BLOW-UP PHENOMENA FOR THE YAMABE EQUATION

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1. INTRODUCTION

Let \((M,g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). The Yamabe problem is concerned with finding metrics of constant scalar curvature in the conformal class of \(g\). This problem can be reduced to a semi-linear elliptic PDE. Indeed, the metric \(u^{\frac{4}{n-2}}g\) has constant scalar curvature \(c\) if and only if

\[
\frac{4(n-1)}{n-2} \Delta_g u - R_g u + c u^{\frac{n+2}{n-2}} = 0,
\]

where \(\Delta_g\) is the Laplace operator with respect to \(g\) and \(R_g\) denotes the scalar curvature of \(g\). Clearly, every solution of (1) is a critical point of the functional

\[
E_g(u) = \frac{\int_M \left( \frac{4(n-1)}{n-2} |du|^2 + R_g u^2 \right) \, dvol_g}{\left( \int_M u^{\frac{2n}{n-2}} \, dvol_g \right)^{\frac{n-2}{n}}}.
\]

It is well known that the PDE (1) has at least one positive solution for any choice of \((M,g)\). If \(n \geq 6\) and \((M,g)\) is not locally conformally flat, this follows from results of T. Aubin [3]. The remaining cases were solved by R. Schoen using the positive mass theorem [16].

Solutions to (1) are not usually unique. As an example, consider the product metric on \(S^1(L) \times S^{n-1}(1)\). If \(L\) is sufficiently small, then the Yamabe PDE has a unique solution. On the other hand, there are many non-minimizing solutions if \(L\) is large. D. Pollack [14] has used gluing techniques to construct high energy solutions on more general background manifolds: given any conformal class with positive Yamabe constant and any positive integer \(N\), there exists a new conformal class which is close to the original one in the \(C^0\)-norm and contains at least \(N\) metrics of constant scalar curvature (see [14], Theorem 0.1).

It is an interesting question whether the set of all solutions to the Yamabe PDE is compact (in the \(C^2\)-topology, say). A well-known conjecture states that this should be true unless \((M,g)\) is conformally equivalent to the round sphere (see [17], [18], [19]). This conjecture has been verified in low dimensions and in the locally conformally flat case: if \((M,g)\) is locally conformally flat, compactness follows from work of R. Schoen [17], [18]. Moreover, Schoen proposed a strategy for proving the conjecture in the non-locally conformally flat case based on the Pohozaev identity.

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In [12], Y.Y. Li and M. Zhu [12] followed this strategy to prove compactness in dimension 3. O. Druet [7] proved the conjecture in dimensions 4 and 5. Recently, F. Marques [13] showed that compactness holds up to dimension 7. The same result was obtained independently by Y.Y. Li and L. Zhang [11]. Moreover, Li and Zhang showed that compactness holds in all dimensions provided that
\[ |W_g(p)| + |\nabla W_g(p)| > 0 \]
for all \( p \in M \). M. Khuri, F. Marques, and R. Schoen [10] proved compactness up to dimension 24, assuming that the positive mass theorem holds.

In this paper, we address the opposite question: is it possible to construct Riemannian manifolds \((M, g)\) such that the set of constant scalar curvature metrics in the conformal class of \( g \) is non-compact? So far, the only known examples where compactness fails involve non-smooth background metrics. The first result in this direction was established by A. Ambrosetti and A. Malchiodi [2]. This result was subsequently improved by M. Berti and A. Malchiodi [6]. Given positive integers \( n \) and \( k \) such that \( k \geq 2 \) and \( n \geq 4k + 3 \), Berti and Malchiodi showed that there exists a Riemannian metric \( g \) on \( S^n \) (of class \( C^k \)) for which the set of solutions to the Yamabe PDE (1) fails to be compact (see [6], Theorem 1.1). A survey of these results can be found in [1]. Recently, O. Druet and E. Hebey [8] showed that blow-up can occur for problems of the form
\[ Lu + cu^{\frac{n+2}{n-2}} = 0, \]
where \( L \) is a lower order perturbation of the conformal Laplacian on \( S^n \).

We improve the results of Berti and Malchiodi by showing that the set of solutions to the Yamabe PDE (1) can fail to be compact even if the background metric \( g \) is \( C^\infty \) smooth. In the examples we construct, the blowing-up sequence develops a singularity consisting of exactly one bubble.

**Theorem.** Assume that \( n \geq 52 \). Then there exists a Riemannian metric \( g \) on \( S^n \) (of class \( C^\infty \)) and a sequence of positive functions \( v_\nu \in C^\infty(S^n) \) (\( \nu \in \mathbb{N} \)) with the following properties:

(i) \( g \) is not conformally flat,
(ii) \( v_\nu \) is a solution of the Yamabe PDE (1) for all \( \nu \in \mathbb{N} \),
(iii) \( E_g(v_\nu) < Y(S^n) \) for all \( \nu \in \mathbb{N} \), and \( E_g(v_\nu) \to Y(S^n) \) as \( \nu \to \infty \),
(iv) \( \sup_{S^n} v_\nu \to \infty \) as \( \nu \to \infty \).

(Here, \( Y(S^n) \) denotes the Yamabe energy of the round metric on \( S^n \).)

Let us sketch the main steps involved in the proof of the Theorem. For convenience, we will work on \( \mathbb{R}^n \) instead of \( S^n \). Let \( g \) be a smooth metric on \( \mathbb{R}^n \) which agrees with the Euclidean metric outside a ball of radius 1. We will assume throughout the paper that \( \det g(x) = 1 \) for all \( x \in \mathbb{R}^n \), so that the volume form associated with \( g \) agrees with the Euclidean volume form.

Our goal is to construct solutions to the Yamabe PDE on \((\mathbb{R}^n, g)\). In Section 2, we show that this problem can be reduced to finding critical points of a certain function \( F_g(\xi, \varepsilon) \), where \( \xi \) is a vector in \( \mathbb{R}^n \) and \( \varepsilon \) is a positive real number. This idea has been used by many authors (see, e.g., [2] or [10]). In Section 3, we show that the function \( F_g(\xi, \varepsilon) \) can be approximated by an auxiliary function \( F(\xi, \varepsilon) \). In Section 4, we prove that the function \( F(\xi, \varepsilon) \) has a critical point, which is a strict local minimum. Finally, in Section 5, we use a perturbation argument to construct critical points of the function \( F_g(\xi, \varepsilon) \). From this the main result follows.

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1T. Aubin has recently claimed a general compactness theorem in all dimensions [3], [5]. We have, however, been unable to verify some of the arguments in [3].
Let
\[ E = \left\{ w \in L^{2n/(n-2)}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |dw|^2 < \infty \right\}. \]

By Sobolev’s inequality, there exists a constant \( K \), depending only on \( n \), such that
\[ \left( \int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq K \int_{\mathbb{R}^n} |dw|^2 \]
for all \( w \in E \). We define a norm on \( E \) by \( \|w\|_E^2 = \int_{\mathbb{R}^n} |dw|^2 \). It is easy to see that \( E \), equipped with this norm, is complete.

Given any pair \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \), we define a function \( u_{(\xi, \varepsilon)} : \mathbb{R}^n \to \mathbb{R} \) by
\[ u_{(\xi, \varepsilon)}(x) = \left( \frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}}. \]
The function \( u_{(\xi, \varepsilon)} \) satisfies the elliptic PDE
\[ \Delta u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)} = 0, \]
It is well known that
\[ \int_{\mathbb{R}^n} u_{(\xi, \varepsilon)}^{2n/(n-2)} = \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}} \]
for all \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \). We next define
\[ \varphi_{(\xi, \varepsilon, 0)}(x) = \left( \frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n+2}{2}} \frac{\varepsilon^2 - |x - \xi|^2}{\varepsilon^2 + |x - \xi|^2} \]
and
\[ \varphi_{(\xi, \varepsilon, k)}(x) = \left( \frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n+2}{2}} \frac{2\varepsilon (x_k - \xi_k)}{\varepsilon^2 + |x - \xi|^2} \]
for \( k = 1, \ldots, n \). It is easy to see that the norm \( \|\varphi_{(\xi, \varepsilon, k)}\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \) is constant in \( \xi \) and \( \varepsilon \). Finally, we define a closed subspace \( E_{(\xi, \varepsilon)} \subset E \) by
\[ E_{(\xi, \varepsilon)} = \left\{ w \in E : \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} w = 0 \quad \text{for} \quad k = 0, 1, \ldots, n \right\}. \]

Clearly, \( u_{(\xi, \varepsilon)} \in E_{(\xi, \varepsilon)} \).

**Proposition 1.** Consider a Riemannian metric on \( \mathbb{R}^n \) of the form \( g(x) = \exp(h(x)) \), where \( h(x) \) is a trace-free symmetric two-tensor on \( \mathbb{R}^n \) satisfying \( |h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq 1 \) for all \( x \in \mathbb{R}^n \) and \( h(x) = 0 \) for \( |x| \geq 1 \). There exists a constant \( C \), depending only on \( n \), such that
\[ \left\| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n}} \right\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \leq C \alpha \]
for all pairs \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \).

**Proof.** Using the pointwise estimate
\[ \left| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n}} \right| \]
\[ \leq C |h| |\partial^2 u_{(\xi, \varepsilon)}| + C |\partial h| |\partial u_{(\xi, \varepsilon)}| + C (|\partial^2 h| + |\partial h|^2) u_{(\xi, \varepsilon)}, \]
we obtain
\[
\left\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^\infty(R^n)} \\
\leq C \|h\|_{L^\infty(R^n)} \|\partial^2 u_{(\xi,\varepsilon)}\|_{L^\infty(R^n)} + C \|\partial h\|_{L^n(R^n)} \|\partial u_{(\xi,\varepsilon)}\|_{L^2(R^n)} \\
+ C \left( \|\partial^2 h\|_{L^\infty(R^n)} + \|\partial h\|_{L^n(R^n)} \right) \|u_{(\xi,\varepsilon)}\|_{L^\infty(R^n)} \\
\leq C \alpha.
\]
This proves the assertion. \qed

**Proposition 2.** There exists a positive constant \( \theta \), depending only on \( n \), such that
\[
\int_{R^n} \left( \left| \frac{dw}{g} \right|^2 - n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w^2 \right) \\
\geq 2 \theta \|w\|_2^2 - \frac{16n^2}{\theta} \left( \int_{R^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{2}} w \right)^2
\]
for all \( w \in \mathcal{E}_{(\xi,\varepsilon)} \).

Proposition 2 follows from an analysis of the eigenvalues of the Laplace operator on \( S^n \). The details can be found in [15].

**Corollary 3.** Consider a Riemannian metric on \( R^n \) of the form \( g(x) = \exp(h(x)) \), where \( h(x) \) is a trace-free symmetric two-tensor on \( R^n \) satisfying \( h(x) = 0 \) for \( |x| \geq 1 \). There exists a positive constant \( \alpha_0 \leq 1 \), depending only on \( n \), with the following property: if \( |h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0 \) for all \( x \in R^n \), then we have
\[
\left( \int_{R^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq 2K \int_{R^n} \left( \left| \frac{dw}{g} \right|^2 + \frac{n-2}{4(n-1)} R_g w^2 \right)
\]
for all \( w \in \mathcal{E} \) and
\[
\int_{R^n} \left| \frac{dw}{g} \right|^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w^2 \\
\geq \theta \|w\|_2^2 - \frac{1}{\theta} \left( \int_{R^n} \left( \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2
\]
for all \( w \in \mathcal{E}_{(\xi,\varepsilon)} \).

**Proof.** Using Proposition 1 and Hölder’s inequality, we obtain
\[
\left| \int_{R^n} \left( \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right| \\
\geq 4n \left| \int_{R^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w \right| - C \alpha_0 \|w\|_\varepsilon.
\]
This implies
\[
\left( \int_{R^n} \left( \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2 \\
\geq 16n^2 \left( \int_{R^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w \right)^2 - \theta^2 \|w\|_2^2
\]
if \( \alpha_0 \) is sufficiently small. Hence, the assertion follows from Proposition 2. \qed
Proposition 4. Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq 1$. Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$ and any function $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, there exists a unique function $w \in \mathcal{E}(\xi, \varepsilon)$ such that

$$
\int_{\mathbb{R}^n} \left( (dw, d\psi)_g + \frac{n-2}{4(n-1)} R_g w \psi - n(n+2) u_{(\xi, \varepsilon)}^\frac{1}{n-2} w \psi \right) = \int_{\mathbb{R}^n} f \psi
$$

for all test functions $\psi \in \mathcal{E}(\xi, \varepsilon)$. Moreover, we have $\|w\|_\varepsilon \leq C \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, where $C$ is a constant that depends only on $n$.

Proof. Suppose that $w \in \mathcal{E}(\xi, \varepsilon)$ and

$$
\int_{\mathbb{R}^n} \left( (dw, d\psi)_g + \frac{n-2}{4(n-1)} R_g w \psi - n(n+2) u_{(\xi, \varepsilon)}^\frac{1}{n-2} w \psi \right) = \int_{\mathbb{R}^n} f \psi
$$

for all test functions $\psi \in \mathcal{E}(\xi, \varepsilon)$. This implies

$$
\int_{\mathbb{R}^n} \left( |dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^\frac{1}{n-2} w^2 \right) = \int_{\mathbb{R}^n} f w
$$

and

$$
\int_{\mathbb{R}^n} \left( \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^\frac{n+2}{n-2} \right) w = -\int_{\mathbb{R}^n} u_{(\xi, \varepsilon)} f.
$$

Using Corollary 3 we obtain

$$
\theta \|w\|_\varepsilon^2 \leq \int_{\mathbb{R}^n} \left( |dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^\frac{1}{n-2} w^2 \right)
$$

$$
+ \frac{1}{\theta} \left( \int_{\mathbb{R}^n} \left( \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^\frac{n+2}{n-2} \right) w \right)^2
$$

$$
\leq \left( \int_{\mathbb{R}^n} |f|_{\frac{2n}{n+2}}^\frac{2n}{n-2} \right)^\frac{n-2}{n} \left( \int_{\mathbb{R}^n} |w|_{\frac{2n}{n-2}}^\frac{2n}{n-2} \right)^\frac{n-2}{n-2}
$$

$$
+ \frac{1}{\theta} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f|_{\frac{2n}{n+2}}^\frac{2n}{n-2} \right)^\frac{n-2}{n} \left( \int_{\mathbb{R}^n} |w|_{\frac{2n}{n-2}}^\frac{2n}{n-2} \right)^\frac{n-2}{n-2}
$$

$$
\leq K^\varepsilon \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|w\|_\varepsilon + \frac{1}{\theta} \left( \frac{Y(S^n)}{4n(n-1)} \right)^\frac{n-2}{n} \|f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^\frac{n-2}{n}.
$$

Hence, it follows from Young’s inequality that

$$
\frac{\theta}{2} \|w\|_\varepsilon^2 \leq K \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2 + \frac{1}{\theta} \left( \frac{Y(S^n)}{4n(n-1)} \right)^\frac{n-2}{n} \|f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^\frac{n-2}{n}.
$$

From this the uniqueness statement follows easily.

In order to prove the existence part, it suffices to minimize the functional

$$
\int_{\mathbb{R}^n} \left( |dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^\frac{1}{n-2} w^2 - 2f w \right)
$$

$$
+ \frac{1}{\theta} \left( \int_{\mathbb{R}^n} \left( \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^\frac{n+2}{n-2} \right) w \right)^2
$$

over all functions $w \in \mathcal{E}(\xi, \varepsilon)$. ∎
Proposition 5. Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. Moreover, let $(\xi, \epsilon) \in \mathbb{R}^n \times (0, \infty)$. There exists a positive constant $\alpha_1 \leq \alpha_0$, depending only on $n$, with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}^n$, then there exists a function $v(\xi, \epsilon) \in \mathcal{V}$ such that $v(\xi, \epsilon) - u(\xi, \epsilon) \in \mathcal{V}(\xi, \epsilon)$ and

$$
\int_{\mathbb{R}^n} \left( \langle dt(\xi, \epsilon), d\psi \rangle + \frac{n - 2}{4(n - 1)} R_g v(\xi, \epsilon) \psi - n(n - 2) |v(\xi, \epsilon)|^{\frac{4}{n - 2}} v(\xi, \epsilon) \psi \right) = 0
$$

for all test functions $\psi \in \mathcal{V}(\xi, \epsilon)$. Moreover, we have the estimate

$$
\|v(\xi, \epsilon) - u(\xi, \epsilon)\|_\epsilon \leq C \left\| \Delta_g u(\xi, \epsilon) - \frac{n - 2}{4(n - 1)} R_g u(\xi, \epsilon) + n(n - 2) u(\xi, \epsilon)^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + n(n - 2) G(\xi, \epsilon) \left( |u(\xi, \epsilon) + w|^\frac{4}{n-2} (u(\xi, \epsilon) + w) - u(\xi, \epsilon)^{\frac{n+2}{n-2}} - n + 2 \frac{u(\xi, \epsilon)}{w} \right).
$$

where $C$ is a constant that depends only on $n$.

Proof. Let $G(\xi, \epsilon) : L^{\frac{2n}{n-2}}(\mathbb{R}^n) \rightarrow \mathcal{V}(\xi, \epsilon)$ be the solution operator constructed in Proposition [4]. We define a non-linear mapping $\Phi(\xi, \epsilon) : \mathcal{V}(\xi, \epsilon) \rightarrow \mathcal{V}(\xi, \epsilon)$ by

$$
\Phi(\xi, \epsilon)(w) = G(\xi, \epsilon) \left( \Delta_g u(\xi, \epsilon) - \frac{n - 2}{4(n - 1)} R_g u(\xi, \epsilon) + n(n - 2) u(\xi, \epsilon)^{\frac{n+2}{n-2}} \right)
$$

$$
+ n(n - 2) G(\xi, \epsilon) \left( |u(\xi, \epsilon) + w|^\frac{4}{n-2} (u(\xi, \epsilon) + w) - u(\xi, \epsilon)^{\frac{n+2}{n-2}} - n + 2 \frac{u(\xi, \epsilon)}{w} \right).
$$

It follows from Proposition [4] that $\|\Phi(\xi, \epsilon)(0)\|_\epsilon \leq C \alpha_1$. Using the pointwise estimate

$$
\left| |u(\xi, \epsilon) + w|^\frac{4}{n-2} (u(\xi, \epsilon) + w) - |u(\xi, \epsilon)|^\frac{4}{n-2} (u(\xi, \epsilon) + \tilde{w}) \right|
$$

$$
\leq C \left( |w|^\frac{4}{n-2} + |\tilde{w}|^\frac{4}{n-2} \right) |w - \tilde{w}|,
$$

we obtain

$$
\|\Phi(\xi, \epsilon)(w) - \Phi(\xi, \epsilon)(\tilde{w})\|_\epsilon \leq C \left( |w|^\frac{4}{n-2} + |\tilde{w}|^\frac{4}{n-2} \right) \|w - \tilde{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}
$$

for all functions $w, \tilde{w} \in \mathcal{V}(\xi, \epsilon)$. This implies

$$
\|\Phi(\xi, \epsilon)(w) - \Phi(\xi, \epsilon)(\tilde{w})\|_\epsilon \leq C \left( |w|^\frac{4}{n-2} + |\tilde{w}|^\frac{4}{n-2} \right) \|w - \tilde{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}
$$

for $w, \tilde{w} \in \mathcal{V}(\xi, \epsilon)$. Hence, if $\alpha_1$ is sufficiently small, then the contraction mapping principle implies that the mapping $\Phi(\xi, \epsilon)$ has a unique fixed point. From this the assertion follows easily. □
We next define a function $F_g : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ by
\[
F_g(\xi, \varepsilon) = \int_{\mathbb{R}^n} \left( |d\psi(\xi, \varepsilon)|^2 + \frac{n-2}{4(n-1)} R_g \psi^2 - (n-2)^2 |\psi(\xi, \varepsilon)|^\frac{2n}{n+2} \right) - 2(n-2) \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{n-2}}.
\]
If we choose $\alpha_1$ small enough, then we obtain the following result:

**Proposition 6.** The function $F_g$ is continuously differentiable. Moreover, if $(\bar{\xi}, \bar{\varepsilon})$ is a critical point of the function $F_g$, then the function $\psi(\bar{\xi}, \bar{\varepsilon})$ is a non-negative weak solution of the equation
\[
\Delta_g \psi(\bar{\xi}, \bar{\varepsilon}) - \frac{n-2}{4(n-1)} R_g \psi(\bar{\xi}, \bar{\varepsilon}) + n(n-2) \frac{\Delta_g}{\psi(\bar{\xi}, \bar{\varepsilon})} = 0.
\]

**Proof.** By definition of $\psi(\xi, \varepsilon)$, we can find real numbers $a_k(\xi, \varepsilon)$, $k = 0, 1, \ldots, n$, such that
\[
\int_{\mathbb{R}^n} \left( (d\psi(\xi, \varepsilon), d\psi)_g + \frac{n-2}{4(n-1)} R_g \psi(\xi, \varepsilon) \psi(\xi, \varepsilon) \psi - n(n-2) \frac{\Delta_g}{\psi(\xi, \varepsilon)} \right) = \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \psi
\]
for all test functions $\psi \in \mathcal{E}$. This implies
\[
\frac{\partial}{\partial \varepsilon} F_g(\xi, \varepsilon) = 2 \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \varepsilon} \psi(\xi, \varepsilon)
\]
and
\[
\frac{\partial}{\partial \xi_j} F_g(\xi, \varepsilon) = 2 \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \xi_j} \psi(\xi, \varepsilon)
\]
for $j = 1, \ldots, n$. On the other hand, we have
\[
\int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) (\psi(\xi, \varepsilon) - u(\xi, \varepsilon)) = 0
\]
since $\psi(\xi, \varepsilon) - u(\xi, \varepsilon) \in \mathcal{E}(\xi, \varepsilon)$. This implies
\[
0 = \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi(\xi, \varepsilon, k) (\psi(\xi, \varepsilon) - u(\xi, \varepsilon)) + \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \varepsilon} (\psi(\xi, \varepsilon) - u(\xi, \varepsilon))
\]
\[
= \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) (\psi(\xi, \varepsilon) - u(\xi, \varepsilon)) + \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \varepsilon} \psi(\xi, \varepsilon)
\]
\[
+ \frac{n-2}{2(n+1)} \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{n-2}} \varepsilon^{-1} \delta_{jk}
\]
and
\[
0 = \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi(\xi, \varepsilon, k) (\psi(\xi, \varepsilon) - u(\xi, \varepsilon)) + \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \xi_j} (\psi(\xi, \varepsilon) - u(\xi, \varepsilon))
\]
\[
= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi(\xi, \varepsilon, k) (\psi(\xi, \varepsilon) - u(\xi, \varepsilon)) + \int_{\mathbb{R}^n} \varphi(\xi, \varepsilon, k) \frac{\partial}{\partial \xi_j} \psi(\xi, \varepsilon)
\]
\[
- \frac{n-2}{2(n+1)} \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{n-2}} \varepsilon^{-1} \delta_{jk}
\]
for $j = 1, \ldots, n$. Putting these facts together, we obtain
\[ -\frac{n - 2}{n + 1} \left( \frac{Y(S^n)}{4n(n - 1)} \right)^{\frac{2}{n}} a_0(\xi, \epsilon) \]
\[ = \epsilon \frac{\partial}{\partial \xi_j} F_g(\xi, \epsilon) + 2\epsilon \sum_{k=0}^{n} a_k(\xi, \epsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi(\xi, \epsilon, k) (v(\xi, \epsilon) - u(\xi, \epsilon)) \]
and
\[ -\frac{n - 2}{n + 1} \left( \frac{Y(S^n)}{4n(n - 1)} \right)^{\frac{2}{n}} a_j(\xi, \epsilon) \]
\[ = \epsilon \frac{\partial}{\partial \xi_j} F_g(\xi, \epsilon) + 2\epsilon \sum_{k=0}^{n} a_k(\xi, \epsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi(\xi, \epsilon, k) (v(\xi, \epsilon) - u(\xi, \epsilon)) \]
for $j = 1, \ldots, n$. Hence, if $(\xi, \epsilon)$ is a critical point of $F_g$, then we have
\[ \sum_{k=0}^{n} |a_k(\xi, \epsilon)| \leq C \|v(\xi, \epsilon) - u(\xi, \epsilon)\|_{L^\frac{2n}{n-2}(\mathbb{R}^n)} \sum_{k=0}^{n} |a_k(\xi, \epsilon)|, \]
where $C$ is a constant that depends only on $n$. On the other hand, we have
\[ \|v(\xi, \epsilon) - u(\xi, \epsilon)\|_{L^\frac{2n}{n-2}(\mathbb{R}^n)} \leq C \alpha_1. \]
Hence, if we choose $\alpha_1$ sufficiently small, then we must have $a_k(\xi, \epsilon) = 0$ for $k = 0, 1, \ldots, n$. Thus, we conclude that
\[ \int_{\mathbb{R}^n} \left( \langle dv(\xi, \epsilon), d\psi \rangle_g + \frac{n - 2}{4(n - 1)} R_g v(\xi, \epsilon) \psi - n(n - 2) |v(\xi, \epsilon)|^\frac{2n}{n-2} v(\xi, \epsilon) \psi \right) = 0 \]
for all test functions $\psi \in \mathcal{E}$. It remains to show that the function $v(\xi, \epsilon)$ is non-negative. To that end, we put $\psi = \min\{v(\xi, \epsilon), 0\}$. Since $v(\xi, \epsilon) \in \mathcal{E}$, we conclude that $\psi \in \mathcal{E}$. This implies
\[ \int_{\{v(\xi, \epsilon) < 0\}} \left( |dv(\xi, \epsilon)|^2_g + \frac{n - 2}{4(n - 1)} R_g v(\xi, \epsilon) \right) \]
\[ = n(n - 2) \int_{\{v(\xi, \epsilon) < 0\}} |v(\xi, \epsilon)|^\frac{2n}{n-2}. \]
Moreover, we have
\[ \left( \int_{\{v(\xi, \epsilon) < 0\}} |v(\xi, \epsilon)|^\frac{2n}{n-2} \right)^\frac{n-2}{n} \]
\[ \leq 2K \int_{\{v(\xi, \epsilon) < 0\}} \left( |dv(\xi, \epsilon)|^2_g + \frac{n - 2}{4(n - 1)} R_g v(\xi, \epsilon) \right) \]
by Corollary [5]. From this we deduce that either $v(\xi, \epsilon) \geq 0$ almost everywhere or
\[ \left( \int_{\{v(\xi, \epsilon) < 0\}} |v(\xi, \epsilon)|^\frac{2n}{n-2} \right)^\frac{n-2}{n} \geq \frac{1}{2n(n-2)K}. \]
On the other hand, we have
\[
\left( \int_{\{v_{(\xi,\varepsilon)}<0\}} |v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq \left( \int_{\mathbb{R}^n} |v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C \alpha_1.
\]
Hence, if \( \alpha_1 \) is sufficiently small, then we have \( v_{(\xi,\varepsilon)} \geq 0 \) almost everywhere. \( \square \)

3. An estimate for the energy of a “bubble”

Throughout this paper, we fix a multi-linear form \( W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \). We assume that \( W_{ijkl} \) satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of \( W \) are non-zero, so that
\[
\sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{iklj})^2 > 0.
\]
For abbreviation, we put
\[
H_{ik}(x) = \sum_{p,q=1}^{n} W_{ipkq} x_p x_q
\]
and
\[
\overline{H}_{ik}(x) = (1 - |x|^2) H_{ik}(x).
\]
It is easy to see that \( H_{ik}(x) \) is trace-free, \( \sum_{i=1}^{n} x_i H_{ik}(x) = 0 \), and \( \sum_{i=1}^{n} \partial_i H_{ik}(x) = 0 \) for all \( x \in \mathbb{R}^n \).

We consider a Riemannian metric of the form \( g(x) = \exp(h(x)) \), where \( h(x) \) is a trace-free symmetric two-tensor on \( \mathbb{R}^n \) satisfying \( h(x) = 0 \) for \( |x| \geq 1 \),
\[
|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1
\]
for all \( x \in \mathbb{R}^n \), and
\[
h_{ik}(x) = \mu (\lambda^2 - |x|^2) H_{ik}(x)
\]
for \( |x| \leq \rho \). We assume that the parameters \( \lambda, \mu, \) and \( \rho \) are chosen such that \( \mu \leq 1 \) and \( \lambda \leq \rho \leq 1 \). Note that \( \sum_{i=1}^{n} x_i h_{ik}(x) = 0 \) and \( \sum_{i=1}^{n} \partial_i h_{ik}(x) = 0 \) for \( |x| \leq \rho \).

Given any pair \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \), there exists a unique function \( v_{(\xi,\varepsilon)} \) such that \( v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)} \) and
\[
\int_{\mathbb{R}^n} \left( (dv_{(\xi,\varepsilon)}, dv\psi) + \frac{n-2}{4(n-1)} R_g v_{(\xi,\varepsilon)} \psi - n(n-2) |v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} v_{(\xi,\varepsilon)} \psi \right) = 0
\]
for all test functions \( \psi \in \mathcal{E}_{(\xi,\varepsilon)} \) (see Proposition). For abbreviation, let
\[
\Omega = \left\{ (\xi, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < 1, \frac{n-8}{3(n+4)} < \varepsilon^2 < \frac{2(n-8)}{3(n+4)} \right\}.
\]

Proposition 7. For every pair \( (\xi, \varepsilon) \in \lambda \Omega \), we have
\[
\left\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n-2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu + C \left( \frac{\lambda}{\rho} \right)^{\frac{n+2}{n-2}}
\]

and

\[ \| \Delta_g u(\xi, \varepsilon) - \frac{n-2}{4(n-1)} R_g u(\xi, \varepsilon) + n(n-2) u_{(\xi, \varepsilon)} \|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \]

\[ + \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon) \leq C \lambda^8 \mu^2 + C \left( \frac{\lambda}{\rho} \right)^{\frac{n+2}{2n}}. \]

Proof. For abbreviation, we define two functions \( A_1 \) and \( A_2 \) by

\[ A_1 = \Delta_g u(\xi, \varepsilon) - \frac{n-2}{4(n-1)} R_g u(\xi, \varepsilon) + n(n-2) u_{(\xi, \varepsilon)} \]

and

\[ A_2 = \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon). \]

Using Proposition 26 and the identity \( \sum_{i=1}^n \partial_i h_{ik}(x) = 0 \), we obtain

\[ |R_g(x)| \leq C |h(x)|^2 |\partial^2 h(x)| + C |\partial h(x)|^2 \leq C \mu^2 (\lambda + |x|)^6 \]

for \( |x| \leq \rho \). This implies

\[ |A_1| = \left| \sum_{i,k=1}^n \partial_i [(g^{ik} - \delta_{ik}) \partial_k u(\xi, \varepsilon)] - \frac{n-2}{4(n-1)} R_g u(\xi, \varepsilon) \right| \leq C \lambda^{\frac{n+2}{2n}} (\lambda + |x|)^{4-n} \]

and

\[ |A_1 + A_2| = \left| \sum_{i,k=1}^n \partial_i [(g^{ik} - \delta_{ik} + h_{ik}) \partial_k u(\xi, \varepsilon)] - \frac{n-2}{4(n-1)} R_g u(\xi, \varepsilon) \right| \leq C \lambda^{\frac{n+2}{2n}} \mu^2 (\lambda + |x|)^{8-n} \]

for \( |x| \leq \rho \). Hence, we obtain

\[ \| A_1 \|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \lambda^{\frac{n+2}{2n}} \mu \left( \int_{\mathbb{R}^n} (\lambda + |x|)^{\frac{2n(n+1)}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \lambda^4 \mu \]

and

\[ \| A_1 + A_2 \|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \lambda^{\frac{n+2}{2n}} \mu^2 \left( \int_{\mathbb{R}^n} (\lambda + |x|)^{\frac{2n(n+4)}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \lambda^8 \mu^2. \]

On the other hand, we have

\[ |A_1(x)| \leq C \lambda^{\frac{n+2}{2n}} |x|^{-n} \]

for \( \rho \leq |x| \leq 1 \) and

\[ |A_2(x)| \leq C \lambda^{\frac{n+2}{2n}} \mu |x|^{4-n} \]

for \( |x| \geq \rho \). Since the function \( A_1(x) \) vanishes for \( |x| \geq 1 \), we conclude that

\[ \| A_1 \|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \lambda^{\frac{n+2}{2n}} \left( \int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{-\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \left( \frac{\lambda}{\rho} \right)^{\frac{n+2}{2n}}. \]
and
\[ \|A_2\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \lambda^{\frac{n-2}{8}} \mu \left( \int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{\frac{2n(n-4)}{n+2}} \right)^{\frac{n+2}{8}} \leq C \rho^4 \left( \frac{\lambda^2}{\rho} \right)^{\frac{n-2}{8}}. \]

Putting these facts together, the assertion follows. \(\square\)

**Corollary 8.** The function \(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\) satisfies the estimate
\[ \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu + C \left( \frac{\lambda}{\rho} \right)^{\frac{n-2}{4}} \]
for \((\xi, \varepsilon) \in \lambda \Omega\).

**Proof.** It follows from Proposition 5 that
\[ \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \left\| \Delta u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \]
where \(C\) is a constant that depends only on \(n\). Hence, the assertion follows from Proposition 7. \(\square\)

We now prove a more refined estimate for the difference \(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\). Using Proposition 4 with \(h = 0\), we conclude that there exists a unique function \(w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}\) such that
\[ \int_{\mathbb{R}^n} \left( (d w_{(\xi,\varepsilon)}, d \psi) - n(n+2) u_{(\xi,\varepsilon)} w_{(\xi,\varepsilon)} \psi \right) \]
\[ = - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik} (x) \partial_i \partial_k u_{(\xi,\varepsilon)} \psi \]
for all test functions \(\psi \in \mathcal{E}_{(\xi,\varepsilon)}\).

**Proposition 9.** The function \(w_{(\xi,\varepsilon)}\) is smooth. Moreover, if \((\xi, \varepsilon) \in \lambda \Omega\), then we have
\[ |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\frac{n-2}{8}} \mu (\lambda + |x|)^{6-n}, \]
\[ |\partial w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\frac{n-2}{8}} \mu (\lambda + |x|)^{5-n}, \]
\[ |\partial^2 w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\frac{n-2}{8}} \mu (\lambda + |x|)^{4-n} \]
for all \(x \in \mathbb{R}^n\).

**Proof.** Let \(\varphi_{(\xi,\varepsilon,k)}\) be the functions defined in Section 2. We can find real numbers \(b_k(\xi, \varepsilon), k = 0, 1, \ldots, n\), such that
\[ \int_{\mathbb{R}^n} \left( (d w_{(\xi,\varepsilon)}, d \psi) - n(n+2) u_{(\xi,\varepsilon)} w_{(\xi,\varepsilon)} \psi \right) \]
\[ = - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik} (x) \partial_i \partial_k u_{(\xi,\varepsilon)} \psi + \sum_{k=0}^n b_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)} \psi \]
for all test functions \(\psi \in \mathcal{E}\). It follows from standard elliptic regularity theory that \(w_{(\xi,\varepsilon)}\) is smooth.
In the next step, we establish quantitative estimates for \( w(\xi, \varepsilon) \). To that end, we consider a pair \((\xi, \varepsilon) \in \Lambda \). A straightforward calculation yields

\[
\left\| \sum_{i,k=1}^{n} \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon) \right\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu.
\]

From this we deduce that \( \| w(\xi, \varepsilon) \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu \) and \( \sum_{k=0}^{n} |b_k(\xi, \varepsilon)| \leq C \lambda^4 \mu \). This implies

\[
\begin{aligned}
|\Delta w(\xi, \varepsilon) + n(n+2) u^{\frac{4}{n-2}} \| w(\xi, \varepsilon) | & \\
= & \sum_{i,k=1}^{n} \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon) - \sum_{k=0}^{n} b_k(\xi, \varepsilon) \varphi(\xi, \varepsilon) \\
\leq & C \lambda^{\frac{n+2}{2}} \mu (\lambda + |x|)^{4-n}
\end{aligned}
\]

for all \( x \in \mathbb{R}^n \). We claim that

\[
\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n+2}{2}} |w(\xi, \varepsilon)(x)| \leq C \lambda^4 \mu.
\]

To show this, we fix a point \( x_0 \in \mathbb{R}^n \) and put \( r = \frac{1}{2} (\lambda + |x_0|) \). Clearly, \( \lambda + |x| \geq r \) for all \( x \in B_r(x_0) \). This implies

\[
u(\xi, \varepsilon)(x)^{\frac{4}{n-2}} \leq C r^{-2}
\]

and

\[
|\Delta w(\xi, \varepsilon) + n(n+2) u^{\frac{4}{n-2}} w(\xi, \varepsilon) | \leq C \lambda^{\frac{n+2}{2}} \mu r^{4-n}
\]

for all \( x \in B_r(x_0) \). Using standard interior estimates, we obtain

\[
r^{\frac{n+2}{2}} \| w(\xi, \varepsilon)(x_0) \| \leq C \| w(\xi, \varepsilon) \|_{L^{\frac{2n}{n-2}}(B_r(x_0))} + C r^{\frac{n+2}{2}} \| \Delta w(\xi, \varepsilon) + n(n+2) u^{\frac{4}{n-2}} w(\xi, \varepsilon) \|_{L^{\infty}(B_r(x_0))} \\
\leq C \lambda^4 \mu + C \lambda^{\frac{n+2}{2}} \mu r^{-\frac{n-10}{2}} \\
\leq C \lambda^4 \mu.
\]

Thus, we conclude that

\[
\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n+2}{2}} |w(\xi, \varepsilon)(x)| \leq C \lambda^4 \mu,
\]

as claimed. Since \( \sup_{x \in \mathbb{R}^n} |x|^{\frac{n+2}{2}} |w(\xi, \varepsilon)(x)| < \infty \), we can express the function \( w(\xi, \varepsilon) \) in the form

\[
w(\xi, \varepsilon)(x) = -\frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |x-y|^{2-n} \Delta w(\xi, \varepsilon)(y) dy
\]

for all \( x \in \mathbb{R}^n \).

We can now use a bootstrap argument to prove the desired estimate for \( w(\xi, \varepsilon) \). It follows from (5) that

\[
\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta} |w(\xi, \varepsilon)(x)| \leq C \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta + 2} |\Delta w(\xi, \varepsilon)(x)|
\]
for all $0 < \beta < n - 2$. Since
\[
|\Delta w_{(\xi,\varepsilon)}(x)| \leq n(n+2) u_{(\xi,\varepsilon)}(x) \frac{\lambda}{|x|^n} |w_{(\xi,\varepsilon)}(x)|
+ C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{4-n}
\]
for all $x \in \mathbb{R}^n$, we conclude that
\[
\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^\beta |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\beta} \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta-2} |w_{(\xi,\varepsilon)}(x)|
+ C \lambda^{\beta} \frac{n-2}{2} \mu
\]
for all $0 < \beta \leq n - 6$. Iterating this inequality, we obtain
\[
\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{n-6} |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\frac{n-2}{2}} \mu.
\]
The estimates for the first and second derivatives of $w_{(\xi,\varepsilon)}$ follow now from standard interior estimates.

Corollary 10. The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}$ satisfies the estimate
\[
\left\| v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} \right\|_{L^{\alpha_\mu_0/2}(\mathbb{R}^n)} \leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left( \frac{\lambda}{\rho} \right)^{\frac{n-2}{2}}
\]
for $(\xi, \varepsilon) \in \lambda \Omega$.

Proof. Consider the functions
\[
B_1 = \sum_{i,k=1}^n \partial_i \left[ (g^{ik} - \delta_{ik}) \partial_k w_{(\xi,\varepsilon)} \right] - \frac{n-2}{4(n-1)} R_g w_{(\xi,\varepsilon)}
\]
and
\[
B_2 = \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}.
\]
Using (3), we obtain
\[
\int_{\mathbb{R}^n} \left( dw_{(\xi,\varepsilon)}(x), d\psi \right)_g + \frac{n-2}{4(n-1)} R_g w_{(\xi,\varepsilon)} \psi - n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w_{(\xi,\varepsilon)} \psi
\]
\[
= - \int_{\mathbb{R}^n} (B_1 + B_2) \psi
\]
for all functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Since $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$, it follows that
\[
w_{(\xi,\varepsilon)} = -G_{(\xi,\varepsilon)} (B_1 + B_2).
\]
Moreover, we have
\[
v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)} (B_3 + n(n-2) B_4),
\]
where
\[
B_3 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}
\]
and
\[
B_4 = \left| v_{(\xi,\varepsilon)} \right|^{\frac{4}{n-2}} v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}).
\]
Thus, we conclude that
\[
v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)} (B_1 + B_2 + B_3 + n(n-2) B_4),
\]
where $G_{(\xi, \varepsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to \mathcal{E}_{(\xi, \varepsilon)}$ denotes the solution operator constructed in Proposition 4. In particular, we have

$$
\|v(\xi, \varepsilon) - u(\xi, \varepsilon) - w(\xi, \varepsilon)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \|B_1 + B_2 + B_3 + n(n-2) B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}
$$

by Proposition 4. Using Proposition 9, we obtain

$$
|B_1(x)| \leq C \lambda^{\frac{n+2}{n-2}} \mu^2 (\lambda + |x|)^{8-n}
$$

for $|x| \leq \rho$ and

$$
|B_1(x)| \leq C \lambda^{\frac{n+2}{n-2}} \mu |x|^{4-n}
$$

for $\rho \leq |x| \leq 1$. Since the function $B_1(x)$ vanishes for $|x| \geq 1$, we conclude that

$$
\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^8 \mu^2 + C \rho^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{n+2}}.
$$

Moreover, we have

$$
\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^8 \mu^2 + C \left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{n-2}}
$$

by Proposition 9. Finally, the function $B_4$ satisfies a pointwise estimate of the form

$$
|B_4| \leq C |v(\xi, \varepsilon) - u(\xi, \varepsilon)|^{\frac{n+2}{n-2}},
$$

where $C$ is a constant that depends only on $n$. Hence, it follows from Corollary 9 that

$$
\|B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \|v(\xi, \varepsilon) - u(\xi, \varepsilon)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{n+2}{n-2}} \leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{n-2}}.
$$

Putting these facts together, we obtain

$$
\|v(\xi, \varepsilon) - u(\xi, \varepsilon) - w(\xi, \varepsilon)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{n-2}},
$$

as claimed. \(\square\)

**Proposition 11.** We have

$$
\left| \int_{\mathbb{R}^n} \left( |dv(\xi, \varepsilon)|^2 - |du(\xi, \varepsilon)|^2 + \frac{n - 2}{4(n-1)} R_g (v^2(\xi, \varepsilon) - u^2(\xi, \varepsilon)) \right) \right|
$$

$$
+ \int_{\mathbb{R}^n} n(n-2) (|v(\xi, \varepsilon)|^{\frac{n+2}{n-2}} - u(\xi, \varepsilon)) u(\xi, \varepsilon) v(\xi, \varepsilon)
$$

$$
- \int_{\mathbb{R}^n} n(n-2) (|v(\xi, \varepsilon)|^{\frac{2n}{n+2}} - u(\xi, \varepsilon))
$$

$$
- \int_{\mathbb{R}^n} \sum_{i,k=1}^{n} \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon) w(\xi, \varepsilon)
$$

$$
\leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2}
$$

for $(\xi, \varepsilon) \in \lambda \Omega$. 

Proof. Using Proposition 5 with \( \psi = v(\xi, \varepsilon) - u(\xi, \varepsilon) \), we obtain
\[
\int_{\mathbb{R}^n} \left( |dv(\xi, \varepsilon)|^2 - |du(\xi, \varepsilon)|^2 + \frac{n-2}{4(n-1)} R_g (v(\xi, \varepsilon) - u(\xi, \varepsilon)) \right)
- \int_{\mathbb{R}^n} n(n-2) |v(\xi, \varepsilon)| \frac{1}{n} (v(\xi, \varepsilon) - u(\xi, \varepsilon))^2 = 0.
\]
Moreover, it follows from Proposition 7 and Corollary 8 that
\[
\int_{\mathbb{R}^n} \left( |dv(\xi, \varepsilon)|^2 - |du(\xi, \varepsilon)|^2 + \frac{n-2}{4(n-1)} R_g (v(\xi, \varepsilon) - u(\xi, \varepsilon)) \right)
- \int_{\mathbb{R}^n} n(n-2) |v(\xi, \varepsilon)| \frac{n+2}{n} (v(\xi, \varepsilon) - u(\xi, \varepsilon))^2
- \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k (v(\xi, \varepsilon) - u(\xi, \varepsilon))
\leq \left| \Delta_g u(\xi, \varepsilon) - \frac{n-2}{4(n-1)} R_g u(\xi, \varepsilon) + n(n-2) u(\xi, \varepsilon) \right|
+ \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k (v(\xi, \varepsilon) - u(\xi, \varepsilon))
\leq C \lambda^{12} \mu^3 + C \lambda^4 \mu \left( \frac{\lambda}{\rho} \right)^{\frac{n+2}{n}} + C \left( \frac{\lambda}{\rho} \right)^{n-2}.
\]
Finally, we have
\[
\int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k (v(\xi, \varepsilon) - u(\xi, \varepsilon) - w(\xi, \varepsilon))
\leq C \lambda^4 \mu \left( \frac{\lambda}{\rho} \right)^{\frac{n+2}{n}} + C \lambda^4 \mu \left( \frac{\lambda}{\rho} \right)^{n-2}
\]
by (41) and Corollary 10. Putting these facts together, the assertion follows. \( \square \)

**Proposition 12.** We have
\[
\left| \int_{\mathbb{R}^n} \left( |v(\xi, \varepsilon)|^{-\frac{4}{n-2}} - u(\xi, \varepsilon)^{\frac{4}{n-2}} \right) |v(\xi, \varepsilon)| v(\xi, \varepsilon) - \frac{2}{n} \int_{\mathbb{R}^n} \left( |v(\xi, \varepsilon)|^{-\frac{2n}{n-2}} - u(\xi, \varepsilon)^{\frac{2n}{n-2}} \right) \right|
\leq C \lambda^{n\frac{2n}{n-2}} \mu + C \left( \frac{\lambda}{\rho} \right)^n
\]
for \((\xi, \varepsilon) \in \lambda \Omega\).

**Proof.** We have the pointwise estimate
\[
\left| \left( |v(\xi, \varepsilon)|^{-\frac{4}{n-2}} - u(\xi, \varepsilon)^{\frac{4}{n-2}} \right) |v(\xi, \varepsilon)| v(\xi, \varepsilon) - \frac{2}{n} \left( |v(\xi, \varepsilon)|^{-\frac{2n}{n-2}} - u(\xi, \varepsilon)^{\frac{2n}{n-2}} \right) \right|
\leq C |v(\xi, \varepsilon) - u(\xi, \varepsilon)|^{\frac{2n}{n-2}},
\]
where \( C \) is a constant that depends only on \( n \). This implies
\[
\left| \int_{\mathbb{R}^n} (\|u_{(\xi,\varepsilon)}\|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^2) u_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} (\|u_{(\xi,\varepsilon)}\|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^2) \right| \\
\leq C \|u_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{4n}{2n-\varepsilon}}(\mathbb{R}^n)}^{\frac{4n}{2n-\varepsilon}} \\
\leq C \lambda^{\frac{2n}{2n-\varepsilon}} \mu^{\frac{2n}{2n-\varepsilon}} + C \left( \frac{\lambda}{\rho} \right)^n
\]
by Corollary \( \S \). \( \square \)

**Proposition 13.** We have
\[
\left| \int_{\mathbb{R}^n} (\|du_{(\xi,\varepsilon)}\|^2 + \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)}^2 - n(n-2) u_{(\xi,\varepsilon)}^2 - 4 \int_{B_\varepsilon(0)} 1 \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \\
+ \int_{B_\varepsilon(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_i h_{ik})^2 u_{(\xi,\varepsilon)}^2 \right| \\
\leq C \lambda^{12} \mu^3 + C \left( \frac{\lambda}{\rho} \right)^{n-2}
\]
for all \((\xi,\varepsilon) \in \lambda \Omega\).

**Proof.** Note that
\[
\left| g^{ik}(x) - \delta_{ik} + h_{ik}(x) - \frac{1}{2} \sum_{i=1}^n h_{il}(x) h_{kl}(x) \right| \\
\leq C |h(x)|^3 \leq C \mu^3 (\lambda + |x|)^{12}
\]
for \(|x| \leq \rho\). This implies
\[
\left| \int_{\mathbb{R}^n} (|du_{(\xi,\varepsilon)}|^2 - |du_{(\xi,\varepsilon)}|^2) + \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \\
- \int_{B_\varepsilon(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \right| \\
\leq C \lambda^{n-2} \mu^3 \int_{B_\varepsilon(0)} (\lambda + |x|)^{14-2n} + C \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\lambda + |x|)^{2-2n} \\
\leq C \lambda^{12} \mu^3 + C \left( \frac{\lambda}{\rho} \right)^{n-2}.
\]
By Proposition \( \S \), the scalar curvature of \( g \) satisfies the estimate
\[
\left| R_g(x) + \frac{1}{4} \sum_{i,k,l=1}^n (\partial_i h_{ik}(x))^2 \right| \\
\leq C |h(x)|^2 |\partial^2 h(x)| + C |h(x)| |\partial h(x)|^2 \\
\leq C \mu^3 (\lambda + |x|)^{10}
\]
for $|x| \leq \rho$. This implies

$$\left| \int_{\mathbb{R}^n} R_g u_{(\xi, \varepsilon)}^2 + \int_{B_\rho(0)} \frac{1}{4} \sum_{i,k,l=1}^n (\partial h_{ik})^2 u_{(\xi, \varepsilon)}^2 \right|$$

$$\leq C \lambda^{12} \mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{14-2n} + C \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{4-2n}$$

$$\leq C \lambda^{12} \mu^3 + C \rho^2 \left(\frac{\lambda}{\rho}\right)^{n-2}.$$ 

At this point, we use the formula

$$\partial_i u_{(\xi, \varepsilon)} \partial_k u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} \partial_i \partial_k (u_{(\xi, \varepsilon)}^2)$$

$$= \frac{1}{n} \left( |du_{(\xi, \varepsilon)}|^2 - \frac{n-2}{4(n-1)} \Delta (u_{(\xi, \varepsilon)}^2) \right) \delta_{ik}.$$ 

Since $h_{ik}$ is trace-free, we obtain

$$\sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi, \varepsilon)} \partial_k u_{(\xi, \varepsilon)} = \frac{n-2}{4(n-1)} \sum_{i,k=1}^n h_{ik} \partial_i \partial_k (u_{(\xi, \varepsilon)}^2);$$

hence

$$\int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi, \varepsilon)} \partial_k u_{(\xi, \varepsilon)} = \int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} \sum_{i,k=1}^n \partial_i \partial_k h_{ik} u_{(\xi, \varepsilon)}^2.$$ 

Since $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$, it follows that

$$\left| \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi, \varepsilon)} \partial_k u_{(\xi, \varepsilon)} \right| \leq C \int_{\mathbb{R}^n \setminus B_\rho(0)} u_{(\xi, \varepsilon)}^2 \leq C \rho^2 \left(\frac{\lambda}{\rho}\right)^{n-2}.$$ 

Putting these facts together, the assertion follows.

**Corollary 14.** The function $F_g(\xi, \varepsilon)$ satisfies the estimate

$$\left| F_g(\xi, \varepsilon) - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi, \varepsilon)} \partial_k u_{(\xi, \varepsilon)} \right.$$ 

$$+ \int_{B_\rho(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial h_{ik})^2 u_{(\xi, \varepsilon)}^2$$

$$- \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi, \varepsilon)} u_{(\xi, \varepsilon)}$$

$$\leq C \lambda^{\frac{3n-2}{2}} \mu_{\frac{n-2}{2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2}$$

for $(\xi, \varepsilon) \in \lambda \Omega$.

**Proof.** This follows by combining Proposition 11, Proposition 12 and Proposition 13. \[\square\]
4. Finding a critical point of an auxiliary function

We define a function \( F : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \) as follows: given any pair \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \), we define

\[
F(\xi, \varepsilon) = \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^{n} H_{il}(x) H_{kl}(x) \partial_i u(\xi, \varepsilon)(x) \partial_k u(\xi, \varepsilon)(x)
- \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^{n} (\partial_i H_{ik}(x))^2 u(\xi, \varepsilon)(x)^2
+ \int_{\mathbb{R}^n} \sum_{i,k=1}^{n} H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon)(x) z(\xi, \varepsilon)(x),
\]

where \( z(\xi, \varepsilon) \in \mathcal{E}(\xi, \varepsilon) \) satisfies the relation

\[
\int_{\mathbb{R}^n} \left( \langle dz(\xi, \varepsilon), d\psi \rangle - n(n+2) u(\xi, \varepsilon)(x) \frac{n+4}{n+2} z(\xi, \varepsilon) \psi \right)
= - \int_{\mathbb{R}^n} \sum_{i,k=1}^{n} H_{ik}(x) \partial_i \partial_k u(\xi, \varepsilon) \psi
\]

for all test functions \( \psi \in \mathcal{E}(\xi, \varepsilon) \). Our goal in this section is to show that the function \( F(\xi, \varepsilon) \) has a critical point.

**Proposition 15.** The function \( F(\xi, \varepsilon) \) satisfies \( F(-\xi, \varepsilon) = F(\xi, \varepsilon) \) for all \( (\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty) \). Consequently, we have \( \partial_{\xi_p} F(0, \varepsilon) = 0 \) and \( \partial^2_{\varepsilon \xi_p} F(0, \varepsilon) = 0 \) for all \( \varepsilon > 0 \) and \( p = 1, \ldots, n \).

**Proof.** This follows immediately from the relation \( \mathbf{H}_{ik}(-x) = \mathbf{H}_{ik}(x) \). \( \square \)

**Proposition 16.** We have

\[
\int_{\partial B_r(0)} \sum_{i,k,l=1}^{n} (\partial_i H_{ik}(x))^2 x_p x_q
= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilik}) r^{n+3}
+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+3}
\]

and

\[
\int_{\partial B_r(0)} \sum_{i,k=1}^{n} H_{ik}(x)^2 x_p x_q
= \frac{2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilik}) r^{n+5}
+ \frac{1}{2n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+5}.
\]
Proof. By definition of $H_{ik}(x)$, we have

$$\int_{\partial B_r(0)} \sum_{i,k,l=1}^{n} (\partial_l H_{ik}(x))^2 x_p x_q$$

$$= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^{n} (W_{ijkl} + W_{iljk}) (W_{imkl} + W_{ilmk}) x_j x_m x_p x_q$$

$$= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilqk}) r^{n+3}$$

$$+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+3}.$$ 

Moreover, it follows from Corollary 29 that

$$\int_{\partial B_r(0)} \sum_{i,k=1}^{n} H_{ik}(x)^2 x_p x_q$$

$$= \int_{\partial B_r(0)} \sum_{i,j,k,l,m,s=1}^{n} W_{ijkl} W_{imks} x_j x_l x_m x_s x_p x_q$$

$$= \frac{2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilqk}) r^{n+5}$$

$$+ \frac{1}{2n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+5}.$$ 

This completes the proof. \hfill \Box

Proposition 17. We have

$$\int_{\partial B_r(0)} \sum_{i,k,l=1}^{n} (\partial_l \overline{H}_{ik}(x))^2 x_p x_q$$

$$= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilqk})$$

$$\times \left[ r^{n+3} - 2(n+8) \frac{n+8}{n+4} r^{n+5} + \frac{n+16}{n+4} r^{n+7} \right]$$

$$+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq}$$

$$\times \left[ r^{n+3} - 2(n+6) \frac{n+6}{n+4} r^{n+5} + \frac{n+10}{n+4} r^{n+7} \right].$$ 

Proof. Using the identity

$$\partial_l \overline{H}_{ik}(x) = (1 - |x|^2) \partial_l H_{ik}(x) - 2 H_{ik}(x) x_l$$
and Euler’s theorem, we obtain
\[
\sum_{i,k,l=1}^{n} (\partial_{x}^{3} H_{ik}(x) )^{2} = (1 - |x|^{2})^{2} \sum_{i,k,l=1}^{n} (\partial_{x}^{2} H_{ik}(x) )^{2} - 4 (1 - |x|^{2}) \sum_{i,k,l=1}^{n} H_{ik}(x) x_{i} \partial_{x}^{2} H_{ik}(x) + 4 |x|^{2} \sum_{i,k=1}^{n} H_{ik}(x) ^{2} = (1 - |x|^{2})^{2} \sum_{i,k,l=1}^{n} (\partial_{x}^{2} H_{ik}(x) )^{2} - 4 (2 - 3 |x|^{2}) \sum_{i,k=1}^{n} H_{ik}(x) ^{2}.
\]

Hence, the assertion follows from the previous proposition. \qed

Corollary 18. We have
\[
\int_{\partial B_{r}(0)} \sum_{i,k,l=1}^{n} (\partial_{x}^{3} H_{ik}(x) )^{2} = \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{iklj})^{2} \cdot \left[ r^{n+1} - \frac{2(n+4)}{n+2} r^{n+3} + \frac{n+8}{n+2} r^{n+5} \right].
\]

Proposition 19. We have
\[
F(0, \varepsilon) = - \frac{(n-2)(n+4)}{16n(n-1)(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{iklj})^{2} \cdot \left[ \frac{n-8}{n+4} \varepsilon^{4} - 2 \varepsilon^{6} + \frac{n+8}{n-10} \varepsilon^{8} \right] \int_{0}^{\infty} (1 + r^{2})^{2-n} r^{n+3} dr.
\]

Proof. Note that \( z_{(0,\varepsilon)}(x) = 0 \) for all \( x \in \mathbb{R}^{n} \). This implies
\[
F(0, \varepsilon) = - \int_{\mathbb{R}^{n}} \frac{n-2}{16(n-1)} \varepsilon^{n-2} (\varepsilon^{2} + |x|^{2})^{2-n} \sum_{i,k,l=1}^{n} (\partial_{x}^{2} H_{ik}(x) )^{2}.
\]

Using Corollary 18, we obtain
\[
\int_{\mathbb{R}^{n}} \varepsilon^{n-2} (\varepsilon^{2} + |x|^{2})^{2-n} \sum_{i,k,l=1}^{n} (\partial_{x}^{3} H_{ik}(x) )^{2} = \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{iklj})^{2} \cdot \int_{0}^{\infty} (1 + r^{2})^{2-n} \left[ \varepsilon^{4} r^{n+1} - \frac{2(n+4)}{n+2} \varepsilon^{6} r^{n+3} + \frac{n+8}{n+2} \varepsilon^{8} r^{n+5} \right] dr.
\]

Moreover, we have
\[
\int_{0}^{\infty} (1 + r^{2})^{2-n} r^{n+1} dr = \frac{n-8}{n+2} \int_{0}^{\infty} (1 + r^{2})^{2-n} r^{n+3} dr
\]
and
\[
\int_{0}^{\infty} (1 + r^{2})^{2-n} r^{n+5} dr = \frac{n+4}{n-10} \int_{0}^{\infty} (1 + r^{2})^{2-n} r^{n+3} dr
\]
by Proposition 27. From this the assertion follows. \qed
Corollary 20. Assume that \( n \geq 52 \). Moreover, suppose that \( \varepsilon_* > 0 \) is defined by
\[
3 + \sqrt{9 - \frac{8(n + 8)(n - 8)}{(n + 4)(n - 10)}} \varepsilon_*^2 = \frac{2(n - 8)}{n + 4}.
\]
Then \( (0, \varepsilon_*) \) is a critical point of the function \( F(\xi, \varepsilon) \). Moreover, we have \( \frac{\partial^2}{\partial \varepsilon^2} F(0, \varepsilon_*) > 0 \).

In the next step, we show that \( (0, \varepsilon_*) \) is a strict local minimum of the function \( F \). To that end, we compute the Hessian of \( F \) at a point \( (0, \varepsilon) \).

Proposition 21. The second order partial derivatives of the function \( F(\xi, \varepsilon) \) are given by
\[
\frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) = \int_{\mathbb{R}^n} \left( n - 2 \right)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i=1}^n \mathcal{H}_{pl}(x) \mathcal{H}_{ql}(x)
- \int_{\mathbb{R}^n} \left( \frac{n-2}{4} \right)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \mathcal{H}_{i k}(x))^2 x_p x_q
+ \int_{\mathbb{R}^n} \left( \frac{n-2}{8(n-1)} \right)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \mathcal{H}_{i k}(x))^2 \delta_{pq}.
\]

Proof. Using the identity
\[
\sum_{i,k,l=1}^n \mathcal{H}_{il}(x) \mathcal{H}_{kl}(x) \partial_i u(\xi, \varepsilon) \partial_k u(\xi, \varepsilon)(x)
= (n - 2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \mathcal{H}_{il}(x) \mathcal{H}_{kl}(x) (x_i - \xi_i) (x_k - \xi_k)
= (n - 2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \mathcal{H}_{il}(x) \mathcal{H}_{kl}(x) \xi_i \xi_k,
\]
we obtain
\[
\frac{\partial^2}{\partial \xi_p \partial \xi_q} \left( \frac{1}{2} \sum_{i,k,l=1}^n \mathcal{H}_{il}(x) \mathcal{H}_{kl}(x) \partial_i u(\xi, \varepsilon) \partial_k u(\xi, \varepsilon)(x) \right) \bigg|_{\xi=0}
= (n - 2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \mathcal{H}_{il}(x) \mathcal{H}_{ql}(x).
\]

Moreover, we have
\[
\frac{\partial^2}{\partial \xi_p \partial \xi_q} \left( \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \mathcal{H}_{i k}(x))^2 u(\xi, \varepsilon)(x)^2 \right) \bigg|_{\xi=0}
= \frac{(n-2)^2}{4} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \mathcal{H}_{i k}(x))^2 x_p x_q
- \frac{(n-2)^2}{8(n-1)} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \mathcal{H}_{i k}(x))^2 \delta_{pq}.
\]
Finally, we have
\[ \sum_{i,k=1}^{n} \mathcal{H}_{i,k}(x) \partial_{i} \partial_{k} u(\xi, \varepsilon)(x) \]
\[ = n(n-2) \varepsilon^{\frac{n+4}{2}} (\varepsilon^2 + |x|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^{n} \mathcal{H}_{i,k}(x) (x_i - \xi_i) (x_k - \xi_k) \]
\[ = n(n-2) \varepsilon^{\frac{n+4}{2}} (\varepsilon^2 + |x|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^{n} \mathcal{H}_{i,k}(x) \xi_i \xi_k \]
since \( \mathcal{H}_{i,k}(x) \) is trace-free. Thus, we conclude that
\[ \frac{\partial^2}{\partial \xi_p \partial \xi_q} \left( \sum_{i,k=1}^{n} \mathcal{H}_{i,k}(x) \partial_{i} \partial_{k} u(\xi, \varepsilon)(x) z(\xi, \varepsilon)(x) \right) \bigg|_{\xi=0} \]
\[ = 2n(n-2) \varepsilon^{\frac{n-2}{2}} (\varepsilon^2 + |x|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^{n} \mathcal{H}_{i,k}(x) z(0, \varepsilon)(x) = 0. \]
From this the assertion follows. \( \square \)

**Proposition 22.** The second order partial derivatives of the function \( F(\xi, \varepsilon) \) are given by
\[ \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) \]
\[ = \frac{4(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{dlkp}) (W_{iqkl} + W_{ilkq}) \]
\[ \cdot \left[ \varepsilon^4 - \frac{3(n+6)}{2(n-8)} \varepsilon^6 \right] \int_{0}^{\infty} (1 + r^2)^{-n} r^{n+5} dr \]
\[ + \frac{(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \]
\[ \cdot \left[ \varepsilon^4 - \frac{n+7}{n-8} \varepsilon^6 \right] \int_{0}^{\infty} (1 + r^2)^{-n} r^{n+5} dr. \]

**Proof.** Using the identity
\[ \int_{\partial B_r(0)} \sum_{l=1}^{n} \mathcal{H}_{pl}(x) \mathcal{H}_{ql}(x) \]
\[ = \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^{n} W_{ipkl} W_{jqml} x_i x_j x_k x_m (1 - |x|^2)^2 \]
\[ = \frac{1}{n(n+2)} |S^{n-1}| \]
\[ \cdot \sum_{i,j,k,l,m=1}^{n} W_{ipkl} W_{jqml} (\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) r^{n+3} (1 - r^2)^2 \]
\[ = \frac{1}{2n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{dlkp}) (W_{iqkl} + W_{ilkq}) r^{n+3} (1 - r^2)^2, \]
we obtain

$$\int_{\mathbb{R}^n} \varepsilon^{-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^{n} \overline{H}_{pl}(x) \overline{H}_{ql}(x)$$

$$= \frac{1}{2n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilikq})$$

$$\cdot \int_{0}^{\infty} (1 + r^2)^{-n} \left[ \varepsilon^2 r^{n+3} - 2 \varepsilon^4 r^{n+5} + \varepsilon^6 r^{n+7} \right] dr.$$

Similarly, it follows from Proposition 17 that

$$\int_{\mathbb{R}^n} \varepsilon^{-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^{n} \left( \partial_l \overline{H}_{ik}(x) \right)^2 x_p x_q$$

$$= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^{n} (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilikq})$$

$$\cdot \int_{0}^{\infty} (1 + r^2)^{-n} \left[ \varepsilon^2 r^{n+3} - \frac{2(n+8)}{n+4} \varepsilon^4 r^{n+5} + \frac{n+16}{n+4} \varepsilon^6 r^{n+7} \right] dr$$

$$+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilikj})^2 \delta_{pq}$$

$$\cdot \int_{0}^{\infty} (1 + r^2)^{-n} \left[ \varepsilon^2 r^{n+3} - \frac{2(n+6)}{n+4} \varepsilon^4 r^{n+5} + \frac{n+10}{n+4} \varepsilon^6 r^{n+7} \right] dr.$$

Moreover, we have

$$\int_{\mathbb{R}^n} \varepsilon^{-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^{n} \left( \partial_l \overline{H}_{ik}(x) \right)^2 \delta_{pq}$$

$$= \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilikj})^2 \delta_{pq}$$

$$\cdot \int_{0}^{\infty} (1 + r^2)^{1-n} \left[ \varepsilon^2 r^{n+1} - \frac{2(n+4)}{n+2} \varepsilon^4 r^{n+3} + \frac{n+8}{n+2} \varepsilon^6 r^{n+5} \right] dr.$$

by Corollary 18. Using Proposition 21 and the identity

$$\int_{0}^{\infty} (1 + r^2)^{1-n} r^{n+1} dr = \frac{2(n-1)}{n+2} \int_{0}^{\infty} (1 + r^2)^{-n} r^{n+3} dr,$$
we obtain
\[
\frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) = \frac{4(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})(W_{ijkl} + W_{iljk}) \cdot \int_0^\infty (1 + r^2)^{-n} \left[ \varepsilon^4 r^{n+5} - \frac{3}{2} \varepsilon^6 r^{n+7} \right] dr
\]
\[
+ \frac{(n-2)^2}{4n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{iljk})^2 \delta_{pq} \cdot \int_0^\infty (1 + r^2)^{-n} \left[ \frac{2(n+6)}{n+4} \varepsilon^4 r^{n+5} - \frac{n+10}{n+4} \varepsilon^6 r^{n+7} \right] dr
\]
\[
- \frac{(n-2)^2}{8n(n-1)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{iljk})^2 \delta_{pq} \cdot \int_0^\infty (1 + r^2)^{1-n} \left[ \frac{2(n+4)}{n+2} \varepsilon^4 r^{n+3} - \frac{n+8}{n+2} \varepsilon^6 r^{n+5} \right] dr.
\]
Hence, the assertion follows from the identities
\[
\int_0^\infty (1 + r^2)^{-n} r^{n+7} dr = \frac{n+6}{n-8} \int_0^\infty (1 + r^2)^{-n} r^{n+5} dr,
\]
\[
\int_0^\infty (1 + r^2)^{1-n} r^{n+3} dr = \frac{2(n-1)}{n+4} \int_0^\infty (1 + r^2)^{-n} r^{n+5} dr,
\]
\[
\int_0^\infty (1 + r^2)^{1-n} r^{n+5} dr = \frac{2(n-1)}{n-8} \int_0^\infty (1 + r^2)^{-n} r^{n+5} dr.
\]

\[\square\]

**Corollary 23.** Assume that \( n \geq 52 \) and \( \varepsilon_* > 0 \) is defined by (6). Then the function \( F(\xi, \varepsilon) \) has a strict local minimum at the point \((0, \varepsilon_*)\).

**Proof.** It follows from Corollary 20 that \((0, \varepsilon_*)\) is a critical point of the function \( F(\xi, \varepsilon) \). Moreover, we have \( \frac{\partial^2}{\partial \varepsilon^2} F(0, \varepsilon_*) > 0 \). Since \( n \geq 52 \), we have
\[
\frac{6}{n+4} < \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}}.
\]
This implies
\[
\frac{3(n+6)}{n+4} \varepsilon_*^2 < \left( 3 + \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}} \right) \varepsilon_*^2 = \frac{2(n-8)}{n+4} \varepsilon_*^2.
\]
Thus, we conclude that
\[
\frac{n+7}{n-8} \varepsilon_*^2 < \frac{3(n+6)}{2(n-8)} \varepsilon_*^2 < 1.
\]
Hence, it follows from Proposition 22 that the matrix \( \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon_*) \) is positive definite. This proves the assertion. \(\square\)
5. Proof of the main theorem

Proposition 24. Assume that \( n \geq 52 \). Moreover, let \( g \) be a smooth metric on \( \mathbb{R}^n \) of the form \( g(x) = \exp(h(x)) \), where \( h(x) \) is a trace-free symmetric two-tensor on \( \mathbb{R}^n \) such that \( |h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1 \) for all \( x \in \mathbb{R}^n \), \( h(x) = 0 \) for \( |x| \geq 1 \), and \( h_{ik}(x) = \mu (\lambda^2 - |x|^2) H_{ik}(x) \) for \( |x| \leq \rho \). As above, we assume that \( \lambda \leq \rho \leq 1 \) and \( \mu \leq 1 \). If \( \alpha \) and \( \rho^{2-n} \mu^{-2} \lambda^{n-10} \) are sufficiently small, then there exists a positive function \( v \) such that

\[
\Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n-2) v^{\frac{n+2}{n-2}} = 0,
\]

and \( \sup_{|x|\leq \lambda} v(x) \geq c \lambda^{\frac{2-n}{2}} \). Here, \( c \) is a positive constant that depends only on \( n \).

Proof. By Corollary [23], the function \( F(\xi, \varepsilon) \) has a strict local minimum at \( (0, \varepsilon_+) \). Hence, we can find an open set \( \Omega' \subset \Omega \) such that \( (0, \varepsilon_+) \in \Omega' \) and

\[
F(0, \varepsilon_+) < \inf_{(\xi, \varepsilon) \in \partial \Omega'} F(\xi, \varepsilon) < 0.
\]

Using Corollary [14], we obtain

\[
|F_g(\lambda \xi, \lambda \varepsilon) - \lambda^8 \mu^2 F(\xi, \varepsilon)| \\
\leq C \lambda^{\frac{8}{2-n}} \mu \frac{\lambda^2}{\rho^2} + C \lambda^4 \mu \left( \frac{\lambda}{\rho} \right)^{\frac{n}{2}} + C \left( \frac{\lambda}{\rho} \right)^{-2}
\]

for all \( (\xi, \varepsilon) \in \Omega \). This implies

\[
|\lambda^{-8} \mu^{-2} F_g(\lambda \xi, \lambda \varepsilon) - F(\xi, \varepsilon)| \\
\leq C \lambda^{\frac{8}{2-n}} \mu \frac{\lambda^2}{\rho^2} + C \rho^{\frac{2-n}{2}} \mu^{-1} \lambda^{\frac{n}{2}} + C \rho^{\frac{n}{2}} \mu^{-2} \lambda^{-10}
\]

for all \( (\xi, \varepsilon) \in \Omega \). Hence, if \( \rho^{2-n} \mu^{-2} \lambda^{n-10} \) is sufficiently small, then we have

\[
F_g(0, \lambda \varepsilon_+) < \inf_{(\xi, \varepsilon) \in \partial \Omega'} F_g(\lambda \xi, \lambda \varepsilon) < 0.
\]

Consequently, there exists a point \( (\xi, \varepsilon) \in \Omega' \) such that

\[
F_g(\lambda \xi, \lambda \varepsilon) = \inf_{(\xi, \varepsilon) \in \Omega'} F_g(\lambda \xi, \lambda \varepsilon) < 0.
\]

By Proposition [6], the function \( v = v(\lambda \xi, \lambda \varepsilon) \) is a non-negative weak solution of the partial differential equation

\[
\Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n-2) v^{\frac{n+2}{n-2}} = 0.
\]

Using a result of N. Trudinger, we conclude that \( v \) is smooth (see [20], Theorem 3 on p. 271). Moreover, we have

\[
2(n-2) \int_{\mathbb{R}^n} v^{\frac{2n}{n+2}} = 2(n-2) \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{n-2}} + F_g(\lambda \xi, \lambda \varepsilon) \\
< 2(n-2) \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{n-2}}.
\]
Finally, it follows from Proposition 5 that \( \| v - u_{(\lambda \xi, \lambda \psi)} \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \alpha \). This implies
\[
|B_\lambda(0)|^{\frac{n-2}{2}} \sup_{|x| \leq \lambda} v(x) \geq \| v \|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} \geq \| u_{(\lambda \xi, \lambda \psi)} \|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} - C \alpha.
\]
Hence, if \( \alpha \) is sufficiently small, then we obtain \( \lambda^{\frac{n-2}{2}} \sup_{|x| \leq \lambda} v(x) \geq c \).  

**Proposition 25.** Let \( n \geq 52 \). Then there exists a smooth metric \( g \) on \( \mathbb{R}^n \) with the following properties:

(i) \( g_{ik}(x) = \delta_{ik} \) for \( |x| \geq \frac{1}{2} \),
(ii) \( g \) is not conformally flat,
(iii) there exists a sequence of non-negative smooth functions \( v_\nu \) (\( \nu \in \mathbb{N} \)) such that
\[
\Delta_g v_\nu - \frac{n-2}{4(n-1)} R_g v_\nu + n(n-2) v_\nu^{\frac{n+2}{n-2}} = 0
\]
for all \( \nu \in \mathbb{N} \),
\[
\int_{\mathbb{R}^n} v_\nu^{\frac{2n}{n-2}} \leq \left( \frac{Y(S^n)}{4n(n-1)} \right)^{\frac{2}{n}}
\]
for all \( \nu \in \mathbb{N} \), and \( \sup_{|x| \leq 1} v_\nu(x) \to \infty \) as \( \nu \to \infty \).

**Proof.** Choose a smooth cutoff function \( \eta : \mathbb{R} \to \mathbb{R} \) such that \( \eta(t) = 1 \) for \( t \leq 1 \) and \( \eta(t) = 0 \) for \( t \geq 2 \). We define a trace-free symmetric two-tensor on \( \mathbb{R}^n \) by
\[
h_{ik}(x) = \sum_{N=N_0}^{\infty} \eta(4N^2 |x-y_N|) 2^{-N} (2^{-N} - |x-y_N|^2) H_{ik}(x-y_N),
\]
where \( y_N = (\frac{1}{N}, 0, \ldots, 0) \in \mathbb{R}^n \). It is straightforward to verify that \( h(x) \) is \( C^\infty \) smooth.

Let \( \alpha \) be the constant appearing in Proposition 24. If \( N_0 \) is sufficiently large, then we have \( |h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \) for all \( x \in \mathbb{R}^n \) and \( h(x) = 0 \) for \( |x| \geq \frac{1}{2} \). Moreover, we have \( h_{ik}(x) = 2^{-N} (2^{-N} - |x-y_N|^2) H_{ik}(x-y_N) \) provided that \( N \geq N_0 \) and \( |x-y_N| \leq \frac{1}{2N} \). Hence, we can apply Proposition 24 with \( \lambda = 2^{-N/2}, \mu = 2^{-N}, \) and \( \rho = \frac{1}{2N} \). From this the assertion follows.

**Appendix A. An asymptotic expansion for the scalar curvature**

Suppose that \( h(x) \) is a trace-free symmetric two-tensor defined on \( \mathbb{R}^n \) satisfying \( |h(x)| \leq 1 \) for all \( x \in \mathbb{R}^n \). We define a Riemannian metric \( g \) on \( \mathbb{R}^n \) by \( g(x) = \exp(h(x)) \). In this section, we derive an approximate formula for the scalar curvature of this metric. A similar formula is derived in [2].

**Proposition 26.** Let \( R_g \) be the scalar curvature of \( g \). There exists a constant \( C \), depending only on \( n \), such that
\[
\left| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_l h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{4} \partial_i h_{ik} \partial_j h_{jk} \right| \\
\leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2.
\]

**Proof.** The Riemann curvature tensor is defined as
\[
\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^l_{jk} \Gamma^m_{il} - \Gamma^l_{ik} \Gamma^m_{jl}.
\]
Hence, the scalar curvature of \( g \) is given by

\[
R_g = g^{jk} (\partial_j \Gamma_{ik}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^l \Gamma^i_{il} - \Gamma_{jk}^l \Gamma^i_{jl}).
\]

Since \( h \) is trace-free, we have \( \det g(x) = 1 \) for all \( x \in \mathbb{R}^n \). This implies \( \Gamma_{ik}^l = \frac{1}{2} g^{il} \partial_k g_{dl} = \frac{1}{2} \partial_k \log \det g = 0 \). Therefore, we obtain

\[
R_g = g^{jk} \partial_j \Gamma_{ik}^l - g^{jk} \Gamma_{ik}^l \Gamma_{jl}^l = \partial_l (g^{jk} \Gamma_{jk}^l) + g^{jk} \Gamma_{ik}^l \Gamma_{jl}^l.
\]

Note that

\[
g^{jk} \Gamma_{jk}^l = g^{il} \partial_k g_{jl}.
\]

From this it follows that

\[
\left| \partial_l (g^{jk} \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \frac{1}{2} \partial_i (h_{il} \partial_k h_{kl}) + \frac{1}{2} \partial_i (h_{kl} \partial_k h_{il}) \right| \leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2,
\]

hence

\[
\left| \partial_l (g^{jk} \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_k h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{2} \partial_i h_{kl} \partial_k h_{il} \right| \leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2.
\]

Moreover, we have

\[
\left| g^{jk} \Gamma_{ik}^i \Gamma_{jl}^l + \frac{1}{4} \partial_i h_{ik} \partial_k h_{ik} - \frac{1}{2} \partial_i h_{il} \partial_k h_{il} \right| \leq C |h| |\partial h|^2.
\]

Putting these facts together, we obtain

\[
\left| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_k h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{4} \partial_i h_{kl} \partial_k h_{ik} \right| \leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2.
\]

This completes the proof. \( \square \)

**Appendix B. Some useful identities**

**Proposition 27.** Suppose that \( \alpha \) and \( \beta \) are real numbers satisfying \( 2\alpha - 2 > \beta + 1 > 0 \). Then

\[
\int_0^\infty (1 + r^2)^{1-\alpha} r^\beta \ dr = \frac{2\alpha - 2}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} r^\beta \ dr
\]

and

\[
\int_0^\infty (1 + r^2)^{-\alpha} r^{\beta+2} \ dr = \frac{\beta + 1}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} r^\beta \ dr.
\]

**Proof.** Using the fundamental theorem of calculus, we obtain

\[
0 = \int_0^\infty \frac{d}{dr} \left[ (1 + r^2)^{1-\alpha} r^{\beta+1} \right] \ dr
\]

\[
= (\beta + 1) \int_0^\infty (1 + r^2)^{1-\alpha} r^\beta \ dr - (2\alpha - 2) \int_0^\infty (1 + r^2)^{-\alpha} r^{\beta+2} \ dr.
\]

From this the assertion follows. \( \square \)

**Proposition 28.** Suppose that \( p(x) \) is a homogenous polynomial of degree \( d \). Then

\[
\int_{\partial B_1(0)} p(x) = \frac{1}{d(n + d - 2)} \int_{\partial B_1(0)} \Delta p(x).
\]
Proof. Using the divergence theorem, we obtain
\[
\int_{\partial B(1)} \Delta p(x) = (n + d - 2) \int_{B(1)} \Delta p(x) \\
= (n + d - 2) \int_{\partial B(1)} \sum_{k=1}^{n} x_k \partial_k p(x) \\
= d(n + d - 2) \int_{\partial B(1)} p(x).
\]
\[\square\]

Corollary 29. We have
\[
\int_{\partial B(1)} x_i x_j = \frac{1}{n} |S^{n-1}| \delta_{ij},
\]
\[
\int_{\partial B(1)} x_i x_j x_k x_l = \frac{1}{n(n + 2)} |S^{n-1}| (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]
and
\[
\int_{\partial B(1)} x_i x_j x_k x_l x_p x_q = \frac{1}{n(n + 2)(n + 4)} |S^{n-1}| (\delta_{ij} \delta_{kl} \delta_{pq} + \delta_{ij} \delta_{kp} \delta_{lq} + \delta_{ij} \delta_{kq} \delta_{lp} \\
+ \delta_{ik} \delta_{jl} \delta_{pq} + \delta_{ik} \delta_{jp} \delta_{lq} + \delta_{ik} \delta_{jq} \delta_{lp} \\
+ \delta_{il} \delta_{jk} \delta_{pq} + \delta_{il} \delta_{jp} \delta_{kq} + \delta_{il} \delta_{jq} \delta_{kp} \\
+ \delta_{ip} \delta_{jk} \delta_{lq} + \delta_{ip} \delta_{jl} \delta_{kq} + \delta_{ip} \delta_{jq} \delta_{kl} \\
+ \delta_{iq} \delta_{jk} \delta_{lp} + \delta_{iq} \delta_{jl} \delta_{kp} + \delta_{iq} \delta_{jp} \delta_{kl}).
\]

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References


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