos Aires, Ciudad Universitaria Pab 1, 1428 Buenos Aires, Argentina, and Departamento Álgebra, Faculdad de Ciencias, Prado de la Magdalena s/n, 47005 Valladolid, Spain

E-mail address: gcorti@agt.uva.es

URL: http://mate.dm.uba.ar/~gcorti

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Current address: Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 SEO, 851 South Morgan Street, Chicago, Illinois 60607-7045

E-mail address: chh@math.uiuc.edu

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08901

E-mail address: weibel@math.rutgers.edu