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ERRATA TO "TOTALLY POSITIVE TOEPLITZ MATRICES AND QUANTUM COHOMOLOGY OF PARTIAL FLAG VARIETIES"

KONSTANZE RIETSCH

Correction to the Proof of Theorem 4.2. The text in [38] from Remark 4.3 at the bottom of p. 374 to line 8 on p. 375 should be replaced with the following. We use the same notation as in [38].

Remark 4.3. If P is the parabolic subgroup, then G_j^m is a well-defined (regular) function on the Bruhat cell $B^+w_PB^-/B^-$ precisely in the case $m \in I^P = \{n_1, \dots, n_k\}$.

Proof of Theorem 4.2. (1) is proved in [33]. See also Lemma 2.3 in [35]. We will deduce (2) very explicitly from the ASK presentation. We begin by defining a particular system of coordinates on the affine space $B^+w_PB^-/B^-$. For indexing purposes introduce sets $\Omega, \Omega_1, \Omega_2$ defined by

$$\begin{aligned} \Omega &:= \{(r,m) \in \mathbb{Z}^2 \mid n_1 \le r < n, \text{ and } 1 \le m \le n_l \text{ if } n_l \le r < n_{l+1}\}, \\ \Omega_1 &:= \{(n_l, m) \in \mathbb{Z}^2 \mid \text{where } l \in \{1, \dots, k\} \text{ and } 1 \le m \le n_l\}, \end{aligned}$$

and $\Omega_2 := \Omega \setminus \Omega_1$. Consider the polynomial rings $\mathbb{C}[\Omega] := \mathbb{C}[g_m^r; (r,m) \in \Omega]$ and $\mathbb{C}[\Omega_i] := \mathbb{C}[g_m^r; (r,m) \in \Omega_i]$ for $i = 1, 2$. We have $n \times (n_{l+1} - n_l)$-matrices $U_{\Omega_1}^{(l)}$ over $\mathbb{C}[\Omega_1]$ defined by

$$U_{\Omega_1}^{(0)} = \begin{pmatrix} 1 & g_1^{n_1} & g_2^{n_1} & \cdots & g_{n_1-1}^{n_1} \\ 0 & 1 & g_1^{n_1} & \ddots & \vdots \\ \vdots & & \ddots & & g_2^{n_1} \\ & & & \ddots & g_1^{n_1} \\ & & & & 1 \\ & & & & 0 \\ & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}, \quad U_{\Omega_1}^{(l)} = \begin{pmatrix} g_{n_l}^{n_l} & & & \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ g_2^{n_l} & & & g_{n_l}^{n_l} \\ g_1^{n_l} & \ddots & & \vdots \\ 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & g_1^{n_l} & g_2^{n_l} \\ \vdots & & 1 & g_1^{n_l} \\ & & & 1 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

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where $1 \leq l \leq k$. Furthermore we define $n \times (n_{l+1} - n_l)$-matrices $U^{(l)}_{\Omega_2}$ over $\mathbb{C}[\Omega_2]$ by

$$U^{(l)}_{\Omega_2} = \begin{pmatrix} 0 & g^{n_l+1}_{n_l} & g^{n_l+2}_{n_l} & \cdots & g^{n_{l+1}-1}_{n_l} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & g^{n_l+1}_{2} & g^{n_l+2}_{2} & \cdots & g^{n_{l+1}-1}_{2} \\ 0 & g^{n_l+1}_{1} & g^{n_l+2}_{1} & \cdots & g^{n_{l+1}-1}_{1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \\ \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

for $1 \leq l \leq k$ and $U^{(0)}_{\Omega_2} = 0$. The matrices defined above combine to $n \times n$ matrices

$$u_{\Omega_i} = \left(\begin{array}{c|c|c|c} U^{(0)}_{\Omega_i} & U^{(1)}_{\Omega_i} & \cdots & U^{(k)}_{\Omega_i} \end{array} \right),$$

for $i = 1, 2$. Moreover we have

$$u_\Omega := u_{\Omega_1} + u_{\Omega_2},$$

which is an element of U^+ over $\mathbb{C}[\Omega]$, or equivalently a morphism $u_\Omega : \mathbb{C}^\Omega \to U^+$. Composing u_Ω with the standard projection $U^+ \to B^+ w_P B^-/B^-$ defines a map

$$\begin{array}{rcl} \mathbb{C}^\Omega & \longrightarrow & B^+ w_P B^-/B^-, \\ z & \mapsto & u_\Omega(z) w_P B^-. \end{array}$$

It is clear that this map is an isomorphism. So we may use it to identify the coordinate ring $\mathbb{C}[B^+ w_P B^-/B^-]$ with $\mathbb{C}[\Omega]$. Note that $G^{n_l}_j$ goes to $g^{n_l}_j$ under this identification. Let $\mathcal{I}_P \subset \mathbb{C}[\Omega]$ denote the defining ideal for $\mathcal{Y}_P$.

Claim. We have $g^r_m \in \mathcal{I}_P$ for all $(r, m) \in \Omega_2$, or equivalently $u_{\Omega_2} \equiv 0 \mod \mathcal{I}_P$.

Let $v_1, \dots, v_n$ be the standard basis of $\mathbb{C}^n$ and $(\ ,\ )$ the bilinear form given by $(v_i, v_j) = \delta^i_j$. From the definition of the Peterson variety it follows that the ideal $\mathcal{I}_P$ is generated by the elements

$$({u_\Omega}^{-1} f u_\Omega {\cdot} v_r, v_h) \in \mathbb{C}[\Omega], \text{ where } (r, h) \in \Omega_2 \text{ or } r = n_{l+1} \text{ and } h \in [1, n_l] \backslash \{n_{l-1} + 1\},$$

for $l = 1, \dots, k$. Let $\prec$ be the lexicographical ordering on Ω_2. We will prove the claim recursively. Consider $(r, m) \in \Omega_2$. Then there is an $l \in \{1, \dots, k\}$ such that $n_l < r < n_{l+1}$ and $1 \leq m \leq n_l$, and we consider the generator

$$({u_\Omega}^{-1} f u_\Omega \cdot v_r, v_{n_l - m+1}) \in \mathcal{I}_P.$$

Note first that $n_l < r < n_{l+1}$ implies $fu_{\Omega_1} \cdot v_r = u_{\Omega_1} \cdot v_{r+1}$. Therefore

$$\begin{aligned}\left(u_\Omega{}^{-1} f u_\Omega \cdot v_r, v_{n_l-m+1}\right) &= \left(u_\Omega{}^{-1} f u_{\Omega_1} \cdot v_r, v_{n_l-m+1}\right) + \left(u_\Omega{}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l-m+1}\right) \\ &= \left(u_\Omega{}^{-1} u_{\Omega_1} \cdot v_{r+1}, v_{n_l-m+1}\right) + \left(u_\Omega{}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l-m+1}\right) \\ &= 0 - \left(u_\Omega{}^{-1} u_{\Omega_2} \cdot v_{r+1}, v_{n_l-m+1}\right) + \left(u_\Omega{}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l-m+1}\right).\end{aligned}$$

Now

$$u_{\Omega_2} \cdot v_r = \begin{cases} 0 & \text{if } r = n_l + 1, \\ g_{n_l}^{r-1} v_1 + g_{n_l-1}^{r-1} v_2 + \ldots + g_1^{r-1} v_{n_l} & \text{otherwise.} \end{cases}$$

Therefore

$$\left(u_\Omega{}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l-m+1}\right) \equiv 0 \mod (g_h^t)_{(t,h)\prec(r,m)}.$$

Since $u_{\Omega_2} \cdot v_{r+1} = g_{n_l}^r v_1 + g_{n_l-1}^r v_2 + \ldots + g_1^r v_{n_l}$ and u_Ω is upper-triangular, we have all in all

$$\begin{aligned}\left(u_\Omega{}^{-1} f u_\Omega \cdot v_r, v_{n_l-m+1}\right) &\equiv -\sum_{j=1}^{m} g_j^r \left(u_\Omega{}^{-1} \cdot v_{n_l-j+1}, v_{n_l-m+1}\right) + 0 \\ &\equiv -g_m^r \left(u_\Omega{}^{-1} \cdot v_{n_l-m+1}, v_{n_l-m+1}\right) \equiv -g_m^r \mod \left(g_h^t\right)_{(t,h)\prec(r,m)}.\end{aligned}$$

For the minimal element $(n_1+1, 1)$ in Ω_2 in particular this implies

$$\left(u_\Omega{}^{-1} f u_\Omega \cdot v_{n_1+1}, v_{n_1}\right) = -g_1^{n_1+1},$$

and so $g_1^{n_1+1}$ lies in $\mathcal{I}_P$. By induction it follows that all g_m^r for $(r,m) \in \Omega_2$ lie in $\mathcal{I}_P$, and the claim is proved.

We have thus shown that $\mathcal{Y}_P$ lies in the subvariety $\mathcal{V}_{\Omega_2}$ of $B^+ w_P B^-/B^-$ defined by the ideal $I_{\Omega_2} = (g_h^t)_{(t,h)\in\Omega_2}$. Its coordinate ring $\mathcal{O}(\mathcal{V}_{\Omega_2}) = \mathbb{C}[\Omega]/I_{\Omega_2}$ can be identified with the polynomial ring $\mathbb{C}[g_h^t; (t,h) \in \Omega_1]$, or equivalently with $\mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}]$, where the $G_j^{n_i}$ now denote restrictions to the affine space $\mathcal{V}_{\Omega_2}$ of the functions defined in (4.4). We furthermore let u be the restriction of u_Ω to $\mathcal{V}_{\Omega_2}$, viewing u as an element of $U^+(\mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}])$. Also, we let $\mathcal{J}_P$ denote the ideal defining $\mathcal{Y}_P$ inside $\mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}]$.

With this, the rest of the proof proceeds as in paragraph 2 on p. 375 in [38]. Only on line 14 of p. 375 we also need to change the morphism $B^+ w_P B^-/B^- \to \mathfrak{gl}_n$ to a morphism $\mathcal{V}_{\Omega_2} \to \mathfrak{gl}_n$.

Further minor corrections.

1. In (3.4) and (3.6) replace the exponent "n_1" with "$n_2 - n_1$" and the exponent "$n_k - n_{k-1}$" with "$n - n_k$". Similarly for the subscripts in the displayed equation after (3.4).
2. In line 2 of paragraph 5 of Section 3.4 replace "at most $(n_j - n_{j-1})$ parts" with "at most $(n_{j+1} - n_j)$ parts".
3. In line 6 of Section 3.7 replace λ_i with λ_{d-i+1}.
4. Replace t with t^{-1} everywhere in displayed equation (5.2).
5. In point (6) of Section 8 replace "$q(u(x_1, \ldots, x_n))$" with "$q(u(x_1, \ldots, x_d))$".
6. In the second displayed equation in the proof of Lemma 8.1, insert exponents:

$$\text{“}\sigma_{w^P}^P = (\sigma_{s_1\cdots s_{n_1}}^P)^{n_2-n_1} \cdot (\sigma_{s_1\cdots s_{n_2}}^P)^{n_3-n_2} \cdot \ldots \cdot (\sigma_{s_1\cdots s_{n_k}}^P)^{n-n_k}.\text{”}$$

(7) Two paragraphs down from (6) in Section 8, add the words "in type A" to read: "Peterson has announced in [33] that all the quantum cohomology rings $qH^*(G/P)$ in type A are reduced."
(8) The statement of Corollary 11.4(2) should be replaced with: "If $y \in \mathcal{X}_{P,>0}$ and $\sigma_w^P(y) \geq 0$ for all $w \in W^P$, then $q_i^P(y) > 0$ for all $i = 1, \dots, k$."
(9) The first line of the proof of Proposition 12.2 should read: "By (5.1) and Lemma 12.1 we have".

Acknowledgements

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Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom
E-mail address: konstanze.rietsch@kcl.ac.uk