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POTENTIALLY SEMI-STABLE DEFORMATION RINGS

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Contents

Introduction

Let $K/\mathbb{Q}_p$ be a finite extension and $G_K = \mathrm{Gal}(\bar{K}/K)$ the Galois group of an algebraic closure $\bar{K}$. Let $\mathbb{F}$ be a finite field of characteristic p, and $V_{\mathbb{F}}$ a finite dimensional $\mathbb{F}$-vector space equipped with a continuous action of G_K. The study of the deformation theory of Galois representations was initiated by Mazur [Ma], who showed that if $V_{\mathbb{F}}$ has no non-trivial endomorphisms, then it admits a universal deformation ring $R_{V_{\mathbb{F}}}$. After the work of Wiles [Wi] and the conjectures of Fontaine-Mazur [FM] it became clear that for arithmetic applications it was important to understand certain quotients of $R_{V_{\mathbb{F}}}$ corresponding to deformations satisfying certain conditions. For example Wiles uses deformations which arise from finite flat group schemes, and the corresponding quotient of $R_{V_{\mathbb{F}}}$ was constructed by Ramakrishna [Ra].

Suppose that L/K is a finite extension and let $a \leqslant b$ be integers. It seems to be a kind of folklore conjecture that there should be a quotient of $R_{V_{\mathbb{F}}}$ whose points in finite extensions of $W(\mathbb{F})[1/p]$ correspond to deformations of $V_{\mathbb{F}}$ which become semi-stable over L and have Hodge-Tate weights in the interval $[a, b]$. This is closely related to the final conjecture of [Fo 3]. Special cases of this, when $V_{\mathbb{F}}$ is 2-dimensional, are conjectured in the papers of Fontaine-Mazur [FM, p.191], Breuil-Conrad-Diamond-Taylor [BCDT, Conj. 1.1.1] and Breuil-Mézard [BM, 2.2.2.4].

The purpose of this paper is to prove such a result.

Theorem. *There is a quotient $R_{V_{\mathbb{F}}}^{[a,b],L}$ of $R_{V_{\mathbb{F}}}$ such that for any finite $W(\mathbb{F})[1/p]$-algebra B, a map $x : R_{V_{\mathbb{F}}} \to B$ factors through $R_{V_{\mathbb{F}}}^{[a,b],L}$ if and only if the induced*

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B-representation of G_K, V_x is semi-stable when restricted to L, and has Hodge-Tate weights in $[a, b]$.

In fact we prove a refinement of this theorem involving V_x of fixed p-adic Hodge type and of Galois type. Such a refinement is crucial for applications to the Fontaine-Mazur conjecture. The former invariant describes the precise shape of the Hodge filtration on the weakly admissible module attached to V_x, while the latter gives the action of the inertia subgroup of $\mathrm{Gal}(L/K)$ (when L is Galois over K) on this module. We also show that $R^{[a,b],L}_{V_{\mathbb{F}}}[1/p]$ is equidimensional and generically formally smooth, and we give a simple formula for its dimension. This is enough to prove Conjecture 2.2.2.4 of [BM].[1] This result should be regarded as the "easy" part of the Breuil-Mézard conjecture, the main point of [BM] being to predict the Hilbert-Samuel multipicities of these rings. The "hard" part is established in [Ki 3] (in many cases) and is used to prove a significant part of the Fontaine-Mazur conjecture for 2-dimensional, odd representations of the absolute Galois group of $\mathbb{Q}$. This application was our main motivation in proving the theorem above. Another application is to the Galois representations associated to Hilbert modular forms.

Corollary. *Let F be a totally real field and π a Hilbert modular eigenform of weight $\underline{k} = (k_1, \ldots, k_n)$ with $k_i \geq 2$, integers all having the same parity. Let ρ_π denote the 2-dimensional p-adic Galois representation of the absolute Galois group G_F of F attached to π, and suppose that the resulting mod p representation $\bar{\rho}_\pi$ is absolutely irreducible.*

If $v|p$ is a prime of F, and $G_{F_v} \subset G_F$ denotes a decomposition group at v, then $\rho_\pi|_{G_{F_v}}$ is potentially semi-stable with p-adic Hodge type corresponding to the weight $\underline{k}$. Moreover the Weil-Deligne group representation attached to $\rho_\pi|_{G_{F_v}}$ via Fontaine's construction corresponds to the local factor π_v of π via the local Langlands correspondence.

The condition on the p-adic Hodge type means the following: If ρ_π is defined over a finite extension $E/\mathbb{Q}_p$, and w denotes the largest of the integers k_i, then for $v|p$ there is a graded $F_v \otimes_{\mathbb{Q}_p} E$-module D_v attached to the Hodge-Tate representation $\rho_\pi|_{G_{F_v}}$. If $\sigma : E \to \bar{F}_v$ is an embedding into a fixed algebraic closure of F_v, then $D_\sigma = D \otimes_{F_v \otimes_{\mathbb{Q}_p} E, \sigma} \bar{F}_v$ is non-zero in degrees $(w - k_\sigma)/2$ and $(w + k_\sigma - 2)/2$, where k_σ is one of the integers k_i. The number of σ for which a given integer equals k_σ is proportional to the number of times it appears in $\underline{k}$.

When $[F : \mathbb{Q}]$ is odd or when $[F : \mathbb{Q}]$ is even and π is a discrete series at some finite place w, the corollary follows from work of Carayol [Ca, Thm. A], who constructed the representations ρ_π in the cohomology of Shimura curves, together with a result of Saito [Sa 2]. When $[F : \mathbb{Q}]$ is even, Taylor [Ta 1] constructed ρ_π by interpolating representations $\rho_{\pi'}$ where π' is special at some $w \nmid p$, and so the corollary follows from the theorem (or more precisely Theorem (2.7.6) below) and the cases established by Carayol. Some cases of the corollary (corresponding to the known cases of the theorem) were previously known, even without the restriction on $\bar{\rho}_\pi$ (see [Ta 2], [Br]). The statement that ρ_π is potentially semi-stable was known in most

[1] As far as it is true. Part (ii) of the conjecture, which asserts that $R^{[a,b],L}_{V_{\mathbb{F}}}[1/p]$ is formally smooth, is not quite right, as can already be seen for ordinary representations with Hodge-Tate weights 0 and 1. The corresponding deformation ring has two components meeting along a codimension 1 subspace which parameterizes representations which are Barsotti-Tate extensions of χ by $\chi(1)$ for χ an unramified character.

cases thanks to the work of Blasius-Rogowski [BR] who gave a motivic construction of ρ_π, however at least the statement that the Weil-Deligne group representation attached to ρ_π is compatible with the local Langlands correspondence does not seem to follow easily from their work.

Let us say a word about what part of the theorem was previously known. It will be useful to distinguish three flavors of the statement that some property $\mathbf{P}$ (e.g. being crystalline, semi-stable, etc.) of p-adic Galois representations cuts out a closed subspace of the generic fiber of $\operatorname{Spec} R_{V_{\mathbb{F}}}$. For a finite extension $E/\mathbb{Q}_p$ and a map $x : R_{V_{\mathbb{F}}} \to E$, we will denote by V_x the corresponding E-representation of G_K:

(1) There is a closed subspace of the p-adic *analytic space* attached to $R_{V_{\mathbb{F}}}[1/p]$ [deJ, §7] such that V_x has $\mathbf{P}$ if and only if $\ker x$ lies on this subspace.
(2) There is a quotient $R^{\mathbf{P}}_{V_{\mathbb{F}}}$ of $R_{V_{\mathbb{F}}}$ such that $R_{V_{\mathbb{F}}} \to E$ factors through $R^{\mathbf{P}}_{V_{\mathbb{F}}}$ if and only if V_x has $\mathbf{P}$.
(3) Let V be a finite dimensional E-representation of G_K, and $L \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. Suppose that for $i \geq 1$ L/p^iL is a subquotient of a lattice in a representation having $\mathbf{P}$. Then V has $\mathbf{P}$.

It is not hard to see that we have the implications (3) $\implies$ (2) $\implies$ (1). The statement (3) when $\mathbf{P}$ is coming from a finite flat group scheme is what is proved by Ramakrishna. When K is unramified over $\mathbb{Q}_p$ and $\mathbf{P}$ is being crystalline and having Hodge-Tate weights in a fixed interval $[a, b]$, (3) was established by Breuil [Br] when $b - a \leqslant p - 1$, and by Berger in general [Be]. When $\mathbf{P}$ is being semi-stable when restricted to an extension L and having Hodge-Tate weights in $[a, b]$, (3) is what is conjectured in [Fo 3]. For this same condition $\mathbf{P}$ (1) is a result of Berger-Colmez [BC]. We do not know how to prove (3) for this $\mathbf{P}$.[2] Fortunately (2) is sufficient for applications to modularity theorems (whereas (1) is not). More precisely, for these applications one needs (2) with $\mathbf{P}$ being potentially semi-stable of some fixed Galois type. It is not clear how to formulate (3) for this $\mathbf{P}$ since it is a condition which only makes sense for representations of some fixed rank.

The key ingredient in the proof of the theorem is the theory developed in our previous paper [Ki 2], in which we gave a new classification of semi-stable G_K-representations. In particular, we showed that there is a close relationship between semi-stable representations and representations of finite E-height. This is a condition which depends only on the restriction of the representation to $K_\infty = K(\sqrt[p^n]{\pi})_{n\geq 1}$ where π is a fixed uniformiser of K. A semi-stable representation has finite E-height, but a representation of $\operatorname{Gal}(\bar{K}/K_\infty)$ of finite E-height does not, in general, give rise to a semi-stable representation. This is true, however, if the E-height is $\leqslant 1$, in which case the representation has Hodge-Tate weights 0 and 1.

In the first section of the paper we build a projective $\operatorname{Spec} R_{V_{\mathbb{F}}}$-scheme $\mathscr{L}^{\leqslant h}$ whose scheme theoretic image $\operatorname{Spec} R^{\leqslant h}_{V_{\mathbb{F}}}$ parameterizes representations of E-height $\leqslant h$. This is a construction similar to that used in [Ki 1] to analyze Barsotti-Tate deformation rings using moduli of finite flat group schemes. Although the quotient $R^{\leqslant h}_{V_{\mathbb{F}}}$ is too large unless the Hodge-Tate weights are $0, 1$, it already carries a vector bundle $\underline{D}$ which is a candidate for a family of weakly admissible modules. In §2 we construct a quotient of $R^{\leqslant h}_{V_{\mathbb{F}}}$ corresponding to semi-stable deformations with Hodge-Tate weights in $[0, h]$, and we use this to show the existence of deformation

[2] After the writing of this paper was completed T. Liu was able to prove Fontaine's conjecture [L] using the theory of [Ki 2].

rings for potentially semi-stable deformations of fixed Galois type. Finally in §3 we analyze the local structure of $\operatorname{Spec} R^{[a,b],L}_{V_{\mathbb{F}}}[1/p]$ using the deformation theory of filtered φ, N-modules. In particular we establish a simple formula for its dimension, which is needed in applications to modularity.

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§1. Representations of finite E-height

(1.1) Let k be a finite field of characteristic $p > 0$, let $W = W(k)$ be its ring of Witt vectors, let $K_0 = W(k)[1/p]$, and let K/K_0 be a finite totally ramified extension of degree e. Fix an algebraic closure $\bar{K}$ of K, and write $G_K = \operatorname{Gal}(\bar{K}/K)$. We begin by recalling the definitions of some p-adic period rings.

Let $\mathcal{O}_{\bar{K}}$ denote the ring of integers of $\bar{K}$. Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ where the transition maps are given by Frobenius. There is a unique surjective map $\theta : W(R) \to \widehat{\mathcal{O}_{\bar{K}}}$ to the p-adic completion $\widehat{\mathcal{O}_{\bar{K}}}$ of $\mathcal{O}_{\bar{K}}$, which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit.

Denote by $\mathfrak{S}$ the power series ring $W[\![u]\!]$ in a variable u. We equip $\mathfrak{S}$ with a Frobenius endomorphism φ which sends u to u^p and acts as the usual Frobenius on W. Fix a uniformiser $\pi \in K$ and elements $\pi_n \in \bar{K}$, for n a non-negative integer, such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. We will denote by $E(u) \in K_0[u]$ the Eisenstein polynomial of π. Write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$, and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative. We embed the W-algebra $W[u] \subset \mathfrak{S}$ into $W(R)$ by $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$ this embedding extends to a continuous embedding $\mathfrak{S} \hookrightarrow W(R)$, and $\theta|_{\mathfrak{S}}$ is the map $\mathfrak{S} \to \mathcal{O}_K$ sending u to π. This embedding is compatible with Frobenius endomorphisms.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with residue field the field of Laurent series $k((u))$. Let $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If $\operatorname{Fr} R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\operatorname{Fr} R)$. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\operatorname{Fr} R)[1/p]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\operatorname{Fr} R)[1/p]$, and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ its ring of integers. Since $\operatorname{Fr} R$ is algebraically closed [Fo 1, A.3.1.6], the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is a separable closure of $k((u))$. We denote by $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, and by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ its field of fractions. $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\operatorname{Fr} R)[1/p]$ equipped with its p-adic topology. We write $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cap W(R) \subset W(\operatorname{Fr} R)$. We regard all these rings as subrings of $W(\operatorname{Fr} R)[1/p]$. They are equipped with a Frobenius operator φ induced from the Frobenius on $W(\operatorname{Fr} R)[1/p]$.

For $n \geq 0$, let $K_{n+1} = K(\pi_n)$, and let $K_\infty = \bigcup_{n \geq 0} K_n$ and $G_{K_\infty} = \operatorname{Gal}(\bar{K}/K_\infty)$. Then G_{K_∞} fixes the subring $\mathfrak{S} \subset W(R)$, and hence it acts on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$.

(1.2) Let V be a finite free $\mathbb{Z}_p$-module of rank d, equipped with a continuous G_K-action. We write V^* for its $\mathbb{Z}_p$-dual. Set $M = (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \hat{\otimes}_{\mathbb{Z}_p} V^*)^{G_{K_\infty}}$. We say that M is of E-height $\leqslant h$ if there exists a finite free $\mathfrak{S}$-submodule $\mathfrak{M} \subset M$ of rank d, such that $\mathfrak{M}$ is stable by φ, spans the $\mathcal{O}_{\mathcal{E}}$-module M, and the cokernel of $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ is killed by $E(u)^h$. We call such a $\mathfrak{S}$-submodule a $\mathfrak{S}$-lattice of E-height

$\leqslant h$. By [Ki 2, 2.1.12] if a $\mathfrak{S}$-lattice of E-height $\leqslant h$ exists, then it is unique. We say that V is of E-height $\leqslant h$ if M is.

If V is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous action of G_{K_∞}, then we say that V is of E-height $\leqslant h$ if V admits a G_{K_∞}-stable $\mathbb{Z}_p$-lattice of E-height $\leqslant h$. If this is the case, then any G_{K_∞}-stable $\mathbb{Z}_p$-lattice in V is of E-height $\leqslant h$ [Ki 2, 2.1.15].

We now introduce coefficients. For any $\mathbb{Z}_p$-algebra R and any finite $\mathbb{Z}_p$-algebra A, we will write $R_A = R \otimes_{\mathbb{Z}_p} A$.

Consider a local Artin ring A with residue field $\mathbb{F}$, a finite extension of $\mathbb{F}_p$, and a representation of G_{K_∞} on a finite free A-module V_A of rank d. Write V_A^* for the A-dual of V_A, and set $M_A = (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V_A^*)^{G_{K_\infty}}$. This is a free $\mathcal{O}_{\mathcal{E},A} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$-module of rank d [Ki 1, 1.2.7(4)], equipped with an isomorphism $\varphi^* M_A \xrightarrow{\sim} M_A$.

For an A-algebra B, we set $M_B = M_A \otimes_A B$, and we extend φ to M_B by B-linearity. Write $\mathfrak{S}_B = \mathfrak{S} \otimes_{\mathbb{Z}_p} B$, and let h be a non-negative integer. A $\mathfrak{S}_B$-lattice of E-height $\leqslant h$ is a $\mathfrak{S}_B$-submodule $\mathfrak{M}_B \subset M_B$ such that

(1) $\mathfrak{M}_B$ is a finite projective $\mathfrak{S}_B$-module of rank d which generates M_B as a $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} B$-module.
(2) $\mathfrak{M}_B$ is stable by φ and the cokernel of $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ is killed by $E(u)^h$.

We denote by $L_{V_A}^{\leqslant h}(B)$ the set of $\mathfrak{S}_B$-lattices of E-height $\leqslant h$.

If $B \to B'$ is a map of A-algebras, and $\mathfrak{M}_B \subset M_B$ is a $\mathfrak{S}_B$-lattice of E-height $\leqslant h$, then $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_B \otimes_B B'$ is a projective $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} B'$-module of rank d, and hence the surjective map $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_B \otimes_B B' \to M_B \otimes_B B'$ is an isomorphism. Hence $\mathfrak{M}_{B'} = \mathfrak{M}_B \otimes_B B' \subset M_B \otimes_B B'$ is a $\mathfrak{S}_{B'}$-lattice of E-height $\leqslant h$. It follows that $L_{V_A}^{\leqslant h}$ is a functor on A-algebras.

Proposition (1.3). *The functor $L_{V_A}^{\leqslant h}$ is represented by a projective A-scheme $\mathscr{L}_{V_A}^{\leqslant h}$. If $A \to A'$ is map of local Artin rings with residue field $\mathbb{F}$ and $V_{A'} = V_A \otimes_A A'$, then there is a canonical isomorphism $\mathscr{L}_{V_A}^{\leqslant h} \otimes_A A' \xrightarrow{\sim} \mathscr{L}_{V_{A'}}^{\leqslant h}$. Moreover $\mathscr{L}_{V_A}^{\leqslant h}$ is equipped with a canonical (i.e. functorial in A) very ample line bundle.*

Proof. The argument is essentially identical to that given in [Ki 1, 2.1.7], so we only sketch it.

First the main result of [BL] implies that $L_{V_A}^{\leqslant h}$ is represented by a closed sub-Ind-scheme $\mathscr{L}_{V_A}^{\leqslant h}$ of the affine Grassmannian for $\mathrm{Res}_{W(k)/\mathbb{Z}_p}\mathrm{GL}_d$ over A. Let $\mathfrak{N}_A \subset M_A$ be a finite projective $\mathfrak{S}_A$-module of rank d, which spans M_A, and r the least integer such that $u^r \mathfrak{N}_A \subset (1 \otimes \varphi)\varphi^*(\mathfrak{N}_A) \subset u^{-r}\mathfrak{N}_A$. Write $\mathfrak{N}_B = \mathfrak{N}_A \otimes_A B$. If $\mathfrak{M}_B \subset M_B$ is a $\mathfrak{S}_B$-lattice of E-height $\leqslant h$, then condition (2) implies that $u^i \mathfrak{N}_B \subset \mathfrak{M}_B \subset u^{-i}\mathfrak{N}_B$ with $i \leqslant \frac{esh+r}{p-1}$, where $e = [K : K_0]$ is the degree of $E(u)$, and s is the least integer such that $p^s = 0$ in A. This implies that $\mathscr{L}_{V_A}^{\leqslant h}$ is actually a projective scheme.

The very ample line bundle in the proposition is the restriction of a canonical line bundle on the affine Grassmannian which is very ample on any closed subscheme of finite type [Fa 1, pp. 42-43]. □

(1.4) We now explain the relation between lattices of height $\leqslant h$, and the representation V_A. As usual we regard A-modules as sheaves on $\operatorname{Spec} A$. Write $\Theta_A : \mathscr{L}_{V_A}^{\leqslant h} \to \operatorname{Spec} A$ for the projective map of Proposition (1.3). We will denote by $\underline{\mathfrak{M}}$ the universal sheaf of $\Theta_A^*(\mathfrak{S}_A)$-modules on $\mathscr{L}_{V_A}^{\leqslant h}$.

Lemma (1.4.1). *Set $\tilde{A} = \Theta_{A*}(\mathcal{O}_{\mathscr{L}_{V_A}^{\leqslant h}})$. Then there is a canonical $\tilde{A}$-linear, G_{K_∞}-equivariant isomorphism*

$$V_{\tilde{A}} = V_A \otimes_A \tilde{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathfrak{S}_{\tilde{A}}^{\mathrm{ur}}).$$

Proof. Let M_A^* be the $\mathcal{O}_{\mathcal{E},A}$-dual of M_A, equipped with the induced Frobenius. By [Fo 1, Prop. A.1.2.6] we have natural G_{K_∞}-equivariant isomorphisms

$$V_{\tilde{A}} \xrightarrow{\sim} (M_{\tilde{A}}^* \otimes_{\mathcal{O}_{\mathcal{E},\tilde{A}}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}})^{\varphi=1} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}},\varphi}(M_{\tilde{A}}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}). \tag{1.4.2}$$

Observe that $\Theta_{A*}(\underline{\mathfrak{M}})$ is a finite $\Theta_{A*}\Theta_A^*(\mathfrak{S}_A) = \mathfrak{S} \otimes_A \tilde{A}$-module, and hence a finite $\mathfrak{S}$-module. It is equipped with a map

$$\varphi^*\Theta_{A*}(\underline{\mathfrak{M}}) \xrightarrow{\sim} \Theta_{A*}(\varphi^*(\underline{\mathfrak{M}})) \to \Theta_{A*}(\underline{\mathfrak{M}})$$

where the first isomorphism follows from the fact that φ is a finite flat map on $\mathfrak{S}$. The cokernel of the above map is killed by $E(u)^h$. It follows from [Fo 1, B.1.8.4] that the natural map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathfrak{S}_{\tilde{A}}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}})$$

is an isomorphism. Taking the $\tilde{A}$-linear maps on both sides yields an isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathfrak{S}_{\tilde{A}}^{\mathrm{ur}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}). \tag{1.4.3}$$

Now by the projection formula we have

$$\Theta_{A*}(\underline{\mathfrak{M}}) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} \Theta_{A*}(\Theta_A^*(M_A)) \xrightarrow{\sim} M_A \otimes_A \tilde{A} = M_{\tilde{A}}$$

so the second term in (1.4.3) is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \Theta_{A*}(\underline{\mathfrak{M}}), \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}},\varphi}(M_{\tilde{A}}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}).$$

Combining this with (1.4.2) proves the proposition. □

(1.5) Suppose now that A is a complete local ring with residue field $\mathbb{F}$ and maximal ideal $\mathfrak{m}_A$, and V_A is a finite free A-module equipped with a continuous action of G_{K_∞}. We again write V_A^* for the A-dual of V_A.

For any $\mathbb{Z}_p$-algebra R, we will denote by R_A the $\mathfrak{m}_A$-adic completion of $R \otimes_{\mathbb{Z}_p} A$. This is compatible with the notation introduced above for finite $\mathbb{Z}_p$-algebras. We set

$$M_A = (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \hat{\otimes}_{\mathbb{Z}_p} V_A^*)^{G_{K_\infty}} \xrightarrow{\sim} \varprojlim (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V_A^* \otimes_A A/\mathfrak{m}_A^i)^{G_{K_\infty}},$$

where $\hat{\otimes}$ means $\mathfrak{m}_A$-adic completion, and the isomorphism follows from [De, 2.1.8]. In particular, M_A is a free $\mathcal{O}_{\mathcal{E},A}$-module of rank d.

If B is an A-algebra such that $\mathfrak{m}_A^i \cdot B = 0$ for some $i \geq 1$, then we define $L_{V_A}^{\leqslant h}(B) = L_{V_{A/\mathfrak{m}_A^i}}^{\leqslant h}(B)$. We have

Corollary (1.5.1). *The functor $L_{V_A}^{\leqslant h}$ on A-algebras B such that $\mathfrak{m}_A^i \cdot B = 0$ for some i, is represented by a projective A-scheme $\mathscr{L}_{V_A}^{\leqslant h}$.*

Proof. By (1.3) the functor is representable by a formal scheme over A, which is equipped with a very ample line bundle. By formal GAGA [Gr, III, 5.4.5] this formal scheme is obtained by completing a projective A-scheme along the ideal $\mathfrak{m}_A$. □

(1.6) We will describe the image of the map obtained from $\mathscr{L}_{V_A}^{\leqslant h} \to \mathrm{Spec}\, A$ by inverting p. To do this we require some preparation.

Lemma (1.6.1). *Let B be a finite $\mathbb{Q}_p$-algebra, and $\mathfrak{M}_B$ a finite $\mathfrak{S}_B = \mathfrak{S} \otimes_{\mathbb{Z}_p} B$-module, which is flat over $\mathfrak{S}[1/p]$ and equipped with a map $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ whose cokernel is killed by $E(u)^h$. (Here φ acts on $\mathfrak{S}_B$ B-linearly). Suppose that $\mathcal{E} \otimes_{\mathfrak{S}[1/p]} \mathfrak{M}_B$ is finite free over $\mathcal{E} \otimes_{\mathbb{Q}_p} B$. Then $\mathfrak{M}_B$ is a finite projective $\mathfrak{S}_B$-module.*

Proof. Let $U \subset \operatorname{Spec} \mathfrak{S}[1/p]$ denote the largest open subscheme over which $\mathfrak{M}_B$ is $\mathfrak{S}_B$-flat, and let Z be its (reduced) complement. Since $\mathcal{E} \otimes_{\mathfrak{S}[1/p]} \mathfrak{M}_B$ is a free $\mathcal{E} \otimes_{\mathbb{Z}_p} B$-module we see that Z does not contain the generic point of $\operatorname{Spec} \mathfrak{S}[1/p]$. So Z is equal to the subscheme $V(g)$ corresponding to some element $g \in \mathfrak{S}[1/p]$.

Since the map $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ becomes an isomorphism after inverting $E(u)$, we see that the roots of g (i.e. the elements x in $\bar{K}$ with $|x| < 1$, and such that $g(x) = 0$) are contained in those of $\varphi(g)E(u)$, while those of $\varphi(g)$ are contained in those of $gE(u)$. Let x be a non-zero root of g such that $|x|$ is as small as possible, and y a root such that $|y|$ is as large as possible. Then $\varphi(g)$ has a root w with $|w| = |y^{1/p}| > |y|$, and every non-zero root of $\varphi(g)$ has absolute value at least $|x^{1/p}| > |x|$. It follows that w and x are roots of $E(u)$, so that $|x| = |\pi| = |w| = |y^{1/p}| > |y|$, which is a contradiction. Hence $u = 0$ is the only root of g, and $Z = V(u)$.

Now to show that $\mathfrak{M}_B$ is finite projective over $\mathfrak{S}_B$ it suffices to check that any non-zero Fitting ideal of $\mathfrak{M}_B$ is equal to $\mathfrak{S}_B$. Let I be such an ideal. Since $I[1/u]$ is the unit ideal in $\mathfrak{S}_B[1/u]$, we have $u^i \in I$ for some $i \geq 0$. Let $K_0[\![u]\!]_B = \mathfrak{S}_B \otimes_{\mathfrak{S}} K_0[\![u]\!]$. Then φ extends to a continuous endomorphism of $K_0[\![u]\!]_B$. It suffices to show that $J = IK_0[\![u]\!]_B \subset K_0[\![u]\!]_B$ is the unit ideal. Since the map $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ becomes an isomorphism after tensoring with $K_0[\![u]\!]$, we see that $\varphi(J)K_0[\![u]\!]_B = J$. It follows from Lemma (1.6.2) below that $J = J_0K_0[\![u]\!]_B$ for some ideal $J_0 \subset B$. Since $u^i \in J$, we see that $J_0 = B$, and J is the unit ideal, as required. □

Lemma (1.6.2). *Let B be a finite $\mathbb{Q}_p$-algebra, and let $J \subset K_0[\![u]\!]_B = K_0[\![u]\!] \otimes_{\mathbb{Q}_p} B$ be an ideal such that $\varphi(J)K_0[\![u]\!]_B = J$, where φ acts B-linearly. Then J is induced by an ideal of B.*

Proof. It suffices to consider the case where B is a local ring with maximal ideal $\mathfrak{m}_B$. Let $J_0 \subset K_0 \otimes_{\mathbb{Q}_p} B$ be the ideal generated by constant terms of elements of J. Since $\varphi(J) \subset J$, J_0 is stable by φ, and for $j \geq 1$ we have

$$J = \varphi^j(J)K_0[\![u]\!]_B \subset J_0K_0[\![u]\!]_B + u^{p^j}K_0[\![u]\!]_B.$$

Since $J_0K_0[\![u]\!]_B \subset K_0[\![u]\!]_B$ is closed for the u-adic topology, we see that $J \subset J_0K_0[\![u]\!]_B$. Similarly $J_0 \subset J + u^{p^j}K_0[\![u]\!]_B$ for $j \geq 1$, so that $J_0 \subset J$. Thus $J = J_0K_0[\![u]\!]_B$. Finally, since J_0 is stable by φ it is induced by an ideal of B. □

Corollary (1.6.3). *Let A be a finite flat $\mathbb{Z}_p$-algebra, and V_A a finite free A-module equipped with a continuous action of G_{K_∞}. Set $M_A = (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \hat{\otimes}_{\mathbb{Z}_p} V_A^*)^{G_{K_\infty}}$. Suppose that V_A, considered as a $\mathbb{Z}_p[G_{K_\infty}]$-module, is of E-height $\leqslant h$, and let $\mathfrak{M}_A \subset M_A$ be the unique $\mathfrak{S}$-lattice of E-height $\leqslant h$.*

Then $\mathfrak{M}_A$ is a $\mathfrak{S}_A$-submodule of M_A, and $\mathfrak{M}_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is finite projective over $\mathfrak{S}_A[1/p]$.

Proof. That $\mathfrak{M}_A$ is a $\mathfrak{S}_A$-submodule follows immediately from the fact that any endomorphism of the $\mathcal{O}_{\mathcal{E}}$-module M which is compatible with φ preserves $\mathfrak{M}$ [Ki 2, 2.1.11].

Since $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}_A = \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} M_A$ is a free $\mathcal{E} \otimes_{\mathbb{Z}_p} A$-module, $\mathfrak{M}_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is finite projective over $\mathfrak{S}_A[1/p]$ by Lemma (1.6.1). □

Proposition (1.6.4). *Let A and V_A be as in Subsection (1.5). Then*

(1) *The map $\Theta_A : \mathscr{L}_A^{\leqslant h} \to \operatorname{Spec} A$ becomes a closed immersion after inverting p.*
(2) *If $A^{\leqslant h}$ denotes the quotient of A corresponding to the scheme-theoretic image of Θ_A, then for any finite $W(\mathbb{F})[1/p]$-algebra B, a map $A \to B$ factors through $A^{\leqslant h}$ if and only if $V_B = V_A \otimes_{\mathbb{Z}_p} B$ is of E-height $\leqslant h$.*

Proof. Consider a finite, local $W(\mathbb{F})[1/p]$-algebra B, with residue field E, and maximal ideal $\mathfrak{m}_B$. Then B is naturally an E-algebra. We denote by $B^\circ \subset B$ the subring consisting of elements which map to the ring of integers $\mathcal{O}_E$ of E under the projection $B \to E$. We denote by Int_B the set of finitely generated $\mathcal{O}_E$-subalgebras C of B° such that $C[1/p] = B$. Since $\mathfrak{m}_B$ is nilpotent, any C in Int_B is finite over $\mathcal{O}_E$, and the union of all C in Int_B is B°.

We remark that any B-valued point of $\operatorname{Spec} A$ is induced by a B°-valued point, and hence by a C-valued point for some $C \in \operatorname{Int}_B$. The same holds for $\mathscr{L}_A^{\leqslant h}$ by the valuative criterion for properness.

Now for any finite flat $W(\mathbb{F})$-algebra C, the uniqueness of the submodule $\mathfrak{M}$ of M mentioned in Subsection (1.2) implies that the map $\mathscr{L}_A^{\leqslant h}(C) \to (\operatorname{Spec} A)(C)$ is injective. It follows from the above discussion that the map $\mathscr{L}_A^{\leqslant h}(B) \to (\operatorname{Spec} A)(B)$ is injective for any finite local $W(\mathbb{F})[1/p]$-algebra B. Applying this with $B = E$ we see that the map obtained from Θ_A by inverting p is quasi-finite, and hence finite, and induces an injection on closed points. Taking $B = E[\epsilon]/(\epsilon^2)$ we see that this map induces a surjection on tangent spaces, and is therefore a closed immersion.

To prove the second part of the proposition, it suffices to consider the case of B local. Suppose first that the map $A \to B$ factors through $A^{\leqslant h}$. This gives a B-valued point of $\mathscr{L}_A^{\leqslant h}$ which is induced by a point of $\mathscr{L}_A^{\leqslant h}(C)$ for some $C \in \operatorname{Int}_B$. Let $V_C = V_A \otimes_A C$, and $M_C = M_A \otimes_A C$. By definition of $\mathscr{L}_A^{\leqslant h}$, we have a finite projective $\mathfrak{S}_C$-submodule $\mathfrak{M}_C \subset M_C$ of rank d, which is stable by φ, spans M_C as an $\mathcal{O}_{\mathcal{E},C}$-module (and hence as an $\mathcal{O}_{\mathcal{E}}$-module), and such that $\mathfrak{M}_C/(1 \otimes \varphi)\varphi^*(\mathfrak{M}_C)$ is killed by $E(u)^h$. In particular, V_C is of finite E-height, and hence so is V_B.

Suppose conversely, that V_B is of E-height $\leq h$. Let $C \in \operatorname{Int}_B$ be such that $A \to B$ factors through C, and let V_C and M_C be as above. By Corollary (1.6.3) the unique $\mathfrak{S}$-lattice of E-height $\leqslant h$, $\mathfrak{M}_C \subset M_C$, is a $\mathfrak{S}_C$-module, and $\mathfrak{M}_C \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is finite projective over $\mathfrak{S}_C[1/p]$.

Let $\mathfrak{M}'_{\mathcal{O}_E}$ be the image of $\mathfrak{M}_C$ under the projection $M_C \to M_{\mathcal{O}_E}$, and set

$$\mathfrak{M}_{\mathcal{O}_E} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'_{\mathcal{O}_E} \cap \mathfrak{M}'_{\mathcal{O}_E}[1/p] \subset \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}'_{\mathcal{O}_E}.$$

Then $\mathfrak{M}_{\mathcal{O}_E}$ is a $\mathfrak{S}$-lattice of E-height $\leqslant h$ in $M_{\mathcal{O}_E}$. Indeed it is clear that $\mathfrak{M}_{\mathcal{O}_E}$ spans $M_{\mathcal{O}_E}$ and has the same $\mathfrak{S}$-rank as $\mathfrak{M}'_{\mathcal{O}_E}$. It is $\mathfrak{S}$-flat, since it is the kernel of a map of flat $\mathfrak{S}$-modules (cf. [Ki 2, 1.3.13]). Since $\mathfrak{M}_{\mathcal{O}_E}$ is finite flat over $\mathfrak{S}$, $\mathfrak{M}_{\mathcal{O}_E}/u\mathfrak{M}_{\mathcal{O}_E}$ is p-torsion free, and hence is a projective $\mathcal{O}_E \otimes_{\mathbb{Z}_p} W(k)$-module. In fact this module is free, since $\mathfrak{M}_{\mathcal{O}_E}/u\mathfrak{M}_{\mathcal{O}_E}[1/p]$ is isomorphic to its pullback by φ, and φ permutes the maximal ideals of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} W(k)$ transitively (cf. [Ki 1, 1.2.2(4)]). Hence $\mathfrak{M}_{\mathcal{O}_E}$ is a finite free $\mathfrak{S}_{\mathcal{O}_E}$-module.

Choose an $\mathfrak{S}_{\mathcal{O}_E}$-basis for $\mathfrak{M}_{\mathcal{O}_E}$, and lift it to a $\mathfrak{S}_C[1/p]$-basis $\mathbf{b}$ of $\mathfrak{M}_C \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The matrix of φ in this basis has entries which are in $B^\circ \otimes_{\mathbb{Z}_p} \mathfrak{S}$. Choose $C' \in \operatorname{Int}_B$ containing C such that these entries are in $\mathfrak{S}_{C'}$, and let $\mathfrak{M}_{C'}$ be the $\mathfrak{S}_{C'}$-submodule of $\mathfrak{M}_C \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ spanned by $\mathbf{b}$. Now $\mathfrak{M}_{C'}$ is stable by φ, and $\mathfrak{M}_{C'}/(1 \otimes$

$\varphi)\varphi^*(\mathfrak{M}_{C'})[1/p]$ is killed by $E(u)^h$ since it is obtained from $\mathfrak{M}_C/(1\otimes\varphi)\varphi^*(\mathfrak{M}_C)$ by applying $\otimes_C C'[1/p]$. It follows that $\mathfrak{M}_{C'}/(1\otimes\varphi)\varphi^*(\mathfrak{M}_{C'})$ is killed by $E(u)^h$, because as an $\mathfrak{S}$-module it is a successive extension of copies of $\mathfrak{M}_{\mathcal{O}_E}/(1\otimes\varphi)\varphi^*(\mathfrak{M}_{\mathcal{O}_E})$ and hence has no p-torsion.

Finally we see that $\mathfrak{M}_{C'} \subset M_{C'} = M_C \otimes_C C'$ corresponds to a C'-valued point of $\mathscr{L}_A^{\leqslant h}$, and it follows that the composite $A \to C \to C' \to B$ factors through $A^{\leqslant h}$. □

Corollary (1.7). *There exists a finite $\mathfrak{S}_{A^{\leqslant h}}$-module $\mathfrak{M}_{A^{\leqslant h}}$ such that*

(1) *$\mathfrak{M}_{A^{\leqslant h}}$ is equipped with a map $\varphi^*(\mathfrak{M}_{A^{\leqslant h}}) \to \mathfrak{M}_{A^{\leqslant h}}$ whose cokernel is killed by $E(u)^h$.*
(2) *$\mathfrak{M}_{A^{\leqslant h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a locally free $\mathfrak{S}_{A^{\leqslant h}}[1/p]$-module.*
(3) *For any finite $W(\mathbb{F})[1/p]$-algebra B, any map $\zeta : A^{\leqslant h} \to B$ and any $C \in \mathrm{Int}_B$ through which ζ factors, there is a canonical, φ-compatible isomorphism of $\mathfrak{S} \otimes_{\mathbb{Z}_p} B$-modules*
$$\mathfrak{M}_{A^{\leqslant h}} \otimes_{A^{\leqslant h}} B \xrightarrow{\sim} \mathfrak{M}_C \otimes_C B.$$
Here $\mathfrak{M}_C$ denotes the unique $\mathfrak{S}$-lattice of E-height $\leqslant h$ in $M_C = M_A \otimes_A C$.
(4) *There is a canonical isomorphism*
$$V_{A^{\leqslant h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{A^{\leqslant h}}[1/p],\varphi}(\mathfrak{M}_{A^{\leqslant h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathfrak{S}^{\mathrm{ur}}_{A^{\leqslant h}}[1/p]).$$

Proof. Denote by $\hat{\mathscr{L}}_A^{\leqslant h}$ the $\mathfrak{m}_A$-adic completion of $\mathscr{L}_A^{\leqslant h}$. Write
$$\tau : \hat{\mathscr{L}}_A^{\leqslant h} \times \mathrm{Spf}\,\mathfrak{S}_A \to \hat{\mathscr{L}}_A^{\leqslant h}$$
for the natural projection (the product of $\mathfrak{m}_A$-adic formal schemes being taken over $\mathrm{Spf}\,A$). By definition of the functor $L_A^{\leqslant h}$, the formal scheme $\hat{\mathscr{L}}_A^{\leqslant h}$ is equipped with a formal sheaf of $\tau_*\mathcal{O}_{\hat{\mathscr{L}}_A^{\leqslant h}\times\mathrm{Spf}\,\mathfrak{S}_A}$-modules. We may regard this as a formal coherent sheaf $\hat{\underline{\mathfrak{M}}}$ on $\hat{\mathscr{L}}_A^{\leqslant h} \times \mathrm{Spf}\,\mathfrak{S}_A$.

By Proposition (1.6.4), we have a projective map of formal schemes
$$\mathscr{L}_A^{\leqslant h} \times_{\mathrm{Spf}\,A} \mathrm{Spf}\,\mathfrak{S}_A \to \mathrm{Spf}\,\mathfrak{S}_A. \tag{1.7.1}$$
By formal GAGA, $\hat{\underline{\mathfrak{M}}}$ is the completion of a coherent sheaf $\underline{\mathfrak{M}}$ on the projective $\mathfrak{S}_A$-scheme $\mathrm{Spec}\,\mathscr{L}_A^{\leqslant h} \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,\mathfrak{S}_A$. Write $\Theta_{\mathfrak{S}_A}$ for the projection of this last scheme onto the factor $\mathrm{Spec}\,\mathfrak{S}_A$. Then the scheme theoretic image of $\Theta_{\mathfrak{S}_A}$ is $\mathrm{Spec}\,\mathfrak{S}_{A^{\leqslant h}}$. Let $\mathfrak{M}_{A^{\leqslant h}} = \Theta_{\mathfrak{S}_A *}(\underline{\mathfrak{M}})$.

Property (1) follows from the corresponding property for $\underline{\mathfrak{M}}$ together with the fact that φ is a flat map on $\mathfrak{S}$, so φ^* commutes with direct images. Condition (2) follows from the fact that, after inverting p, $\mathrm{Spec}\,\mathscr{L}_A^{\leqslant h} \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,\mathfrak{S}_A$ becomes isomorphic to $\mathrm{Spec}\,\mathfrak{S}_{A^{\leqslant h}}[1/p]$. To see (3) first note that by Proposition (1.6.4)(2) M_C is of E-height $\leqslant h$, and hence contains a unique $\mathfrak{S}$-lattice of E-height $\leqslant h$, $\mathfrak{M}_C$. If $\mathfrak{M}'_C$ denotes the image of $\mathfrak{M}_{A^{\leqslant h}} \otimes_{A^{\leqslant h}} C$ in M_C, then $\mathfrak{M}'_C$ is a torsion free, φ-stable, $\mathfrak{S}_C$-submodule of M_C such that the cokernel of $\varphi^*(\mathfrak{M}'_C) \to \mathfrak{M}'_C$ is killed by $E(u)^h$. As in the proof of Proposition (1.6.4) this implies that $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'_C \cap \mathfrak{M}'_C[1/p]$ is a $\mathfrak{S}$-lattice of E-height $\leqslant h$ in M_C, and hence is equal to $\mathfrak{M}_C$.

Finally, to see (4) we use Lemma (1.4.1). We denote by $\hat{\Theta}_{\mathfrak{S}_A}$ the composite of the maps in (1.7.1). Write $\tilde{A} = \hat{\Theta}_{\mathfrak{S}_A *}(\mathcal{O}_{\hat{\mathscr{L}}_A^{\leqslant h}})$. For a $\mathbb{Z}_p$-algebra R, we denote by $R_{\tilde{A}}$ the completion of $R \otimes_{\mathbb{Z}_p} \tilde{A}$ with respect to the Jacobson radical of $\tilde{A}$. (This extends the definition of Subsection (1.5), since $\tilde{A}$ need only be semi-local). Using Lemma

(1.4.1), and the theorem on formal functions, one finds that there is a canonical isomorphism

$$V_{\tilde{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\hat{\Theta}_{\mathfrak{S}_A *}(\underline{\hat{\mathfrak{M}}}), \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}).$$

Inverting p, and using formal GAGA, the definition of $\mathfrak{M}_{A^{\leqslant h}}$, and the fact that $A^{\leqslant h}[1/p] \xrightarrow{\sim} \tilde{A}[1/p]$ by Proposition (1.6.4)(1), one finds a canonical isomorphism

$$V_{A^{\leqslant h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{\tilde{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}}[1/p],\varphi}(\mathfrak{M}_{A^{\leqslant h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}[1/p]). \tag{1.7.2}$$

Now $\tilde{A}$ is a finite $A^{\leqslant h}$-algebra, hence the topologies induced on $\tilde{A}$ by the Jacobson radicals of $\tilde{A}$ and $A^{\leqslant h}$ are equivalent. Since the map $A^{\leqslant h} \to \tilde{A}$ has kernel and cokernel killed by some fixed power of p, we see that the same is true after tensoring both sides by $\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p}$ and taking completions with respect to the radical $\mathfrak{m}_{A^{\leqslant h}}$ of $A^{\leqslant h}$. Hence the previous remark shows that the map $\mathfrak{S}^{\mathrm{ur}}_{A^{\leqslant h}} \to \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}$ has kernel and cokernel killed by a finite power of p, so that $\mathfrak{S}^{\mathrm{ur}}_{A^{\leqslant h}}[1/p] \xrightarrow{\sim} \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}[1/p]$. The same argument shows that $\mathfrak{S}_{A^{\leqslant h}}[1/p] \xrightarrow{\sim} \mathfrak{S}_{\tilde{A}}[1/p]$ (this is also easily seen directly). Now (4) follows from (1.7.2). □

§2. Potentially semi-stable representations

(2.1) In this section we consider complete local rings which carry a deformation of a mod p representation of G_K. We will construct quotients of such rings, or more precisely their generic fibers, over which the given representation is potentially semi-stable. The construction builds on that in the previous section, via the fact that semi-stable representations have finite E-height.

Consider a complete local ring $(A^\circ, \mathfrak{m}_{A^\circ})$ with residue field the finite field $\mathbb{F}$. We will assume that A° is p-torsion free, and we set $A = A^\circ[1/p]$. Note that this notation differs slightly from that of the previous section where A denoted a complete local ring. It will be more convenient here.

For any Noetherian, complete local $\mathbb{Z}_p$-algebra R we write $R_A = R_{A^\circ}[1/p]$, where R_{A° denotes the $\mathfrak{m}_{A^\circ}$-adic completion of $R \otimes_{\mathbb{Z}_p} A^\circ$, as in §1. Since $\mathbb{F}$ is finite this is equal to the completion with respect to the ideal of $R \otimes_{\mathbb{Z}_p} A$ generated by the radicals of R and A. It is also the tensor product in the category of Noetherian, complete semi-local $\mathbb{Z}_p$-algebras. If B is an A-algebra, we write $R_B = R_A \otimes_A B$. In particular, we have $\mathfrak{S}_A = \mathfrak{S}_{A^\circ}[1/p]$ and we extend φ to an A-linear endomorphism of $\mathfrak{S}_A$. There is a canonical isomorphism $\mathfrak{S}_A/u\mathfrak{S}_A \xrightarrow{\sim} W_A = W \otimes_{\mathbb{Q}_p} A$.

We will recall some constructions which were carried out in [Ki 2] in the special case $A^\circ = \mathbb{Z}_p$. Let $\mathcal{O} = \varprojlim_n (W[\![u, u^n/p]\!][1/p])$. Then $\mathcal{O}$ may be thought of as the ring of rigid analytic functions on the open disk of radius 1 (cf. [Ki 2, 1.1.1]). We have an inclusion $\mathfrak{S} \hookrightarrow \mathcal{O}$, and the endomorphism φ has a unique continuous extension to each ring $W[\![u, u^n/p]\!]$ and hence to $\mathcal{O}$. We set $c_0 = E(0)$ and

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(E(u)/c_0) \in \mathcal{O}.$$

We denote by $\widehat{\mathfrak{S}}_0$ the completion of $K_0[u]$ at the ideal $(E(u))$. Write

$$\mathcal{O}_A = \varprojlim_n W[\![u, u^n/p]\!]_A = \varprojlim_n W_{A^\circ}[\![u, u^n/p]\!][1/p].$$

We set $\widehat{\mathfrak{S}}_{0,A}$ equal to the completion of $K_0[u] \otimes_{\mathbb{Q}_p} A$ at the ideal $(E(u))$.

Suppose we are given a finite projective $\mathfrak{S}_A$-module $\mathfrak{M}_A$ of constant rank r, which is equipped with a φ-semi-linear endomorphism $\varphi : \mathfrak{M}_A \to \mathfrak{M}_A$ such that the

induced map $\varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ has cokernel killed by $E(u)^h$. Set $\mathcal{M}_A = \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathcal{O}$, and write $D_A = \mathcal{M}_A/u\mathcal{M}_A$. We again denote by φ the endomorphism of D_A induced by φ on $\mathfrak{M}_A$. We have the following analogue of [Ki 2, 1.2.6].

Lemma (2.2). *There is a unique φ-compatible, W_A-linear map $\xi : D_A \to \mathcal{M}_A$, whose reduction modulo u is the identity.*

The induced map $D_A \otimes_A \mathcal{O} \to \mathcal{M}_A$ has cokernel killed by λ^h, and the image of the map $D_A \otimes_{W_A} \widehat{\mathfrak{S}}_{0,A} \to \mathcal{M}_A \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A}$ is equal to that of

$$\varphi^*(\mathcal{M}_A) \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A} \to \mathcal{M}_A \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A}.$$

Proof. Let $s_0 : D_A \to \mathcal{M}_A$ be any W_A-linear section to the projection $\mathcal{M}_A \to D_A$. Consider the sum

$$s = s_0 + \sum_{i=0}^{\infty} (\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i}).$$

We claim that s converges to a well defined W_A-linear map. Granting this, one sees immediately that $\varphi \circ s = s \circ \varphi$, and that s reduces to the identity modulo u.

Let $D_A^\circ \subset D_A$ be a finitely generated W_{A°-submodule which spans D_A. Similarly, we choose a finitely generated $\mathfrak{S}_{A^\circ}$-submodule $\mathfrak{M}_A^\circ \subset \mathfrak{M}_A$ which spans $\mathfrak{M}_A$. We may choose $\mathfrak{M}_A^\circ$, so that $\varphi \circ s_0 \circ \varphi^{-1} - s_0 : D_A \to u\mathfrak{M}_A$ takes D_A° into $u\mathfrak{M}_A^\circ$. Choose $j \geq 0$, such that φ induces a map $\mathfrak{M}_A^\circ \to p^{-j}\mathfrak{M}_A^\circ$ and φ^{-1} induces a map $D_A^\circ \to p^{-j} D_A$. Then we have

$$\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i} : D_A^\circ \to p^{-2ij} u^{p^i} \mathfrak{M}_A^\circ$$

and we see that for every $n \geq 0$ the series defining s converges to a well defined map

$$D_A^\circ \to \mathfrak{M}_A^\circ \otimes_{\mathfrak{S}_{A^\circ}} W_{A^\circ}[\![u, u^{p^n}/p]\!][1/p] \xrightarrow{\sim} \mathfrak{M}_A \otimes_{\mathfrak{S}_A} W[\![u, u^{p^n}/p]\!]_A,$$

and hence to a map $\xi : D_A \to \mathcal{M}_A$.

The uniqueness of ξ may be seen as in [Ki 2, 1.2.6]: If s' is another such map, one finds that elements in $(s - s')(D_A)$ are infinitely u-divisible, and hence 0.

To check the final claim, we again proceed as in *loc. cit.* First since ξ reduces to the identity modulo u, and has determinant in $W[\![u/p]\!]_A$, Lemma (2.2.1) below implies that for n sufficiently large it induces an isomorphism

$$D_A \otimes_{W_A} W[\![u/p^n]\!]_A \xrightarrow{\sim} \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u/p^n]\!]_A.$$

Denote by ξ_s the map

$$D_A \otimes_{W_A} W[\![u^{p^s}/p^n]\!]_A \longrightarrow \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u^{p^s}/p^n]\!]_A$$

induced by ξ.

Consider the commutative diagram

$$\begin{array}{ccc} \varphi^*(D_A \otimes_{W_A} \mathcal{O}_A) & \xrightarrow{\varphi^*\xi} & \varphi^*\mathcal{M}_A \\ \downarrow{\sim} & & \downarrow{1\otimes\varphi} \\ D_A \otimes_{W_A} \mathcal{O}_A & \xrightarrow{\xi} & \mathcal{M}_A \end{array}$$

Let r be the least integer such that $e < p^r/n$. Tensoring the above diagram by $\otimes_{\mathcal{O}_A} W[\![u, u^{p^s}/p^n]\!]_A$ for $s = 0, \ldots, r-1$ yields a diagram where the right vertical arrow is an isomorphism. Hence we see by induction that ξ_s is an isomorphism for

$s = 0, 1, \ldots, r-1$. Applying the same argument with $s = r$ shows that the cokernel of ξ_r is killed by $E(u)^h$, and that its image coincides with the map obtained from $\varphi^*(\mathcal{M}_A) \to \mathcal{M}_A$ by applying $\otimes_{\mathcal{O}_A} W[\![u, u^{p^r}/p^n]\!]_A$. This proves the last claim of the lemma. Repeating the above argument shows that the cokernel of ξ_s is killed by λ^h for any $s \geq 0$. Using the definition of $\mathcal{O}_A$, one sees that this completes the proof of the lemma. □

Lemma (2.2.1). *Let $I \subset W[\![u/p]\!]_A$ be an ideal such that $IW[\![u/p]\!]_A/uW[\![u/p]\!]_A$ is the unit ideal. Then for n sufficiently large $IW[\![u/p^n]\!]_A$ is the unit ideal.*

Proof. Suppose first that $I = (f)$ is principal. After multiplying f by a unit, we may assume that its image in $W[\![u/p]\!]_A/uW[\![u/p]\!]_A = W_A$ is 1, so that $f \in 1 + uW[\![u/p]\!]_A$. Then $f \in 1 + p^{-j}uW[\![u/p]\!]_{A^\circ}$ for some $j \geq 0$, and f has an inverse in $W[\![u/p^{j+1}]\!]_{A^\circ}$.

In general write $I = (f_1, f_2, \ldots, f_r)$ for $f_1, \ldots, f_r \in W[\![u/p]\!]_A$. Write $\bar{f}_i$ for the image of f_i in $W[\![u/p]\!]_A/uW[\![u/p]\!]_A$. Then $1 = \sum_{i=1}^r \bar{g}_i \bar{f}_i$ for some $\bar{g}_i \in W[\![u/p]\!]_A/u$. Lift $\bar{g}_i$ to $g_i \in W[\![u/p]\!]_A$. Then by the first part $\sum_{i=1}^r g_i f_i$ is a unit in $W[\![u/p^n]\!]$ for n sufficiently large. □

(2.3) As usual, we denote by A_{cris} the p-adic completion of the divided power envelope of the ring $W(R)$ of Subsection (1.1) with respect to $\ker(\theta)$, and by B^+_{dR} the $\ker(\theta)$-adic completion of $W(R)[1/p]$. The inclusion $\mathfrak{S} \subset W(R)$ extends uniquely to a continuous inclusion $\mathcal{O} \subset A_{\mathrm{cris}}$. We denote by $\ell_u \in B^+_{\mathrm{dR}}$ the element

$$\ell_u = \log[\underline{\pi}] := \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\frac{[\underline{\pi}] - \pi}{\pi}\right)^i.$$

We equip the polynomial ring $K_0[\ell_u]$ with an operator N, which acts by formal differentiation of the variable ℓ_u, and a Frobenius φ which acts as the usual Frobenius on K_0, and satisfies $\varphi(\ell_u) = p\ell_u$.

For $\sigma \in G_K$, we have $\sigma(\ell_u) - \ell_u = \beta(\sigma) \in B^+_{\mathrm{cris}}$. The map β is a 1-cocycle, and may be viewed as taking values in $\mathbb{Z}_p(1)$. (It is the Kummer cocycle corresponding to π.) If $\beta(\sigma) \neq 0$, then it generates the maximal ideal of B^+_{dR}, and for any $i \geq 1$ there is a surjection $B^+_{\mathrm{cris}} \to B^+_{\mathrm{dR}}/(\beta(\sigma)^i)$.

We will write $B^+_{\mathrm{cris},A} = A_{\mathrm{cris},A^\circ}[1/p] = A_{\mathrm{cris},A}$ where, as usual, $A_{\mathrm{cris},A^\circ}$ denotes the $\mathfrak{m}_{A^\circ}$-adic completion of $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$. For any A-algebra B, we write $B^+_{\mathrm{cris},B} = B^+_{\mathrm{cris}} \otimes_A B = A_{\mathrm{cris},B}$, and $D_B = D_A \otimes_A B$. We extend φ to D_B by B-linearity.

The ring A_{cris} is equipped with an exhaustive decreasing filtration $\mathrm{Fil}^i A_{\mathrm{cris}}$, induced by that on B^+_{cris}. We denote by $\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$ the $\mathfrak{m}_{A^\circ}$-adic completion of $\mathrm{Fil}^i A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$, and for any A-algebra B, we set $\mathrm{Fil}^i B^+_{\mathrm{cris},B} = B \otimes_{A^\circ} \mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$.

Lemma (2.3.1). *For any A°-module M, denote by $\widehat{M}$ its $\mathfrak{m}_{A^\circ}$-adic completion. If M is a flat A°-module, then*

(1) *For any finite A°-module N, the natural map*

$$N \otimes_{A^\circ} \widehat{M} \to \widehat{N \otimes_{A^\circ} M}$$

is an isomorphism.

(2) *$\widehat{M}$ is flat over A°. If M is faithfully flat over A°, then so is $\widehat{M}$.*

(3) *The functor $M \mapsto \widehat{M}$ preserves short exact sequences of flat A°-modules.*

Proof. Using the Artin-Rees lemma, and the flatness of M, one finds that $N \mapsto \widehat{N \otimes_{A^\circ} M}$ is exact in N. That the map in (1) is an isomorphism is clear if N is finite free over A°, and in general it can be seen by taking a presentation of N by finite free A°-modules, and using the exactness of the above functor. The first part of (2) now follows since for any ideal I of A°, we have that $I \otimes_{A^\circ} \widehat{M} = \widehat{I \otimes_{A^\circ} M}$ injects into $\widehat{M}$. Moreover, if M is faithfully flat over A°, then $M/\mathfrak{m}_{A^\circ}^n M$ is faithfully flat over $A^\circ/\mathfrak{m}_{A^\circ}^n$ for $n \geq 1$. Since A° is $\mathfrak{m}_{A^\circ}$-adically separated, this implies that $\widehat{M}$ is a faithful A°-module. Finally, (3) is clear. □

Lemma (2.3.2). *We have*

(1) *For $i \geq 0$, the ideal $\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$ of $A_{\mathrm{cris},A^\circ}$ is a faithfully flat A°-module.*
(2) *For $i \geq 0$, $\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}/\mathrm{Fil}^{i+1} A_{\mathrm{cris},A^\circ}$ is a faithfully flat A°-module, which is isomorphic to the $\mathfrak{m}_{A^\circ}$-adic completion of $(\mathrm{Fil}^i A_{\mathrm{cris}}/\mathrm{Fil}^{i+1} A_{\mathrm{cris}}) \otimes_{\mathbb{Z}_p} A^\circ$.*
(3) *For any A-algebra B, $i \geq 1$, and $\sigma \in G_K$, $B^+_{\mathrm{cris},B}/(\beta(\sigma)B + \mathrm{Fil}^i B^+_{\mathrm{cris},B})$ is a flat B-module. If $\beta(\sigma) \notin \mathrm{Fil}^i B^+_{\mathrm{cris}}$, then $\beta(\sigma) \notin \mathrm{Fil}^i B^+_{\mathrm{cris},B}$.*
(4) *If B° is a finite, local A°-algebra, with radical $\mathfrak{m}_{B^\circ}$, then $A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} B^\circ$ is canonically isomorphic to the $\mathfrak{m}_{B^\circ}$-adic completion of $A^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} B^\circ$.*
(5) *The map*
$$A_{\mathrm{cris},A^\circ} \to \prod A_{\mathrm{cris},A^\circ/\mathfrak{q}}$$
is injective, where $\mathfrak{q}$ runs over ideals of A° such that $A^\circ/\mathfrak{q}$ is a finite flat $\mathbb{Z}_p$-algebra.
(6) *If $0 \neq f \in A_{\mathrm{cris}}$, then f is not a zero divisor in $A_{\mathrm{cris},A^\circ}$.*

Proof. (1) follows from (2) and (3) of Lemma (2.3.1). The second claim in (2) now follows from the fact that $\mathrm{Fil}^i A_{\mathrm{cris}}/\mathrm{Fil}^{i+1} A_{\mathrm{cris}}$ is a faithfully flat $\mathbb{Z}_p$-module, and the first claim follows from the second and Lemma (2.3.1)(2).

If $\beta(\sigma) = 0$, then (3) follows from (2). If $\beta(\sigma) \neq 0$, then the first part of (3) follows from an argument similar to that for (2), once we note that there is a greatest integer j such that $\beta(\sigma)/p^j \in A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}}$, so that $A_{\mathrm{cris}}/(\beta(\sigma)/p^j \cdot \mathbb{Z}_p + \mathrm{Fil}^i A_{\mathrm{cris}})$ is $\mathbb{Z}_p$-flat. To see the second part, we may assume that B is a finitely generated A-algebra. Then there is a map $B \to B'$ where B' is a finite $W(\mathbb{F})[1/p]$-algebra. If $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris},B}$, then also $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris},B'}$, and this easily implies $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris}}$.

Since the $\mathfrak{m}_{A^\circ}$-adic and $\mathfrak{m}_{B^\circ}$-adic topologies on B° coincide, to prove (4) it suffices to show that $A^+_{\mathrm{cris},A^\circ} \otimes_{A^\circ} B^\circ$ is isomorphic to the $\mathfrak{m}_{A^\circ}$-adic completion of $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} B^\circ$. This follows from Lemma (2.3.1)(1).

For (5) suppose that $0 \neq f \in A^+_{\mathrm{cris},A^\circ}$. Then for some $n \geq 1$, f has non-zero image in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/\mathfrak{m}_{A^\circ}^n$. Since $A^\circ[1/p]$ is a Jacobson ring, and the residue fields at the maximal ideals are finite extensions of $\mathbb{Q}_p$, one easily sees that there exists an ideal $\mathfrak{q} \subset A^\circ$, such that $A^\circ/\mathfrak{q}$ is a finite flat $\mathbb{Z}_p$-module, and $\mathfrak{q} \subset \mathfrak{m}_{A^\circ}^n$.

Finally, to see (6) we may reduce to the case where A° is finite flat over $\mathbb{Z}_p$ using (5). Then $A^+_{\mathrm{cris},A^\circ} = A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$, because A_{cris} is p-adically complete, and (6) follows because A° is flat over $\mathbb{Z}_p$. □

Lemma (2.3.3). *Let M be an A°-module and $x \in A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} M$. The set of A°-submodules $N \subset M$ such that $x \in A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} N$ has a smallest element $N(x)$.*

Proof. If M has finite length, then the lemma is clear because tensoring by the flat A°-module $A_{\mathrm{cris},A^\circ}$ commutes with taking intersections.

Suppose that M is a finite A°-module, and for $n \geq 1$, let x_n denote the image of x in $A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} M/\mathfrak{m}_{A^\circ}^n M$. Let $N(x_n) \subset M/\mathfrak{m}_{A^\circ}^n M$ denote the submodule obtained by applying the lemma to x_n and $M/\mathfrak{m}_{A^\circ}^n M$. Then the image of $N(x_{n+1})$ in $M/\mathfrak{m}_{A^\circ}^n M$ is $N(x_n)$. Let $N(x) = \varprojlim_n N(x_n) \subset M$. By Lemma (2.3.1)(1), a submodule $N \subset M$ satisfies the condition of the lemma, if and only if the image of x in $A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} M/(N + \mathfrak{m}_{A^\circ}^n M)$ vanishes for all $n \geq 1$. That is, if and only if $N(x_n) \subset N/(N \cap \mathfrak{m}_{A^\circ}^n M)$. By the Artin-Rees lemma, this is equivalent to $N(x) \subset N$.

Finally, since $A_{\mathrm{cris},A^\circ}$ is a flat A°-module, the lemma for any A°-module M follows by applying the case of finitely generated modules to any finitely generated submodule of $M' \subset M$ such that $x \in A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} M'$. $\square$

(2.4) Next suppose that V_{A° is a finite free A°-module of rank r, equipped with a continuous action of G_K. Suppose also that $(A^\circ)^{\leqslant h} = A^\circ$ and that, if $\mathfrak{M}_{A^\circ}$ is the finite free $\mathfrak{S}_{A^\circ}$-module given by Corollary (1.7), then $\mathfrak{M}_A = \mathfrak{M}_{A^\circ} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (compatibly with the operator φ). In particular, writing $V_A = V_{A^\circ} \otimes_{A^\circ} A$, we have a canonical G_{K_∞}-equivariant isomorphism

$$V_A \xrightarrow[\iota]{\sim} \mathrm{Hom}_{\mathfrak{S}_A,\varphi}(\mathfrak{M}_A, \mathfrak{S}_A^{\mathrm{ur}})$$

where $\mathfrak{S}_A^{\mathrm{ur}} = \mathfrak{S}_{A^\circ}^{\mathrm{ur}}[1/p]$. This gives us a $\mathfrak{S}_A$-linear, φ-compatible map

$$\mathfrak{M}_A \to \mathrm{Hom}_A(V_A, \mathfrak{S}_A^{\mathrm{ur}}), \quad m \mapsto (v \mapsto \langle m, \iota(v)\rangle). \tag{2.4.1}$$

Tensoring both sides by $\mathcal{O}_A$, and using Lemma (2.2) we obtain φ-compatible maps

$$D_A \xrightarrow{\xi} \mathcal{M}_A \to \mathrm{Hom}_A(V_A, \mathfrak{S}_A^{\mathrm{ur}}) \otimes_{\mathfrak{S}_A} \mathcal{O}_A \to \mathrm{Hom}_A(V_A, B_{\mathrm{cris},A}^+). \tag{2.4.2}$$

Given an A-algebra B, (2.4.2) induces a $B_{\mathrm{cris},B}^+$-linear map

$$D_B \otimes_{W_B} B_{\mathrm{cris},B}^+ \to \mathrm{Hom}_A(V_A, B_{\mathrm{cris},A}^+) \otimes_A B = \mathrm{Hom}_B(V_B, B_{\mathrm{cris},B}^+). \tag{2.4.3}$$

As usual, G_K acts on the right hand side by $f \mapsto \sigma \circ f \circ \sigma^{-1}$. From the definitions one sees immediately, that this map is compatible with the action of G_{K_∞} on the two sides. To equip the left hand side with an action of G_K, suppose we are given a $W_B = W_A \otimes_A B$-linear map $N : D_B \to D_B$ which satisfies $p\varphi N = N\varphi$. We define an action of G_K on $D_B \otimes_{W_B} B_{\mathrm{cris},B}^+$ by setting

$$\sigma(d \otimes b) = \sum_{i=0}^{\infty} \frac{N^i(d)}{i!} \otimes \beta(\sigma)^i \sigma(b) = \exp(N \otimes \beta(\sigma)) \cdot d \otimes \sigma(b)$$

for $\sigma \in G_K$. Using the fact that $\varphi(\beta(\sigma)) = p\beta(\sigma)$ one sees that the action of G_K commutes with φ (cf. [Fa 2, §4]).

Consider the composite of the isomorphisms

$$D_B \otimes_{K_0} K_0[\ell_u] \xrightarrow{\sim} (D_B \otimes_{K_0} K_0[\ell_u])^{N=0} \otimes_{K_0} K_0[\ell_u] \xrightarrow{(\ell_u \mapsto 0)\otimes 1} D_B \otimes_{K_0} K_0[\ell_u], \tag{2.4.4}$$

where the first map is the inverse of the natural isomorphism

$$(D_B \otimes_{K_0} K_0[\ell_u])^{N=0} \otimes_{K_0} K_0[\ell_u] \xrightarrow{\sim} D_B \otimes_{K_0} K_0[\ell_u]$$

induced by multiplication in $K_0[\ell_u]$. Set $B_{\mathrm{st},B}^+ = B_{\mathrm{cris},B}^+ \otimes_{K_0} K_0[\ell_u]$, equipped with the induced operators φ and N. Tensoring (2.4.4) by $\otimes_{W_B} B_{\mathrm{cris},B}^+$ and (2.4.3) by

$\otimes_{K_0} K_0[\ell_u]$, we obtain maps

$$D_B \otimes_{K_0} K_0[\ell_u] \overset{(2.4.4)}{\to} D_B \otimes_{W_B} B^+_{\mathrm{st},B} \overset{(2.4.3)}{\to} \mathrm{Hom}_B(V_B, B) \otimes_B B^+_{\mathrm{st},B}. \tag{2.4.5}$$

The map (2.4.3) respects the action of G_K if and only if (2.4.5) does when G_K is regarded as acting trivially on D_B. This follows from a simple computation using the fact that an explicit inverse to the bijection $(D \otimes_{K_0} K_0[\ell_u])^{N=0} \overset{\ell_u \mapsto 0}{\to} D$ is given by $d \mapsto \exp(-N \otimes \ell_u) \cdot d$. We will study the condition that (2.4.3) and (2.4.5) respect the action of G_K. First we need the following

Lemma (2.4.6). *For each A-algebra B the maps (2.4.3) and (2.4.5) are injective, and their cokernels are flat B-modules.*

Proof. It suffices to prove the assertions for (2.4.3) in the case $B = A$.

To prove the injectivity of (2.4.3), it suffices, by Lemma (2.3.2)(5), to consider the case when A° is finite flat over $\mathbb{Z}_p$. Note that the $\mathfrak{S}^{\mathrm{ur}}_A$-linear map induced by (2.4.1) sits in a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}^{\mathrm{ur}}_A \otimes_{\mathfrak{S}_A} \mathfrak{M}_A & \longrightarrow & \mathrm{Hom}_A(V_A, \mathfrak{S}^{\mathrm{ur}}_A) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}, A^\circ}[1/p] \otimes_{\mathfrak{S}_A} \mathfrak{M}_A & \longrightarrow & \mathrm{Hom}_A(V_A, \mathcal{O}^{\mathrm{ur}}_{\mathcal{E}, A^\circ}[1/p]) \end{array}$$

It follows easily from the remarks of Subsection (1.5) that the lower map is an isomorphism, and the left vertical map is injective, since it is obtained from the inclusion $\mathfrak{S}^{\mathrm{ur}} \hookrightarrow \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ by applying $\otimes_{\mathfrak{S}} \mathfrak{M}_A$, because A° is finite over $\mathbb{Z}_p$. Hence the top map is an injective map of finite free $\mathfrak{S}^{\mathrm{ur}}_A$-modules of the same rank, and it remains an injection after tensoring by $\otimes_{\mathfrak{S}^{\mathrm{ur}}} B^+_{\mathrm{cris}}$. This shows that the map

$$\mathcal{M}_A \otimes_{\mathcal{O}_A} B^+_{\mathrm{cris},A} \xrightarrow{\sim} \mathfrak{M}_A \otimes_{\mathfrak{S}_A} B^+_{\mathrm{cris},A} \to \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A})$$

is an injection. Now the injectivity of (2.4.3) follows from the fact that the determinant of the map

$$D_A \otimes_{W_A} B^+_{\mathrm{cris},A} \overset{\xi \otimes 1}{\to} \mathcal{M}_A \otimes_{\mathcal{O}_A} B^+_{\mathrm{cris},A}$$

is a divisor of $E([\underline{\pi}])^s$ for some $s > 0$, by Lemma (2.2), and $E([\underline{\pi}])$ is not a zero divisor in $B^+_{\mathrm{cris},A}$ by Lemma (2.3.2)(6).

To see that the cokernel of (2.4.3) is A-flat, note that by Lemma (2.3.2)(4), and the injectivity proved above, (2.4.3) remains injective after tensoring by $\otimes_A A/I$ for any ideal I of A. □

Proposition (2.4.7). *The functor which assigns to an A-algebra B the collection of W_B-linear maps $N : D_B \to D_B$ which satisfy $p\varphi N = N\varphi$ and such that (2.4.3) is compatible with the action of G_K, is representable by a quotient A^{st} of A.*

Proof. Consider the functor which assigns to B the collection of W_B-linear maps $N : D_B \to D_B$ which satisfy $p\varphi N = N\varphi$. This is easily seen to be representable by a finitely generated A-algebra A^N.

Write ψ_B for the map of (2.4.3). For any $d \in D_{A^N}$ and $\sigma \in G_K$, let

$$\delta_\sigma(d) = \psi_{A^N}(\sigma(d)) - \sigma(\psi_{A^N}(d)).$$

Then $\delta_\sigma(d) \in Q := \mathrm{Hom}_{A^N}(V_{A^N}, B^+_{\mathrm{cris},A^N})$. Fix a B^+_{cris,A^N}-basis for Q, and let $x_1, \ldots, x_r$ denote the co-ordinates of $\delta_\sigma(d)$ with respect to this basis. Applying

Lemma (2.3.3) with $M = A^N$ and $x = x_i$, $i = 1, \dots, r$, we obtain A°-submodules $N(x_i) \subset A^N$. Let $I_{\sigma,d} \subset A^N$ be the ideal generated by the $N(x_i)$, so that $I_{\sigma,d}$ is the smallest ideal $I \subset A^N$ such that $\delta_\sigma(d) \in IQ$. We take $A^{\mathrm{st}} = A^N / \sum_{\sigma,d} I_{\sigma,d}$, where d runs over all elements of D_{A^N} and σ over G_K.

If B is an A-algebra, then a map $A^N \to B$ factors through A^{st} if and only if its kernel K contains $I_{\sigma,d}$ for each σ, d. Since Q is faithfully flat over A^N by Lemma (2.3.1), this is equivalent to asking that for all σ, d, $I_{\sigma,d}Q \subset KQ$, or that $\delta_\sigma(d) \in KQ$. This last condition just means that ψ_B is compatible with the action of G_K. Hence A^{st} represents the functor of the proposition, and it remains to see that $\operatorname{Spec} A^{\mathrm{st}}$ is a closed subscheme of $\operatorname{Spec} A$. We will show that the map $\operatorname{Spec} A^{\mathrm{st}} \to \operatorname{Spec} A$ is a proper monomorphism.

To show that this map is a monomorphism, we have to show that the map $N : D_B \to D_B$, if it exists, is unique. Suppose that $N, N' : D_B \to D_B$, are two endomorphisms satisfying the hypothesis of the proposition. Then Lemma (2.4.6) implies that N and N' induce the same action of G_K on $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$. Hence, for $\sigma \in G_K$ and $d \in D_B$ we have

$$d = \exp(N \otimes \beta(\sigma)) \cdot \exp(-N' \otimes \beta(\sigma)) \cdot d = \exp((N - N')\beta(\sigma))d \in D_B \otimes_{W_B} B^+_{\mathrm{cris},B}.$$

It follows that modulo the ideal $\mathrm{Fil}^2 B^+_{\mathrm{cris},B}$ of $B^+_{\mathrm{cris},B}$, we have $(N-N)'(d)\beta(\sigma) = 0$. If $\sigma \notin G_{K_\infty}$, then $\beta(\sigma) \notin \mathrm{Fil}^2 B^+_{\mathrm{cris},B}$ by Lemma (2.3.2)(3), and so we find that $N = N'$.

Finally, we check the valuative criterion for properness. Suppose that the A-algebra B is a discrete valuation ring, with uniformiser π_B. Let $N : D_{B[1/\pi_B]} \to D_{B[1/\pi_B]}$ be an endomorphism satisfying the conditions of the proposition. Let $\sigma \in G_K$ be such that $\beta(\sigma) \neq 0$. If $d \in D_B$, then

$$\exp(N \otimes \beta(\sigma)) \cdot d \in D_B \otimes_{W_B} B^+_{\mathrm{cris},B}[1/\pi_B] \cap \mathrm{Hom}_B(V_B, B^+_{\mathrm{cris},B}) \tag{2.4.8}$$

where we regard the left hand side of (2.4.3) as a subspace of the right hand side. Since the cokernel of (2.4.3) is B-flat, it has no π_B-torsion, and so the right hand side of (2.4.8) is equal to $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$.

Hence we find that modulo $\mathrm{Fil}^2 B^+_{\mathrm{cris},B}$,

$$d - \exp(N \otimes \beta(\sigma))d = -N(d) \otimes \beta(\sigma) \in D_B \otimes_{W_B} B^+_{\mathrm{cris},B}/\mathrm{Fil}^2 B^+_{\mathrm{cris},B}.$$

It follows from Lemma (2.3.2)(3) that $N(d) \in D_B$, so that N induces a map $N : D_B \to D_B$. This endomorphism satisfies the conditions of the proposition because it does so after tensoring by $\otimes_B B[1/\pi_B]$. □

(2.5) We will show that A^{st} is the maximal quotient of A over which the representation V_A is semi-stable with Hodge-Tate weights in $[0, h]$ (in a sense to be made precise below). To do this we recall the relationship between weakly admissible filtered (φ, N)-modules, and $\mathfrak{S}$-lattices of finite E-height established in [Ki 2].

Suppose that D is a weakly admissible filtered (φ, N)-module, with $\mathrm{Fil}^0 D = D$, and $\mathrm{Fil}^{h+1} D = 0$. Then there is a finite free $\mathfrak{S}[1/p]$-module $\mathfrak{M}$, equipped with a map $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ whose cokernel is killed by $E(u)^h$, which is functorially attached to D, and which has the following properties [Ki 2, 1.3.15]:

(1) There is a canonical φ-compatible isomorphism $\mathfrak{M}/u\mathfrak{M} \xrightarrow{\sim} D$.
(2) If $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}[1/p]} \mathcal{O}$, then $\mathcal{M}$ admits a unique meromorphic connection

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}}[1/u\lambda]$$

such that $\nabla \circ \varphi = \varphi \circ \nabla$ and which induces a differential operator

$$N_\nabla : \mathcal{M} \to \mathcal{M}, \quad m \mapsto \langle \nabla(m), -u\lambda \frac{d}{du} \rangle$$

such that $N_\nabla|_{u=0} = N$.

(3) $\mathfrak{M}$ admits a finite free φ-stable $\mathfrak{S}$-submodule $\mathfrak{M}^\circ$ which spans $\mathfrak{M}$, and such that the cokernel of $1 \otimes \varphi : \varphi^*(\mathfrak{M}^\circ) \to \mathfrak{M}^\circ$ is killed by $E(u)^h$.

There is a G_{K_∞}-equivariant isomorphism [Ki 2, 2.1.5]

$$\mathrm{Hom}_{\mathfrak{S}[1/p],\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}). \tag{2.5.1}$$

We briefly recall the construction of this map, as we will need a slightly modified formulation of it.

Given any φ-compatible map $g : D \to B^+_{\mathrm{cris}}$, the composite

$$D \hookrightarrow D \otimes_{K_0} K_0[\ell_u] \overset{(2.4.4)}{\to} D \otimes_{K_0} K_0[\ell_u] \overset{g\otimes 1}{\to} B^+_{\mathrm{st}}$$

is compatible with φ and N. We will say that g is compatible with filtrations if the resulting map $D \to B^+_{\mathrm{st}}$ is compatible. Since any map of (φ, N)-modules $D \to B^+_{\mathrm{st}}$ extends to a $K_0[\ell_u]$-linear map $D \otimes_{K_0} K_0[\ell_u] \to B^+_{\mathrm{st}}$, it induces a map

$$D \xrightarrow{\sim} (D \otimes_{K_0} K_0[\ell_u])^{N=0} \to B^+_{\mathrm{cris}}$$

(the inverse of the isomorphism being given by $\ell_u \mapsto 0$), and we see that there are bijections

$$\mathrm{Hom}_{B^+_{\mathrm{cris}},\mathrm{Fil},\varphi}(D \otimes_{K_0} B^+_{\mathrm{cris}}, B^+_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B^+_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}). \tag{2.5.2}$$

The composite is compatible with the action of G_K, where $\sigma \in G_K$ acts on the left hand side as in Subsection (2.4) (with $B = \mathbb{Q}_p$). Again, this is easily seen using the explicit inverse to the map $(D \otimes_{K_0} K_0[\ell_u])^{N=0} \overset{\ell_u \mapsto 0}{\to} D$ given by $d \mapsto \exp(-N \otimes \ell_u) \cdot d$.

Now given a φ-equivariant map $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}[1/p]$, we can tensor it by $\otimes_{\mathfrak{S}[1/p]} \mathcal{O}$, and then compose it with the section $\xi : D \to \mathcal{M}$ of Lemma (2.2) on the left, and with $\mathcal{O} \to B^+_{\mathrm{cris}}$ on the right. It is shown in [Ki 2, 2.1.5] that this induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{S}[1/p],\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B^+_{\mathrm{cris}}), \tag{2.5.3}$$

where the maps on the right respect filtrations in the sense explained above. This map, when composed with (2.5.2), yields the isomorphism (2.5.1).

Proposition (2.5.4). *We continue to assume that $A^\circ = (A^\circ)^{\leqslant h}$. Let B be a finite $\mathbb{Q}_p$-algebra, $\zeta : A \to B$ a map of $\mathbb{Q}_p$-algebras, and $V_B = V_A \otimes_A B$. Then ζ factors through A^{st} if and only if V_B is semi-stable when viewed as a $\mathbb{Q}_p$-representation.*

Proof. Suppose first that ζ factors through A^{st}. Then (2.4.5) is compatible with Galois actions, and injective by Lemma (2.4.6). In particular, if V_B^* denotes the B-dual of V_B, then $(V_B^* \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}})^{G_K}$ is of K_0-dimension at least $\dim_{K_0} D_B = \dim_{\mathbb{Q}_p} V_B$. Hence V_B^* is semi-stable and so is V_B.

Suppose that V_B is semi-stable. Let

$$\tilde{D}_B = \mathrm{Hom}_{B[G_K]}(V_B, B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B)$$

be the weakly admissible filtered (φ, N)-module associated to V_B^*. Let $\tilde{\mathfrak{M}}_B$ be the $\mathfrak{S}[1/p]$-module attached to $\tilde{D}_B$, as discussed in Subsection (2.5), and set $\mathfrak{M}_B =$

$\mathfrak{M}_A \otimes_A B$. Using the map ι of Subsection (2.4) and (2.5.1), we have G_{K_∞}-equivariant isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\mathfrak{M}_B, \mathfrak{S}_B^{\mathrm{ur}}) &\xrightarrow{\sim} V_B \\ &\xrightarrow{\sim} \mathrm{Hom}_{B,\mathrm{Fil},\varphi,N}(\tilde{D}_B, B^+_{\mathrm{st},B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\tilde{\mathfrak{M}}_B, \mathfrak{S}_B^{\mathrm{ur}}).\end{aligned}$$

More precisely the third isomorphism is obtained from (2.5.1) by applying $\otimes_{\mathbb{Q}_p} B$ and taking B-linear maps on both sides.

The uniqueness of $\mathfrak{S}$-lattices of height $\leqslant h$, implies that we may identify $\mathfrak{M}_B$ and $\tilde{\mathfrak{M}}_B$ inside $M_B = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V_B^*)^{G_K}$. Hence by Subsection (2.5)(1), we may identify $\tilde{D}_B$ with $D_B = \mathfrak{M}_B/u\mathfrak{M}_B$, and we equip D_B with the operator N induced from $\tilde{D}_B$. Now consider the commutative diagram, obtained by using the discussion of Subsection (2.5):

$$\begin{tikzcd}
V_B \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\tilde{\mathfrak{M}}_B, \mathfrak{S}_B^{\mathrm{ur}}[1/p]) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Hom}_{B^+_{\mathrm{cris},B},\mathrm{Fil},\varphi}(\tilde{D}_B \otimes_B B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B}) \arrow[d] \\
V_B \arrow[r, "\sim"] & \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\mathfrak{M}_B, \mathfrak{S}_B^{\mathrm{ur}}[1/p]) \arrow[r] & \mathrm{Hom}_{B^+_{\mathrm{cris},B}}(D_B \otimes_B B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B})
\end{tikzcd}$$

By definition the composite of the maps in the top row is compatible with the action of G_K. Hence so is the composite of the maps in the bottom row. The latter composite induces a G_K-equivariant map $D_B \otimes_B B^+_{\mathrm{cris},B} \to \mathrm{Hom}_B(V_B, B^+_{\mathrm{cris},B})$, and an inspection of the definitions shows that this equal to the map of (2.4.3). It follows that ζ factors through A^{st}. $\square$

Theorem (2.5.5). *Let A° be a complete local Noetherian $W(\mathbb{F})$-algebra, and V_{A° a finite free A°-module of rank r, equipped with a continuous action of G_K. If h is a non-negative integer, then there exists a quotient $A^{\mathrm{st},h}$ of $A = A^\circ[1/p]$ with the following properties:*

1. *If B is a finite $\mathbb{Q}_p$-algebra, and $\zeta : A \to B$ a map of $\mathbb{Q}_p$-algebras, then ζ factors through $A^{\mathrm{st},h}$ if and only if $V_B = V_A \otimes_A B$ is semi-stable with Hodge-Tate weights in $[0, h]$.*
2. *There is a projective $W_{A^{\mathrm{st},h}}$-module $D_{A^{\mathrm{st},h}}$ of rank r equipped with a Frobenius semi-linear automorphism φ, and a $W_{A^{\mathrm{st},h}}$-linear automorphism N, such that for ζ which factors through $A^{\mathrm{st},h}$, there is a canonical isomorphism*
$$D_B = D_{A^{\mathrm{st},h}} \otimes_{A^{\mathrm{st},h}} B \xrightarrow{\sim} \mathrm{Hom}_{B[G_K]}(V_B, B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B)$$
respecting the action of φ and N.

Proof. If V_B is semi-stable with Hodge-Tate weights in $[0, h]$, then V_B is of E-height $\leqslant h$, by [Ki 2, 1.2.2, 2.1.5]. Hence we may replace A° by its quotient $(A^\circ)^{\leqslant h}$, and assume that we are in the situation described in Subsection (2.4). We set $A^{\mathrm{st},h}$ equal to the ring A^{st} of Proposition (2.4.7), and $D_{A^{\mathrm{st}}} = \mathfrak{M}_A/u\mathfrak{M}_A \otimes_A A^{\mathrm{st}}$. If V_B is semi-stable, then ζ factors through A^{st} by Proposition (2.5.4). Conversely, if ζ factors through A^{st}, then Proposition (2.5.4) implies that V_B is semi-stable of E-height $\leqslant h$. If $\tilde{D}_B = (V_B^* \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}})^{G_K}$ and $\tilde{\mathfrak{M}}_B$ is the $\mathfrak{S}[1/p]$-module associated to $\tilde{D}_B$, then the uniqueness of $\mathfrak{S}$-lattices of finite E-height implies that $\tilde{\mathfrak{M}}_B$ has E-height $\leqslant h$, and the claim about Hodge-Tate weights now follows from [Ki 2, 1.2.2].

To see (2), note that we have G_K-equivariant isomorphisms

(2.5.6)
$$V_B \xrightarrow{\sim} \mathrm{Hom}_{B^+_{\mathrm{cris},B},\mathrm{Fil},\varphi}(D_B \otimes_B B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B}) \xrightarrow{\sim} \mathrm{Hom}_{B,\mathrm{Fil},N,\varphi}(D_B, B^+_{\mathrm{st},B}).$$

Here the first isomorphism was seen in the proof of Proposition (2.5.4) using the identification of D_B and $\tilde{D}_B$, while the second is deduced from (2.5.2) by applying $\otimes_{\mathbb{Q}_p} B$ and taking B-linear maps on both sides. Now (2) follows by applying $\mathrm{Hom}_{B[G_K]}(\_, B^+_{\mathrm{st},B})$ to both sides of (2.5.6). □

(2.6) It is often convenient to construct a refinement (i.e. a quotient) of $A^{\mathrm{st},h}$ corresponding to semi-stable representations with a given p-adic Hodge type. To recall the meaning of this, let E be a finite extension of $\mathbb{Q}_p$ and suppose that A admits the structure of an E-algebra, which we fix from now on.

Suppose we are given a finite dimensional E-vector space D_E, and a filtration of $D_{E,K} = D_E \otimes_{\mathbb{Q}_p} K$ by $E \otimes_{\mathbb{Q}_p} K$-submodules such that the associated graded is concentrated in degrees in $[0,h]$. We set $\mathbf{v} = \{D_E, \mathrm{Fil}^i D_{E,K},\ i = 0,\dots,h\}$.

If B is a finite E-algebra and V_B is a finite free B-module of rank r, equipped with a continuous action of G_K, which makes V_B into a de Rham representation, then we say that V_B is of p-adic Hodge type $\mathbf{v}$ if V_B has all its Hodge-Tate weights in $[0,h]$ and for $i = 0,1,\dots,h$ there is an isomorphism of $B \otimes_{\mathbb{Q}_p} K$-modules

$$\mathrm{gr}^i \mathrm{Hom}_{B[G_K]}(V_B, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} B) \xrightarrow{\sim} \mathrm{gr}^i D_{E,K} \otimes_E B.$$

We will show that there is a quotient $A^{\mathrm{st},\mathbf{v}}$ of A^{st} corresponding to representations of p-adic Hodge type $\mathbf{v}$. We need the following

Lemma (2.6.1). *For $i \geq 0$, set*

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A) = (1 \otimes \varphi)^{-1}(E(u)^i \mathfrak{M}_A) \subset \varphi^*(\mathfrak{M}_A).$$

Then the following are finite projective $K_A = W_A \otimes_{K_0} K$-modules:

(1) $\mathfrak{M}_A/(1 \otimes \varphi)(\varphi^*(\mathfrak{M}_A))$.
(2) $\varphi^*(\mathfrak{M}_A)/\mathrm{Fil}^h \varphi^*(\mathfrak{M}_A)$.
(3) $\mathrm{Fil}^h \varphi^*(\mathfrak{M}_A)/E(u)^i \mathrm{Fil}^{h-i} \varphi^*(\mathfrak{M}_A)$ *where* $\mathrm{Fil}^j \varphi^*(\mathfrak{M}_A) = \varphi^*(\mathfrak{M}_A)$ *for* $j < 0$.
(4) $\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A))$.

Proof. The K_A-modules in (1)-(4) are easily seen to be finite, and it suffices to show that they are flat W_A-modules.

The argument for (1) is analogous to that in [Ki 1, 1.2.2]. First we remark that for any W-algebra R, $E(u)$ is not a zero-divisor in $R[\![u]\!]$. Now the determinant of $1 \otimes \varphi : \varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ (in any choice of $\mathfrak{S}_A$-bases) is a divisor of $E(u)^h$. Hence, if $I \subset W_A$ is any ideal, then $1 \otimes \varphi$ remains injective after $\otimes_{W_A} W_A/I$, and this shows that the W_A-module in (1) is flat.

The flatness of the W_A-module in (2) now follows from the exact sequence

$$0 \to \varphi^*(\mathfrak{M}_A)/\mathrm{Fil}^h \varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A/E(u)^h \mathfrak{M}_A \to \mathfrak{M}_A/(1 \otimes \varphi)(\varphi^*(\mathfrak{M}_A)) \to 0$$

and that in (3) from the exact sequence

$$\begin{aligned} 0 \to \mathrm{Fil}^h \varphi^*(\mathfrak{M}_A)/E(u)^i \mathrm{Fil}^{h-i} \varphi^*(\mathfrak{M}_A) \\ \to \varphi^*(\mathfrak{M}_A)/E(u)^i \varphi^*(\mathfrak{M}_A) \to \varphi^*(\mathfrak{M}_A)/\mathrm{Fil}^h \varphi^*(\mathfrak{M}_A) \to 0. \end{aligned}$$

Using (3) one sees that $\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/E(u)\mathrm{Fil}^{i-1} \varphi^*(\mathfrak{M}_A)$ is W_A-flat, and this is equal to the module in (4), since $E(u)\mathrm{Fil}^{i-1} \varphi^*(\mathfrak{M}_A) = (E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A))$. □

Corollary (2.6.2). *Fix* $\mathbf{v}$ *as above. There exists a quotient* $A^{\mathrm{st},\mathbf{v}}$ *of* A^{st} *corresponding to a union of connected components of* $\operatorname{Spec} A^{\mathrm{st}}$ *such that, if* B *is a finite* E*-algebra, and* $\zeta : A \to B$ *is a map of* E*-algebras, then* ζ *factors through* $A^{\mathrm{st},\mathbf{v}}$ *if and only if* $V_B = V_A \otimes_A B$ *is semi-stable of* p*-adic Hodge type* $\mathbf{v}$.

Proof. By Lemma (2.6.1) $\mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A))$ is a finite projective K_A-module. Consider the points $\mathfrak{p}$ of $\operatorname{Spec} A$ such that for $i = 0, 1, \ldots, h$, there is an isomorphism of $K_{A_\mathfrak{p}} = K_A \otimes_A A_\mathfrak{p}$-modules

$$(\mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A))) \otimes_A A_\mathfrak{p} \xrightarrow{\sim} \mathrm{Fil}^i D_{E,K} \otimes_E A_\mathfrak{p}.$$

The set of such $\mathfrak{p}$ consists of a union of connected components of $\operatorname{Spec} A$, and corresponds to a quotient $A^{\mathbf{v}}$ of A. We set $A^{\mathrm{st},\mathbf{v}} = A^{\mathrm{st},h} \otimes_A A^{\mathbf{v}}$.

To see that $A^{\mathrm{st},h}$ has the required property consider a map of E-algebras $\zeta : A \to B$, with B local and finite over E. By Lemma (2.2), we may identify $D_B \otimes_{K_0} K$ with $\varphi^*(\mathfrak{M}_B)/E(u)\varphi^*(\mathfrak{M}_B)$, where $\mathfrak{M}_B = \mathfrak{M}_A \otimes_A B$, and we equip $D_B \otimes_{K_0} K$ with a filtration by setting

$$\mathrm{Fil}^i(D_B \otimes_{K_0} K) = \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)) \otimes_A B.$$

To prove the corollary we have to check that this filtration coincides with that induced by the right hand side of the isomorphism in Theorem (2.5.5)(2).

The argument for this is similar to that in Proposition (2.5.4): Let $\tilde{D}_B$ denote D_B equipped with the filtration induced by the right hand side of Theorem (2.5.5)(2), so that $\tilde{D}_B$ is the weakly admissible module corresponding to V_B^*. Let $\tilde{\mathfrak{M}}_B$ be the $\mathfrak{S}[1/p]$-module attached to $\tilde{D}_B$. By [Ki 2, 1.2.8], we may identify $\mathrm{Fil}^i\tilde{D}_B \otimes_{K_0} K$ with the quotient $\mathrm{Fil}^i\varphi^*(\tilde{\mathfrak{M}}_B)/(E(u)\varphi^*(\tilde{\mathfrak{M}}_B) \cap \mathrm{Fil}^i\varphi^*(\tilde{\mathfrak{M}}_B))$, and the uniqueness of lattices of E-height $\leqslant h$ shows that we may identify $\tilde{\mathfrak{M}}_B$ with $\mathfrak{M}_A \otimes_A B$. $\square$

(2.7) Let B be a finite, local $\mathbb{Q}_p$-algebra, and V_B a finite free B-module equipped with a continuous action of G_K. Suppose that V_B is potentially semi-stable. Following [Fo 2] we set

$$D^*_{\mathrm{pst}}(V_B) = \varinjlim_{K'} \mathrm{Hom}_{B[G_{K'}]}(V_B, B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B),$$

where K' runs over finite extensions of K.

Let $\bar{K}_0 \subset \bar{K}$ denote the union of the finite unramified extensions of K_0 contained in $\bar{K}$, and $I_K \subset G_K$ the inertia group. Then $D^*_{\mathrm{pst}}(V_B)$ is a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-module, equipped with a semi-linear Frobenius automorphism φ, a nilpotent endomorphism N, satisfying $p\varphi N = N\varphi$, and a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-linear action of I_K, which has open kernel and commutes with φ and N.

Using the fact that φ is an automorphism one sees that $D^*_{\mathrm{pst}}(V_B)$ is actually a finite free $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-module (cf. [Ki 1, 1.2.7]). Since the action of I_K commutes with that of φ, the traces of elements of I_K are contained in B, and there is a finite étale, local B-algebra B', such that $D^*_{\mathrm{pst}}(V_B) \otimes_B B'$ descends to a representation of I_K on a finite free B'-module $\tilde{P}_{B'}$. Finally, any such representation with open kernel arises by extension of scalars from an E'-representation $P_{B'}$, where E'-denotes the residue field of B' (which is canonically a subfield of B'). This follows easily from the fact that the cohomology of a finite group with coefficients in an E'-representation is trivial in degree > 0.

If $\bar{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$, and $\tau : I_K \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ is a representation with open kernel, then we will say V_B is potentially semi-stable of type τ, if P_B is

equivalent to τ. In other words, for any $\gamma \in I_K$, the trace of $\tau(\gamma)$ is equal to the trace of γ on $D^*_{\mathrm{pst}}(V_B)$.

We will use the above results to construct deformation rings for potentially semi-stable representations of fixed type. We begin with a lemma.

Lemma (2.7.1). *Let A° be as in Subsection (2.1). For $i \geq 0$ there is an isomorphism*

$$W_A \cdot t^i \xrightarrow{\sim} \mathrm{Hom}_{A[G_K]}(A(i), B^+_{\mathrm{st},A}),$$

where $A(i)$ denotes A with G_K acting via the ith-power of the p-adic cyclotomic character χ.

In particular, if $B_{\mathrm{st},A} = B^+_{\mathrm{st},A}[1/t]$, then $B^{G_K}_{\mathrm{st},A} = W_A$.

Proof. First we remark that the lemma is well known if $A^\circ = \mathbb{Z}_p$, and hence it follows easily when A° is finite over $\mathbb{Z}_p$. Suppose that $x = \sum_{i=0}^n a_i \ell_u^i \in B^+_{\mathrm{st},A}$ with $a_i \in B^+_{\mathrm{cris},A}$, and that G_K acts on x via χ^i. Using Lemma (2.3.2)(5), and the case of A° finite over $\mathbb{Z}_p$, we see that $a_i = 0$ if $i > 0$. After multiplying x by a power of p, we may assume that $x = a_0 \in A_{\mathrm{cris},A^\circ}$.

Now let $\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \dots$ be a decreasing sequence of ideals of A, such that $\bigcap_{j=1}^\infty \mathfrak{q}_j = \{0\}$ and $A/\mathfrak{q}_j$ is a finite $W(\mathbb{F})[1/p]$-algebra. Set $\mathfrak{q}_j^\circ = A^\circ \cap \mathfrak{q}_j$. Then for each $m \geq 0$, we have $\mathfrak{q}_j^\circ \subset \mathfrak{m}_{A^\circ}^m$ for sufficiently large j (this holds for any decreasing sequence of ideals of A° with trivial intersection), and $A \xrightarrow{\sim} \varprojlim A/\mathfrak{q}_j^\circ$. Hence we have

$$A_{\mathrm{cris},A^\circ} \xrightarrow{\sim} \varprojlim_j A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/\mathfrak{q}_j^\circ = \varprojlim_j A_{\mathrm{cris},A^\circ/\mathfrak{q}^\circ}$$

and similarly with W in place of A_{cris}.

If r_i is the greatest integer such that $t^i/p^{r_i} \in A_{\mathrm{cris}}$, then for $j \geq 1$, the image of a_0 in $A^+_{\mathrm{cris},A^\circ/\mathfrak{q}_j^\circ}$ is contained in $t^i/p^{r_i} \cdot W_{A^\circ}$. This is easily seen using the case $A^\circ = \mathbb{Z}_p$, as above. Passing to the inverse limit we find that $a_0 \in W_{A^\circ} \cdot t^i/p^{r_i} \subset t^i \cdot W_A$. $\square$

Proposition (2.7.2). *Suppose that $h \geq 0$, and that $A = A^{\mathrm{st},h}$. Then (2.4.5) (with $B = A$) induces an isomorphism*

$$D_A \otimes_{W_A} B_{\mathrm{st},A} \xrightarrow{\sim} \mathrm{Hom}_A(V_A, B_{\mathrm{st},A}). \tag{2.7.3}$$

In particular, we have

$$D_A \xrightarrow{\sim} \mathrm{Hom}_{A[G_K]}(V_A, B^+_{\mathrm{st},A}). \tag{2.7.4}$$

Proof. To see (2.7.3) we make two observations. First if $A^\circ \to A'^\circ$ is a local map of complete local Noetherian rings with residue field $\mathbb{F}$, and $A' = A'^\circ[1/p]$, then the map (2.7.3) for $V_{A'^\circ} = V_{A^\circ} \otimes_{A^\circ} A'^\circ$ is obtained from that for V_{A° by applying $\otimes_{B_{\mathrm{st},A}} B_{\mathrm{st},A'}$ (D_A being replaced by $D_A \otimes_{W_A} W_{A'}$). This is easily seen, directly from the construction of the map. In particular if (2.7.3) is an isomorphism for V_{A°, then it is an isomorphism for $V_{A'^\circ}$.

Next, if V_{A° is an unramified representation, then the proposition is also easily checked, for if $\bar{k}$ is the residue field of $\bar{K}$, then the map (2.4.2) is induced by an isomorphism $D_A \xrightarrow{\sim} \mathrm{Hom}_{A[G_K]}(V_A, W(\bar{k})_A)$, and D_A spans the $W(\bar{k})_A$-module $\mathrm{Hom}_A(V_A, W(\bar{k})_A)$. Hence D_A also spans the right hand side of (2.4.2) as a $B^+_{\mathrm{cris},A}$-module.

Now (2.7.3) is a map of finite free $B_{\mathrm{st},A}$-modules of the same rank, so it suffices to show that it induces an isomorphism on top exterior powers. Hence we may assume

that V_{A° is free of rank 1 over A°. By Corollary (2.6.2), $V_A|_{I_K}$ is locally constant on $\operatorname{Spec} A$, locally equal to $\theta|_{I_K}$, where θ is a product of conjugates of Lubin-Tate characters for K. In particular, θ takes values in a finite extension of $\mathbb{Z}_p$. Working locally on A, we may assume that $V_A|_{I_K} \sim \theta|_{I_K}$. Hence it suffices to consider the two cases where $V_A \sim \theta$ and where V_A is an unramified character. The unramified case follows from our second observation. If $V_A \sim \theta$, then our first observation shows that we may assume A° is the ring of integers in a finite extension of $\mathbb{Q}_p$. In this case the fact that (2.7.3) is an isomorphism follows, for example, from Theorem (2.5.5)(2).

Taking G_K-invariants in (2.7.3), we find that $D_A \xrightarrow{\sim} \operatorname{Hom}_{A[G_K]}(V_A, B_{\mathrm{st},A})$ by Lemma (2.7.1). Since (2.4.5) sends D_A into $\operatorname{Hom}_{A[G_K]}(V_A, B^+_{\mathrm{st},A})$, (2.7.4) follows. □

(2.7.5) Suppose now that E is a finite extension of $\mathbb{Q}_p$ and $\mathbf{v}$ is a p-adic Hodge type as in Subsection (2.6), consisting of an E-vector space D_E of dimension r and a collection of $E \otimes_{\mathbb{Q}_p} K$ submodules of $D_{E,K}$. Let A° be a Noetherian, complete local $\mathcal{O}_E$-algebra and V_{A° a finite free A°-module of rank r equipped with a continuous action of G_K. As usual, we set $A = A^\circ[1/p]$. Finally, we fix a representation

$$\tau : I_K \to \operatorname{End}_E(D_E) \xrightarrow{\sim} \mathrm{GL}_r(E)$$

with open kernel.

Theorem (2.7.6). *There exists a quotient $A^{\tau,\mathbf{v}}$ of A such that for any finite E-algebra B, a map of E-algebras $\zeta : A \to B$ factors through $A^{\tau,\mathbf{v}}$ if and only if $V_B = V_A \otimes_A B$ is potentially semi-stable of type τ and with p-adic Hodge type $\mathbf{v}$.*

Proof. Let L/K be a finite Galois extension such that the inertia subgroup $I_L \subset I_K$ is contained in $\ker \tau$. By Corollary (2.6.2) there exists a quotient $A^{\mathrm{pst},\mathbf{v}}$ of A such that ζ factors through $A^{\mathrm{pst},\mathbf{v}}$ if and only if $V_B|_{G_L}$ is semi-stable with p-adic Hodge type $\mathbf{v}$. We may assume $A = A^{\mathrm{pst},\mathbf{v}}$.

Let W_L denote the ring of integers of L_0, and write $W_{L,A} = (W_L)_A$. By Proposition (2.7.2) we have an isomorphism of finite free $W_{L,A}$-modules,

$$D_A \xrightarrow{\sim} \operatorname{Hom}_{A[G_L]}(V_A, B^+_{\mathrm{st},A})$$

which is compatible with the action of φ. The group $\mathrm{Gal}(L/K)$ acts L_0-semi-linearly on $\operatorname{Hom}_{A[G_L]}(V_A, B^+_{\mathrm{st},A})$, and the inertia subgroup $I_{L/K} \subset \mathrm{Gal}(L/K)$ acts L_0-linearly. Since the action of $\mathrm{Gal}(L/K)$ commutes with φ, if $\sigma \in I_{L/K}$ the trace $\mathrm{tr}(\sigma)$ is in $(W_{L,A})^{\varphi=1} = A$. The discussion of Subsection (2.7) shows that $\mathrm{tr}(\sigma)$ is a locally constant function on $\operatorname{Spec} A$. We denote by $A^{\tau,\mathbf{v}}$ the quotient of A corresponding to the union of components of $\operatorname{Spec} A$ where $\mathrm{tr}(\sigma) = \mathrm{tr}(\tau(\sigma))$ for all $\sigma \in I_K$. One sees from the definitions that this quotient has the required property. □

Corollary (2.7.7). *There exists a quotient $A^{\tau,\mathbf{v}}_{\mathrm{cr}}$ of A such that for any finite E-algebra B, a map of E-algebras $\zeta : A \to B$ factors through $A^{\tau,\mathbf{v}}_{\mathrm{cr}}$ if and only if $V_B = V_A \otimes_A B$ is potentially crystalline of type τ and with p-adic Hodge type $\mathbf{v}$.*

Proof. By Theorem (2.5.5), there is a finite projective $W_{A^{\tau,\mathbf{v}}}$-module $D_{A^{\tau,\mathbf{v}}}$ equipped with a linear endomorphism N such that V_B is potentially crystalline if and only if the specialization of N by ζ vanishes. Hence we may take $A^{\tau,\mathbf{v}}_{\mathrm{cr}}$ to be the quotient of $A^{\tau,\mathbf{v}}$ defined by the equation $N = 0$. □

§3. The local structure of potentially semi-stable deformation rings

(3.1) Using the results of the previous section one can immediately show the existence of deformation rings for potentially semi-stable Galois representations corresponding to a given type and p-adic Hodge type, and we will do this below. Most of the effort in this section is directed toward establishing results about the generic fibers of these rings. Specifically we will show that the components of the generic fiber are generically formally smooth, and we will compute their dimensions. In fact our results allow one, in principle, to determine the local structure of these generic fibers even at non-smooth points. However, we do not make this explicit.

(3.1.1) Fix a finite Galois extension L of K, and denote by L_0 the maximal unramified subfield of L. We also fix a positive integer d. Write $G_{L/K} = \mathrm{Gal}(L/K)$. It will be convenient to use the language of groupoids. What we need is contained in the appendix to [Ki 1]. We begin by defining two groupoids on the category of $\mathbb{Q}_p$-algebras A. For such an algebra, we extend the action of φ to $L_0 \otimes_{\mathbb{Q}_p} A$ by A-linearity. We will again denote by A the corresponding groupoid on $\mathbb{Q}_p$-algebras.

Let $\mathfrak{Mod}_N$ be the groupoid whose fiber over a $\mathbb{Q}_p$-algebra A consists of finite projective $L_0 \otimes_{\mathbb{Q}_p} A$-modules D_A of rank d, equipped with a semi-linear action of $G_{L/K}$, and a nilpotent linear operator N. We require that the action of N commutes with $G_{L/K}$, and that D_A is free over $L_0 \otimes_{\mathbb{Q}_p} A$, locally on $\mathrm{Spec}\, A$. The last condition means that for any prime $\mathfrak{p}$ of A, $(D_A)_{\mathfrak{p}}$ is free over $L_0 \otimes_{\mathbb{Q}_p} A_{\mathfrak{p}}$.

Next let $\mathfrak{Mod}_{\varphi,N}$ be the groupoid whose fiber over a $\mathbb{Q}_p$-algebra A consists of a module D_A in $\mathfrak{Mod}_N$ equipped with a Frobenius semi-linear automorphism φ such that $p\varphi N = N\varphi$. We have an obvious morphism $\mathfrak{Mod}_{\varphi,N} \to \mathfrak{Mod}_N$.

Given D_A in $\mathfrak{Mod}_{\varphi,N}$ we set $\mathrm{ad}D_A = \mathrm{Hom}_{L_0 \otimes_{\mathbb{Q}_p} A}(D_A, D_A)$. We equip $\mathrm{ad}D_A$ with an operator φ given by sending a map f to $\varphi \circ f \circ \varphi^{-1}$ and an operator N given by sending f to $N \circ f - f \circ N$. These satisfy $p\varphi N = N\varphi$. We also equip $\mathrm{ad}D_A$ with an action of $G_{L/K}$, with $\gamma \in G_{L/K}$ sending f to $\gamma \circ f \circ \gamma^{-1}$. Consider the anti-commutative diagram

$$\begin{array}{ccc} (\mathrm{ad}D_A)^{G_{L/K}} & \xrightarrow{1-\varphi} & (\mathrm{ad}D_A)^{G_{L/K}} \\ \downarrow N & & \downarrow N \\ (\mathrm{ad}D_A)^{G_{L/K}} & \xrightarrow{p\varphi-1} & (\mathrm{ad}D_A)^{G_{L/K}} \end{array}$$

We denote by $C^\bullet(D_A)$ the total complex of this double complex, concentrated in degrees $0, 1$ and 2, and we write $H^\bullet(D_A)$ for the cohomology of $C^\bullet(D_A)$. The following result describes how this complex controls the deformation theory of $\mathfrak{Mod}_{\varphi,N}$ (cf. [FP, I. §1.4] and [EK, §3]).

Proposition (3.1.2). *Let A be a local $\mathbb{Q}_p$-algebra with maximal ideal $\mathfrak{m}_A$, and $I \subset A$ an ideal with $I\mathfrak{m}_A = 0$. Let $D_{A/I}$ be in $\mathfrak{Mod}_{\varphi,N}(A/I)$ and set $D_{A/\mathfrak{m}_A} = D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A$.*

(1) *If $H^2(D_{A/\mathfrak{m}_A}) = 0$, then there exists a module D_A in $\mathfrak{Mod}_{\varphi,N}(A)$ whose reduction modulo I is isomorphic to $D_{A/I}$.*
(2) *The set of isomorphism classes of liftings of $D_{A/I}$ to D_A in $\mathfrak{Mod}_{\varphi,N}(A)$ is either empty or is a torsor under $H^1(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$. (Two liftings D_A*

and D'_A are isomorphic if there exists a map $D_A \to D'_A$ respecting φ, N and $G_{L/K}$, and reducing to the identity modulo I.)

Proof. Since A is local, $D_{A/I}$ is a free $A/I \otimes_{\mathbb{Q}_p} L_0$-module. Let D_A be a free $A \otimes_{\mathbb{Q}_p} L_0$-module, equipped with an isomorphism $D_A \otimes_A A/I \xrightarrow{\sim} D_{A/I}$. Since the cohomology of the finite group $G_{L/K}$ with coefficients in any $\mathbb{Q}_p$-representation vanishes in degree > 0, the action of $G_{L/K}$ on $D_{A/I}$ lifts to an $A \otimes_{\mathbb{Q}_p} L_0$-semi-linear action on D_A, and any two such liftings differ by an automorphism of D_A which reduces to the identity modulo I. Similarly, a $G_{L/K}$-invariant map in $\mathrm{Hom}_{A/I \otimes_{\mathbb{Q}_p} L_0}(D_{A/I}, D_{A/I})$ lifts to a $G_{L/K}$-invariant map in $\mathrm{Hom}_{A \otimes_{\mathbb{Q}_p} L_0}(D_A, D_A)$, so we can lift N to an endomorphism $\tilde{N}$ of D_A. In the same way, by lifting the map $\varphi^*(D_{A/I}) \to D_{A/I}$ to a $G_{L/K}$-equivariant map $\varphi^*(D_A) \to D_A$, we obtain a Frobenius semi-linear endomorphism $\tilde{\varphi}$ of D_A which commutes with the action of $G_{L/K}$.

Now $D_{A/I}$ can be lifted to an object of $\mathfrak{Mod}_{\varphi,N}(A)$ if and only if $\tilde{N}$ and $\tilde{\varphi}$ can be chosen so that $h = \tilde{N} - p\tilde{\varphi}\tilde{N}\tilde{\varphi}^{-1} = 0$. Observe that h induces a $G_{L/K}$-equivariant map $h : D_{A/\mathfrak{m}_A} \to D_{A/\mathfrak{m}_A} \otimes_A I$. Suppose that $H^2(D_{A/\mathfrak{m}_A}) = 0$, and write $h = N(f) + (p\varphi - 1)(g)$ where $f, g \in \mathrm{ad}(D_{A/\mathfrak{m}_A})^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I$. Set $\tilde{N}' = \tilde{N} + g$ and $\tilde{\varphi}' = \tilde{\varphi} - f \circ \tilde{\varphi}$. A straightforward calculation show that $\tilde{N}'\tilde{\varphi}' = p\tilde{\varphi}'\tilde{N}'$. This proves (1).

To show (2) suppose that $\tilde{N}\tilde{\varphi} = p\tilde{\varphi}\tilde{N}$ above. Let f, g be any two elements of $\mathrm{ad}(D_{A/\mathfrak{m}_A})^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I$ and define $\tilde{N}'$ and $\tilde{\varphi}'$ as before. Then $\tilde{N}'\tilde{\varphi}' = p\tilde{\varphi}'\tilde{N}'$ if and only if $N(f) + (p\varphi - 1)(g) = 0$. That is, if and only if $(f, g) \in \ker(d^1)$, where d^i denotes the differential in $H^\bullet(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$.

The lifting $(D_A, \tilde{N}', \tilde{\varphi}')$ is isomorphic to $(D_A, \tilde{N}, \tilde{\varphi})$ if and only if there exists a $j \in (\mathrm{ad}D_A)^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I$ such that $(1+j)\circ\tilde{N} = \tilde{N}'\circ(1+j)$ and $(1+j)\circ\tilde{\varphi} = \tilde{\varphi}'\circ(1+j)$. Since $\mathfrak{m}_A \cdot I = 0$, we have $(\mathrm{ad}D_A)^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I \xrightarrow{\sim} (\mathrm{ad}D_{A/\mathfrak{m}_A})^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I$, so these conditions reduce to asking that we have $\tilde{N} - \tilde{N}' = N \circ j - j \circ N$ and $\tilde{\varphi} - \tilde{\varphi}' = \varphi \circ j - j \circ \varphi = (1-\varphi)(-j)\circ\varphi$, that is, $(\tilde{N}' - \tilde{N}, (\tilde{\varphi} - \tilde{\varphi}')\circ\varphi^{-1}) = d^0(-j)$. □

Corollary (3.1.3). *Let A be a $\mathbb{Q}_p$-algebra and let D_A be in $\mathfrak{Mod}_{\varphi,N}(A)$. Suppose that the morphism $A \to \mathfrak{Mod}_{\varphi,N}$ corresponding to D_A is formally smooth, and that $H^2(D_A) = 0$. Then A is formally smooth over $\mathbb{Q}_p$.*

Proof. Let B be a local $\mathbb{Q}_p$-algebra with maximal ideal $\mathfrak{m}_B$, and $I \subset B$ an ideal with $\mathfrak{m}_B \cdot I = 0$. Given a map $A \to B/I$, let $D_{B/I} = D_A \otimes_A B/I$. Then $H^2(D_{B/I}) = 0$ so that $D_{B/I}$ lifts to a module D_B in $\mathfrak{Mod}_{\varphi,N}(B)$. Since $A \to \mathfrak{Mod}_{\varphi,N}$ is formally smooth, D_B is induced from a map $A \to B$ lifting $A \to B/I$. □

(3.1.4) Let $E/\mathbb{Q}_p$ be a finite extension, and D_E a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module of rank d equipped with a semi-linear action of $G_{L/K}$. Consider the functor which assigns to an E-algebra A the set of pairs (φ, N), where φ is a semi-linear endomorphism of $D_A = D_E \otimes_E A$ and N is a linear endomorphism of D_A such that (φ, N) makes D_A into an object of $\mathfrak{Mod}_{\varphi,N}(A)$. It is clear that this functor is represented by a E-algebra $B_{\varphi,N}$, such that $X_{\varphi,N} = \mathrm{Spec}\, B_{\varphi,N}$ is a locally closed subscheme of the vector space $\mathrm{Hom}_E(D_E, D_E)^2$.

Similarly the functor which assigns to A the set of endomorphisms N of D_A, such that N makes D_A into an object of $\mathfrak{Mod}_N$, is representable by a $\mathbb{Q}_p$-algebra B_N. Set $X_N = \mathrm{Spec}\, B_N$. We have a morphism $X_{\varphi,N} \to X_N$ given by forgetting φ.

Lemma (3.1.5). *Let $D_{B_{\varphi,N}} = D_E \otimes_{\mathbb{Q}_p} B_{\varphi,N}$, which we regard as a vector bundle on $X_{\varphi,N}$.*

(1) *The morphism of groupoids on E-algebras $B_{\varphi,N} \to \mathfrak{Mod}_{\varphi,N}$ is formally smooth.*
(2) *There is a dense open subset $U \subset X_{\varphi,N}$ such that $H^2(D_{B_{\varphi,N}})|_U = 0$.*

Proof. (1) is immediate from the definition of $X_{\varphi,N}$.

Let $U \subset X_{\varphi,N}$ denote the complement of the support of $H^2(D_{B_{\varphi,N}})$. To show U is dense, it suffices to show that it is dense in each fiber of $X_{\varphi,N} \to X_N$. Let $y \in X_N$, and let D_y denote the pullback to y of the tautological vector bundle on X_N. Then $(X_{\varphi,N})_y$ may be identified with an open subset of the $\kappa(y)$-vector space of maps $\varphi : D_y \to D_y$ which are φ-semilinear, commute with $G_{L/K}$ and satisfy $N\varphi = p\varphi N$. (Here $\kappa(y)$ denotes the residue field of y.) More precisely, $X_{\varphi,N}$ is defined by the non-vanishing of $\det \varphi$ (viewed as a $\kappa(y)$-linear map) inside this vector space. In particular, we see that $(X_{\varphi,N})_y$ is smooth and connected, so it suffices to show that, if $(X_{\varphi,N})_y$ is non-empty, then there is a point of $(X_{\varphi,N})_y$ at which $H^2(D_{B_{\varphi,N}}) = 0$.

The remark on the vanishing of cohomology of $G_{L/K}$ made in the proof of Proposition (3.1.2) shows that the category of semi-linear representations of $G_{L/K}$ on finite dimensional L_0-vector spaces is semi-simple. Hence there is a decomposition

$$D_y \xrightarrow{\sim} \oplus_\tau \mathrm{Hom}_{L_0[G_{L/K}]}(\tau, D_y) \otimes_{K_0} \tau$$

where τ runs over the irreducible semi-linear representations of $G_{L/K}$ on L_0-vector spaces. The action of N on D_y induces a nilpotent operator N of

$$D_{y,\tau} = \mathrm{Hom}_{L_0[G_{L/K}]}(\tau, D_y),$$

and the above isomorphism is then compatible with the action of N, which acts on the right hand side by $N \otimes 1$.

For $s \geq 1$ let D_s denote $(L_0 \otimes_{\mathbb{Q}_p} \kappa(y))^s$ equipped with a linear operator N which is given on the canonical basis $(e_1, \ldots, e_s)$ of D_s by $N(e_i) = e_{i+1}$ for $i = 1, \ldots, s-1$ and $N(e_s) = 0$. We claim that $D_{y,\tau}$ with its operator N is isomorphic to a direct sum of copies of the D_s for various s.

To see this note that $L_0 \otimes_{\mathbb{Q}_p} \kappa(y)$ is a product of fields. If z is a maximal ideal of $L_0 \otimes_{\mathbb{Q}_p} \kappa(y)$ and $\kappa(z)$ is the residue field at z, we denote by $D_{z,\tau}$ the restriction of $D_{y,\tau}$ to z. Since $(X_{\varphi,N})_y$ is non-empty, there exists some Frobenius semi-linear isomorphism $\varphi : D_y \to D_y$ satisfying $N\varphi = p\varphi N$. This induces an isomorphism $\varphi : D_{y,\tau} \to D_{y,\tau}$ for each τ. In particular one sees that the $\kappa(z)$-vector spaces $D_{z,\tau}$ equipped with the nilpotent operator N are all isomorphic (that is, they are independent of z). Using this one sees that $D_{y,\tau}$ is isomorphic to a sum of copies of D_s, as claimed. Fix such an isomorphism for each τ.

Now consider the Frobenius semi-linear endomorphism of D_s given by setting $\varphi(e_i) = p^{s-i}e_i$ for $i = 1, \ldots, s$. It satisfies $N\varphi = p\varphi N$. We equip $D_{y,\tau}$ and then D_y with the induced operator φ. A simple computation shows that with this choice of φ we have $H^2(D_y) = 0$. □

Proposition (3.1.6). *Let A be a Noetherian $\mathbb{Q}_p$-algebra and let D_A be in $\mathfrak{Mod}_{\varphi,N}$. If $A \to \mathfrak{Mod}_{\varphi,N}$ is formally smooth, and $U \subset \mathrm{Spec}\, A$ denotes the complement of the support of $H^2(D_A)$ in $\mathrm{Spec}\, A$, then U is dense in $\mathrm{Spec}\, A$, and formally smooth over $\mathbb{Q}_p$.*

Proof. By Corollary (3.1.3) we have to show that the support of $H^2(D_A)$ is a nowhere dense subset of $\operatorname{Spec} A$. We may assume that A is a complete local ring. Denote by κ its residue field. We fix a coefficient field for A, and regard A as a κ-algebra. Let $D_\kappa = D_A \otimes_A \kappa$. As in Subsection (2.7), there is a subfield $E \subset \kappa$ which is a finite extension of $\mathbb{Q}_p$, and a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module D_E equipped with a semi-linear action of $G_{L/K}$ such that there exists a $G_{L/K}$-equivariant isomorphism $D_E \otimes_E \kappa \xrightarrow{\sim} D_\kappa$. Similarly, there exists a $G_{L/K}$-equivariant isomorphism $D_\kappa \otimes_\kappa A \xrightarrow{\sim} D_A$ lifting the identity on D_κ, and combining these two isomorphisms we may identity D_A with $D_E \otimes_E A$. We will apply Lemma (3.1.5) with this choice of D_E.

By definition of $B_{\varphi,N}$, D_A is induced by a morphism $B_{\varphi,N} \to A$. Let $B = B_{\varphi,N} \otimes_E \kappa$ and let $\mathfrak{p} \subset B$ be the preimage of the radical of A under the induced map $B_{\varphi,N} \otimes_E \kappa \to A$. Write $\hat{B}$ for the completion of $B_\mathfrak{p}$. We may extend the induced map $B_\mathfrak{p} \to A$ to a surjection $\hat{B}[\![x_1, \dots, x_r]\!] \to A$ for some $r > 0$. Since $A \to \mathfrak{Mod}_{\varphi,N}$ is formally smooth this map has a section $A \to \hat{B}[\![x_1, \dots, x_r]\!]$ such that there is an isomorphism

$$D_A \otimes_A \hat{B}[\![x_1, \dots, x_r]\!] \xrightarrow{\sim} D_{B_{\varphi,N}} \otimes_{B_{\varphi,N}} \hat{B}[\![x_1, \dots, x_r]\!] \tag{3.1.7}$$

in $\mathfrak{Mod}_{\varphi,N}$.

If $\mathfrak{q}$ denotes a minimal prime of A, then $\mathfrak{q}$ is the preimage of a minimal prime $\tilde{\mathfrak{q}}$ of $\hat{B}[\![x_1, \dots, x_r]\!]$. Let $\kappa(\mathfrak{q})$ and $\kappa(\tilde{\mathfrak{q}})$ denote the residue fields of $\mathfrak{q}$ and $\tilde{\mathfrak{q}}$ respectively. By Lemma (3.1.5) and (3.1.7), $H^2(D_A) \otimes_A \kappa(\tilde{\mathfrak{q}}) = 0$, and hence $H^2(D_A) \otimes_A \kappa(\mathfrak{q}) = 0$, so that $\mathfrak{q}$ is not in the support of $H^2(D_A)$. □

(3.2) We now introduce filtrations. We denote by $\mathfrak{Mod}_{F,\varphi,N}$ the groupoid on $\mathbb{Q}_p$-algebras such that $\mathfrak{Mod}_{F,\varphi,N}(A)$ consists of an object D_A in $\mathfrak{Mod}_{\varphi,N}$ together with an exhaustive, separated, decreasing filtration of $D_{A,L} = D_A \otimes_{L_0} L$ by $L \otimes_{\mathbb{Q}_p} A$-submodules, which are stable under the action of $G_{L/K}$, and such that the associated graded is a projective $L \otimes_{\mathbb{Q}_p} A$-module. Here $G_{L/K}$ acts L-semi-linearly on $D_{A,L}$ via its natural action on L. We remark that this gives a well defined groupoid, since if $A \to B$ is a map of $\mathbb{Q}_p$-algebras, and D_A is in $\mathfrak{Mod}_{F,\varphi,N}(A)$, then the projectivity condition on $\operatorname{gr}^\bullet D_{A,L}$ implies that applying $\otimes_A B$ to the filtration on $D_{A,L}$ gives a well defined filtration on $D_{A,L} \otimes_A B$.

The filtration on $D_{A,L}$ induces a filtration on $(\operatorname{ad} D_A) \otimes_{L_0} L$. We denote by $C^\bullet_F(D_A)$ the total complex of the double complex

$$\begin{array}{ccccc} (\operatorname{ad} D_A)^{G_{L/K}} & \xrightarrow{1-\varphi, N} & (\operatorname{ad} D_A)^{G_{L/K}} \oplus (\operatorname{ad} D_A)^{G_{L/K}} & \xrightarrow{N, p\varphi - 1} & (\operatorname{ad} D_A)^{G_{L/K}} \\ \downarrow & & & & \\ (\operatorname{ad} D_{A,L}/\operatorname{Fil}^0 \operatorname{ad} D_{A,L})^{G_{L/K}} & & & & \end{array}$$

where the top line is the complex $C^\bullet(D_A)$ of (3.1.1). We denote by $H^\bullet_F(D_A)$ the cohomology of $C^\bullet_F(D_A)$.

Lemma (3.2.1). *The morphism of groupoids $\mathfrak{Mod}_{F,\varphi,N} \to \mathfrak{Mod}_{\varphi,N}$, obtained by forgetting filtrations, is formally smooth.*

Let A be a local Artin ring with maximal ideal $\mathfrak{m}_A$, and $I \subset A$ an ideal with $I\mathfrak{m}_A = 0$. Let $D_{A/I}$ be in $\mathfrak{Mod}_{F,\varphi,N}(A/I)$ and set $D_{A/\mathfrak{m}_A} = D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A$.

(1) *If $H^2_F(D_{A/\mathfrak{m}_A}) = 0$, then there exists a module D_A in $\mathfrak{Mod}_{F,\varphi,N}(A)$ whose reduction modulo I is isomorphic to $D_{A/I}$.*

(2) *The set of isomorphism classes of liftings of* $D_{A/I}$ *to* D_A *in* $\mathfrak{Mod}_{F,\varphi,N}(A)$ *is either empty or a torsor under* $H^1_F(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$.

Proof. The first claim, regarding formal smoothness is easily seen, since if A is as above, and D_A is an object of $\mathfrak{Mod}_{\varphi,N}$ lifting $D_{A/I}$, then one can always lift the filtration on $D_{A/I,L}$ to a filtration by projective $L \otimes_{\mathbb{Q}_p} A$-modules on $D_{A,L}$. (This follows from the smoothness of the corresponding Grassmannian variety, which is well known.) The claim in (1) follows from this and Proposition (3.1.2)(1), since $H^2_F(D_{A/\mathfrak{m}_A}) = H^2(D_{A/\mathfrak{m}_A})$.

The argument for (2) is similar to that for Proposition (3.1.2)(2). Suppose we have a lifting of $D_{A/I}$ to an object D_A of $\mathfrak{Mod}_{F,\varphi,N}$. We saw in the proof of Proposition (3.1.2) that to any other such lifting D'_A and an isomorphism $\varepsilon : D_A \xrightarrow{\sim} D'_A$ of the underlying $L_0 \otimes_{\mathbb{Q}_p} A$-modules which induces the identity on $D_{A/I}$, one can associate a pair of elements of $(\mathrm{ad}D_{A/\mathfrak{m}_A})^{G_{L/K}} \otimes_{A/\mathfrak{m}_A} I$. Since ε respects filtrations modulo I, it induces an element of

$$\mathrm{Hom}_{L\otimes_{\mathbb{Q}_p}A}(D_{A,L}, D'_{A,L})/\mathrm{Fil}^0\mathrm{Hom}_{L\otimes_{\mathbb{Q}_p}A}(D_{A,L}, D'_{A,L}) \otimes_A I \xrightarrow{\sim} \mathrm{ad}D_{A/\mathfrak{m}_A,L}/\mathrm{Fil}^0\mathrm{ad}D_{A/\mathfrak{m}_A,L} \otimes_{A/\mathfrak{m}_A} I.$$

D'_A is isomorphic to D_A in $\mathfrak{Mod}_{F,\varphi,N}$ if and only if the automorphism $1+j$ in the proof of Proposition (3.1.2) can be chosen to take the filtration on D_A into that on D'_A. That is, if and only if the image of j in $\mathrm{ad}D_{A/\mathfrak{m}_A,L}/\mathrm{Fil}^0\mathrm{ad}D_{A/\mathfrak{m}_A,L} \otimes_{A/\mathfrak{m}_A} I$ is equal to that of ε. □

(3.3) Suppose now that we are in the situation of Subection (2.7.5). In particular, we have the p-adic Hodge type $\mathbf{v}$ and the Galois type τ, which we assume factors through the inertia subgroup $I_{L/K}$ of $G_{L/K}$.

We again denote by $\mathbb{F}$ the residue field of A°. Let $V_{\mathbb{F}} = V_{A^\circ} \otimes_{A^\circ} \mathbb{F}$. Let $D_{V_{\mathbb{F}}}$ be the groupoid on the category of complete local $\mathcal{O}_E$-algebras with residue field $\mathbb{F}$, whose fiber over such an algebra B consists of deformations of $V_{\mathbb{F}}$ to a G_K-representation on a finite free B-module V_B.

Suppose that $\mathfrak{m}$ is a maximal ideal of $A^{\tau,\mathbf{v}}$, and denote by E' its residue field. For $i \geq 1$, the G_K-representation $V_{A^\circ} \otimes_{A^\circ} A^{\tau,\mathbf{v}}/\mathfrak{m}^i A^{\tau,\mathbf{v}}$ is potentially semi-stable of type τ and Hodge-type $\mathbf{v}$. Hence, it gives rise to an object of $\mathfrak{Mod}_{F,\varphi,N}(A^{\tau,\mathbf{v}}/\mathfrak{m}^i A^{\tau,\mathbf{v}})$. Passing to the limit with i yields a morphism of groupoids on E-algebras

$$\hat{A}^{\tau,\mathbf{v}}_{\mathfrak{m}} \to \mathfrak{Mod}_{F,\varphi,N}.$$

Proposition (3.3.1). *Suppose that the morphism* $A^\circ \to D_{V_{\mathbb{F}}}$ *is formally smooth. Then the morphism* $\hat{A}^{\tau,\mathbf{v}}_{\mathfrak{m}} \to \mathfrak{Mod}_{F,\varphi,N}$ *of groupoids on* E*-algebras is formally smooth.*

Proof. Let B be an E-algebra, $I \subset B$ an ideal with $I^2 = 0$, and $h : \hat{A}^{\tau,\mathbf{v}}_{\mathfrak{m}} \to B/I$ a map of E-algebras. Let $D_{B/I}$ be the object of $\mathfrak{Mod}_{F,\varphi.N}(B/I)$ induced by h, and let D_B in $\mathfrak{Mod}_{F,\varphi,N}(B)$ be an object equipped with an isomorphism $D_B \otimes_B B/I \xrightarrow{\sim} D_{B/I}$. We have to show that D_B is induced by a map $\hat{A}^{\tau,\mathbf{v}}_{\mathfrak{m}} \to B$ lifting h.

We may replace B/I by the image of h, and B by the preimage of $h(\hat{A}^{\tau,\mathbf{v}}_{\mathfrak{m}})$. Then B is the limit of subalgebras B' which surject onto B/I and such that $I \cap B'$ is a finitely generated B/I-module. D_B is induced from an object of $\mathfrak{Mod}_{F,\varphi,N}(B')$ for one of these B', so we may assume that I is a finitely generated B/I-module. In

particular, we may assume that B is a Noetherian, complete local E'-algebra, with residue field E'.

Let $\mathfrak{m}_B$ denote the maximal ideal of B. If $i \geq 1$, then $D_B \otimes_B B/\mathfrak{m}_B^i$ is in $\mathfrak{Mod}_{F,\varphi,N}(B/\mathfrak{m}_B^i)$ and it is a weakly admissible (and hence admissible) module, because it is a successive extension of the admissible module $D_{B/I} \otimes_{B/I} B/\mathfrak{m}_B$ [Fo 2, 4.4.4]. Hence one can associate to $D_B \otimes_B B/\mathfrak{m}_B^i$ a finite free B-module $V_{B/\mathfrak{m}_B^i}$ of rank d, equipped with a continuous action of G_K. Note that by Corollary (2.6.2) $V_{B/\mathfrak{m}_B^i}$ is of type τ and p-adic Hodge type $\mathbf{v}$, since this is true when $i = 1$.

For $i \geq 1$, we now construct maps $\tilde{h}_i : \hat{A}_{\mathfrak{m}}^{\tau,\mathbf{v}} \to B/\mathfrak{m}_B^i$ such that the composite of $\tilde{h}_i$ with the projection $B/\mathfrak{m}_B^i \to B/\mathfrak{m}_B^{i-1}$ is $\tilde{h}_{i-1}$ and the composite with $B/\mathfrak{m}_B^i \to B/(\mathfrak{m}_B^i + I)$ is the reduction of h modulo $\mathfrak{m}_B^i$. When $i = 1$, we take $\tilde{h}_1$ to be the reduction of h modulo $\mathfrak{m}_B$. Given $\tilde{h}_{i-1}$ consider the composite

$$\hat{A}_{\mathfrak{m}}^{\tau,\mathbf{v}} \to B/(\mathfrak{m}_B^{i-1} \cap I) \to B/(\mathfrak{m}_B^{i-1} \cap (\mathfrak{m}_B^i + I)) \tag{3.3.2}$$

where the first map is the unique morphism which reduces to $\tilde{h}_{i-1}$ modulo $\mathfrak{m}_B^{i-1}$ and to h modulo I. We again write $\mathfrak{m}$ for the maximal ideal of A induced by $\mathfrak{m}$. By [Ki 1, 2.3.2, 2.3.3] the formal smoothness of $A^\circ \to D_{V_{\mathbb{F}}}$ implies that the composite of $\hat{A}_{\mathfrak{m}} \to \hat{A}_{\mathfrak{m}}^{\tau,\mathbf{v}}$ and (3.3.2) lifts to a map $\hat{A}_{\mathfrak{m}} \to B/\mathfrak{m}_B^i$ inducing $V_{B/\mathfrak{m}_B^i}$. Since $V_{B/\mathfrak{m}_B^i}$ is of type τ and p-adic Hodge type $\mathbf{v}$, this map factors through $\hat{A}_{\mathfrak{m}}^{\tau,\mathbf{v}}$ by Theorem (2.7.6) and yields the required map $\tilde{h}_i$. Finally, passing to the limit with i gives a lifting of h to B. $\square$

(3.3.3) We now fix an $\mathbb{F}$-basis for $V_{\mathbb{F}}$, and we denote by $D_{V_{\mathbb{F}}}^{\square}$ the groupoid on the category of complete local $\mathcal{O}_E$-algebras with residue field $\mathbb{F}$, whose fiber over such an algebra B consists of an object V_B of $D_{V_{\mathbb{F}}}(B)$, together with a lifting of the given basis of $V_{\mathbb{F}}$ to a B-basis for V_B.

We denote by $|D_{V_{\mathbb{F}}}^{\square}|$ the functor which assigns to B as above the set of isomorphism classes of $D_{V_{\mathbb{F}}}^{\square}(B)$, and similarly for $|D_{V_{\mathbb{F}}}|$. The functor $|D_{V_{\mathbb{F}}}^{\square}|$ is always representable by a complete local $\mathcal{O}_E$-algebra $R_{V_{\mathbb{F}}}^{\square}$. If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, then $|D_{V_{\mathbb{F}}}|$ is representable by a complete local $\mathcal{O}_E$-algebra $R_{V_{\mathbb{F}}}$.

We set $\mathrm{ad}D_{E,K} = \mathrm{Hom}_{E\otimes_{\mathbb{Q}_p} K}(D_{E,K}, D_{E,K})$, where D_E is the E-vector space underlying τ as in (2.7.5), and $D_{E,K} = D_E \otimes_{\mathbb{Q}_p} K$ carries the filtration giving rise to $\mathbf{v}$. The $E \otimes_{\mathbb{Q}_p} K$-module $\mathrm{ad}D_{E,K}$ is naturally equipped with a filtration.

Theorem (3.3.4). $\mathrm{Spec}\,(R_{V_{\mathbb{F}}}^{\square}[1/p])^{\tau,\mathbf{v}}$ *is equi-dimensional of dimension*

$$d^2 + \dim_E \mathrm{ad}D_{E,K}/\mathrm{Fil}^0\mathrm{ad}D_{E,K},$$

and admits a formally smooth, dense open subscheme.

If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, *then the same is true for* $\mathrm{Spec}\,(R_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$, *except that its dimension is given by* $1 + \dim\, \mathrm{ad}D_{E,K}/\mathrm{Fil}^0\mathrm{ad}D_{E,K}$.

Proof. Let $A = (R_{V_{\mathbb{F}}}^{\square}[1/p])^{\tau,\mathbf{v}}$. We saw in the proof of Theorem (2.7.6) that A is canonically equipped with a module D_A in $\mathfrak{Mod}_{\varphi,N}(A)$. It follows from Proposition (3.1.6), Lemma (3.2.1) and Proposition (3.3.1) that there is a formally smooth, dense open subscheme U of $\mathrm{Spec}\, A$ such that the support of $H_F^2(D_A)$ does not meet U.

To compute the dimension of A, let E' be a finite extension of E, and x a closed point of U with residue field E'. Write $\mathfrak{m}$ for the corresponding maximal ideal of A, let V_x be the representation of G_K obtained by specializing the universal

representation over A by x, and D_x the object of $\mathfrak{Mod}_{F,\varphi,N}(E')$ to which x gives rise via the morphism of Proposition (3.3.1). Since A is formally smooth at x, we need to compute the dimension of its tangent space at $\mathfrak{m}$. By Theorem (2.7.6) and [Ki 1, 2.3.5] this is equal to

$$\dim_{E'} \mathrm{Ext}^1_{\mathrm{pst}}(V_x, V_x) + d^2 - \dim_{E'}(\mathrm{ad}_{E'} V_x)^{G_K} \tag{3.3.5}$$

where $\mathrm{Ext}^1_{\mathrm{pst}}$ means potentially semi-stable extensions (cf. [Ki 1, 2.3.11] and its proof). It follows from [Fo 2, §5.6] that $\mathrm{Ext}^1_{\mathrm{pst}}(V_x, V_x) \xrightarrow{\sim} \mathrm{Ext}^1(D_x, D_x)$ where the Ext on the right is computed in the category of $\mathfrak{Mod}_{F,\varphi,N}(E')$. Now using Lemma (3.2.1) and the fact that $H^2_F(D_x) = 0$ we find that

$$\begin{aligned} \dim_{E'} \mathrm{Ext}^1_{\mathrm{pst}}(V_x, V_x) &= \dim_{E'} H^1_F(D_x) \\ &= \dim_{E'}(\mathrm{ad}D_{x,L}/\mathrm{Fil}^0\mathrm{ad}D_{x,L})^{G_{L/K}} + \dim_{E'} H^0_F(D_x), \end{aligned} \tag{3.3.6}$$

where $D_{x,L} = D_x \otimes_{L_0} L$. By Hilbert's theorem 90, and because V_x is of p-adic Hodge type $\mathbf{v}$, we have

$$\dim_{E'}(\mathrm{ad}D_{x,L}/\mathrm{Fil}^0\mathrm{ad}D_{x,L})^{G_{L/K}} = \dim_E \mathrm{ad}D_{E,K}/\mathrm{Fil}^0\mathrm{ad}D_{E,K}. \tag{3.3.7}$$

Since $\dim_{E'}(\mathrm{ad}_{E'} V_x)^{G_K} = \dim_{E'} H^0_F(D_x)$ the theorem for $(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ follows by combining (3.3.5), (3.3.6), and (3.3.7). If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, then $(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ is formally smooth over $(R_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ of relative dimension $d^2 - 1$, so the final claim also follows. □

Theorem (3.3.8). $\mathrm{Spec}\,(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}_{\mathrm{cr}}$ *is formally smooth and equi-dimensional of dimension*

$$d^2 + \dim_E \mathrm{ad}D_{E,K}/\mathrm{Fil}^0\mathrm{ad}D_{E,K}.$$

If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, *then the same is true for* $\mathrm{Spec}\,(R_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$, *except that its dimension is given by* $1 + \dim \mathrm{ad}D_{E,K}/\mathrm{Fil}^0\mathrm{ad}D_{E,K}$.

Proof. Since this is similar, and in fact somewhat easier, than the proof of Theorem (3.3.4) we only sketch the details.

Let $\mathfrak{Mod}_{\varphi}$ and $\mathfrak{Mod}_{F,\varphi}$ denote the full subgroupoids of $\mathfrak{Mod}_{\varphi,N}$ and $\mathfrak{Mod}_{F,\varphi,N}$, respectively, consisting of objects such that $N = 0$. Then the first part of the proof of Proposition (3.1.2) shows that $\mathfrak{Mod}_{\varphi}$ is formally smooth, and it follows from Proposition (3.2.1) that $\mathfrak{Mod}_{F,\varphi}$ is also formally smooth. By Proposition (3.3.1), the completion of $\mathrm{Spec}\,(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}_{\mathrm{cr}}$ at any maximal ideal admits a formally smooth map to $\mathfrak{Mod}_{F,\varphi}$, and hence is formally smooth.

It remains to prove the claims regarding dimension. For this, let E' be a finite extension of E, x a closed point of $\mathrm{Spec}\,(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}_{\mathrm{cr}}$ with residue field E', and D_x the corresponding object of $\mathfrak{Mod}_{F,\varphi}(E')$. A variant of the arguments of Proposition (3.1.2) and Lemma (3.2.1) shows that $\mathrm{Ext}^1(D_x, D_x)$ computed in $\mathfrak{Mod}_{F,\varphi}(E')$ is canonically isomorphic to the cokernel of the map

$$(\mathrm{ad}D_x)^{G_{L/K}} \to (\mathrm{ad}D_x)^{G_{L/K}} \oplus (\mathrm{ad}D_{x,L}/\mathrm{Fil}^0\mathrm{ad}D_{x,L})^{G_{L/K}}$$

for which the first components is given by $1 - \varphi$ and the second by the natural map. A computation as in the proof of Theorem (3.3.4) gives the formulas for the dimensions of $\mathrm{Spec}\,(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}_{\mathrm{cr}}$ and $\mathrm{Spec}\,(R_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}_{\mathrm{cr}}$. □

§4. Hilbert modular forms

(4.1) Let F be a totally real field, and fix an algebraic closure $\bar{F}$ of F. For any finite set S of primes of F, we will denote by $G_{F,S}$ the quotient of $\mathrm{Gal}(\bar{F}/F)$ corresponding to the maximal extension of F unramified outside S. If v is a finite prime of F, we denote by $G_{F_v} \subset G_{F,S}$ a decomposition group at v.

Let f be a Hilbert modular eigenform of weight $\underline{k} = (k_1, \ldots, k_n)$ with $k_i \geq 2$ all having the same parity, and $\pi = \bigotimes_v \pi_v$ the algebraic automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ generated by f.

Let $U = \prod_v U_v \subset \prod_v \mathrm{GL}_2(\mathcal{O}_{F_v})$ be a compact open subgroup, where v runs over the finite primes. We assume that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if π_v is spherical and that f is U-invariant. This last condition can always be satisfied if we replace f by a suitable non-zero vector in π.

Denote by S_0 the set of finite primes where π is spherical. The eigenvalues of the operators $T_v = U_v \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) U_v$ and $S_v = U_v \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix}\right) U_v$ for $v \in S_0$, acting on f, generate a number field E_π. Denote these eigenvalues by t_v and s_v respectively.

We now fix an extension λ of the p-adic valuation on $\mathbb{Q}$ to the field of algebraic numbers in $\mathbb{C}$. In particular, we may consider the completion $E_{\pi,\lambda}$ of E_π with respect to this valuation. Let S be the union of the primes not in S_0 and the primes of F of residue characteristic p. By a result of Carayol and Taylor [Ca], [Ta 1], there exists a continuous representation

$$\rho_{\pi,\lambda} : G_{F,S} \to \mathrm{GL}_2(E_{\pi,\lambda})$$

such that for $v \notin S$ and a *geometric* Frobenius $\mathrm{Frob}_v \in G_{F,S}$ at v, the characteristic polynomial of $\rho_{\pi,\lambda}(\mathrm{Frob}_v)$ is given by $X^2 - t_v X + \mathbf{N}(v) s_v$. Here $\mathbf{N}(v)$ is the order of the residue field at v.

Now let $\mathcal{O}_{\pi,\lambda}$ denote the ring of integers of $E_{\pi,\lambda}$ and $\mathbb{F}_{\pi,\lambda}$ its residue field. After conjugating $\rho_{\pi,\lambda}$, we may assume that it takes values in $\mathrm{GL}_2(\mathcal{O}_{\pi,\lambda})$ and we denote by $\bar{\rho}_{\pi,\lambda} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F}_{\pi,\lambda})$ the induced mod p representation. The semi-simplification of $\bar{\rho}_{\pi,\lambda}$ is independent of the choice of conjugation made above.

(4.2) We return for a moment to the situation of Subsection (2.7) and we use the notation introduced there. Suppose that $E/\mathbb{Q}_p$ is a finite extension and that V is a finite dimensional E-vector space equipped with a continuous action of G_K, which makes V into a potentially semi-stable G_K-representation. We let

$$D_{\mathrm{pst}}(V) = \varinjlim_{K'} (V \otimes B_{\mathrm{st}})^{G_{K'}}$$

where K' runs over finite extensions of K. Then $D_{\mathrm{pst}}(V)$ is the $E \otimes_{\mathbb{Q}_p} \bar{K}_0$-dual of the I_K-representation $D^*_{\mathrm{pst}}(V)$ of Subsection (2.7). (It will be more convenient to use the covariant functor here.)

Following Fontaine, we can extend the action of I_K on $D_{\mathrm{pst}}(V)$ to an action of the Weil-Deligne group WD_K as follows. Let $\sigma \in \mathrm{Gal}(\bar{k}/\mathbb{F}_p)$ denote the absolute Frobenius given by $x \mapsto x^p$. For w an element of the Weil group W_K, we define an integer by $\nu(w) \in \mathbb{Z}$ by requiring that w act on $\bar{k}$ as $\sigma^{-\nu(w)}$.

Now the action of G_K on V and B_{st} induces a semi-linear action of W_K on $D_{\mathrm{pst}}(V)$. For $w \in W_K$, denote by $\rho_{\mathrm{sl}}(w)$ the corresponding semi-linear endomorphism of $D_{\mathrm{pst}}(V)$. The operator φ on B_{st} induces a Frobenius semi-linear operator φ on $D_{\mathrm{pst}}(V)$, and we define a *linear* action of W_K on $D_{\mathrm{pst}}(V)$ by letting $w \in W_K$ act as $\rho_{\mathrm{sl}}(w) \circ \varphi^{\nu(w)}$. We extend this to an action of the Weil-Deligne group WD_K by letting N act on $D_{\mathrm{pst}}(V)$ via its action on B_{st}.

The same argument as in Subsection (2.7) shows that after replacing E by a finite extension, the WD_K-representation $D_{\mathrm{pst}}(V)$ descends to a representation on an E-vector space, which we denote by $\sigma(V)$. Finally we remind the reader that attached to $\sigma(V)$ we have its Frobenius semi-simplification $\sigma^{\mathrm{ss}}(V)$. As a W_K-representation this is the semi-simplification of $\sigma(V)|_{W_K}$, but the Jordan form of N acting on $\sigma^{\mathrm{ss}}(V)$ is the same as that of N acting on $\sigma(V)$.

Theorem (4.3). *Suppose that $\bar{\rho}_{\pi,\lambda}$ is absolutely irreducible. If $v|p$ is a prime of F, then $\rho_{\pi,\lambda}|_{G_{F_v}}$ is potentially semi-stable with p-adic Hodge type corresponding to $\underline{k}$. Moreover, $\sigma^{\mathrm{ss}}(\rho_{\pi,\lambda}|_{G_{F_v}})$ corresponds to π_v via the local Langlands correspondence.*

Proof. If π is discrete series at some finite place, then the theorem follows from work of Saito [Sa 2] building on the results of Carayol mentioned above. In particular, we may assume that π_v is not special at any $v|p$. In this case the assertion of the theorem is that $\rho_{\pi,\lambda}|_{G_{F_v}}$ is potentially crystalline of p-adic Hodge type (corresponding to) $\underline{k}$ (the meaning of this is explained in the Introduction), and that the W_{F_v}-representation $\sigma^{\mathrm{ss}}(V_{\pi,\lambda})$ is associated to π_v. Here $V_{\pi,\lambda}$ denotes the underlying $E_{\pi,\lambda}$-vector space of $\rho_{\pi,\lambda}$.

Let σ_v denote the W_{F_v}-representation attached to π_v by the local Langlands correspondence. We have to show that $\sigma^{\mathrm{ss}}(V_{\pi,\lambda})$ is potentially crystalline, and that for $w \in W_{F_v}$ we have

$$\mathrm{tr}(w|\sigma^{\mathrm{ss}}(V_{\pi,\lambda})) = \mathrm{tr}(w|\sigma_v). \tag{4.3.1}$$

The argument of [Sa 1, Lem. 1] shows that it suffices to show (4.3.1) for w such that $\nu(w) > 0$. Fix such a w. Then there exists a finite extension F'_v/F_v such that the restrictions of $\sigma^{\mathrm{ss}}(V_{\pi,\lambda})$ and σ_v to $W_{F'_v} \subset W_{F_v}$ are unramified, $w \in W_{F'_v}$ and the image of w in $W_{F'_v}/I_{F'_v} \xrightarrow{\sim} \mathbb{Z}$ is a generator.

It follows that after making a base change to a finite, totally real extension F'/F, we may assume that σ_v is unramified and that w maps to a generator in W_{F_v}/I_{F_v}. It suffices to show that $V_{\pi,\lambda}$ is crystalline and that

$$\mathrm{tr}(\varphi^{\nu(w)}|\sigma^{\mathrm{ss}}(V_{\pi,\lambda})) = \mathrm{tr}(w|\sigma_v) = t_w.$$

The proof of this is identical to that of [Ki 1, 3.4.2]. For the convenience of the reader, we sketch the argument.

The representation $\bar{\rho}_{\pi,\lambda}$ corresponds to a maximal ideal $\mathfrak{m}$ in a certain Hecke algebra $\mathbb{T}$, which is a finite flat $\mathcal{O}_{\pi,\lambda}$-algebra. There is representation $\rho_{\mathbb{T}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$ which gives rise to $\rho_{\pi,\lambda}$ via a map $\theta_\pi : \mathbb{T}_{\mathfrak{m}} \to \mathcal{O}_{\pi,\lambda}$, and is characterized by the condition that the characteristic polynomial of $\rho_{\mathbb{T}}(\mathrm{Frob}_{v'})$ is $X^2 - T_{v'}X + \mathbf{N}(v')S_{v'}$ for $v' \notin S$. Although $\rho_{\mathbb{T}}$ is defined up to conjugation, we fix a choice of this representation from now on.

Now $\mathbb{T}_{\mathfrak{m}}[1/p]$ is étale over $E_{\pi,\lambda}$, and any maximal ideal of $\mathbb{T}_{\mathfrak{m}}[1/p]$ corresponds to a Hilbert modular form with associated automorphic representation π'. That is, $\rho_{\pi',\lambda}$ is obtained from $\rho_{\mathbb{T}}$ via a map $\mathbb{T}_{\mathfrak{m}} \to \mathcal{O}_{\pi',\lambda}$.

Let $V_{\mathbb{F}}$ denote the underlying $\mathbb{F}_{\pi,\lambda}$-vector space of $\bar{\rho}_{\pi,\lambda}|_{G_{F_v}}$. Let R be the quotient of $R^{\square}_{V_{\mathbb{F}}}$ (cf. Subsection (3.3.3)) corresponding to crystalline representations of p-adic Hodge type $\underline{k}$, given by Theorem (2.7.6). For any finite extension E of $W(\mathbb{F})[1/p]$, and a map $x : R^{\square}_{V_{\mathbb{F}}} \to E$, denote by V_x the G_{F_v}-representation obtained by specializing the universal representation over $R^{\square}_{V_{\mathbb{F}}}$ by this map. Let W be the ring of integers in the maximal absolutely unramified subfield of F_v. By Theorem (2.5.5) there is a vector bundle D_R over $W \otimes_{\mathbb{Z}_p} R[1/p]$ equipped with a Frobenius

semi-linear map endomorphism φ, such that for any $x : R \to E$, the specialization of D_R by x is canonically isomorphic to $D_{\mathrm{cris}}(V_x)$. Let $T_\varphi \in R[1/p]$ denote the trace of $\varphi^{\nu(w)}$ on D_R as a $W \otimes_{\mathbb{Z}_p} R[1/p]$-module.

Choose $r > 0$ so that $p^r T_\varphi \in R$. We have to show that

$$R^{\square}_{V_{\mathbb{F}}} \to \mathbb{T}_{\mathfrak{m}} \xrightarrow{\theta_\pi} \mathcal{O}_{\pi,\lambda} \tag{4.3.2}$$

factors through R, and maps $p^r T_\varphi$ to $\theta_\pi(p^r T_v) \in \mathcal{O}_{\pi,\lambda}$. If π were special at some finite place this would follow from the results of Carayol and Saito already cited above.

To construct $\rho_{\pi,\lambda}$ Taylor [Ta 1] considers a Hecke algebra $\mathbb{T}^{\mu}_{\mathfrak{m}}$ corresponding to Hilbert modular forms which have an auxiliary prime $\mu \nmid p$ in the level. This has $\mathbb{T}_{\mathfrak{m}}$ as a quotient, corresponding to oldforms, and a quotient $\mathbb{T}^{\mu\text{-new}}_{\mathfrak{m}}$ corresponding to forms which are new at μ. Taylor shows that for any $s > 0$ one can choose μ so that the composite

$$\mathbb{T}^{\mu}_{\mathfrak{m}} \to \mathbb{T}_{\mathfrak{m}} \xrightarrow{\theta_\pi} \mathcal{O}_{\pi,\lambda}/p^s\mathcal{O}_{\pi,\lambda}$$

factors through $\mathbb{T}^{\mu\text{-new}}_{\mathfrak{m}}$. Hence it suffices to show that the map $R^{\square}_{V_{\mathbb{F}}} \to \mathbb{T}^{\mu\text{-new}}_{\mathfrak{m}}$ factors through R, and sends T_φ to $T_v \in \mathbb{T}^{\mu\text{-new}}_{\mathfrak{m}}$. This follows from the results of Carayol and Saito because a maximal ideal of $\mathbb{T}^{\mu\text{-new}}_{\mathfrak{m}}[1/p]$ corresponds to a Hilbert modular form whose associated automorphic representation is special at μ. $\square$

Errata for [Ki 2]

In this section references are to [Ki 2], unless explicitly stated otherwise. We will freely use the notation of those sections of that paper to which we refer.

(E.1) The formula for the section s in (1.3.5) does not converge in general, unless $N = 0$. To define a section fix an $\mathcal{O}$-basis for $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}', \mathcal{M})$ and denote by N minus the residue matrix for the logarithmic connection on this space. Then a section lifting the identity is given by $s = \sum_{i=0}^{\infty} s_i u^i$, where s_0 is any lifting of the identity on $\mathcal{M}/u\mathcal{M}$, $s_1 = -(1-N)^{-1}\nabla(\frac{d}{du})(s_0)$ (which is well defined because s_0 lifts the identity so $N(s_0)(\mathcal{M}') \subset u\mathcal{M}$), and $s_{i+1} = -(1+i-N)^{-1}(\nabla(\frac{d}{du}) + N/u)(s_i)$ for $i \geq 1$. Since N is nilpotent one easily sees that the series converges for $|u|$ sufficiently small. A simple calculation shows that $\nabla(\frac{d}{du})(s) = 0$.

As remarked in [Ki 2], one can weaken the assumption that N is nilpotent on $\mathcal{M}/u\mathcal{M}$. More precisely, a theorem of Manin asserts that over $K_0[\![u]\!]$ the $\mathcal{O}$-module $\mathcal{M}$ with its logarithmic connection is isomorphic to $\mathcal{M}' = \mathcal{M}/u\mathcal{M} \otimes \mathcal{O}$ equipped with the connection $\nabla(m \otimes f) = -N(m)/u \otimes f du + m \otimes df$. This isomorphism is defined over a small disk around the origin, and is given by the same formula as above, provided that that the residue matrix of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{M}')$ has no eigenvalues which are p-adic Liouville numbers, in which case some of the terms $(1+i-N)^{-1}$ grow too quickly. This condition is equivalent to asking that no two eigenvalues of the residue matrix of $\mathcal{M}$ differ by a Liouville number.

(E.2) In the proof of (1.3.14), it is unnecessary to replace $\mathfrak{M}'$ by $\mathfrak{M}$. (This is not an error, but may cause confusion.) The $\mathfrak{S}$-module $\mathfrak{M}'$ is the kernel of a map of flat $\mathfrak{S}$-modules, and hence is flat. The $\mathfrak{S}$-modules $\mathfrak{M}$ and $\mathfrak{M}'$ are therefore equal, as in the proof of (1.6.4) of the present paper.

(E.3) In (2.1.1) $\mathcal{E}$ embeds in $W(\mathrm{Fr}\, R)[1/p]$ (not $W(\mathrm{Fr}\, R)$) and $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is the closure (taken for the p-adic topology) of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}\, R)[1/p]$ (not $W(\mathrm{Fr}\, R)$). $\mathfrak{S}^{\mathrm{ur}}$ should be defined as $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cap W(R)$ (not $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$).

(E.4) There is a gap in the proof of (2.1.12). (We are grateful to Brian Conrad for pointing this out.) It is not clear that the natural map $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_3 \to \mathcal{M}_3$ is an isomorphism, so one cannot apply the case where h is an isomorphism to conclude that $F(\mathfrak{M}_3) = F(\mathfrak{M}'_3)$. The proof of the case where h an isomorphism is correct however, and can be adapted to establish the general case:

We want to show that $h : \mathcal{M}_1 \to \mathcal{M}_2$ induces a map $\mathfrak{M}_1 \to \mathfrak{M}_2$. We may replace $\mathfrak{M}_2$ and $\mathcal{M}_2$ by $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ and $\mathcal{M}_1 \oplus \mathcal{M}_2$, respectively, and h by its graph, in which case h is injective. Then the map $V_{\mathfrak{S}}(\mathfrak{M}_2) \to V_{\mathfrak{S}}(\mathfrak{M}_1)$ is a surjection by (2.1.4), and (2.1.10) implies that we may view $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as submodules of $\mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}_2), \mathfrak{S}^{\mathrm{ur}})$ which is a finite $\mathfrak{S}$-module of rank $d = \mathrm{rk}_{\mathcal{O}_{\mathcal{E}}} \mathcal{M}_2$.

In particular, $\mathfrak{M}_3 = \mathfrak{M}_1 + \mathfrak{M}_2 \subset \mathcal{M}_2$ is a finite $\mathfrak{S}$-module of rank d, which is stable under the action of φ, and $\mathfrak{M}_3/\varphi^*(\mathfrak{M}_3)$ is killed by a power of $E(u)$. Since $F(\mathfrak{M}_3)$ has rank d, we have $F(\mathfrak{M}_3) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} \mathcal{M}_2$, and hence $F(\mathfrak{M}_3) = \mathfrak{M}_2$ by (2.1.9). It follows that $\mathfrak{M}_1 \subset \mathfrak{M}_3 \subset F(\mathfrak{M}_3) = \mathfrak{M}_2$.

(E.5) Replace "homotopy equivalences" by "homotopies" in (2.3.3).

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